Topic: Bernoulli's Equation: Theory and Problem-Solving

Statement of Bernoulli's Equation

A fundamental principle that explains how energy behaves in fluid systems.

In fluid dynamics, **Bernoulli's Equation** states that:

In a steady, incompressible flow, the sum of pressure energy, kinetic energy, and potential energy per unit volume remains constant.

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

where:

- P: Static pressure (Pa),
- ρ: Fluid density (kg/m^3),
- v: Flow velocity (m/s),
- g: Acceleration due to gravity (m/s^2),
- h: Height above a reference point (m).

Problem 1: Velocity and Pressure in a Pipe

Water flows through a horizontal pipe of diameter $D_1 = 0.1\,m$ at velocity $v_1 = 2\,m/s$. The pipe narrows to $D_2 = 0.05\,m$. Calculate the velocity and pressure difference between the wide and narrow sections. Assume water density $\rho = 1000\,kg/m^3$.

Solution:

1. **Find the velocity in the narrow section:** Using the **continuity equation**,

$$A_1 v_1 = A_2 v_2,$$

where $A = \frac{\pi D^2}{4}$, we find:

$$v_2 = v_1 \frac{A_1}{A_2} = v_1 \frac{D_1^2}{D_2^2}.$$

Substituting values:

$$v_2 = 2\,m/s \times \left(\frac{0.1}{0.05}\right)^2 = 8\,m/s.$$

2. **Calculate the pressure difference:** Apply Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

Rearranging for $(P_1 - P_2)$:

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2).$$

Substituting values:

$$P_1 - P_2 = \frac{1}{2} \times 1000\,(8^2 - 2^2) = 1000 \times 30 = 30,000\,\mathrm{Pa}.$$

Answer: The velocity in the narrow section is $8\,m/s$, and the pressure difference is $30,000\,\mathrm{Pa}$.

Problem 2: Height and Velocity in a Tank

Water flows out of a large open tank through a small hole at the bottom. If the water level is $h = 5\,m$ above the hole, find the velocity of the water exiting the hole. Assume negligible velocity at the tank surface and $\rho = 1000\,kg/m^3$.

Solution:

Simplifying Bernoulli's equation: At the water surface ($P_1 = P_{\text{atm}}, v_1 \approx 0, h_1 = h$) and at the hole ($P_2 = P_{\text{atm}}, h_2 = 0$):

$$\frac{1}{2}\rho v_2^2 = \rho g h.$$

$$v_2 = \sqrt{2gh}.$$

Substituting $g = 9.8\,m/s^2$ and $h = 5\,m$:

$$v_2 = \sqrt{2 \times 9.8 \times 5} = \sqrt{98} \approx 9.9\,m/s.$$

Answer: The velocity of water exiting the hole is approximately $9.9\,m/s$.

Problem 3: Water Flow Over a Weir

Water flows over a weir (a small dam) and falls freely. At the top of the weir, the water velocity is $v_1 = 1.5\,m/s$ and the height above the base is $h_1 = 4\,m$. At the bottom, the height is $h_2 = 0$ and the velocity is v_2. Assuming atmospheric pressure at both points, calculate v_2.

Solution:

Applying Bernoulli's equation between the top (1) and the bottom (2):

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2.$$

Since $P_1 = P_2$ (atmospheric pressure) and $h_2 = 0$:

$$\frac{1}{2}\rho v_1^2 + \rho g h_1 = \frac{1}{2}\rho v_2^2.$$

$$v_2^2 = v_1^2 + 2gh_1.$$

Substituting $v_1 = 1.5\,m/s$, $g = 9.8\,m/s^2$, and $h_1 = 4\,m$:

$$v_2^2 = (1.5)^2 + 2 \times 9.8 \times 4 = 2.25 + 78.4 = 80.65.$$

$$v_2 = \sqrt{80.65} \approx 8.98\,m/s.$$

Answer: The velocity at the bottom of the weir is approximately $8.98\,m/s$.

Problem 4: Airflow Through a Tube

Air flows through a tube where the cross-sectional area reduces from $A_1 = 0.05\,m^2$ to $A_2 = 0.02\,m^2$. The velocity at the wide section is $v_1 = 3\,m/s$, and the density of air is $\rho = 1.2\,kg/m^3$. Calculate the pressure difference between the wide and narrow sections.

Solution:

Using the continuity equation:

$$A_1 v_1 = A_2 v_2.$$

Substituting:

$$v_2 = v_1 \frac{A_1}{A_2} = 3 \times \frac{0.05}{0.02} = 7.5\,m/s.$$

Applying Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

Rearranging for $(P_1 - P_2)$:

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2).$$

Substituting $\rho = 1.2\,kg/m^3$, $v_1 = 3\,m/s$, and $v_2 = 7.5\,m/s$:

$$P_1 - P_2 = \frac{1}{2} \times 1.2 \times (7.5^2 - 3^2).$$

$$P_1 - P_2 = 0.6 \times (56.25 - 9) = 0.6 \times 47.25 = 28.35\,\text{Pa}.$$

Answer: The pressure difference between the wide and narrow sections is $28.35\,\text{Pa}$.

Problem 5: Flow Through a Nozzle

Water flows through a nozzle of a garden hose. At the inlet, the velocity is $v_1 = 2\,m/s$ and the pressure is $P_1 = 33000\,\text{Pa}$. The nozzle's exit diameter is smaller than the inlet, causing the velocity at the nozzle's exit to be $v_2 = 8\,m/s$. If the height difference between the inlet and the exit is negligible, calculate the pressure at the nozzle's exit, P_2.

Solution:

Applying Bernoulli's equation between the inlet and the nozzle exit:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

Rearranging for P_2:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2).$$

Substituting the known values: $\rho = 1000\,\text{kg/m}^3$, $P_1 = 33000\,\text{Pa}$, $v_1 = 2\,m/s$, and $v_2 = 8\,m/s$:

$$P_2 = 3000 + \frac{1}{2} \times 1000 \times (2^2 - 8^2) = 3000 + 500 \times (4 - 64) = 33000 - 30,000 = 3000\,\text{Pa}.$$

Answer: The pressure at the nozzle exit is $P_2 = 3000\,\text{Pa}$.