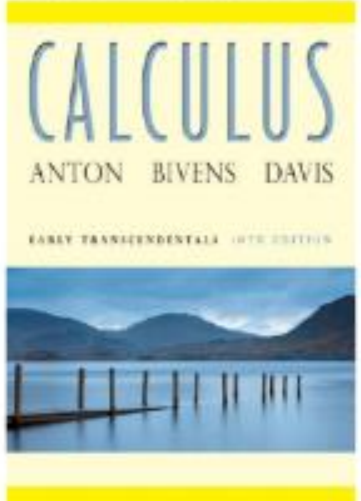
LIMIT AND CONTINUITY

Chapter 1

Limits and Continuity



LIMITS

LET US FOCUS ON THE LIMIT CONCEPT ITSELF

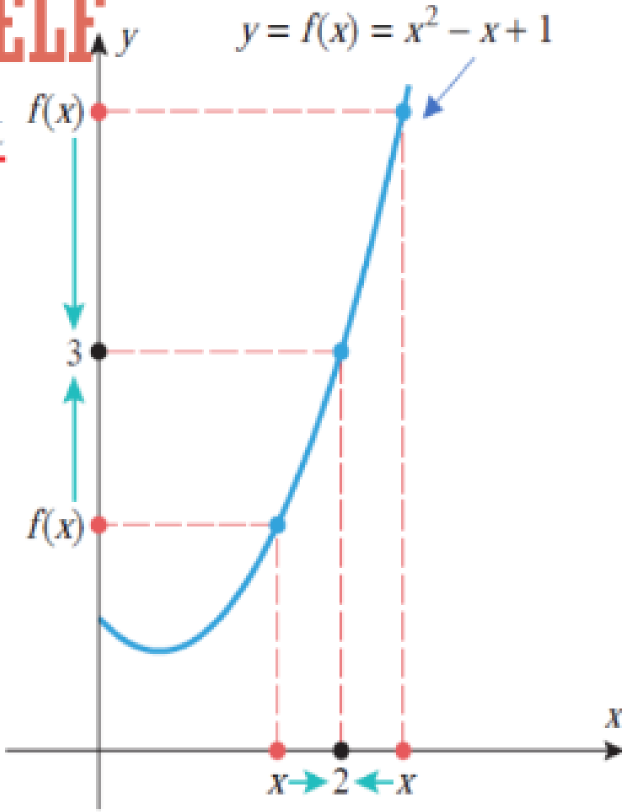
The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value.

For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 8 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

Figure 8

LIMITS

GENERAL IDEA

LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a$$

ONE-SIDED LIMITS

GENERAL IDEA

ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

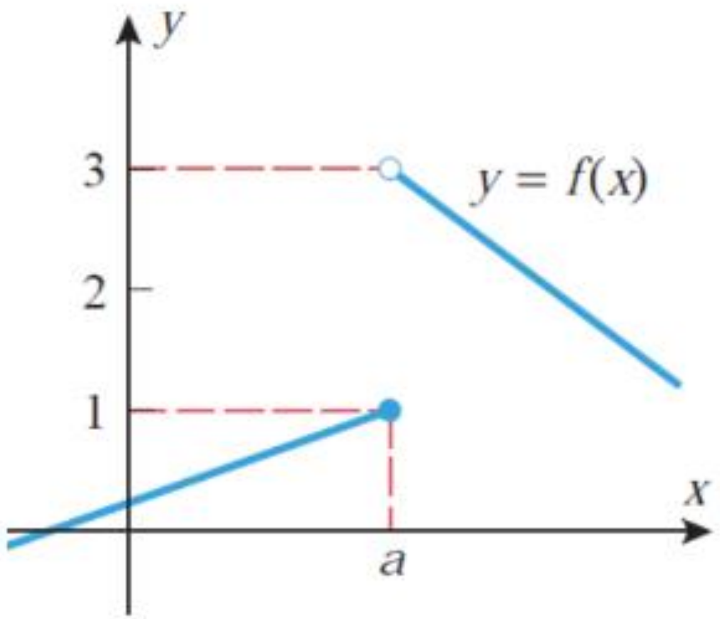
$$\lim_{x \to a^-} f(x) = L$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

