Integration

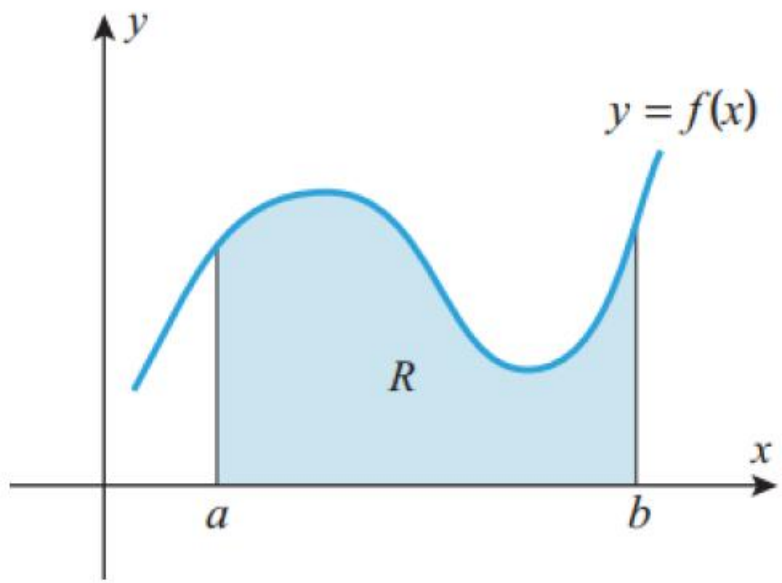
Key Word

Area Problem

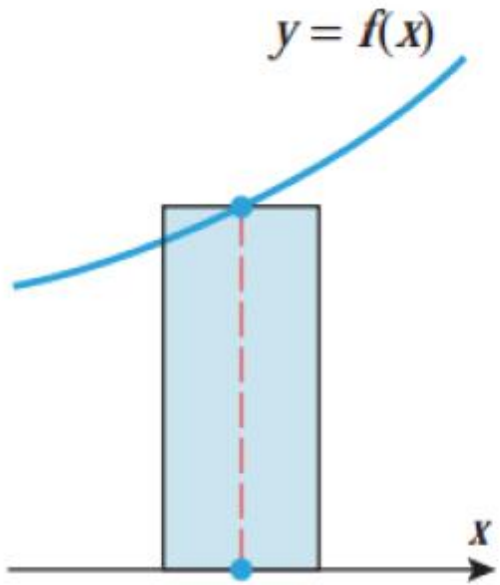
Derivative of a Function

AREA PROBLEM

Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis.



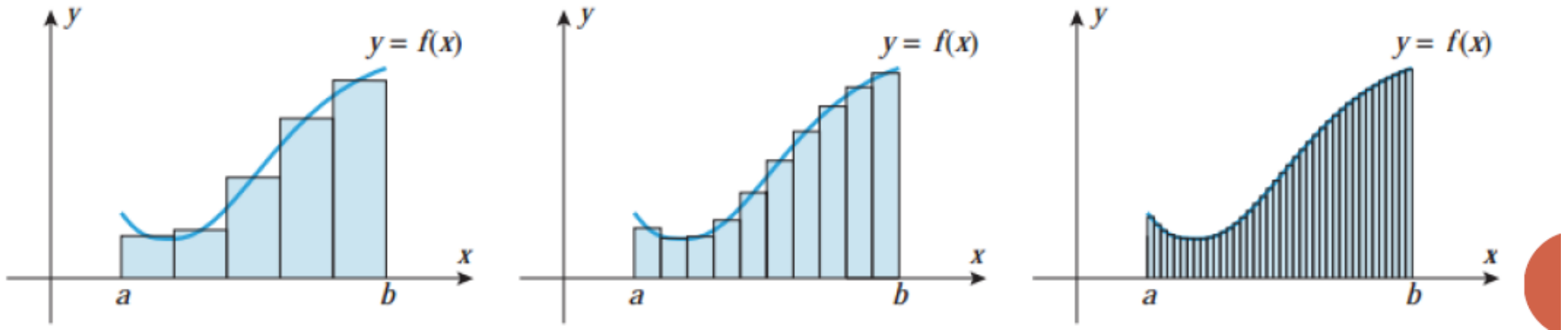
Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x –axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter - it can be above the center, above an endpoint, or above any other point in the subinterval.



