Domain and Range

If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the ***domain*** of f, and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f.

Function	Domain
$y = x^2$	All real values of x or $-\infty < x < \infty$ or $(-\infty, \infty)$
$y = \dfrac{1}{x}$	All real values of x except $x = 0$ or $\{x : x = 0\}$ or $(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	All non-negative real values of x or $x \geq 0$ or $[0, \infty)$

Natural Domain of A Function

DEFINITION If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

Example Find the natural domain of

(a) $f(x) = x^3$ (b) $f(x) = 1/[(x-1)(x-3)]$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

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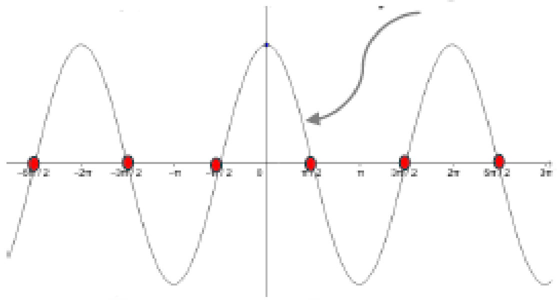
Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

Natural Domain of A Function



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Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$.

Thus, the natural domain consists of all real numbers except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$$x^2 - 5x + 6$$
$$= x^2 - 3x - 2x + 6$$
$$= x(x-3) - 2(x-3)$$
$$= (x-3)(x-2)$$

Solution (d). The function f has real values, except when the expression inside the radical is negative.

x	x-3	x-2	product	R or I
-5	-8	-7	56	R
2	-1	0	0	R
2.5	-0.5	0.5	-0.25	I
3	0	1	0	R
5	2	3	6	R

Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x-3)(x-2) \geq 0$$

This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

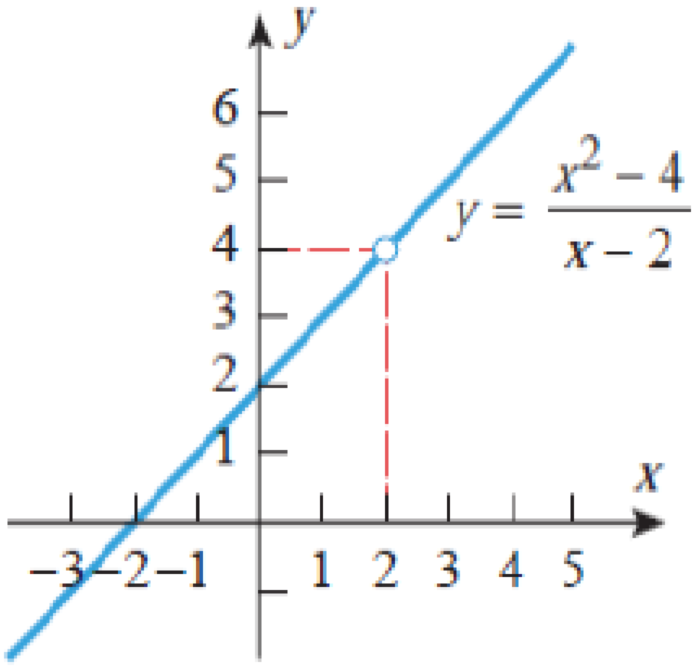
$$(-\infty, 2] \cup [3, +\infty)$$

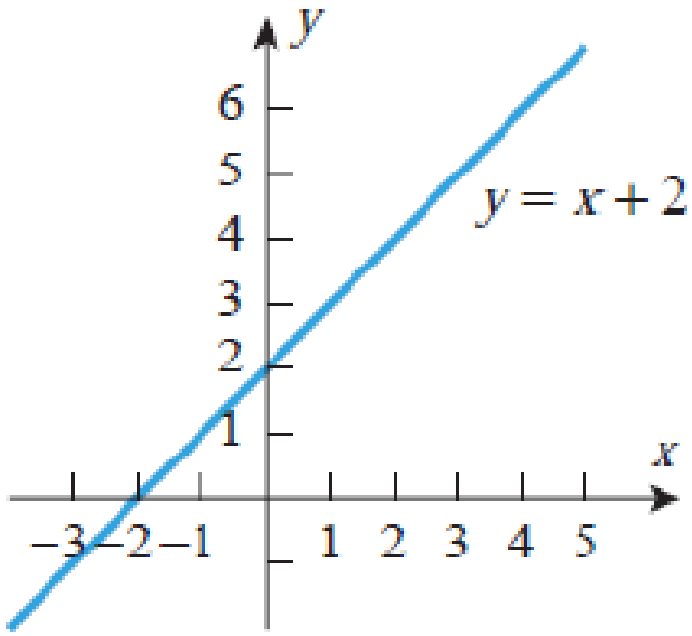
Effect of Algebraic Operations on the Domain

Example The natural domain of the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ consists of all real x except $x = 2$.

