Planar Graphs and Partial k-Trees

by

Philip Thomas Henderson

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Abstract

It is well-known that many $\mathcal{NP}$-hard problems can be solved efficiently on graphs of bounded treewidth. We begin by showing that Knuth's results on nested satisfiability are easily derived from this fact since nested satisfiability graphs have treewidth at most three. Noting that nested satisfiability graphs have a particular form of planar drawing, we define a more general form of graph drawing while maintaining planarity and bounded treewidth. We name all graphs with such drawings partial Matryoshka graphs—a reference to the nesting Russian dolls—and demonstrate that they encompass other important graph classes of treewidth at most three, such as Halin and IO-graphs.

Based on decompositions derived for partial Matryoshka graphs, we then proceed to explore edge-decompositions into graphs of bounded treewidth. We prove that any Hamiltonian planar graph on n vertices can be decomposed into a forest and a graph of $O(\log n)$ treewidth, and provide an efficient algorithm for constructing this decomposition. Similarly, we show that graphs of maximum degree three can be decomposed into a matching and an SP-graph (i.e. a graph of treewidth two). Furthermore, we prove that this manner of decomposition cannot be extended to graphs of higher degree by giving a method for constructing a graph of maximum degree four that cannot be decomposed into a matching and a graph of treewidth at most k, for any constant k. Lastly, we motivate such decompositions by producing an approximation algorithm for weighted vertex cover and weighted independent set on all graphs that can be decomposed into a forest and a graph of bounded treewidth.

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I greatly appreciate Rick Mabry's correspondence regarding the decomposition of planar graphs into two bipartite graphs, and for sending me the results of his discussions with Paul Seymour on this subject, as well as many other useful references.

I have also had many helpful discussions with other graduate students. Thanks to Reinhold Burger, Niel de Beaudrap, Magdalena Georgescu, Graeme Kemkes, Brendan Lucier, Zach Olesh, Alex Stewart, and Luke Tanur, among others. In particular, I would like to highlight Magda and Reinhold's help regarding partial $(k, 1)$-trees, and to thank Graeme for referring me to existing research regarding dynamic programming and approximation schemes.

Dedication

This is dedicated to my family, friends, and all others who help to make this world a worthwhile place to live.

I am especially thankful to Mom, Dad, my sister Tamara, as well as Luke, Magda, Reinhold, Andrew, Jon, and Eddy. You all helped to support me during my most challenging times.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Since the concept of treewidth was introduced by Robertson and Seymour [89], there have been many important implications in both the fields of computer science and graph theory, particularly with respect to algorithm design and computational complexity. One of the more common applications of treewidth is in constructing fixed-parameter tractable algorithms for $\mathcal{NP}$-hard problems on graphs of bounded treewidth [12]. As a result, many algorithms have been devised to recognize graphs of bounded treewidth and construct their tree decomposition [25, 28, 94], or at least identify bounds on the treewidth of a graph.

Planar graphs, an important class of graphs in terms of practical applications (e.g. circuit design and layout, visual presentation of flow charts, networks, database and web design, etc. [15, 16, 29, 77]), are known to have unbounded treewidth in general. However, several important types of planar graph—including forests, outerplanar graphs, SP-graphs, Halin graphs, and IO-graphs—are known to have bounded treewidth. Planar graphs with bounded outerplanarity or diameter are also known to have bounded treewidth [26, 37, 46, 89].

Our thesis begins by identifying a form of SAT that corresponds to a subset of planar partial 3-trees. This observation leads to generalizations of the results by Knuth [74] and Kratochvíl and Křivánek [75]. We also define a new graph type—partial Matryoshka graphs—that generalizes the structure of these SAT problems, identifying a relationship between graph drawings and a subset of planar partial 3-trees. We relate this graph class to existing planar graph classes of bounded treewidth and examine some important properties of such graphs.

We then proceed to examine decompositions of planar graphs into graphs of bounded treewidth. These questions relate to the open conjecture of Chartrand, Geller, and Hedetniemi that every planar graph can be decomposed into two outerplanar graphs [30]. We also demonstrate how such decompositions can be used to create approximation algorithms for graph classes that do not have bounded treewidth. Finally, we identify many open questions that remain in these research areas.

1.1 Graph-Theoretic Background

In the remainder of this introductory chapter we formally define the basic graph-theoretic terminology and notation that we will be using throughout the thesis. We also review some well-known graph-theoretic results, and the definitions of some important graph classes. Theorems and remarks that appear without citation are standard graph theory knowledge, and can be found in any textbook on the subject (e.g. [39, 60]).

A *graph* $G = (V, E)$ is a tuple consisting of V, a finite set of objects called *vertices* and E, a set of *edges*, where an edge is an unordered pair of distinct vertices. Thus the graphs we consider are always simple (no loops or multi-edges) and undirected. Given a graph G, we can also denote the vertices and edges of G as $V(G)$ and $E(G)$ respectively. The variables n and m are used to denote $|V(G)|$ and $|E(G)|$ respectively.

Given an edge $e = (u, v)$, we say that u and v are the *endpoints* of e. A vertex is *incident* to an edge if it is one of the endpoints of that edge. Given two distinct vertices in a graph, we say that they are *adjacent* if they are both incident to a common edge. Given two distinct edges in a graph, we say that they are *adjacent* if they are both incident to a common vertex.

The *neighbour set* of a vertex v is the set of vertices adjacent to v. The *degree* of a vertex v, denoted $deg_G(v)$ (or simply $deg(v)$ if G is implicit), is the cardinality of its neighbour set. An *isolated vertex* is a vertex of degree zero, and a *leaf* is a vertex of degree one. The *maximum degree* of a graph G, denoted $\Delta(G)$, equals $max_{v \in V(G)} deg_G(v)$. Similarly, the *minimum degree* of a graph G, denoted $\delta(G)$, equals $min_{v \in V(G)} deg_G(v)$. A graph G is called *regular* if $\Delta(G) = \delta(G)$ and k-regular if $\Delta(G) = \delta(G) = k$. For the special cases where $k = 3, 4$ we call k-regular graphs *cubic* and *quartic*, respectively.

Given a graph $G = (V, E)$, a vertex $v \in V$, and an edge $e \in E$, then $G - v$ denotes the graph $(V \setminus \{v\}, E \setminus \{e \in E : e \text{ is incident to } v\})$ and $G - e$ denotes the graph $(V, E \setminus \{e\})$. These two operations are called *deleting a vertex* and *deleting an edge*, respectively. Any graph that can be obtained via these two operations is a *subgraph* of G.

Given a graph G, a vertex set $V_1 \subseteq V(G)$, and an edge set $E_1 \subseteq E(G)$, then $G[V_1]$ denotes the graph $(V_1, \{e \in E(G) : \text{both endpoints of } e \text{ are in } V_1\})$ and $G[E_1]$ denotes the graph $(V(G), E_1)$. $G[V_1]$ and $G[E_1]$ are the *induced subgraphs* of G on V_1 and E_1, respectively. We can also delete vertex sets and edge sets to obtain induced subgraphs (i.e. $G - V_1 = G[V(G) \setminus V_1]$ and $G - E_1 = G[E(G) \setminus E_1]$).

A *complete graph* is a graph such that all pairs of vertices are adjacent. The complete graph on n vertices is denoted K_n. A *clique* is a set of vertices such that every pair of vertices in the set are adjacent (i.e. a set of vertices that induces a complete subgraph). A *simplicial vertex* is a vertex whose neighbour set induces a clique. An *almost simplicial vertex* is a vertex whose neighbour set minus a single vertex induces a clique. An *independent set* is a set of vertices such that no two are adjacent. A *matching* is a set of edges such that no two are adjacent. A matching is called a *perfect matching* if the number of edges in the set is $\lfloor \frac{n}{2} \rfloor$. Finally, a *vertex cover* is a set of vertices such that every edge in the graph is incident to some vertex in the cover. These concepts form the foundation for many $\mathcal{NP}$-hard problems, such as finding the maximum clique, maximum independent set, and minimum vertex cover of a graph [49].

A *walk* is a sequence of vertices such that each consecutive pair of vertices is adjacent (since the graph is simple, the edges between vertices are implicit). A *path* is a walk with distinct vertices. A *cycle* is a walk such that the first and last vertices are equal, and all other vertices are distinct. The *length* of a walk/path/cycle is the number of edges that it implicitly contains (i.e. one less than the length of the sequence).

The *distance* between two variables u and v in graph G, denoted $d_G(u, v)$ (or simply $d(u, v)$ if G is implicit), is the length of the shortest walk containing both u and v. The *eccentricity* of a vertex v in graph G is the maximum possible distance from v to any other vertex in G. In other words, $ecc_G(v) = max_{u \in V(G)} d(u, v)$. The *radius* of a graph G is the value $min_{v \in V(G)} ecc_G(v)$, while the *diameter* of G is $max_{v \in V(G)} ecc_G(v) = max_{u,v \in V(G)} d(u, v)$. The following remark demonstrates the relationship between these values:

Remark 1.1 *Let G be a graph, with radius r and diameter d. Then $r \leq d \leq 2r$.*

A cycle of length $k \geq 3$ is denoted by C_k. A *Hamiltonian cycle* is a cycle containing all vertices in the graph, and a graph is *Hamiltonian* if it contains a Hamiltonian cycle. Given a cycle C in graph G, a *chord* of C is an edge $e \in E(G)$ whose endpoints are two non-consecutive vertices appearing in C. A graph is *chordal* if every cycle of length greater than three has a chord.

Two vertices are *connected* if there exists a walk containing both vertices. A *connected component* of a graph is a maximal set of vertices such that every pair of vertices is connected. Clearly the vertices of a graph can be partitioned into connected components. A graph is *connected* if it has a single connected component.

A *separating set* is a set of vertices whose deletion increases the number of connected components. Likewise, an *edge cut* is a set of edges whose deletion increases the number of connected components. The *connectivity* of a graph is the size of the smallest separating set. A graph is *k-connected* if it has connectivity at least k. A *cut-vertex* is a separating set of size one. Likewise, a *bridge* is an edge cut of size one. A *k-connected component* of graph G is a maximal subgraph H with connectivity at least k.

A *forest* is a graph containing no cycles, and a *tree* is a connected forest. It follows that each edge is a bridge, and so we have the following basic fact about forests:

Remark 1.2 *The number of edges in any forest's component is exactly one less than its number of vertices.*

An *edge-decomposition* of a graph G is a set of subgraphs $G[E_1], \ldots, G[E_k]$ such that $E(G)$ is the disjoint union of $E_1, \ldots, E_k$. We say that G can be *decomposed* into graphs $G[E_1], \ldots, G[E_k]$. Likewise, a *vertex-decomposition* of a graph G is a set of subgraphs $G[V_1], \ldots, G[V_k]$ such that V is the disjoint union of $V_1, \ldots, V_k$. Throughout this thesis, a decomposition means an edge-decomposition unless explicitly stated otherwise.

A *k-vertex-colouring* is a vertex-decomposition $(V_1, \ldots, V_k)$ such that each subgraph only has isolated vertices (i.e. V_i is an independent set for all $i \in \{1, \ldots, k\}$). A *bipartite* graph is one with a 2-vertex-colouring, (V_1, V_2), which we call a *bipartition*.

Remark 1.3 *A graph is bipartite if and only if it has no cycles of odd length.*

A *complete bipartite graph* is one with bipartition (V_1, V_2) such that for all $u \in V_1, v \in V_2$, u and v are adjacent. We denote such a graph as $K_{i,j}$, where $i = |V_1|$ and $j = |V_2|$.

A *planar embedding* of a graph is a drawing in the plane with vertices as (distinct) points and edges as continuous curves joining their two endpoints, such that each edge intersects vertices only at its endpoints, and edges only intersect at vertices. A graph is *planar* if it has a planar embedding.

Planar graphs have many important applications, such as in the fields of circuit design and graph drawing [15, 16, 29, 77]. They can be recognized in linear time [69].

Remark 1.4 *Every planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges.*

Theorem 1.1 *Every 4-connected planar graph has a Hamiltonian cycle [101].*

Theorem 1.2 *Four Colour Theorem: Every planar graph has a 4-vertex-colouring [4, 5, 87].*

Given a planar embedding, a *face* is a minimal region of the plane bounded by edges. The *outer-face* is the (unique) face containing the point at infinity. A vertex is *incident* to a face f (or equivalently, *on* f) if it is incident to an edge on the *boundary* of f. A planar graph is *triangulated* if exactly three vertices are incident to every face.

Theorem 1.3 *(Euler's formula) For a planar graph G, where n is the number of vertices, m the number of edges, and f the number of faces, the following equation always holds: $n - m + f = 2$.*

Given a planar embedding of graph G, the *dual* of graph G, denoted G^*, is the graph defined by representing each face in the embedding with a single vertex, and adding an edge between two (face) vertices for every edge their boundaries have in common. Thus, in Euler's formula, the values of n and f are reversed, while the edges of G and G^* are in one-to-one correspondence. Note that the dual of a graph is not necessary simple, as both multi-edges and loops can be introduced (i.e. if two faces have more than one edge in common, or there exists a bridge).

Remark 1.5 *The dual of a planar graph is itself planar. Furthermore, if G is a triangulated planar graph other than K_3, then G^* is also simple.*

Given a planar graph G with a fixed planar embedding, the vertices incident to the outer-face are said to be on the first layer, L_1. Maintaining the same planar embedding, we recursively define L_i to be the set of vertices incident to the outer-face on $G - L_1 - \cdots - L_{i-1}$. The *layer-depth* of a given planar embedding for graph G is the maximum i such that $L_i \neq \emptyset$. An *outerplanar graph* is a graph with a planar embedding of layer-depth one. More generally, a *k-outerplanar graph* is a graph with a planar embedding of layer-depth at most k. The *outerplanarity* of a graph is the minimum layer-depth over all of its planar embeddings.

Remark 1.6 *An outerplanar graph on n vertices has at most $2n - 3$ edges.*

Remark 1.7 *The dual of an outerplanar graph G is a tree plus a single vertex adjacent to all leaves in the tree (this last vertex represents the outer-face of G).*

We now introduce the concept of treewidth, an important idea by Robertson and Seymour (see [9, 24, 27, 89, 90, 91, 92] among others) that provides the foundation for the rest of this thesis:

A *tree decomposition* of a graph $G = (V, E)$ is a tree $T = (W, F)$ such that each vertex $w \in W$ is *labelled* with $X_w \subseteq V$ such that

- $\bigcup_{w \in W} X_w = V$,
- for all edges $(u, v) \in E$ there exists a vertex $w \in W$ such that $u, v \in X_w$, and
- for any $v \in V$, the vertices of T containing the label v induce a tree (i.e. a connected subgraph)

The *width* of a tree decomposition is the maximum value in the set $\{|X_w|-1 : w \in W\}$. The *treewidth* of a graph is the minimum width over all its tree decompositions, and is denoted by $tw(G)$. The *induced graph* of a tree decomposition is the graph containing all vertices that appear as labels, with two vertices adjacent if and only if they appear as labels in a common node.

