Integration

Key Word

Area Problem

Derivative of a Function

INTEGRATION BY SUBSTITUTION

In calculus, **integration by substitution**, also known as ***u*-substitution** or **change of variables**, is a method for evaluating integrals and antiderivatives.

EXAMPLE

Evaluate $\int x^2(x^3-3)^{20}dx$.

Consider $u = x^3 - 3$

$$\Rightarrow \frac{du}{dx} = 3x^2$$

$$\Rightarrow x^2 dx = \frac{1}{3} du$$

$$\int x^2(x^3-3)^{20}dx = \int (x^3-3)^{20}x^2dx$$

$$= \int (u^{20})\left(\frac{1}{3}du\right)$$

$$= \frac{1}{3}\int u^{20}du$$

$$= \frac{1}{3} \times \frac{u^{20+1}}{20+1} + c$$

$$= \frac{u^{21}}{63} + c$$

$$= \frac{(x^3-3)^{21}}{63} + c$$

Problems

Evaluate the followings:

1. $\int 3x(2-x^2)^5 dx$	**Ans:** $-\frac{(2-x^2)^6}{4}+c$
2. $\int x\sqrt[3]{x^2+4}dx$	**Ans:** $\frac{3}{8}(x^2+4)^{\frac{4}{3}}+c$
3. $\int (3x^2-2)(x^3-2x)^{11}dx$	**Ans:** $\frac{(x^3-2x)^{12}}{12}+c$
4. $\int e^{5x+7}dx$	**Ans:** $\frac{1}{5}e^{5x+7}+c$
5. $\int xe^{2x^2-3}dx$	**Ans:** $\frac{1}{4}e^{2x^2-3}+c$
6. $\int e^{\cos x}\sin x\, dx$	**Ans:** $-e^{\cos x}+c$
7. $\int x\cos(x^2-5)\, dx$	**Ans:** $\frac{1}{2}\sin(x^2-5)+c$

Home-Work

Evaluate the followings:

1. $\int 2x\sqrt{x^2+7}dx$

Ans: $\frac{2}{3}(x^2+7)^{\frac{3}{2}}+c$

2. $\int 6\sqrt[3]{6x-5}dx$

Ans: $\frac{3}{4}(6x-5)^{\frac{4}{3}}+c$

3. $\int x(x^2-1)^{12}dx$

Ans: $\frac{1}{26}(x^2-1)^{13}+c$

4. $\int (2x^3-x)(x^4-x^2)^7 dx$

Ans: : $\frac{1}{16}(x^4-x^2)^8+c$

5. $\int (x^2-1)\sqrt[5]{x^3-3x}dx$

Ans: : $\frac{5}{18}(x^3-3x)^{\frac{6}{5}}+c$

Home-Work

Evaluate the followings:

6. $\int xe^{4x^2+5}dx$

Ans: $\frac{1}{8}e^{4x^2+5}+c$

7. $\int (x^2-1)\,e^{1-3x+x^3}dx$

Ans: $\frac{1}{3}e^{1-3x+x^3}+c$

8. $\int (sec^2x)e^{1-\tan x}dx$

Ans: $-\,e^{1-\tan x}+c$

9. $\int x\sin(2x^2+6)\,dx$

Ans: $-\frac{1}{4}\cos(2x^2+6)+c$

POLYNOMIAL LONG DIVISION FOR INTEGRATION

Polynomial long division is necessary for integration, when the function holds the following two conditions.

- It is a rational function of the form $\frac{p(x)}{q(x)}$.
- If the degree of $p(x)$ is m and the degree of $q(x)$ is n, where $m \geq n$.

