Gamma and Beta Function

Key Word

Integration

GAMMA FUNCTION

The integral $\int_0^\infty x^{n-1} e^{-x}\, dx,\ n > 0$ is known as **Gamma Function**. It is denoted by $\Gamma(n)$.

i.e.

$$\boxed{\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx, n > 0}$$

Formula

- $\Gamma 1 = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(n+1) = n!$
- $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$
- $\int_0^{\frac{\pi}{2}} sin^m x \, cos^n x \; dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

Formula

Extension of Gamma function from factorial notation:

- $\Gamma(n+1) = n!$ [When n is positive integer]

  For example,

$$\Gamma 3 = \Gamma(2+1) = 2! = 2 \times 1 = 2$$

$$\Gamma 6 = ???$$

$$\Gamma 6 = \Gamma(5+1) = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Formula

- $\Gamma n = (n-1)(n-2)$ ... ... upto a positive number in Γ function [When $\boldsymbol{n}$ is a positive rational number]

For example, $\Gamma\frac{7}{2} = \left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$

$$= \frac{5}{2}\Gamma\frac{5}{2}$$

$$= \frac{5}{2}\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$$

$$= \frac{5}{2}.\frac{3}{2}\Gamma\frac{3}{2}$$

$$= \frac{5}{2}.\frac{3}{2}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$= \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{15}{8}\sqrt{\pi}$$

Evaluate $\boldsymbol{\Gamma\frac{5}{2}}$

Ans: $\boldsymbol{=\frac{3}{4}\sqrt{\pi}}$

Formula

- $\Gamma n = \frac{\Gamma(n+1)}{n}$ [When $\boldsymbol{n}$ is a negative rational number]

For example, $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$

$$= -\frac{2}{1}\sqrt{\pi}$$

$$= -2\sqrt{\pi}$$

Evaluate $\boldsymbol{\Gamma\left(-\frac{3}{2}\right)}$

Ans: $\boldsymbol{= \frac{4}{3}\sqrt{\pi}}$

Gamma Function Related Examples

1. Evaluate $\int_0^{\pi/2} \sin^{\frac{3}{2}}\theta \cos^3\theta \, d\theta$
2. Show that: $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta = \frac{5\pi}{192}$
3. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} \, d\theta = \frac{\pi}{\sqrt{2}}$

Example 1

Evaluate $\int_0^{\pi/2} \sin^{3/2}\theta \cos^3\theta \, d\theta$

Solution:

$$\int_0^{\pi/2} \sin^{3/2}\theta \cos^3\theta \, d\theta = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(2)}{2\Gamma\left(\frac{13}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{4}\right).1}{2.\frac{9}{4}\Gamma\left(\frac{9}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{4}\right)}{\frac{9}{2}\cdot\frac{5}{4}\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{8}{45}$$

$$\int_0^{\frac{\pi}{2}} sin^m x \; cos^n x \;\; dx$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Example 2

Show that $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta = \frac{5\pi}{192}$

Solution:

Let, $z = 3\theta$. Then, $dz = 3d\theta$

Where, Limit $z = \begin{cases} \frac{\pi}{2} & when, \theta = \frac{\pi}{6} \\ 0 & when, \theta = 0 \end{cases}$

$$\therefore \int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta$$

$$= \int_0^{\pi/6} \cos^4 3\theta \, (2 \sin 3\theta \cos 3\theta)^2 d\theta$$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2 z \, \cos^6 z \, dz$$

Example 2

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 z \ \cos^6 z \, dz$$

$$= \frac{4}{3}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}$$

$$= \frac{2}{3}\frac{\frac{1}{2}.\sqrt{\pi}.\frac{5}{2}.3.3.\frac{1}{2}.\sqrt{\pi}}{4.3.2.1}$$

$$= \frac{5\pi}{192}$$

Example 3

Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

Solution:

$$L.H.S. = \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}} (\cos\theta)^{-\frac{1}{2}}\, d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$

$$= \frac{\pi}{2.1.\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} = R.H.S.$$

Gamma Function Related Home-Works

(1) Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$

Hint: Similar to Example 1 and consider $1 = \cos^0 x$ $\left[\boldsymbol{Ans}: \frac{5\pi}{8} \right]$

(2) Show that $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \int_0^{\pi/2} \cos^5 x \sin^4 x \, dx = \frac{8}{315}$