MECHANICS

Scalars and Vectors

In physics and mathematics, quantities are categorized into two types: **scalars and vectors**. Understanding the distinction between these two is fundamental to various fields of science and engineering.

1 Scalars

A **scalar** is a quantity that is fully described by its magnitude alone. It has no direction and follows the rules of ordinary algebra.

1.1 Examples of Scalars

Some common examples of scalar quantities include:

- Temperature ($25°C$)
- Mass (5 kg)
- Energy (100 J)
- Time (60 s)

1.2 Properties of Scalars

- Scalars can be added, subtracted, multiplied, and divided using ordinary arithmetic.

2 Vectors

A **vector** is a quantity that has both magnitude and direction. Vectors follow the rules of vector algebra.

2.1 Examples of Vectors

Some common examples of vector quantities include:

- Displacement (5 m north)
- Velocity (10 m/s east)
- Acceleration (9.8 m/s^2 downward)
- Force (50 N at 30° from horizontal)
- Momentum ($p = mv$)

2.2 Properties of Vectors

- Vectors obey the rules of vector addition and subtraction.
- They can be represented graphically using arrows.

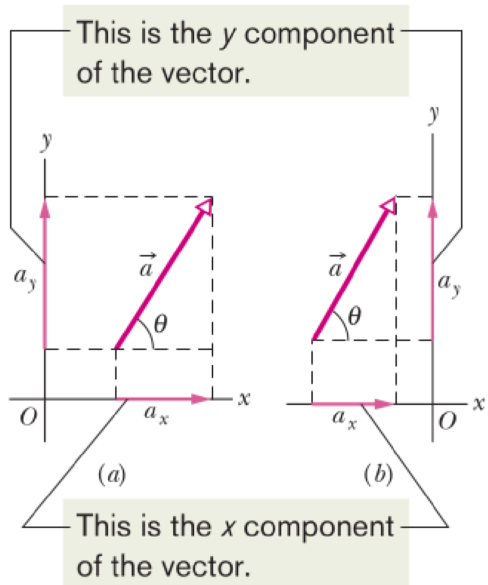
3 Vector Representation

A vector is typically represented in component form:

$$\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}, \tag{1}$$

where A_x, A_y, and A_z are the components of the vector along the x, y, and z axes, respectively, and $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors.

3.1 Components of Vectors



Let a vector $\vec{a}$ be aligned at an angle θ with the x-axis. The components of the vector along the x-axis and y-axis can be determined using trigonometry.

The component along the x-axis is given by:

$$a_x = a\cos\theta$$

The component along the y-axis is given by:

$$a_y = a\sin\theta$$

Thus, the vector $\vec{a}$ can be expressed in terms of its components as:

$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

Substituting the values of a_x and a_y, we obtain:

$$\vec{a} = a\cos\theta\hat{i} + a\sin\theta\hat{j}$$

where:

- a is the magnitude of the vector $\vec{a}$.
- θ is the angle the vector makes with the x-axis.
- $\hat{i}$ and $\hat{j}$ are the unit vectors along the x-axis and y-axis, respectively.

Determining the Magnitude of the Vector

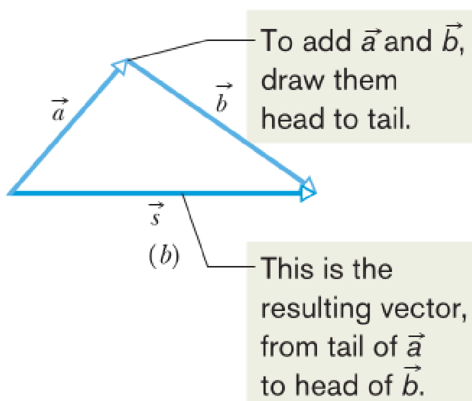
The magnitude of a vector is given by the Euclidean norm:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2}$$

4 Vector Operations

4.1 Addition and Subtraction

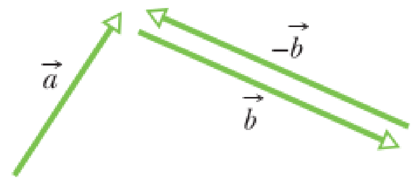
Two vectors $\vec{a}$ and $\vec{b}$ may be added geometrically by drawing them to a common scale and placing them head to tail. The vector connecting the tail of the first vector to the head of the second is the **vector sum** $\vec{s}$, given by:



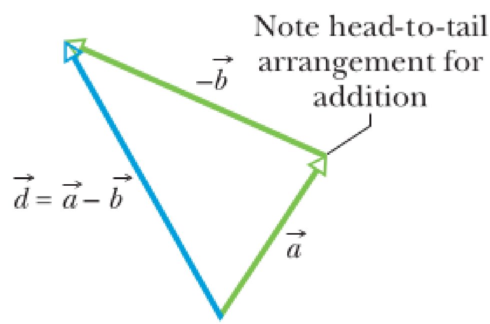
$$\vec{s} = \vec{a} + \vec{b}$$

To subtract $\vec{b}$ from $\vec{a}$, we first reverse the direction of $\vec{b}$ to get $-\vec{b}$, and then add $-\vec{b}$ to $\vec{a}$:

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



(a)



(b)

Vectors are added using the head-to-tail method or by adding their respective components:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \Rightarrow (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}. \tag{2}$$

4.2 Vector Multiplication: Scalar and Vector Products

4.2.1 Multiplying a Vector by a Scalar

A vector can be multiplied by a scalar to produce a new vector. Let $\mathbf{a}$ be a vector and s be a scalar. The product of $\mathbf{a}$ and s is written as:

$$\mathbf{a}' = s\mathbf{a}$$

where $\mathbf{a}'$ is the resulting vector. This multiplication affects both the magnitude and direction of the vector:

- The magnitude of the new vector is given by:

$$|\mathbf{a}'| = |s||\mathbf{a}|$$

  That is, the magnitude of the resulting vector is the product of the absolute value of the scalar s and the magnitude of the original vector $\mathbf{a}$.

- The direction of the new vector depends on the sign of the scalar s:
  - If $s > 0$, the direction of $\mathbf{a}'$ is the same as that of $\mathbf{a}$.
  - If $s < 0$, the direction of $\mathbf{a}'$ is opposite to that of $\mathbf{a}$.

To divide a vector $\mathbf{a}$ by a scalar s, we multiply $\mathbf{a}$ by the reciprocal of s:

$$\mathbf{a}' = \frac{1}{s}\mathbf{a}, \quad \text{for } s \neq 0.$$

4.2.2 Multiplying a Vector by a Vector

There are two ways to multiply two vectors:

- **The Scalar Product (Dot Product):** This operation results in a scalar quantity.
- **The Vector Product (Cross Product):** This operation results in a new vector.

4.3 The Scalar Product (Dot Product)

The scalar product, also known as the dot product, of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between the two vectors. The result of this operation is a scalar quantity. The dot product has the following properties:

- It is commutative:
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
- It distributes over vector addition:
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

4.4 The Vector Product (Cross Product)

The vector product, also known as the cross product, of two vectors **a** and **b** is defined as:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \mathbf{n}$$

where:

- θ is the angle between the two vectors,
- **n** is a unit vector perpendicular to the plane containing **a** and **b**, with direction determined by the right-hand rule.

The cross product has the following properties:

- It is anti-commutative:
$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$
- It distributes over vector addition:
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

5 Position, Displacement, Velocity, Acceleration

5.1 Position

The location of a particle relative to the origin of a coordinate system is given by a position vector $\mathbf{r}$, which in unit-vector notation is:

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}. \tag{3}$$

5.2 Displacement

If a particle moves so that its position vector changes from $\mathbf{r}_1$ to $\mathbf{r}_2$, the particle's displacement $\Delta\mathbf{r}$ is:

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1. \tag{4}$$

The displacement can also be written as:

$$\Delta\mathbf{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}. \tag{5}$$

5.3 Velocity

The velocity vector can be written as:

$$\vec{v} = \frac{displacement}{time} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Alternatively, using component notation:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

where the scalar components of velocity are:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

5.4 Acceleration

The acceleration vector is defined as the time derivative of the velocity:

$$\vec{a} = \frac{velocity}{time} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

In component form:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

where the scalar components of acceleration are:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \quad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

6 Problem : Rabbit's Motion in a Parking Lot

1. A rabbit runs across a parking lot on which a set of coordinate axes has been drawn. The coordinates (in meters) of the rabbit's position as functions of time t (in seconds) are given by:

$$x = -0.31t^2 + 7.2t + 28, \tag{6}$$

$$y = 0.22t^2 - 9.1t + 30. \tag{7}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation?

