Topic: Potential Energy, Momentum, Impulse, Collision

Conservative and Nonconservative Force:

Conservative force:

A force is a conservative force if **the net work** it does on a particle moving around any closed path, from an initial point and then back to that point, **is zero**. That means for conservative force

$$W_1 = -W_2 \tag{1}$$

Or,

$$W_1 + W_2 = 0 \tag{2}$$

Example

- The gravitational force
- The spring force

Non-Conservative force:

A force is a conservative force if **the net work** it does on a particle moving around any closed path, from an initial point and then back to that point, **is not zero**. In other words a force that is not conservative is called a nonconservative force. That means for non conservative force

Example

- Frictional force
- The Drag force

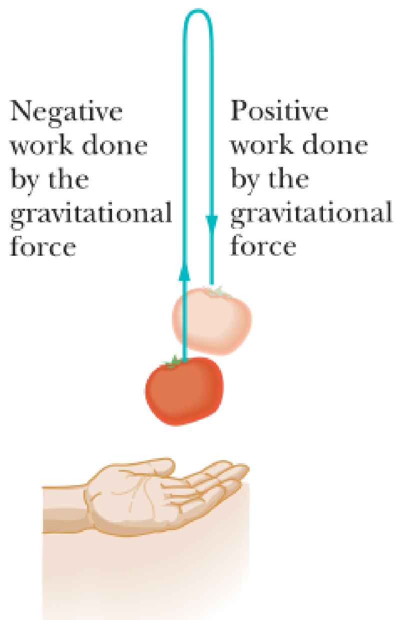
Work and Potential Energy

Let us discuss the relation between work and a change in potential energy.

Consider a tomato thrown upward. As it rises, the work W_g done by the gravitational force is negative, transferring energy from the tomato's kinetic energy to the gravitational potential energy of the tomato-Earth system. The tomato slows, stops, and then falls back due to gravity. During the fall, the transfer is reversed: the work W_g is now positive, converting gravitational potential energy back into kinetic energy.

For either rise or fall, the change in gravitational potential energy ΔU is defined as the negative of the work done by the gravitational force:

$$\Delta U = -W. \tag{3}$$



Determining Potential Energy Values

Consider a particle-like object in a system where a conservative force $\mathbf{F}$ acts. When this force does work W on the object, the change in the system's potential energy ΔU is given by

$$\Delta U = -W. \tag{4}$$

For a varying force, the work done as the object moves from x_i to x_f is

$$W = \int_{x_i}^{x_f} F(x)\,dx. \tag{5}$$

Since the force is conservative, the work done is independent of the path taken. Substituting this into the expression for ΔU, we obtain

$$\Delta U = -\int_{x_i}^{x_f} F(x)\,dx. \tag{6}$$

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Gravitational Potential Energy

Consider a particle of mass m moving vertically along the y-axis. The gravitational force $\mathbf{F}_g$ does work as the particle moves from y_i to y_f. Using the gravitational force $F_g = -mg$, we obtain

$$\Delta U = -\int_{y_i}^{y_f} (-mg)\,dy = mg\int_{y_i}^{y_f} dy. \tag{7}$$

Evaluating the integral,

$$\Delta U = mg(y_f - y_i) = mg\Delta y. \tag{8}$$

To define a gravitational potential energy function, we rewrite this as

$$U - U_i = mg(y - y_i). \tag{9}$$

Choosing $U_i = 0$ at $y_i = 0$, we get the gravitational potential energy:

$$U(y) = mgy. \tag{10}$$

The gravitational potential energy associated with a particle – Earth system depends only on the vertical position y (or height) of the particle relative to the reference position y= 0, not on the horizontal position.

Elastic Potential Energy

Consider a block-spring system where the block moves on a spring with force constant k. The spring force is given by $F_x = -kx$. The change in elastic potential energy:

$$\Delta U = -\int_{x_i}^{x_f} (-kx)\, dx = k \int_{x_i}^{x_f} x\, dx. \tag{11}$$

