Topic: Concepts of Fluid, Pressure and Density, Measurement of Pressure, General Concepts of Fluid Flow, Continuity Equation

What is a Fluid?

- A fluid is a substance that can flow, unlike a solid.
- It takes the shape of its container because it cannot resist sideways forces.
- Fluids cannot hold their shape against a force but can push back in the direction straight out from their surface.
- Some materials, like pitch, take a long time to flow but are still considered fluids.
- Both liquids and gases are fluids because their particles are not arranged in a fixed pattern like solids.

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Density

To find the density ρ of a fluid at any point, we isolate a small volume element ΔV around that point and measure the mass Δm of the fluid contained within that element. The density is then given by:

$$\rho = \frac{\Delta m}{\Delta V} \tag{1}$$

In practice, we assume that a fluid sample is large relative to atomic dimensions and thus is 'smooth' (with uniform density) rather than 'lumpy' with atoms. This assumption allows us to write the density in terms of the total mass m and total volume V of the sample:

$$\rho = \frac{m}{V} \quad \text{(uniform density)} \tag{2}$$

Density is a scalar property, meaning it has magnitude but no direction. Its SI unit is the kilogram per cubic meter (kg/m^3).

Pressure

Pressure is the force exerted per unit area on a surface. If a force F acts perpendicularly on a surface of area A, the pressure P is given by:

$$P = \frac{F}{A} \tag{3}$$

In SI units, pressure is measured in pascals (Pa). One pascal is defined as one newton per square meter:

$$1 \text{ Pa} = 1\frac{N}{m^2} \tag{4}$$

This means that if a force of one newton is applied uniformly over an area of one square meter, the resulting pressure is one pascal.

Pressure is a scalar quantity. It is defined as the force applied per unit area, and it acts equally in all directions at a given point in a fluid. Since it does not have a specific direction (only magnitude), it is considered a scalar, not a vector.

Problem 1 :

A living room has floor dimensions of 3.5 m and 4.2 m and a height of 2.4 m.

(a) What does the air in the room weigh when the air pressure is 1.0 atm? The density of air at 1.0 atm as $\rho = 1.21\,\mathbf{kg/m}^3$

Solution:

Key Ideas:

1. The air's weight is equal to mg, where m is its mass.
2. Mass m is related to the air density ρ and the air volume V by the equation $\rho = \frac{m}{V}$.

From the two ideas above, the air weight can be found by:

$$mg = (\rho V)g$$

The volume of the room is:

$$V = (3.5\,\text{m}) \times (4.2\,\text{m}) \times (2.4\,\text{m}) = 35.28\,\text{m}^3$$

Now, we calculate the weight:

$$mg = (1.21\,\text{kg/m}^3) \times (35.28\,\text{m}^3) \times (9.8\,\text{m/s}^2) = 418.1\,\text{N}$$

Thus, the weight of the air is approximately:

$$\boxed{418\,\text{N}}$$

Problem 2:

What is the magnitude of the atmosphere's downward force on the top of your head, which we take to have an area of 0.040 m^2? We approximate that the air pressure $p = 1.0\,\text{atm}$.

Solution :

When the fluid pressure P on a surface of area A is uniform, the fluid force on the surface can be obtained from the equation:

$$F = PA$$

Given that the air pressure $p = 1.0\,\text{atm}$. Converting atm to N/m^2:

$$1\,\text{atm} = 1.01 \times 10^5\,\text{N/m}^2$$

Thus, the force is:

$$F = (1.0\,\text{atm}) \times (0.040\,\text{m}^2) = (1.01 \times 10^5\,\text{N/m}^2) \times (0.040\,\text{m}^2) = 4.04 \times 10^3\,\text{N}$$

Therefore, the downward force on the top of your head is:

$$\boxed{4.0 \times 10^3\,\text{N}}$$

Measurement of Pressure:

Fluid at Rest:

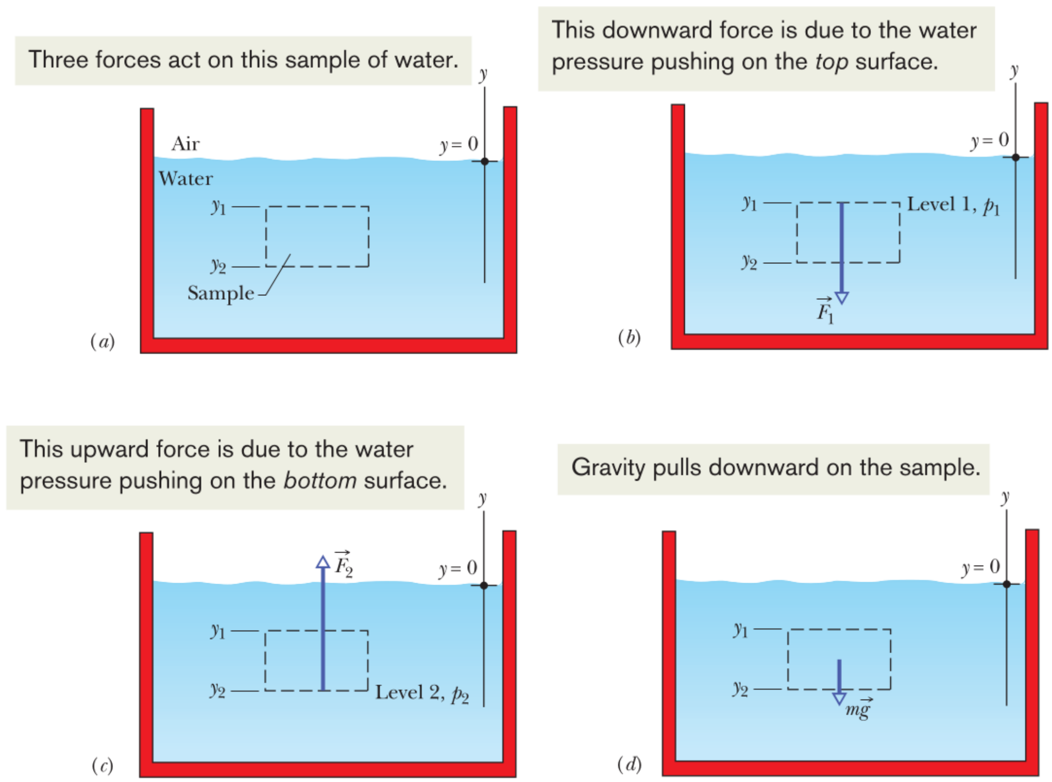


Figure 1: A tank of water in which a sample of water is contained in an imaginary cylinder of horizontal base area A.

The water in the cylinder is in static equilibrium, meaning it is stationary and the forces on it balance. Three forces act on the water vertically:

- Force F_1 acts at the top surface of the cylinder and is due to the water above the cylinder.
- Force F_2 acts at the bottom surface of the cylinder and is due to the water just below the cylinder.
- The gravitational force on the water is mg, where m is the mass of the water in the cylinder.

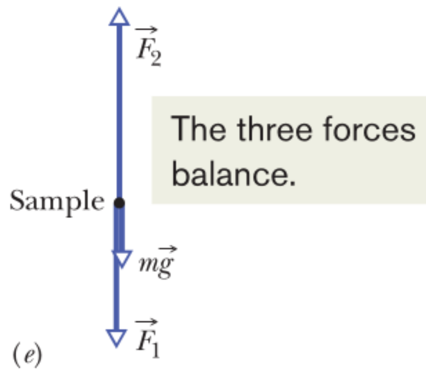


Figure 2: Free-Body Diagram for the Water in the Cylinder

The balance of these forces is given by:

$$F_2 = F_1 + mg \tag{1}$$

To involve pressures, we use the equation for the force due to pressure:

$$F_1 = p_1 A \quad \text{and} \quad F_2 = p_2 A \tag{2}$$

The mass m of the water in the cylinder is given by:

$$m = \rho V$$

where the volume V of the cylinder is the product of its face area A and its height $y_1 - y_2$, so:

$$m = \rho A(y_1 - y_2) \tag{3}$$

Substituting equation (2) and (3) into equation (1) , we get:

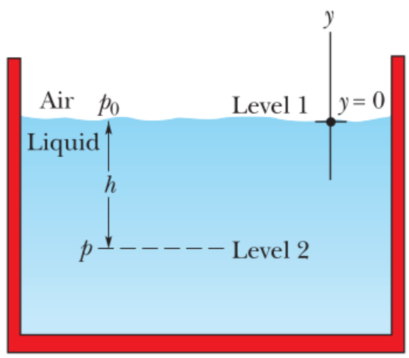
$$p_2 A = p_1 A + \rho A g(y_1 - y_2)$$

Simplifying:

$$p_2 = p_1 + \rho g(y_1 - y_2) \tag{4}$$

This equation can be used to find the pressure both in a liquid (as a function of depth) and in the atmosphere (as a function of altitude or height).

Definitions of Different Types of Pressure:



- **Absolute Pressure:** The total pressure exerted at a point, including both the atmospheric pressure and the pressure due to the liquid (or any other medium) above it. In this figure, p is the absolute pressure at level 2, given by:

$$p = p_0 + \rho g h$$

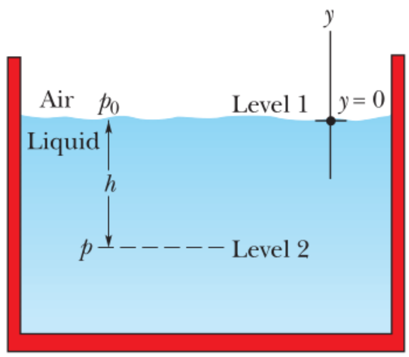
  where p_0 is the atmospheric pressure, ρ is the density of the liquid, g is the gravitational acceleration, and h is the depth below the surface.

- **Atmospheric Pressure:** The pressure exerted by the weight of the atmosphere at the surface of the Earth. It is denoted by p_0 and is generally considered constant at sea level.

- **Gauge Pressure:** The pressure measured relative to atmospheric pressure. It is the difference between the absolute pressure and the atmospheric pressure. The gauge pressure is what is usually measured by pressure gauges. For example, in Fig., the gauge pressure at level 2 is given by:

$$p_{\text{gauge}} = \rho g h$$

  where $\rho g h$ is the pressure due to the liquid above level 2.



Case 1:

For the case of pressure at a depth h below the liquid surface, we set level 1 to be the surface and level 2 to be a distance h below it. We then substitute:

$$y_1 = 0, \quad p_1 = p_0, \quad y_2 = -h, \quad p_2 = p$$

into equation (4), which becomes:

$$p = p_0 + \rho g h$$

This gives the pressure at depth h in the liquid.

Case 2:

For example, to find the atmospheric pressure at a distance d above level 1 in Fig. , we substitute:

$$y_1 = 0$$

Then, with $\rho = \rho_{\text{air}}$, we obtain:

$$p_1 = p_0$$

and

$$y_2 = d$$

Thus, the pressure p at a distance d above level 1 is:

$$p = p_0 - \rho_{\text{air}} g d$$

Definitions of Different Types of Fluid Flow:

The motion of real fluids is very complicated and not yet fully understood. Instead, we shall discuss the motion of an ideal fluid, which is simpler to handle mathematically and yet provides useful results.

- **Steady Flow:** In steady (or laminar) flow, the velocity of the moving fluid at any fixed point does not change with time. The flow remains smooth, and the particles move along well-defined paths. For example, the gentle flow of water near the center of a quiet stream is steady.
- **Unsteady Flow:** When the flow speed changes at a certain point, the flow transitions to nonsteady (or nonlaminar or turbulent) flow.
- **Incompressible Flow:** In incompressible flow, the fluid's density remains constant and uniform throughout the flow. This assumption is made to simplify the analysis of fluid motion. For many practical fluids, such as water, this assumption holds true under normal conditions, as the change in volume with respect to pressure is negligible.
- **Compressible Flow:** In compressible flow, the fluid's density doesn't remain constant.
- **Nonviscous Flow:** In nonviscous flow, the fluid experiences no resistance to flow. Viscosity, which is a measure of a fluid's resistance to flow (like honey being more viscous than water), is assumed to be zero in an ideal fluid.

