# Mathematical Induction

Discrete Mathematics

# What is induction?

- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: assume it is true for any given element
    - **Must be clearly labeled!!!**
  - Show that if it true for the next highest element

# Induction example

- Show that the sum of the first *n* odd integers is $n^2$
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, Show $\forall n \; P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 == n^2$
- Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 == 1^2$$
$$= 1 == 1$$

# Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{\boxed{i=1}}^{\boxed{k}} 2i-1 == \boxed{k}^2$$

  - Note: we don't yet know if this is true or not!
- Inductive step: show true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i-1 == \boxed{(k+1)}^2$$

# Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i-1 == k^2$

- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

$$2(k+1)-1+\boxed{\sum_{i=1}^{k} 2i-1} == k^2+2k+1$$

$$2(k+1)-1+\boxed{k^2} == k^2+2k+1$$

$$k^2+2k+1 == k^2+2k+1$$

# What did we show

- Base case: P(1)
- If P($k$) was true, then P($k$+1) is true
  - i.e., P($k$) → P($k$+1)

- We know it's true for P(1)
- Because of P($k$) → P($k$+1), if it's true for P(1), then it's true for P(2)
- Because of P($k$) → P($k$+1), if it's true for P(2), then it's true for P(3)
- Because of P($k$) → P($k$+1), if it's true for P(3), then it's true for P(4)
- Because of P($k$) → P($k$+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of $n$

- In other words, we showed that:

$$\left[\mathrm{P}(1) \wedge \forall k \left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$$

# The idea behind inductive proofs

- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

# Second induction example

- Rosen, section 3.3, question 2:
  - Show the sum of the first *n* positive even integers is $n^2 + n$
  - Rephrased: $$\forall n \, \mathrm{P}(n) \text{ where} \mathrm{P}(n) = \sum_{i=1}^{n} 2i == n^2 + n$$
- The three parts:
  - Base case
  - Inductive hypothesis
  - Inductive step

# Second induction example, continued

- Base case: Show P(1):

$$P(1) = \sum_{i=1}^{1} 2(i) == 1^2 + 1$$
$$= 2 == 2$$

- Inductive hypothesis: Assume

$$P(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

- Inductive step: Show

$$P(k+1) = \sum_{i=1}^{k+1} 2i == (k+1)^2 + (k+1)$$

# Second induction example, continued

- Recall our inductive hypothesis: $P(k) = \sum_{i=1}^{k} 2i == k^2 + k$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

# Notes on proofs by induction

- We manipulate the *k*+1 case to make part of it look like the *k* case
- We then replace that part with the other side of the *k* case

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

# Third induction example

- Rosen, question 7: Show $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- Base case: *n* = 1 $$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$ $$1^2 = \frac{6}{6}$$ $$1 = 1$$
- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

# Third induction example

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

# Third induction again: what if your inductive hypothesis was wrong?

- Show: $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$$
- Base case: $n = 1$:

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$$

$$1^2 = \frac{7}{6}$$

$$1 \neq \frac{7}{6}$$

- But let's continue anyway…
- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