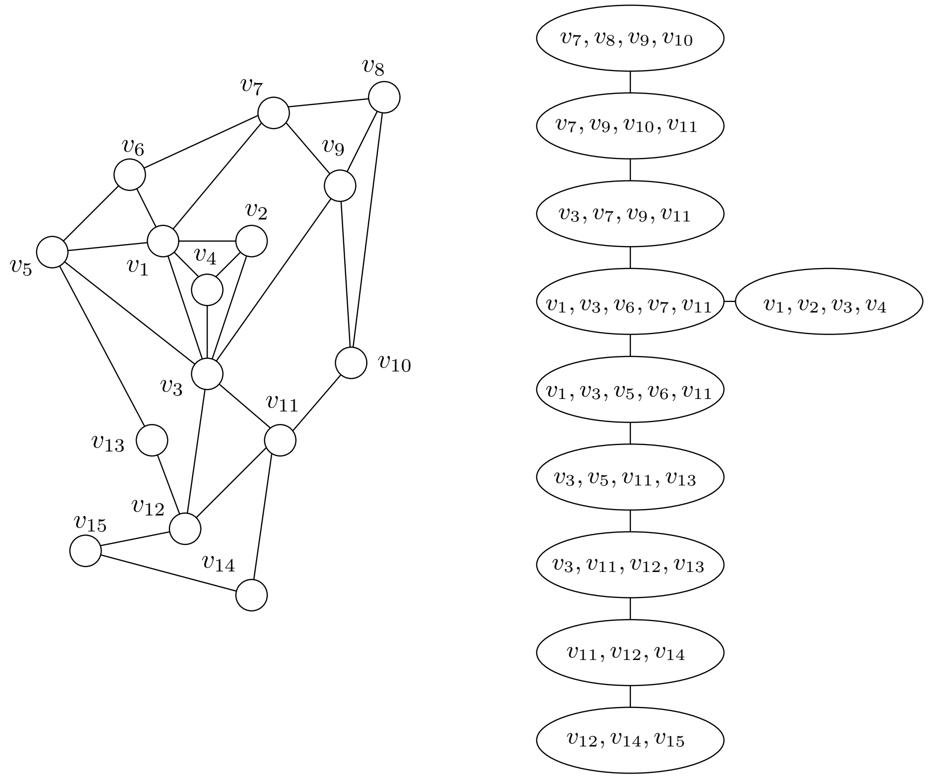


Figure 1.1: A Graph and its Tree Decomposition

Figure 1.1 demonstrates a graph and one possible tree decomposition for this graph. Another tree decomposition for the same graph is shown in Figure 1.2, and we note that the first decomposition has width four, while the second has width three. Thus the graph in Figure 1.1 has treewidth at most three, and indeed, we can prove that this is also a lower bound on the treewidth by using the following observation:

Remark 1.8 *The treewidth of a graph G is greater or equal to the size of G's largest clique minus one.*

This observation follows from the definition of a tree decomposition and Helly's Theorem applied to subtrees of trees [66], since all the vertices composing a clique must

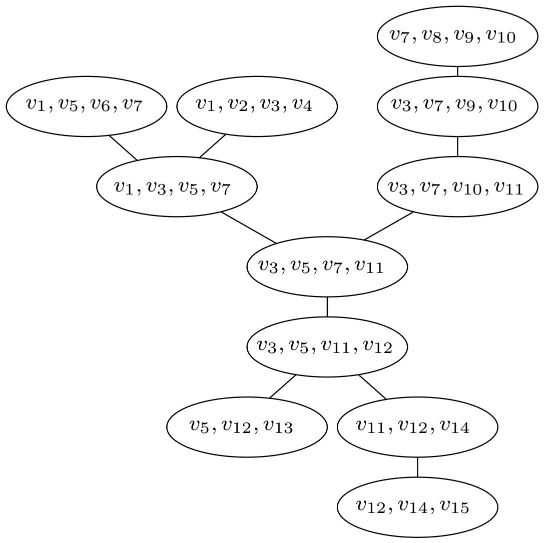


Figure 1.2: Another Tree Decomposition for the Graph in Figure 1.1

appear in a common node of the tree. Hence we conclude that the graph in Figure 1.1 has treewidth three.

Given an ordered list of vertices $L = v_1, \ldots, v_n$, the *index* of a vertex is its position in the list (i.e. v_i has index i). Given an ordered list of vertices, the *predecessors* of a vertex are the subset of its neighbour set with a smaller index than itself. Similarly, the *successors* of a vertex are its neighbours with a higher index than itself. A *perfect elimination order* is an ordered list of the vertices of a graph such that each vertex's predecessors induce a clique.

Theorem 1.4 *A graph has a perfect elimination ordering if and only if it is chordal [47, 53, 52].*

A *k-tree* is a chordal graph with a perfect elimination order such that all vertices with index greater than k have exactly k predecessors. A graph G is a *partial k-tree* if it is a subgraph of a k-tree.

Theorem 1.5 *A graph is a partial k-tree if and only if it has treewidth at most k [26].*

The importance of treewidth lies in the observation that one can solve many $\mathcal{NP}$-hard problems on partial k-trees in $O(f(k)n)$ time, where $f(k)$ is an exponential function such as 2^k or 2^{k+1} [6, 7, 9, 8, 23]. This is done by performing dynamic programming on the tree decomposition, and since the number of nodes in a tree decomposition is $O(n)$

(where n is the number of vertices in the original graph) and the number of labels within a node is at most $k+1$ (by definition of treewidth), then for each node we must compute roughly 2^{k+1} states. Thus, if we restrict ourselves to partial k-trees for any constant k, then many $\mathcal{NP}$-hard problems are solvable in linear time!

We shall now examine how graph operations affect the treewidth of a graph. Given a graph $G = (V, E)$ and an edge $e = (u, v) \in E$, the *contraction* of edge e results in the graph $(V \setminus \{v\}, E \cup \{(u, w) : (v, w) \in E\} \setminus \{f \in E : f \text{ is incident to } v\})$. Given a graph G, a *minor* of G is any graph that can be obtained via the operations of edge contraction, and vertex and edge deletion.

Theorem 1.6 *If H is a minor of graph G, then $tw(H) \leq tw(G)$ [39, 91].*

Given two (disjoint) graphs G_1, G_2 and two vertices $v_1 \in V(G_1), v_2 \in V(G_2)$, then $G_3 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ is the *union* of G_1 and G_2. The graph created by *identifying* v_1 with v_2 is constructed by computing the union G_3, adding edge (v_1, v_2) to $E(G_3)$, and then contracting this edge. Given that graphs G_1 and G_2 both have cliques of size c, the *clique sum* of these two graphs is equivalent to identifying these two cliques (with a one-to-one correspondence between vertices in the cliques). This operation is useful with regards to treewidth, because of the following fact:

Remark 1.9 *If G is the clique sum of graphs G_1 and G_2, then $tw(G) = max\{tw(G_1), tw(G_2)\}$.*

We will now present certain graph classes with bounded treewidth. These graph classes will arise in future discussions and proofs within the thesis.

Remark 1.10 *A graph has treewidth one if and only if it is a forest.*

Theorem 1.7 *A k-outerplanar graph has treewidth at most $3k-1$ [26].*

2-*terminal series-parallel graphs* are defined recursively as follows:

- Base case: A single edge is a 2-terminal series-parallel graph, with each of the endpoints being one terminal.
- Series case: Given 2-terminal series-parallel graphs G_1, G_2 with terminals s_1, t_1 and s_2, t_2 respectively, the graph obtained by identifying t_1 with s_2 is a 2-terminal series-parallel graph with terminals s_1 and t_2.
- Parallel case: Given 2-terminal series-parallel graphs G_1, G_2 with terminals s_1, t_1 and s_2, t_2 respectively, the graph obtained by identifying s_1 with s_2 and t_1 with t_2 is a 2-terminal series-parallel graph with terminals $s_1 = s_2$ and $t_1 = t_2$.

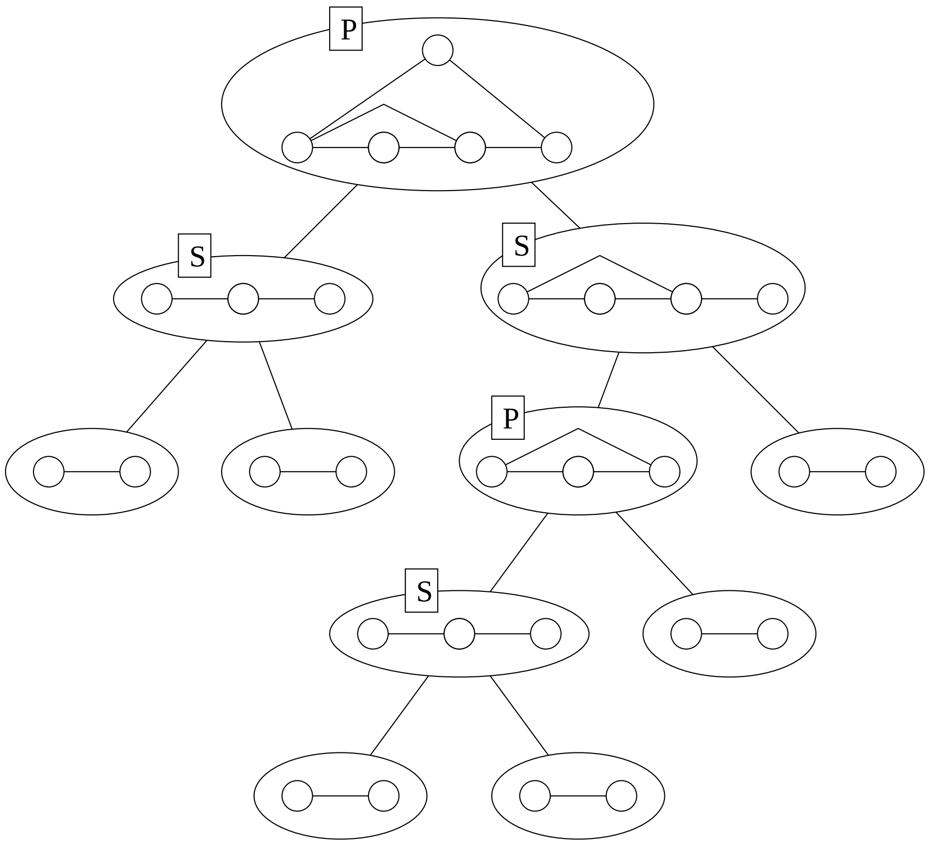


Figure 1.3: 2-Terminal SP-Graph Construction

A *series-parallel (SP-)graph* is a graph such that every 2-connected component is a 2-terminal series-parallel graph. SP-graphs, by their very construction, are always planar (see Figure 1.3).

Theorem 1.8 *A graph is a partial* 2*-tree if and only if it is a series-parallel graph [26].*

We note that since outerplanar graphs have treewidth at most two (by Theorem 1.7), it follows that they are SP-graphs. Indeed, we can use the dual tree (see Remark 1.7) of a 2-connected internally-triangulated outerplanar graph to construct its tree decomposition, as shown in Figure 1.4. Since every outerplanar graph is the subgraph of some 2-connected internally-triangulated outerplanar graph, we can easily construct a tree decomposition for any outerplanar graph.

A *Halin graph* (defined in [57]) is a 2-connected planar graph with a planar embedding such that the deletion of all edges on the outer-face results in a tree (see Figure 1.5).

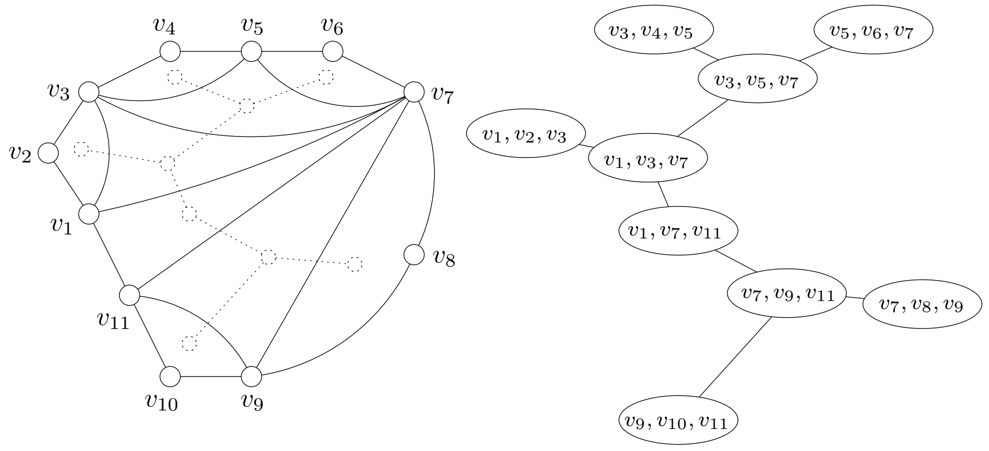


Figure 1.4: Constructing an Outerplanar Graph's Tree Decomposition

An *IO-graph* (defined by El-Mallah and Colbourn [42]) is a 2-connected planar graph with a planar embedding such that the deletion of all vertices on the outer-face leaves a (possibly empty) independent set (i.e. a graph with no edges).

Theorem 1.9 *Halin graphs and IO-graphs are both planar partial* 3*-trees [42].*

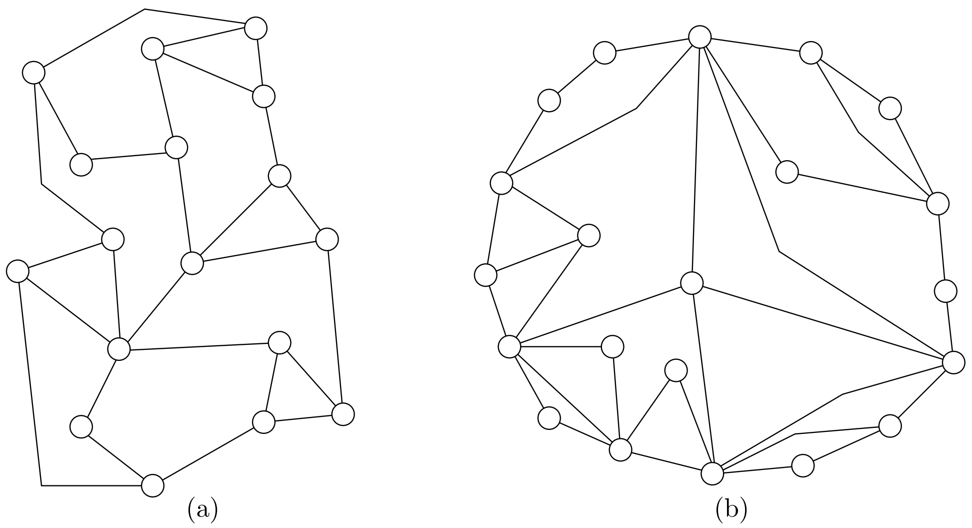


Figure 1.5: A Halin Graph (a) and an IO-Graph (b)

Lastly, we mention Robertson and Seymour's theorem relating to graph minors:

Theorem 1.10 *Graph Minors Theorem: If a set of graphs S is closed under the taking of minors, then there exists a finite set of graphs F, called the* forbidden minors *of S,* *such that graph G is in set S if and only if it does not contain any graph in F as a minor [88].*

We can see that if a graph is planar, then the operations of vertex deletion, edge deletion, and edge contraction cannot make it non-planar. Thus the set of planar graphs is closed under taking minors, and as such, it has a finite set of forbidden minors:

Theorem 1.11 *Kuratowski's Theorem: A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

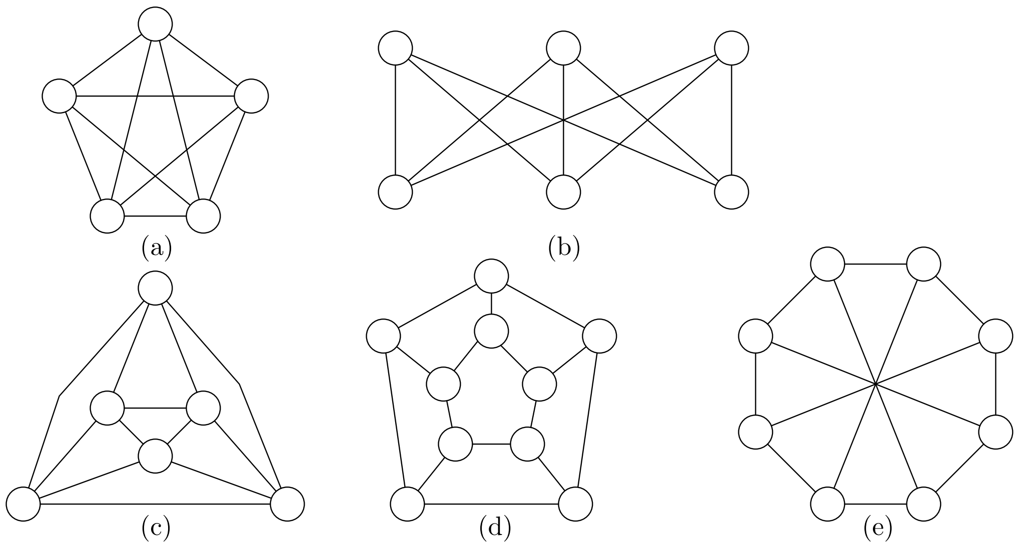

(a) (b) (c) (d) (e)

Figure 1.6: Forbidden Minors: (a) K_5 (b) $K_{3,3}$ (c) P_6 (d) $C_{5,5}$ (e) C_8 with 4 cross edges

Applying Theorem 1.6, it also follows that partial k-trees have a finite set of forbidden minors for every positive integer k. We list the forbidden minors for small values of k, as we shall be using this information later on:

Theorem 1.12 *The forbidden minors of SP-graphs and outerplanar graphs are K_4, and $K_{2,3}$ and K_4, respectively. The forbidden minors of partial 3-trees are K_5, the octahedral graph P_6, the pentagonal prism $C_{5,5}$, and C_8 with four additional edges connecting opposite vertices. The forbidden minors of planar partial 3-trees are $K_5, P_6, C_{5,5}$, and $K_{3,3}$ [11, 43, 95]. See Figure 1.6.*

However, it is still an open question as to what the forbidden minors of partial k-trees are for all $k \geq 4$.

