# GAMMA AND BETA FUNCTION

**Key Word**

Integration

# Gamma Function

The integral $\int_0^\infty x^{n-1}e^{-x}\,dx,\ n>0$ is known as **Gamma Function**. It is denoted by $\Gamma(n)$.

*i.e.*

$$\boxed{\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx,\ n>0}$$

# Formula

- $\Gamma 1 = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(n+1) = n!$
- $\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$
- $\int_0^{\frac{\pi}{2}} sin^m x\ cos^n x\ \ dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

# Formula

**Extension of Gamma function from factorial notation:**

- $\Gamma(n+1) = n!$ [When n is positive integer]

  For example,

$$\Gamma 3 = \Gamma(2+1) = 2! = 2 \times 1 = 2$$

$$\Gamma 6 = ???$$

$$\Gamma 6 = \Gamma(5+1) = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

# Formula

- $\Gamma n = (n-1)(n-2) \ldots \ldots \ldots$ upto a positive number in $\Gamma$ function [When $\boldsymbol{n}$ is a positive rational number]

For example, $\Gamma \frac{7}{2} = \left(\frac{7}{2} - 1\right) \Gamma \left(\frac{7}{2} - 1\right)$

$$= \frac{5}{2} \Gamma \frac{5}{2}$$

$$= \frac{5}{2} \left(\frac{5}{2} - 1\right) \Gamma \left(\frac{5}{2} - 1\right)$$

$$= \frac{5}{2} . \frac{3}{2} \Gamma \frac{3}{2}$$

$$= \frac{5}{2} . \frac{3}{2} \left(\frac{3}{2} - 1\right) \Gamma \left(\frac{3}{2} - 1\right)$$

$$= \frac{5}{2} . \frac{3}{2} . \frac{1}{2} \Gamma \frac{1}{2}$$

$$= \frac{15}{8} \sqrt{\pi}$$

**Evaluate $\boldsymbol{\Gamma \frac{5}{2}}$**

**Ans: $\boldsymbol{= \frac{3}{4} \sqrt{\pi}}$**

# Formula

- $\Gamma n = \frac{\Gamma(n+1)}{n}$ [When $\boldsymbol{n}$ is a negative rational number]

For example, $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$

$$= -\frac{2}{1}\sqrt{\pi}$$

$$= -2\sqrt{\pi}$$

**Evaluate $\boldsymbol{\Gamma\left(-\frac{3}{2}\right)}$**

**Ans: $\boldsymbol{= \frac{4}{3}\sqrt{\pi}}$**

# Gamma Function Related Examples

1. Evaluate $\int_0^{\pi/2} \sin^{\frac{3}{2}}\theta \cos^3\theta \, d\theta$

2. Show that: $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta = \frac{5\pi}{192}$

3. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} \, d\theta = \frac{\pi}{\sqrt{2}}$

4. Evaluate $\int_0^{\infty} e^{-3x} x^{10} dx$

5. Evaluate $\int_0^{\infty} e^{-y^2} y^5 dy$ [Hints: $y^2 = \mathrm{t}$ ]

# Example 1

Evaluate $\int_0^{\pi/2} \sin^{3/2}\theta \cos^3\theta \, d\theta$

**Solution:**

$$\int_0^{\pi/2} \sin^{3/2}\theta \cos^3\theta \, d\theta = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(2)}{2\Gamma\left(\frac{13}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{4}\right).1}{2.\frac{9}{4}\Gamma\left(\frac{9}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{4}\right)}{\frac{9}{2}\cdot\frac{5}{4}\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{8}{45}$$

$$\int_0^{\frac{\pi}{2}} sin^m x \; cos^n x \;\; dx$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

# Example 2

Show that $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta \ = \frac{5\pi}{192}$

**Solution:**

Let, $z = 3\theta$. Then, $dz \ = \ 3d\theta$

Where, Limit $z = \begin{cases} \frac{\pi}{2} & when, \theta = \frac{\pi}{6} \\ 0 & when, \theta = 0 \end{cases}$

$$\therefore \int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta$$

$$= \int_0^{\pi/6} \cos^4 3\theta \, (2 \sin 3\theta \cos 3\theta)^2 d\theta$$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2 z \ \cos^6 z \, dz$$

# Example 2

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 z \ \cos^6 z \, dz$$

$$= \frac{4}{3}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}$$

$$= \frac{2}{3}\frac{\frac{1}{2}.\sqrt{\pi}.\frac{5}{2}.3.3.\frac{1}{2}.\sqrt{\pi}}{4.3.2.1}$$

$$= \frac{5\pi}{192}$$

# Example 3

Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

**Solution:**

$$L.H.S. = \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}} (\cos\theta)^{-\frac{1}{2}}\, d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$

$$= \frac{\pi}{2.1.\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} = R.H.S.$$

