Topic: Electric flux, Gauss' law

Electric Flux

The electric flux Φ_E through a surface is the amount of electric field $\mathbf{E}$ that pierces the surface.

The area vector $d\mathbf{A}$ for an infinitesimal area element (patch element) on a surface is a vector that is **perpendicular to the element** and has a magnitude equal to the area dA of the element.

The electric flux $d\Phi_E$ through a patch element with area vector $d\mathbf{A}$ is given by the dot product:

$$d\Phi_E = \mathbf{E} \cdot d\mathbf{A}$$

To find the total flux Φ_E, the integration is carried out over the entire surface S:

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A}$$

For a closed surface, the net flux is given by:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A}$$

This is the total electric flux through a closed surface, which is used in Gauss's Law. By Gauss's Law, the net flux is proportional to the enclosed charge Q_{enc}:

$$\Phi_E = \frac{Q_{\text{enc}}}{\epsilon_0}$$

where ϵ_0 is the permittivity of free space.

Problem 1: Consider a Gaussian surface in the form of a closed cylinder (a Gaussian cylinder or G-cylinder) of radius R. It lies in a uniform electric field E, with the cylinder's central axis (along the length of the cylinder) parallel to the field. What is the net flux of the electric field through the cylinder?

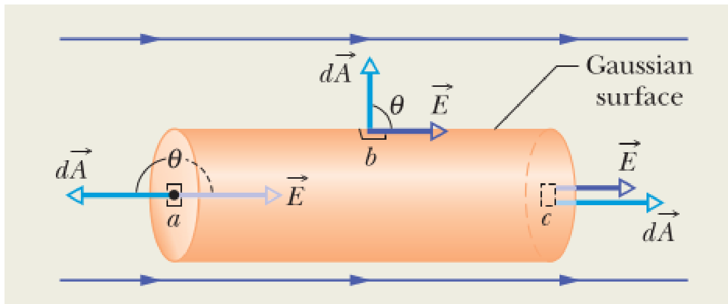


Figure 1: A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction

Solution:

The electric flux Φ_E through a surface is given by the integral:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A}$$

For this problem, we assume that the electric field $\mathbf{E}$ is uniform and parallel to the axis of the cylinder. The Gaussian surface consists of three parts: - The two circular end caps of the cylinder. - The curved side of the cylinder.

Electric flux through the end caps

For one of the end caps, the area vector $d\mathbf{A}$ is parallel to the electric field, so the angle $\theta = 0°$. The flux through this end cap is:

$$\Phi_{\text{cap1}} = \mathbf{E} \cdot d\mathbf{A} = E \cdot \pi R^2$$

For the other end cap, the area vector $d\mathbf{A}$ is in the opposite direction to the electric field, so the angle $\theta = 180°$. The flux through this cap is:

$$\Phi_{\text{cap2}} = \mathbf{E} \cdot d\mathbf{A} = E \cdot \pi R^2 \cdot \cos 180° = -E \cdot \pi R^2$$

Electric flux through the curved side

For the curved side of the cylinder, the electric field is parallel to the axis of the cylinder, while the area vector $d\mathbf{A}$ of each patch is perpendicular to the electric field. Therefore, the flux through the side is zero:

$$\Phi_{\text{side}} = 0$$

Net flux through the cylinder

The total flux through the Gaussian cylinder is the sum of the flux through the two end caps and the flux through the side:

$$\Phi_E = \Phi_{\text{cap1}} + \Phi_{\text{cap2}} + \Phi_{\text{side}}$$

Substituting the values:

$$\Phi_E = E \cdot \pi R^2 + (-E \cdot \pi R^2) + 0 = 0$$

Thus, the net electric flux through the Gaussian cylinder is:

$$\Phi_E = 0$$

Gauss's Law

Gauss's Law relates the electric flux through a closed surface to the charge enclosed by that surface. It states that the net electric flux Φ_E through a closed surface is proportional to the total charge Q_{enc} enclosed within the surface.