Chapter 2

Nested Satisfiability

This chapter generalizes the results of Knuth [74], and Kratochvíl and Křivánek [75] regarding a subset of satisfiability problems known as Nested SAT. We first presented these results in a technical report [21], and explore them again here in greater detail. We begin by reviewing the satisfiability problem, including some important variations of this problem, and then proceed with Knuth's definition of nested satisfiability. Knuth proved that any Nested SAT problem could be solved in linear-time [74], and Kratochvíl and Křivánek extended these results to Nested MaxSAT and co-Nested SAT [75].

In this chapter we show that Nested SAT problems correspond to a subset of planar SAT problems whose graphs have treewidth at most three, and provide an algorithm to produce the corresponding tree decomposition. It is known that SAT problems whose graph is of bounded treewidth are solvable in parameterized linear-time [54, 97]. However, the fact that Nested SAT graphs have bounded treewidth does not appear in the literature, and is the main new contribution of this chapter. In consequence, our results can be used to obtain (and generalize) those of Knuth, and Kratochvíl and Křivánek.

2.1 Satisfiability

Satisfiability (SAT) is an historically important decision problem, as it was the first problem shown to be $\mathcal{NP}$-complete [35]. The input for SAT is a 2-tuple (X, F) composed of a set of boolean variables $X = \{x_1, \ldots, x_n\}$ and a boolean formula F constructed from these variables and the logical operations AND (conjunction, $\wedge$), OR (disjunction, $\vee$), and NOT (negation, $\neg$), as well as brackets to indicate order of operations. If there exists an assignment of boolean values to the variables X such that the formula F evaluates to true, then we say that the problem is *satisfiable.* The decision problem *accepts* the input (X, F) if and only if (X, F) is satisfiable, otherwise it *rejects* the input.

Regarding the composition of boolean formulas, a *literal* is an instance of a variable or its logical negation. The *positive* and *negative* literals of variable X_i are denoted as x_i and $\neg x_i$, respectively. A *clause* is a disjunction of literals (e.g. $C_1 = (\neg x_1 \vee \neg x_3 \vee x_7 \vee x_8)$). Finally, a boolean formula is in *conjunctive normal form (CNF)* if it is the conjunction of a set of clauses. 3-SAT corresponds to the subset of SAT problems such that the boolean formula F is in conjunctive normal form with each clause containing exactly three literals. Every SAT problem can be reduced to a 3-SAT problem, and thus 3-SAT is also $\mathcal{NP}$-complete [49, 72, 78].

Given a formula in conjunctive normal form, we can assume without loss of generality that every clause has two or more literals, since any clause containing a single literal forces a particular boolean value for the corresponding variable, and thus the variable can be eliminated from the formula (along with all clauses that consequently evaluate to true). Similarly, we can assume that each literal within a given clause belongs to a distinct variable, since two or more occurences of the same literal is redundant, while the presence of both the positive and negative literals for a variable would imply the clause itself was irrelevant to the problem (i.e. it would be satisfied by all boolean variable truth assignments). These properties will be used when defining Nested SAT. Finally, from now on we will only be considering satisfiability problems in conjunctive normal form, and hence we will denote them by the 2-tuple (X, C) instead, where $X = \{X_1, \ldots, X_n\}$ represents the set of variables and $C = \{C_1, \ldots, C_m\}$ is the set of clauses in the boolean formula F. Similarly, since we know that clauses are the disjunction of literals, we will just denote each clause as a set of literals.

MaxSAT is a variation on the SAT decision problem. The input includes the set of variables X, a set of clauses C, and an integer k. MaxSAT accepts the input (X, C, k) if and only if there exists an assignment of boolean values to the variables X such that k or more clauses in C evaluate to true. The optimization version of this decision problem is to maximize the number of clauses that are satisfied. MaxSAT is $\mathcal{NP}$-complete, even when restricted to boolean formulas containing only two literals per clause [49, 51].

Given a SAT problem (X, C), the corresponding *SAT graph* is defined as $G = (V, E)$ where $V = X \cup C$ and $E = \{(X_i, C_j) : x_i \in C_j \text{ or } \neg x_i \in C_j\} \cup \{(X_i, X_{i+1}) : 1 \leq i < n\}$. For instance, if $X = \{X_1, X_2, X_3\}$ and $C = \{C_1 = \{x_1, x_2, \neg x_3\}, C_2 = \{x_1, \neg x_2, \neg x_3\}, C_3 = \{\neg x_1, \neg x_2, x_3\}\}$ then we obtain the satisfiability graph shown in Figure 2.1. *Planar SAT* refers to the subset of SAT problems whose corresponding graphs are planar, and this subset of the decision problem is also $\mathcal{NP}$-complete, as is planar 3-SAT [79]. We can now proceed with the introduction of nested satisfiability.

2.2 Defining Nested SAT

The concept of Nested SAT was introduced by Knuth [74], who developed a linear-time algorithm to solve such problems. Kratochvíl and Křivánek later described how to identify

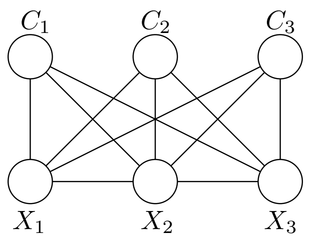


Figure 2.1: The Graph of a 3-SAT Problem

Nested SAT problems in linear time, and improved the algorithm to also solve Nested MaxSAT in linear time [75]. We will now present Knuth's definition of Nested SAT.

Given a SAT problem (X, C) in CNF, we can impose an order on the variables in X. We write this as $X_1 < X_2 < \cdots < X_n$, and from now on we will assume that the indices indicate the ordering (i.e. $X_i < X_j$ if and only if $1 \leq i < j \leq n$). In a similar manner, we define the relations $\leq$, $>$, and $\geq$ on X as well. Since the variables now have a defined order, we can use the $<$ relation to order the literals in a clause (according to the variable of each literal). This ordering is well-defined since we can assume that no clause contains more than one literal for any variable in X. Finally, this ordering allows us to identify three types of relationships among clauses:

Let $C_1 = (v_1, \ldots, v_{n_1})$ and $C_2 = (w_1, \ldots, w_{n_2})$ be clauses on the variables X, where v_i, w_i represent literals (positive or negative) ordered by the relation $<$. Then we write $C_1 \leq C_2$ if $v_{n_1} \leq w_1$. Obviously $C_1 \leq C_2$ implies that $v_1 < \cdots < v_{n_1} \leq w_1 < \cdots < w_{n_2}$. If $C_1 \leq C_2$ or $C_2 \leq C_1$, then we say that these clauses are *disjoint* (see Figure 2.2 (a)).

The second relationship occurs when $v_i \leq w_1$ and $w_{n_2} \leq v_{i+1}$ for some $i \in \{1, \ldots, n_1 - 1\}$. In this case we say that C_1 *straddles* C_2. If C_1 straddles C_2, then it is not the case that C_2 straddles C_1 unless C_1 and C_2 are clauses on the same two variables. In this special case we say that C_1 and C_2 are *concurrent* clauses. Examples are given in Figure 2.2 (b) and (e).

Finally, if there exist i, j such that $v_1 < w_i < v_{n_1}$ and $w_1 < v_j < w_{n_2}$, then we say that C_1 and C_2 *overlap.* See Figure 2.2 (c) and (d).

Knuth showed that for any SAT problem (X, C) in CNF, any pair of clauses must either be disjoint, overlap, or straddle (one-way or concurrently). *Nested SAT* is simply the subset of SAT problems (in conjunctive normal form) for which there exists a variable ordering such that no two clauses overlap.

It can easily be seen that disjointness and straddling are transitive properties (e.g. if C_1 straddles C_2 and C_2 straddles C_3, then C_1 straddles C_3). Furthermore, except for concurrent clauses, disjointness and enclosure are acyclic properties (i.e. there does not exist $C_1, \ldots, C_k$ such that $C_1 \leq C_2 \leq \cdots \leq C_k \leq C_1$, and similarly for straddling). If we impose a straddling hierarchy on concurrent clauses, then we can use the ordering on the variables to arrange the clauses in a forest-like hierarchy, with parents straddling

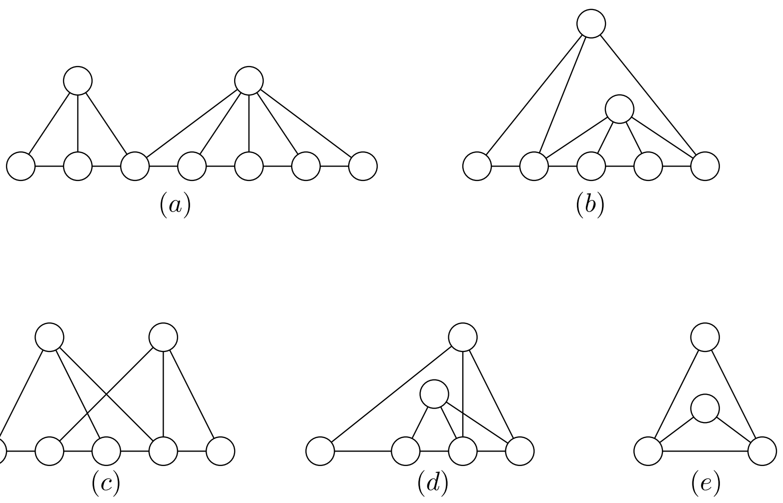


Figure 2.2: Clause Relationships: (a) Disjoint, (b) Straddling, (c) Overlapping, (d) Overlapping, (e) Concurrent

their children, and children ordered left to right according to disjointness [74]. In order to attain a tree-like hierarchy of clauses, we can add an extra clause containing only variables X_1 and X_n (i.e. a clause that straddles all other clauses) in the same manner as Knuth [74]. We shall call this extra clause the *root clause* since it is the root of this tree-like hierarchy.

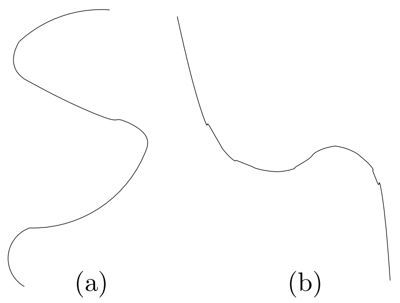


Figure 2.3: Monotonicity: (a) a y-monotonic curve (b) not a y-monotonic curve

A curve drawn on the Cartesian plane is *monotonic* in the y-direction if the y-coordinate of the curve never decreases while progressing from one endpoint of the curve to the other (or equivalently, by progressing in the opposite direction, the y-coordinate never increases; see Figure 2.3). An *LH-drawing* is a planar drawing in the Cartesian plane with vertices positioned on a horizontal line (e.g. $y = 0$) or in the half-space above the line, with edges drawn as curves monotonic in the y-direction [19]. One can show that a SAT problem is a Nested SAT problem if and only if its corresponding SAT graph has an LH-drawing with variables on the line and clauses in the half-space (see Figure

2.4). An equivalent observation is made by Kratochvíl and Křivánek [75], and hence we omit any formal proof.

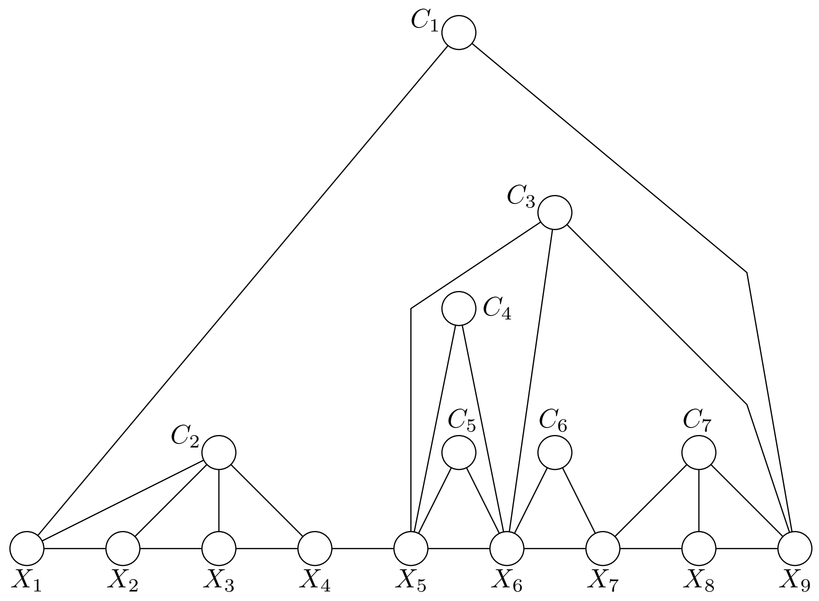


Figure 2.4: A Nested SAT Graph with an LH-drawing

We will now use the structure of a Nested SAT graph to define the terminology for constructing our tree decomposition. Given an LH-drawing of a Nested SAT graph, we can define the *left boundary variable* of a clause to be its neighbour (variable) of lowest index, and the *right boundary variable* to be its neighbour of highest index. The *range* of a clause is the set of variables between its left and right boundary variables (inclusive). For instance, the range of clause C_3 in Figure 2.4 is $\{X_5, X_6, X_7, X_8, X_9\}$. We say a variable is *visible* to a clause if the variable is in the clause's range and they share a common interior face. Thus X_8 is visible to C_7 and X_5 is visible to C_1, C_3, C_4, and C_5. However, X_8 is not visible to C_1 or C_3, and X_5 is not visible to C_2 because of face and range requirements, respectively. Lastly, we can use the LH-drawing to define the clause tree hierarchy more formally as follows: if clauses C_1 and C_2 share a common face such that the range of C_1 is a subset of the range of C_2 and the y-coordinate of C_1 is less than the y-coordinate of C_2, then we say that C_1 is a *child* of C_2.

Relating these terms to Knuth's original terminology, we see that a pair of clauses is disjoint when their ranges share at most one boundary variable, and they straddle when one is a descendant of the other within the clause hierarchy. The parent-child relationship as defined on the graph is thus guaranteed to be acyclic, with the condition relating to y-coordinates imposing an order on pairs of concurrent clauses (such as C_4 and C_5), thereby avoiding any ambiguity in the clause hierarchy. We note that this clause hierarchy forms the tree structure which we will use to prove that these graphs have small treewidth (see Figure 2.5).

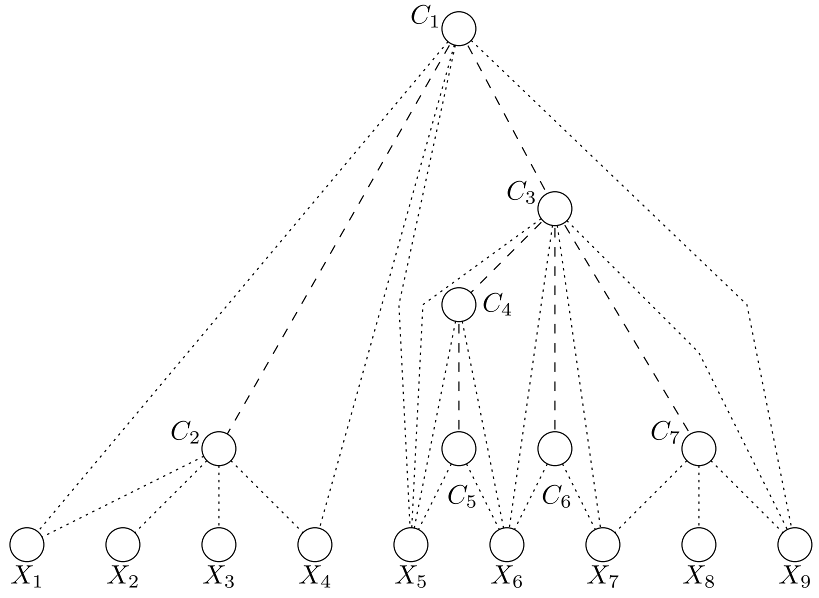


Figure 2.5: Clause Hierarchy (dashed) and Visible Variables (dotted)

2.3 Nested SAT Graphs are Partial 3-Trees

As mentioned in the previous section, a Nested SAT graph imposes a tree structure on the clauses of the Nested SAT problem. In order to construct a tree decomposition of width at most three for any Nested SAT problem, we shall substitute a *clause path* for each node in this clause tree structure. The following algorithm defines and constructs a clause path for an arbitrary clause in a Nested SAT graph.

```
Algorithm 1: Constructing a Clause Path
  Input: Clause $C$ with visible variables $X_{i_1}, \ldots, X_{i_q}$ $(i_1 < \ldots < i_q)$
  Output: The clause path of $C$
  Set the first node of the clause path to contain labels $\{C, X_{i_1}, X_{i_q}\}$
  for $j \leftarrow 2$ to $q-1$ do
      Append to the clause path a node containing labels $\{C, X_{i_{j-1}}, X_{i_j}, X_{i_q}\}$
```

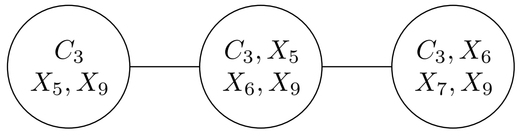


Figure 2.6: Clause Path for C_3 in Figure 2.4

We note that this clause path is a valid tree decomposition for the subgraph induced by clause C and its visible variables, allowing edges (C, X_{i_j}) for $j \in \{1, \ldots, q\}$ and edges $(X_{i_j}, X_{i_{j+1}})$ for $j \in \{1, \ldots, q-1\}$ (see Figure 2.6). Furthermore, this tree decomposition has width at most three. Thus, we only need to show that these paths can be combined to form a valid tree decomposition for the Nested SAT graph.

