Topic: Kinetic Energy and Work

Kinetic Energy

DEFINATION: Kinetic energy, denoted as K, is a form of energy that is associated with the motion of an object.

1. It is a **scalar quantity** that depends on the mass and velocity of the object.
2. The faster an object moves, the greater its kinetic energy.
3. When an object is at rest, meaning it has no motion, its kinetic energy is zero.

In classical mechanics, for an object with mass m moving with a velocity v (where the speed is much less than the speed of light), the kinetic energy is mathematically expressed as:

$$K = \frac{1}{2}mv^2 \tag{1}$$

This equation implies that the kinetic energy of an object is *directly proportional to its mass and to the square of its velocity.* This means that if the velocity of an object doubles, its kinetic energy increases by a factor of four.

Kinetic energy plays a fundamental role in the study of mechanical systems. Understanding kinetic energy is essential in analyzing the motion and interactions of objects in physics.

Work

Work is defined as the product of force and displacement in the direction of the force. Mathematically, it is given by:

$$W = \mathbf{F} \cdot d, \tag{2}$$

where $\mathbf{F}$ is the force applied, and $d\mathbf{r}$ is the infinitesimal displacement vector. For a constant force applied along a straight line, the work simplifies to:

$$W = Fd\cos\theta, \tag{3}$$

where F is the magnitude of the force, d is the displacement, and θ is the angle between the force and displacement vectors.

Work and Kinetic Energy

Finding an Expression for Work

To derive an expression for work, consider a bead that can slide along a frictionless wire stretched along a horizontal x-axis. A constant force $\mathbf{F}$, acting at an angle ϕ to the wire, accelerates the bead.

Using Newton's second law along the x-axis, we write:

$$F_x = ma_x, \tag{4}$$

where m is the mass of the bead. As the bead moves through a displacement d, its velocity changes from an initial value v_0 to some final value v. Since the force is constant, the acceleration is also constant, allowing us to use the kinematic equation:

$$v^2 = v_0^2 + 2a_x d. \tag{5}$$

Solving for a_x, we get:

$$a_x = \frac{v^2 - v_0^2}{2d}. \tag{6}$$

Substituting this into Newton's second law:

$$F_x = m\frac{v^2 - v_0^2}{2d}. \tag{7}$$

Multiplying both sides by d:

$$F_x d = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2. \tag{8}$$

Rearranging:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = F_x d. \tag{9}$$

Here, the term $\frac{1}{2}mv^2$ represents the final kinetic energy K_f of the bead, while $\frac{1}{2}mv_0^2$ represents the initial kinetic energy K_i. The equation thus expresses the work-energy theorem:

$$W = F_x d. \tag{10}$$

This equation shows that the work done on the bead by the force is equal to the change in its kinetic energy. Given values for F_x and d, we can compute the work W using this relation.

The Work–Kinetic Energy Theorem

The Work–Kinetic Energy Theorem states that the change in kinetic energy of an object is equal to the net work done on it. For particle-like objects, we can generalize this equation as:

$$\Delta K = K_f - K_i = W, \tag{11}$$

which states that

change in kinetic energy of a particle = net work done on the particle.

We can also write it as:

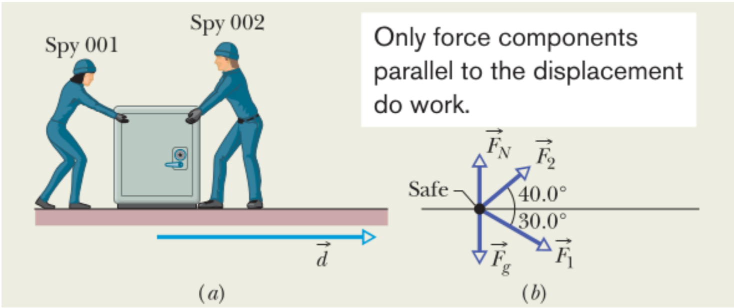
$$K_f = K_i + W, \tag{12}$$

which means that

kinetic energy after net work is done = kinetic energy before net work is done + net work done.

Problem 1:

Two industrial spies slide an initially stationary 225 kg floor safe a displacement d of magnitude 8.50 m. The push F_1 of spy 001 is 12.0 N at an angle of $30.03°$ downward from the horizontal; the pull F_2 of spy 002 is 10.0 N at $40.03°$ above the horizontal. The magnitudes and directions of these forces do not change as the safe moves, and the floor and safe make frictionless contact.



(a) What is the net work done on the safe by forces F_1 and F_2 during the displacement d?

(b) During the displacement, what is the work W_g done on the safe by the gravitational force F_g and what is the work W_N done on the safe by the normal force F_N from the floor?