Example

$$\int \frac{x^2 - 4x + 1}{x + 1} dx$$

$$= \int \left(x - 5 + \frac{6}{x + 1}\right) dx$$

$$= \int x \, dx - 5 \int dx + 6 \int \frac{1}{x + 1} dx$$

$$= \frac{x^2}{2} - 5x + 6 \ln(x + 1) + C$$

$$\begin{array}{r@{\,}l}
x + 1) & x^2 - 4x + 1 \;(x - 5 \\
 & \underline{x^2 + \;\; x \qquad} \\
 & \qquad - 5x + 1 \\
 & \qquad \underline{- 5x - 5} \\
 & \qquad\qquad\quad 6
\end{array}$$

Practice Problems

Evaluate the followings:

1. $\int \frac{3x-5}{x+2} dx$

Ans: $3x - 11\ln|x+2| + c$

2. $\int \frac{6x+3}{2x-1} dx$

Ans: $3x + 3\ln|2x-1| + c$

3. $\int \frac{-4x}{x+2} dx$

Ans: $-4x + 8\ln|x+2| + c$

4. $\int \frac{x^2}{x^2+1} dx$

Ans: $x - \ tan^{-1}x + c$

Practice Problems

Evaluate the followings:

5. $\int \frac{2x^3+18x+1}{x^2+9}\,dx$

Ans: $x^2+\frac{1}{3}tan^{-1}\left(\frac{x}{3}\right)+c$

6. $\int \frac{2x^2+2x-5}{x^2-4}\,dx$

Ans: $2x+\frac{7}{4}\ln|x-2|+\frac{1}{4}\ln|x+2|+c$

7. $\int \frac{x^4-x^3+5x^2-2x+2}{x^3+2x}\,dx$

Ans: $\frac{x^2}{2}-x+\ln|x|+\ln|x^2+2|+c$

Home-Work

Evaluate the followings:

1. $\int \frac{8x-5}{4x+3}dx$

Ans: $2x-\frac{11}{4}\ln|4x+3|+c$

2. $\int \frac{3x}{2-x}dx$

Ans: $-6\ln|2-x|-3x+c$

3. $\int \frac{63-32x+4x^2-2x^3}{x^2+16}dx$

Ans $-x^2+4x-\frac{1}{4}tan^{-1}\left(\frac{x}{4}\right)+c$

4. $\int \frac{3x^3+2x^2-3x}{x^2-1}dx$

Ans: $\frac{3x^2}{2}+2x+\ln|x-1|-\ln|x+1|+c$

5. $\int \frac{6x^5+3x^3+9x^2-9x}{2x^3+3x+1}dx$

Ans: $x^3-3x+\ln|2x^3+3x+1|+c$

Partial Fraction for Integration

Partial fraction is necessary for integration, when the function holds the following three conditions.

- It is a rational function of the form $\frac{p(x)}{q(x)}$.
- If the degree of $p(x)$ is m and the degree of $q(x)$ is n, where $m < n$.
- $q(x)$ can be factored as some functions of x.

With the help of partial fraction the factors of $q(x)$ can be separated.

How?

$\frac{m}{(ax+b)(cx+d)}$	$\frac{A}{ax+b}+\frac{B}{cx+d}$
$\frac{m}{(ax+b)(cx+d)(ex+f)}$	$\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{ex+f}$
$\frac{m}{(ax+b)^2(cx+d)}$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{cx+d}$
$\frac{m}{(ax+b)^3(cx+d)}$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\frac{D}{cx+d}$
$\frac{m}{(ax^2+b)(cx+d)}$	$\frac{Ax+B}{ax^2+b}+\frac{C}{cx+d}$
$\frac{m}{(ax^2+b)^2(cx+d)}$	$\frac{Ax+B}{ax^2+b}+\frac{Cx+D}{(ax^2+b)^2}+\frac{E}{cx+d}$

