Planar Graphs and Partial k-Trees

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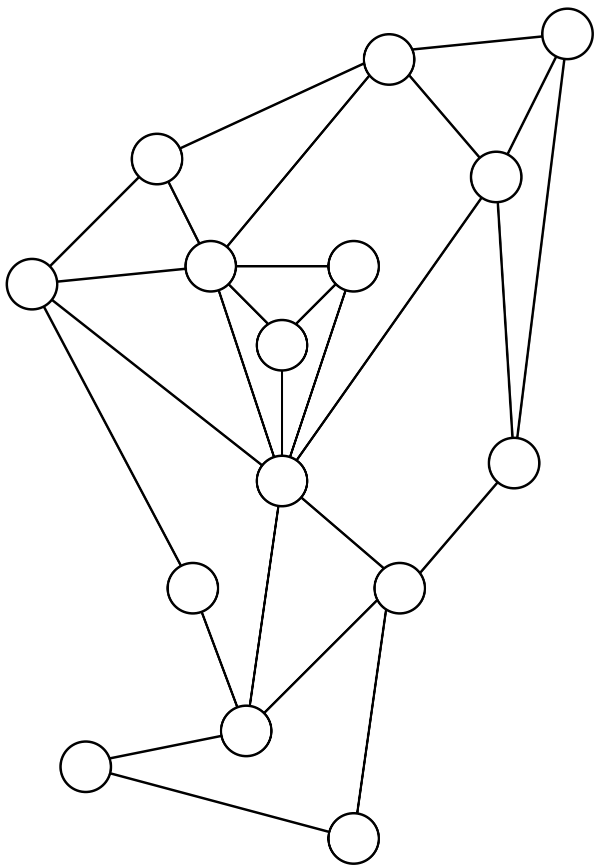
Overview

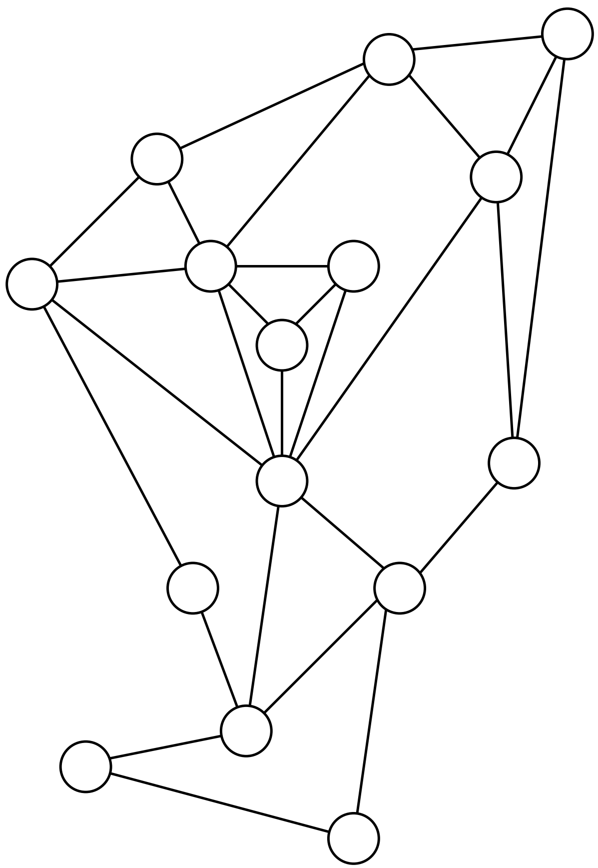
- Terminology
- Nested Satisfiability
- Matryoshka Graphs
- Decompositions of Bounded Treewidth

Planar Graphs



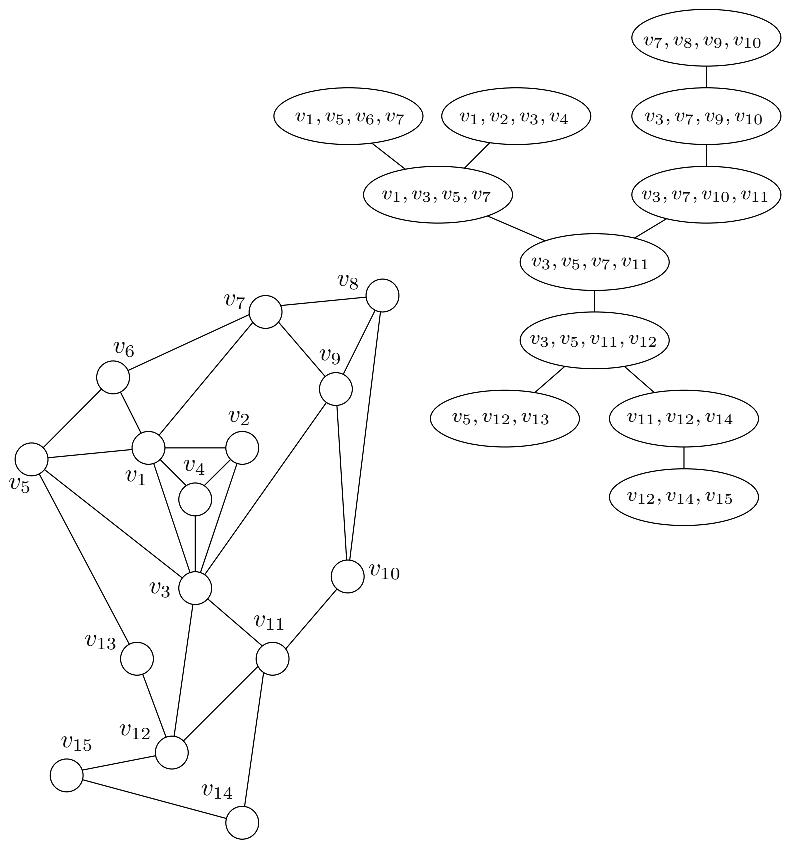
- Vertices as points
- Edges as continuous curves
- Can be drawn in the plane without edge crossing

Partial k-Trees



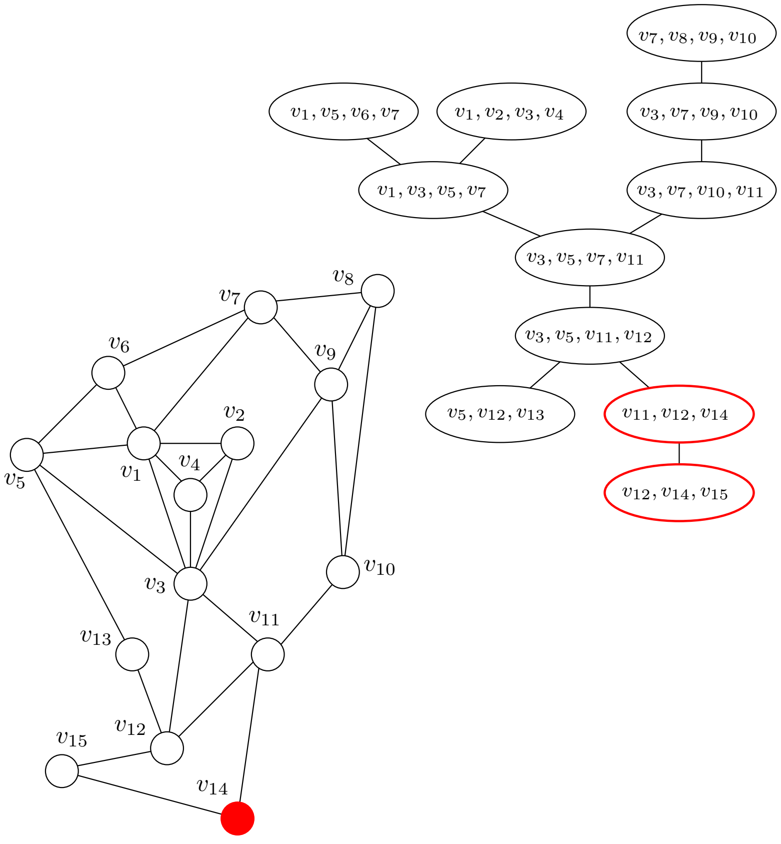
- Subclass of all graphs: every graph is a partial k-tree for some k
- Many NP-hard problems (e.g. SAT, independent set, etc.) are solvable in linear time for constant k

Tree Decomposition



A tree decomposition of graph G is a tree with labels on nodes such that:

Tree Decomposition



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- every vertex appears in a label

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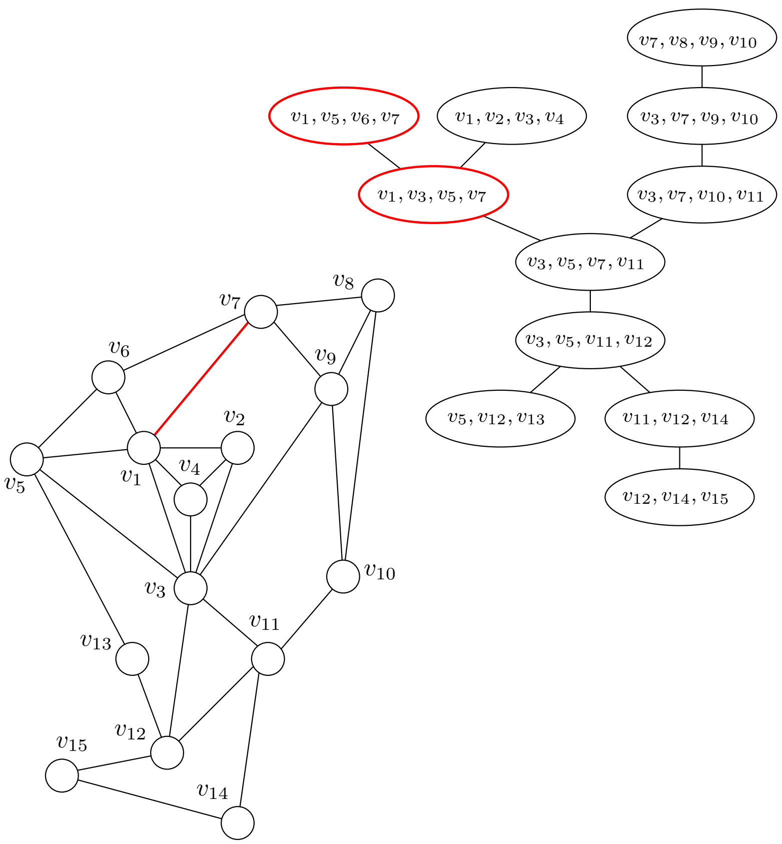

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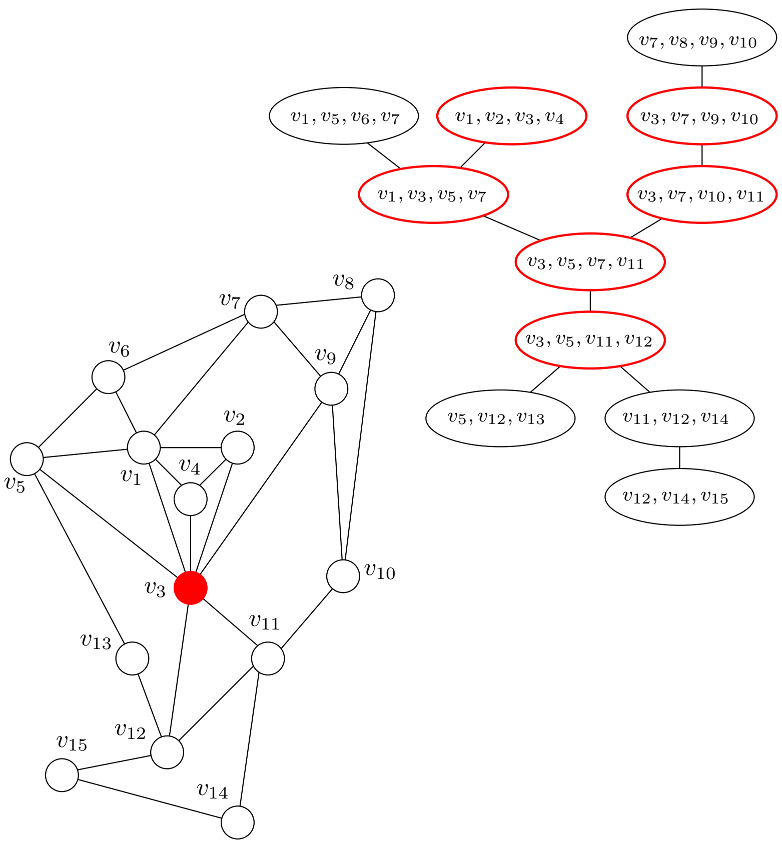
- for any edge (u, v), u and v are in a label