(c) The safe is initially stationary. What is its speed v_f at the end of the 8.50 m displacement?

Solution

(a) Net Work Done by F_1 and F_2

The horizontal components of the forces contribute to the work done on the safe. The horizontal components are:

$$F_{1x} = F_1 \cos 30.03^\circ = 12.0 \times \cos 30.03^\circ = 10.39 \text{ N} \tag{13}$$

$$F_{2x} = F_2 \cos 40.03^\circ = 10.0 \times \cos 40.03^\circ = 7.66 \text{ N} \tag{14}$$

The net force in the horizontal direction is:

$$F_{\text{net}} = F_{1x} + F_{2x} = 10.39 + 7.66 = 18.05 \text{ N} \tag{15}$$

The work done is given by:

$$W = F_{\text{net}} d = (18.05)(8.50) = 153.4 \text{ J} \tag{16}$$

(b) Work Done by Gravity and Normal Force

The gravitational force is:

$$F_g = mg = (225)(9.81) = 2207.25 \text{ N} \tag{17}$$

Since gravity acts perpendicular to the displacement, the work done by gravity is:

$$W_g = F_g d \cos 90^\circ = 0 \text{ J} \tag{18}$$

Similarly, the normal force is perpendicular to the displacement, so:

$$W_N = 0 \text{ J} \tag{19}$$

(c) Final Speed of the Safe

Using the work-energy theorem:

$$W_{\text{net}} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \tag{20}$$

Since the safe is initially stationary, $v_i = 0$, so:

$$v_f = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(153.4)}{225}} \tag{21}$$

$$v_f = \sqrt{1.363} = 1.17 \text{ m/s} \tag{22}$$

So, the final speed of the safe is 1.17 m/s.

Problem 2:

During a storm, a crate of crepe is sliding across a slick, oily parking lot through a displacement

$$\mathbf{d} = (-3.0 \text{ m})\hat{\imath}. \tag{23}$$

while a steady wind pushes against the crate with a force

$$\mathbf{F} = (2.0 \text{ N})\hat{\imath} + (-6.0 \text{ N})\hat{\jmath}. \tag{24}$$

(a) How much work does this force do on the crate during the displacement?

(b) If the crate has a kinetic energy of 10 J at the beginning of displacement **d**, what is its kinetic energy at the end of **d**?

Solution

(a) Work Done by the Force

The work done by a force during a displacement is given by the dot product:

$$W = \mathbf{F} \cdot \mathbf{d}. \tag{25}$$

Expanding the dot product:

$$W = (2.0\hat{\imath} + (-6.0)\hat{\jmath}) \cdot (-3.0\hat{\imath}). \tag{26}$$

Since $\hat{\imath} \cdot \hat{\imath} = 1$ and $\hat{\jmath} \cdot \hat{\imath} = 0$, we get:

$$W = (2.0)(-3.0) + (-6.0)(0) = -6.0 \text{ J}. \tag{27}$$

Thus, the force does -6.0 J of work on the crate, meaning it removes energy from the system.

(b) Final Kinetic Energy

Using the work-energy theorem:

$$K_f = K_i + W. \tag{28}$$

Given $K_i = 10$ J:

$$K_f = 10 + (-6.0) = 4.0 \text{ J}. \tag{29}$$

Thus, the kinetic energy of the crate at the end of the displacement is 4.0 J.

The work W_g done by the gravitational force

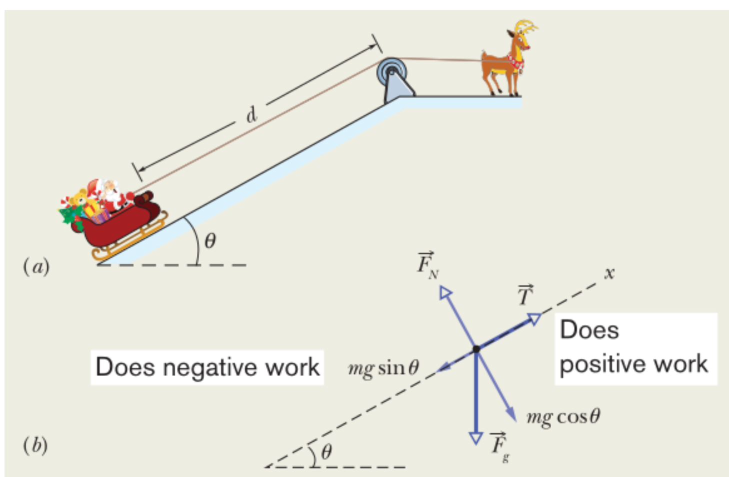
The work W_g done by the gravitational force $\mathbf{F}_g$ on a particle-like object of mass m as the object moves through a displacement $\mathbf{d}$ is given by:

$$W_g = mgd\cos\theta, \tag{30}$$

in which θ is the angle between $\mathbf{F}_g$ and $\mathbf{d}$.