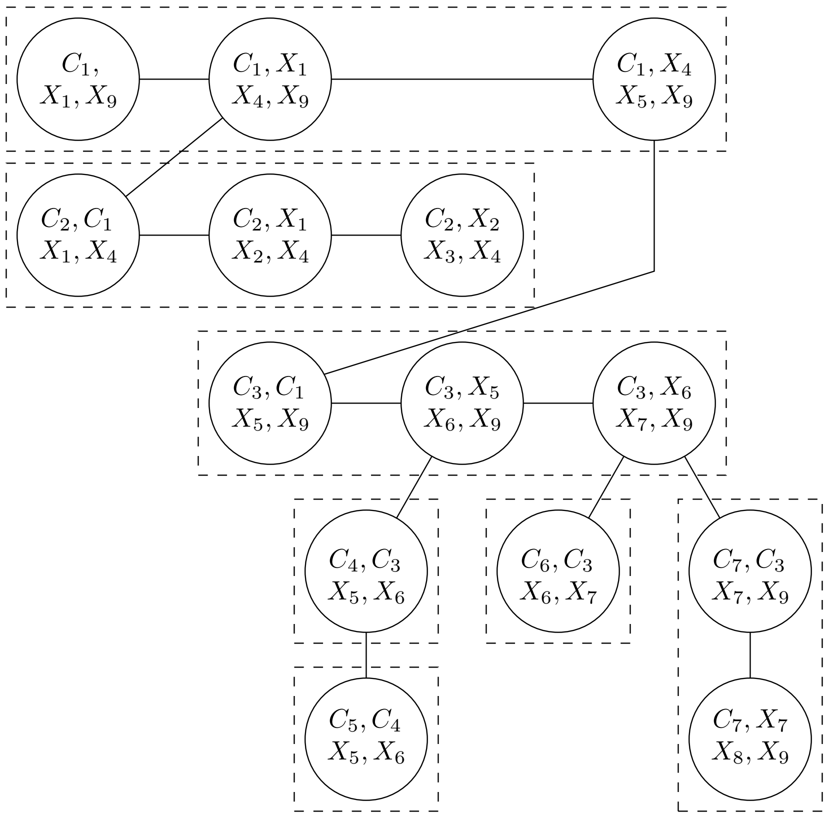


Figure 2.7: Tree Decomposition for the Nested SAT Graph in Figure 2.4

Let C_p be a clause and let C_i be one of its children in the clause hierarchy. Then the visible variables of C_p include the left and right boundary variables of C_i, but no other variables in the range of C_i (since a common interior face is impossible). Thus, the boundary variables of C_i are consecutive visible variables for C_p, and hence the clause path of C_p will include a node containing both of these variables. If we add an edge from this node to the first node of C_i's clause path, then the resulting structure is still a valid tree decomposition (i.e. the connected subtree condition is maintained). Furthermore, repeating this process cannot introduce cycles since, as we noted before, the parent-child relationship is acyclic. Since each of the clause paths has width at most three, the same holds of their combination. Thus, we can combine clause paths to form a tree decomposition for any Nested SAT graph (see Figure 2.7 for such a tree decomposition).

Hence, Nested SAT problems are a subset[1] of SAT problems with treewidth at most three. Furthermore, this bound on treewidth cannot be reduced to two since we can construct a Nested SAT graph containing K_4 as a minor using two clauses and three variables (see Figure 2.8).

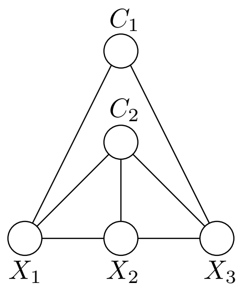


Figure 2.8: Nested SAT Graph Containing K_4 as a Minor

It is worth noting that we can add parent-child edges between clauses without increasing the treewidth. This results in the following modification to the algorithm for constructing clause paths:

```
Algorithm 2: Constructing the Enhanced First Node of a Clause Path
  Input: Clause $C$ with visible variables $X_{i_1}, \ldots, X_{i_q}$ $(i_1 < \ldots < i_q)$
  Output: The first node of the clause path of $C$
  if clause $C$ has a parent $C_p$ then
      Set the first node of the clause path to contain the labels $\{C, C_p, X_{i_1}, X_{i_q}\}$
  else
      Set the first node of the clause path to contain the labels $\{C, X_{i_1}, X_{i_q}\}$
```

The method for combining these enhanced clause paths is no different, and still results in a valid tree decomposition.

Theorem 2.1 *Let G be the graph of a Nested SAT problem, and let G^+ denote the graph obtained from G by adding a root clause, all parent-child clause edges, and all edges from clauses to their (non-adjacent) visible variables. Then G^+ has treewidth at most three.*

From now on we refer to graphs in the form of G^+ (i.e. Nested SAT graphs with clauses adjacent to all of their visible variables, as well as parent-child clause edges added) as *extended Nested SAT graphs.*

[1]They are a strict subset, since a 3-SAT problem on three variables and three clauses has a graph of treewidth 3, yet it cannot be nested since it may contain $K_{3,3}$ as a subgraph. See Figure 2.1.

2.3.1 Alternative Proofs

We note that another proof of this theorem can be obtained by using the simplicial and almost-simplicial rules developed by Bodlaender *et al.* [22]. Recall from Chapter 1 that a simplicial vertex is a vertex whose neighbours form a clique, while an almost-simplicial vertex is a vertex whose neighbour set minus one vertex forms a clique.

Theorem 2.2 *The Simplicial Rule: Given a graph G and a simplicial vertex v in G, then $tw(G) = max\{tw(G - v), deg_G(v)\}$ [22].*

Theorem 2.3 *The Almost-Simplicial Rule: Given a graph G and an almost-simplicial vertex $v \in V(G)$, if $deg_G(v) \leq tw(G)$, then $tw(G) = tw(H)$ where H is the graph that results from making v simplicial in G and then deleting v [22].*

We can now proceed with the alternative proof of Theorem 2.1:

Alternative Proof 1 of Theorem 2.1: In order to prove that Nested SAT graphs have treewidth at most three, then by Theorem 1.6 we need only consider extended Nested SAT graphs. Given an extended Nested SAT graph G, if no clauses exist then we have a forest of variables, and hence we are done by applying Remark 1.10. Thus, we can assume there exists one or more clauses and select a clause C with no children clauses. Such a clause exists since the clause hierarchy is a tree (i.e. we choose a leaf clause). If C has more than two adjacent variables, then at least one of them is not a boundary variable, and we denote it as X. X must be an almost-simplicial vertex since it is adjacent to two other variables as well as C and, since X is not a boundary variable, its neighbouring variables are adjacent to C. Hence we can apply the almost-simplicial rule, eliminating X and connecting X's neighbouring variables with an edge, resulting in a new extended Nested SAT graph where clause C has one less internal variable (see Figure 2.9(b)). We can then repeat this process until clause C has no internal variables and exactly two boundary variables.

At this stage, if clause C has no parent clause, then it is a simplicial vertex of degree two, and can simply be deleted. If instead clause C has a parent clause C_p, then it is a simplicial vertex of degree three, since both its boundary variables must be adjacent to its parent clause. Thus, we can eliminate C via the simplicial rule, thereby reducing the number of clauses by one (see Figure 2.9(a)). Removing this simplicial vertex results in a new extended Nested SAT graph with one less clause. Thus, we can repeat this process until no clauses remain, resulting in a forest of variables. Since the degree of every vertex eliminated by the simplicial and almost-simplicial rules is at most three, the conclusion follows immediately using Theorems 2.2 and 2.3. $\square$

We note that the tree decomposition is not immediately apparent from this alternative proof, and hence we provided our constructive algorithm instead. However, the above

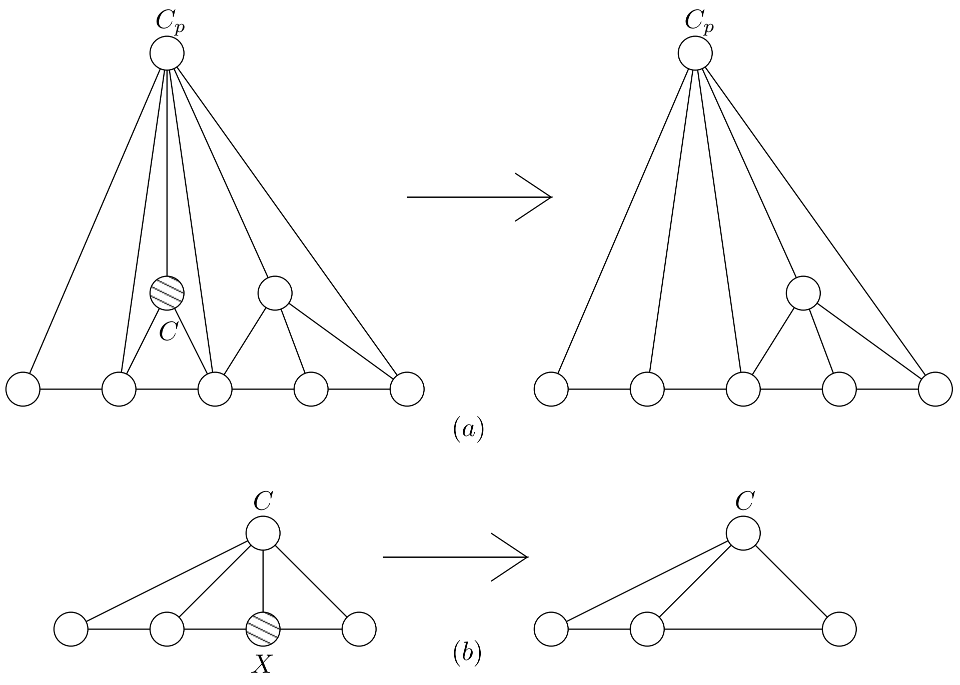


Figure 2.9: The Simplicial (a) and Almost-Simplicial (b) Rules

method does give a perfect elimination ordering (i.e. the reverse order in which the vertices are eliminated).

Another alternative proof can be obtained by observing that Nested SAT graphs are minors of IO-graphs:

Alternative Proof 2: Let G be a Nested SAT graph. Without loss of generality, we can add a parent clause to G to ensure that the clause hierarchy is a tree rather than a forest (in the same manner as Knuth [74]). Thus, the edges between variables and the two edges to the topmost clause form a cycle, and this is the boundary of the outer-face. Deleting all vertices on this boundary leaves only clause vertices, and since no two clause vertices are adjacent, this results in an independent set, thus satisfying the definition of IO-graphs. Since IO-graphs have treewidth at most three (see Theorem 1.9) and the original graph G is a subgraph of an IO-graph, then by Theorem 1.6 we conclude that Nested SAT graphs have treewidth at most three. $\square$

However, this observation does not apply to extended Nested SAT graphs, as the clauses no longer form an independent set. Thus, Theorem 2.1 is a stronger result.

Finally, a simple but loose bound on the treewidth of extended Nested SAT graphs can be obtained using Theorem 1.7. Since extended Nested SAT graphs are 2-outerplanar (all variables are on the outer-face, and their deletion leaves a forest which is outerplanar), we easily conclude that the treewidth of extended Nested SAT graphs is at most five.

2.4 Algorithmic Consequences

Since SAT problems of bounded treewidth are solvable in parameterized linear time [54, 97], Knuth's result [74] is an immediate corollary of Theorem 2.1.

Corollary 2.1 *Nested SAT is solvable in linear-time.*

The results of Knuth were expanded by Kratochvíl and Křivánek [75] in two ways. First, they considered co-nested SAT: Nested SAT with the role of variables and clauses reversed (i.e. with clauses on the line, and variables in the half-space above). Secondly, they gave a linear-time algorithm for solving MaxSAT on nested and co-nested graphs. By reversing the role of variables and clauses in our Nested SAT tree decomposition, we obtain the following result:

Corollary 2.2 *Let G be the graph of a co-nested SAT problem. Then G has treewidth at most three.*

Since MaxSAT is also solvable in parameterized linear time on graphs of bounded treewidth (which can be shown by modifying the algorithm for SAT suitably [54, 97]), this implies a generalization of the result by Kratochvíl and Křivánek:

Corollary 2.3 *MaxSAT on nested and co-nested graphs is solvable in linear time.*

Chapter 3

Matryoshka Graphs

In this chapter we define a new class of graphs—Matryoshka graphs and partial Matryoshka graphs—that generalize the structure of extended Nested SAT graphs. Matryoshka graphs take their names from the nested Russian dolls, as we explain in Section 3.1.2. Since partial Matryoshka graphs are also a strict subset of planar partial 3-trees, we explore their characteristics, and determine how they relate to planar partial 3-trees in general, as well as other graph classes of treewidth three, such as Halin graphs and IO-graphs.

3.1 Defining Matryoshka Graphs

3.1.1 Observations and Constraints

We begin by noting that in the tree decomposition of Nested SAT graphs (see Figure 2.7), aside from parent-child clause edges being induced, edges between non-consecutive variables were also implied. In particular, for each clause, every one of its variables is adjacent to the right boundary variable of the clause, in addition to its immediate neighbours on the line (see Figure 3.1). Obviously these edges can be added to the graph without increasing treewidth, since they are already included within the tree decomposition. Furthermore, as we will show, these edges can be added without causing the graph to become non-planar. We begin by proving the result for a single clause path:

Lemma 3.1 *Let P be a clause path tree decomposition for clause C with parent C_p, and variables $X_{i_1}, \ldots, X_{i_q}$ ($i_1 < \ldots < i_q$), as constructed by Algorithms 1 and 2. Then the graph induced by P is a planar triangulated graph G of treewidth at most three with C_p, X_{i_1}, and X_{i_q} on one face.*

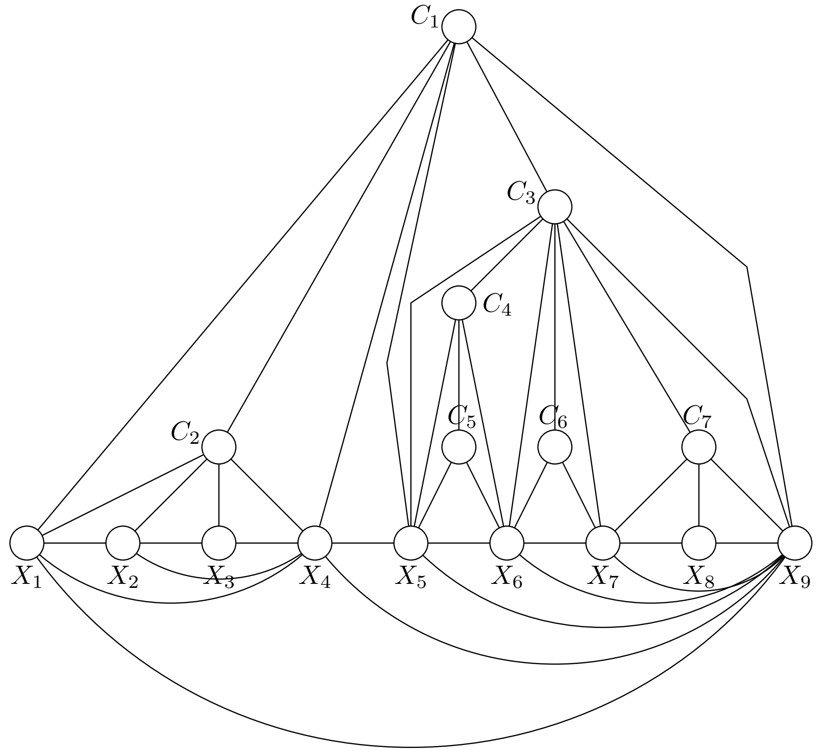


Figure 3.1: Induced Graph of the Tree Decomposition in Figure 2.7

Proof: The treewidth of the induced graph G must be at most three by definition, since P is a tree decomposition of width three. Thus, we need only show that the induced graph G is planar and triangulated. Due to the enhancement of Algorithm 2, the first node in P must contain the four labels C, C_p, X_{i_1}, and X_{i_q}. Thus the first node of P induces the triangulated planar graph K_4. Algorithm 1 clearly constructs a clause path such that every successive node introduces one new vertex label and has three vertex labels in common with the previous node. Since these three shared vertex labels form a face in the graph induced by the previous nodes of the tree decomposition, then each successive node in the clause path simply introduces a new label that replaces an existing triangulated face with an instance of K_4. In other words, each successive node in the tree decomposition subdivides an existing triangulated face into three triangulated faces. Thus, the induced graph remains planar and triangulated after the addition of each successive node by Algorithm 1. It follows that the property will hold for graph G. $\square$

We have proven the result holds for a clause path when the clause has a parent. We now extend this observation to clause paths for which no clause parent exists:

Lemma 3.2 *Let P be a clause path tree decomposition for clause C with no parent clause* *and variables $X_{i_1}, \ldots, X_{i_q}$ ($i_1 < \ldots < i_q$), as constructed by Algorithms 1 and 2. Then the graph induced by P is a planar triangulated graph G of treewidth at most three.*

Proof: The existence of clause path P constructed by Algorithms 1 and 2 ensures that the treewidth of G is at most three, so we need only prove that G is planar and triangulated. The first node of P will only contain the three labels C, X_{i_1}, and X_{i_q} according to Algorithm 2, so the first node of P induces the planar triangulated graph K_3. Each successive node in the clause path will have three labels in common with the previous one (see Algorithm 1), and thus at each stage we simply replace an existing triangulated face with three triangulated faces and a new vertex. Hence the induced graph remains planar and triangulated for each successive node, and thus the conclusion must hold for the graph G. □

We can combine the above results to show that the entire tree decomposition produced by Algorithms 1 and 2 induces a planar triangulated graph of treewidth at most three.

