The Derivative

Key Word

Tangent Line
Slope of a line
Limit

SECANT LINE AND TANGENT LINE

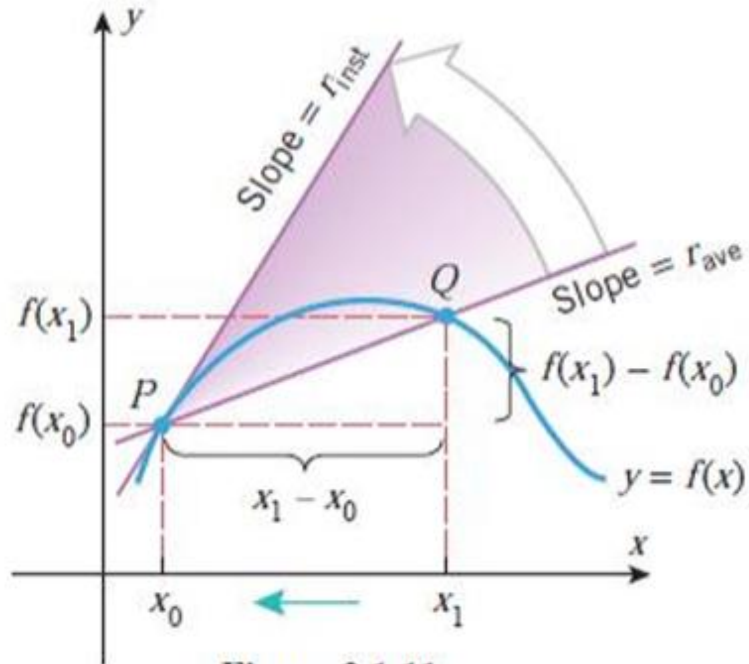


Figure 2.1.11

AVERAGE AND INSTANTANEOUS RATE OF CHANGE

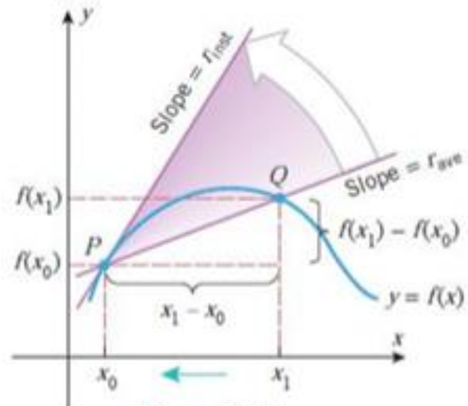


Figure 2.1.11

If $y = f(x)$, then we define the ***average rate of change of y with respect to x over t interval*** $[x_0, x_1]$ to be

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and we define the ***instantaneous rate of change of y with respect to x at*** x_0 to be

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Geometrically, the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is the slope of the secant line through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$ (Figure 2.1.11), and the instantaneous rate of change of y with respect to x at x_0 is the slope of the tangent line at the point $P(x_0, f(x_0))$ (since it is the limit of the slopes of the secant lines through P).

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If desired, we can let $h = x_1 - x_0$,

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

EXAMPLE 1

Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval [3, 5].

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

Solution (a).

$f(x) = x^2 + 1$, $x_0 = 3$, and $x_1 = 5$

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

Thus, y increases an average of 8 units per unit increase in x over the interval [3, 5].

Solution (b).

$f(x) = x^2 + 1$ and $x_0 = -4$.

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{f(x_1) - f(-4)}{x_1 - (-4)} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} \frac{(x_1 + 4)(x_1 - 4)}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Thus, a small increase in x from $x = -4$ will produce approximately an 8-fold decrease in y. ◀

Practice Problems

The Derivatives

1. Let $y = x^2 - 2x + 3$.

(a) Find the average rate of change of y with respect to x over the interval $[1, 3]$.

(b) Find the instantaneous rate of change of y with respect to x when $x = 2$.

2. Let $y = 5 + 3x - x^2$.

(a) Find the average rate of change of y with respect to x over the interval $[-1,1]$.

(b) Find the instantaneous rate of change of y with respect to x when $x = 0$.

DEFINITION OF DERIVATIVE FUNCTION

The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

DIFFERENTIABILITY

It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined. To account for this possibility we make the following definition.

DEFINITION

A function f is said to be ***differentiable at*** $\boldsymbol{x_0}$ if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on*** $(\boldsymbol{a}, \boldsymbol{b})$, and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is ***differentiable everywhere***.