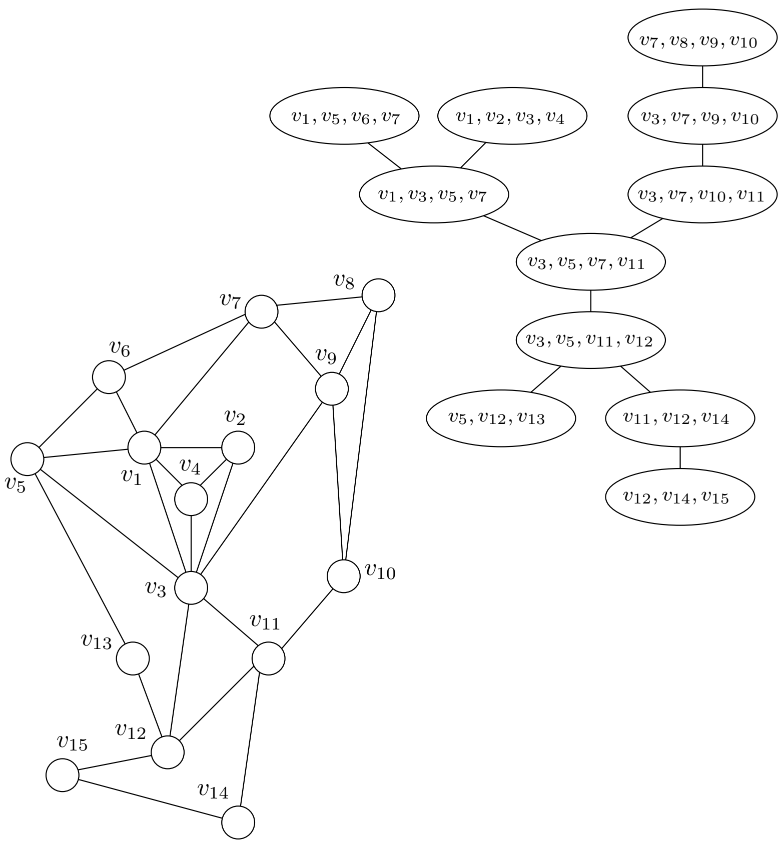
Tree Decomposition



A tree decomposition of graph G is a tree with labels on nodes such that:

- every vertex appears in a label
- for any edge (u, v), u and v are in a label
- for any vertex v, the labels containing v form a tree

Partial k-Trees

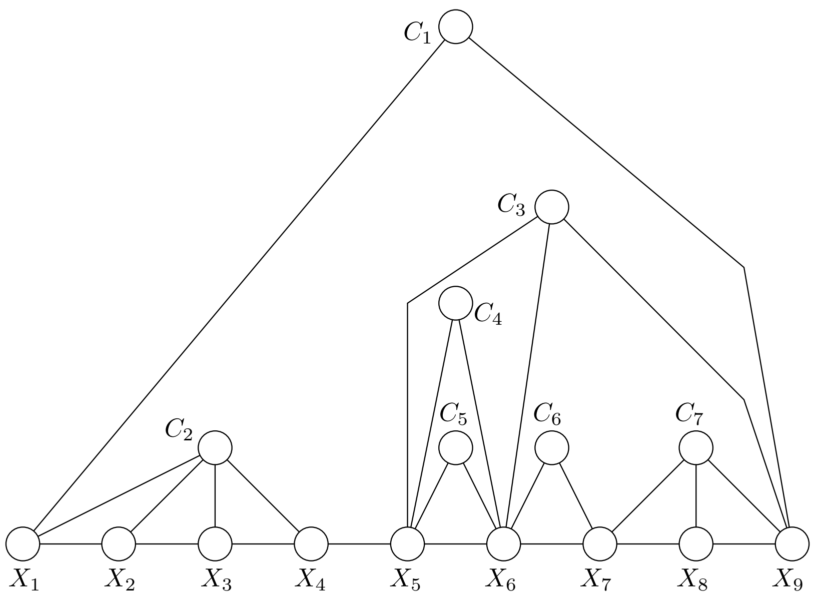


- Width $=$ maximum label size -1
- Partial k-tree: graph has tree decomposition of width $\leq k$
- Equivalently: graph has treewidth $\leq k$

Part I: Nested SAT

- SAT = satisfiability: classic NP-hard problem
- Knuth ’90: Nested SAT can be solved in linear-time
- Kratochvíl and Křivánek ’93: extends to MaxSAT and co-nested SAT
- We show: Nested SAT graphs are partial 3-trees
- This unifies and generalizes the results above

Nested Satisfiability



SAT (in CNF):

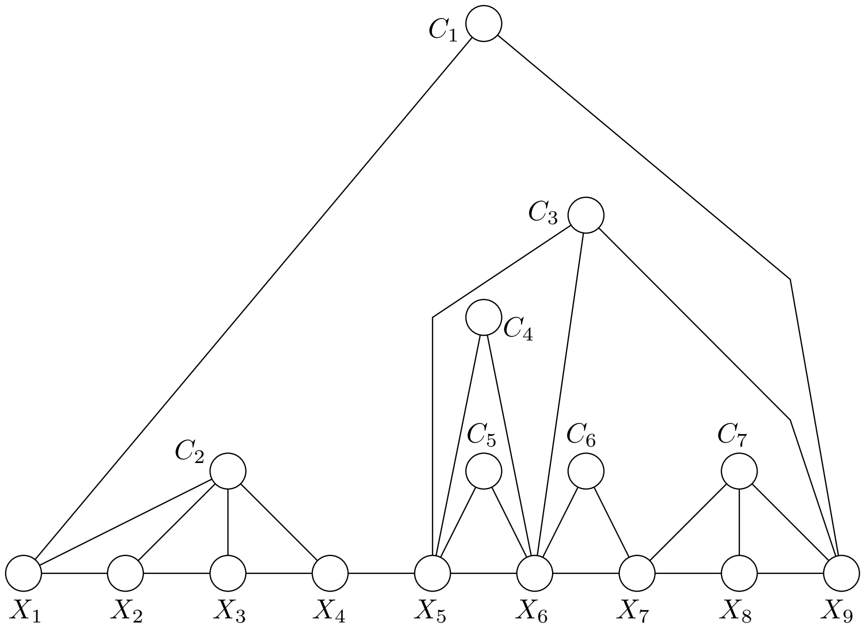
- variables $X_1, \dots, X_n$
- clauses $C_1, \dots, C_m$

SAT graph:

- vertex for each X_i and C_j
- edge (X_i, C_j) iff $X_i \in C_j$ or $\overline{X_i} \in C_j$
- edges (X_i, X_{i+1})

Nested Satisfiability

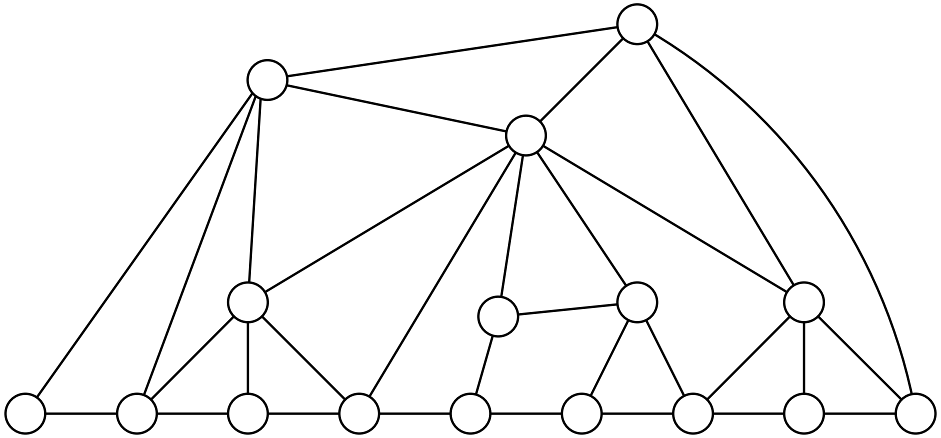
Nested SAT: SAT that has LH-drawing with variables on the line and clauses in the half-space



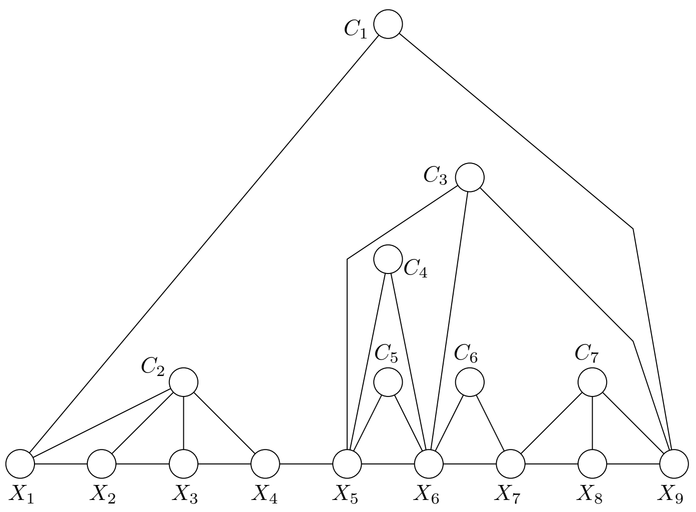
LH-Drawing

Given: graph G with vertex partition $V_1 \cup V_2$, an LH-drawing is a planar drawing such that

- vertices in V_1 are on the line
- vertices in V_2 are in the half-space above
- all edges are y-monotonic curves



Clause Hierarchy and Visible Variables



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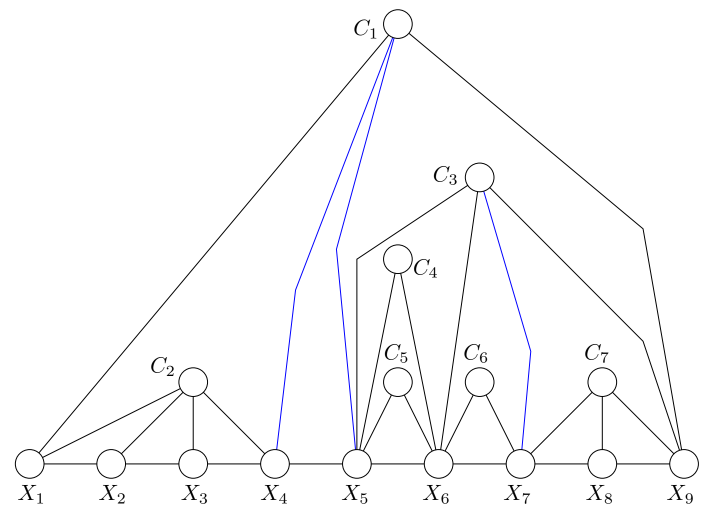
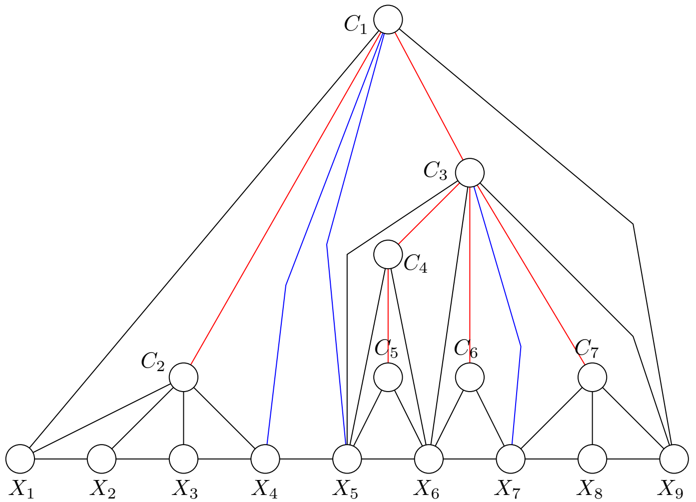


