Propositional Logic, Discrete Mathematics

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Propositional Logic

- Propositional logic is a branch of logic that deals with propositions, which are statements that are either true or false.
- A proposition is a statement or sentence, that is either true or false.
- A proposition is denoted by letters such as p, q, and r.
- Example of propositions:
  - p: "It is raining."
  - q: "The sky is clear."
  - s: "The instructor of Discrete mathematics course is Dr. Mohammad Salah Uddin."
  - d: "Dhaka is the capital city of Canada."

Why a Question is Not a Proposition

- A proposition is a declarative sentence that is either **true** or **false**.
- Example of propositions:
  - "The sky is cloudy." (This can be evaluated as True or False)
  - "$2 + 2 = 4$." (True statement)
- A question cannot be a proposition because it does not have a truth value.
- Example of a question:
  - "What is the time?" (Cannot be classified as True or False)
  - "Are you coming?" (This is an inquiry, not a statement that can be evaluated)
- Only declarative sentences with definitive truth values qualify as propositions.

Non-Propositions

- A statement with variables, like $2x + 3 = 8$, is **not a proposition** because its truth value depends on the value of x.
- **Example of a non-proposition:**

$$2x + 3 = 8$$

- When $x = 2.5$, the equation becomes:

$$2(2.5) + 3 = 8$$

This can now be evaluated as true, making it a proposition.

Logical Connectives: Definition and Explanation

- **Logical Connectives** are used to connect propositions (statements that can either be true or false) to form new compound propositions.
- The truth value of the compound proposition depends on the truth values of elementary propositions and the logical connective that link them.
- There are six main types of logical connectives:
  1. **Negation** ($\neg$)
  2. **Conjunction** ($\wedge$)
  3. **Disjunction** ($\vee$)
  4. **Implication** ($\rightarrow$)
  5. **Biconditional** ($\leftrightarrow$)
  6. **XOR: Exclusive OR** ($\oplus$)

1. Negation ($\neg$)

- The **negation** of a proposition p, denoted $\neg p$, means "not p."
- It reverses the truth value of the proposition.
- If p is true, then $\neg p$ is false; if p is false, then $\neg p$ is true.

Example:

- p: "It is raining."
- $\neg p$: "It is not raining."

Truth Table of Negation

Description: The negation of a proposition P, denoted as $\neg P$, is true when P is false and false when P is true. This table illustrates the relationship between a proposition and its negation.

P	$\neg P$
True	False
False	True

2. Conjunction ($\wedge$)

- The **conjunction** of two propositions p and q, denoted $p \wedge q$, means "p and q."
- The conjunction is true if and only if both p and q are true.
- If either p or q is false, then $p \wedge q$ is false.

Example:

- p: "It is raining."
- q: "The sky is cloudy."
- $p \wedge q$: "It is raining and the sky is cloudy."

Conjunction ($\wedge$)

Conjunction (AND)

P	Q	$P \wedge Q$
True	True	True
True	False	False
False	True	False
False	False	False

3. Disjunction ($\vee$)

- The **disjunction** of two propositions p and q, denoted $p \vee q$, means "p or q."
- The disjunction is true if either p or q is true, or both are true.
- It is only false if both p and q are false.

Example:

- p: "It is raining."
- q: "The sky is cloudy."
- $p \vee q$: "It is raining or the sky is cloudy."

Disjunction ($\vee$)

Disjunction (OR)

P	Q	$P \vee Q$
True	True	True
True	False	True
False	True	True
False	False	False

4. Implication ($\rightarrow$)

- The **implication** $p \rightarrow q$ is read as "if p, then q."
- It means that if p is true, then q must also be true for the implication to be true.
- The implication is false only if p is true and q is false. In all other cases, it is true.

Example:

- p: "It is raining."
- q: "The road is wet."
- $p \rightarrow q$: "If it is raining, then the road is wet."

Implication ($\rightarrow$)

Implication (IF...THEN)

P	Q	$P \rightarrow Q$
True	True	True
True	False	False
False	True	True
False	False	True

Hypothesis and Conclusion in Implication

In logic, an implication is expressed as $P \to Q$, where:

- **Hypothesis:** The statement P is known as the hypothesis. It represents the condition that must be satisfied.
- **Conclusion:** The statement Q is known as the conclusion. It represents the outcome that follows if the hypothesis is true.

Example:
If P is "It is raining," and Q is "The road is wet," then:

$P \to Q$ translates to "If it is raining, then the road is wet."

5. Bi-conditional ($\leftrightarrow$)

- The **biconditional** $p \leftrightarrow q$ is read as "p if and only if q."
- It means that p and q must either both be true or both be false for the biconditional to be true.
- If p and q have different truth values, the bi-conditional is false.

Example:

- p: "The light is on."
- q: "The switch is up."
- $p \leftrightarrow q$: "The light is on if and only if the switch is up."