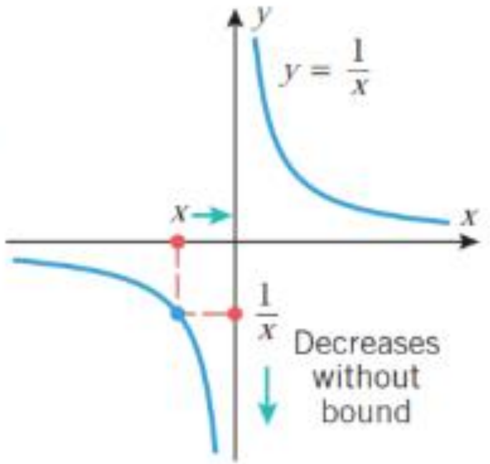


Explain why $\lim_{x \to a} f(x)$ does not exist.

INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound.

For example, consider the behavior of $f(x) = 1/x$ for values of x near 0.

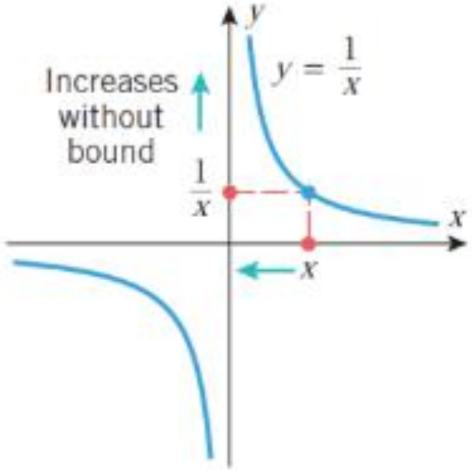


x	-1	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	-1	-10	-100	-1000	$-10{,}000$		10,000	1000	100	10	1

Left side → ← Right side

We describe these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$



INFINITE LIMITS

INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

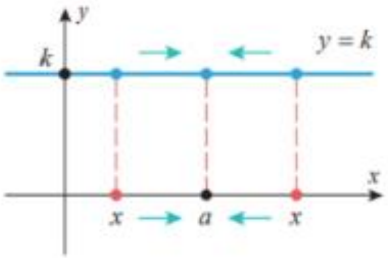
$$\lim_{x \to a} f(x) = -\infty$$

TECHNIQUES FOR COMPUTING LIMITS

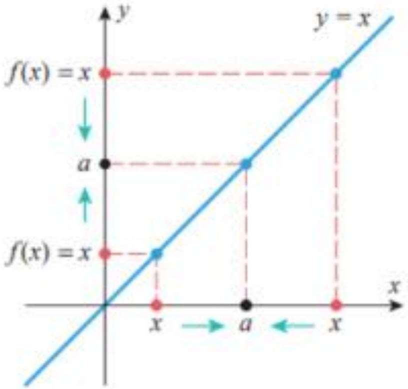
SOME BASIC LIMITS

THEOREM *Let a and k be real numbers.*

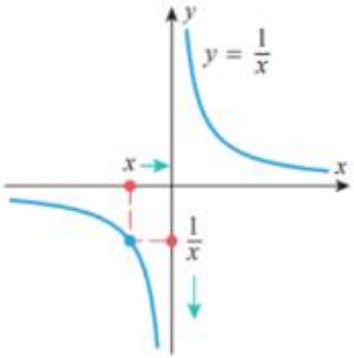
$$(a)\ \lim_{x \to a} k = k \qquad (b)\ \lim_{x \to a} x = a \qquad (c)\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad (d)\ \lim_{x \to 0^+} \frac{1}{x} = +\infty$$



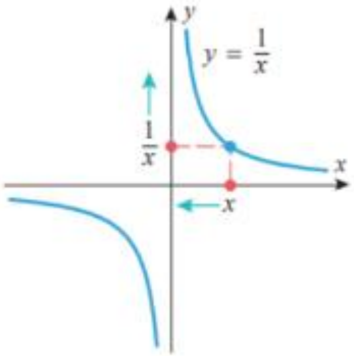
$\lim_{x \to a} k = k$



$\lim_{x \to a} x = a$



$\lim_{x \to 0^-} \frac{1}{x} = -\infty$



$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

TECHNIQUES FOR COMPUTING LIMITS

SOME BASIC LIMITS

THEOREM *Let a be a real number, and suppose that*

$$\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(*a*) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$

(*b*) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$

(*c*) $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$

(*d*) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L_1}{L_2}$, *provided* $L_2 \neq 0$

(*e*) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, *provided* $L_1 > 0$ *if* n *is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF POLYNOMIALS AS $x \to a$

THEOREM *For any polynomial*

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Example 3 Find $\lim_{x \to 5} (x^2 - 4x + 3)$.