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

where:

- $\mathbf{E}$ is the electric field vector,
- $d\mathbf{A}$ is an infinitesimal area element on the closed surface S,
- Q_{enc} is the total charge enclosed within the surface S,
- ϵ_0 is the permittivity of free space.

Problem 2: Electric Field outside a Uniformly Charged Cylindrical Rod

Consider an infinitely long cylindrical plastic rod with a uniform charge density λ (charge per unit length). We want to find an expression for the electric field magnitude E at a radius r from the central axis of the rod, outside the rod.

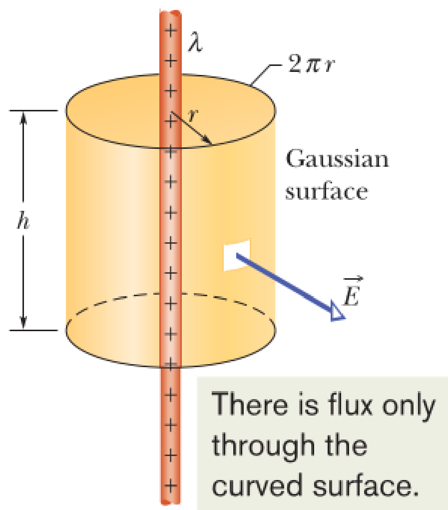


Figure 2: A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindri- cal plastic rod.

Solution:

We will apply Gauss's Law to find the electric field outside the rod. First, let's set up the problem using cylindrical symmetry.

Gauss's Law is given by:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

where: - $\mathbf{E}$ is the electric field vector, - $d\mathbf{A}$ is the differential area element on a Gaussian surface, - Q_{enc} is the total charge enclosed by the Gaussian surface, - ϵ_0 is the permittivity of free space.

To exploit the cylindrical symmetry, we choose a Gaussian surface **in the form of a coaxial cylinder** with the charged rod. The Gaussian cylinder will have: - Radius r (the distance from the axis of the rod at which we want to find the electric field), - Length h, where h is an arbitrary length of the Gaussian cylinder.

The electric flux through the cylindrical surface is:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A}$$

Since the electric field $\mathbf{E}$ is radially symmetric, the dot product simplifies, and we get:

$$\Phi_E = E \cdot (2\pi r h) \cdot$$

The total charge enclosed by the Gaussian surface is:

$$Q_{\text{enc}} = \lambda h$$

where λ is the charge density (charge per unit length) and h is the length of the Gaussian surface.

Applying Gauss's Law

Substitute the flux and the enclosed charge into Gauss's Law:

$$E(r) \cdot (2\pi r h) = \frac{\lambda h}{\epsilon_0}$$

Simplifying, we get the electric field:

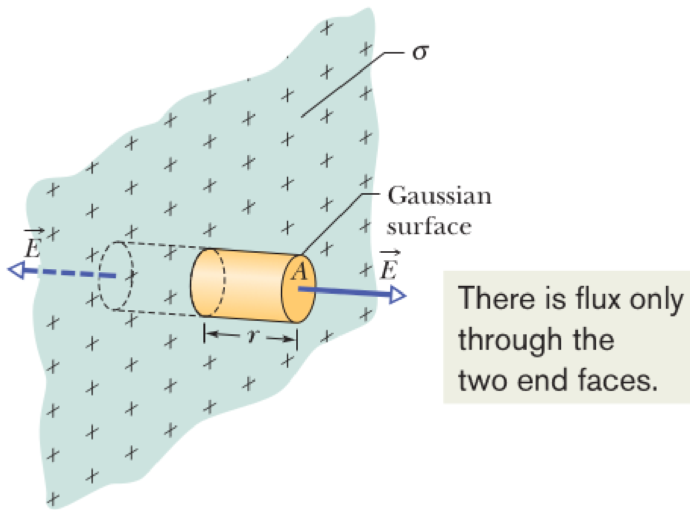
$$E(r) = \frac{\lambda}{2\pi \epsilon_0 r}$$

Thus, the magnitude of the electric field at a distance r from the axis of the uniformly charged cylindrical rod (outside the rod) is:

$$E(r) = \frac{\lambda}{2\pi \epsilon_0 r}$$

Problem 3: Electric Field due to an Infinite Uniformly Charged Sheet

Consider a thin, infinite, nonconducting sheet with a uniform positive surface charge density σ (charge per unit area). We wish to find the electric field E at a distance r from the sheet, in front of it.