Bi-conditional ($\leftrightarrow$)

Bi-conditional (IF AND ONLY IF)

P	Q	$P \leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

6. XOR: Exclusive OR ($\oplus$)

- XOR (exclusive or) is a logical connective that outputs true if exactly one of the inputs is true, but not both.
- Denoted by: $p \oplus q$
- Can be represented by the logical expression:

$$p \oplus q = (p \vee q) \wedge \neg(p \wedge q)$$

Truth Table for XOR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

In XOR, two truth values are not agree.

Example of XOR

- Let p: "It is raining."
- Let q: "The sun is shining."
- $p \oplus q$: "It is raining or the sun is shining, but not both."
- This is true only if one of p or q is true, but not both.

Building Compound Propositions

- Using logical connectives, we can combine simple propositions to form compound propositions.
- The truth value of the compound proposition is determined by the truth values of the simple propositions and the logical connectives used.

Constructing Truth Table: $(p \vee q) \wedge \neg r$

p	q	r	$\neg r$	$p \vee q$	$(p \vee q) \wedge \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	F	F

Constructing a Truth Table (cont.)

Let's create a truth table for the following compound proposition:

$$(p \to q) \leftrightarrow (\neg p \vee q)$$

Where:

- p: Proposition 1
- q: Proposition 2

Logical Connectives:

- $\to$ is the implication (if-then): False only when the first is true and the second is false.
- $\leftrightarrow$ is the biconditional (if and only if): True when both propositions are either true or false.
- $\neg$ is the NOT operator.
- $\vee$ is the OR operator.

Constructing a Truth Table (cont.)

p	q	$\neg p$	$p \to q$	$\neg p \lor q$	$(p \to q) \leftrightarrow (\neg p \lor q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Explanation:

- Begin by listing all possible truth values for p and q.
- Calculate $\neg p$ (NOT p).
- Determine $p \to q$ (implication from p to q).
- Compute $\neg p \lor q$.
- Finally, evaluate the biconditional statement $(p \to q) \leftrightarrow (\neg p \lor q)$, comparing the truth values of the two expressions.

In this case, the compound proposition is true for all possible combinations of p and q.

XOR Truth Table

Consider the compound proposition:

$$p \oplus q$$

Where:

- $\oplus$ is XOR, which is true only when exactly one of p or q is true.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Explanation:

- XOR is true only when p and q have different truth values (one true and one false).

Complex Proposition

Let's consider a more complex proposition:

$$(p \vee \neg q) \wedge (q \rightarrow p)$$

Where:

- $\vee$: OR
- $\neg$: NOT
- $\wedge$: AND
- $\rightarrow$: Implication

p	q	$\neg q$	$p \vee \neg q$	$q \rightarrow p$	$(p \vee \neg q) \wedge (q \rightarrow p)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	F	F	F
F	F	T	T	T	T

Tautology, Contradiction, and Contingency

Tautology:

- A proposition that is always true, regardless of the truth values of the individual propositions involved.

Example:

$$p \vee \neg p$$

This is true whether p is true or false.

Contradiction:

- A proposition that is always false, no matter what the truth values of the individual propositions are.

Example:

$$p \wedge \neg p$$

This is false for all values of p.

Contingency:

- A proposition that can be either true or false depending on the truth values of the individual propositions involved.

Computer Representation of True and False

In Boolean Logic:

- Computers represent logical values using bits (binary digits).
- **True** is represented as **1**.
- **False** is represented as **0**.

In Programming Languages:

- Most programming languages use these binary values to represent Boolean conditions.
- Example in C/C++:
  - `true` is equivalent to **1**.
  - `false` is equivalent to **0**.

Performing bitwise operations

- **AND** operation ($x\&y$):

$$\begin{array}{r} 0110 \\ \&1001 \\ \hline 0000 \end{array}$$

- **OR** operation ($x|y$):

$$\begin{array}{r} 0110 \\ |1001 \\ \hline 1111 \end{array}$$

- **XOR** operation ($x \oplus y$):

$$\begin{array}{r} 0110 \\ \oplus 1001 \\ \hline 1111 \end{array}$$

- **NOT** operation ($\neg x$):

$$\begin{array}{r} \neg 0110 \\ \hline 1001 \end{array}$$

Translating a Logical Statement

Statement: You can have free coffee if you are a senior citizen and it is Tuesday.

Step 1: Identify Propositions

- p: You are a senior citizen.
- q: It is Tuesday.
- r: You can have free coffee.

Step 2: Determine Logical Connectives

- AND is represented by $\wedge$.
- The conditional if-then is represented by $\rightarrow$.

Step 3: Translate into Logical Symbols

$$(p \wedge q) \rightarrow r$$

Translating a Logical Statement (cont.)

Statement: You will pass the exam if you study hard or attend all the classes.

Step 1: Identify Propositions

- p: You study hard.
- q: You attend all the classes.
- r: You will pass the exam.

Step 2: Determine Logical Connectives

- OR is represented by $\vee$.
- The conditional if-then is represented by $\rightarrow$.

Step 3: Translate into Logical Symbols

$$(p \vee q) \rightarrow r$$

Translation Exercises

Exercise 1: Translate the following statement into logical symbols:

If you drive over the speed limit, then you will receive a ticket.

