Increasing, Decreasing and Concavity of A Function

Key Word

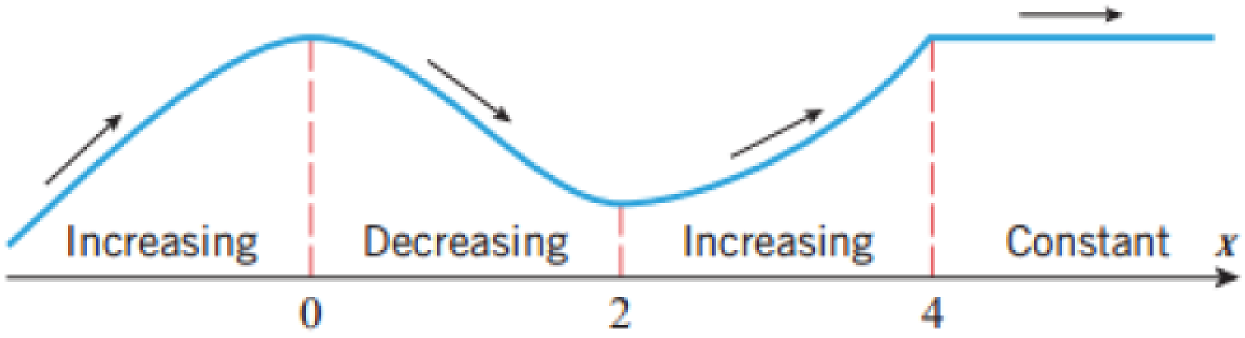
Graph of Function

Differentiation

Behavior of a Function

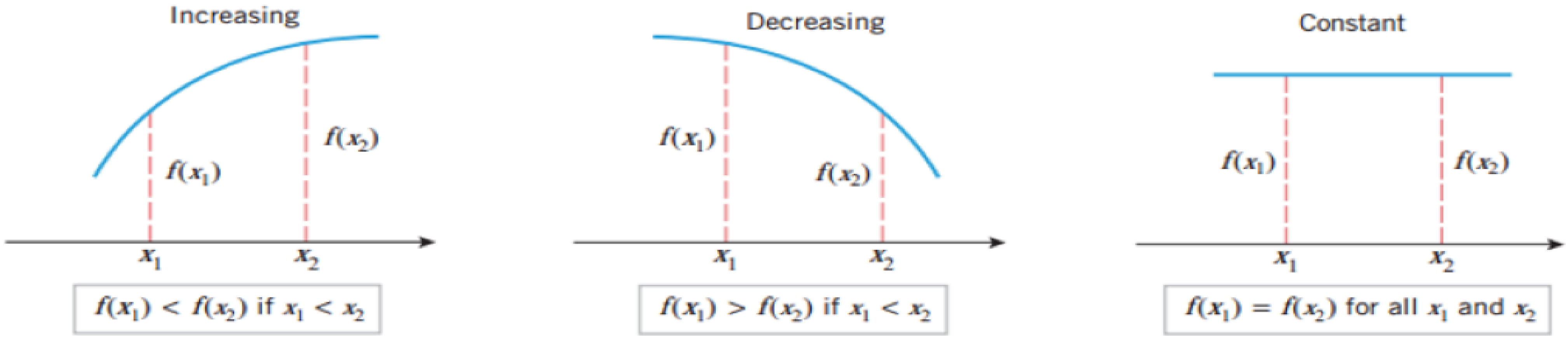
The terms increasing, decreasing, and constant are used to describe the behavior of a function as we travel left to right along its graph.

For example, the function graphed in following figure can be described as increasing to the left of $x = 0$, decreasing from $x = 0$ to $x = 2$, increasing from $x = 2$ to $x = 4$, and constant to the right of $x = 4$.



Definition of Increasing, Decreasing and Constant Functions

The definitions of "increasing," "decreasing," and "constant" describe the behavior of a function on an interval and not at a point.