EXAMPLE

Evaluate $\int \frac{2x+1}{x^2-2x-3}\,dx.$

$$x^2-2x-3$$
$$= x^2-3x+x-3$$
$$= x(x-3)+(x-3)$$
$$= (x-3)(x+1)$$

$$\frac{2x+1}{(x-3)(x+1)} = \frac{A}{x-3}+\frac{B}{x+1}$$

$$\Rightarrow 2x+1 = A(x+1)+B(x-3)$$

$$\begin{aligned} A+\ B &= 2 \\ A-3B &= 1 \end{aligned} \quad \Rightarrow A=\tfrac{7}{4}, B=\tfrac{1}{4}$$

$$\therefore \frac{2x+1}{(x-3)(x+1)} = \frac{7}{4(x-3)}+\frac{1}{4(x+1)}$$

$$\int \frac{2x+1}{x^2-2x-3}\,dx = \int \left[\frac{7}{4(x-3)}+\frac{1}{4(x+1)}\right] dx$$

$$= \frac{7}{4}\int \frac{dx}{x-3}+\frac{1}{4}\int \frac{dx}{x+1}$$

$$= \frac{7}{4}\ln(x-3)+\frac{1}{4}\ln(x+1)+C$$

Problems

Evaluate the followings:

1. $\int \frac{3x-5}{x^2-4} dx$

Ans: $\frac{1}{4} \ln|x-2| + \frac{11}{4} \ln|x+2| + C$

2. $\int \frac{3-2x}{2x^2+x} dx$

Ans: $3\ln|x| - 4\ln|2x+1| + C$

3. $\int \frac{5}{2x^2+x-1} dx$

Ans: $\frac{5}{3} \ln|2x-1| - \frac{5}{3} \ln|x+1| + C$

4. $\int \frac{x-2}{x^2-x-6} dx$

Ans: $\frac{1}{5} \ln|x-3| + \frac{4}{5} \ln|x+2| + C$

5. $\int \frac{x^2+2x-3}{x^3-2x^2+x-2} dx$

Ans: $\ln|x-2| + 2arctan(x) + C$

Home-Work

Evaluate the followings:

1. $\int \frac{-3}{x^2+4x} dx$

Ans: $-\frac{3}{4}\ln|x| + \frac{3}{4}\ ln\ |\ x+4\ | + C$

2. $\int \frac{2x-3}{10-3x-x^2} dx$

Ans: $-\frac{13}{7}\ ln\ |\ x+5\ | - \frac{1}{7}\ ln\ |\ x-2\ | + C$

3. $\int \frac{x+2}{6x^2-x-1} dx$

Ans: $-\frac{1}{3}\ ln\ |\ 3x+1\ | + \frac{1}{2}\ ln\ |\ 2x-1\ | + C$

4. $\int \frac{1-3x}{x^3+3x^2+3x+9} dx$

Ans: $\frac{5}{6}\ ln\ |\ x+3\ | - \frac{5}{12} ln\ |\ x^2+3\ | - \frac{1}{2\sqrt{3}}\tan^{-1}(\frac{x}{\sqrt{3}}) + C$

5. $\int \frac{x^3-x^2+x-1}{x^4+3x^2+2} dx$

Ans: $\frac{1}{2} ln\ |\ x^2+2\ | - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

Home-Work

Evaluate the followings:

6. $\int \left[\frac{2x+1}{3x^2+3x+7} - 5x^2(x^3-4)^{18}\right] dx$

Ans: $\frac{1}{3}\ \ln|3x^2+3x+7| - \frac{5}{57}\ (x^3-4)^{19} + C$

7. $\int \frac{x^2+3}{x^3+x}\, dx$

Ans: $-\ \ln|x^2+1| + 3ln\ |\ x\ | + C$

8. $\int \frac{-x^3+5x^2+5x-27}{x^2-5}\, dx$

Ans: $-\frac{x^2}{2} + 5x - \frac{1}{\sqrt{5}} ln\ |\ \frac{x-\sqrt{5}}{x+\sqrt{5}}\ | + C$

9. $\int \frac{x+8}{2x^2+8x-10}\, dx$

Ans: $-\frac{1}{4}\ ln\ |\ x+5\ | + \frac{3}{4} ln\ |\ x-1\ | + C$

10. $\int \frac{12x^2+34x+27}{x(2x+3)^2}\, dx$

Ans: $3ln\ |\ x\ | + \frac{1}{2x+3}\ + C$

11. $\int \frac{x^3+6x^2+5x+3}{x^2+5x}\, dx$

Ans: $\frac{x^2}{2} + x - 6ln\ |\ x\ | + C$