Problem 3:

An object is pulled along a ramp, but the object starts and ends at rest. Figure shows the situation. A rope pulls a 200 kg sleigh up a slope at an incline angle $\theta = 30°$, through a distance $d = 20$ m. The sleigh and its contents have a total mass of 200 kg. The snowy slope is so slippery that we take it to be frictionless. How much work is done by each force acting on the sleigh?



Solution

The work W_N done by the normal force :

The work W_N done by the normal force is zero since the normal force is perpendicular to the displacement of the sleigh.

$$W_N = F_N d\cos 90° = 0. \tag{31}$$

The work W_g done by the gravitational force :

The angle between $\mathbf{F}_g$ and $\mathbf{d}$ is 120° (sum of the incline angle 30° and 90°).

$$W_g = F_g d \cos 120^\circ = mgd \cos 120^\circ, \tag{32}$$

$$W_g = (200 \text{ kg})(9.8 \text{ m/s}^2)(20 \text{ m}) \cos 120^\circ, \tag{33}$$

$$W_g = -1.96 \times 10^4 \text{ J}. \tag{34}$$

The work W_T done by the tension force :

Total Work

$$W = W_N + W_g + W_T. \tag{35}$$

Since the change in kinetic energy is zero (as the initial and final kinetic energies are both zero), applying work energy theorem:

$$0 = W_N + W_g + W_T. \tag{36}$$

Substituting the known values:

$$0 = 0 + (-1.96 \times 10^4 \text{ J}) + W_T. \tag{37}$$

Solving for W_T:

$$W_T = 1.96 \times 10^4 \text{ J}. \tag{38}$$

Problem 4:

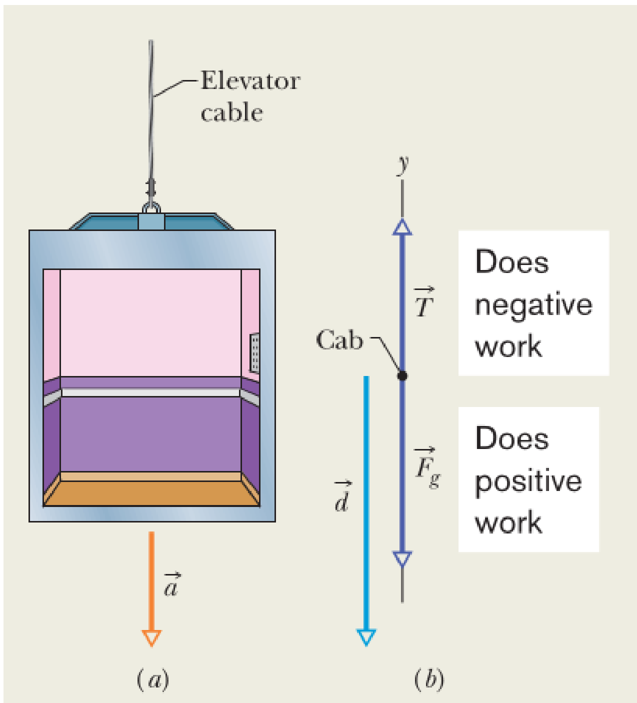
Problem: An elevator cab of mass $m = 500\,\text{kg}$ is descending with speed $v_i = 4.0\,\text{m/s}$ when its supporting cable begins to slip, allowing it to fall with constant acceleration $a = \frac{g}{5}$.

(a) During the fall through a distance $d = 12\,\text{m}$, what is the work W_g done on the cab by the gravitational force F_g?

(b) During the 12 m fall, what is the work W_T done by the tension in the cable?

(c) What is the net work W done on the cab during the fall?

(d) What is the cab's kinetic energy at the end of the 12 m fall?



Solution:

(a) During the fall through a distance $d = 12\,\text{m}$, we need to calculate the work W_g done on the cab by the gravitational force F_g. The angle between the direction of F_g and the cab's displacement d is $0°$, so the work done is given by:

$$W_g = F_g \cdot d \cdot \cos 0°.$$

Since $F_g = mg$, the work done is:

$$W_g = mgd \cdot \cos 0° = (500\,\text{kg})(9.8\,\text{m/s}^2)(12\,\text{m})(1).$$

Calculating the value:

$$W_g = 5.88 \times 10^4\,\mathrm{J} = 59\,\mathrm{kJ}.$$

Thus, the work done by the gravitational force is $W_g = 5.88 \times 10^4$ J or 59 kJ.