Solution

To find the position vector $\vec{r}$ at $t = 15$ s, we substitute $t = 15$ into the given equations.

For the x-coordinate

$$\begin{aligned} x &= -0.31(15)^2 + 7.2(15) + 28 \\ &= -0.31(225) + 108 + 28 \\ &= -69.75 + 108 + 28 \\ &= 66.25 \text{ m} \end{aligned}$$

For the y-coordinate

$$
\begin{aligned}
y &= 0.22(15)^2 - 9.1(15) + 30 \\
&= 0.22(225) - 136.5 + 30 \\
&= 49.5 - 136.5 + 30 \\
&= -57 \text{ m}
\end{aligned}
$$

Position Vector

The position vector $\vec{r}$ is given by:

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{r} = (66.25\hat{i} - 57\hat{j}) \text{ m}$$

Answer:

$$\vec{r} = 66.25\hat{i} - 57\hat{j} \text{ m.}$$

2. The position of a rabbit in a parking lot is given by the parametric equations:

$$x = -0.31t^2 + 7.2t + 28, \tag{8}$$
$$y = 0.22t^2 - 9.1t + 30. \tag{9}$$

(a) Find the velocity vector $\vec{v}$ and its magnitude at $t = 15$ s.

Solution

The velocity vector is given by the time derivative of the position vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

Calculating v_x

$$v_x = \frac{dx}{dt} = \frac{d}{dt}\left(-0.31t^2 + 7.2t + 28\right)$$

$$v_x = -0.62t + 7.2$$

Substituting $t = 15$:

$$v_x = -0.62(15) + 7.2$$

$$v_x = -9.3 + 7.2$$

$$v_x = -2.1 \text{ m/s}$$

Calculating v_y

$$v_y = \frac{dy}{dt} = \frac{d}{dt}\left(0.22t^2 - 9.1t + 30\right)$$

$$v_y = 0.44t - 9.1$$

Substituting $t = 15$:

$$v_y = 0.44(15) - 9.1$$

$$v_y = 6.6 - 9.1$$

$$v_y = -2.5 \text{ m/s}$$

Velocity Vector

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$\vec{v} = (-2.1\hat{i} - 2.5\hat{j}) \text{ m/s}$$

Magnitude of Velocity

The magnitude of the velocity vector is given by:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

$$|\vec{v}| = \sqrt{(-2.1)^2 + (-2.5)^2}$$

$$|\vec{v}| = \sqrt{4.41 + 6.25}$$

$$|\vec{v}| = \sqrt{10.66}$$

$$|\vec{v}| \approx 3.27 \text{ m/s}$$

Answer:

$$\vec{v} = -2.1\hat{i} - 2.5\hat{j} \text{ m/s}, \quad |\vec{v}| \approx 3.27 \text{ m/s}.$$

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Newton's Laws of Motion

Newton's First Law

Newton's First Law: If no net force acts on a body ($\vec{F}_{\text{net}} = 0$), the body's velocity cannot change; that is, the body cannot accelerate. This means that an object at rest remains at rest, and an object in motion continues in uniform motion unless acted upon by an external force.

Newton's Second Law

Newton's Second Law: The net force on a body is equal to the product of the body's mass and its acceleration. In equation form:

$$\vec{F}_{\text{net}} = m\vec{a}$$

where:

- $\vec{F}_{\text{net}}$ is the net force acting on the object,
- m is the mass of the object,
- $\vec{a}$ is the acceleration of the object.

Newton's Third Law

Newton's Third Law: If a body A exerts a force on body B, then body B exerts an equal and opposite force on body A. In equation form:

$$\vec{F}_{A\to B} = -\vec{F}_{B\to A}$$

This means that forces always come in action-reaction pairs.

7 Application of Newton's Law

Problem

A box of mass $m = 0.20$ kg moves along a frictionless ice surface along the x-axis under the influence of forces. The forces involved are:

- $F_1 = 4.0$ N (acting in the positive x-direction)
- $F_2 = 2.0$ N (acting in the negative x-direction)
- $F_3 = 1.0$ N (acting at an angle $\theta = 30°$)

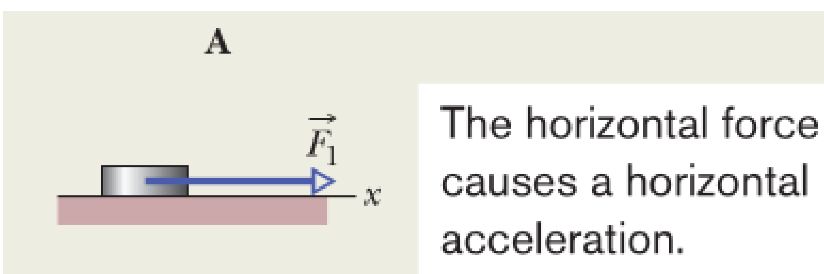
Using Newton's second law,

$$\vec{F}_{\text{net}} = m\vec{a}$$

Determine the acceleration a_x of the puck in three different situations.