For each n, the total area of the rectangles can be viewed as an **approximation** to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit. That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n \to \infty} A_n$$

We will call this the **rectangle method** for computing A.



INDEFINITE INTEGRAL

The process of finding antiderivatives is called **antidifferentiation** or **integration**. Thus, if

$$\frac{d}{dx}[F(x)] = f(x)$$

then **integrating** (or **antidifferentiating**) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, above equation is recast using **integral notation**,

$$\int f(x)\,dx = F(x) + C$$

where C is understood to represent an arbitrary constant.

FORMULAE

1. $\int dx = x + C$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
3. $\int e^{mx} dx = \frac{1}{m} e^{mx} + C$
4. $\int \frac{f'(x)}{f(x)} dx = ln\,|f(x)| + C$
5. $\int \frac{1}{x} dx = ln|x| + C$
6. $\int a^x dx = \frac{a^x}{lna} + C$
7. $\int cosmxdx = \frac{1}{m} sinmx + C$
8. $\int sinmxdx = -\frac{1}{m} cosmx + C$
9. $\int sec^2 xdx = tanx + C$
10. $\int cosec^2 xdx = -cotx + C$
11. $\int secxtanxdx = secx + C$
12. $\int cotxcosecxdx = -cosecx + C$
13. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1} \frac{x}{a} + C$
14. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} ln \frac{x-a}{x+a} + C$
15. $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1} x + C$
16. $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}|x| + C$

Problems

Evaluate the followings:

1. $\int (x - \cos 2x + e^{-x})dx$

Ans: $\frac{x^2}{2} - \frac{Sin\ 2x}{2} - e^{-x} + c$

2. $\int \left(\sqrt{x} - \frac{6}{x} + \frac{1}{1+x^2}\right) dx$

Ans: $\frac{2}{3}x^{\frac{3}{2}} - 6\ln x + tan^{-1}x + c$

3. $\int \left(\frac{sinx}{cos^2 x} + \frac{8}{4+x^2} - 2\right) dx$

Ans: $secx + 4tan^{-1}\frac{x}{2} - 2x + c$

4. $\int \left(cosec^2 x - e^{-2x} + \frac{1}{\sqrt{1-x^2}}\right) dx$

Ans: $-cotx + \frac{1}{2}e^{-2x} + sin^{-1}x + c$

5. $\int \left(\frac{2}{x^2-4} - \frac{1}{x^2} + \frac{2}{1+x^2}\right) dx$

Ans: $\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right| + \frac{1}{x} + 2tan^{-1}x + c$

6. $\int \left(\sqrt{x} - \frac{1}{x^2+9} + \frac{\sin 2x}{cosx}\right) dx$

Ans: $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}tan^{-1}\left(\frac{x}{3}\right) - 2cosx + c$

Home-Work

Evaluate the followings:

1. $\int \left(\sqrt[3]{x} - 3^x + e^{-x/2}\right)dx$

Ans: $\frac{3}{4}x^{\frac{4}{3}} - \frac{3^x}{\ln(3)} - 2e^{-\frac{x}{2}} + c$

2. $\int \left(1 - \frac{3x^2-5}{x^3} + \frac{1}{4-x^2}\right)dx$

Ans: $x - 3\ln|x| + \frac{5}{2x^2} - \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + c$

3. $\int \left(cosec^2 x + \frac{2}{\sqrt{1-x^2}} - 6x\right)dx$

Ans: $-\cot x + 2sin^{-1}x - 3x^2 + c$

4. $\int (sec^2 x - 3x)dx$

Ans: $\tan x - \frac{3x^2}{2} + c$

5. $\int \left(\frac{3}{x^2-9} - \frac{x+1}{x^2-1} + \frac{2}{4+x^2}\right)dx$

Ans: $\frac{1}{2}\ln\left|\frac{x-3}{x+3}\right| - \ln(x-1) + tan^{-1}\left(\frac{x}{2}\right) + c$

6. $\int \left(2x + 5 - \frac{1}{x^2+100}\right)dx$

Ans: $x^2 + 5x - \frac{1}{10}tan^{-1}\left(\frac{x}{10}\right) + c$

Integration by Parts

In calculus, **integration by parts** or **partial integration** is a process that finds the integral of a product of functions in terms of the integral of the product of their derivative and antiderivative.