(b)

We can calculate work W_T by writing the force equation for the vertical components in Fig.:

$$F_{\mathrm{net},y} = ma_y \quad \text{for the components.}$$
$$T - F_g = ma \quad .$$

Solving for T, substituting mg for F_g, and then substituting the result in Eq. 7-7, we obtain:

$$W_T = Td\cos\theta = m(a+g)d\cos\theta$$

Next, substituting $-\frac{g}{5}$ for the (downward) acceleration a and 180° for the angle θ between the directions of forces T and F_g, we find:

$$W_T = m\left(\frac{g}{5} - g\right)d\cos 180^\circ = -\frac{mgd\cos 180^\circ}{5}.$$

Substituting the known values:

$$W_T = -\frac{(500\,\mathrm{kg})(9.8\,\mathrm{m/s^2})(12\,\mathrm{m})\cos 180^\circ}{5} = -4.70 \times 10^4\,\mathrm{J} = -47\,\mathrm{kJ}.$$

(c)

The net work is the sum of the works done by the forces acting on the cab:

$$W = W_g + W_T = 5.88 \times 10^4\,\mathrm{J} - 4.70 \times 10^4\,\mathrm{J} = 1.18 \times 10^4\,\mathrm{J} = 12\,\mathrm{kJ}.$$

(d)

We write the initial kinetic energy as:

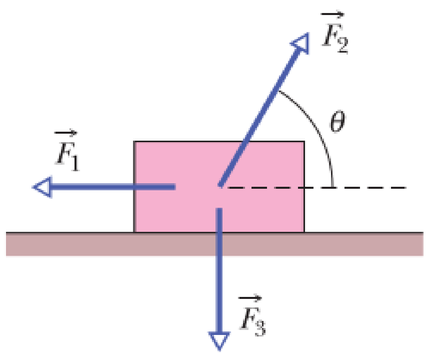
$$K_i = \frac{1}{2}mv_i^2.$$

We then write the final kinetic energy as:

$$K_f = K_i + W = \frac{1}{2}mv_i^2 + W = \frac{1}{2}(500\,\mathrm{kg})(4.0\,\mathrm{m/s})^2 + 1.18\times10^4\,\mathrm{J} = 1.58\times10^4\,\mathrm{J} = 16\,\mathrm{kJ}.$$

Problem 5:

Figure shows three forces applied to a trunk that moves leftward by 3.00 **m over a frictionless floor. The force magnitudes are:** $F_1 = 5.00$ **N**, $F_2 = 9.00$ **N**, $F_3 = 3.00$ **N**, **and the indicated angle is:** $\theta = 60.0°$.



During the displacement:

What is the net work done on the trunk by the three forces? Does the kinetic energy of the trunk increase or decrease?

Solution:

The forces are constant, so the work done by any one of them is given by

$$W = \vec{F} \cdot \vec{d},$$

where $\vec{d}$ is the displacement. Force $\vec{F}_1$ is in the direction of the displacement, so

$$W_1 = F_1 d \cos\phi_1 = (5.00\,\mathrm{N})(3.00\,\mathrm{m})\cos 0° = 15.0\,\mathrm{J}.$$

Force $\vec{F}_2$ makes an angle of $120°$ with the displacement, so

$$W_2 = F_2 d \cos\phi_2 = (9.00\,\mathrm{N})(3.00\,\mathrm{m})\cos 120° = -13.5\,\mathrm{J}.$$

Force $\vec{F}_3$ is perpendicular to the displacement, so

$$W_3 = F_3 d \cos\phi_3 = 0 \quad (\text{since } \cos 90° = 0).$$

The net work done by the three forces is

$$W = W_1 + W_2 + W_3 = 15.0\,\mathrm{J} - 13.5\,\mathrm{J} + 0 = +1.50\,\mathrm{J}.$$