Derivation of the Equation of Continuity

The equation of continuity is based on the principle of conservation of mass. It states that the mass flow rate of a fluid remains constant as it moves through a pipe or a channel with varying cross-sectional area.

Mass Flow Rate

Consider a fluid flowing through a pipe with different cross-sectional areas at two points.

- At section 1:
  - Cross-sectional area = A_1
  - Velocity of fluid = v_1
  - Density of fluid = ρ_1
- At section 2:

- Cross-sectional area = A_2
- Velocity of fluid = v_2
- Density of fluid = ρ_2

The mass flow rate $\dot{m}$ is given by:

$$\dot{m} = \text{Density} \times \text{Volume flow rate} = \rho A v \tag{5}$$

Applying this at sections 1 and 2:

$$\dot{m}_1 = \rho_1 A_1 v_1 \tag{6}$$

$$\dot{m}_2 = \rho_2 A_2 v_2 \tag{7}$$

Since mass is conserved,

$$\dot{m}_1 = \dot{m}_2 \tag{8}$$

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \tag{9}$$

For Incompressible Fluids

For an incompressible fluid, the density remains constant ($\rho_1 = \rho_2 = \rho$), so it cancels out:

$$A_1 v_1 = A_2 v_2 \tag{10}$$

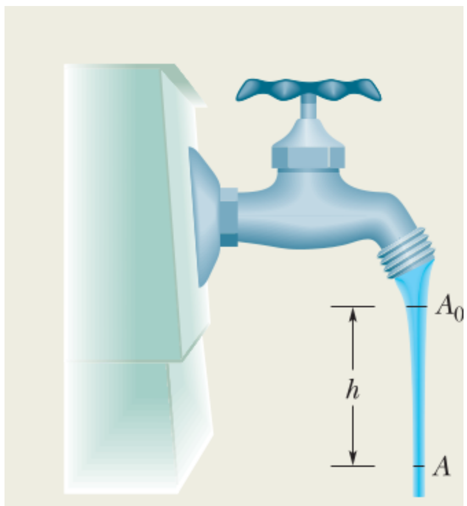
This is the equation of continuity for incompressible fluids. It shows that if the cross-sectional area of the flow decreases, the velocity must increase, and vice versa.

Problem 3:

Water flows out of a faucet and narrows as it falls due to gravity. The cross-sectional areas of the stream at two points are:

- **At the faucet:** $A_0 = 1.2\,\mathbf{cm}^2$**,**
- **After falling a distance of** $h = 45\,\mathbf{mm}$**:** $A = 0.35\,\mathbf{cm}^2$**.**

What is the volume flow rate of water from the faucet?



Solution

The volume flow rate Q is constant across both cross-sections:

$$Q = A_0 v_0 = Av,$$

where:

- v_0 is the velocity of water at the top,
- v is the velocity of water at the bottom.

Rearranging for v:

$$v = \frac{A_0}{A} v_0.$$

The velocities at the top and bottom are related through the kinematic equation:

$$v^2 = v_0^2 + 2gh,$$

where $h = 45\,\text{mm} = 0.045\,\text{m}$.

Substituting $v = \frac{A_0}{A} v_0$ into the kinematic equation:

$$\left(\frac{A_0}{A} v_0\right)^2 = v_0^2 + 2gh.$$

Expanding and solve for v_0^2:

$$\frac{A_0^2}{A^2} v_0^2 = v_0^2 + 2gh,$$

$$\left(\frac{A_0^2}{A^2} - 1\right) v_0^2 = 2gh.$$

Rearranging:

$$v_0^2 = \frac{2gh}{\frac{A_0^2}{A^2} - 1}.$$

Taking the square root to find v_0:

$$v_0 = \sqrt{\frac{2gh}{\frac{A_0^2}{A^2} - 1}}.$$

Now,

$$A_0 = 1.2\,\mathrm{cm}^2 = 1.2 \times 10^{-4}\,\mathrm{m}^2, \quad A = 0.35\,\mathrm{cm}^2 = 0.35 \times 10^{-4}\,\mathrm{m}^2.$$