Evaluating the integral,

$$\Delta U = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2. \tag{12}$$

Defining the potential energy function with $U = 0$ at $x = 0$, we obtain:

$$U(x) = \frac{1}{2}kx^2. \tag{13}$$

Conservation of Mechanical Energy

The mechanical energy E_{mec} of a system is the sum of its kinetic energy K and potential energy U:

$$E_{\mathrm{mec}} = K + U. \tag{14}$$

When only conservative forces act within an isolated system, energy transfers occur between K and U, but the total mechanical energy remains constant. The change in kinetic energy due to work W is

$$\Delta K = W, \tag{15}$$

and the corresponding change in potential energy is

$$\Delta U = -W. \tag{16}$$

Thus, we obtain

$$\Delta K = -\Delta U, \tag{17}$$

which implies that any increase in one form of energy results in an equal decrease in the other. This can be rewritten as

$$K_2 - K_1 = -(U_2 - U_1), \tag{18}$$

or equivalently,

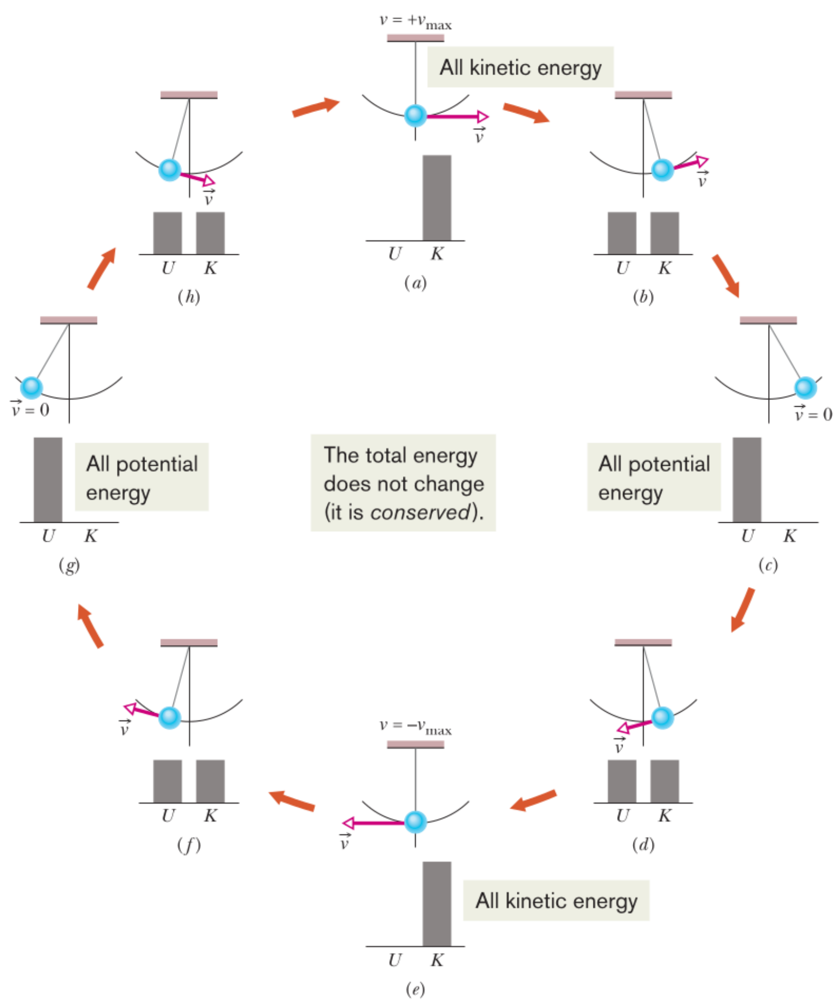
$$K_1 + U_1 = K_2 + U_2. \tag{19}$$

(The sum of K and U for any state of a system) = (The sum of K and U for any other state of the system). (20)

This expresses the principle of conservation of mechanical energy.

So,

In an isolated system with only conservative forces, the kinetic and potential energies may change, but their sum remains constant.



Problem 1:

A child of mass m is released from rest at the top of a frictionless water slide at height $h = 8.5$ m above the bottom. Assuming no friction, find the child's speed at the bottom of the slide.

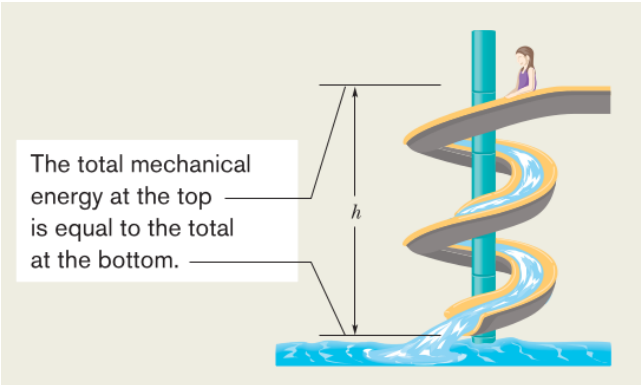


Figure 1: A child slides down a water slide as she descends a height h.