- Visible variables: can add clause-variable edge and still have an LH-drawing

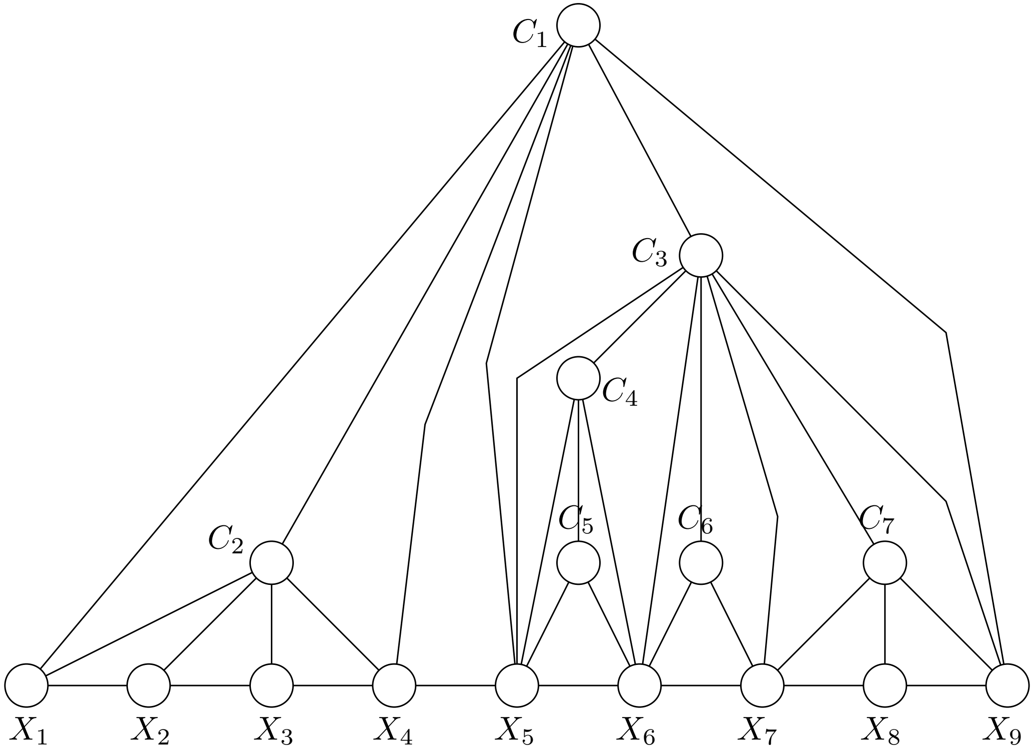
Clause Hierarchy and Visible Variables



- Visible variables: can add clause-variable edge and still have an LH-drawing
- Clause hierarchy: LH-drawing imposes tree structure on clauses

Clause Paths

Define tree decomposition for single clause and its visible variables



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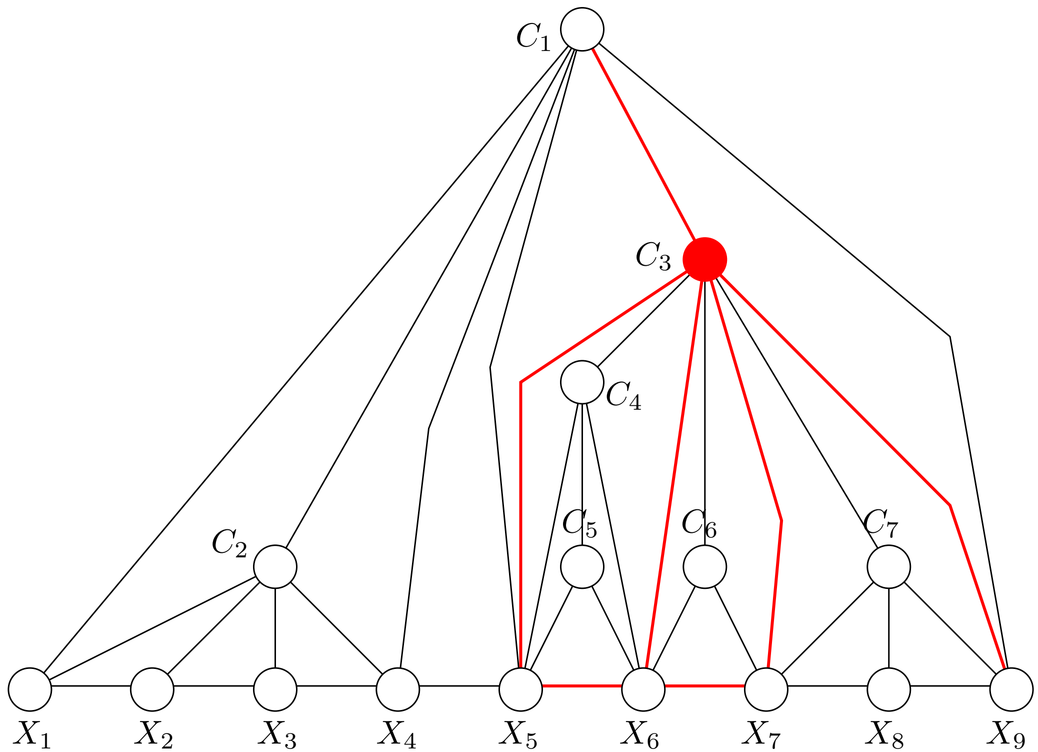

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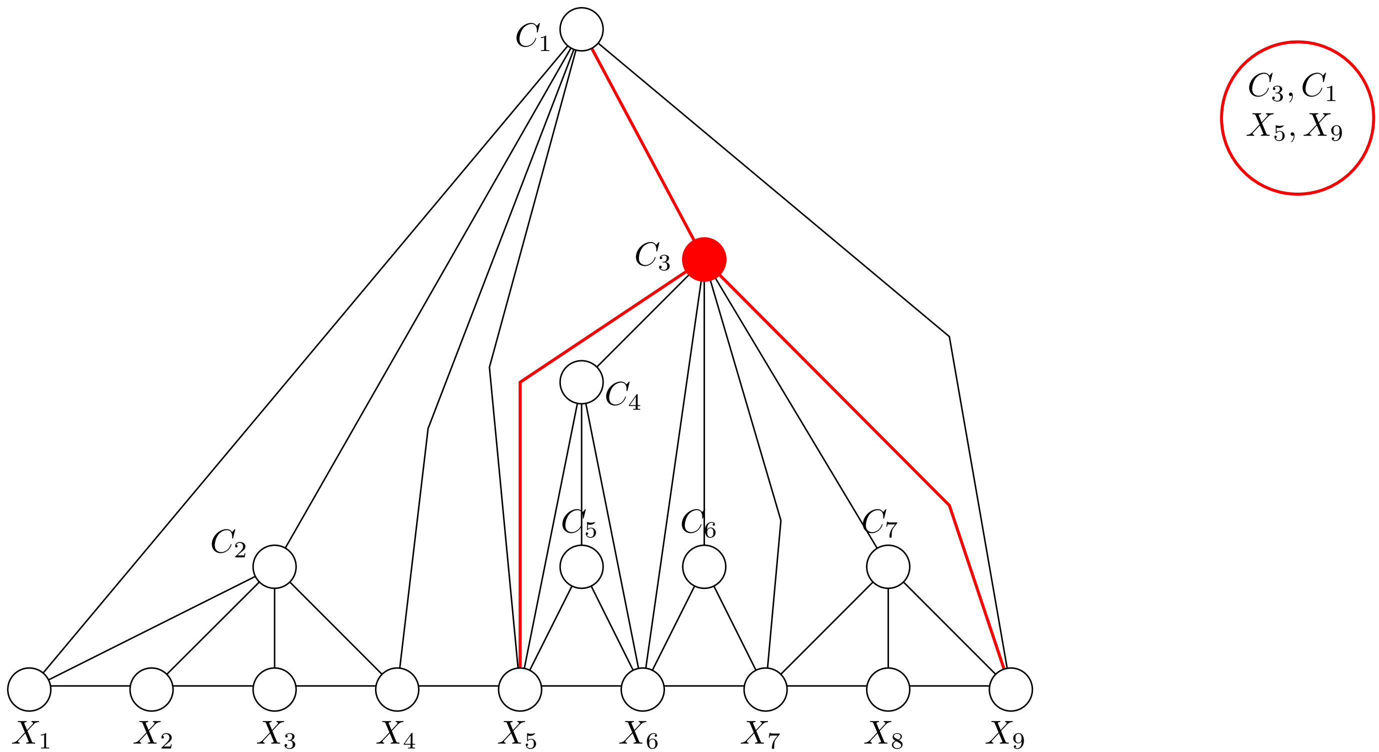

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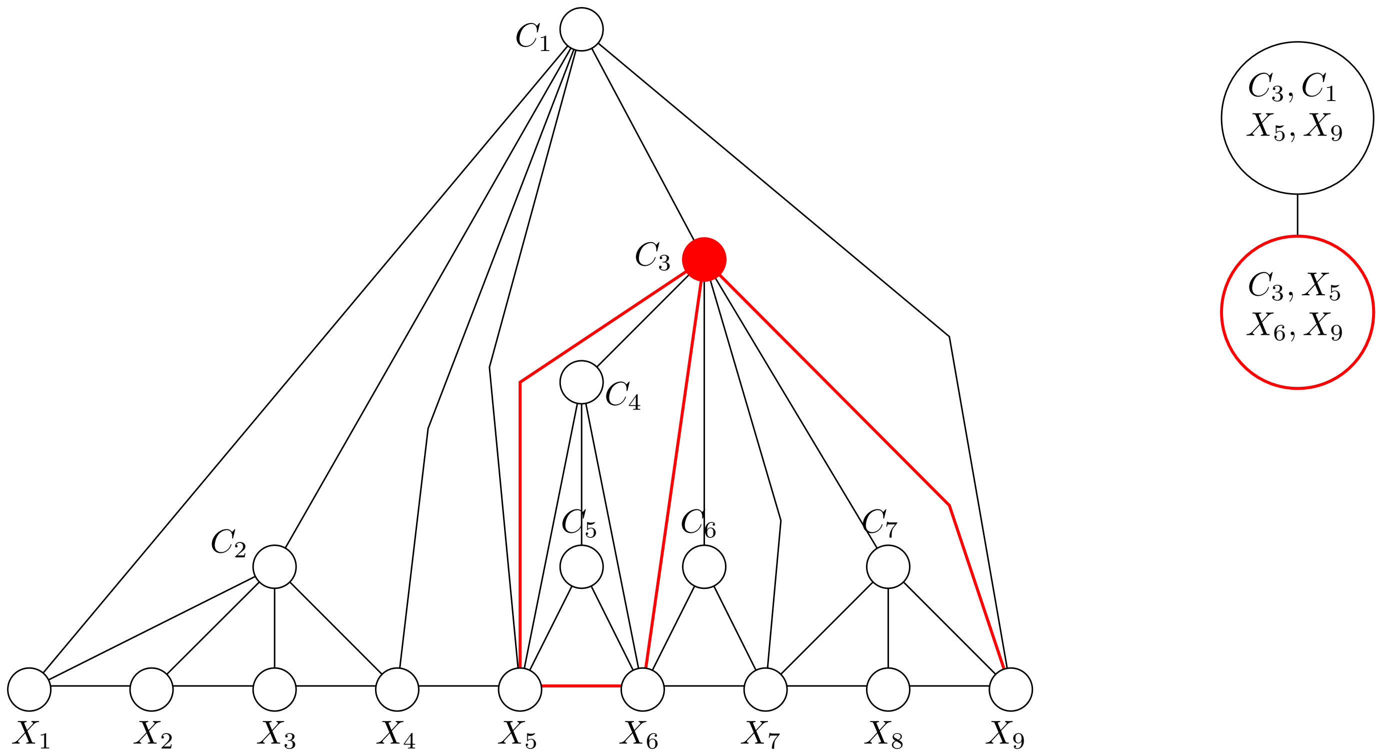

