Relative Extrema

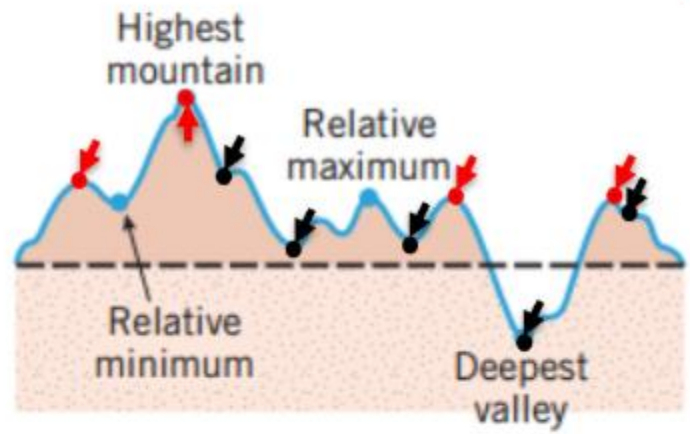
Key Word

Graph of Function

Differentiation

RELATIVE MAXIMA AND MINIMA

If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima".



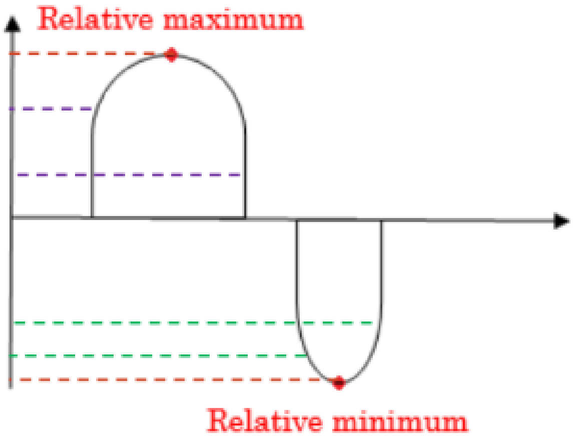
A relative maximum need not be the highest point in the entire mountain range, and a relative minimum need not be the lowest point—they are just high and low points relative to the nearby terrain.

DEFINITION

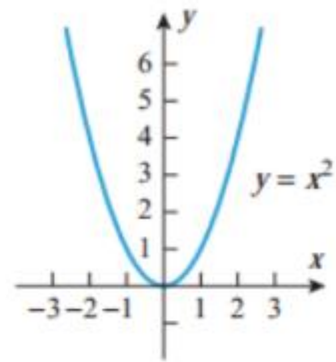
A function f is said to have a **relative maximum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval.

Similarly, f is said to have a **relative minimum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval.

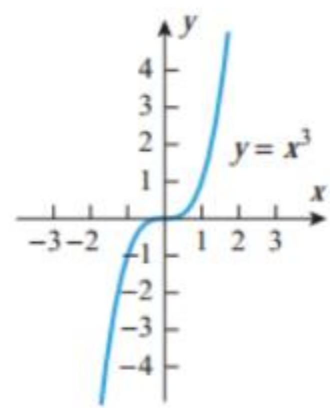
If f has either a relative maximum or a relative minimum at x_0, then f is said to have a **relative extremum** at x_0.



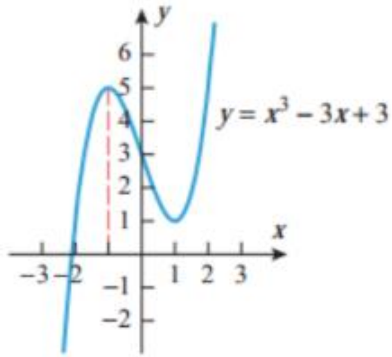
EXAMPLES



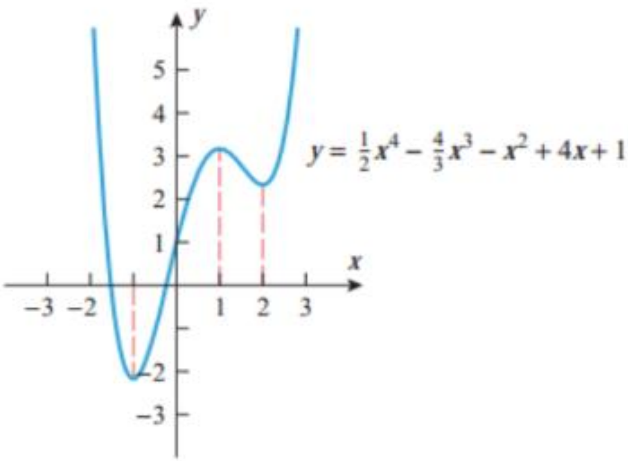
f has a relative minimum at $x = 0$ but no relative maxima



