ROLLE'S THEOREM; MEAN-VALUE THEOREM

Keywords:

1. Derivative of a Function
2. Tangent of a Curve

Rolle's Theorem

If the graph of a differentiable function intersects the x-axis at two places, a and b, then somewhere between a and b there must be at least one place where the tangent line is horizontal (Figure 1).

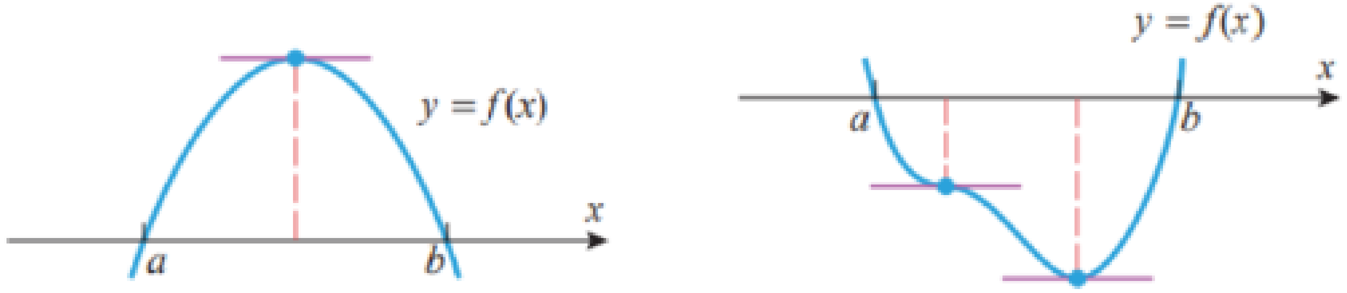

Figure 1

THEOREM ***(Rolle's Theorem)***

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = 0$ and $f(b) = 0$ then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

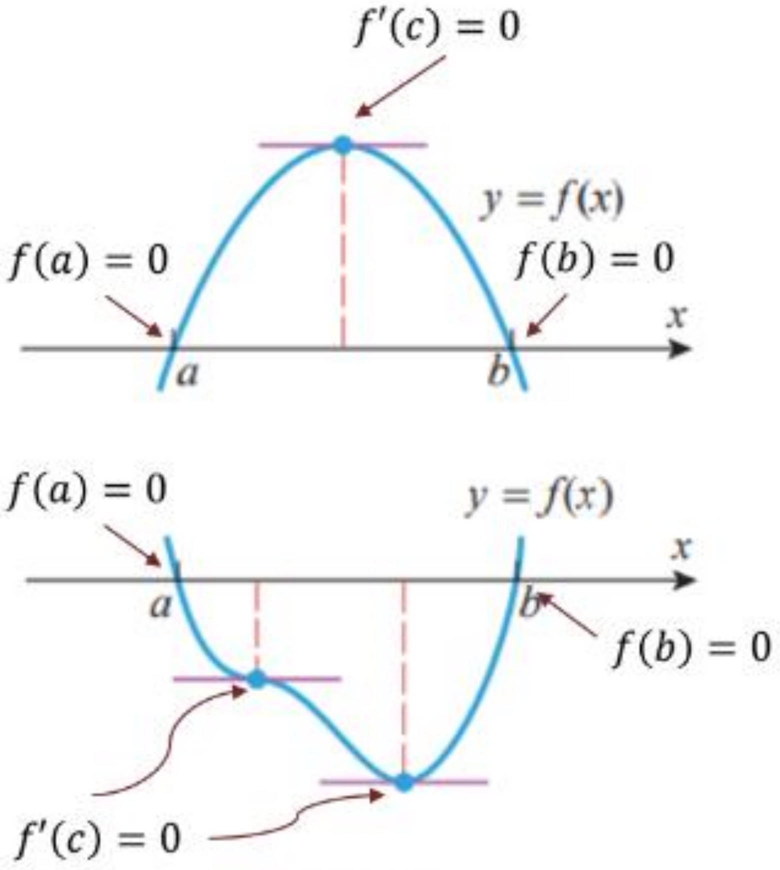

Figure 1

Problem

Example 1:

1. Verify the truth of Rolle's theorem for the function $f(x) = x^2 - 3x + 2$ in the interval $(1,2)$.

Solution: Given that

$$f(x) = x^2 - 3x + 2$$

Clearly, $f(x)$ is continuous in $1 \leq x \leq 2$ and $f'(x)$ exists in $1 < x < 2$.