Lemma 3.3 *Let T be a tree decomposition for a Nested SAT graph, as constructed by Algorithms 1 and 2. Then the graph induced by T is a triangulated planar graph of treewidth at most three.*

Proof: We will prove by induction that for any clause C, the clause paths of C and its descendants in the clause tree hierarchy induce a triangulated planar graph of treewidth at most three. Furthermore, if C has a parent, then the triangle consisting of C's parent and C's left and right boundary variables is a face.

For the base case, we consider clauses with no descendants in the clause tree hierarchy (i.e. clauses of height zero in the clause tree hierarchy). Our induction hypothesis holds for these clauses by Lemmas 3.1 and 3.2. Assume we have proven the claim for all clauses of height at most $k, k \geq 0$ in the clause tree hierarchy, and let us consider a clause C_p of height $k+1$. Then the induction hypothesis must hold for every child clause of C_p. Furthermore, the clause path of C_p induces a triangulated planar graph of treewidth three and, for every consecutive pair of C_p's visible variables, the induced graph contains a face incident to C_p and this pair of variables. Thus, for any child clause C_i of C_p, we can insert the induced graph of C_i and its descendants into the appropriate face of C_p's induced graph, thereby maintaining a planar triangulated graph. Finally, we note that the width of the tree decomposition cannot increase, since Algorithms 1 and 2 simply combine clause paths of width three, without introducing any new nodes or labels. Thus, by the principle of mathematical induction, the claim holds for any clause in such a tree decomposition. In particular, this statement holds for the root clause, from which the result follows immediately. □

Based on this observation, we will now explore generalizations of Nested SAT graphs that neither increase treewidth nor introduce non-planarity. We will then examine how these graphs compare to planar partial 3-trees, and explore some of their characteristics.

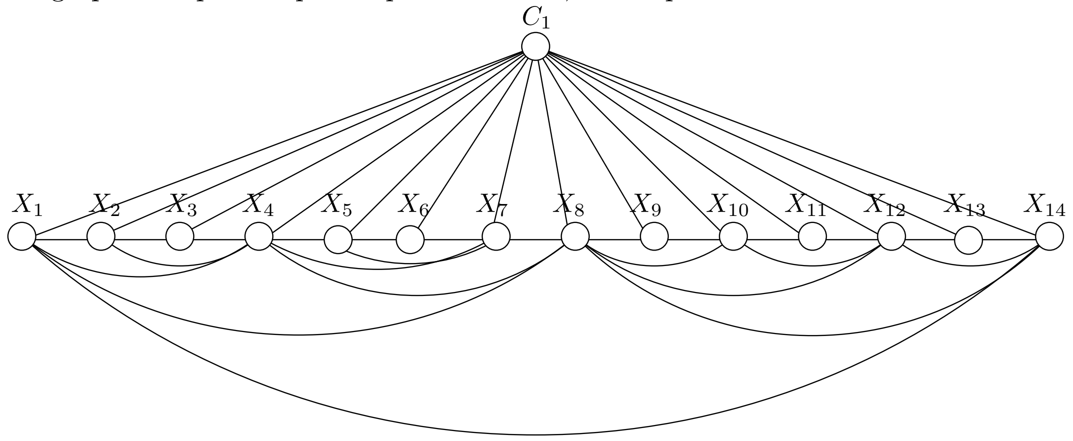


Figure 3.2: A Clause with an Outerplanar Graph on its Visible Variables

When constructing the tree decomposition of Nested SAT graphs, Algorithms 1 and 2 represent a clause and its neighbour variables using a path within the tree decomposition. The use of a path decomposition results in the clause's variable edges having a particular structure, namely that each clause's variables are adjacent to the rightmost variable of the clause, as well as their neighbour clauses on the line. However, as we shall see, using *clause trees* instead of clause paths will allow us greater flexibility over the edges between a clause's variables. We will now construct tree decompositions for clauses whose visible variables induce a triangulated outerplanar subgraph, as shown in Figure 3.2.

Algorithm 3: Constructing a Clause Tree

```
Input: Clause $C$ with its visible variables $X_{i_1}, \ldots, X_{i_q}$ $(i_1 < \ldots < i_q)$ inducing a
       triangulated outerplanar subgraph $U$, where $(X_{i_1}, X_{i_q})$ is an edge
Output: The clause tree of $C$

Construct the tree decomposition $T$ for the outerplanar graph $U$
Add clause $C$ to every node in tree $T$
if clause $C$ has a parent clause $C_p$ then
    Add a root node with labels $\{C, C_p, X_{i_1}, X_{i_q}\}$ to tree $T$
else
    Add a root node with labels $\{C, X_{i_1}, X_{i_q}\}$ to tree $T$
Return $T$
```

Algorithm 3 first constructs a tree decomposition for the outerplanar subgraph on clause C's visible variables. By Theorem 1.7 and Remark 1.10, this tree decomposition will have width exactly two. By adding clause C to every node, we obtain a tree decomposition of width three. Lastly, there must exist a node in tree T containing labels C, X_{i_1}, and X_{i_q} as a subset, since these three vertices form a clique. Thus, we can connect an additional node containing just these three labels or, if appropriate, these three labels and clause C's parent clause. In either case, we get a valid tree decomposition of width three for clause C and its visible variables.

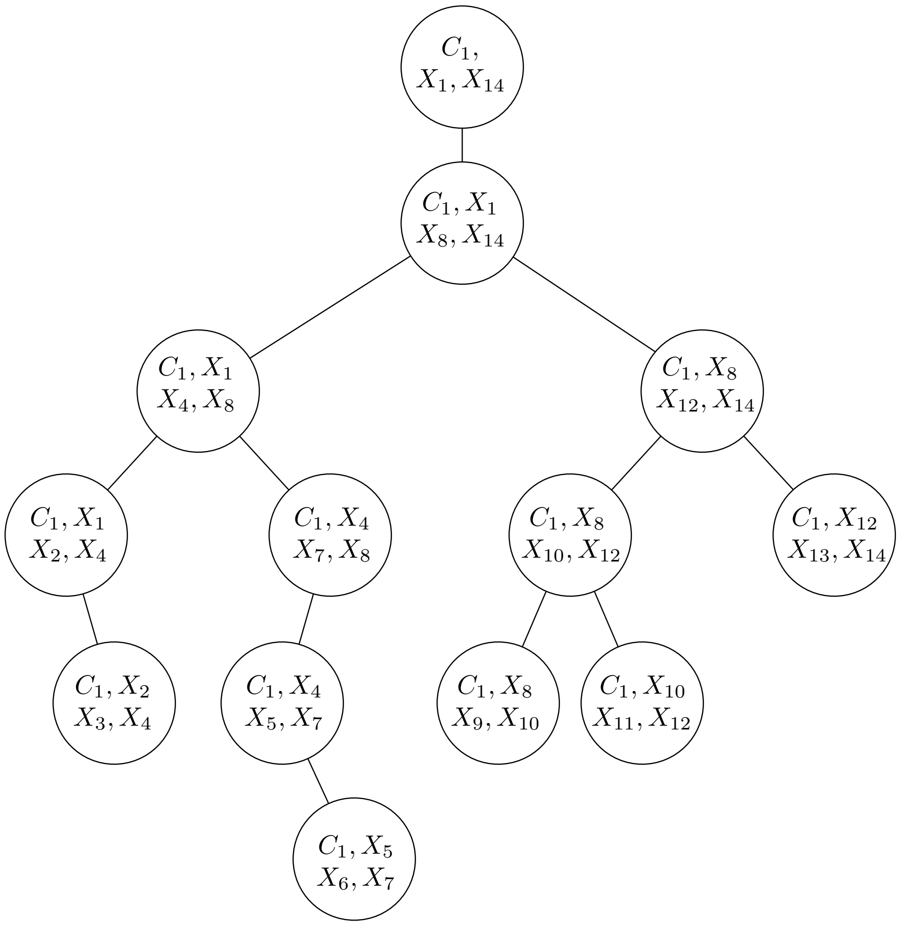


Figure 3.3: Clause Tree for Figure 3.2

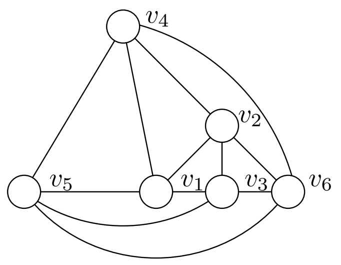
Running Algorithm 3 on the graph in Figure 3.2 results in the tree decomposition shown in Figure 3.3, assuming C_1 has no parent clause (otherwise we get an additional label in the topmost node). We must now confirm that we can combine these clause trees in the same manner as we combined clause paths for extended Nested SAT graphs.

Lemma 3.4 *Let T be a tree decomposition obtained by combining clause trees (as constructed in Algorithm 3) according to a tree-like hierarchy of clauses. Then the graph induced by T is a triangulated planar graph of treewidth at most three.*

Proof: As in Lemma 3.3, we will prove by induction that for any clause C, the clause trees of C and its descendants in the hierarchy induce a triangulated planar graph of treewidth at most three. Furthermore, if C has a parent clause, then the triangle consisting of C's parent and C's left and right boundary variables is a face.

The argument is exactly the same as in Lemma 3.3, since each individual clause tree induces a triangulated planar graph of treewidth three that contains a face for every consecutive pair of its visible variables (i.e. a face composed of the clause and this pair of variables, that allow this clause graph's combination with the graphs of its children clauses). Furthermore, if clause C has a parent clause, then its clause tree's induced graph contains a face composed of its parent and two boundary variables (i.e. a face that allows its planar insertion into the graph of its parent clause). Hence, for every clause C, the clause trees of C and its descendants in the clause tree hierarchy induce a triangulated planar graph of treewidth at most three. □

Thus, we have generalized extended Nested SAT graphs such that edges between vertices on the line can be added in a more flexible form without increasing treewidth or introducing non-planarity. However, this modification is still subject to the constraint that any two adjacent variable vertices have a common clause neighbour. Thus, the natural question is to ask whether we can generalize this further, or whether this property is necessary in order to guarantee treewidth at most three. We now show that removing this restriction can result in greater treewidth:



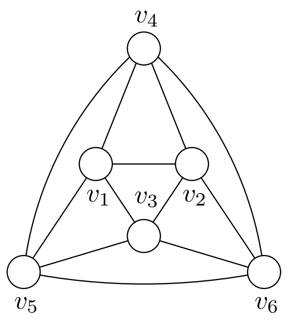


Figure 3.4: Unrestricted Variable Edges and Treewidth Greater Than Three

Lemma 3.5 *Given a Nested SAT graph and its LH-drawing with variables on the line and clauses in the half-space, if we triangulate this graph such that two vertices on the line are made adjacent without a common neighbour in the half-space, then the graph may have treewidth greater than three.*

Proof: Recall that the octahedral graph is a forbidden minor for graphs of treewidth three, by Theorem 1.12. In Figure 3.4, we can clearly see that allowing a single edge between variables with no common half-space neighbour allows the creation of the octahedral graph as a subgraph, and hence as a minor. Thus, we conclude that two adjacent vertices on the line may result in treewidth greater than three if they do not have a common neighbour in the half-space. □

We also note that with our generalized construction (i.e. clause trees instead of clause paths), we have kept the constraint that all faces contain at least one vertex on the line. We now show that this constraint cannot be eliminated without increasing treewidth:

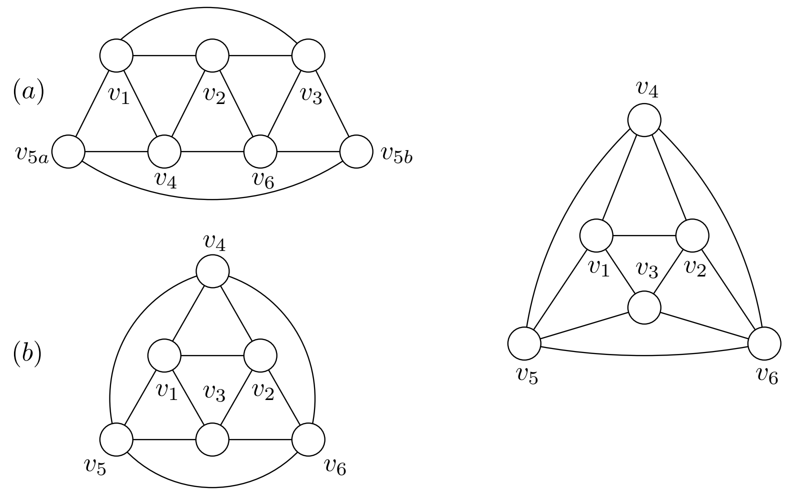


Figure 3.5: Clause Cycles and Treewidth Greater Than Three

Lemma 3.6 *Given a Nested SAT graph and its LH-drawing with variables on the line and clauses in the half-space, if we triangulate this graph such that there exists a face with no vertex on the line, then the graph may have treewidth greater than three.*

Proof: We consider two cases: a cycle among sibling clauses, and a cycle among one parent clause and two of its children clauses. In the first case, we can obtain the octahedral graph as a minor as shown in Figure 3.5(a), by contracting the edge between v_{5a} and v_{5b}. In the second case, the octahedral graph may be a subgraph, and hence a minor, as shown in Figure 3.5(b). Thus, in both cases the octahedral graph is a minor, and so by Theorem 1.12 we must conclude that the graphs shown in Figure 3.5 have treewidth greater than three. □

Thus, based on Lemmas 3.5 and 3.6, we will retain both of these restrictions in the definition of our new graph class.

3.1.2 Introducing Matryoshka Graphs

We can now formally define Matryoshka graphs:

Definition 3.1 *A graph is a* Matryoshka graph *if it is a planar triangulated graph with a planar drawing such that every vertex is on a horizontal line or in the half-space above the line and satisfies the following conditions:*

- *every face is incident to a vertex on the line,*
- *edges incident to vertices in the half-space are drawn as curves monotonic in the y-direction,*
- *edges with both endpoints on the line lie on or below the line, and*
- *any two adjacent vertices on the line have a common neighbour in the half-space.*

Given such a planar drawing (which we shall denote as a *Matryoshka drawing*), we can identify a tree-like hierarchy of the variables in the half-space using the order of vertices on the line, and the relationships of disjointness and straddling, as we did with Nested SAT graphs. Since each half-space vertex's neighbours on the line induce an outerplanar graph, we can construct a clause tree for each half-space vertex and its neighbours on the line using Algorithm 3. Finally, we can combine these clause trees using the tree-like hierarchy. Thus, by Lemma 3.4 Matryoshka graphs are triangulated planar graphs of treewidth at most three.

We define a *partial Matryoshka graph* to be any subgraph of a Matryoshka graph. We note that Matryoshka graphs are obviously a generalization of extended Nested SAT graphs, with the addition of edges below the line connecting vertices on the line (subject to the condition that they have a common neighbour in the half-space). Thus we can make the following simple observation:

Remark 3.1 *Extended Nested SAT graphs and Nested SAT graphs are partial Matryoshka graphs.*

The following property regarding Matryoshka graphs also follows directly from the definition:

Remark 3.2 *Every Matryoshka graph has a vertex-decomposition into a forest and an outerplanar graph, where the forest is composed of vertices in the half-space, and the outerplanar graph is induced by the vertices on the line.*

We will now show that there exists an edge-decomposition for all Matryoshka graphs into a forest and an SP-graph. As we shall see in Chapter 4, this type of decomposition is already known to exist for all partial 3-trees. However, the decomposition we describe below (see Figure 3.6) uses the structure of a Matryoshka graph more clearly. We note that both of these decomposition results must of course hold for all partial Matryoshka graphs as well.