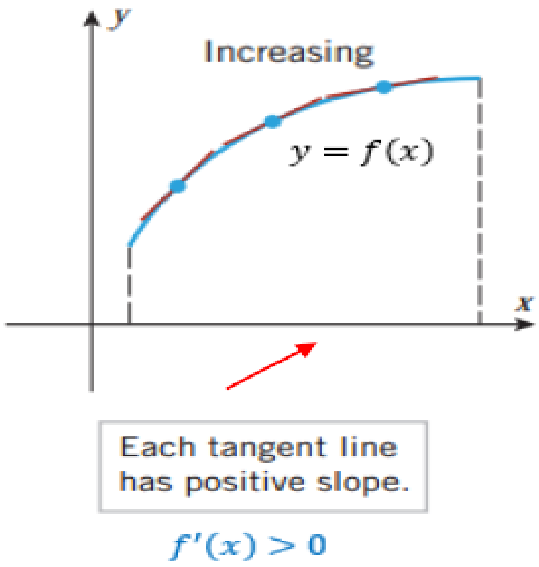
Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

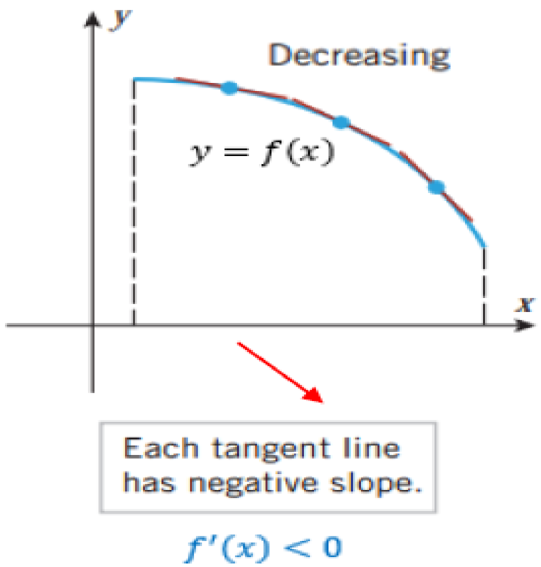
I. f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

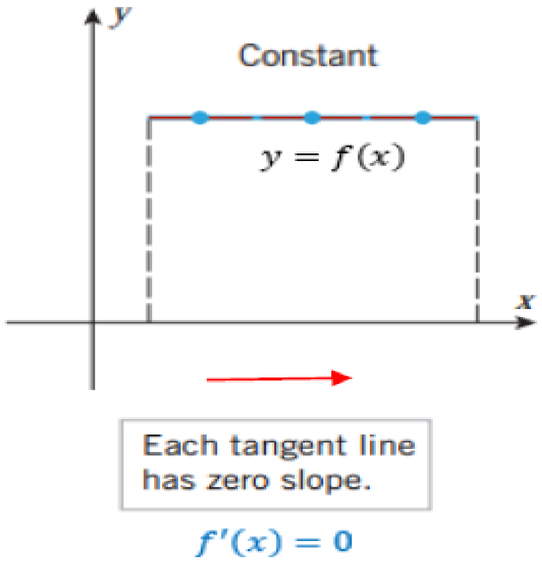
II. f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

III. f is constant on the interval if $f(x_1) = f(x_2)$) for all points x_1 and x_2.

Slope of Tangent and Behavior of a Function







Theorem 1

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

- **I.** If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.
- **II.** If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.
- **III.** If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

Example 1

a. Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

b. Draw the graph of the function and graphically show that your result is correct.

Solution (a):

