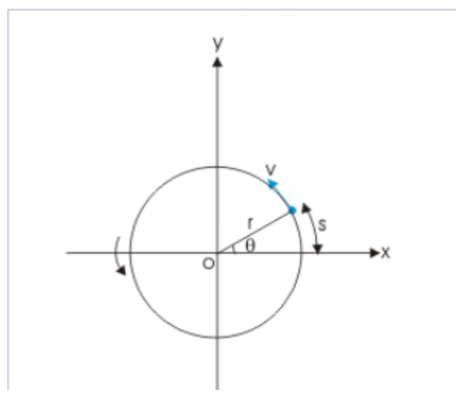
Topic: Rotational Motion

For the rotation of a rigid body it rotates about a fixed axis, called **the rotation axis**.

Angular position

Angular position θ is measured in radians,

$$\theta = \frac{s}{r}, \tag{1}$$



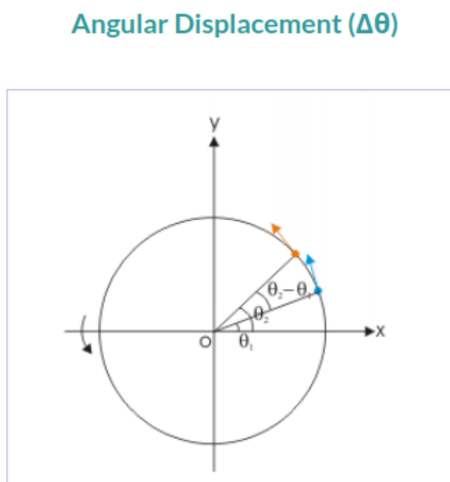
where s is the arc length of a circular path of radius r and angle θ.

Angular Displacement

The angular displacement of the body is given by:

$$\Delta\theta = \theta_2 - \theta_1, \tag{2}$$

where $\Delta\theta$ is **positive for counterclockwise** rotation and **negative for clockwise rotation**.



Angular Velocity

If a body rotates through an angular displacement $\Delta\theta$ in a time interval Δt, its average angular velocity is given by:

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}. \tag{3}$$

The instantaneous angular velocity is defined as:

$$\omega = \frac{d\theta}{dt}. \tag{4}$$

Angular Acceleration

If the angular velocity of a body changes from ω_1 to ω_2 in a time interval $\Delta t = t_2 - t_1$, the average angular acceleration is given by:

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}. \tag{5}$$

The instantaneous angular acceleration is:

$$\alpha = \frac{d\omega}{dt}. \tag{6}$$

Both α_{avg} and α are vectors.
The relationship between different units of angular measurement is given by:

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}. \tag{7}$$

Problem 1:

A child's top is spun with angular acceleration

$$\alpha = 5t + 4, \tag{8}$$

with t in seconds and α in radians per second-squared. At $t = 0$, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad. Find :

- **Expression for Angular Velocity ,** $\omega(t)$
- **Expression for Angular Position ,** $\theta(t)$

(a) Expression for Angular Velocity

The angular velocity is found by integrating the angular acceleration:

$$\omega(t) = \int (5t + 4) dt. \tag{9}$$

Evaluating the integral,

$$\omega(t) = \frac{5}{2}t^2 + 4t + C. \tag{10}$$

Using the initial condition $\omega(0) = 5$:

$$5 = \frac{5}{2}(0)^2 + 4(0) + C \Rightarrow C = 5. \tag{11}$$

Thus, the angular velocity is:

$$\omega(t) = \frac{5}{2}t^2 + 4t + 5. \tag{12}$$

(b) Expression for Angular Position

The angular position is found by integrating angular velocity:

$$\theta(t) = \int \left(\frac{5}{2}t^2 + 4t + 5\right) dt. \tag{13}$$

Evaluating the integral,

$$\theta(t) = \frac{5}{6}t^3 + 2t^2 + 5t + C. \tag{14}$$

Using the initial condition $\theta(0) = 2$:

$$2 = \frac{5}{6}(0)^3 + 2(0)^2 + 5(0) + C \Rightarrow C = 2. \tag{15}$$

Thus, the angular position is:

$$\theta(t) = \frac{5}{6}t^3 + 2t^2 + 5t + 2. \tag{16}$$

Kinematic Equations

For **constant angular acceleration** ($\alpha =$ constant) the appropriate kinematic equations are

$$\omega = \omega_0 + \alpha t, \tag{17}$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2, \tag{18}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0), \tag{19}$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t, \tag{20}$$

$$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2. \tag{21}$$

Problem 2:

A grindstone rotates with a constant angular acceleration of

$$\alpha = 0.35 \ \mathbf{rad/s}^2.$$

At time $t = 0$, its initial angular velocity is

$$\omega_0 = -4.6 \ \mathbf{rad/s},$$

and a reference line on it is initially at the angular position

$$\theta_0 = 0.$$

1. **At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0$ revolutions?**
2. **At what time t does the grindstone momentarily stop?**

Solution

(a) Time for the reference line to reach 5.0 **revolutions**

We use the kinematic equation for rotational motion:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$$

Since $\theta_0 = 0$ and $\theta = 5.0$ revolutions (where 1 revolution $= 2\pi$ radians), we have:

$$\theta = 5.0 \times 2\pi = 10\pi \text{ rad}.$$

Substituting values:

$$10\pi = -4.6t + \frac{1}{2}(0.35)t^2.$$

$$10\pi = -4.6t + 0.175t^2.$$

Rearranging:

$$0.175t^2 - 4.6t + 10\pi = 0.$$

Solving t= 32 s

(b) Time when the grindstone momentarily stops

The grindstone stops when its angular velocity becomes zero:

$$\omega = \omega_0 + \alpha t.$$

Setting $\omega = 0$:

$$0 = -4.6 + (0.35)t.$$

Solving for t:

$$t = \frac{4.6}{0.35} \approx 13.14 \text{ s}.$$

Thus, the grindstone momentarily stops at $t \approx 13.14$ s.

Relation between Linear and Angular Variables:

- A point in a rigid rotating body, at a perpendicular distance r from the rotation axis, moves in a circle with radius r. If the body rotates through an angle θ, the point moves along an arc with length s given by

  $$s = \theta r \quad \text{(radian measure)}, \tag{22}$$

  where θ is in radians.

- The linear velocity **v** of the point is tangent to the circle; the point's linear speed v is given by

  $$v = \omega r \quad \text{(radian measure)}, \tag{23}$$

  where ω is the angular speed (in radians per second) of the body.

- The linear acceleration **a** of the point has both tangential and radial components. The tangential component is

$$a_t = \alpha r \quad \text{(radian measure)}, \tag{24}$$

where α is the magnitude of the angular acceleration (in radians per second squared) of the body.

The radial component of **a** is

$$a_r = \frac{v^2}{r} = \omega^2 r \quad \text{(radian measure)}. \tag{25}$$

- If the point moves in uniform circular motion, the period T of the motion for the point and the body is

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} \quad \text{(radian measure)}. \tag{26}$$

Kinetic Energy of Rotation

A rotating rigid body has kinetic energy:

$$K = \sum \frac{1}{2} m_i v_i^2. \tag{27}$$

Since $v_i = r_i \omega$, we get:

$$K = \sum \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2. \tag{28}$$

Defining the moment of inertia:

$$I = \sum m_i r_i^2, \tag{29}$$

This quantity tells us how the mass of the rotating body is distributed about its axis of rotation. We call that quantity the rotational inertia (or moment of inertia) I of the body with respect to the axis of rotation. we obtain the rotational kinetic energy:

$$K = \frac{1}{2} I \omega^2. \tag{30}$$

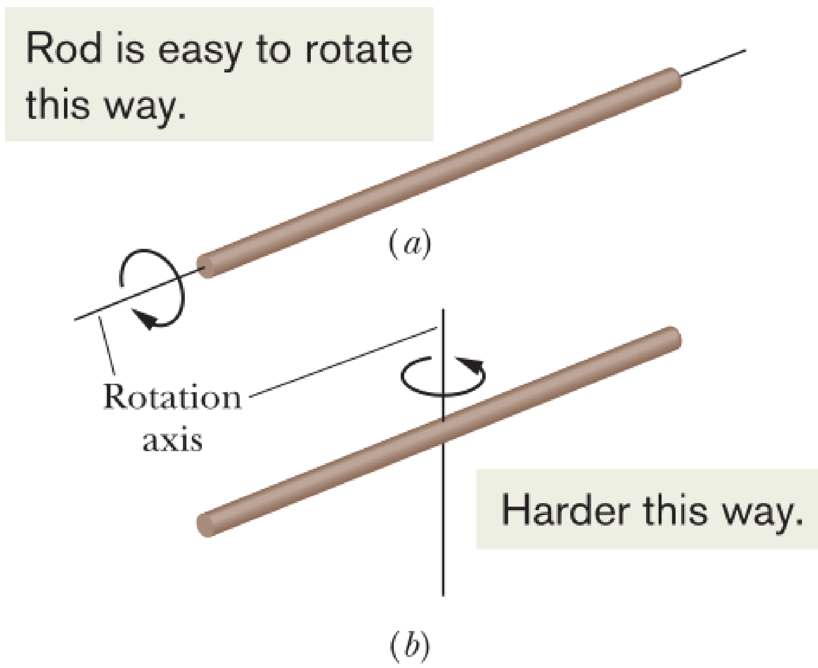


Figure 1: A long rod is much easier to rotate about (a) its central (longitudinal) axis than about (b) an axis through its center and perpendicular to its length. The reason for the difference is that the mass is distributed closer to the rotation axis in (a) than in (b)

Torque

Torque (τ) is the rotational equivalent of force, defined as:

$$\tau = rF\sin\theta, \tag{31}$$

where r is the position vector, F is the applied force, and θ is the angle between them.

In vector form:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}. \tag{32}$$

Newton's Second Law in Angular Form

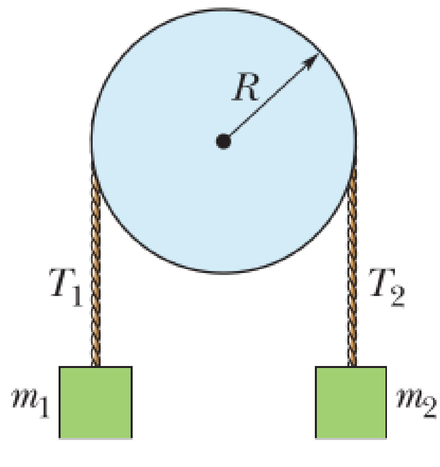
For a rigid body with moment of inertia I and angular acceleration α:

$$\tau = I\alpha. \tag{33}$$

This is the rotational analog of Newton's second law.

Problem 3:

Block 1 has a mass of $m_1 = 460$ g, and block 2 has a mass of $m_2 = 500$ g. The pulley, which is mounted on a horizontal axle with negligible friction, has a radius $R = 5.00$ cm. When released from rest, block 2 falls a distance of 75.0 cm in 5.00 s without the cord slipping on the pulley.



Find:

1. **The acceleration of the blocks.**
2. **The tensions T_1 and T_2.**
3. **The angular acceleration of the pulley.**
4. **The rotational inertia of the pulley.**

Solution

(a) Using kinematic equations, we find the acceleration. Since the block falls a distance $y = 0.750$ m in time $t = 5.00$ s, using

$$y = \frac{1}{2}at^2, \tag{34}$$

we solve for a:

$$a = \frac{2y}{t^2} = \frac{2(0.750 \text{ m})}{(5.00 \text{ s})^2} = 6.00 \times 10^{-2} \text{ m/s}^2. \tag{35}$$

Block 1 has an acceleration of 6.00×10^{-2} m/s^2 upward.

(b) Using Newton's second law for block 2:

$$m_2 g - T_2 = m_2 a, \tag{36}$$

solving for T_2:

$$T_2 = m_2(g - a) = (0.500 \text{ kg}) \times (9.8 - 0.060) \text{ m/s}^2 = 4.87 \text{ N}. \tag{37}$$

(c) Using Newton's second law for block 1:

$$T_1 - m_1 g = m_1 a, \tag{38}$$

solving for T_1:

$$T_1 = m_1(g + a) = (0.460 \text{ kg}) \times (9.8 + 0.060) \text{ m/s}^2 = 4.54 \text{ N}. \tag{39}$$

(d) Since the cord does not slip, the tangential acceleration of the pulley's rim equals the acceleration of the blocks:

$$\alpha = \frac{a}{R} = \frac{6.00 \times 10^{-2} \text{ m/s}^2}{5.00 \times 10^{-2} \text{ m}} = 1.20 \times 10^0 \text{ rad/s}^2. \tag{40}$$