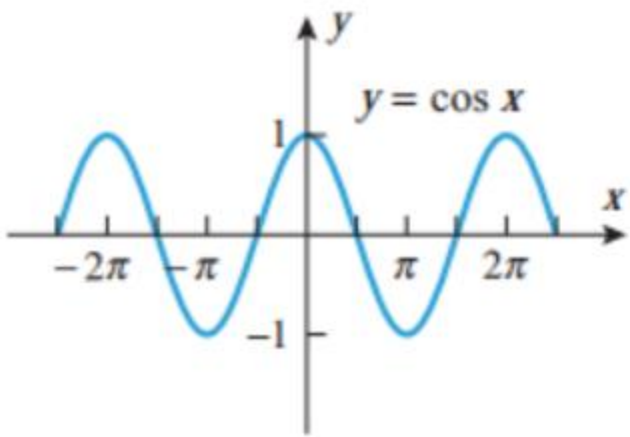
f has no relative extrema



f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$

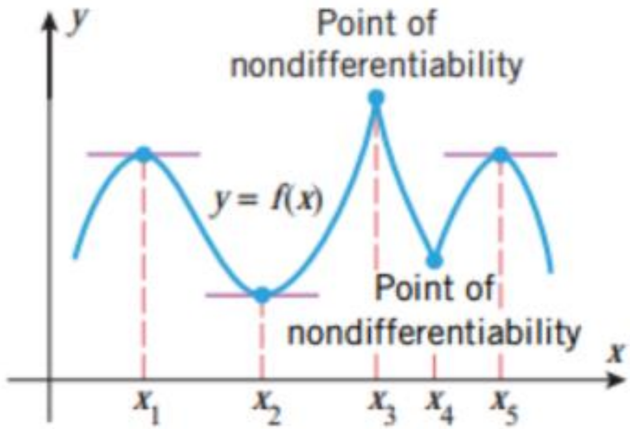


f has relative minimum at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$



f has relative maximum at $x = -2\pi$, $x = 0$ and $x = 2\pi$ and relative minimum at $x = -\pi$ and $x = \pi$

CRITICAL POINT AND STATIONARY POINT



The relative extrema for the five functions occur at points where the graphs of the functions have horizontal tangent lines. Above figure illustrates that a relative extremum can also occur at a point where a function is not differentiable. In general, we define a critical point for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable.

To distinguish between the two types of critical points we call x a stationary point of f if $f'(x) = 0$.

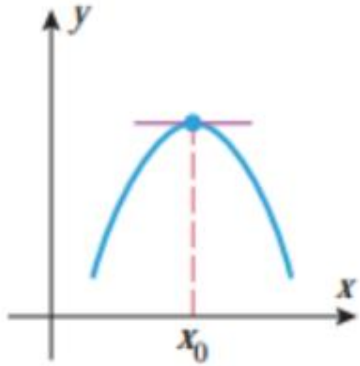
THEOREM 1

Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f ; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.

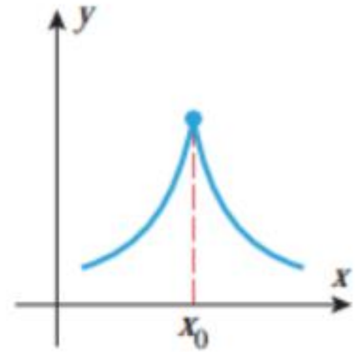
NOTE

Relative extrema must occur at critical points, but it does not say that a relative extremum occurs at every critical point.