Also, $f(1) = 0 \ and \ f(2) = 0$,

Therefore $f(1) = f(2)$

Now, $f'(x) = 2x - 3$

$\therefore f'(x) = 0$

$\therefore 2x - 3 = 0$

$\therefore x = \dfrac{3}{2}$

Which is lies between 1 and 2.

Thus there exists a point $x = \frac{3}{2}$ within the interval (1,2) such that $f'\left(\frac{3}{2}\right) = 0$.

Therefore, Rolle's theorem is verified.

Practice Problem (Rolle's Theorem)

- Verify that the hypothesis of Rolle's theorem are satisfied on the given interval and find all the value of c in that interval and satisfy the condition of the theorem

  a) $f(x) = x^3 - 7x^2 + 36; [-2, 3]$

  b) $f(x) = cos\ x;\ [\frac{\pi}{2}, \frac{3\pi}{2}]$

  c) $f(x) = \frac{x}{2} - \sqrt{x};\ [0, 4]$

  d) $f(x) = \frac{1}{x^2} - \frac{4}{3x} + \frac{1}{3};\ [1, 3]$

Slope of a Secant Line

The secant line joining $A(a, f(a))$ and $B(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}$$

Mean-Value Theorem

THEOREM ***(Mean-Value Theorem)*** *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

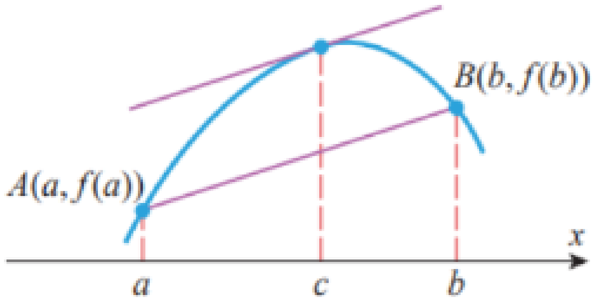

Figure 3 (a)

Problem

Example 1:

Examples:

1. Verify Mean value theorem for the function $f(x) = 2x - x^2$ in the interval $(0,1)$.

Solution: Given that

$$f(x) = 2x - x^2 \text{ in the interval } (0,1).$$

Clearly, $f(x)$ is continuous in $0 \leq x \leq 1$ and $f'(x)$ exists in the open interval $0 < x < 1$.

By Mean value theorem, we have

$$f'(p) = \frac{f(1) - f(0)}{1 - 0}, \qquad where\ 0 < p < 1$$

$$\therefore 2 - 2p = \frac{1 - 0}{1 - 0}$$

$$\Rightarrow \ -2p = 1 - 2$$

$$\Rightarrow \ p = \frac{1}{2}$$

Since $0 < p = \frac{1}{2} < 1$.

Hence the mean value theorem is verified.

Practice Problem (Mean Value Theorem)

- Verify that the hypothesis of Mean Value theorem are satisfied on the given interval and find all the value of c in that interval and satisfy the condition of the theorem

    a) $f(x) = x(x-1)(x-3); [0,4]$

    b) $f(x) = \sqrt{x+1};\ [0,3]$

    c) $f(x) = \sqrt{25-x^2};\ [-5,3]$

    d) $f(x) = \frac{1}{x^2} - \frac{4}{3x} + \frac{1}{3};\ [1,3]$