Algebraric Opertion $$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$$

The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.





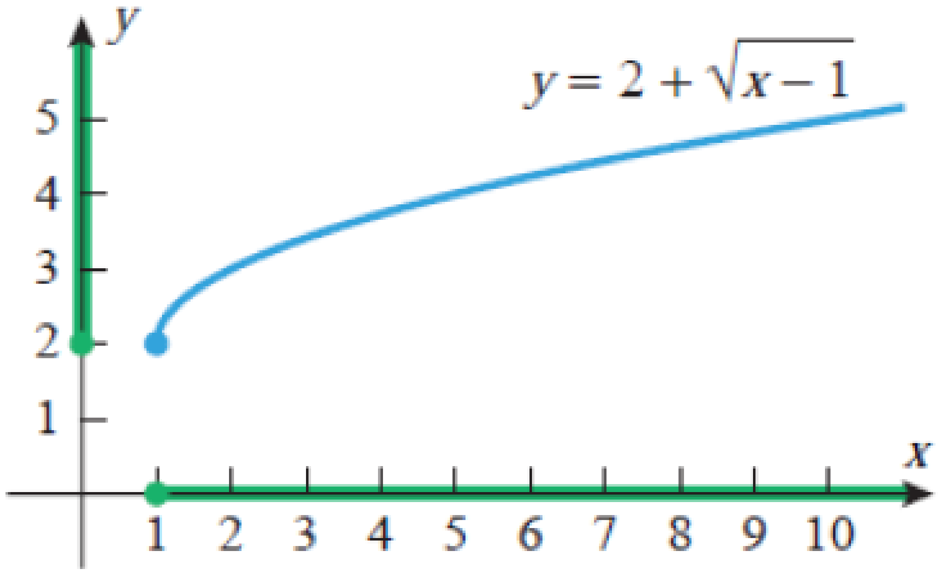
Domain and Range

Example Find the domain and range of

(a) $f(x) = 2 + \sqrt{x-1}$ (b) $f(x) = (x+1)/(x-1)$

Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain, $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f.

x	1	2	5	10
$y = 2 + \sqrt{x-1}$	2	3	4	5



Domain and Range

Example Find the domain and range of

(a) $f(x) = 2 + \sqrt{x-1}$ (b) $f(x) = (x+1)/(x-1)$

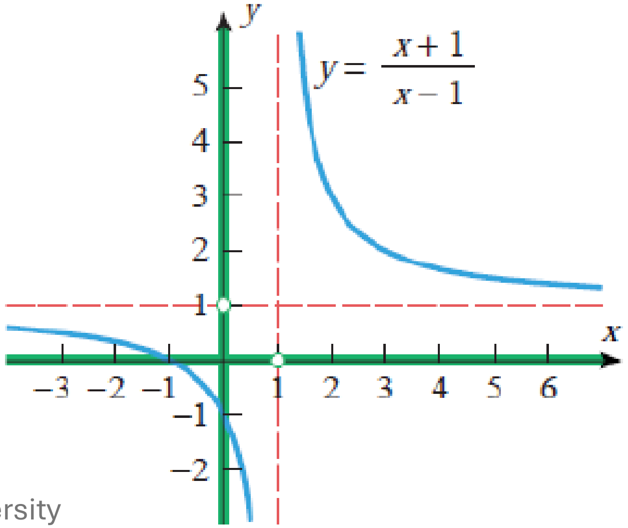
Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically.

$$\begin{aligned} y &= \frac{x+1}{x-1} \\ (x-1)y &= x+1 \\ xy - y &= x+1 \\ xy - x &= y+1 \\ x(y-1) &= y+1 \\ x &= \frac{y+1}{y-1} \end{aligned}$$

x	-3	-1	0	1.5	2	5
$y = \dfrac{x+1}{x-1}$	0.5	0	-1	5	3	1.5



Practice Problem

2. Find the domain of the following functions.

a) $f(x) = x^4 - 2x + 4$ Ans: $(-\infty, \infty)$

b) $f(x) = \frac{3x+2}{x}$ Ans:$(-\infty, 0) \cup (0, \infty)$

c) $f(x) = \sqrt{x^2 - 6x + 8}$ Ans: $(-\infty, 2] \cup [4, \infty)$

d) $f(x) = \sqrt{2x^3 + 5x^2 - 3x}$ Ans:$(-\infty,3] \cup [3,0] \cup [\frac{1}{2}, \infty)$

e) $f(x) = \sqrt{-(x+2)(x+5)(x-1)}$ Ans:$(-\infty, -5] \cup [-2,1]$

Practice Problem

3. Find the domain and range of the following function

a) $f(x) = 3$

Ans: D=$(-\infty, \infty)$ or $\{x|x \in R\}, R = \{y|y = 7\}$

b) $f(x) = \sqrt{x^2 - 4x}$

Ans: D=$(-\infty, 0] \cup [4, \infty), R = [0, \infty)$

c) $f(x) = \sqrt{2 - x} + 1$

Ans: D=$(-\infty, 2], R = [1, \infty)$

d) $f(x) = \frac{5x-6}{x+3}$

Ans: D=$(-\infty, -3) \cup (-3, \infty)$,
$R = (-\infty, 5) \cup (5, \infty)$

Composition of Functions

Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

Example Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). $f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$