Solution

Situation A: Single Force F_1



Only one force F_1 acts in the positive x-direction:

$$F_{\text{net},x} = F_1$$

Using Newton's second law:

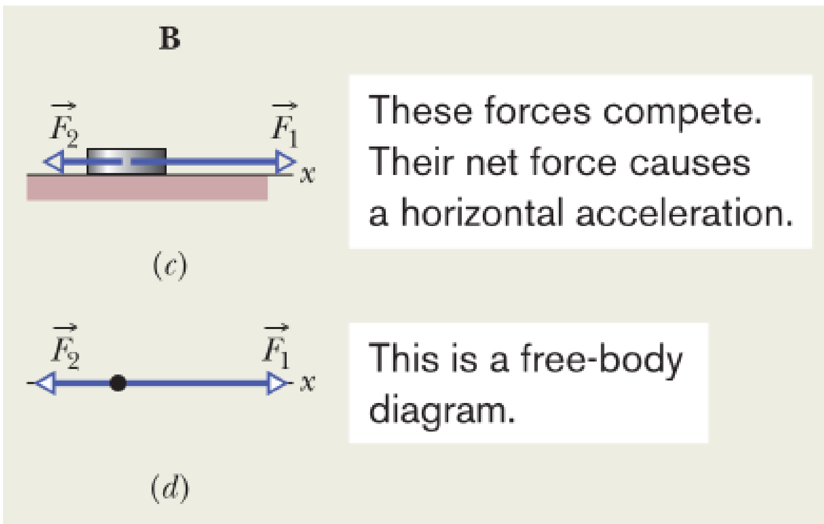
$$F_1 = ma_x$$

Solving for a_x:

$$a_x = \frac{F_1}{m} = \frac{4.0 \text{ N}}{0.20 \text{ kg}} = 20 \text{ m/s}^2$$

Thus, the acceleration is in the positive x-direction.

Situation B: Opposing Forces F_1 and F_2



Here, two forces act in opposite directions: F_1 in the positive x-direction and F_2 in the negative x-direction.

$$F_{\text{net},x} = F_1 - F_2$$

Applying Newton's second law:

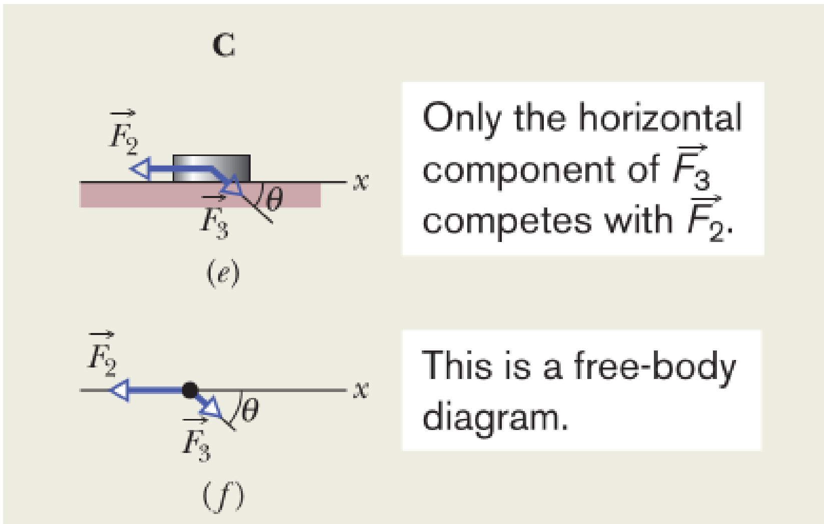
$$F_1 - F_2 = ma_x$$

Solving for a_x:

$$a_x = \frac{F_1 - F_2}{m} = \frac{4.0 \text{ N} - 2.0 \text{ N}}{0.20 \text{ kg}} = 10 \text{ m/s}^2$$

Thus, the acceleration is in the positive x-direction.