The given function is $f(x) = x^2 - 4x + 3$

The derivative of $f(x)$ is $f'(x) = 2x - 4 = 2(x - 2)$

It follows that,

$$f'(x) < 0 \quad if \quad x < 2$$
$$f'(x) > 0 \quad if \quad x > 2$$

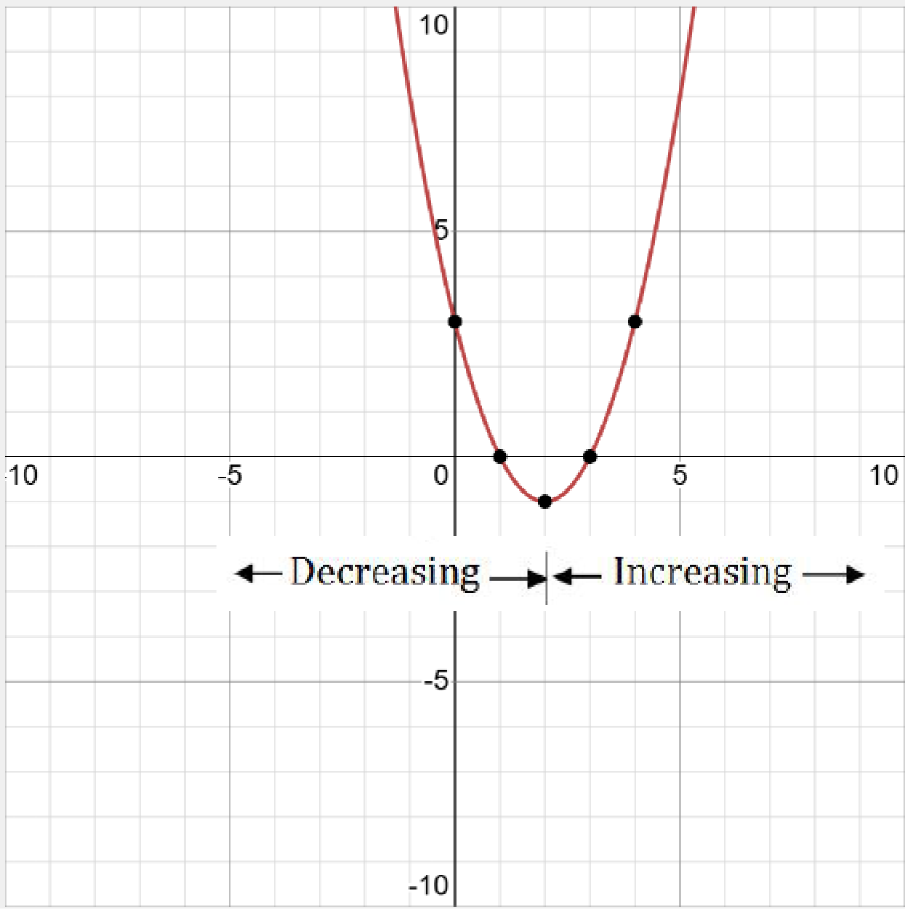
Since $f(x)$ is continuous everywhere, it follows that

f is decreasing on $(-\infty, 2]$

f is increasing on $[2, +\infty)$

Solution (b):

The graph of the given function $f(x) = x^2 - 4x + 3$ is given below:



The figure suggests that

f is decreasing on $(-\infty, 2]$

f is increasing on $[2, +\infty)$

Example 2

Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and the intervals on which it is decreasing.

Solution:

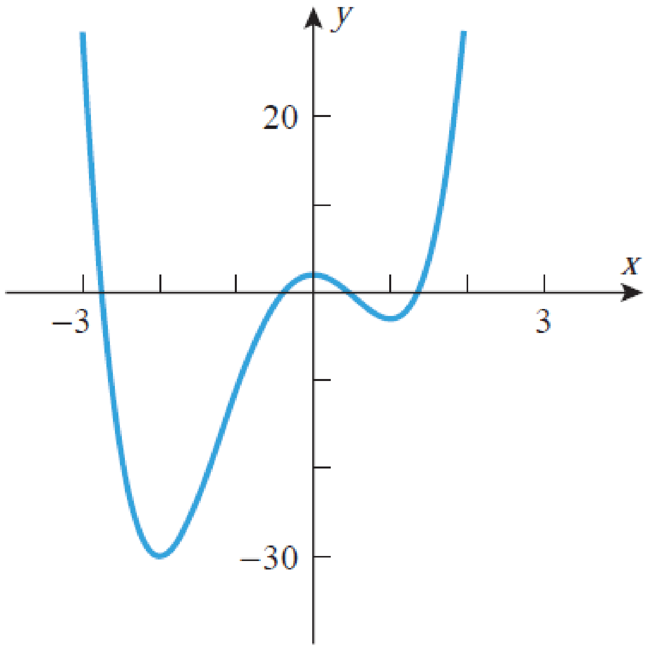
The given function is $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

The derivative of $f(x)$ is $f'(x) = 12x^3 + 12x^2 - 24x$

$$= 12x(x^2 + x - 2)$$

$$= 12x(x - 1)(x + 2)$$

INTERVAL	$(12x)(x+2)(x-1)$	$f'(x)$	CONCLUSION
$x < -2$	$(-)\,(-)\,(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)\,(+)\,(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)\,(+)\,(-)$	$-$	f is decreasing on $[0, 1]$
$1 < x$	$(+)\,(+)\,(+)$	$+$	f is increasing on $[1, +\infty)$