# Third induction again: what if your inductive hypothesis was wrong?

- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$

$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

# Fourth induction example

- Rosen, question 14: show that $n! < n^n$ for all $n > 1$

- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$

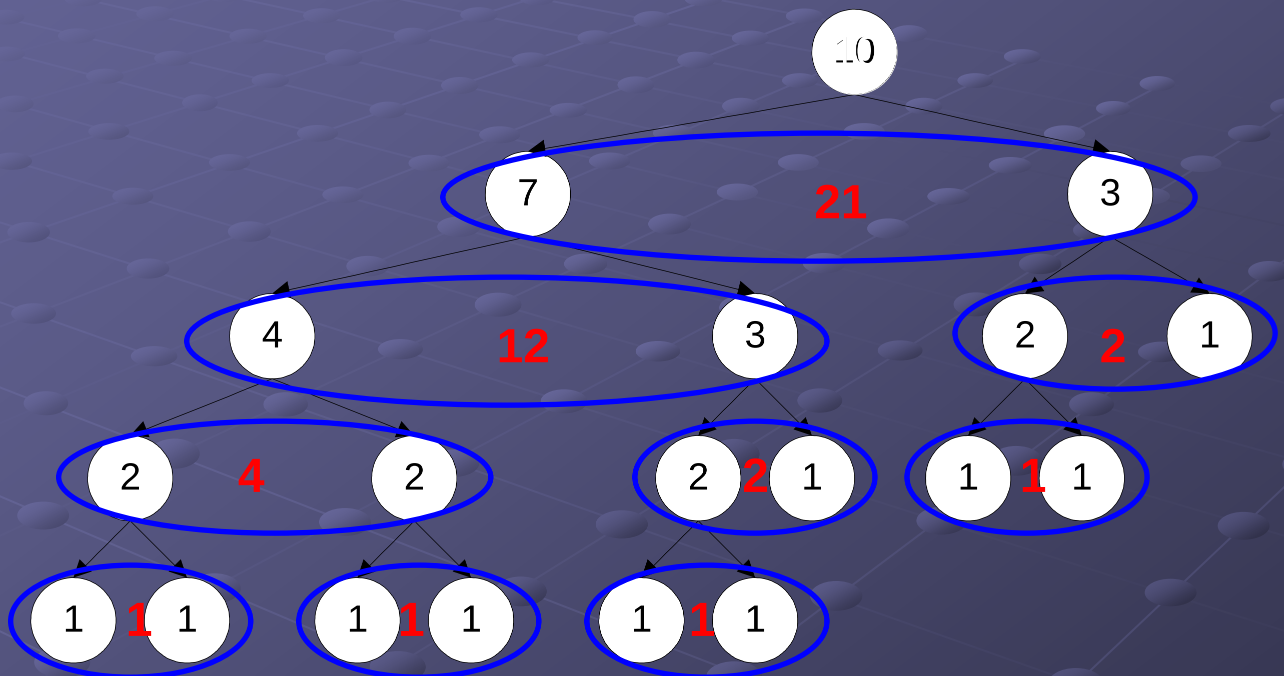
- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = (k+1)\boxed{k!} < (k+1)\boxed{k^k} < (k+1)(k+1)^k = (k+1)^{k+1}$$

# Question 40

- Take a pile of *n* stones
  - Split the pile into two smaller piles of size *r* and *s*
  - Repeat until you have *n* piles of 1 stone each
- Take the product of **all** the splits
  - So all the *r*'s and *s*'s from **each** split
- Sum up each of these products
- Prove that this product equals $\frac{n(n-1)}{2}$

# Question 40




$$\frac{n(n-1)}{2}$$

$$21+12+2+4+2+1+1+1+1=45=\frac{10*9}{2}$$

# Question 40

- We will show it is true for a pile of $k$ stones, and show it is true for $k+1$ stones
  - So $P(k)$ means that it is true for $k$ stones

- Base case: $n = 1$
  - No splits necessary, so the sum of the products = 0
  - $1*(1-1)/2 = 0$
  - Base case proven

# Question 40

- Inductive hypothesis: assume that P(1), P(2), …, P($k$) are all true
  - This is strong induction!
- Inductive step: Show that P($k$+1) is true
  - We assume that we split the $k$+1 pile into a pile of $i$ stones and a pile of $k$+1-$i$ stones
  - Thus, we want to show that ($i$)*($k$+1-$i$) + P($i$) + P($k$+1-$i$) = P($k$+1)
  - Since $0 < i < k+1$, both $i$ and $k$+1-$i$ are between 1 and $k$, inclusive

# Question 40

- Thus, we want to show that

($i$)*($k$+1-$i$) + P($i$) + P($k$+1-$i$) = P($k$+1)

$$P(i) = \frac{i^2 - i}{2}$$

$$P(k+1-i) = \frac{(k+1-i)(k+1-i-1)}{2} = \frac{k^2 + k - 2ki - i + i^2}{2}$$

$$P(k+1) = \frac{(k+1)(k+1-1)}{2} = \frac{k^2 + k}{2}$$

$$(i)*(k+1-i) + P(i) + P(k+1-i) = P(k+1)$$

$$ki + i - i^2 + \frac{i^2 - i}{2} + \frac{k^2 + k - 2ki - i - i^2}{2} = \frac{k^2 + k}{2}$$

$$2ki + 2i - 2i^2 + i^2 - i + k^2 + k - 2ki - i + i^2 = k^2 + k$$

$$k^2 + k = k^2 + k$$