Solution:

We will apply Gauss's Law to find the electric field at a distance from an infinite sheet of charge.

Gauss's Law states:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

where: - $\mathbf{E}$ is the electric field vector, - $d\mathbf{A}$ is the differential area element on the Gaussian surface, - Q_{enc} is the total charge enclosed by the Gaussian surface, - ϵ_0 is the permittivity of free space.

Symmetry and Choice of Gaussian Surface

Due to **the planar symmetry** of the problem, we choose a Gaussian surface in the form of a **"pillbox"** (a small cylindrical surface) with its flat faces parallel to the sheet.

The pillbox will: - Have two faces, one in front of the sheet and one behind it, each with area A, - Be symmetric with respect to the sheet, so the electric field will have the same magnitude on both faces and point perpendicular to the surface (normal to the sheet).

Thus, the electric field will only depend on the distance r from the sheet and will point away from the sheet on both sides.

The flux through the pillbox is:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A}$$

For the two flat faces of the pillbox, the electric field is perpendicular to the surface and points outward. The flux through each face is:

$$\Phi_E = E \cdot A + E \cdot A = 2EA$$

where A is the area of each face of the pillbox.

Enclosed Charge

The total charge enclosed by the Gaussian surface is the charge on the sheet that lies inside the pillbox. The surface charge density is σ, so the total charge enclosed is:

$$Q_{\text{enc}} = \sigma A$$

where A is the area of the pillbox face.

Applying Gauss's Law

Substituting the flux and enclosed charge into Gauss's Law:

$$2EA = \frac{\sigma A}{\epsilon_0}$$

Simplifying, we get the electric field:

$$E = \frac{\sigma}{2\epsilon_0}$$

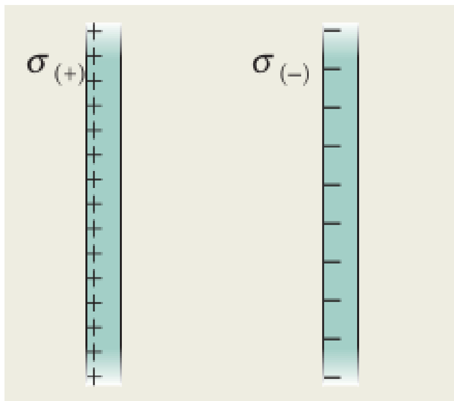
Thus, the electric field at a distance r from the sheet (in front of the sheet) is:

$$E = \frac{\sigma}{2\epsilon_0}$$

This result shows that the electric field due to an infinite sheet of charge is **constant and independent of the distance from the sheet**. The electric field points away from the positively charged sheet on both sides, and its magnitude is given by $\frac{\sigma}{2\epsilon_0}$.

Problem 4: Electric Field due to Two Parallel Sheets (Opposite Charge Densities)

We consider two large, parallel, non-conducting sheets: - The positively charged sheet has a surface charge density $\sigma_1 = 6.8\,\mathrm{C/m^2}$, - The negatively charged sheet has a surface charge density $\sigma_2 = 4.3\,\mathrm{C/m^2}$. We are tasked with finding the electric field E to the left of the sheets.



Solution

The electric field due to an infinite sheet of charge with surface charge density σ is given by:

$$E = \frac{\sigma}{2\epsilon_0},$$

where ϵ_0 is the permittivity of free space.