# Example 4

Evaluate $\int_0^\infty e^{-3x}\, x^{10}\, dx$

**Solution:**

Let, $3x = t \Rightarrow 3dx = dt \Rightarrow dx = \frac{dt}{3}$

| $x$ | 0 | $\infty$ |
|---|---|---|
| $t$ | 0 | $\infty$ |

$$\therefore \int_0^\infty e^{-3x}\, x^{10}\, dx = \int_0^\infty e^{-t} \left(\frac{t}{3}\right)^{10} . \frac{dt}{3}$$

$$= \frac{1}{3} . \frac{1}{3^{10}} . \int_0^\infty e^{-t}\, t^{10}\, dt$$

$$= \frac{1}{3^{11}} \int_0^\infty e^{-t}\, t^{11-1}\, dt$$

$$= \frac{1}{3^{11}} \Gamma(11) = \frac{10!}{3^{10}}$$

So, $\int_0^\infty e^{-3x}\, x^{10}\, dx = \frac{10!}{3^{10}}$

# Example 5

Evaluate $\int_0^{\infty} e^{-y^2} y^5 \, dy$

**Solution:**

Let, $y^2 = t \Rightarrow 2y dy = dt$

| $y$ | 0 | $\infty$ |
|---|---|---|
| $t$ | 0 | $\infty$ |

$$\therefore \int_0^{\infty} e^{-y^2} y^5 \, dy = \int_0^{\infty} e^{-t} \left(\sqrt{t}\right)^4 . \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\infty} e^{-t} t^2 \, dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{-t} t^{3-1} \, dt$$

$$= \frac{1}{2} \Gamma(3) = \frac{2!}{2} = \frac{2}{2} = 1$$

$So,$ $\int_0^{\infty} e^{-y^2} y^5 \, dy = 1$

# Gamma Function Related Home-Works

(1) Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$

Hint: Similar to Example 1 and consider $1 = \cos^0 x$ $\left[ \boldsymbol{Ans}: \frac{5\pi}{32} \right]$

(2) Show that $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \int_0^{\pi/2} \cos^5 x \sin^4 x \, dx = \frac{8}{315}$

(3) Evaluate $\int_0^{\infty} e^{-5x} x^7 dx$

(4) Evaluate $\int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx$

# Beta Function

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m > 0$ and $n > 0$ is known as **Beta Function**. It is denoted by $\mathrm{B}(m,n)$.

*i.e.*

$$\boxed{\mathrm{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, m > 0 \text{ and } n > 0}$$

# FORMULAE

1. $$\mathrm{B}(m,n) = \mathrm{B}(n,m)$$

2. $$\mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

# Gamma Beta Function Related Examples

Evaluate the followings:

1. Prove that $B\left(\frac{1}{3},\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}}$

2. Prove that $B\left(\frac{1}{5},\frac{4}{5}\right)=\frac{\pi}{\sin\frac{\pi}{5}}$

3. $\int_0^1 x^2\sqrt{1-x}\,dx$ $\left[\boldsymbol{Ans}: \frac{16}{105}\right]$

4. $\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}}dx$ $\left[\boldsymbol{Ans}: \frac{3\pi}{256}\right]$

5. $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$ $[\boldsymbol{Ans}: 0]$

# Example 1

Evaluate $\beta\left(\frac{1}{3},\frac{2}{3}\right)$

**Solution:**

$$\beta\left(\frac{1}{3},\frac{2}{3}\right)=\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)}$$

$$=\frac{\pi}{\sin\frac{1}{3}\pi.1}=\frac{\pi}{\frac{\sqrt{3}}{2}}=\frac{2\pi}{\sqrt{3}}$$

$$Ans:\beta\left(\frac{1}{3},\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}}$$

# Example 2

Evaluate: $\beta\left(\frac{1}{5},\frac{4}{5}\right)$

**Solution:**

$$\beta\left(\frac{1}{5},\frac{4}{5}\right)=\frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{1}{5}+\frac{4}{5}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(1-\frac{1}{5}\right)}{\Gamma\left(\frac{5}{5}\right)}$$

$$=\frac{\pi}{\sin\frac{1}{5}\pi.\Gamma(1)}=\frac{\pi}{\sin\frac{\pi}{5}}$$

$Ans: \beta\left(\frac{1}{5},\frac{4}{5}\right)=\frac{\pi}{\sin\frac{\pi}{5}}$

# Example 3

Evaluate $\int_0^1 x^2\sqrt{1-x}\,dx$

**Solution:**

$$\int_0^1 x^2\sqrt{1-x}\,dx = \int_0^1 x^2(1-x)^{\frac{1}{2}}\,dx$$

$$= \int_0^1 x^{3-1}(1-x)^{1-\frac{1}{2}}\,dx$$

$$= \beta(3,1)$$

$$= \frac{\Gamma(3)\Gamma(1)}{\Gamma(3+1)} = \frac{2!}{3!} = \frac{2}{6} = \frac{1}{3}$$

So, $\int_0^1 x^2\sqrt{1-x}\,dx = \frac{1}{3}$

# Example 4

Evaluate $\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}}dx$

**Solution:**

$$\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}}\,dx = \int_0^1 x^{\frac{7}{2}-1}(1-x)^{\frac{5}{2}-1}\,dx$$

$$= \beta\left(\frac{7}{2},\frac{5}{2}\right) = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5+7}{2}\right)}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}.\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{5!} = \frac{\frac{45}{32}(\sqrt{\pi})^2}{120}$$

$$= \frac{45}{32}\times\frac{1}{120}\pi = \frac{3\pi}{256}$$

$$So, \int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}}dx = \frac{3\pi}{256}$$

# Example 5

Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

**Solution:**

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8-x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(15,9)$$

$$= \beta(9,15) - \beta(9,15) \qquad [\beta(m,n) = \beta(n,m)]$$

$$= 0$$

So, $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$

# Gamma Beta Function Related Home-Work

Evaluate the followings:

1. Prove that $B\left(\frac{1}{4},\frac{3}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}$

2. $\int_0^1 x^6\sqrt{1-x^2}dx$ $\left[\boldsymbol{Ans}\text{: } \frac{\pi}{256}\right]$

3. $\int_0^\infty \frac{x^4(1+x^2)}{(1+x)^{15}}\,dx$ $\left[\boldsymbol{Ans}\text{: } \frac{1}{24}\right]$