Solution (b). $g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$

Even and Odd Functions

A function f is said to be an ***even function*** if

$$f(-x) = f(x)$$

and is said to be an ***odd function*** if

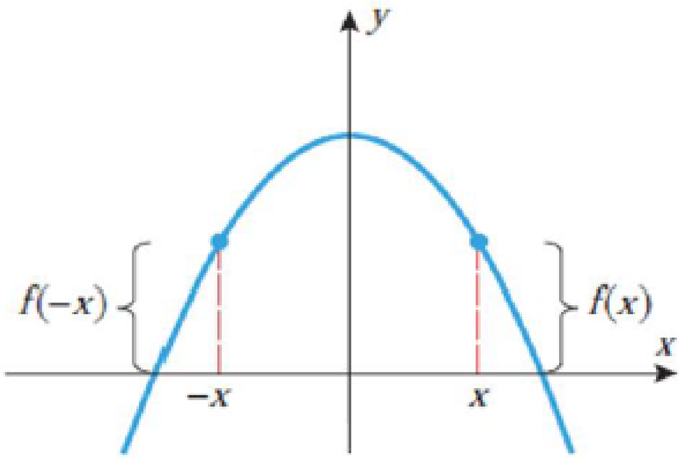
$$f(-x) = -f(x)$$

EVEN FUNCTION

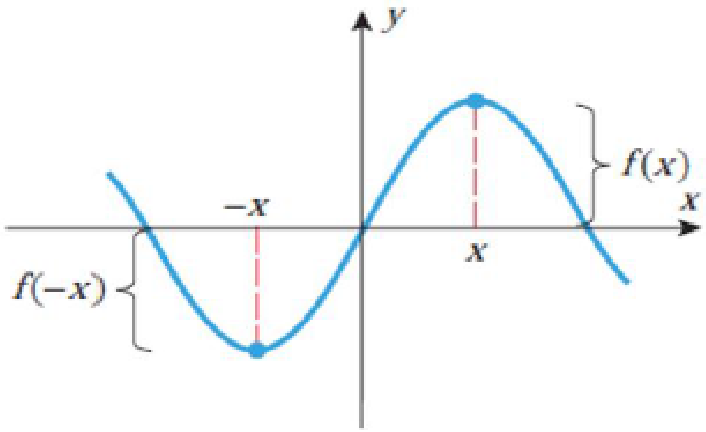
x^2, x^4, x^6, and $\cos x$

ODD FUNCTION

x^3, x^5, x^7, and $\sin x$.



This is the graph of an even function since $f(-x) = f(x)$.



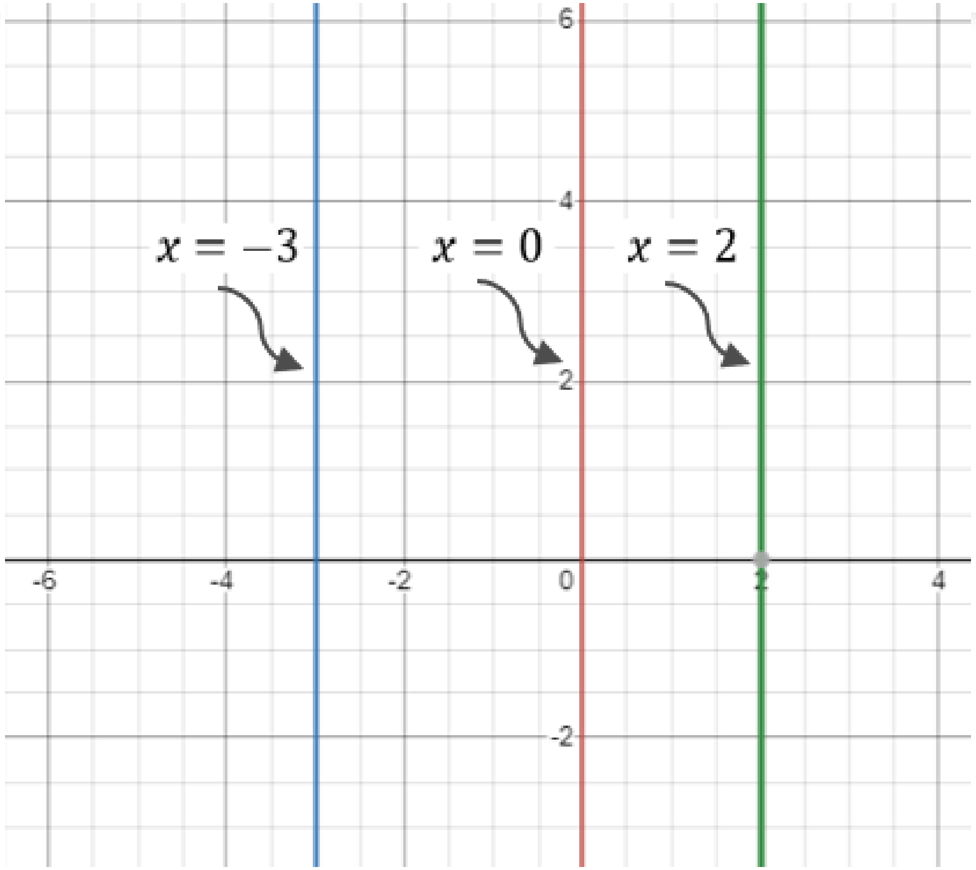
This is the graph of an odd function since $f(-x) = -f(x)$.