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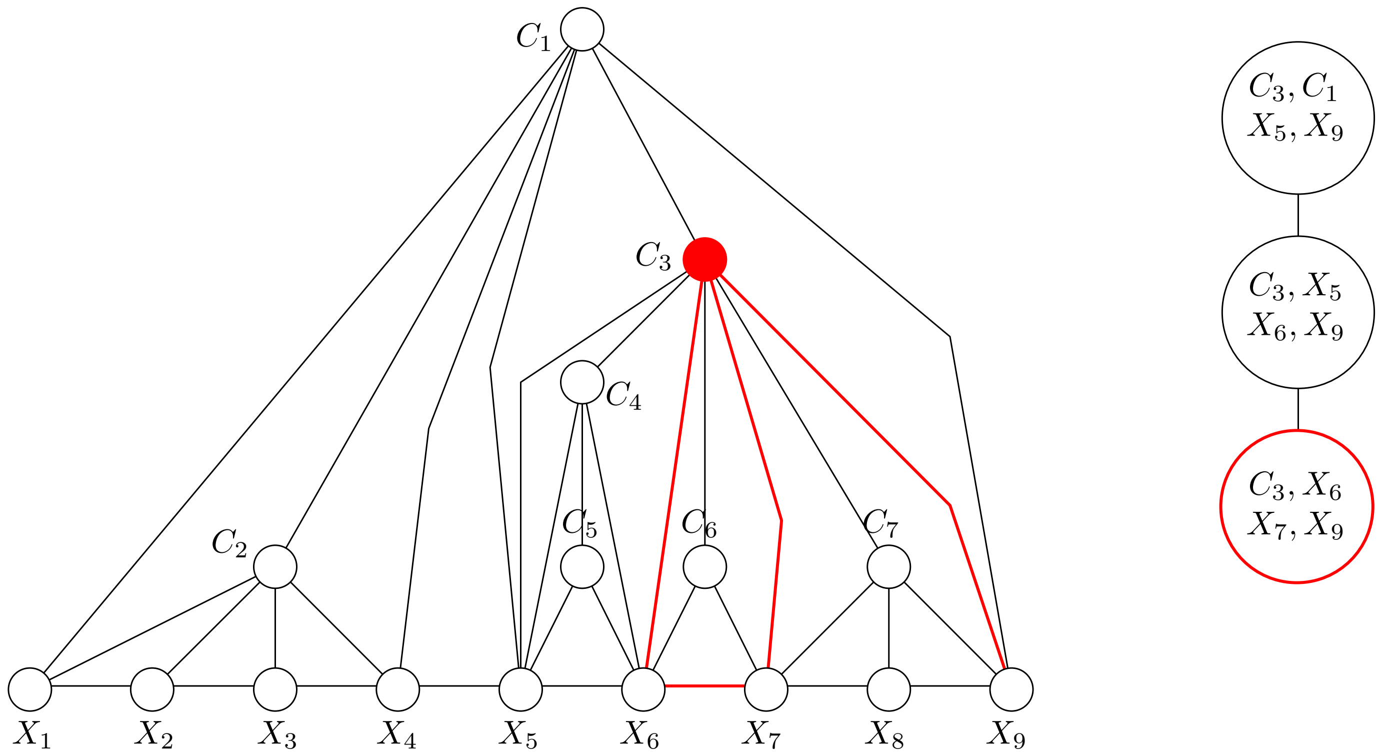
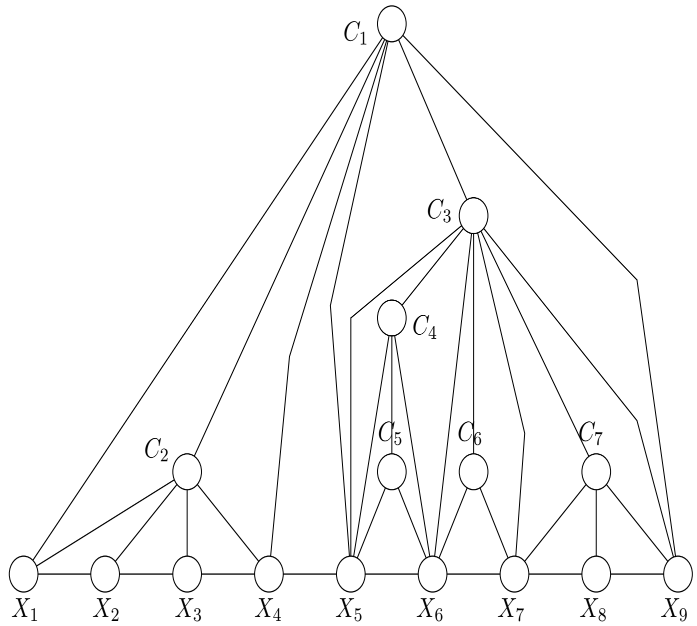


Nested SAT Tree Decomposition

Combine clause paths using clause hierarchy tree



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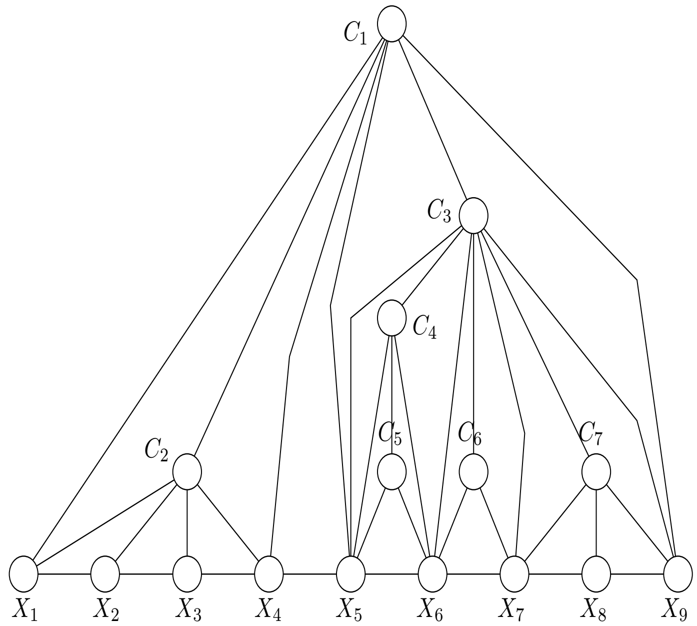

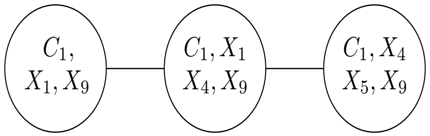

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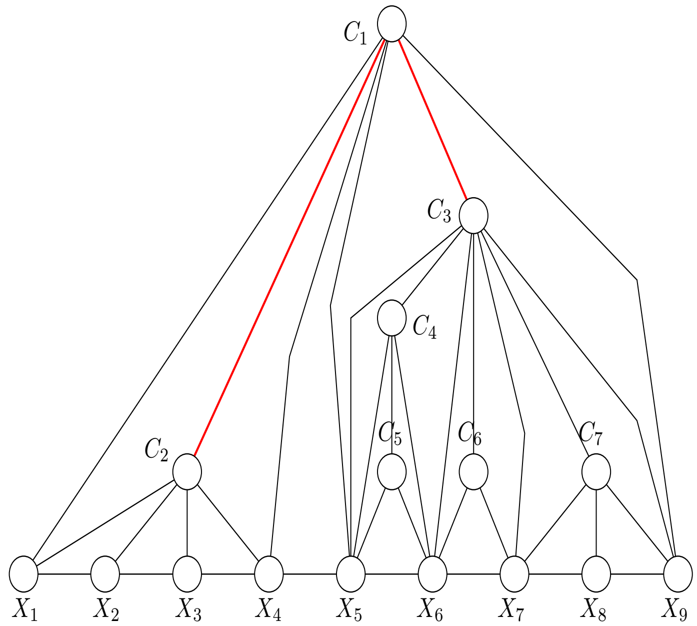

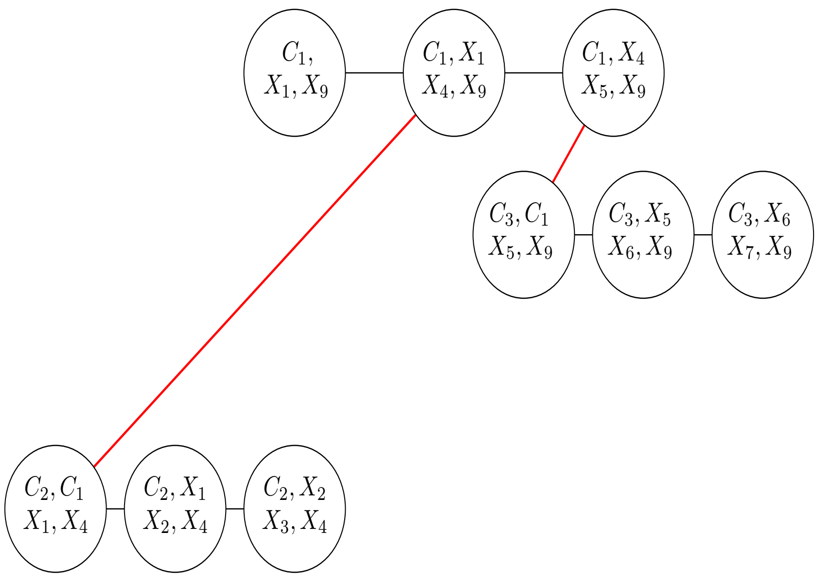

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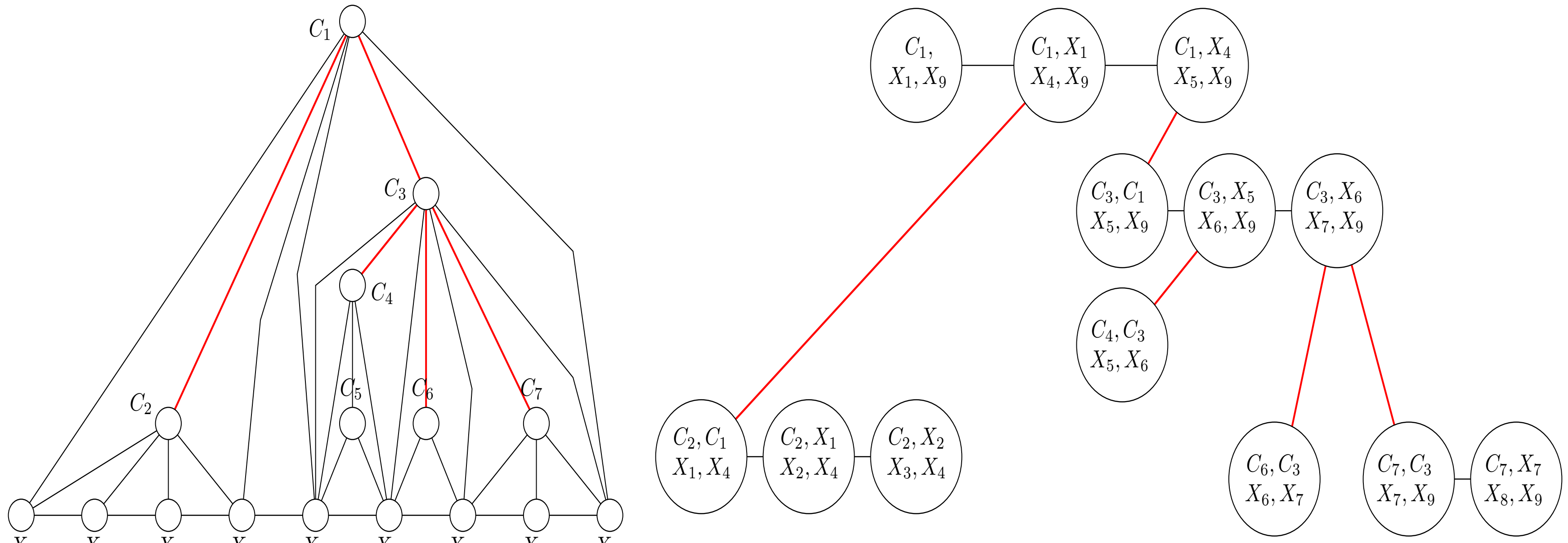

