LEIBNITZ THEOREM

KEYWORD:

DERIVATIVE OF A FUNCTION

Leibnitz Formula for the n^{th} Derivative of A Product

This formula expresses the n-th derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If u and v are functions of x, we have,

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Differentiating again with respect to x,

$$\frac{d^2}{dx^2}(uv) = \frac{d^2u}{dx^2}v + \frac{du}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

$$= \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Similarly,

$$\frac{d^3}{dx^3}(uv) = \frac{d^3u}{dx^3}v + \frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{du}{dx}\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}$$

$$= \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}$$

However far this process may be continued, it will be seen that the numerical coefficients follow the same law as those of the Binomial Theorem, and the indices of the derivatives correspond to the exponents of the Binomial Theorem. Reasoning then by mathematical induction from the m-th to the (m + 1)-st derivative of the product, we can prove Leibnitz's Formula

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n}\binom{n}{k}u^{n-k}v^k$$

$u^{n-k} = (n-k)th\ derivative\ of\ u$

$v^0 = only\ v$

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} v^k$$

$$\frac{d^n}{dx^n}(uv) = \binom{n}{0} u^{n-0} v^0 + \binom{n}{1} u^{n-1} v^1 + \binom{n}{2} u^{n-2} v^2 + \ \dots \ \dots$$

$$+ \binom{n}{n-1} u^1 v^{n-1} + \binom{n}{n} u^0 v^n$$

$$\frac{d^n}{dx^n}(uv) = \binom{n}{0}\left(\frac{d^n u}{dx^n}\right)(v) + \binom{n}{1}\left(\frac{d^{n-1} u}{dx^{n-1}}\right)\left(\frac{dv}{dx}\right) + \binom{n}{2}\left(\frac{d^{n-2} u}{dx^{n-2}}\right)\left(\frac{d^2 u}{dx^2}\right)$$

$$+ \ \dots \ \dots \ + \binom{n}{n-1}\left(\frac{du}{dx}\right)\left(\frac{d^{n-1} v}{dx^{n-1}}\right) + \binom{n}{n}(u)\left(\frac{d^n v}{dx^n}\right)$$

Leibnitz's Theorem

Leibnitz's Theorem:

If u and v are two functions of x, each possessing derivatives upto nth order, then the nth derivative of their product, i.e.,

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + {}^n c_3 u_{n-3} v_3 + \ldots\ldots + {}^n c_r u_{n-r} v_r + \ldots\ldots + u v_n ,$$

where the suffixes of u and v denote the order of differentiations of u and v with respect to x.

$$^n c_r = \frac{n!}{r!(n-r)!}$$

Problems

Example 1

Given $y = e^x lnx$, find $\frac{d^3y}{dx^3}$ by Liebnitz's Theorem.

Example 2

Given $y = x^5e^{2x}$, find $\frac{d^5y}{dx^5}$ by Liebnitz's Theorem.

1. If $y=\sin(m\sin^{-1}x)$, prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$.

Solution: Given that $y=\sin(m\sin^{-1}x)$

Differentiating both sides with respect to x, we get

$$y_1 = \cos(m\sin^{-1}x).m.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (\sqrt{1-x^2})y_1 = m\cos(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\{1-\sin^2(m\sin^{-1}x)\}$$

$$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2-m^2y^2$$

Again differentiating both sides with respect to x, we get

$$(1-x^2)2y_1y_2+y_1^2(0-2x)=0-m^2 2y\,y_1$$

$$\Rightarrow (1-x^2)2y_1y_2-2xy_1^2=-m^2 2y\,y_1$$

$$\Rightarrow (1-x^2)\,y_2-xy_1=-m^2\,y$$

$$\Rightarrow (1-x^2)\,y_2-xy_1+m^2\,y=0$$

$$(uv)_n=u_nv+{}^nc_1u_{n-1}v_1+{}^nc_2u_{n-2}v_2+{}^nc_3u_{n-3}v_3+\dots\dots.+{}^nc_ru_{n-r}v_r+\dots\dots.+uv_n$$

Differentiating n times with the help of Leibnitz's theorem, we get

$$(1-x^2)y_{n+2}+{}^nc_1 y_{n+1}(0-2x)+{}^nc_2 y_n(-2)-(xy_{n+1}+{}^nc_1 y_n.1)+m^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}+ny_{n+1}(0-2x)+\frac{n(n-1)}{2}y_n(-2)-(xy_{n+1}+ny_n)+m^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n+m^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-2nxy_{n+1}-(n^2-n)y_n-xy_{n+1}-ny_n+m^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-2nxy_{n+1}-n^2 y_n+ny_n-xy_{n+1}-ny_n+m^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-2nxy_{n+1}-n^2 y_n-xy_{n+1}+m^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-2nxy_{n+1}-xy_{n+1}+m^2 y_n-n^2 y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$$

Practice Problems (Leibnitz Theorem)

1 If $y = e^{\cos^{-1} x}$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0.$

2 If $y = a\cos(log x) + bsin(log x)$, prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$

3 If $y = \ln\left\{x + \sqrt{1+x^2}\right\}$, show that

(i) $(1+x^2)y_2 + xy_1 = 0$

(ii) $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$