Graph of Common Functions

- **Functions**
  - **Algebraic Function**
    - $x + 2y = 4$ (Straight Line)
    - $y^2 = x - 1$ (Parabola)
    - $x^2 + \frac{y^2}{4} = 1$ (Ellipse)
    - $y = \frac{x+3}{x-2}$ (Rational Function)
  - **Trigonometric Function**
    - $y = \sin x$ (Sine Function)
    - $y = 1 + \cos x$ (Cosine Function)
  - **Exponential Function**
    - $y = e^x$
  - **Logarithmic Function**
    - $y = \ln x$

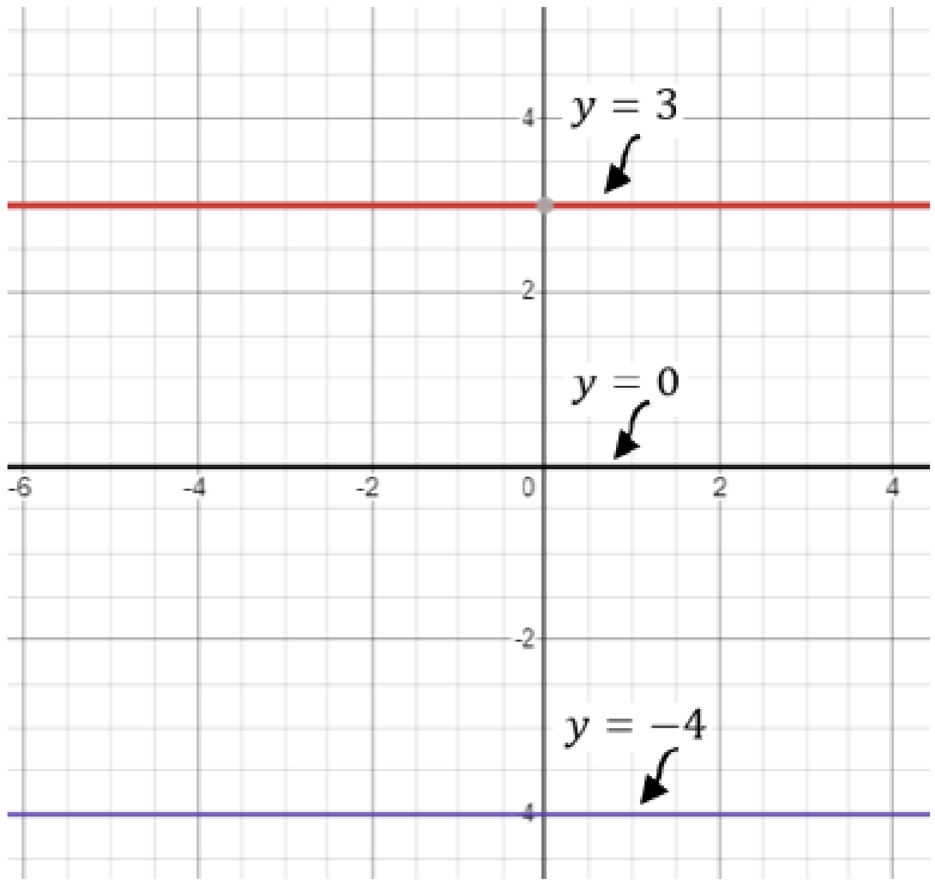
Graphs of straight line parallel to an axis.

$x = a$ (a is a constant) is a straight line parallel to $y -$ axis.



Graphs of straight line parallel to an axis.

$y = b$ (b is a constant) is a straight line parallel to $x -$ axis.