Formula

$$\int uvdx = u\int v\,dx - \int \left\{\frac{du}{dx}\int vdx\right\}dx$$

EXAMPLE

Evaluate $\int xe^{2x}dx$. Consider $u = x$ and $v = e^{2x}$

$$\int xe^{2x}dx = x\int e^{2x}\,dx - \int\left\{\frac{dx}{dx}\int e^{2x}dx\right\}dx$$

$$= x\left(\frac{1}{2}e^{2x}\right) - \int\left\{(1)\left(\frac{1}{2}e^{2x}\right)\right\}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

FORMULA $\int uvdx = u\int v\,dx - \int\left\{\frac{du}{dx}\int vdx\right\}dx$

LIATE RULE

A rule of thumb has been proposed, consisting of choosing as u the function that comes first in the following list:

L	logarithmic functions	$\ln x$, $\log_b x$, etc.
I	inverse trigonometric functions	$\sin^{-1} x$, $\sec^{-1} x$, etc.
A	algebraic functions	x^2, $-5x^{11} + 3x$, etc.
T	trigonometric functions	$\cos x$, $\cot x$, etc.
E	exponential functions	e^{5x}, 4^x, etc.

Problems

Evaluate the followings:

1. $\int x \cos x \, dx$

Ans: $x \sin x + \cos x + c$

2. $\int (3x - 2)e^{-x} dx$

Ans: $-3xe^{-x} - e^{-x} + c$

3. $\int (x^2 + x - 2) \sin(3x) \, dx$

Ans: $-\frac{1}{3}x^2 \cos(3x) + \frac{2}{9}x \sin(3x) - \frac{1}{3}xcos(3x) + \frac{20}{27}\cos(3x) + \frac{1}{9}\sin(3x) + C$

4. $\int e^x \cos x \, dx$

Ans: $\frac{e^x(sin\, x + \cos x)}{2} + c$

Home-Work

Evaluate the followings:

1. $\int x^2 \sin 2x \, dx$

Ans: $-\frac{1}{2}x^2\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + c$

2. $\int (x+4)e^{3x} dx$

Ans: $\frac{xe^{3x}}{3} + \frac{11e^{3x}}{9} + c$

3. $\int (x-5)\cos x \, dx$

Ans: $x \sin x - 5\sin x + \cos x + c$

4. $\int e^{2x} \cos 2x \, dx$

Ans: $\frac{e^{2x}(sin\ 2x + \cos 2x)}{4} + c$

Home-Work

Evaluate the followings:

5. $\int \ln x \, dx$

Ans: $x \ln x - x + c$

6. $\int \left(\sqrt[3]{x} + \frac{16}{x^2-16} - \ln x \right) dx$

Ans: $\frac{3}{4} x^{\frac{4}{3}} + 2 \ln \left| \frac{x-4}{x+4} \right| - x \ln x + x + c$

7. $\int \left(\frac{2}{x^2+1} - \frac{3x-2}{x^2} \right) dx$

Ans: $2\, tan^{-1}x \; - 3 + \frac{2}{x} + c$

8. $\int \left((x^2-2) \cos 3x - x^2 + 3 - xe^{-5x} \right) dx$

Ans: $\frac{x^2}{3} \sin(3x) + \frac{2x}{9} \cos(3x) - \frac{20}{27} \sin(3x) - \frac{x^3}{3} + 3x + \frac{x\, e^{-5x}}{5} + \frac{e^{-5x}}{25} + c$