Situation C: Force F_3 at an Angle



Only the x-component of F_3 contributes to acceleration:

$$F_{3,x} = F_3 \cos\theta$$

Thus, the net force is:

$$F_{\text{net},x} = F_{3,x} - F_2$$

Applying Newton's second law:

$$F_3 \cos\theta - F_2 = ma_x$$

Solving for a_x:

$$a_x = \frac{F_3 \cos\theta - F_2}{m} = \frac{(1.0\text{ N})\cos 30^\circ - 2.0\text{ N}}{0.20\text{ kg}}$$

$$a_x = \frac{0.866 - 2.0}{0.20} = \frac{-1.134}{0.20} = -5.7\text{ m/s}^2$$

The negative sign indicates that the acceleration is in the negative x-direction.

Final Answers

- $a_x = 20\ \text{m/s}^2$ (Situation A)
- $a_x = 10\ \text{m/s}^2$ (Situation B)
- $a_x = -5.7\ \text{m/s}^2$ (Situation C)

Some Particular Forces

1. The Gravitational Force

The gravitational force $\vec{F}_g$ is a fundamental force that pulls objects toward a massive body, such as Earth. In most practical cases, this force acts downward toward the center of the Earth.

For objects near the Earth's surface, we assume that the ground is an inertial frame of reference, meaning that it does not accelerate relative to the falling object. The gravitational force is responsible for the natural downward pull on objects, causing them to fall when no other forces are acting.

2. Free Fall and Newton's Second Law

When an object of mass m is in free fall under the influence of gravity alone (neglecting air resistance), the only force acting on it is the gravitational force $\vec{F}_g$.

Applying Newton's second law:

$$\vec{F}_{\text{net}} = m\vec{a}$$

Since the acceleration due to gravity is g and acts downward, we define a vertical y-axis with the positive direction upward. The equation for the net force along this axis is:

$$F_{\text{net},y} = ma_y$$

Since the acceleration in free fall is $a_y = -g$, we write:

$$-F_g = m(-g)$$

Rearranging, we get:

$$F_g = mg$$

The equation $F_g = mg$ tells us that the gravitational force on an object is directly proportional to its mass, with g as the proportionality constant. The acceleration due to gravity near the Earth's surface is approximately:

$$g \approx 9.8 \text{ m/s}^2$$

This means that in free fall, all objects, regardless of their mass, experience the same acceleration if air resistance is negligible.

allows us to determine the weight of an object, where mg is the force with which the Earth pulls the object downward.

3. Normal Force, F_N

When a body presses against a surface, the surface (even if it appears rigid) deforms and exerts a normal force F_N on the body. This normal force is always perpendicular to the surface.

Example: A block of mass m presses downward on a table, causing the table to deform slightly due to the gravitational force F_g acting on the block. The table, in response, pushes upward on the block with a normal force F_N.

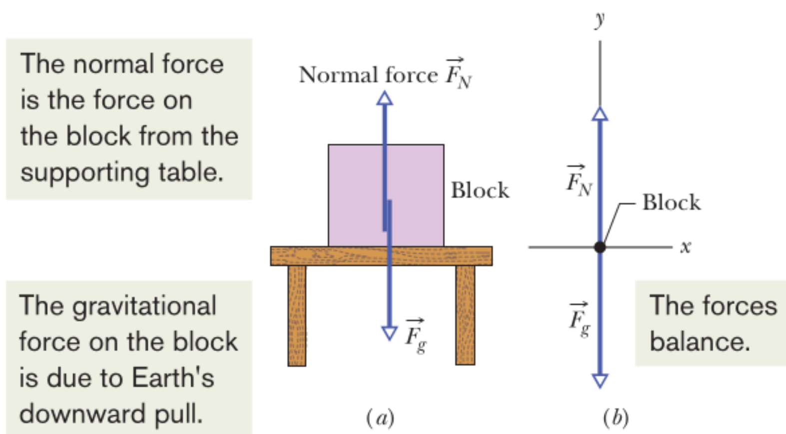


Figure 1: (a) A block resting on a table experiences a normal force FN perpendicular to the tabletop. (b) The free-body diagram for the block.

Free-Body Diagram and Newton's Second Law

In this situation, the only two forces acting on the block are the gravitational force F_g (acting downward) and the normal force F_N (acting upward).