Solution

By the principle of conservation of mechanical energy,

$$E_{\text{mec},b} = E_{\text{mec},t}. \tag{21}$$

Since mechanical energy is the sum of kinetic and potential energy,

$$K_b + U_b = K_t + U_t. \tag{22}$$

Substituting the expressions for kinetic and gravitational potential energy,

$$\frac{1}{2}mv_b^2 + mgy_b = \frac{1}{2}mv_t^2 + mgy_t. \tag{23}$$

Dividing by m and rearranging,

$$v_b^2 = v_t^2 + 2g(y_t - y_b). \tag{24}$$

Given that the child starts from rest, $v_t = 0$, and the height difference is $y_t - y_b = h$, we obtain

$$v_b = \sqrt{2gh}. \tag{25}$$

Substituting $g = 9.8$ m/s^2 and $h = 8.5$ m,

$$v_b = \sqrt{2(9.8)(8.5)} = \sqrt{166.6} \approx 12.9 \text{ m/s}. \tag{26}$$

Thus, the child's speed at the bottom of the slide is approximately **12.9 m/s**.

Linear Momentum

The linear momentum of a particle is a vector quantity defined as:

$$\mathbf{p} = m\mathbf{v}$$

where m is the mass of the particle and $\mathbf{v}$ is its velocity. Since mass is always a positive scalar, $\mathbf{p}$ and $\mathbf{v}$ have the same direction. The SI unit of momentum is $\mathrm{kg} \cdot \mathrm{m/s}$.

Linear Momentum of a System of Particles

For a system of n particles, each with mass m_i and velocity $\mathbf{v}_i$, the total linear momentum $\mathbf{P}$ is given by the vector sum of the individual momenta:

$$\mathbf{P} = \sum_{i=1}^{n} m_i \mathbf{v}_i.$$

Comparing this with the equation for the center of mass velocity, we obtain:

$$\mathbf{P} = M\mathbf{v}_{\mathrm{com}},$$

where M is the total mass of the system and $\mathbf{v}_{\mathrm{com}}$ is the velocity of the center of mass.

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

Impulse

The momentum $\vec{p}$ of any particle-like body cannot change unless a net external force changes it. Let the projectile be a ball and the target be a bat. The collision is brief, and the ball experiences a force that is great enough to slow, stop, or even reverse its motion.

The ball experiences a force $\vec{F}(t)$ that varies during the collision and changes the linear momentum $\vec{p}$ of the ball. That change is related to the force by Newton's second law written in the form:

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{27}$$

By rearranging this expression, we see that, in the time interval dt, the change in the ball's momentum is:

$$d\vec{p} = \vec{F}(t)dt. \tag{28}$$

We can find the net change in the ball's momentum due to the collision if we integrate both sides from a time t_i just before the collision to a time t_f just after the collision:

$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{29}$$

The left side of this equation gives us the change in momentum:

$$\vec{p}_f - \vec{p}_i = \Delta\vec{p}. \tag{30}$$

The right side, which is a measure of both the magnitude and the duration of the collision force, is called the impulse $\vec{J}$ of the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{31}$$

Thus, the change in an object's momentum is equal to the impulse on the object:

$$\Delta\vec{p} = \vec{J}. \tag{32}$$

This expression can also be written in the vector form:

$$\vec{p}_f - \vec{p}_i = \vec{J}. \tag{33}$$

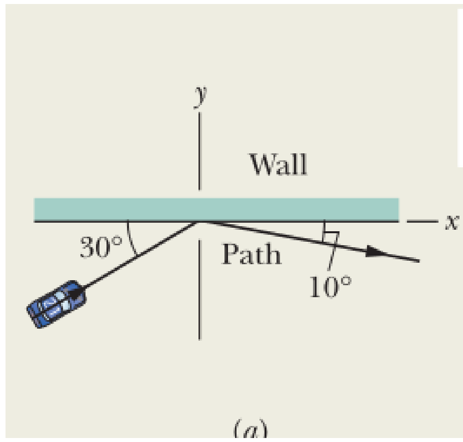
In component form, we can write:

$$\Delta p_x = J_x, \tag{34}$$

or explicitly,

$$p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t)dt. \tag{35}$$

Problem 2:



Above figure is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70$ m/s along a straight line at $30°$ from the wall. Just after the collision, he is traveling at speed $v_f = 50$ m/s along a straight line at $-10°$ from the wall. His mass m is 80 kg.