Geometrically, a function f is differentiable at x_0 if the graph of f has a tangent line at x_0. Thus, f is not differentiable at any point x_0 where the secant lines from $P(x_0, f(x_0))$ to points $Q(x, f(x))$ distinct from P do not approach a unique *nonvertical* limiting position as $x \to x_0$. Figure 2.2.6 illustrates two common ways in which a function that is continuous at x_0 can fail to be differentiable at x_0. These can be described informally as

- corner points
- points of vertical tangency

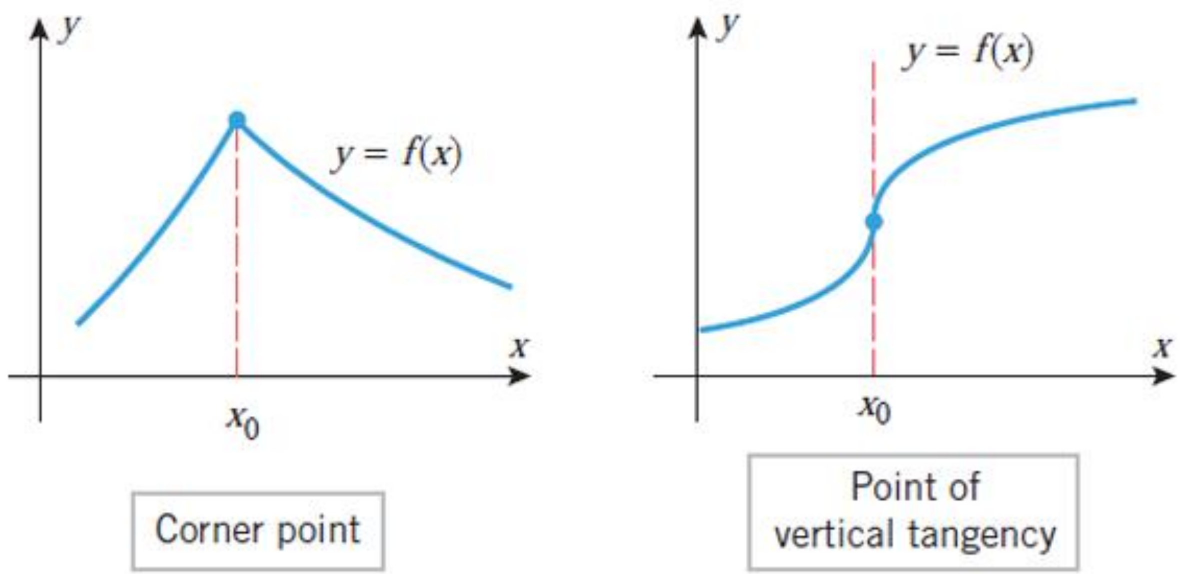


Figure 2.2.6

Relationship between Differentiability and Continuity

If a function f is differentiable at x_0, then f is continuous at x_0.

FORMULAE

1. $\frac{d}{dx}[c] = 0$, where c is a constant
2. $\frac{d}{dx}[x^n] = nx^{n-1}$, where n is a real number
3. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$, where c is a constant and $f(x)$ is differentiable at x
4. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$, where $f(x)$ and $g(x)$ are differentiable at x
5. $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$, where $f(x)$ and $g(x)$ are differentiable at x