(e) The net torque on the pulley is:

$$\tau = (T_2 - T_1)R. \tag{41}$$

Using $\tau = I\alpha$, we solve for I:

$$I = \frac{(T_2 - T_1)R}{\alpha} = \frac{(4.87 - 4.54) \text{ N} \times (5.00 \times 10^{-2} \text{ m})}{1.20 \text{ rad/s}^2}. \tag{42}$$

Simplifying,

$$I = 1.38 \times 10^{-2} \text{ kg} \cdot \text{m}^2. \tag{43}$$

Problem 4:

During the launch from a board, a diver's angular speed about her center of mass changes from zero to 6.20 rad/s in 220 ms. Her rotational inertia about her center of mass is 12.0 kg·m^2. Find:

1. **The magnitude of her average angular acceleration.**
2. **The magnitude of the average external torque on her from the board.**

Solution

(a) The angular acceleration is given by:

$$\alpha = \frac{\omega - \omega_0}{t} \tag{44}$$

where $\omega_0 = 0$, $\omega = 6.20$ rad/s, and $t = 220$ ms $= 220 \times 10^{-3}$ s. Substituting the values:

$$\alpha = \frac{6.20 \text{ rad/s} - 0}{220 \times 10^{-3} \text{ s}} = 28.2 \text{ rad/s}^2. \tag{45}$$

(b) The average external torque is given by:

$$\tau = I\alpha, \tag{46}$$

where $I = 12.0$ kg·m^2 and $\alpha = 28.2$ rad/s^2. Substituting:

$$\tau = (12.0 \text{ kg} \cdot \text{m}^2)(28.2 \text{ rad/s}^2). \tag{47}$$

Simplifying,

$$\tau = 3.38 \times 10^2 \text{ N} \cdot \text{m}. \tag{48}$$

Problem 5

A pulley, with a rotational inertia of 1.0×10^{-3} kg·m^2 about its axle and a radius of 10 cm, is acted on by a force applied tangentially at its rim. The force magnitude varies in time as

$$F = 0.50t + 0.30t^2, \tag{49}$$

with F in newtons and t in seconds. The pulley is initially at rest.

Find:

1. **The angular acceleration at $t = 3.0$ s.**
2. **The angular speed at $t = 3.0$ s.**

Solution

(a) The torque exerted by the force about the axle is:

$$\tau = Fr = (0.50t + 0.30t^2)(0.10 \text{ m}). \tag{50}$$

Since torque and rotational inertia are related by

$$\tau = I\alpha, \tag{51}$$

we solve for angular acceleration α:

$$\alpha = \frac{\tau}{I} = \frac{(0.50t + 0.30t^2)(0.10)}{1.0 \times 10^{-3}}. \tag{52}$$

Simplifying,

$$\alpha = 50t + 30t^2 \quad (\text{rad/s}^2). \tag{53}$$

At $t = 3.0$ s:

$$\alpha = 50(3.0) + 30(3.0)^2 = 150 + 270 = 4.2 \times 10^2 \ \text{rad/s}^2. \tag{54}$$

(b) The angular speed ω is obtained by integrating α with respect to time:

$$\omega = \int \alpha dt = \int (50t + 30t^2) dt. \tag{55}$$

Evaluating the integral:

$$\omega = \left(25t^2 + 10t^3\right)\Big|_0^3. \tag{56}$$

Substituting $t = 3.0$ s:

$$\omega = 25(3.0)^2 + 10(3.0)^3 = 225 + 270 = 5.0 \times 10^2 \ \text{rad/s}. \tag{57}$$

Problem 6:

A torque of 32.0 N·m applied to a wheel causes an angular acceleration of 25.0 rad/s^2.

Find: *The rotational inertia of the wheel.*

Solution

The rotational inertia I is related to the torque τ and angular acceleration α by the equation:

$$\tau = I\alpha. \tag{58}$$

Rearranging for I:

$$I = \frac{\tau}{\alpha}. \tag{59}$$

Substituting the given values:

$$I = \frac{32.0 \ \text{N} \cdot \text{m}}{25.0 \ \text{rad/s}^2}. \tag{60}$$

Simplifying,

$$I = 1.28 \ \text{kg} \cdot \text{m}^2. \tag{61}$$

Thus, the rotational inertia of the wheel is 1.28 kg·m^2.