Solution.

$$\begin{aligned}\lim_{x \to 5} (x^2 - 4x + 3) &= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3 \\ &= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3 \\ &= 5^2 - 4(5) + 3 \\ &= 8\end{aligned}$$

Find $\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35}$.

Answer: 0

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF RATIONAL FUNCTIONS AS $x \to a$

THEOREM *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(*a*) *If* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a)$.

(*b*) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

Example 4 Find $\lim_{x \to 2} \frac{5x^3 + 4}{x - 3}$.

Solution.

$$\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} = \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)}$$

$$= \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$$

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF RATIONAL FUNCTIONS AS $x \to a$

Example 5 Find $\lim_{x \to 4} \frac{2-x}{(x-4)(x+2)}$

Let $f(x) = \frac{p(x)}{q(x)}$.

If $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

Solution.

At $x = 4$, $(x-4)(x+2) = 0$

but $2 - x \neq 0$

$\therefore \lim_{x \to 4} \frac{2-x}{(x-4)(x+2)}$ does not exist.

Alternative Method

If we substitute $x = 4$, then the denominator becomes zero.

$$\lim_{x \to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty$$

$$\lim_{x \to 4^-} \frac{2-x}{(x-4)(x+2)} = +\infty$$

$$\therefore \lim_{x \to 4} \frac{2-x}{(x-4)(x+2)} \text{ does not exist.}$$

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF RATIONAL FUNCTIONS AS $x \to a$

Example 6 Find (a) $\lim\limits_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3}$ (b) $\lim\limits_{x \to -4} \dfrac{2x + 8}{x^2 + x - 12}$ (c) $\lim\limits_{x \to 5} \dfrac{x^2 - 3x - 10}{x^2 - 10x + 25}$

Try to solve the problem

Solution (a). $\lim\limits_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x-3)^2}{x - 3} = \lim\limits_{x \to 3} (x - 3) = 0$

Solution (b). $\lim\limits_{x \to -4} \dfrac{2x + 8}{x^2 + x - 12} = \lim\limits_{x \to -4} \dfrac{2(x+4)}{(x+4)(x-3)} = \lim\limits_{x \to -4} \dfrac{2}{x - 3} = -\dfrac{2}{7}$

Solution (c). $\lim\limits_{x \to 5} \dfrac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim\limits_{x \to 5} \dfrac{(x-5)(x+2)}{(x-5)(x-5)} = \lim\limits_{x \to 5} \dfrac{x + 2}{x - 5}$

However, $\lim\limits_{x \to 5} (x + 2) = 7 \neq 0$ and $\lim\limits_{x \to 5} (x - 5) = 0$

$$\lim_{x \to 5^+} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5^+} \frac{x + 2}{x - 5} = +\infty$$

$$\lim_{x \to 5^-} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5^-} \frac{x + 2}{x - 5} = -\infty$$

so $\lim\limits_{x \to 5} \dfrac{x^2 - 3x - 10}{x^2 - 10x + 25}$

$= \lim\limits_{x \to 5} \dfrac{x + 2}{x - 5}$ does not exist.

TECHNIQUES FOR COMPUTING LIMITS

LIMITS INVOLVING RADICALS

Example 7 Find $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Solution. $\dfrac{x-1}{\sqrt{x}-1} = \dfrac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$

$$= \frac{(x-1)(\sqrt{x}+1)}{x-1}$$

$$= \sqrt{x}+1$$

Therefore, $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x}-1} = \lim\limits_{x \to 1} (\sqrt{x}+1) = 2$

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF PIECEWISE-DEFINED FUNCTION

Example 8 Let $f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \le 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$

Find (a) $\lim_{x \to -2} f(x)$ (b) $\lim_{x \to 0} f(x)$ (c) $\lim_{x \to 3} f(x)$

Solution (a).

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

$\lim_{x \to -2} f(x)$ does not exist.

Solution (b).