(a) What is the impulse $\vec{J}$ on the driver due to the collision?

(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

Solution

From the impulse-momentum theorem:

$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i. \tag{36}$$

We will evaluate the components separately.

X-component of Impulse

$$\begin{aligned} J_x &= m(v_{fx} - v_{ix}) && (37)\\ &= (80 \text{ kg}) \left[(50 \text{ m/s}) \cos(-10°) - (70 \text{ m/s}) \cos(30°)\right] && (38)\\ &= (80 \text{ kg}) \left[50 \times 0.9848 - 70 \times 0.8660\right] && (39)\\ &= (80 \text{ kg}) \left[49.24 - 60.62\right] && (40)\\ &= (80 \text{ kg}) \times (-11.38) && (41)\\ &= -910 \text{ kg} \cdot \text{m/s}. && (42) \end{aligned}$$

Y-component of Impulse

$$
\begin{aligned}
J_y &= m(v_{fy} - v_{iy}) && (43)\\
&= (80 \text{ kg}) \left[(50 \text{ m/s}) \sin(-10^\circ) - (70 \text{ m/s}) \sin(30^\circ)\right] && (44)\\
&= (80 \text{ kg}) \left[50 \times (-0.1736) - 70 \times 0.5\right] && (45)\\
&= (80 \text{ kg}) \left[-8.68 - 35\right] && (46)\\
&= (80 \text{ kg}) \times (-43.68) && (47)\\
&= -3495 \text{ kg} \cdot \text{m/s} \approx -3500 \text{ kg} \cdot \text{m/s}. && (48)
\end{aligned}
$$

Magnitude of Impulse

The total impulse magnitude is:

$$
\begin{aligned}
J &= \sqrt{J_x^2 + J_y^2} && (49)\\
&= \sqrt{(-910)^2 + (-3500)^2} && (50)\\
&= \sqrt{828100 + 12250000} && (51)\\
&= \sqrt{13078100} && (52)\\
&= 3617 \text{ kg} \cdot \text{m/s}. && (53)
\end{aligned}
$$

The collision lasts for $t = 14$ ms $= 0.014$ s. What is the magnitude of the average force on the driver during the collision?

Solution (b)

The average force $\vec{F}_{\text{avg}}$ during the collision is given by the impulse-momentum theorem:

$$\vec{J} = \vec{F}_{\text{avg}} \Delta t. \tag{54}$$

Solving for F_{avg}:

$$F_{\text{avg}} = \frac{J}{\Delta t}. \tag{55}$$

From the previous calculation, the magnitude of impulse is:

$$J = 3616 \text{ kg} \cdot \text{m/s}. \tag{56}$$

Substituting the values:

$$
\begin{aligned}
F_{\text{avg}} &= \frac{3616 \text{ kg} \cdot \text{m/s}}{0.014 \text{ s}} && (57)\\
&= 2.583 \times 10^5 \text{ N} && (58)\\
&\approx 2.6 \times 10^5 \text{ N}. && (59)
\end{aligned}
$$

Law of Conservation of Linear Momentum

Suppose that the net external force $\vec{F}_{\text{net}}$ (and thus the net impulse $\vec{J}$) acting on a system of particles is zero, meaning the system is *isolated*, and no particles enter or leave the system, meaning it is *closed*. Setting $\vec{F}_{\text{net}} = 0$ in Newton's second law:

$$\frac{d\vec{P}}{dt} = 0, \tag{60}$$

implies that the total linear momentum $\vec{P}$ remains constant:

$$\vec{P} = \text{constant}. \tag{61}$$

This result, known as the **law of conservation of linear momentum**, states:

> If no net external force acts on a system of particles, the total linear momentum of the system cannot change.