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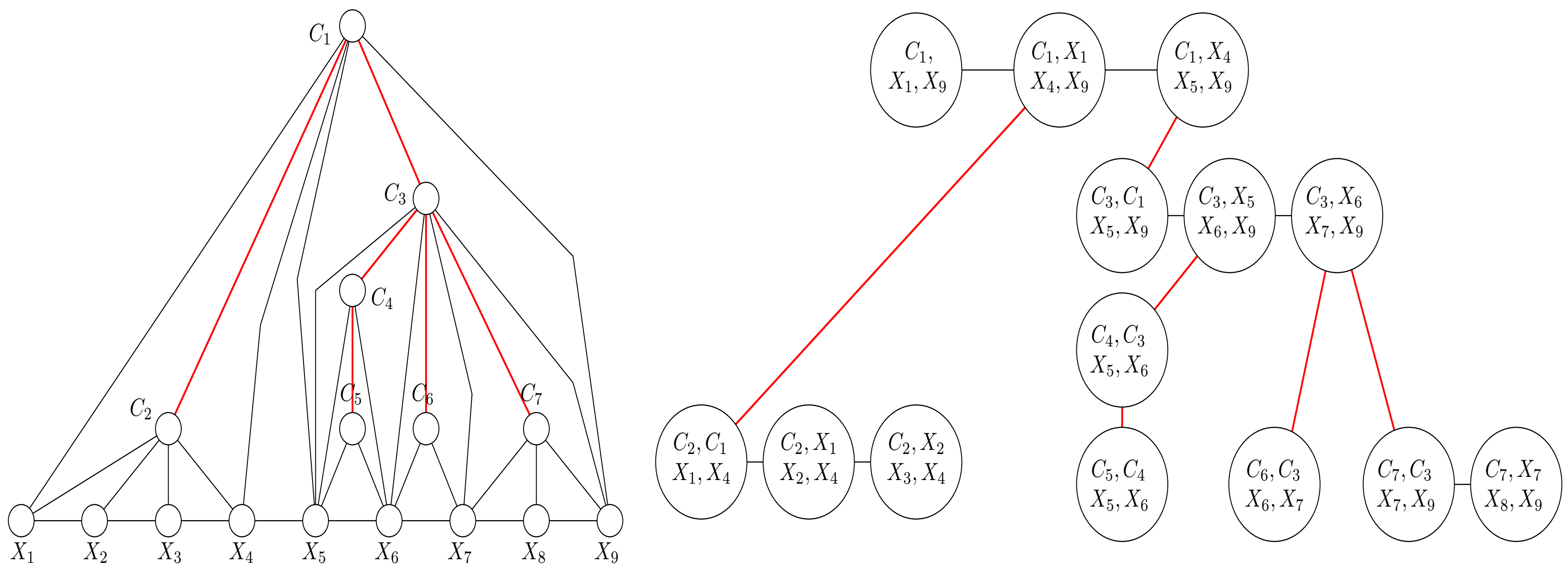

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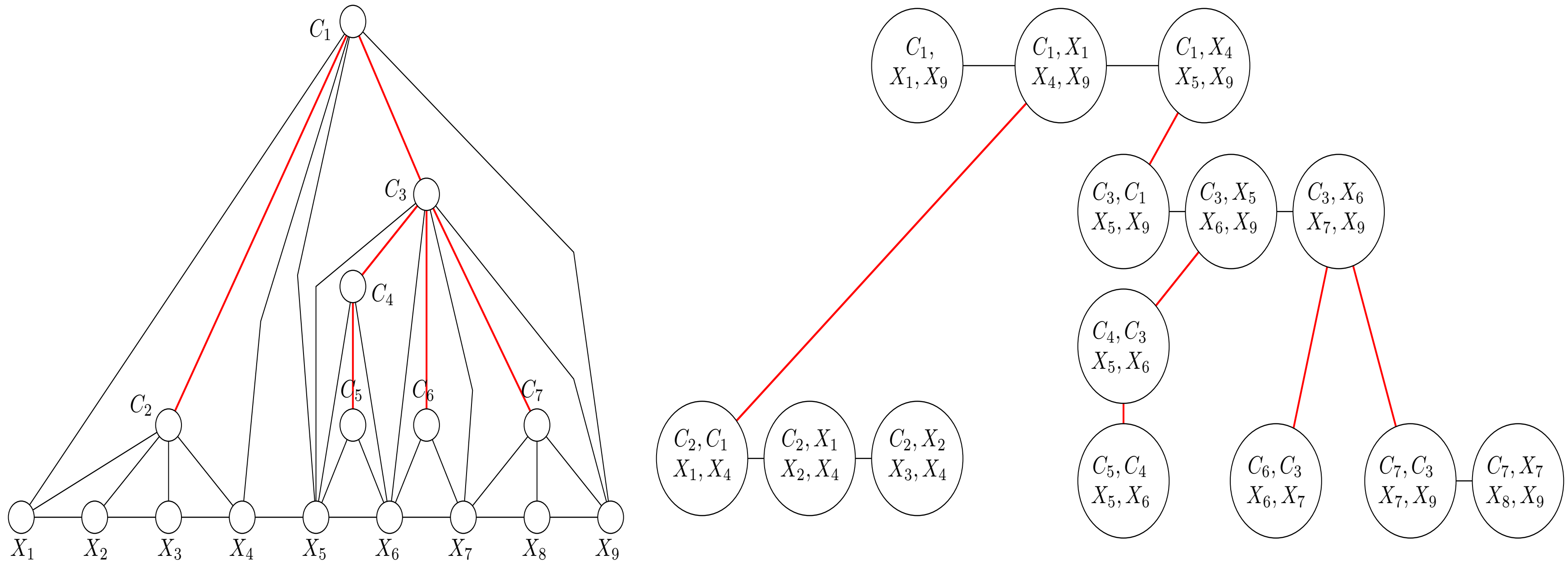
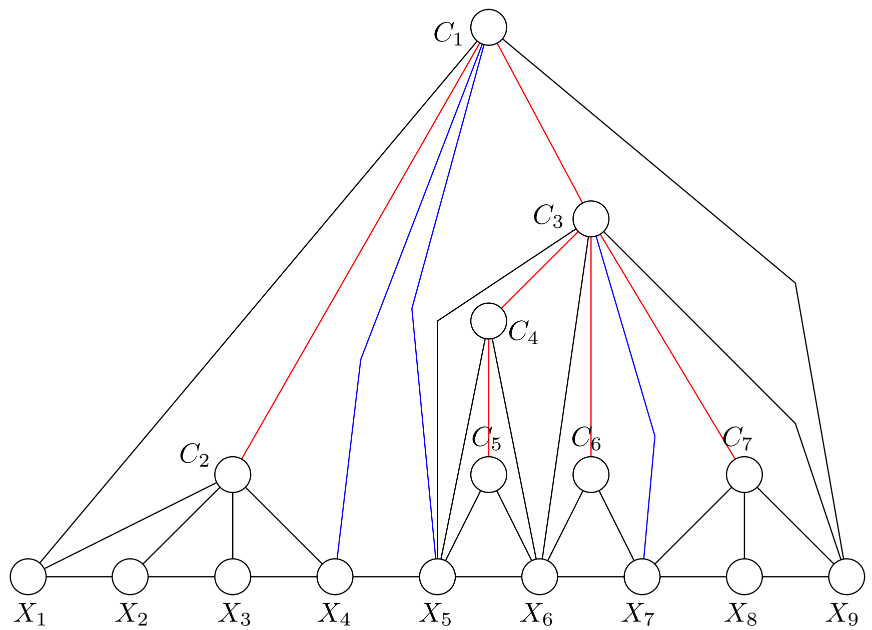


$\Rightarrow$ Nested SAT graphs are partial 3-trees

Part II: Generalizing Nested SAT Graphs

Can add edges and remain a partial 3-tree:



- Edges to visible variables
- Edges in clause hierarchy

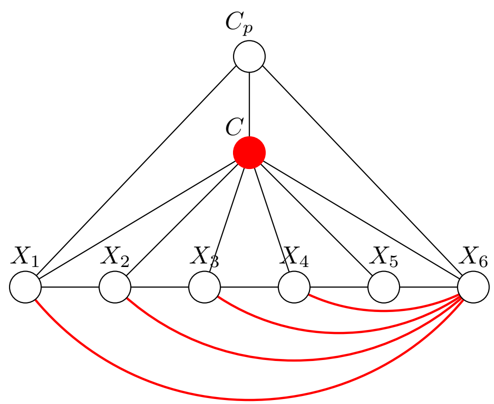
LH-drawing satisfies:

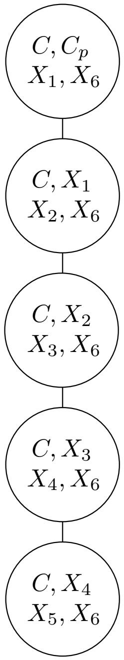
- All faces incident to the line

Can we push this farther?