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 5)$$

$$= 0^2 - 5$$

$$= -5$$

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF PIECEWISE-DEFINED FUNCTION

Example 8 Let $f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \le 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$

Find (a) $\lim_{x \to -2} f(x)$ (b) $\lim_{x \to 0} f(x)$ (c) $\lim_{x \to 3} f(x)$

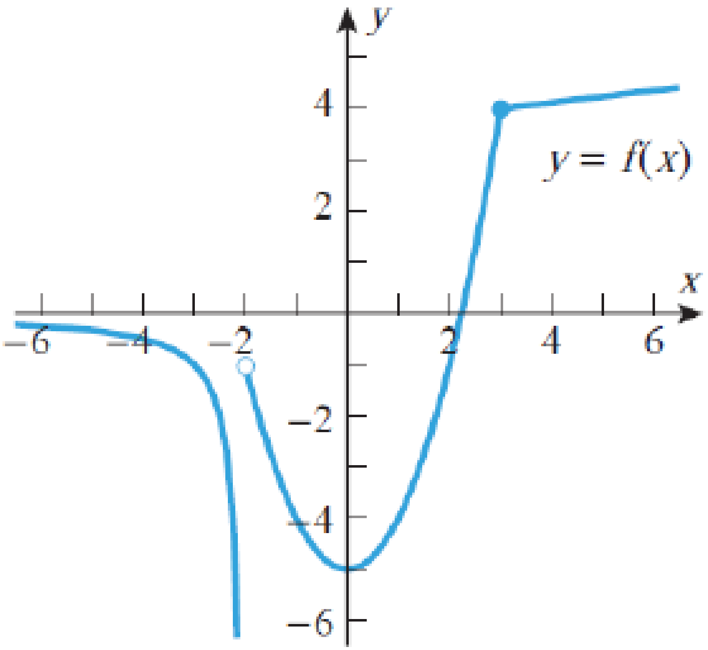
Solution (c).

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5) = 3^2 - 5 = 4$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+13} = \sqrt{\lim_{x \to 3^+} (x+13)}$$

$$= \sqrt{3+13} = 4$$

Therefore, $\lim_{x \to 3} f(x) = 4$



TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF POLYNOMIALS FUNCTIONS AS $x \to \pm\infty$

If $c_n \neq 0$, then

$$\lim_{x \to -\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to -\infty} c_n x^n$$

$$\lim_{x \to +\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to +\infty} c_n x^n$$

Example 9

$$\lim_{x \to -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \to -\infty} 7x^5 = -\infty$$

$$\lim_{x \to -\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x \to -\infty} -4x^8 = -\infty$$

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF RATIONAL FUNCTIONS AS $x \to \pm\infty$

Divide both numerator and denominator by the highest power of x in the denominator.

Example 10 Find $\displaystyle\lim_{x\to+\infty} \frac{3x+5}{6x-8}$.

Solution.

$$\lim_{x\to+\infty} \frac{3x+5}{6x-8} = \lim_{x\to+\infty} \frac{3+\frac{5}{x}}{6-\frac{8}{x}}$$

$$= \frac{\displaystyle\lim_{x\to+\infty}\left(3+\frac{5}{x}\right)}{\displaystyle\lim_{x\to+\infty}\left(6-\frac{8}{x}\right)}$$

$$= \frac{\displaystyle\lim_{x\to+\infty} 3 + \lim_{x\to+\infty} \frac{5}{x}}{\displaystyle\lim_{x\to+\infty} 6 - \lim_{x\to+\infty} \frac{8}{x}}$$

$$= \frac{3+5\displaystyle\lim_{x\to+\infty}\frac{1}{x}}{6-8\displaystyle\lim_{x\to+\infty}\frac{1}{x}}$$

$$= \frac{3+0}{6-0}$$

$$= \frac{1}{2}$$

TECHNIQUES FOR COMPUTING LIMITS

LIMITS OF RATIONAL FUNCTIONS AS $x \to \pm\infty$

Example 11 Find (a) $\displaystyle\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5}$ (b) $\displaystyle\lim_{x \to +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x}$

Solution (a).

$$\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x \to -\infty} \frac{\frac{4}{x} - \frac{1}{x^2}}{2 - \frac{5}{x^3}}$$

$$= \frac{\displaystyle\lim_{x \to -\infty} \frac{4}{x} - \lim_{x \to -\infty} \frac{1}{x^2}}{\displaystyle\lim_{x \to -\infty} 2 - \lim_{x \to -\infty} \frac{5}{x^3}}$$

$$= \frac{0 - 0}{2 - 0}$$

$$= 0$$

Solution (b).