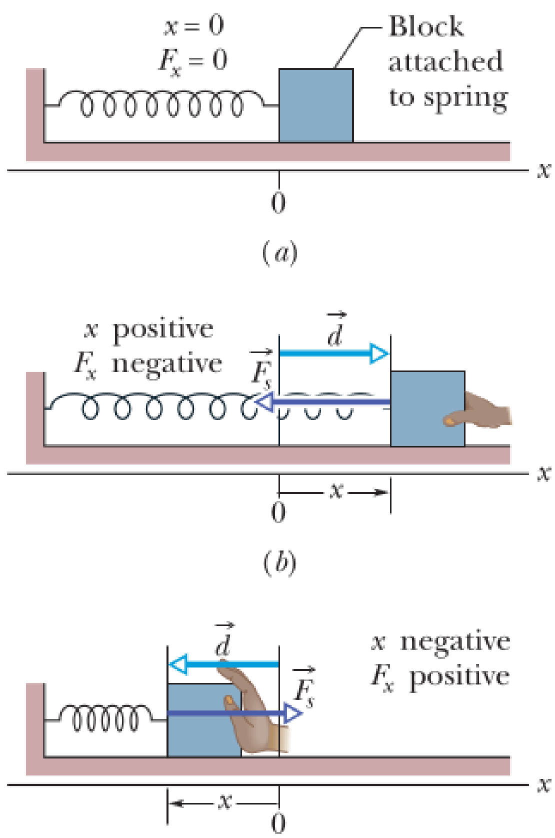
(b) If no other forces do work on the box, its kinetic energy increases by $1.50\,\mathrm{J}$ during the displacement.

The work done by a spring force

The spring force is given by **Hooke's law**:

$$F_s = -kd \quad ,$$

where the minus sign indicates that the direction of the spring force is always opposite to the direction of the displacement of the spring's free end.



The constant k is called the spring constant (or force constant), and it is a measure of the **stiffness of the spring**. A larger k indicates a stiffer spring, meaning the spring exerts a stronger pull or push for a given displacement. The SI unit for k is the newton per meter (N/m).

The work done by a spring force is given by:

$$W_s = \int_{x_i}^{x_f} F_x \, dx.$$

From , the force magnitude F_x is kx. Substituting this into the integral:

$$W_s = \int_{x_i}^{x_f} kx\,dx = k\int_{x_i}^{x_f} x\,dx.$$

Evaluating the integral:

$$W_s = \left(\frac{1}{2}k\right)\left[x^2\right]_{x_i}^{x_f} = \left(\frac{1}{2}k\right)\left(x_f^2 - x_i^2\right).$$

This simplifies to:

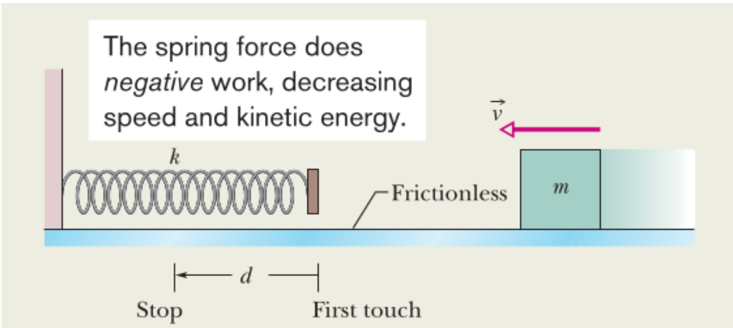
$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2.$$

The work done by the spring force, W_s, can have a positive or negative value depending on whether the net transfer of energy is to or from the block as it moves from x_i to x_f. If $x_i = 0$ and we call the final position $x_f = x$, then Eq. becomes:

$$W_s = -\frac{1}{2}kx^2(SpringForce)$$

Problem 6:

A box of mass $m = 0.40\,\mathrm{kg}$ slides across a horizontal frictionless counter with speed $v = 0.50\,\mathrm{m/s}$. It then runs into and compresses a spring of spring constant $k = 750\,\mathrm{N/m}$. When the box is momentarily stopped by the spring, by what distance d is the spring compressed?



Solution:

The work W_s done on the box by the spring force is :

$$W_s = -\frac{1}{2}kd^2,$$

where d replaces x.

The work W_s is also related to the kinetic energy of the box by:

$$K_f - K_i = W.$$

The box's kinetic energy has an initial value of $K_i = \frac{1}{2}mv^2$ and a value of zero when the box is momentarily at rest, so:

$$K_f - K_i = -\frac{1}{2}kd^2,$$

$$0 - \frac{1}{2}mv^2 = -\frac{1}{2}kd^2.$$

Simplifying, solving for d, and substituting known data:

$$d = \sqrt{\frac{mv^2}{k}}.$$

Substituting the values:

$$d = \sqrt{\frac{(0.40\,\text{kg})(0.50\,\text{m/s})^2}{750\,\text{N/m}}} = \sqrt{\frac{0.40 \times 0.25}{750}} = \sqrt{\frac{0.10}{750}} \approx \sqrt{0.0001333} \approx 0.0115\,\text{m}.$$

Thus, the spring is compressed by a distance $d \approx 0.0115\,\text{m}$ or $1.15\,\text{cm}$.