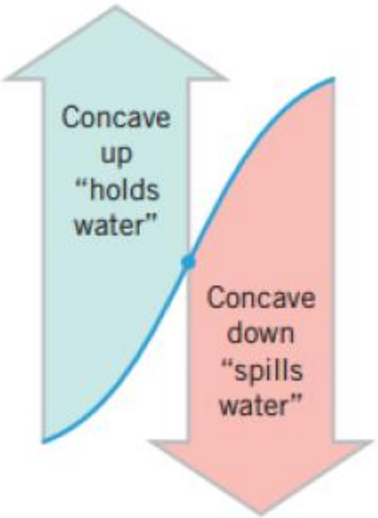


$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

CONCAVITY

Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of curvature.

For example, the graph is increasing on both sides of the point in figure, but on the left side it has an upward curvature ("holds water") and on the right side it has a downward curvature ("spills water").



On intervals where the graph of f has upward curvature we say that f is **concave up**, and on intervals where the graph has downward curvature we say that f is **concave down**.

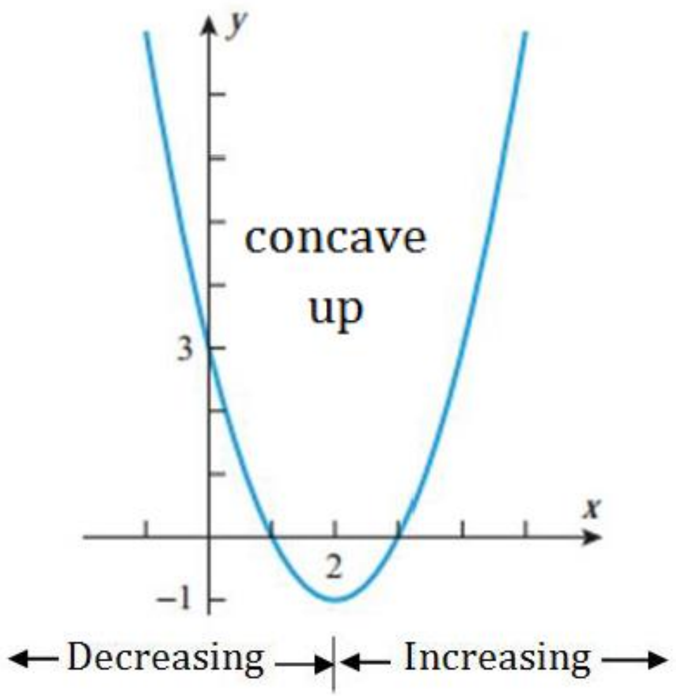
If f is differentiable on an open interval, then f is said to be **concave up** on the open interval if f' is increasing on that interval, and f is said to be **concave down** on the open interval if f' is decreasing on that interval.

THEOREM 2

Let f be twice differentiable on an open interval.

- If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.
- If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.

As for example the function $f(x) = x^2 - 4x + 3$ is concave up on the interval $(-\infty, +\infty)$. [see the figure]



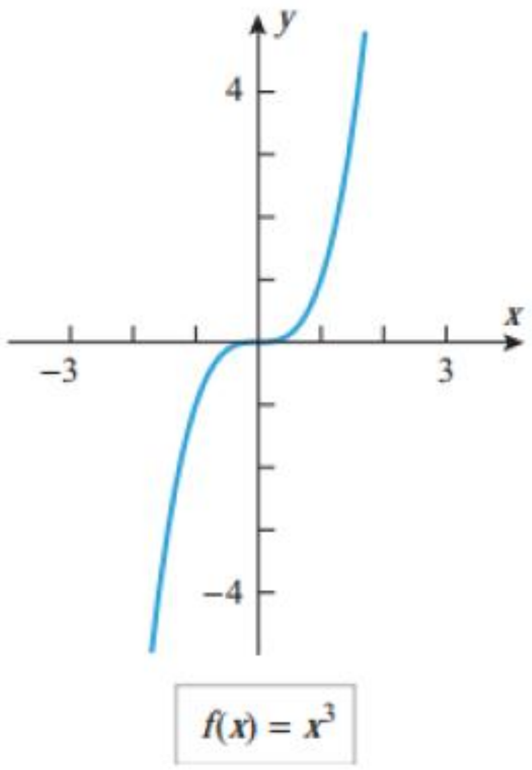
Mathematically we can show the same by using Theorem 2 that