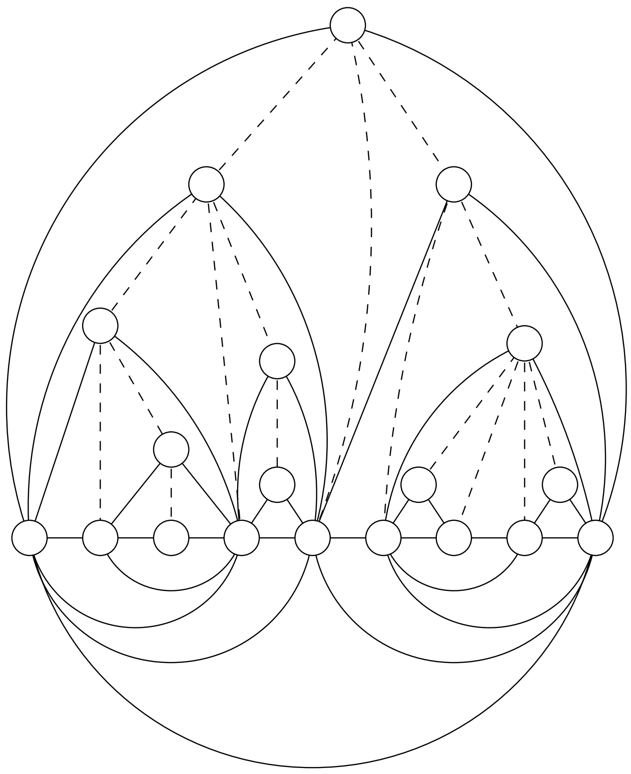


Figure 3.6: Matryoshka Graph Edge-Decomposition into an SP-Graph and a Forest

Lemma 3.7 *Every Matryoshka graph has an edge-decomposition into an SP-graph and a forest.*

Proof: We first note that each vertex in the half-space is adjacent to a range of vertices on the line. For a vertex in the half-space, we will call its leftmost and rightmost neighbours on the line its *boundary vertices*, and we will call the rest of its neighbours on the line its *internal vertices*. We note that a vertex on the line is the internal vertex of at most one vertex in the half-space, since otherwise a planar Matryoshka drawing would be impossible (i.e. overlapping clauses cannot occur in Nested SAT, and similarly, no two vertices in the half-space can have overlapping ranges). Using these definitions, we construct the forest F as follows: assign all edges incident to two half-space vertices to the set F, and for each half-space vertex, assign its edges to internal vertices to set F as well. We will now prove that set F is a forest, and that the remaining edges form a graph of treewidth two.

By definition the edges only incident to half-space vertices cannot contain a cycle, since otherwise there would be a face not incident to any vertex on the line. Furthermore, since each edge to an internal vertex merely adds a vertex of degree one, then these edges cannot be part of any cycle either. Thus we conclude that set F induces a forest. The edges not in F include all edges incident to two vertices on the line, as well as the edges joining half-space vertices to their boundary vertices. By definition the edges only incident to vertices on the line induce an outerplanar graph, which has treewidth two by Theorem 1.7. Thus we need only show that the boundary vertex edges do not increase treewidth. However, since every vertex in the half-space has exactly two boundary vertices, and since these two boundary vertices are adjacent by the definition of a Matryoshka graph, then the boundary vertex edges simply add several simplicial vertices of degree two. Applying the simplicial rule from Theorem 2.2, we conclude that these edges do not increase the treewidth. Finally, since a graph has treewidth two if and only if it is an SP-graph (Theorem 1.8), then the conclusion follows. □

It remains an open question as to whether or not all Matryoshka graphs have an edge-decomposition into an outerplanar graph and a forest. However, our investigations lead us to believe that this is not the case.

Finally, we note that the name Matryoshka finds its root in the name of the Russian nesting dolls. Since Matryoshka graphs are a generalization of Nested SAT graphs, and since Matryoshka graphs contain Matryoshka graphs within themselves (i.e. child clause trees nested within parent clause trees, each one inducing a Matryoshka graph), we thought this name was an appropriate analogy.

3.2 Properties of Matryoshka Graphs

We now explore various characteristics of Matryoshka and partial Matryoshka graphs, and identify how they relate to other planar graph classes of bounded treewidth.

3.2.1 Superclasses of Partial Matryoshka Graphs

Recall that every partial Matryoshka graph is planar and has treewidth at most three, and so is a planar partial 3-tree. We shall now prove that partial Matryoshka graphs are a strict subset of planar partial 3-trees. However, in order to do so, we first need to introduce some new notation. Given two planar triangulated graphs G and H with a fixed planar embedding, then $G \leftarrow H$ represents the graph that results when every face of G is replaced with a copy of graph H. In other words, the outer-face boundary of H is identified with the boundary of G's face for each replacement (recalling that all of these faces are triangles). An example of this process is shown in Figure 3.7, where we substitute K_4 for every face of K_4. In other words, we subdivide every triangulated face of K_4 into three triangulated faces and an additional vertex. Similarly, we shall define $G \leftarrow_{int} H$ as the graph that results when every *internal* face of G is replaced with a copy of graph H.

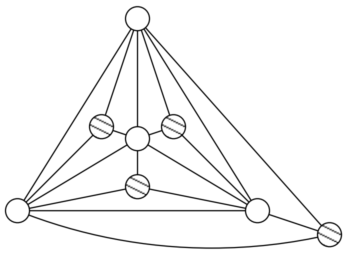


Figure 3.7: $K_4 \leftarrow K_4$

We now proceed with our proof that not every planar partial 3-tree is a partial Matryoshka graph.

Lemma 3.8 *The graph $K_4 \leftarrow K_4$ is a triangulated planar graph of treewidth three.*

Proof: Figure 3.7 clearly illustrates that the graph is planar and triangulated. Furthermore, applying Bodlaender's simplicial rule (Theorem 2.2) repeatedly, we can reduce $K_4 \leftarrow K_4$ to K_4, by eliminating four simplicial vertices (e.g. the shaded vertices in Figure 3.7). Thus, by applying Theorem 2.2 and Remark 1.8, the result follows. □

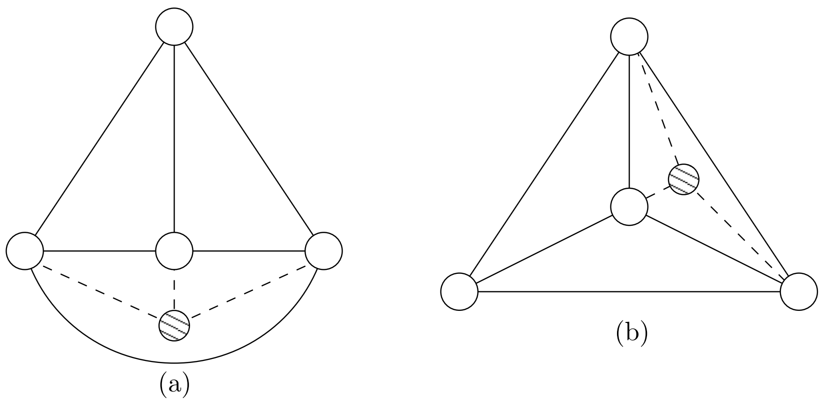


Figure 3.8: LH-Drawings of K_4

Lemma 3.9 *$K_4 \leftarrow K_4$ is not a partial Matryoshka graph.*

Proof: Any partial Matryoshka graph has a planar drawing with vertices on the line and in the above half-space, with every face incident to some vertex on the line. We shall show that these conditions cannot occur for any drawing of $K_4 \leftarrow K_4$ by analyzing the number of non-shaded vertices (see Figure 3.7) above the line.

If three or more non-shaded vertices are above the line, then we clearly have a cycle of length three above the line. Hence there is a face not incident to any vertices on the line, which is a contradiction. If all of the non-shaded vertices are on the line, then this implies that K_4 is an outerplanar graph, by Remark 3.2. However, if K_4 is outerplanar, then adding a vertex adjacent to all vertices in K_4 would result in a planar graph, which contradicts Theorem 1.11. Hence not all non-shaded vertices are on the line.

Thus, we only need to consider the cases where one or two non-shaded vertices are above the line. If exactly one non-shaded vertex is above the line, then the three non-shaded vertices on the line form a cycle on and below the line. Thus the shaded vertex adjacent to these three vertices must be within this cycle and hence below the line, contradicting the definition of a Matryoshka drawing (see Figure 3.8(a)). Finally, assume exactly two of the non-shaded vertices are above the line. If we consider the triangle formed by these two vertices and one of the other non-shaded vertices, then since edges connecting line vertices to half-space vertices are monotonic in the y-direction, the shaded vertex adjacent to these three non-shaded vertices must also be in the half-space (see Figure 3.8(b)). However, this creates a cycle of length three above the line, implying that there exists a face not incident to the line. Thus we conclude that no such drawing exists for $K_4 \leftarrow K_4$. In other words, $K_4 \leftarrow K_4$ is not a partial Matryoshka graph. □

Theorem 3.1 *Partial Matryoshka graphs are a strict subset of planar graphs of treewidth three.*

Proof: From Lemmas 3.8 and 3.9 we can see that there exists at least one planar triangulated partial 3-tree that is not a Matryoshka graph. The result for all partial Matryoshka graphs follows immediately. □

Graphs of treewidth at most three, both planar and non-planar, have been well-characterized and can be identified in linear time [10, 11, 28, 43, 95]. Since partial Matryoshka graphs are a strict subset of planar partial 3-trees, we hope to characterize this difference, and thus give an efficient recognition algorithm for partial Matryoshka graphs. We begin by noting that any planar triangulated graph of treewidth three can be created by starting with K_4, and then repeatedly selecting a face which we replace with K_4. This observation can easily be derived from Bodlaender's Simplicial Rule (i.e. Theorem 2.2) and the fact that every planar triangulated graph of treewidth three must have a simplicial vertex of degree three (i.e. the last vertex in the perfect elimination order).

Before demonstrating some of the restrictions on partial Matryoshka graphs compared to planar partial 3-trees in general, we first need to refine our tree decompositions:

Theorem 3.2 *Given a graph G of treewidth k, there exists a tree decomposition for G such that every node contains $k+1$ labels and each pair of adjacent nodes in the tree has k labels in common [26].*

We shall call such a tree decomposition a *standardized tree decomposition*, and the k common labels between adjacent nodes will be called the *label intersection* of these two nodes. We now prove that any planar partial 3-tree has a standardized tree decomposition with maximum degree at most four.

Lemma 3.10 *Let T be a standardized tree decomposition of width three. If any node in T has the same label intersection with two or more of its neighbours, then the graph implied by T is non-planar.*

Proof: Let x be a node in T with the same label intersection with two of its neighbours, and let y and z denote two of x's neighbours with the same label intersection. Then the subtree formed by nodes x, y, and z must induce a graph containing $K_{3,3}$ as a subgraph (see Figure 3.9). Thus, the result follows by Theorem 1.11. □

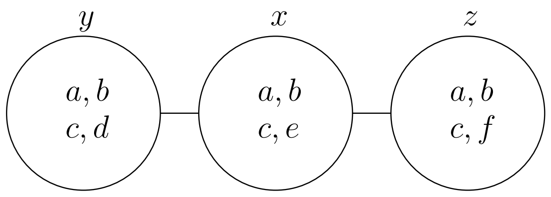


Figure 3.9: Standardized Tree Decomposition for $K_{3,3}$

Corollary 3.1 *Let T be a standardized tree decomposition of width three that induces a planar graph. Then T has maximum degree at most four.*

Proof: By Lemma 3.10, we know that no node in T has two neighbours with the same label intersection. Since T has width three, each node has four labels, and hence four possible label intersections. Thus each node in T has at most four neighbours, which concludes the proof. □

We shall now show that every Matryoshka graph has a standardized tree decomposition of width three with maximum degree at most three (i.e. not four, as with planar partial 3-trees in general).

Lemma 3.11 *Every clause tree has a standardized tree decomposition of width three and maximum degree three.*

Proof: Using Algorithm 3 we can construct a standardized tree decomposition of width three for each variable in the half-space and its neighbours on the line, with the minor modification that we shall omit the addition of a root node to the clause tree if it has no parent clause (to ensure it remains a standardized tree decomposition). Thus we only need to show that the maximum degree of this tree decomposition is three. Since the initial framework for this tree decomposition is the standardized tree decomposition for an outerplanar graph, then each node initially has at most three labels (i.e. width two) and, rooting the tree decomposition at the appropriate node, each node has at most two children, and hence maximum degree three. Adding the clause label to all nodes increases the width of the tree decomposition to three, but it remains standardized and the number of children nodes does not increase. Finally, since the root node has degree two, we can add the root node (if the clause has a parent clause) without increasing the maximum degree. Hence the result follows. □

Lemma 3.12 *Every Matryoshka graph has a standardized tree decomposition of width three with maximum degree three.*

Proof: By Lemma 3.11 we can create a standardized (rooted) tree decomposition for each clause and its visible variables of width three where each node has at most two children. Combining these clause trees will not increase the width of the tree decomposition, and the tree decomposition remains standardized since the node in the parent clause tree contains all but one of the labels in the root node of the child clause tree. Thus, we need only show that the combination of these clause trees does not increase the maximum number of children nodes. If a clause tree node connects to the root of a child clause tree, then two of the labels in that node are consecutive visible variables for the parent clause. Hence, this node cannot have two children in the clause tree. Furthermore, for each pair of consecutive visible variables, the clause has at most one child clause. Thus, connecting these two clause trees does not increase the maximum degree of the tree decomposition. □

It follows that every partial Matryoshka graph has a standardized tree decomposition of width three with maximum degree three, since any graph's tree decomposition is valid for all of its subgraphs. We also note that Lemma 3.12 provides an alternative proof that $K_4 \leftarrow K_4$ is not a partial Matryoshka graph, since it cannot have a tree decomposition of width three and degree less than four.

Recalling that any planar triangulated graph of treewidth three can be created by starting with K_4 and repeatedly subdividing faces (i.e. replacing existing faces with K_4), we now define the graph class of *binary planar* 3*-trees.* A graph is a binary planar 3-tree if it can be constructed by starting with K_4, subdividing at most two of the three internal faces (by introducing one new vertex per subdivided face), and then recursively, for each new vertex introduced by the previous round of subdivisions, subdividing at most two of the three internal faces incident to that vertex. A *binary planar partial* 3*-tree* is any graph that is a subgraph of a binary planar 3-tree. The relevance of this graph class is that a planar triangulated graph of treewidth three is a binary planar 3-tree if and only it has a standardized tree decomposition of width three and maximum degree three. Thus, we shall show that partial Matryoshka graphs are a subset of binary planar partial 3-trees.

Lemma 3.13 *A triangulated planar graph is a binary planar* 3*-tree if and only if it has a standardized tree decomposition of width three and maximum degree three.*

Proof: A binary planar 3-tree is constructed by starting with K_4, and then subdividing at most two of the three internal faces at each level of subdivision. Thus, we can construct our root node of width three to represent the original K_4 and, for each internal subdivision at the first level, add a child node of width three containing the three vertices of the subdivided face (i.e. three common labels with the parent node) and the new vertex introduced into the subdivided face. For each child node, the new vertex is incident to three internal faces, at most two of which will be subdivided. Thus we can repeat the process, treating each child node as we did the original parent node, and constructing a standardized tree decomposition of width three and maximum degree three.