Let:

- p: You drive over the speed limit.
- q: You will receive a ticket.

Translation:

$$p \to q$$

Translation Exercises

Exercise 2: Translate the following statement into logical symbols:

You will not receive a ticket if you do not drive over the speed limit.

Let:

- p: You drive over the speed limit.
- q: You will receive a ticket.

Translation:

$$\neg p \to \neg q$$

Translation Exercises

Exercise 3: Translate the following statement into logical symbols:
You will receive a ticket if and only if you are driving blindly or over the speed limit.

Let:

- p: You are driving blindly.
- q: You drive over the speed limit.
- r: You will receive a ticket.

Translation:

$$r \leftrightarrow (p \vee q)$$

Equivalence

Definition: Two logical expressions are said to be **equivalent** if they have the same truth values for all possible combinations of truth values of their variables.

Example: Prove the equivalence of the expressions $p \to q$ and $\neg p \vee q$.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Conclusion: Since the columns for $p \to q$ and $\neg p \vee q$ are identical, we conclude that:

$$p \to q \equiv \neg p \vee q$$

Logical Equivalence: Definition and Laws

Definition: Two propositions P and Q are said to be **logically equivalent** if they have the same truth value for every possible truth assignment.

$$P \equiv Q \iff (P \leftrightarrow Q) \text{ is a tautology.}$$

Important Logical Equivalence Laws:

- **Identity Law:**

$$P \wedge \text{True} \equiv P \quad \text{and} \quad P \vee \text{False} \equiv P$$

- **Domination Law:**

$$P \vee \text{True} \equiv \text{True} \quad \text{and} \quad P \wedge \text{False} \equiv \text{False}$$

- **Idempotent Law:**

$$P \vee P \equiv P \quad \text{and} \quad P \wedge P \equiv P$$

Logical Equivalence: Laws

- **Double Negation Law:**

$$\neg(\neg P) \equiv P$$

- **Commutative Law:**

$$P \vee Q \equiv Q \vee P \quad \text{and} \quad P \wedge Q \equiv Q \wedge P$$

- **Associative Law:**

$$(P \vee Q) \vee R \equiv P \vee (Q \vee R) \quad \text{and} \quad (P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$$

- **Distributive Law:**

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Logical Equivalence: Laws

- **De Morgan's Laws:**

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

- **Absorption Law:**

$$P \vee (P \wedge Q) \equiv P \quad \text{and} \quad P \wedge (P \vee Q) \equiv P$$

- **Negation Laws:**

$$P \vee \neg P \equiv \text{True} \quad \text{and} \quad P \wedge \neg P \equiv \text{False}$$

**Refer the book for more details about logical equivalence.

Proof Using Logical Equivalence

Statement to Prove:

$$(P \wedge Q) \vee (\neg P) \equiv Q \vee \neg P$$

Step-by-step proof:

$$\begin{aligned}
(P \wedge Q) \vee \neg P &\equiv (\neg P) \vee (P \wedge Q) && \text{(Commutative Law)} \\
&\equiv (\neg P \vee P) \wedge (\neg P \vee Q) && \text{(Distributive Law)} \\
&\equiv \text{True} \wedge (\neg P \vee Q) && \text{(Negation Law: } \neg P \vee P \equiv \text{True)} \\
&\equiv \neg P \vee Q && \text{(Identity Law: True} \wedge X \equiv X)
\end{aligned}$$

Conclusion:

$$(P \wedge Q) \vee \neg P \equiv \neg P \vee Q$$

Hence, the original statement is true using logical equivalence.

Proof Using Logical Equivalence Laws

Example: Prove that $(P \wedge Q) \rightarrow P$ is a tautology.

Step-by-step proof:

$$
\begin{aligned}
(P \wedge Q) \rightarrow P &\equiv \neg(P \wedge Q) \vee P && \text{(Implication Law)} \\
&\equiv (\neg P \vee \neg Q) \vee P && \text{(De Morgan's Law)} \\
&\equiv \neg P \vee (P \vee \neg Q) && \text{(Associative Law)} \\
&\equiv \text{True} \vee \neg Q && \text{(Negation Law: } P \vee \neg P \equiv \text{True)} \\
&\equiv \text{True} && \text{(Domination Law: True} \vee X \equiv \text{True)}
\end{aligned}
$$

Since the final result is True, we conclude that $(P \wedge Q) \rightarrow P$ is a tautology.

Exercises on Logical Equivalence

Exercise 1: Prove the following equivalence:

$$(P \lor Q) \land \neg Q \equiv P$$

Exercise 2: Use logical equivalences to simplify:

$$\neg(P \land Q) \lor (\neg P \land R)$$

Exercise 3: Prove that:

$$(P \land Q) \to R \equiv \neg R \to \neg(P \land Q)$$

Exercise 4: Show that the following statement is a contradiction:

$$(P \land \neg P) \lor (Q \land \neg Q)$$