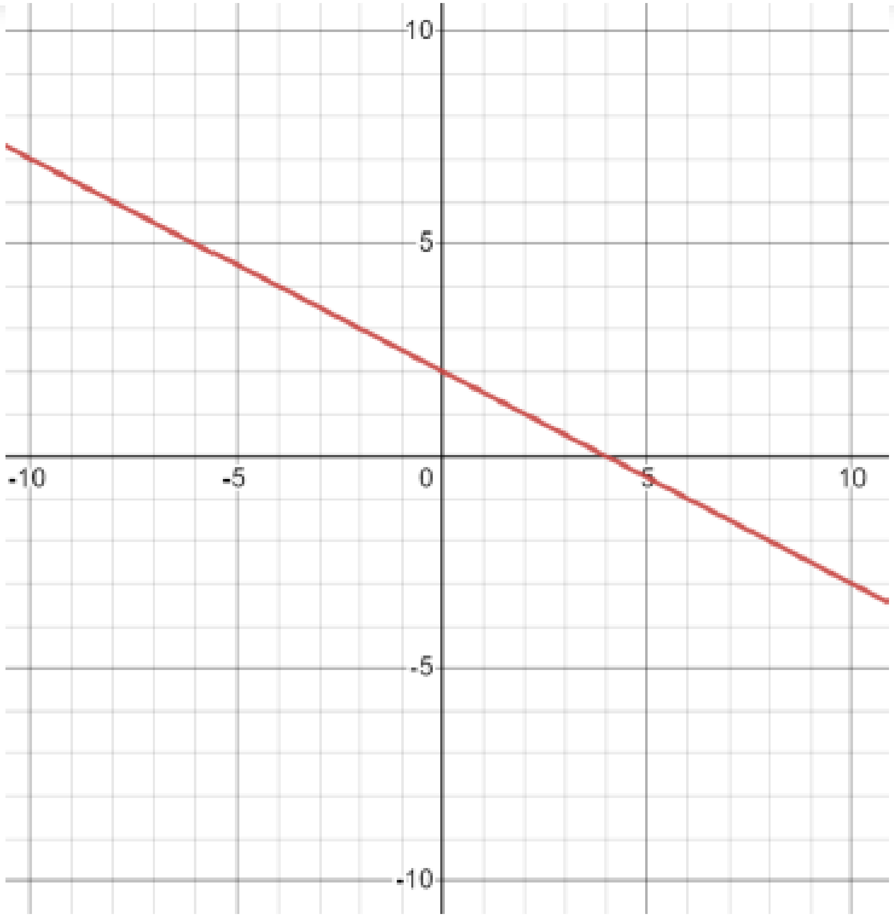


Draw the graphs of the given equations.

$ax + by = c$ $(a, b, c$ are constants) represents a straight line.

$x + 2y = 4$

x	0	4
y	2	0



Draw the graphs of the given equations.

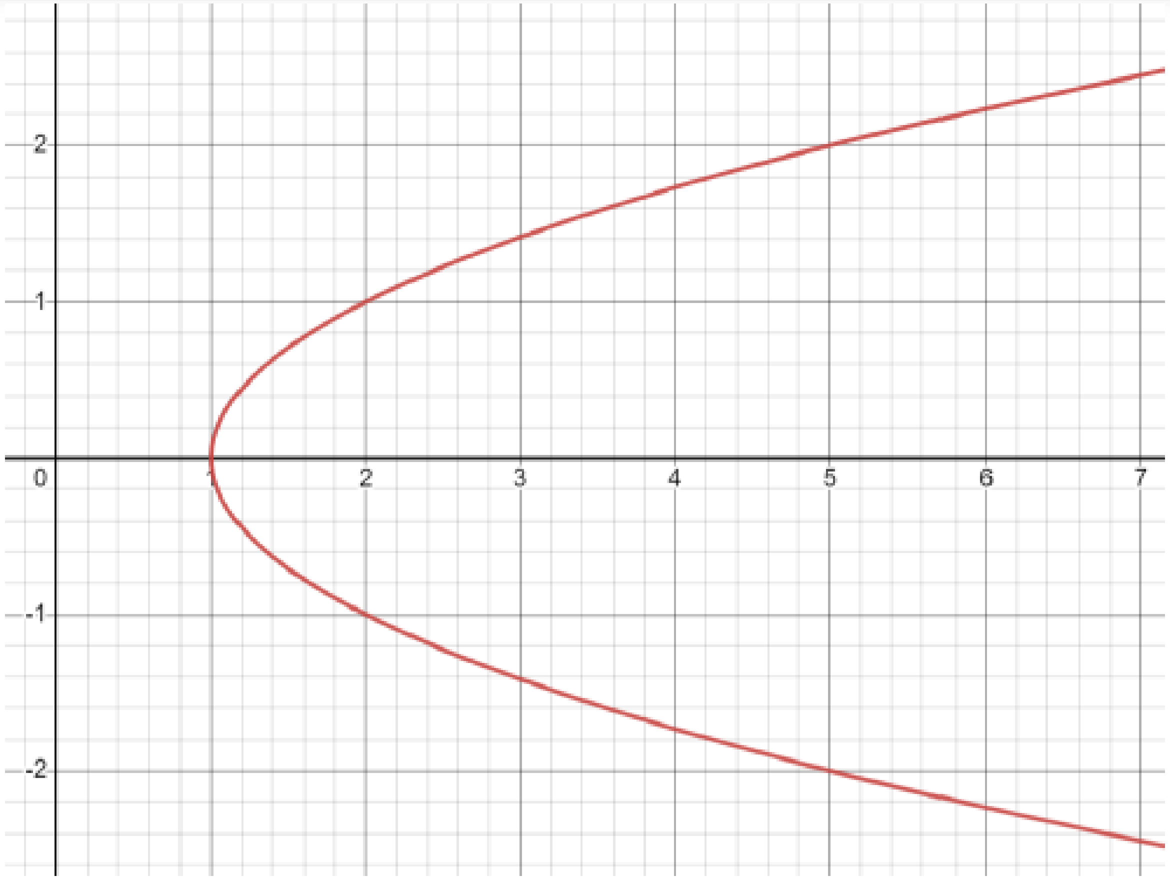
$y^2 = 4ax$ and $x^2 = 4ay$ (a is a constant) represent parabolla.