For the other direction, say planar triangulated graph G has a rooted standardized tree decomposition of width three such that every node has at most two children (i.e. by rooting the tree decomposition at a node of degree less than three, such as a leaf). The root of this tree decomposition must have four labels, and hence induces K_4. Since the induced graph is planar, there does not exist a node x with two neighbours y, z such that there are three labels in common for x, y, and z. Thus, each child node has a different set of three common labels, and thus subdivides a distinct face incident to the vertex introduced by its parent node. Since each node has at most two children, the result follows. □

Corollary 3.2 *Every Matryoshka graph is a binary planar* 3*-tree.*

Proof: Follows directly from Lemmas 3.12 and 3.13. □

The restricted structure of binary planar 3-trees ensures that they (and hence, Matryoshka graphs) are Hamiltonian:

Theorem 3.3 *Every binary planar* 3*-tree is Hamiltonian.*

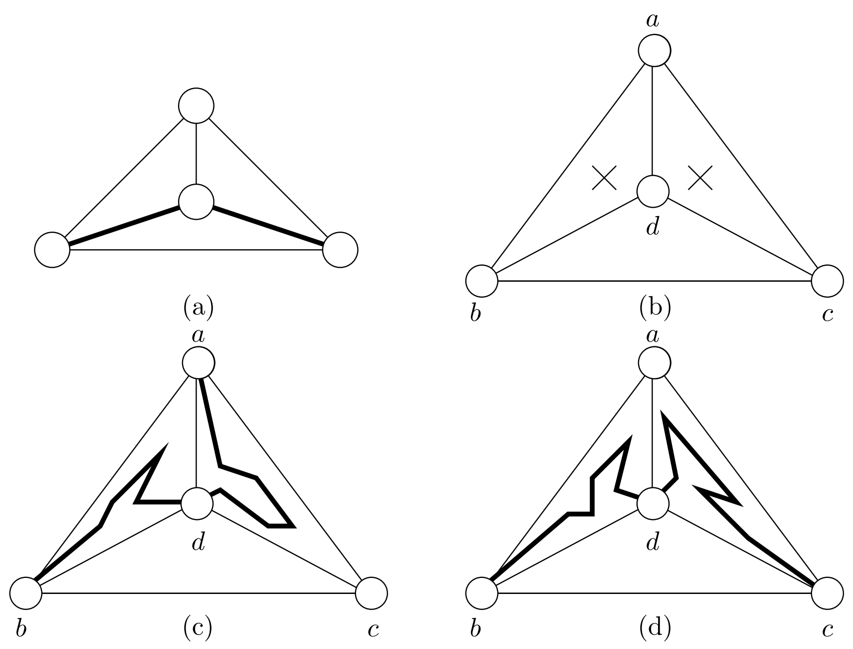


Figure 3.10: Binary Planar 3-Trees are Hamiltonian

Proof: We begin by proving that for any two vertices on the outer-face of a binary planar 3-tree (i.e. on the original K_4), there exists a path between these two vertices that visits every vertex except for the third vertex on the outer-face. We prove this by induction, and the base case is K_4, for which the statement obviously holds (see Figure 3.10 (a)). Assume the statement holds for all binary planar 3-trees on at most $k \geq 4$ vertices, and let us consider a binary planar 3-tree G on $n = k + 1$ vertices. Let a, b, c be the vertices on the outer-face of the original K_4 for G, let d be the internal K_4 vertex, and without loss of generality say at most two of the internal faces a, b, d and a, c, d are subdivided (as in Figure 3.10 (b)). Now we note that the graph within the face a, b, d is a binary planar 3-tree, as is the graph within the face a, c, d. Thus, if we wish to construct a path from vertex a to vertex b for graph G, then we can use the path from vertex a to vertex d from the graph within a, c, d (i.e. applying the induction hypothesis) and append the path from vertex d to vertex b in the graph of a, b, d (see Figure 3.10 (c)). The situation is symmetric if we wish to construct a path from vertex a to vertex c. Finally, if we wish to construct a path from vertex b to vertex c, then we can use the paths as shown in Figure 3.10 (d). We note that if one of these faces were not subdivided, then we could simply use the appropriate edge in the original K_4. Thus for any two outer-face vertices in graph G's K_4, we can construct a path between these two vertices that visits every vertex in G except for the third outer-face vertex. By the principle of mathematical induction, the result follows. Thus, for any binary planar 3-tree, we can construct a Hamiltonian cycle by creating this path between two of its outer-face vertices, and then adding the two (outer-face) edges joining the omitted outer-face vertex to the endpoints of this path. □

Corollary 3.3 *Every Matryoshka graph is Hamiltonian.*

Proof: Follows directly from Corollary 3.2 and Theorem 3.3. □

We note that this property is significant since it does not hold for all triangulated planar partial 3-trees:

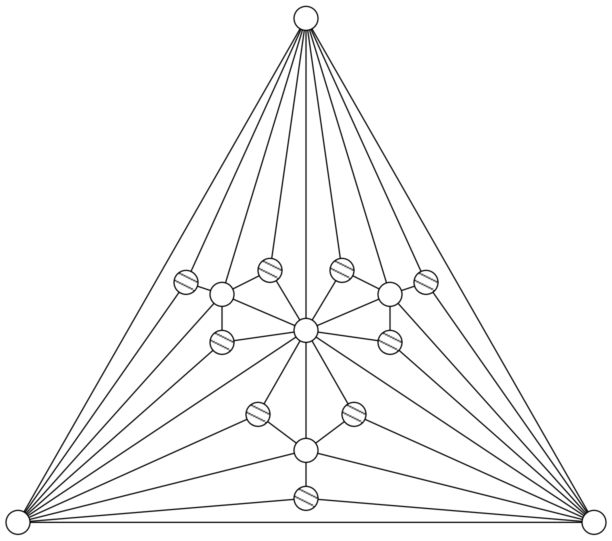


Figure 3.11: $K_4 \leftarrow_{int} K_4 \leftarrow_{int} K_4$

Lemma 3.14 *The graph $K_4 \leftarrow_{int} K_4 \leftarrow_{int} K_4$ is a planar triangulated partial 3-tree with no Hamiltonian cycle.*

Proof: The planar drawing of $K_4 \leftarrow_{int} K_4 \leftarrow_{int} K_4$ in Figure 3.11 proves that the graph is planar and triangulated, so we only need to prove it has treewidth at most three and no Hamiltonian cycle. Applying Bodlaender's simplicial rule (Theorem 2.2) repeatedly we can eventually reduce this graph to K_4. Thus, by Theorem 2.2 and Remark 1.8 the graph $K_4 \leftarrow_{int} K_4 \leftarrow_{int} K_4$ must have treewidth three. Finally, we note that the shaded vertices form an independent set. Thus, if a Hamiltonian cycle exists, these vertices cannot be adjacent within the cycle. However, there are 16 vertices in total, and nine in the independent set. Thus the independent set exceeds half of the vertices in the graph, and hence no Hamiltonian cycle is possible. □

This provides yet another proof that partial Matryoshka graphs are a strict subset of planar partial 3-trees. Indeed, as we shall now show, partial Matryoshka graphs are also a strict subset of binary planar partial 3-trees. We begin by first proving that $K_4 \leftarrow_{int} K_4$ is a partial Matryoshka graph with a very restricted Matryoshka drawing.

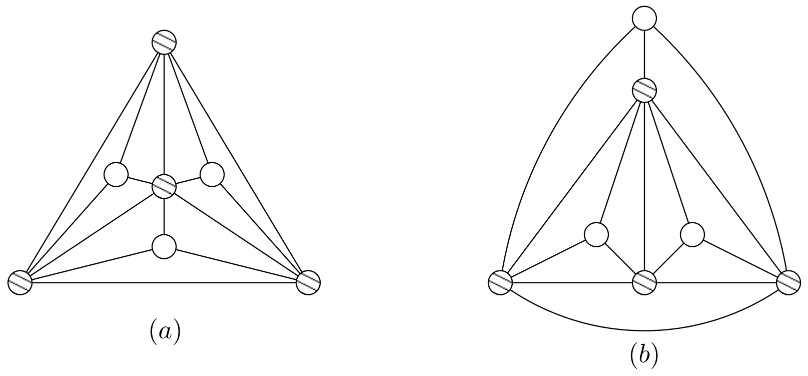


Figure 3.12: $K_4 \leftarrow_{int} K_4$

Lemma 3.15 *The graph $K_4 \leftarrow_{int} K_4$ (see Figure 3.12 (a)) is a partial Matryoshka graph. Furthermore, its Matryoshka drawing must have the three shaded outer-face vertices on the line and its internal shaded vertex in the half-space.*

Proof: Given the planar drawing in Figure 3.12 (b), it is clear that $K_4 \leftarrow_{int} K_4$ is a Matryoshka graph. Thus we need only show that its shaded vertices are restricted as claimed. At most two of the shaded vertices can be in the half-space, since three or more would create a cycle in the half-space. Also, K_4 is not outerplanar, hence at least one shaded vertex must be in the half-space. If two of the shaded vertices are in the half-space, then they have a common non-shaded neighbour. This non-shaded vertex must be within the triangle formed by its neighbours: the two shaded vertices in the half-space and a third shaded vertex on the line. Hence, this non-shaded vertex must be in the half-space, thus producing a cycle in the half-space, which is a contradiction. Thus, the last possibility is that exactly one of the shaded vertices is in the half-space, and the rest are on the line. If the three shaded vertices on the line have a common non-shaded neighbour, then this non-shaded vertex must also be on the line. However, since K_4 is not outerplanar, this is impossible. Thus, the three shaded vertices on the line cannot have a common non-shaded neighbour. We conclude that the shaded vertices on the line must be the three shaded vertices on the outer-face in Figure 3.12 (a). $\square$

We will now use the restricted Matryoshka drawing of $K_4 \leftarrow_{int} K_4$ to construct a binary planar partial 3-tree that is not a partial Matryoshka graph.

Theorem 3.4 *Partial Matryoshka graphs are a strict subset of binary planar partial* 3-*trees.*

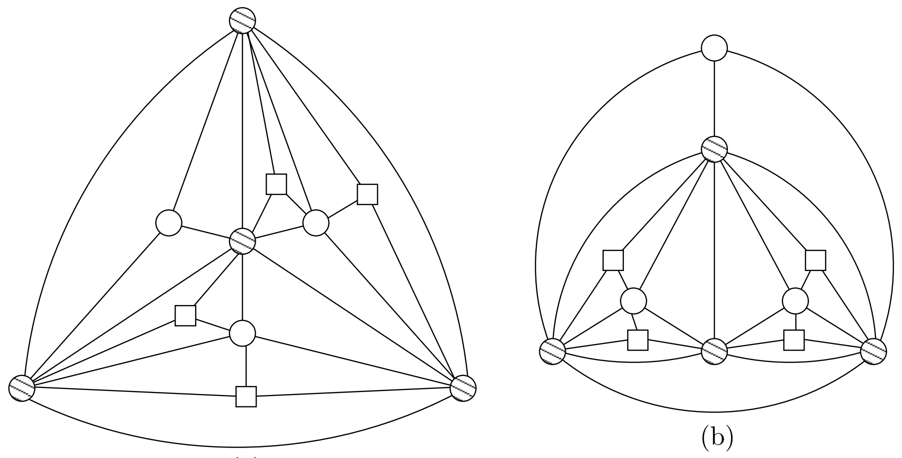


Figure 3.13: Binary Planar Partial 3-Tree that is not a Partial Matryoshka Graph

Proof: The graph shown in Figure 3.13 (b) is a binary planar 3-tree, since we can start with the K_4 formed by the three outer-face vertices and the topmost shaded vertex, and then subdivide at most two of the three internal faces at each stage from then on. Furthermore, because this graph contains $K_4 \leftarrow_{int} K_4$ as a subgraph (see Figure 3.13 (a)), then by applying Lemma 3.15 we conclude that any Matryoshka drawing of this graph must have the three outer-face shaded vertices on the line, and the internal shaded vertex in the half-space. However, no matter how we order the three shaded vertices on the line, at least one of the unshaded circular vertices with boxed neighbours must appear within a triangle formed by two shaded vertices on the line and one in the half-space. One of this circular vertex's two boxed neighbours must will share its two shaded neighbours on the line. Hence, if the drawing is planar, this circular vertex must be in the half-space, above this boxed neighbour. However, this implies that the other boxed neighbour lies within a triangle formed by the circular vertex in the half-space, the shaded vertex in the half-space, and a shaded vertex on the line. In other words, there must exist a face not incident to the line composed of the boxed vertex, its unshaded neighbour, and the shaded vertex in the half-space. Thus, the graph shown in Figure 3.13 (a) cannot be a partial Matryoshka graph. □

3.2.2 Subclasses of Partial Matryoshka Graphs

As mentioned before in Remark 3.1, (extended) Nested SAT graphs are partial Matryoshka graphs. Indeed, by construction, Matryoshka graphs are a generalization of their structure. In this section, we explore how partial Matryoshka graphs relate to other well-known planar graph classes of treewidth three or less. In particular, we look at Halin graphs, IO-graphs, partial 2-trees, and partial 1-trees. We begin by making the following basic observation:

Lemma 3.16 *Outerplanar graphs are partial Matryoshka graphs.*

Proof: Without loss of generality, we assume the graph is maximally outerplanar (i.e. internally triangulated). There exists a planar embedding for this outerplanar graph with all vertices on the line. If we add a vertex in the half-space adjacent to all vertices in the outerplanar graph (on the line), then it is clear that we obtain a Matryoshka graph. Hence the original outerplanar graph (and any of its subgraphs) is a partial Matryoshka graph. □

It follows from this that all graphs of treewidth one (i.e. forests) are partial Matryoshka graphs. However, it remains uncertain as to whether or not all graphs of treewidth two (i.e. SP-graphs) are partial Matryoshka graphs. We now turn our attention to some graph classes of treewidth three, namely Halin graphs and IO-graphs.

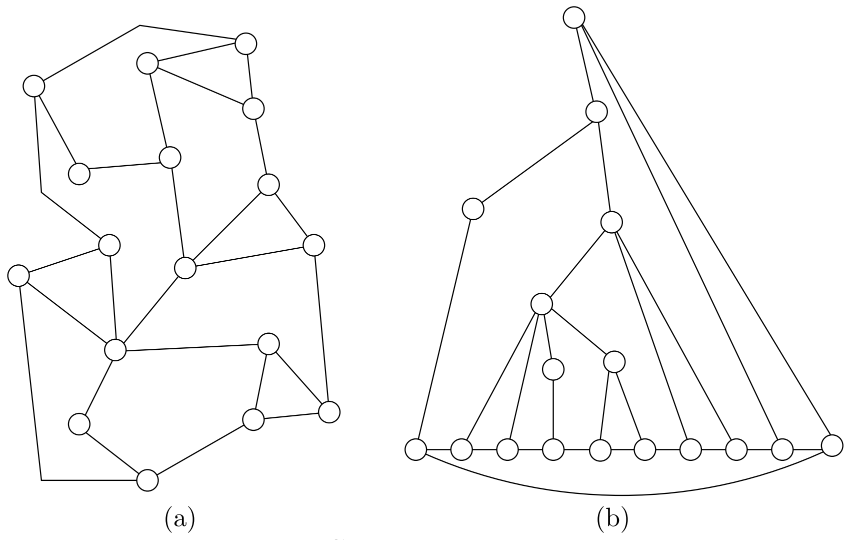


Figure 3.14: Halin Graph and its Matryoshka Drawing

Lemma 3.17 *Halin graphs are partial Matryoshka graphs.*

Proof: Recall from page 9 that a Halin graph has a planar embedding such that the outer-face is a cycle, and all edges not on the outer-face induce a tree. Hence there exists a planar embedding such that all cycle vertices are on the line (ordered according to the cycle), and all other vertices appear in the half-space above. Obviously no cycle exists in the half-space, since by definition a forest contains no cycles. It follows that every two consecutive vertices in the cycle have a common visible vertex in the half-space. Thus, edges can be added such that two vertices are adjacent in the cycle only if they have a common neighbour in the half-space. Since a cycle is obviously outerplanar, the result follows. □

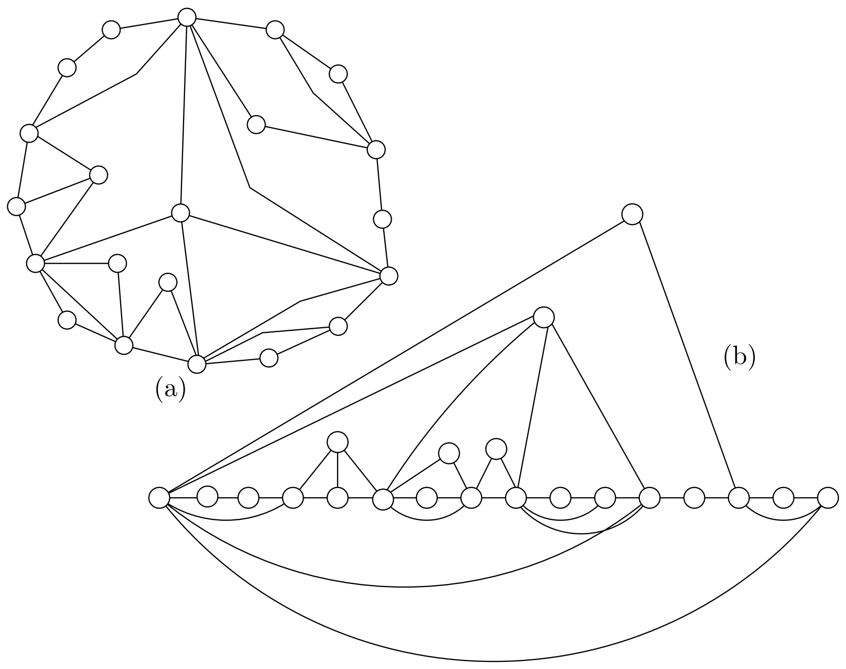


Figure 3.15: IO-Graph and its Matryoshka Drawing

Lemma 3.18 *IO-graphs are partial Matryoshka graphs.*

Proof: By definition, an IO-graph has a planar embedding such that all vertices not on the outer-face form an independent set. Since an IO-graph is 2-connected by definition, then we have a cycle on the outer-face, with all other edges between cycle vertices on the inside of this cycle. In order to simplify our proof, we construct a new graph G from this IO-graph by adding a duplicate edge for each edge between two vertices on the cycle, and then sub-dividing each of these duplicate edges once (i.e. replacing each duplicate edge with two edges and a new vertex in between). This graph is obviously still planar, since each edge and its duplicate can be drawn alongside one another arbitrarily closely. Furthermore, if we draw each duplicate edge and its vertex along the inside face, then each new vertex is not on the outer-face. Finally, each new vertex is only adjacent to two vertices on the cycle. Hence, this resulting graph is also an IO-graph (i.e. the removal of the outer-face vertices still results in an independent set, and the graph is planar and 2-connected). Now we shall show that this new IO-graph G is a partial Matryoshka graph, thereby proving that the original IO-graph (a subgraph) is also a partial Matryoshka graph.