Case 1: Electric Field Due to the Positively Charged Sheet

The positively charged sheet creates an electric field E_1 that points away from the sheet (to the right). Its magnitude is:

$$E_1 = \frac{\sigma_1}{2\epsilon_0}.$$

Case 2: Electric Field Due to the Negatively Charged Sheet

The negatively charged sheet creates an electric field E_2 that points toward the sheet (to the left). Its magnitude is:

$$E_2 = \frac{|\sigma_2|}{2\epsilon_0}.$$

Net Electric Field:

Since the electric fields point in opposite directions, the net electric field to the left of the sheets is the difference:

$$E_{\text{left}} = E_1 - E_2.$$

Substituting the expressions for E_1 and E_2:

$$E_{\text{left}} = \frac{\sigma_1}{2\epsilon_0} - \frac{|\sigma_2|}{2\epsilon_0}.$$

Factoring out $\frac{1}{2\epsilon_0}$:

$$E_{\text{left}} = \frac{\sigma_1 - |\sigma_2|}{2\epsilon_0}.$$

$$E_{\text{left}} = \frac{6.8\,\text{C/m}^2 - 4.3\,\text{C/m}^2}{2\epsilon_0}.$$

Simplify:

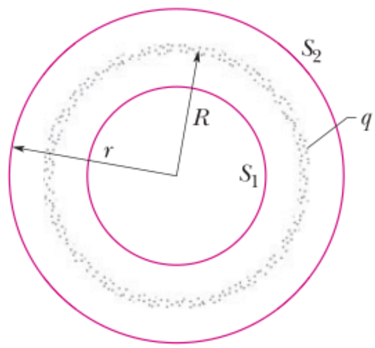
$$E_{\text{left}} = \frac{2.5\,\text{C/m}^2}{2\epsilon_0}.$$

Final Expression:

$$E_{\text{left}} = \frac{2.5\,\text{C/m}^2}{2\epsilon_0}.$$

Problem 5: Electric Field Due to a Charged Spherical Shell

Consider a spherical shell with a total charge q and radius R.Find the electric field for $r \geq R$ (outside the shell) and $r < R$ (inside the shell).



Solution:

Two concentric spherical Gaussian surfaces S_1 and S_2 are placed as follows:

- S_1 is a spherical surface with radius $r < R$ (inside the shell), - S_2 is a spherical surface with radius $r \geq R$ (outside the shell).

We want to apply Gauss's Law to find the electric field for $r \geq R$ (outside the shell) and $r < R$ (inside the shell).

We apply Gauss's Law in both regions, inside and outside the spherical shell.

Gauss's Law states:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

where: - $\mathbf{E}$ is the electric field, - $d\mathbf{A}$ is the differential area element, - Q_{enc} is the total enclosed charge by the Gaussian surface, - ϵ_0 is the permittivity of free space.

Case 1: For $r \geq R$ (Outside the Shell, Gaussian Surface S_2)

For a Gaussian surface S_2 with radius $r \geq R$, the enclosed charge Q_{enc} is the total charge on the spherical shell, which is q. Since the spherical symmetry of the charge distribution ensures that the electric field is radial and has the same magnitude at every point on the surface, we can calculate the electric flux as:

$$\oint_{S_2} \mathbf{E} \cdot d\mathbf{A} = E(r) \cdot 4\pi r^2$$

By Gauss's Law:

$$E(r) \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

Solving for the electric field:

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2}$$

Thus, for $r \geq R$, the electric field is:

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2}$$

This is the same as the electric field due to a point charge q located at the center of the spherical shell.

Case 2: For $r < R$ (Inside the Shell, Gaussian Surface S_1)

For a Gaussian surface S_1 with radius $r < R$, the enclosed charge Q_{enc} is zero because there is no charge inside the spherical shell. Hence, by Gauss's Law:

$$\oint_{S_1} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0} = 0$$

Since the electric field $\mathbf{E}$ is radial and the flux is zero, it follows that the electric field inside the spherical shell is:

$$E(r) = 0 \quad \text{for} \quad r < R$$

Answer: - For $r \geq R$ (outside the shell), the electric field is:

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2}$$

- For $r < R$ (inside the shell), the electric field is:

$$E(r) = 0$$