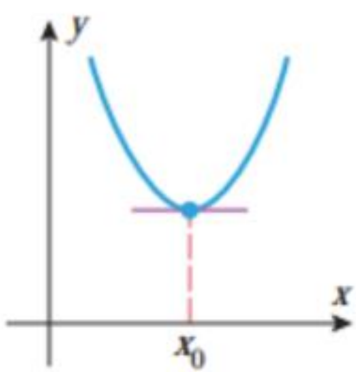
EXAMPLES



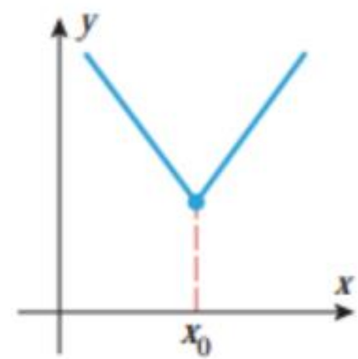
Critical point
Stationary point
Relative maximum



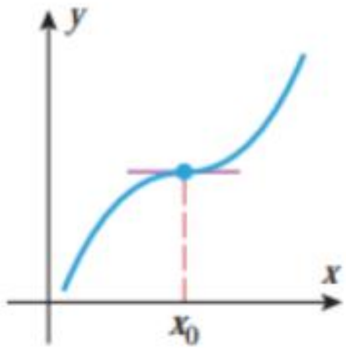
Critical point
Not a stationary point
Relative maximum



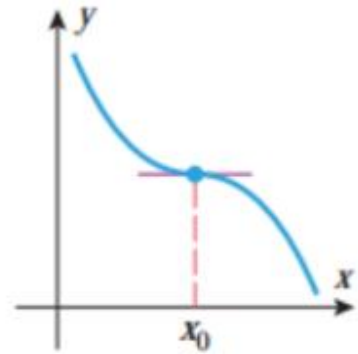
Critical point
Stationary point
Relative minimum



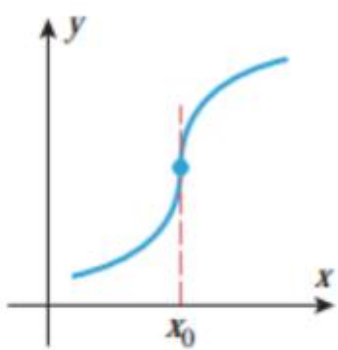
Critical point
Not a stationary point
Relative minimum



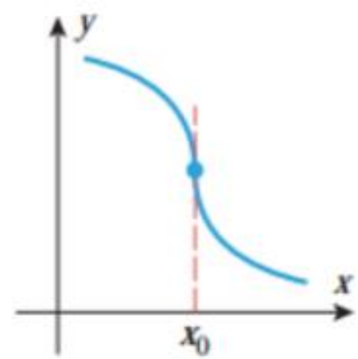
Critical point
Stationary point
Inflection point
Not a relative extremum



Critical point
Stationary point
Inflection point
Not a relative extremum



Critical point
Not a stationary point
Inflection point
Not a relative extremum



Critical point
Not a stationary point
Inflection point
Not a relative extremum

Conclusion

A function f has a relative extremum at those critical points where f' changes sign.

Theorem 2
First Derivative Test

Suppose that f is continuous at a critical point x_0.

(a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.

(b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.

(c) If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.

Note: *If at the critical point there is no relative extremum, then the point is called **saddle point**.*

EXAMPLE 1

Use first derivative test to determine where does the function $f(x) = 2x^3 + 3x^2 - 12x$ have relative maximum and relative minimum?

Solution:

The given function is $f(x) = 2x^3 + 3x^2 - 12x$

The first derivative of $f(x)$ is $f'(x) = 6x^2 + 6x - 12$

Now, $f'(x) = 0$

$\Longrightarrow 6x^2 + 6x - 12 = 0$

$\Longrightarrow x^2 + x - 2 = 0$

$\Longrightarrow x = -2, 1$

Interval	$(x+2)(x-1)$	$f'(x)$
$x < -2$	$(-)(-)$	$+$
$-2 < x < 1$	$(+)(-)$	$-$
$x > 1$	$(+)(+)$	$+$

The sign of $f'(x)$ changes from + to − at x = -2, so there is a relative minimum at that point. The sign changes from − to + at x = 1, so there is a relative maximum at that point.

Example 2

Use first derivative test to determine where does the function $f(x) = 3x^{5/3} - 15x^{2/3}$ have relative maximum and relative minimum?