In problem-solving, we usually express this law as:

$$\vec{P}_i = \vec{P}_f, \tag{62}$$

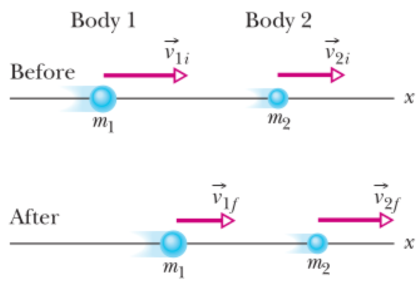
where the total momentum at some initial time t_i is equal to the total momentum at some later time t_f, provided the system remains closed and isolated.

Momentum and Kinetic Energy in Collisions

Key Points

- In a **closed, isolated system**, the total linear momentum $\vec{P}$ is conserved:
$$\vec{P}_i = \vec{P}_f. \tag{63}$$
- The total kinetic energy of a system may or may not be conserved in a collision:
  - **Elastic collision**: Kinetic energy is conserved.
$$K_i = K_f. \tag{64}$$
  - **Inelastic collision**: Some kinetic energy is lost to other forms (e.g., heat, sound).
  - **Completely inelastic collision**: The maximum kinetic energy loss occurs, and the bodies stick together.
- Everyday collisions (e.g., cars, baseball and bat) are typically inelastic.
- In a completely inelastic collision, momentum is still conserved, but kinetic energy is not.

One-Dimensional Inelastic Collision



In the case of a one-dimensional collision between two bodies, as shown in Figure , the law of conservation of linear momentum states that the total momentum before the collision is equal to the total momentum after the collision. This is expressed as:

$$\vec{p}^{\,i}_{\text{total}} = \vec{p}^{\,f}_{\text{total}}. \tag{65}$$

In one dimension, this simplifies to:

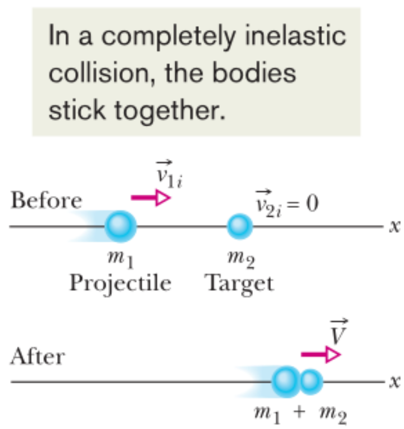
$$p_{1i} + p_{2i} = p_{1f} + p_{2f}. \tag{66}$$

Using $p = mv$, we can rewrite this as:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}. \tag{67}$$

If the masses, initial velocities, and one of the final velocities are known, we can solve for the remaining unknown final velocity.

One-Dimensional Completely Inelastic Collision



In a completely inelastic collision, two bodies stick together after the collision. The body with mass m_2 is initially at rest ($v_{2i} = 0$), while the incoming body with mass m_1 is moving with velocity v_{1i}. After the collision, the combined mass moves with velocity V.

The conservation of linear momentum in this case is expressed as:

$$m_1 v_{1i} = (m_1 + m_2)V. \tag{68}$$

Solving for the final velocity V:

$$V = \frac{m_1 v_{1i}}{m_1 + m_2}. \tag{69}$$

If the masses and the initial velocity of the projectile are known, we can calculate the final velocity V of the combined bodies.

Elastic Collisions in One Dimension

In an elastic collision, the total kinetic energy of the system is conserved. That is, the total kinetic energy before and after the collision remains the same.

Conservation of Linear Momentum

The linear momentum of the system is conserved in an isolated system. Therefore, the equation for the conservation of linear momentum is:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

where:

- m_1 and m_2 are the masses of the two bodies,
- v_{1i} and v_{2i} are the initial velocities of the two bodies (with $v_{2i} = 0$ in this case),
- v_{1f} and v_{2f} are the final velocities of the two bodies after the collision.

Conservation of Kinetic Energy

In addition to momentum, the total kinetic energy of the system is also conserved in an elastic collision. The conservation of kinetic energy can be expressed as:

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

where the terms represent:

- $\frac{1}{2}m_1 v_{1i}^2$ is the initial kinetic energy of body 1,
- $\frac{1}{2}m_1 v_{1f}^2$ is the final kinetic energy of body 1,
- $\frac{1}{2}m_2 v_{2f}^2$ is the final kinetic energy of body 2.