$$y^2 = x - 1$$

$$\Rightarrow y = \pm\sqrt{x-1}$$

x	1	2	2	5	5
y	0	1	-1	2	-2

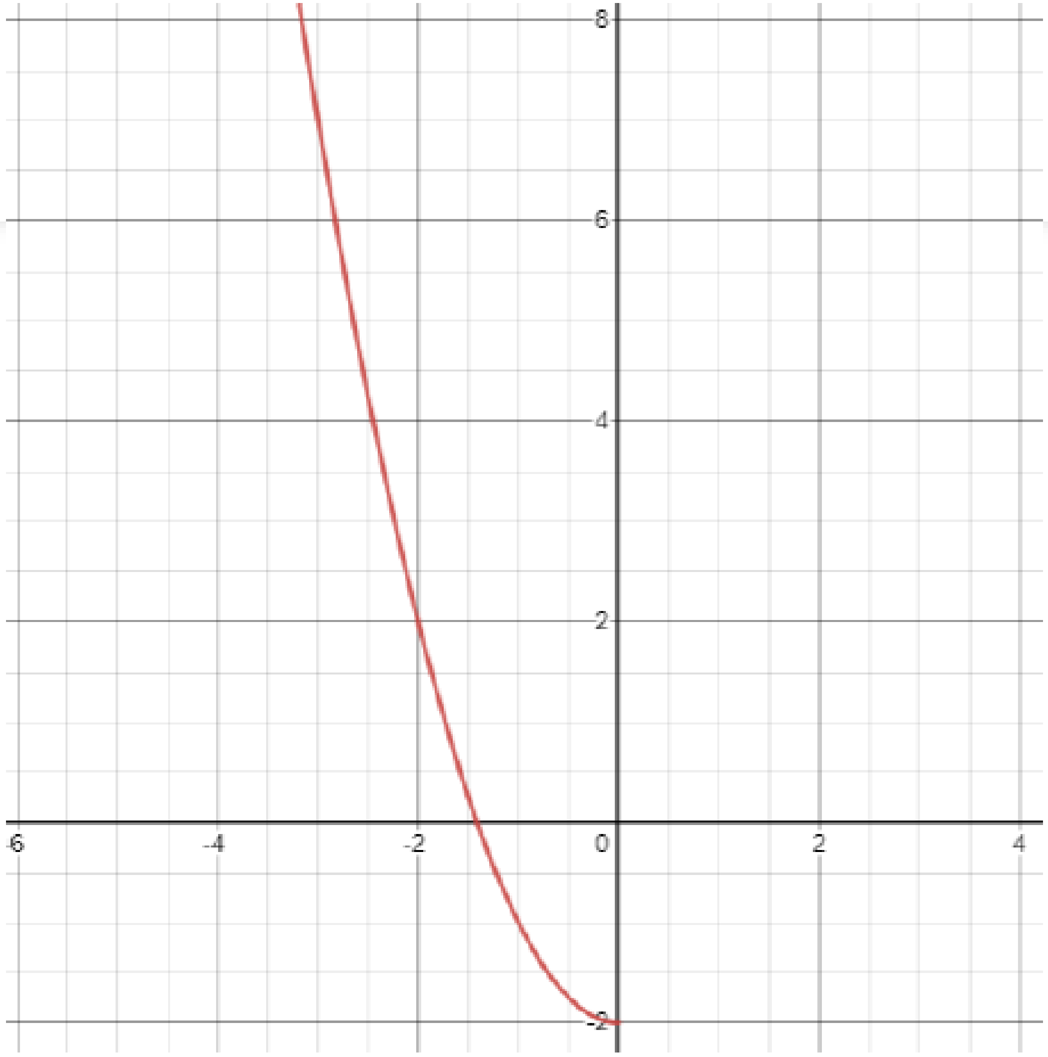
Try to draw $x^2 + y - 2 = 0$.