Solution:

The given function is $f(x) = 3x^{5/3} - 15x^{2/3}$

The first derivative of $f(x)$ is $f'(x) =$

Now, $f'(x) = 0$

$$\Rightarrow \frac{5(x-2)}{x^{1/3}} = 0$$

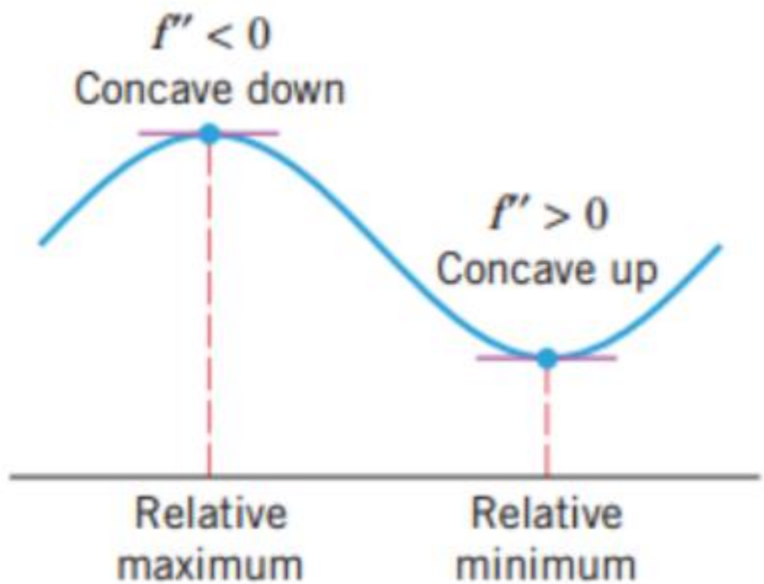
$$\Rightarrow x = 0, 2$$

Interval	$\frac{5(x-2)}{x^{1/3}}$	$f'(x)$
$x < 0$	$(-)(-)$	$+$
$0 < x < 2$	$(-)(+)$	$-$
$x > 2$	$(+)(+)$	$+$

The sign of $f'(x)$ changes from + to − at $x = 0$, so there is a relative maximum at that point. The sign changes from − to + at $x = 2$, so there is a relative minimum at that point.

SECOND DERIVATIVE TEST

There is another test for relative extrema that is based on the following geometric observation: A function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing that point, and it has a relative minimum if it is concave up.



THEOREM 3
SECOND DERIVATIVE TEST

Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.
(b) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.
(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.

Example 3

Let $f(x) = 3x^5 - 5x^3$.

(a) Use second derivative test to find the relative extrema of $f(x)$.

(b) Use first derivative test to solve the problem if the second derivative test is inconclusive.

Solution (a):

The given function is $f(x) = 3x^5 - 5x^3$

The first derivative of $f(x)$ is $f'(x) = 15x^4 - 15x^2$

The second derivative of $f(x)$ is $f''(x) = 60x^3 - 30x$

$$= 30x(2x^2 - 1)$$

Now for stationary point, $f'(x) = 15x^4 - 15x^2 = 0$

$$\Longrightarrow 15x^2(x^2 - 1) = 0$$

$$\Longrightarrow x = 0, 1, -1$$

Stationary Point	$30x(2x^2 - 1)$	$f''(x)$	Second Derivative Test
$x = -1$	-30	$+$	f has a relative maximum
$x = 0$	0	$-$	Inconclusive
$x = 1$	30	$+$	f has a relative maximum

The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f is given in the following table:

Interval	$15x^2(x^2 - 1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

Since there is no sign change in $f'(x)$ at $x = 0$, there is neither a relative maximum nor a relative minimum at that point.

EXAMPLE 4 [Do it by yourself]

Let $f(x) = 3x^4 - 4x^3 - 12x^2$.

(a) Use second derivative test to find the relative extrema of $f(x)$.

(b) Use first derivative test to check that your result is correct.

Result

f has a relative minimum at $x = -1$

f has a relative maximum at $x = 0$

f has a relative minimum at $x = 2$

Practice Problems

3. (a) Use both the first and second derivative tests to show that $f(x) = 3x^2 - 6x + 1$ has a relative minimum at $x = 1$.

(b) Use both the first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

4. (a) Use both the first and second derivative tests to show that $f(x) = \sin^2 x$ has a relative minimum at $x = 0$.

(b) Use both the first and second derivative tests to show that $g(x) = \tan^2 x$ has a relative minimum at $x = 0$.

Practice Problems

25–32 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

25. $f'(x) = x^2(x^3 - 5)$

29. $f'(x) = xe^{1-x^2}$

30. $f'(x) = x^4(e^x - 3)$

31. $f'(x) = \ln\left(\dfrac{2}{1 + x^2}\right)$