$$f'(x) = 2x - 4 \text{ and } f''(x) = 2$$

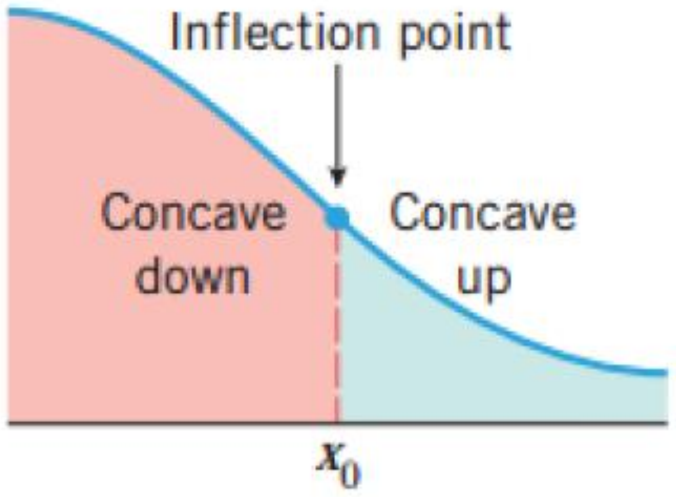
So, $f''(x) > 0$ on the interval $(-\infty, +\infty)$.

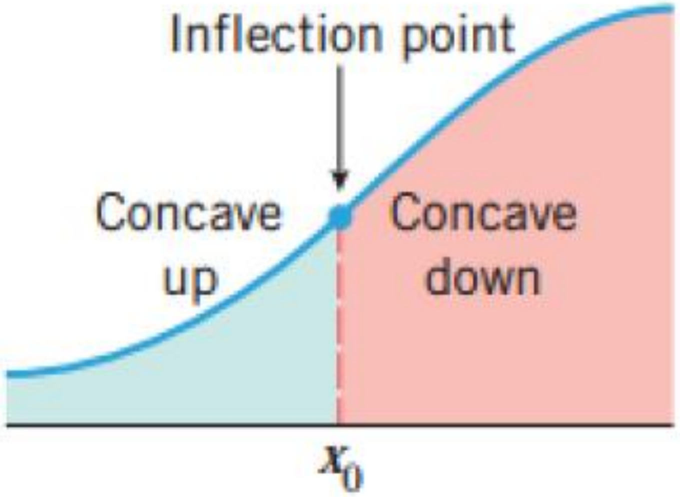
Inflection Point

We see from the figure that the graph of $f(x) = x^3$ changes from concave down to concave up at $x = 0$. Points where a curve changes from concave up to concave down or vice versa are of special interest, so there is some terminology associated with them.



If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an inflection point at x_0, and we call the point $(x_0, f(x_0))$on the graph of f an inflection point of f.





EXAMPLE 3

Let $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution:

The given function is $f(x) = x^3 - 3x^2 + 1$

The first derivative of $f(x)$ is $f'(x) = 3x^2 - 6x$

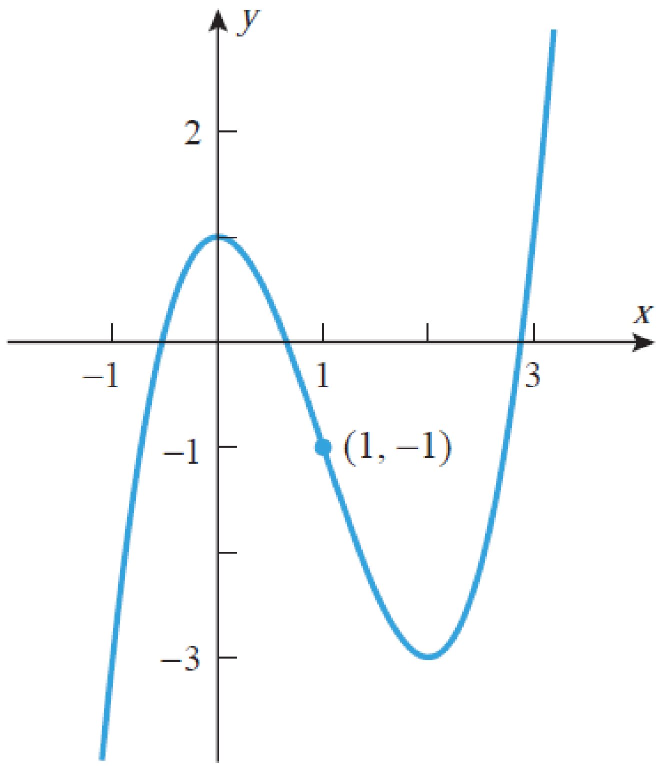
The second derivative of $f(x)$ is $f''(x) = 6x - 6 = 6(x - 1)$