Since both forces are vertical, we can apply Newton's second law in the y-direction, using a positive-upward y-axis:

$$F_{\text{net},y} = ma_y$$

The net force in the y-direction is:

$$F_N - F_g = ma_y$$

Thus, the equation of motion for the block is:

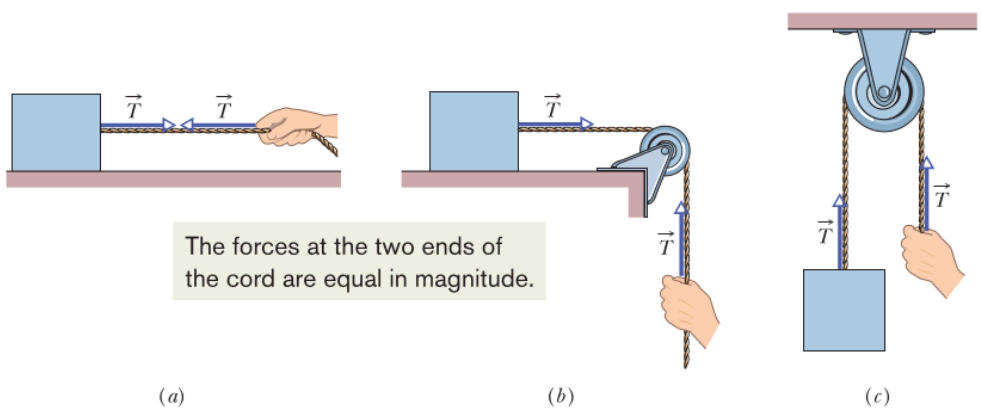
$$F_N - F_g = ma_y$$

Note:

The normal force F_N exerted by the table is the force that counteracts the gravitational force F_g, keeping the block in equilibrium (or in uniform motion if $a_y = 0$).

4. Tension,T

When a cord, rope, cable, or similar object is attached to a body and pulled taut, it exerts a force T on the body. This force is directed away from the body and along the length of the cord, as shown in Figure. The force exerted by the cord on the body is referred to as a ***tension force***. The reason it is called a tension force is that the cord is said to be in a state of tension (or under tension), meaning it is being pulled taut and stretched.



The tension in the cord is the magnitude of the force, T, applied by the cord on the body. For example, if the force exerted by the cord on the body has a magnitude of $T = 50\,\text{N}$, we say that the tension in the cord is 50 N. This tension force is always directed along the cord and away from the object that is being pulled.

Characteristics of the Cord

In many problems, we make a simplifying assumption about the cord. The cord is often treated as *massless*, meaning that its mass is negligible compared to the mass of the objects it connects. This assumption allows us to focus on the forces acting on the bodies without needing to consider the mass of the cord itself.

Additionally, the cord is often considered *unstretchable*, meaning that it does not stretch under the forces acting on it. This means the tension is assumed to be the same throughout the length of the cord, and there is no elongation or deformation of the cord itself.

Force Transmission

The cord exists only as a means of transmitting forces between two objects. Because the cord is massless and unstretchable, it exerts the same force magnitude T on both bodies it connects. This means that if a force is applied at one end of the cord, the same force is transmitted to the other end, assuming the cord is ideal. Thus, the magnitude of the tension in the cord is constant throughout its length and affects both bodies equally.

In summary, the tension in a cord refers to the force it exerts when pulled taut, and the magnitude of this force is constant throughout a massless and unstretchable cord.

5. Friction, f

When a body is either sliding or attempting to slide over a surface, the motion is resisted by a force due to the bonding between the body and the surface. This resisting force is called ***friction***, and it acts along the surface in the opposite direction of the intended motion, as shown in Figure.

In many cases, to simplify the analysis, friction is assumed to be negligible. In such cases, the surface (or the body) is said to be *frictionless*, meaning that no resistance is considered in the motion.

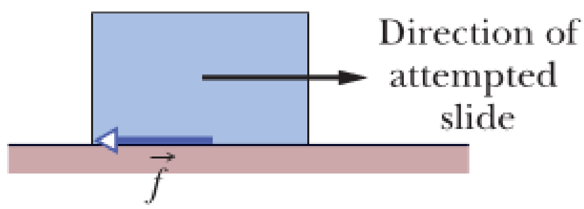


Figure 5-8 A frictional force $\vec{f}$ opposes the attempted slide of a body over a surface.