Collisions in Two Dimensions

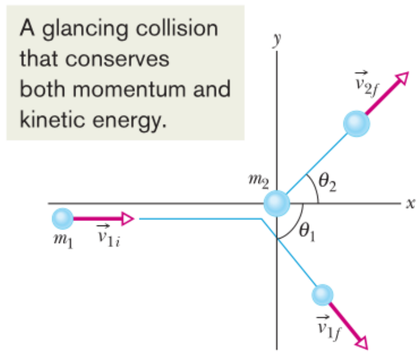


Figure 2: An elastic collision between two bodies in which the collision is not head- on. The body with mass m2 (the target) is initially at rest.

In a two-dimensional elastic collision, both momentum and kinetic energy are conserved. The linear momentum conservation equation is:

$$\mathbf{P}_1 i + \mathbf{P}_2 i = \mathbf{P}_1 f + \mathbf{P}_2 f$$

which can be written in terms of components along the x- and y-axes:

Along the x-axis:

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$$

Along the y-axis:

$$0 = m_1 v_{1f} \sin\theta_1 - m_2 v_{2f} \sin\theta_2$$

For the special case of an elastic collision, the total kinetic energy is conserved:

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Problem 3:

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Collision Problem

Two bodies, A and B, each with a mass of 2.0 kg, collide. The velocities before the collision are:

$$\mathbf{v}_A = (15\hat{\imath} + 30\hat{\jmath})\,\text{m/s}, \quad \mathbf{v}_B = (-10\hat{\imath} + 5\hat{\jmath})\,\text{m/s}$$

After the collision, the velocity of body A is given by:

$$\mathbf{v}'_A = (-5\hat{\imath} + 20\hat{\jmath})\,\text{m/s}$$

We are asked to find:

(a) The final velocity of body B ($\mathbf{v}'_B$).

(b) The change in the total kinetic energy of the system (including sign).

Solution

(a) Final velocity of body B

From the principle of conservation of linear momentum:

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B$$

Since $m_A = m_B = m = 2.0\,\text{kg}$, the masses cancel out, and the equation becomes:

$$\mathbf{v}_A + \mathbf{v}_B = \mathbf{v}'_A + \mathbf{v}'_B$$

Substituting the given values:

$$(15\hat{\imath} + 30\hat{\jmath}) + (-10\hat{\imath} + 5\hat{\jmath}) = (-5\hat{\imath} + 20\hat{\jmath}) + \mathbf{v}'_B$$

Simplifying:

$$(5\hat{\imath} + 35\hat{\jmath}) = (-5\hat{\imath} + 20\hat{\jmath}) + \mathbf{v}'_B$$

Now, solve for $\mathbf{v}'_B$:

$$\mathbf{v}'_B = (5\hat{\imath} + 35\hat{\jmath}) - (-5\hat{\imath} + 20\hat{\jmath})$$

$$\mathbf{v}'_B = (10\hat{\imath} + 15\hat{\jmath})\,\text{m/s}$$

Thus, the final velocity of body B is:

$$\mathbf{v}'_B = (10\hat{\imath} + 15\hat{\jmath})\,\text{m/s}$$

(b) Change in Total Kinetic Energy

The total kinetic energy before and after the collision is given by:

$$K_{\text{initial}} = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$K_{\text{final}} = \frac{1}{2}m_A v_A'^2 + \frac{1}{2}m_B v_B'^2$$

The change in the total kinetic energy is:

$$\Delta K = K_{\text{final}} - K_{\text{initial}}$$

Initial Kinetic Energy:

$$K_{\text{initial}} = \frac{1}{2}(2.0)\left(15^2 + 30^2\right) + \frac{1}{2}(2.0)\left((-10)^2 + 5^2\right)$$

$$K_{\text{initial}} = (1.0)\,(225 + 900) + (1.0)\,(100 + 25)$$

$$K_{\text{initial}} = (1.0) \times 1125 + 125 = 1250\,\text{J}$$

Final Kinetic Energy:

$$K_{\text{final}} = \frac{1}{2}(2.0)\left((-5)^2 + 20^2\right) + \frac{1}{2}(2.0)\left(10^2 + 15^2\right)$$

$$K_{\text{final}} = (1.0)\,(25 + 400) + (1.0)\,(100 + 225)$$

$$K_{\text{final}} = (1.0) \times 425 + 325 = 750\,\text{J}$$

Change in Kinetic Energy:

$$\Delta K = 750\,\text{J} - 1250\,\text{J} = -500\,\text{J}$$

Thus, the change in kinetic energy is:

$$\Delta K = -500\,\text{J}$$