The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$

The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f.



$f(x) = x^3 - 3x^2 + 1$

Example 4

Let $f(x) = xe^{-x}$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, concave down. Also locate all inflection points.

Solution:

The given function is $f(x) = xe^{-x}$

The first derivative of $f(x)$ is $f'(x) = (1 - x)\, e^{-x}$

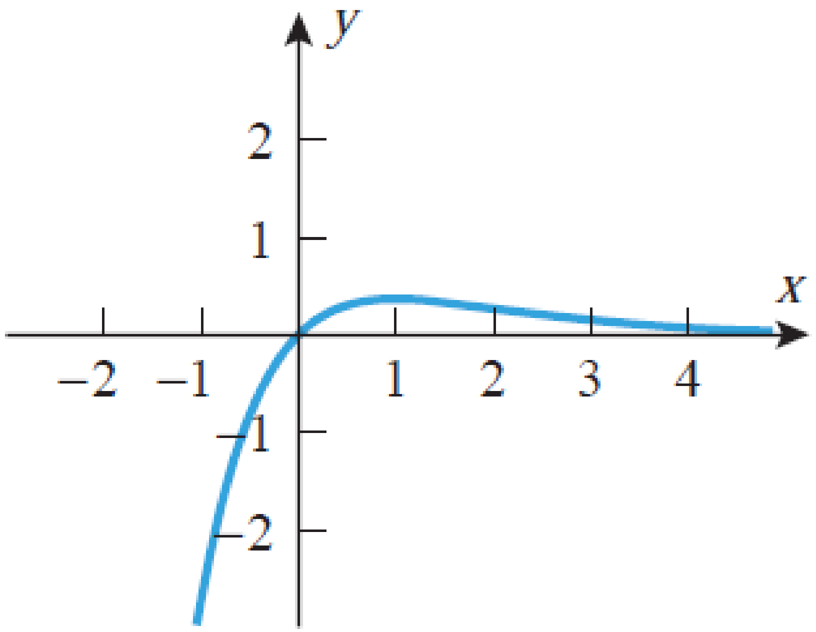
The second derivative of $f(x)$ is $f''(x) = (x - 2)\, e^{-x}$

Keeping in mind that e^{-x} is positive for all x, the sign analysis of these derivatives is easily determined:

INTERVAL	$(1-x)(e^{-x})$	$f'(x)$	CONCLUSION
$x < 1$	$(+)(+)$	$+$	f is increasing on $(-\infty, 1]$
$x > 1$	$(-)(+)$	$-$	f is decreasing on $[1, +\infty)$

INTERVAL	$(x-2)(e^{-x})$	$f''(x)$	CONCLUSION
$x < 2$	$(-)(+)$	$-$	f is concave down on $(-\infty, 2)$
$x > 2$	$(+)(+)$	$+$	f is concave up on $(2, +\infty)$

The second table shows that there is an inflection point at $x = 2$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f.



$f(x) = xe^{-x}$

Example 5

Find the inflection points of $f(x) = x^4$

Solution:

The given function is $f(x) = x^4$
The first derivative of $f(x)$ is $f'(x) = 4x^3$
The second derivative of $f(x)$ is $f''(x) = 12x^2$

Since $f(x)$ is positive for $x < 0$ and for $x > 0$, the function f is concave up on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$.

Thus, there is no change in concavity and hence no inflection point at $x = 0$, even though $f''(0) = 0$.

Practice Problems (Increasing, decreasing and concavity of a function)

- Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

a) $f(x) = 5 - 4x - x^2$

b) $f(x) = 5 + 12x - x^3$

c) $f(x) = x^{\frac{2}{3}} - x$

d) $f(x) = xe^{x^2}$