Example 2

Find $\frac{dy}{dx}$

(a) $y = 2x^2 - \sqrt{x} + 3$

(b) $y = \frac{x^2+1}{5}$

Example 3

Find $f'(x)$

(a) $f(x) = -3x^{-5} + 2\sqrt[3]{x}$

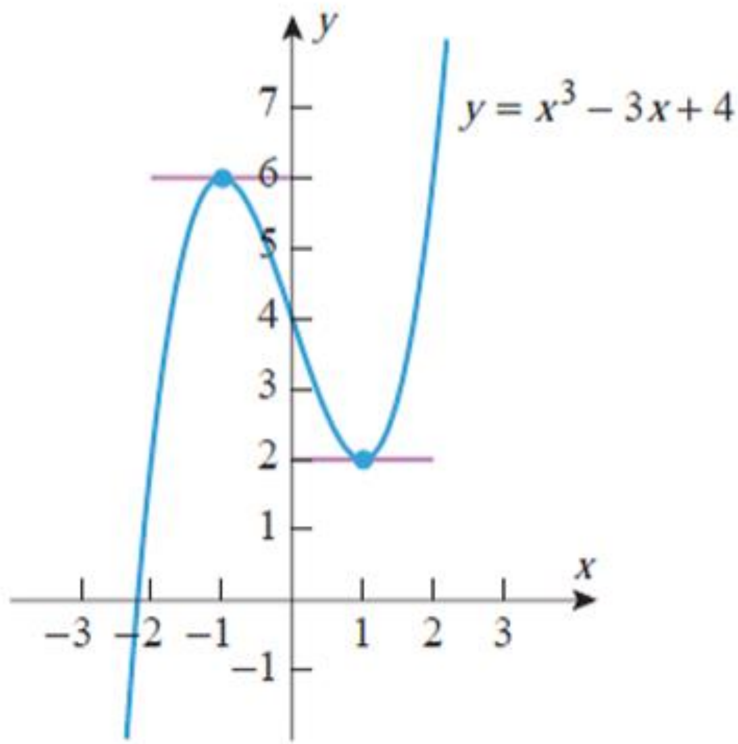
(b) $f(x) = (2x^2 - 3)^2$

Example 4

If $f(x) = x^2(x^4 - x)$, evaluate $f''(x)$.

EXAMPLE 5

At what point, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?



Example 6

Find $\left.\frac{dy}{dx}\right|_{x=1}$, if $y = \frac{1+x+x^2+x^3+x^4+x^5+x^6}{x^3}$.

Example 7

Find y''', when $y = ax^4 + bx^2 + c$.

FORMULAE

1. $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$, where $f(x)$ and $g(x)$ are differentiable at x

2. $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$, where $f(x)$ and $g(x)$ are differentiable at x and $g(x) \neq 0$

EXAMPLE 8

Find $f'(x)$

(a) $f(x) = (x + 2)(2x^2 - 3)$

(b) $f(x) = (2x^5 - x^2)(2x + 1)$

EXAMPLE 9

Find $\frac{dy}{dx}$

(a) $y = \frac{x^2+3}{x-1}$

(b) $y = (2x + 1)\left(1 + \frac{1}{x}\right)$

FORMULAE

1. $\frac{d}{dx}(e^{mx}) = me^{mx}$, where m is a constant.

2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example 10

Find $\frac{dy}{dx}$.

(a) $y = 2e^{x} - e^{4x} + \frac{1}{e^{3x}}$

(b) $y = \ln x + 3\ln(2x) - 7\ln\left(\frac{x}{7}\right)$

(c) $y = 33 - 3e^{-\frac{x}{6}} - \ln(4x)$

FORMULAE

1. $\frac{d}{dx}[sinx] = cosx$

2. $\frac{d}{dx}[cosx] = -sinx$

3. $\frac{d}{dx}[tanx] = sec^2x$

4. $\frac{d}{dx}[cotx] = -cosec^2x$

5. $\frac{d}{dx}[secx] = secxtanx$

6. $\frac{d}{dx}[cosecx] = -cosecxcotx$

EXAMPLE 11

Find $\frac{dy}{dx}$.

(a) $y = xsinx$

(b) $y = \frac{sinx}{1+cosx}$

(c) $y = tanx + 2secx - x^2$

Practice Problems

3. Find $\frac{dy}{dx}$.

(a) $y = \frac{1}{5}x^5 - 5 + \sqrt[3]{x}$

(b) $y = \frac{4x^3 - 12x + 1}{8}$

(c) $y = (2x - 3)(x^2 + 5)$

(d) $y = x\left(3 - \frac{2}{x}\right)$

(e) $y = \left(\frac{1}{x} + 2\right)(x - 3)$

(f) $y = \frac{x-3}{x+2}$

(g) $y = (2x^2 - 7x + 6)^8$

(h) $y = \frac{3-5x}{4+x^2}$

(i) $y = (x - 5)\sin x$

(j) $y = \frac{\cos x}{3x-4}$

(k) $y = 4\tan x + \ln(5x) + \sqrt[5]{x}$

(l) $y = e^{-x}\sin(3x - 2)$

4. Find $f'(x)$.

(a) $f(x) = (2x + 1)^{23}$

(b) $f(x) = x^{-4} + x^4 + 4$

Practice Problems

5. If $f(x) = x^2 - 7x + \frac{1}{x}$ then find the value of $f'''(x)$ at $x = -1$.

6. Use product rule to find $\frac{d^2y}{dx^2}$ at $x = 1$, when $y = x^3(3x - x^2)$.

7. Find $\frac{dy}{dx}$ at $x = -2$, if $y = \frac{2-x+3x^2+x^3-2x^4}{x^2}$.

8. Evaluate $f'''(x) = \ln x + 3x - e^{2x}$ at $x = 1$.