Draw the graphs of the given equations.

$x = -\sqrt{2+y}$

x	0	-1	-2	-3
y	-2	-1	2	7



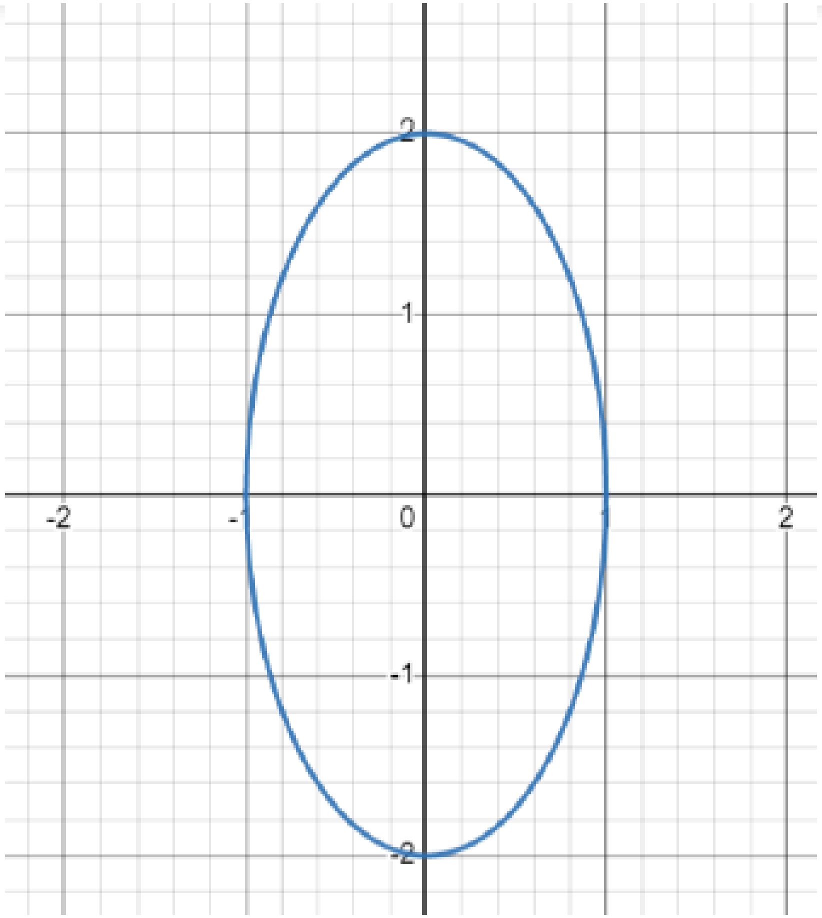
Draw the graphs of the given equations.

$x^2 + y^2 = Constant$ represents either a circle or an ellipse.

$$x^2 + \frac{y^2}{4} = 1$$

$$\Rightarrow y = \pm 2\sqrt{1 - x^2} \quad \text{or} \quad x = \pm\sqrt{1 - \frac{y^2}{4}}$$