Clause Trees

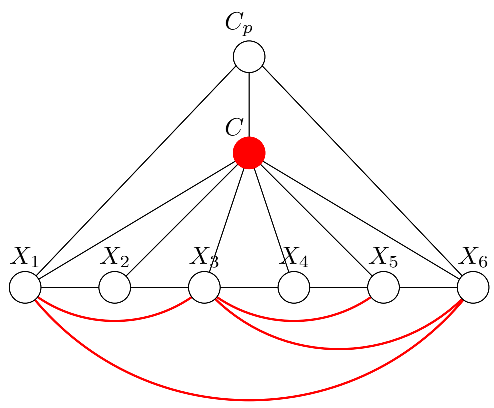
- For each clause, can add edges from visible variables to rightmost variable

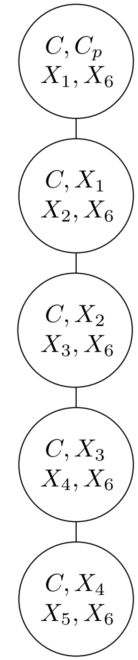




Clause Trees

- Modification: each clause can have an outerplanar† graph on its visible variables





† Outerplanar = planar and all vertices are on the outer-face

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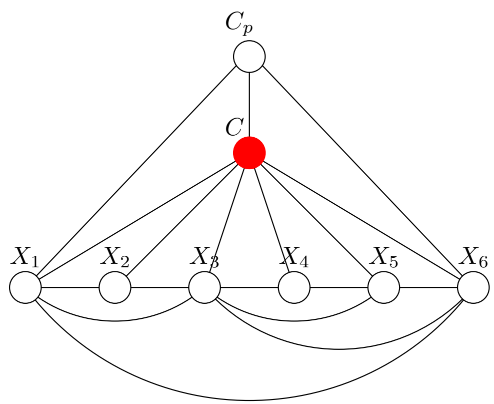

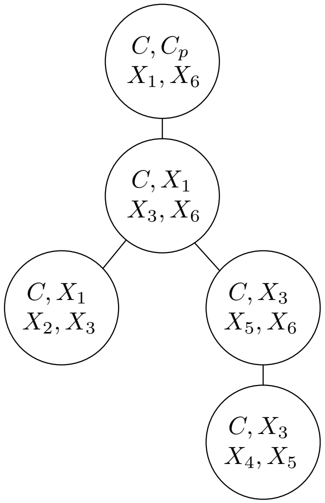

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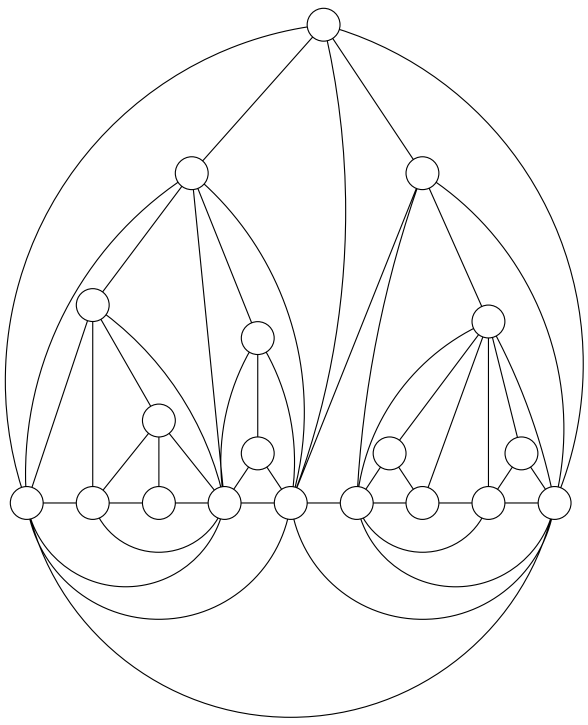
Matryoshka Graphs



Maximal Matryoshka graph has planar drawing with:

- vertices on or above line
- every face incident to line
- edges above: y-monotone
- edges below: endpoints have common neighbour in half-space

Matryoshka graph: subgraph of maximal Matryoshka graph

Characterizing Matryoshka Graphs

From above: Matryoshka graphs are partial 3-trees

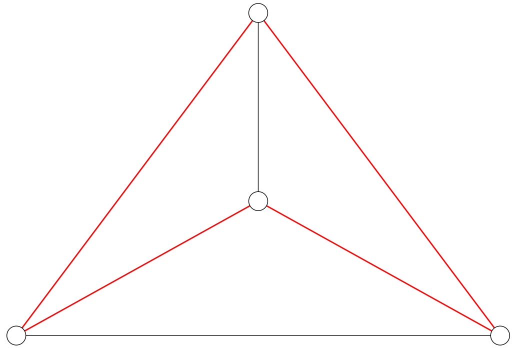
- Can we characterize them?
- How do they differ from planar partial 3-trees?

Theorem All maximal Matryoshka graphs have a Hamiltonian cycle
$\Rightarrow$ Not all planar partial 3-trees are Matryoshka graphs

Binary Planar 3-Trees

Proof by picture for binary planar 3-trees:

- Start with K_4
- Repeatedly subdivide at most two faces of a K_4 in graph



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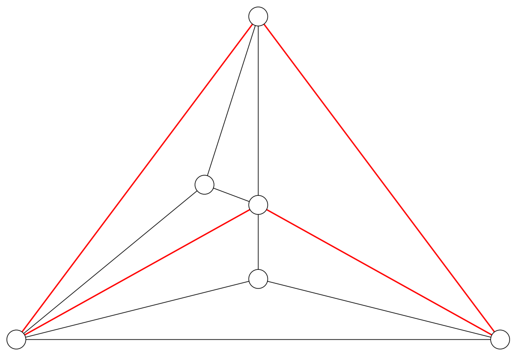

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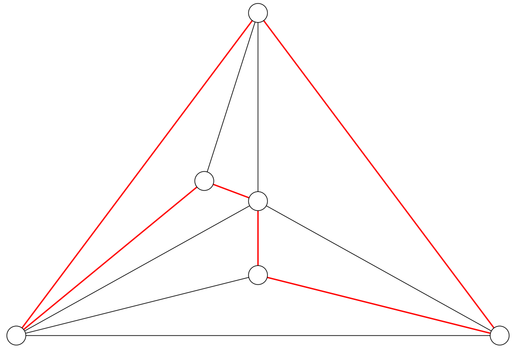

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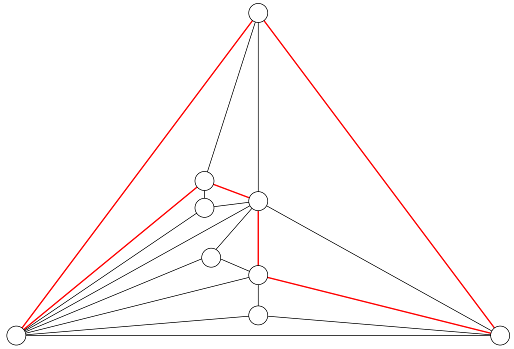

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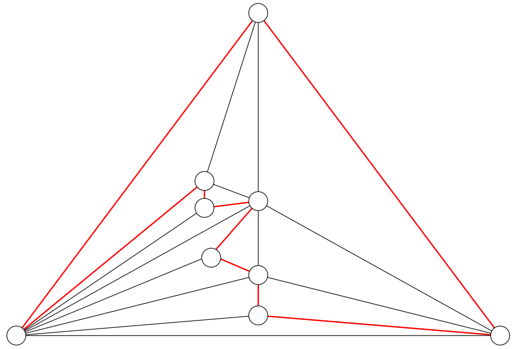
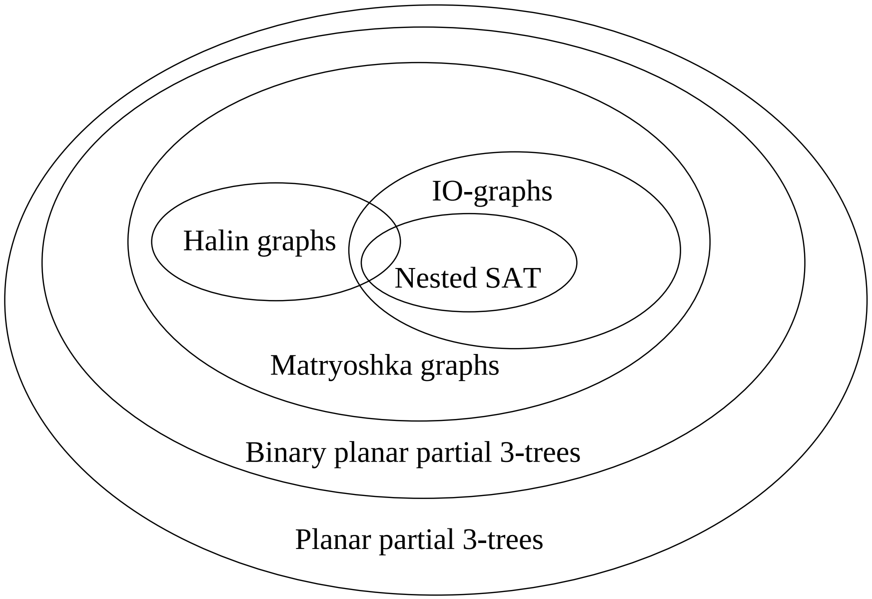


Every maximal Matryoshka graph is a binary planar 3-tree

Superclasses and Subclasses

- Matryoshka graphs $\subset$ binary planar 3-trees
- Matryoshka graphs $\supset$ Halin and IO-graphs
- Matryoshka graphs $\not\supset$ planar partial 2-trees†



† Recent result, does not appear in thesis

Summary of Matryoshka Graphs

- Strict subset of binary planar 3-trees
- Generalizes Halin graphs, IO-graphs, Nested SAT graphs
- No efficient algorithm for recognition yet exists
- Not closed under minors, so minor theory does not help

Part III: Planar Graph Decompositions

- Chartrand-Geller-Hedetniemi conjecture ’71: Planar graphs can be decomposed into two outerplanar graphs
- Know they can be decomposed as:
  - two partial 2-trees (SP-graphs)
  - three partial 1-trees (forests)

We study: Can any planar graph be decomposed into a forest and a partial k-tree?

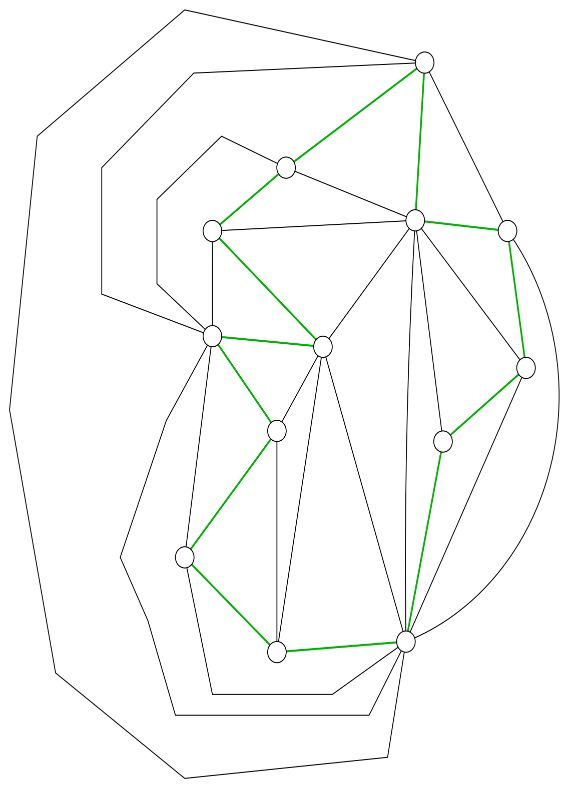
Motivation

Motivation I: Matryoshka graphs are partial $(1,2)$-trees (forest plus SP-graph)

Motivation II: Could lead to better algorithms

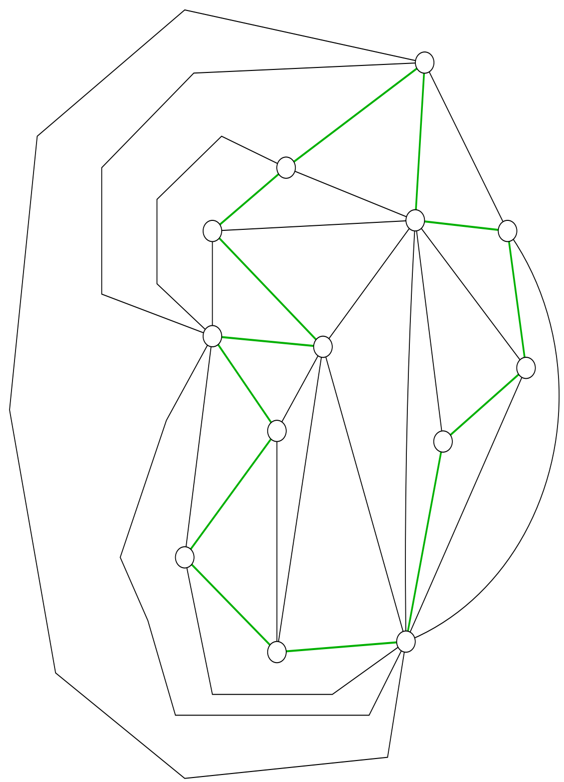
- Can efficiently solve many NP-hard problems on the partial k-tree, then modify solution using forest
- Gives a simple 2-approximation algorithm for weighted vertex cover and weighted independent set

Hamiltonian Planar Graphs



Theorem Sub-Hamiltonian planar graphs are partial $(O(\log n), 1)$-trees.