$$\lim_{x \to +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x} = \lim_{x \to +\infty} \frac{\underline{5x^2 - 2x} + \frac{1}{x}}{\frac{1}{x} - 3}$$

$$\lim_{x \to +\infty} 5x^2 - 2x = +\infty,$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0,$$

$$\lim_{x \to +\infty} \left(\frac{1}{x} - 3\right) = -3$$

Therefore, $\displaystyle\lim_{x \to +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x} = -\infty$

TECHNIQUES FOR COMPUTING LIMITS

Example 12:

A function $f(x)$ is defined as follows:

$$f(x) = \tan\frac{x}{2} \quad \text{when } x < \frac{\pi}{2}$$

$$= 3 - \frac{\pi}{2} \quad \text{when } x = \frac{\pi}{2}$$

$$= \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}} \quad \text{when } x > \frac{\pi}{2}.$$

Prove that $\lim_{x \to \frac{\pi}{2}} f(x)$ does not exist.

Solution: Given that

$$f(x) = \tan\frac{x}{2} \quad \text{when } x < \frac{\pi}{2}$$

$$= 3 - \frac{\pi}{2} \quad \text{when } x = \frac{\pi}{2}$$

$$= \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}} \quad \text{when } x > \frac{\pi}{2}.$$

TECHNIQUES FOR COMPUTING LIMITS

$$L.H.L. = \lim_{x \to \frac{\pi}{2}^-} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \tan \frac{x}{2}$$

$$= \tan \frac{\frac{\pi}{2}}{2}$$

$$= \tan \frac{\pi}{4}$$

$$= 1$$

$$R.H.L. = \lim_{x \to \frac{\pi}{2}^+} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{x^3 - \left(\frac{\pi}{2}\right)^3}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\left(x - \frac{\pi}{2}\right)\left\{x^2 + x.\frac{\pi}{2} + \left(\frac{\pi}{2}\right)^2\right\}}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\left(x - \frac{\pi}{2}\right)\left(x^2 + \frac{\pi x}{2} + \frac{\pi^2}{4}\right)}{x - \frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}.\frac{\pi}{2} + \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4}$$

$$= \frac{\pi^2 + \pi^2 + \pi^2}{4}$$

$$= \frac{3\pi^2}{4}$$

Since $L.H.L. \neq R.H.L.$

Hence $\lim_{x \to \frac{\pi}{2}} f(x)$ does not exist. (Proved)

Practice Problem

1. Find the limits.

(a) $\lim_{x\to -1}(-2x^5 + x^3 - x + 1)$

(b) $\lim_{x\to 0}(5x^2 + 3x - 1)^{11}$

(c) $\lim_{x\to 2}\sqrt[3]{x^5 - 2x^4 + 5x + 17}$

(d) $\lim_{x\to -5}\frac{2x+5}{x+4}$

(e) $\lim_{x\to -4}\frac{2x+8}{x+4}$

(f) $\lim_{x\to 2}\frac{x^2-x-2}{x^2-2x}$

(g) $\lim_{x\to -5}\frac{x^2+6x+9}{x^2+2x-3}$

(h) $\lim_{x\to \infty}(x^6 - 2x^4 + 6x^3 - x^2)$

(i) $\lim_{x\to -\infty}(-5x^4 + 3x^2 - 11)^{35}$

Practice Problem

2. If $f(x) = \begin{cases} 2x-1, & x < -5 \\ \frac{5}{2x+5}, & -5 \le x \le 0 \\ x^2 - 3x + 1, & 0 < x < 5 \\ \sqrt{x+4}, & x > 5 \end{cases}$, then find **(a)** $\lim_{x \to -5} f(x)$. **(b)** $\lim_{x \to -1} f(x)$. **(c)** $\lim_{x \to 0} f(x)$. **(d)** $\lim_{x \to 2} f(x)$. **(e)** $\lim_{x \to 5} f(x)$ and **(f)** $\lim_{x \to 10} f(x)$.

3. A function $f(x)$ as follows:

$$f(x) = x^2 \quad \text{when } x < 1$$

$$= 2.5 \quad \text{when } x = 1$$

$$= x^2 + 2 \quad \text{when } x > 1.$$

Does $\lim_{x \to 1} f(x)$ exist?