So, the ratio of areas:

$$\frac{A_0}{A} = \frac{1.2}{0.35} \approx 3.4286, \quad \frac{A_0^2}{A^2} = (3.4286)^2 \approx 11.7561.$$

Substituting into the equation for v_0:

$$v_0 = \sqrt{\frac{2 \cdot 9.8 \cdot 0.045}{11.7561 - 1}} = \sqrt{\frac{0.882}{10.7561}} = \sqrt{0.082}.$$

$$v_0 \approx 0.981\,\mathrm{m/s}.$$

Calculation of Q

The volume flow rate is:

$$Q = A_0 v_0.$$

Substituting $A_0 = 1.2 \times 10^{-4}\,\mathrm{m}^2$ and $v_0 = 0.981\,\mathrm{m/s}$:

$$Q = 1.2 \times 10^{-4} \cdot 0.981 = 1.177 \times 10^{-4}\,\mathrm{m}^3/\mathrm{s}.$$

Converting to cm^3/s:

$$Q = 117.7\,\mathrm{cm}^3/\mathrm{s}.$$

Answer:

The volume flow rate is:

$$\boxed{117.7\,\text{cm}^3/\text{s}}.$$

Problem 4:

Water flows through a horizontal pipe of diameter $8\,\text{cm}$ at a velocity of $4\,\text{m/s}$. The pipe narrows to a diameter of $4\,\text{cm}$. Find the velocity of water in the narrower section.

Solution:

$$A_1 v_1 = A_2 v_2$$
$$\pi \left(\frac{d_1}{2}\right)^2 v_1 = \pi \left(\frac{d_2}{2}\right)^2 v_2$$
$$\left(\frac{8}{2}\right)^2 \cdot 4 = \left(\frac{4}{2}\right)^2 \cdot v_2$$
$$16 \cdot 4 = 4 \cdot v_2 \implies v_2 = 16\,\text{m/s}$$

Problem 5:

The water flow rate in a pipe is $0.02\,\text{m}^3/\text{s}$. (a) If the pipe's diameter at one point is $10\,\text{cm}$, calculate the velocity of water at that point. (b) If the pipe narrows to a diameter of $5\,\text{cm}$, find the velocity in the narrower section.

Solution:

(a) Flow rate is given by:

$$Q = Av \implies v = \frac{Q}{A}$$
$$A_1 = \pi \left(\frac{d_1}{2}\right)^2 = \pi \left(\frac{0.1}{2}\right)^2 = 7.85 \times 10^{-3}\,\text{m}^2$$
$$v_1 = \frac{0.02}{7.85 \times 10^{-3}} = 2.55\,\text{m/s}$$

(b) For the narrower section:

$$A_2 = \pi \left(\frac{d_2}{2}\right)^2 = \pi \left(\frac{0.05}{2}\right)^2 = 1.96 \times 10^{-3}\,\text{m}^2$$
$$v_2 = \frac{0.02}{1.96 \times 10^{-3}} = 10.2\,\text{m/s}$$

Problem 6

Water flows through a pipe of cross-sectional area $25\,\mathrm{m}^2$ at $3\,\mathrm{m/s}$. It splits into two smaller pipes with areas $10\,\mathrm{m}^2$ and $5\,\mathrm{m}^2$. If the velocity in the $10\,\mathrm{m}^2$ pipe is $4\,\mathrm{m/s}$, find the velocity in the $5\,\mathrm{m}^2$ pipe.

Solution:

The total flow rate is conserved:

$$Q = A_1 v_1 = A_2 v_2 + A_3 v_3$$

$$25 \cdot 3 = 10 \cdot 4 + 5 \cdot v_3$$

$$75 = 40 + 5 \cdot v_3 \implies v_3 = 7\,\mathrm{m/s}$$