Hamiltonian Planar Graphs



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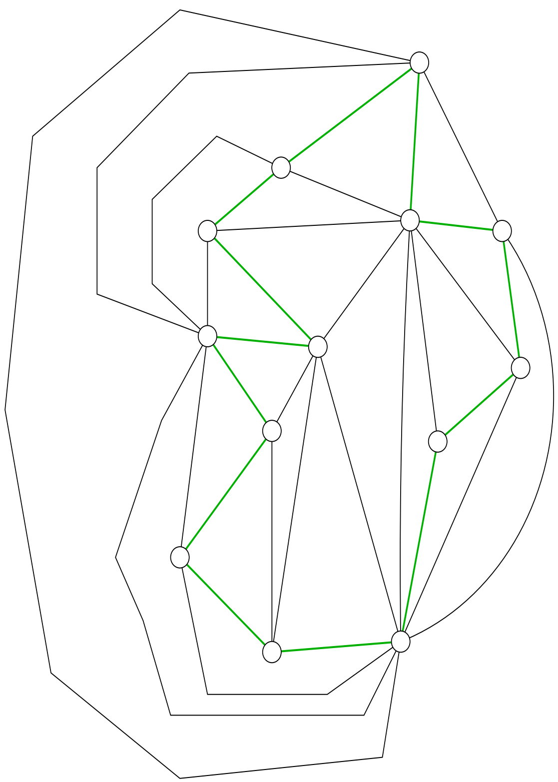
- Inside: outerplanar

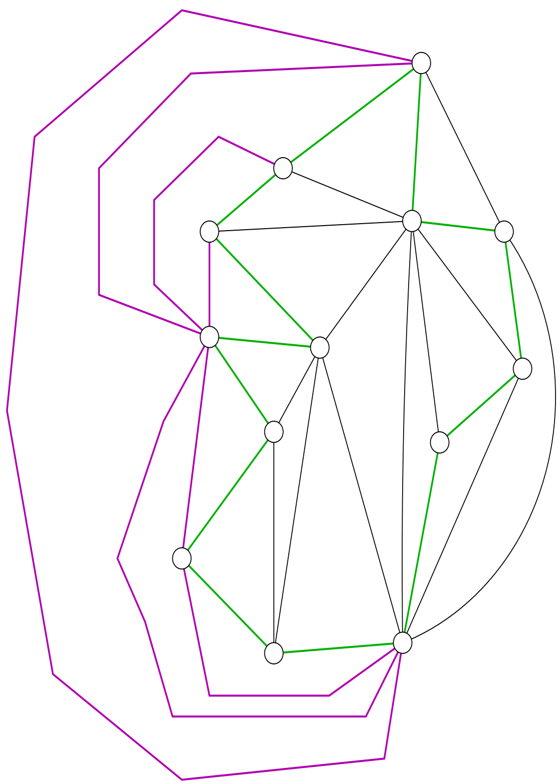
Hamiltonian Planar Graphs



Theorem Sub-Hamiltonian planar graphs are partial $(O(\log n), 1)$-trees.

- Inside: outerplanar $\Rightarrow$ partial 2-tree

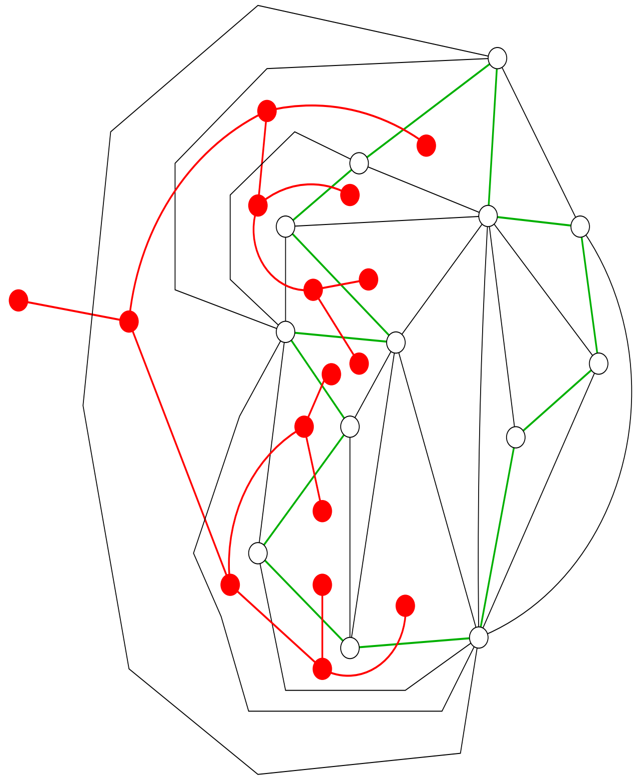
Hamiltonian Planar Graphs



Theorem Sub-Hamiltonian planar graphs are partial $(O(\log n), 1)$-trees.

- Inside: outerplanar $\Rightarrow$ partial 2-tree
- Outside: 3 outerplanar sections

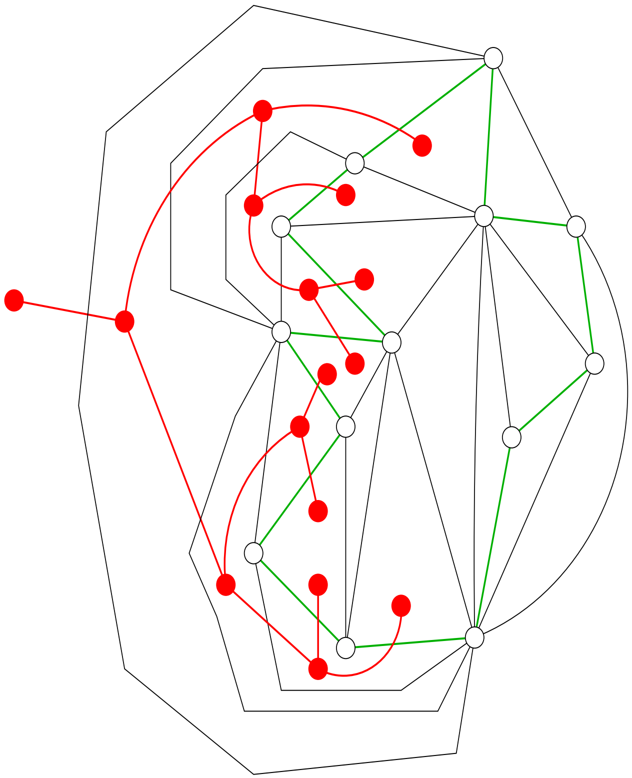
Hamiltonian Planar Graphs



Theorem Sub-Hamiltonian planar graphs are partial $(O(\log n), 1)$-trees.

- Inside: outerplanar $\Rightarrow$ partial 2-tree
- Outside: 3 outerplanar sections
- Each outerplanar section has dual binary tree

Dual Tree Contraction



Theorem A sub-Hamiltonian planar graph G is a partial $(O(\log n), 1)$-tree.

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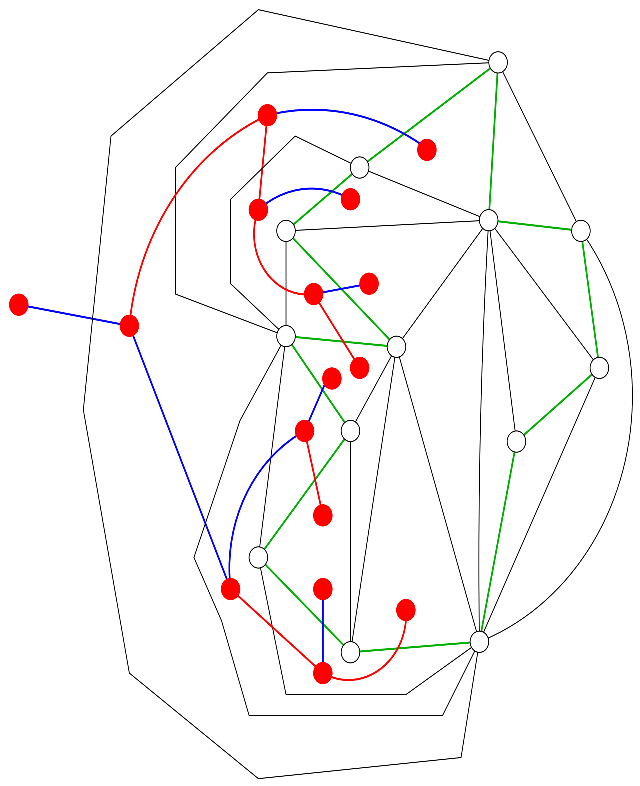

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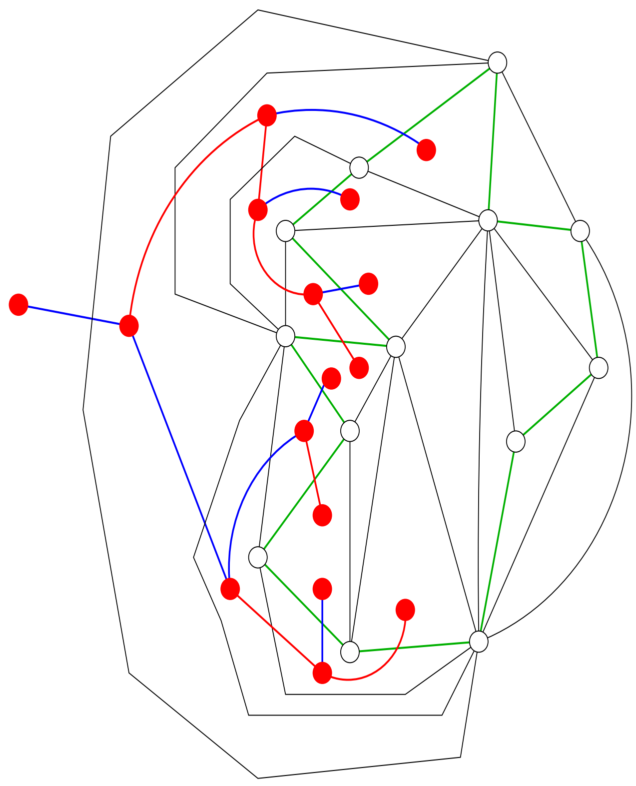
Proof idea: In dual tree, contract edge to taller child

Dual Tree Contraction



Theorem A sub-Hamiltonian planar graph G is a partial $(O(\log n), 1)$-tree.