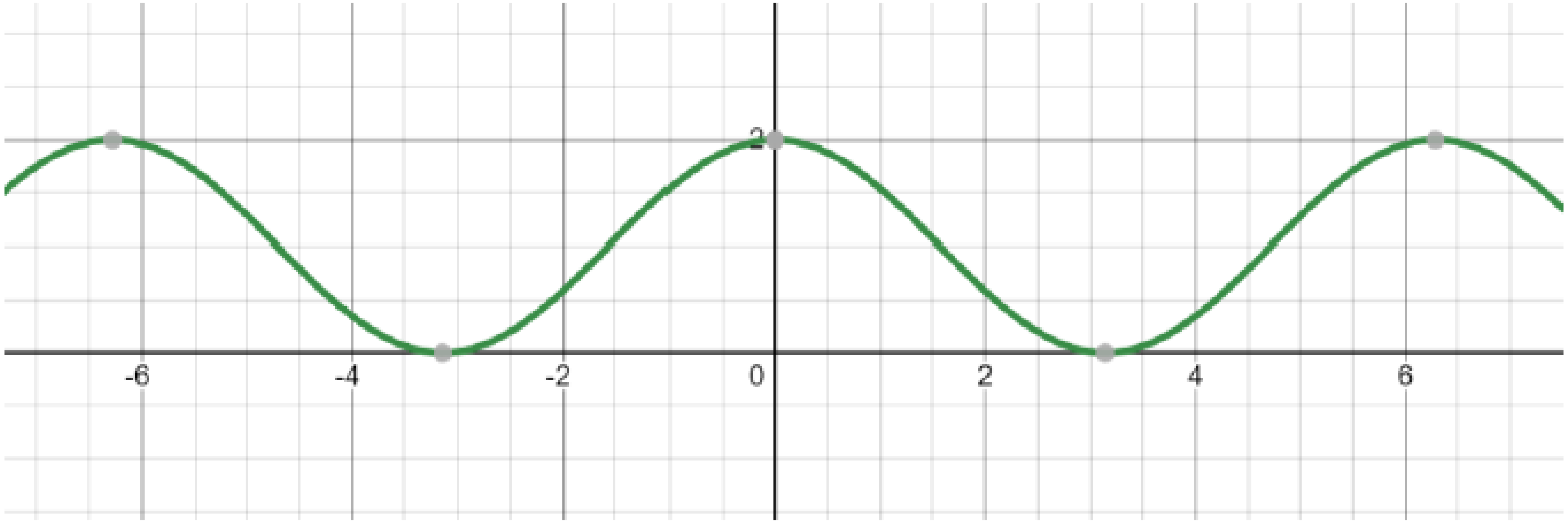
x	0	0	1	-1
y	2	-2	0	0



Draw the graphs of the given equations.

$y = 1 + \cos x$

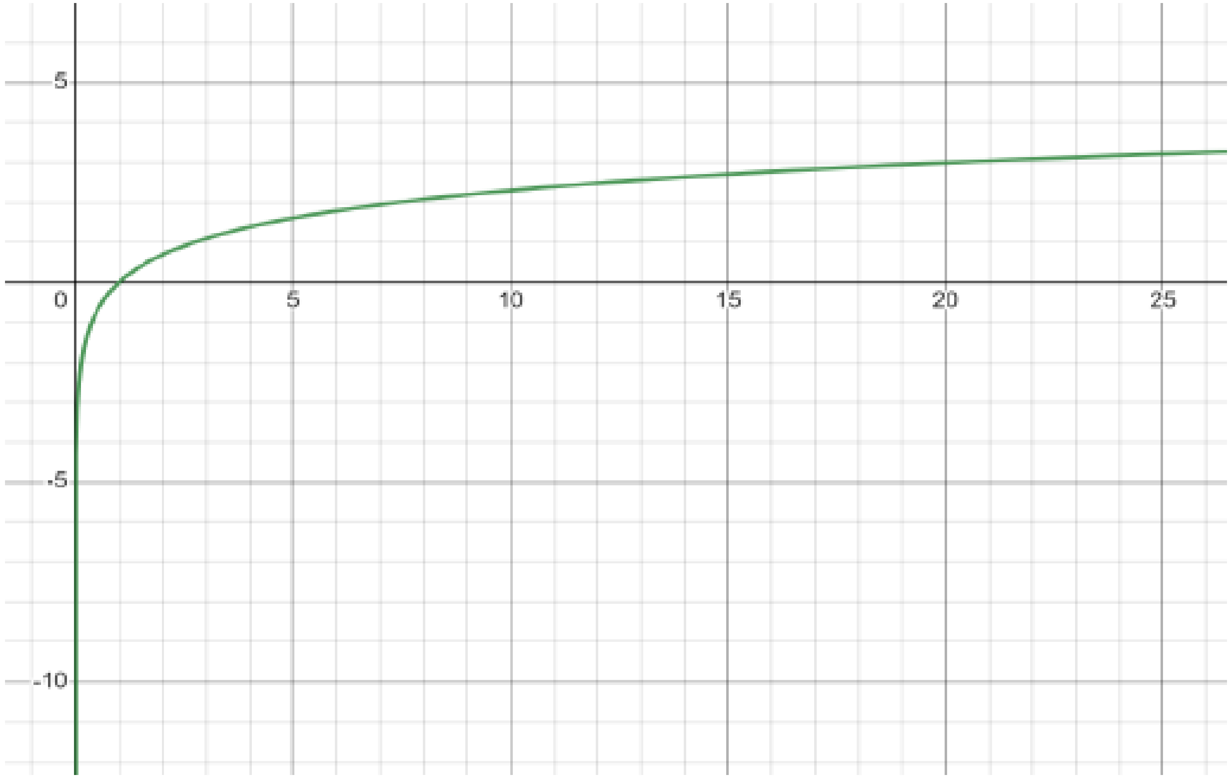
x	-6	-3	0	3	6
y	2.0	0.0	2.0	0.0	2.0



Draw the graphs of the given equations.

$$y = \ln x$$

x	0.01	1	5	10	20
y	-4.6	0.0	1.6	2.3	3.0



Draw the graphs of the given equations.

$$y = e^x$$

x	-10	-5	0	1	3
y	0.00005	0.007	1.000	2.718	20.086