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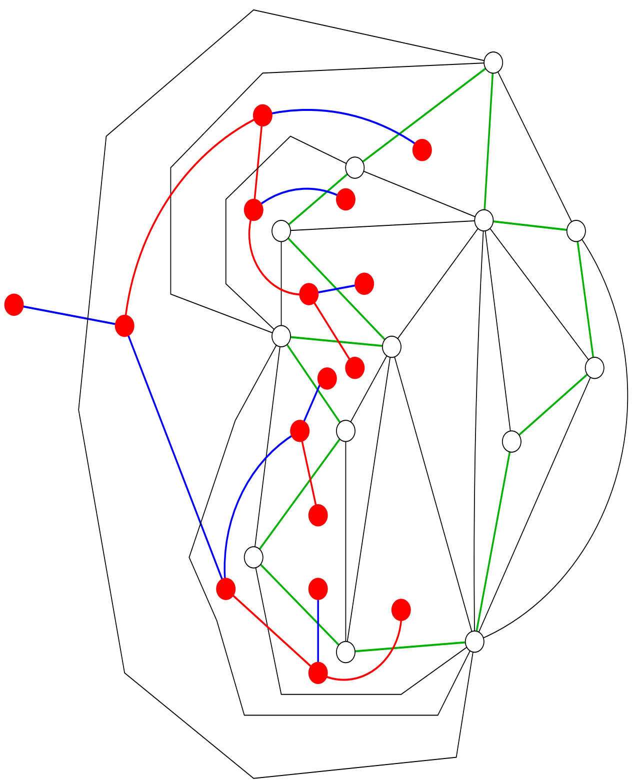
- This deletes a forest in G
- Contracted dual tree has $O(\log n)$ height

Dual Tree Contraction



Theorem A sub-Hamiltonian planar graph G is a partial $(O(\log n), 1)$-tree.

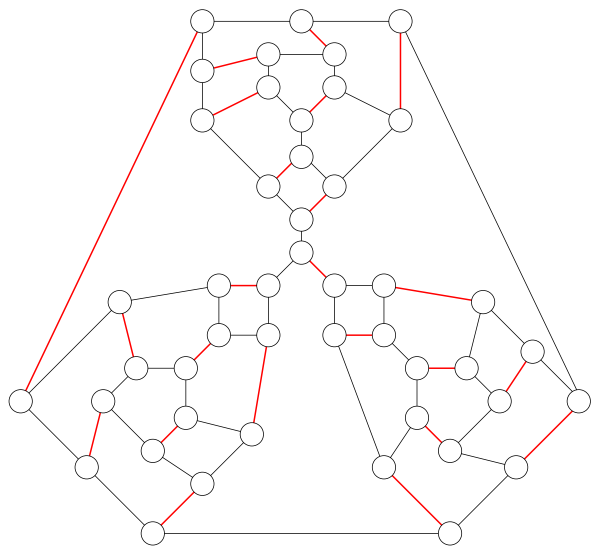
Proof idea: In dual tree, contract edge to taller child

- This deletes a forest in G
- Contracted dual tree has $O(\log n)$ height

$\Rightarrow$ Rest of G is partial $O(\log n)$-tree

Special Case: Bounded Degrees

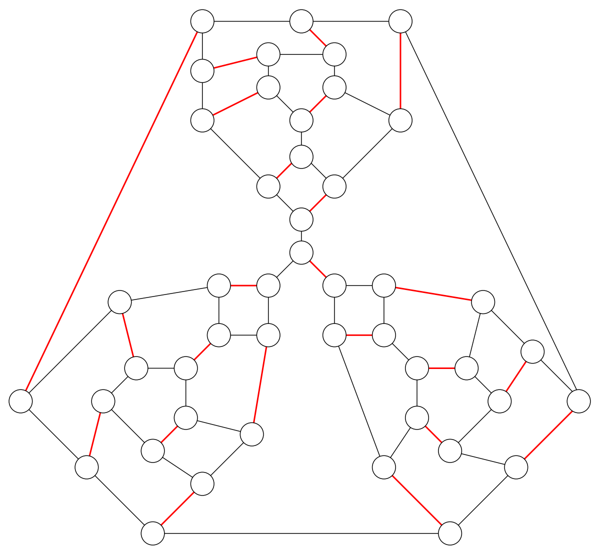
Theorem Every max-deg-3 graph can be decomposed into a matching and a partial 2-tree.



- A matching is a set of edges, no two of which are adjacent
- Petersen's Theorem handles 2-edge-connected 3-regular graphs: a perfect matching exists
- Remaining graph has max-deg-2: partial 2-tree

Special Case: Bounded Degrees

Theorem Every max-deg-3 graph can be decomposed into a matching and a partial 2-tree.



- Bridges don't hurt: handle 2-edge-connected components separately
- For each component: can find a matching that reaches all deg-3 vertices
- Remaining graph has max-deg-2: partial 2-tree

Does This Generalize?

- Can every max-deg-d graph be decomposed into a matching and a partial $O(1)$-tree?

Does This Generalize?

- Can every max-deg-d graph be decomposed into a matching and a partial $O(1)$-tree?
- No!

Does This Generalize?

- Can every max-deg-d graph be decomposed into a matching and a partial $O(\log n)$-tree?

Does This Generalize?

- Can every max-deg-d graph be decomposed into a matching and a partial $O(\log n)$-tree?
- No!

Does This Generalize?

- Can every max-deg-d graph be decomposed into a matching and a partial $o(\sqrt{n})$-tree?

Does This Generalize?

- Can every max-deg-d graph be decomposed into a matching and a partial $o(\sqrt{n})$-tree?
- No!

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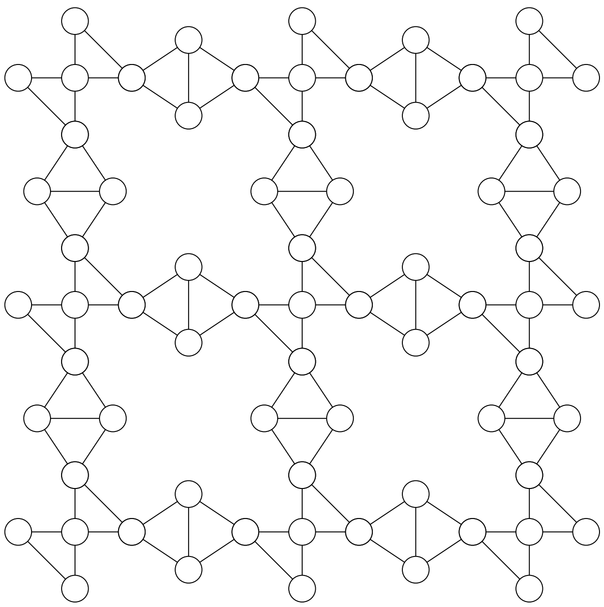

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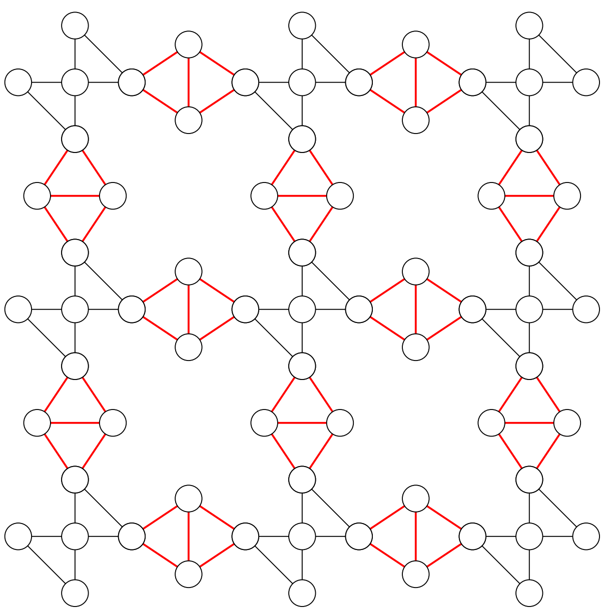

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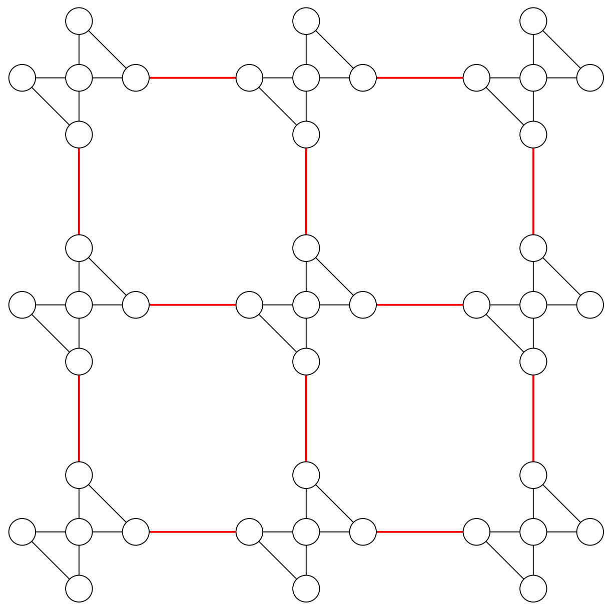

For any matching:

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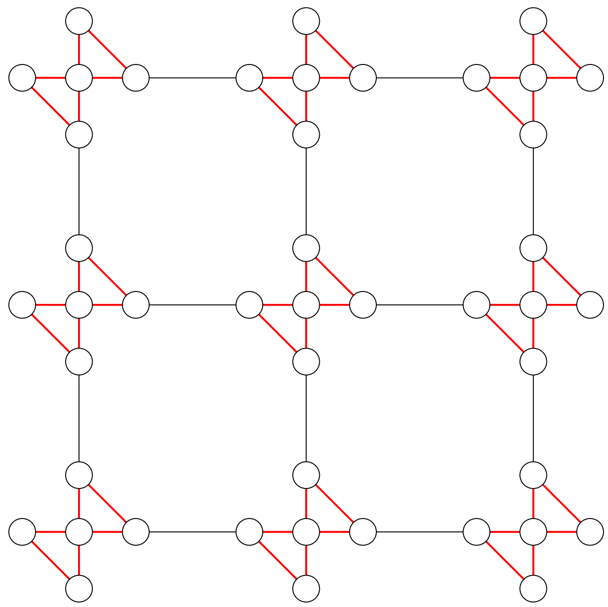

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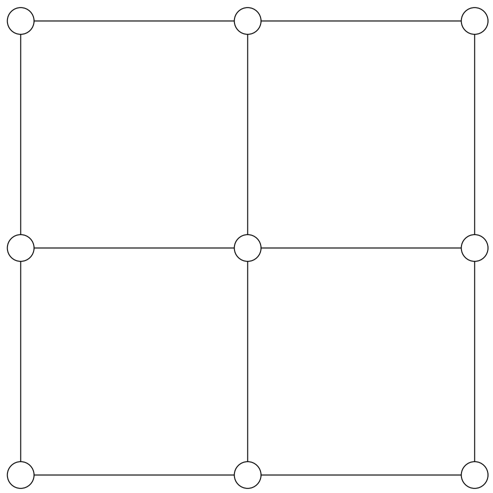

For any matching:

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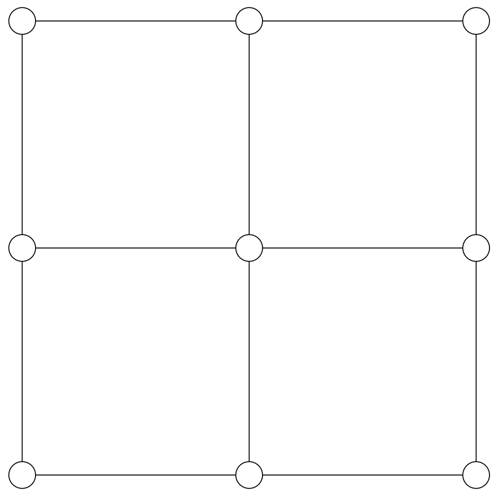

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Obtain $k \times k$ grid $\Rightarrow$ not a partial $(k-1)$-tree

Summary

Planar graphs as partial $(k, 1)$-trees

- Sub-Hamiltonian graphs are partial $(O(\log n), 1)$-trees
  - Does it hold for some constant $k \geq 3$?
- Cannot easily extend to all planar graphs
  - Does it hold for some $k \in o(\sqrt{n})$?

Any Questions?

Thanks to:

- Angèle Hamel for her help during Therese's sabbatical
- Rick Mabry for sending me correspondence with Paul Seymour on related material
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