Since G is an IO-graph, we can select a planar embedding that assigns the outer-face vertices to the line, and the independent set vertices to the half-space above the line. By definition, no two vertices in the independent set are adjacent, and thus every face is incident to a vertex on the line. Furthermore, all edges joining two vertices on the line can be drawn on or below the line without any edges crossing. Finally, every pair of adjacent vertices on the line has a common neighbour in the half-space (i.e. the new vertex we introduced via the duplicate edges). Thus G satisfies the definition of a partial Matryoshka graph, as does the original IO-graph. It follows that all IO-graphs are partial Matryoshka graphs. □

We note that there exists an infinite number of partial Matryoshka graphs that are not contained within any of the above graph classes:

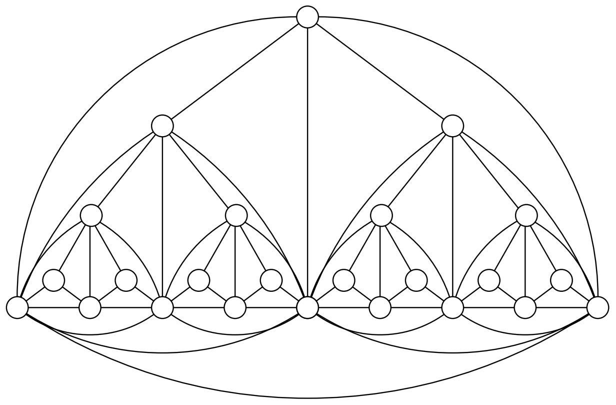


Figure 3.16: Partial Matryoshka Graph with Outerplanarity Greater than Two

Lemma 3.19 *There exist partial Matryoshka graphs that are not Halin graphs, IO-graphs, or outerplanar graphs.*

Proof: Halin graphs have outerplanarity two, since they are composed of a cycle (that can be placed on the outer-face) and a tree. Likewise, by definition IO-graphs and outerplanar graphs have outerplanarity at most two and one, respectively. However, there exist partial Matryoshka graphs of outerplanarity greater than two (see, for example, Figure 3.16). The result follows immediately. □

We summarize these relationships in Figure 3.17.

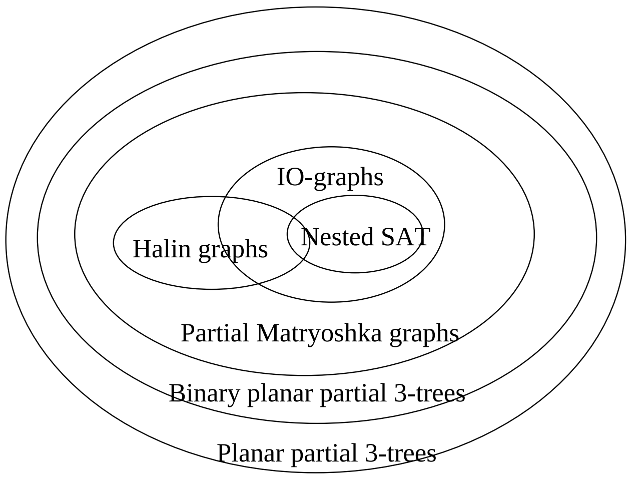


Figure 3.17: Planar Graph Classes of Treewidth Three

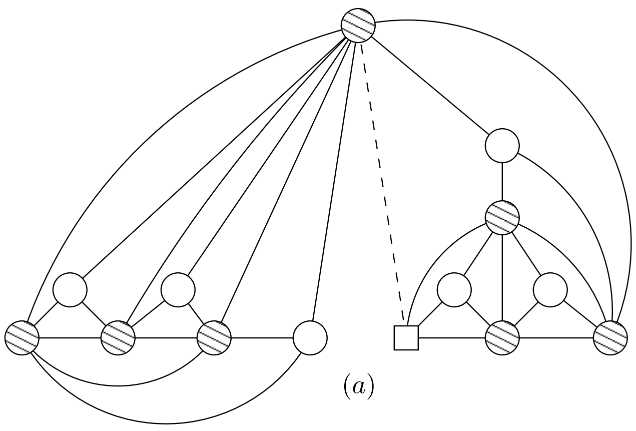
3.2.3 Summary

In summary, although we know that partial Matryoshka graphs are a strict subset of binary planar partial 3-trees, and that they generalize Halin graphs, IO-graphs, and extended Nested SAT graphs, we still lack a nice characterization of this graph class. In order to recognize this graph class efficiently, a reasonable approach would be to use the concept of forbidden minors (as in Theorems 1.11 and 1.12). However, such an approach cannot work since partial Matryoshka graphs are not closed under minors.

Theorem 3.5 *Partial Matryoshka graphs are not closed under minors.*

Proof: The graph shown in Figure 3.18 (a) is clearly a partial Matryoshka graph. By contracting the edge to the boxed vertex, we obtain the graph shown in Figure 3.18 (b). We note that this graph is identical to two copies of $K_4 \leftarrow_{int} K_4$ being combined by identifying one copy's internal shaded vertex with one of the outer-face shaded vertices of the other copy. By applying Lemma 3.15 twice, we must conclude that this common vertex is both in the half-space and on the line, which is a contradiction. In other words, this minor cannot be a partial Matryoshka graph. Thus, partial Matryoshka graphs are not closed under minors. □

Thus, it remains an open question as to whether or not we can efficiently recognize partial Matryoshka graphs (or even just Matryoshka graphs) and produce their corresponding Matryoshka drawing.


(*a*)

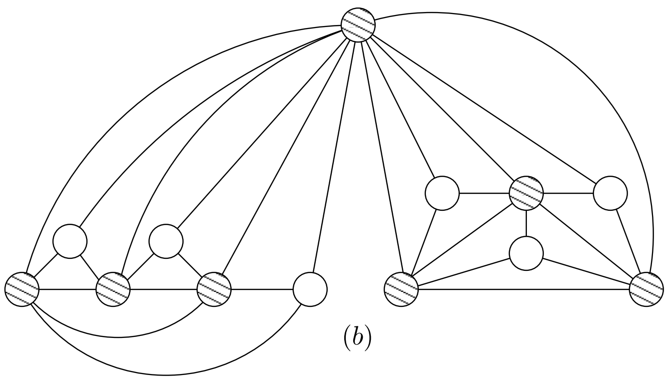

(*b*)

Figure 3.18: Partial Matryoshka Graphs are Not Closed Under Minors

Chapter 4

Decompositions of Bounded Treewidth

4.1 Background

Thus far we have been looking at planar graphs of bounded treewidth and, in particular, planar partial 3-trees. In Chapter 3 we also noted that partial Matryoshka graphs had a particular decomposition into graphs of bounded treewidth (see Lemma 3.7). In this chapter we shall explore the topic of graph (edge) decompositions into subgraphs of bounded treewidth for arbitrary planar graphs and graphs of bounded degree.

For conciseness, any graph that can be decomposed into graphs of treewidth $t_1, \ldots, t_i$ will be denoted as a *partial* $(t_1, \ldots, t_i)$*-tree*[1]. As we shall see in Chapter 5, graph decompositions of bounded treewidth have applications with regards to constructing approximation algorithms for $\mathcal{NP}$-hard problems. Although such algorithms serve no purpose for planar partial 3-trees (since $\mathcal{NP}$-hard problems can be solved on them in linear-time), it provides a useful approach for all planar graphs, which have unbounded treewidth in general.

Furthermore, several important conjectures and existing concepts relate to decompositions of bounded treewidth. For instance, the open Chartrand-Geller-Hedetniemi conjecture that every planar graph can be decomposed into two outerplanar graphs is related to treewidth by Theorem 1.7. Also, the concept of arboricity—the decomposition of a graph into the minimum number of forests—is connected to decompositions of bounded treewidth by Remark 1.10.

[1]This notation is inherited from El-Mallah and Colbourn's paper [42].

4.1.1 Decomposing Partial 3-Trees

It is known that a graph of treewidth $p+q$ is always a partial (p,q)-tree [31, 40]. Using this result repeatedly, we can see that a partial k-tree is always a partial $(k_1,\ldots,k_j)$-tree where $\sum_{i=1}^{j} k_i = k$. We begin by demonstrating that these decompositions are tight in general.

Remark 4.1 *Let G be a graph of treewidth k on n vertices, such that $n \geq k^2$. Then G is a partial (i,j)-tree if and only if $i+j \geq k$.*

Proof: A partial k-tree has at most $m = \binom{k}{2} + k(n-k) = kn - k^2/2 - k/2$ edges. If every partial k-tree is a partial (i,j)-tree, then we must have the property that $(i+j)n - (i^2+j^2)/2 - (i+j)/2 \geq kn - k^2/2 - k/2$ for all n. This implies that:

$$\begin{aligned}
(i+j)n - (i^2+j^2+i+j)/2 &\geq kn - k(k+1)/2 \\
(i+j)n - \frac{(i+j)(i+j+1)-2ij}{2} &\geq kn - k(k+1)/2 \\
k(k+1)/2 - \frac{(i+j)(i+j+1)-2ij}{2} &\geq (k-i-j)n
\end{aligned}$$

Letting $h = k-i-j \geq 0$ we conclude that:

$$\begin{aligned}
\frac{k(k+1)-(k-h)(k-h+1)+2ij}{2} &\geq hn \\
\frac{2hk-h^2+h+2ij}{2} &\geq hn \\
hk+ij &\geq hn
\end{aligned}$$

Since $k = h+i+j$ then $k^2 = hk+ik+jk > hk+ij$. Since k is a fixed constant, the last inequality can only hold for all n if $h=0$ (indeed, we only require that $n \geq k^2$ to eliminate the possibility that $h \geq 1$). $\square$

Having shown that this decomposition of partial k-trees cannot be tightened in the general case, we conclude that decomposing graphs of bounded treewidth into graphs of smaller treewidth is essentially a solved problem[2].

[2]However, the decomposition of a particular partial k-tree into two graphs of bounded treewidth while minimizing the sum of their treewidths is an interesting open problem for which no algorithm currently exists.

A direct consequence of this fact is that all partial 3-trees can be decomposed into an SP-graph and a forest. Thus, our edge-decomposition of a partial Matryoshka graph into an SP-graph and a forest (see Lemma 3.7) is a special case of this more general result. However, we shall now try to identify tighter decompositions for three subclasses of planar partial 3-trees: Halin graphs, IO-graphs, and partial Matryoshka graphs. In particular, we shall try to decompose these graphs into an outerplanar graph and a forest, recalling that outerplanar graphs are a strict subset of SP-graphs (by Theorems 1.7 and 1.8, or simply Theorem 1.12).

By its very definition (see page 9) any Halin graph can be decomposed into a tree and a cycle. An IO-graph's decomposition is not as obvious, but it can always be decomposed into a forest and an outerplanar graph:

Lemma 4.1 *Let G be an IO-graph. Then G can be decomposed into an outerplanar graph and a forest.*

Proof: Recalling the definition of an IO-graph, G has a planar embedding such that the vertices not on the outer-face form an independent set. Using this same planar embedding, we assign all edges in G not incident to the independent set vertices to set A. Using a cyclic labelling of the vertices on the outer-face (say, increasing in clockwise order), we define the range of an independent set vertex v to be the interval (l_v, h_v), where l_v is the lowest label vertex to which the independent set vertex v is adjacent, and h_v is the highest. Given these ranges for all independent set vertices, we obtain a tree-like hierarchy using the property of range enclosure (i.e. since the graph is planar, there can be no overlap of ranges, only enclosure and disjointness as with Nested SAT). Now for each vertex v in the independent set, we assign to set A its edge to the vertex h_v. Since the independent set vertices have degree one in edge set A and they are incident to vertices on the outer-face, then we can shift them to the outer-face without obscuring other vertices. Hence, it follows that the edges in set A induce an outerplanar graph. Thus, it only remains to show that $E(G) \setminus A$ is a forest.

We assume for contradiction that $E(G) \setminus A$ contains a cycle C. Since $E(G) \setminus A$ contains no edges between vertices on the outer-face, and G contains no edges between independent set vertices (by definition), then C must be a cycle of even length that alternates between vertices on the outer-face and vertices in the independent set. Let v be an independent set vertex in C such that v is not an ancestor of any other independent set vertex in C according to the range hierarchy on independent set vertices. Such a v must exist since enclosure is an acyclic property (i.e. the range hierarchy is a tree). Based on this, we can see that vertex v can only share l_v and/or h_v as common neighbours with other independent set vertices in cycle C. However, v is not adjacent to h_v in $E(G) \setminus A$, and thus l_v is a cut-vertex separating v from all other independent set vertices in C. This contradicts the fact that v is in a cycle C with these vertices, and hence we conclude that such a cycle cannot exist. Thus, $E(G) \setminus A$ is a forest. □

See Figure 4.1 for an example of such a decomposition. Finally, as mentioned in Chapter 3, finding a tighter edge-decomposition for partial Matryoshka graphs remains an open problem.

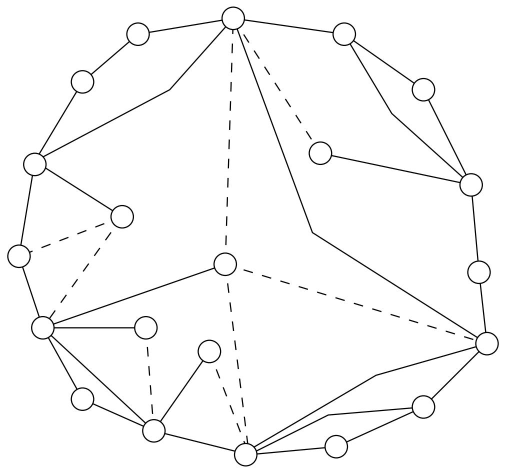


Figure 4.1: IO-Graph Decomposition into an Outerplanar Graph and a Forest

We shall now turn our attention to decomposing graphs of unbounded treewidth. To ensure that our results are not trivial expressions of Remark 4.1, we will only be considering graph classes where k-trees (and other partial k-trees containing most edges implied by its tree decomposition) are excluded for large enough k. In particular, planar graphs and regular graphs enforce a linear relationship between m and n, and hence the additive relationship for bounded treewidth decompositions does not apply to them.

4.1.2 Decomposing Planar Graphs into Graphs of Bounded Treewidth

Every graph has an edge-decomposition into forests. The arboricity of a graph G, denoted $\Upsilon(G)$, is the minimum number of forests required to express G in this fashion. By Remark 1.10 it follows that the arboricity of a graph indicates the smallest k such that G is a partial $\underbrace{(1,\ldots,1)}_{k}$-tree.

Nash-Williams proved that for any non-trivial graph G, $\Upsilon(G) = max_k\{\left\lceil \frac{q_k}{k-1} \right\rceil\}$, where q_k denotes the maximum number of edges in a subgraph of G containing k vertices [83, 84, 102]. By the Nash-Williams formula above and Remark 1.4, it is obvious that planar graphs have arboricity at most three, and that this bound is tight. In other words, planar graphs are a (strict) subset of partial $(1, 1, 1)$-trees but not a subset of partial $(1, 1)$-trees. Moreover, for this special case the decomposition can be found in linear time [36, 55, 96] and tightened with regards to the maximum degree of one forest [13].

We also note that since any graph of treewidth $p+q$ is a partial (p,q)-tree, then we can apply this result repeatedly to conclude that the arboricity of a graph is never greater than its treewidth:

Remark 4.2 *Let G be a graph of treewidth k. Then $\Upsilon(G) \leq k$.*

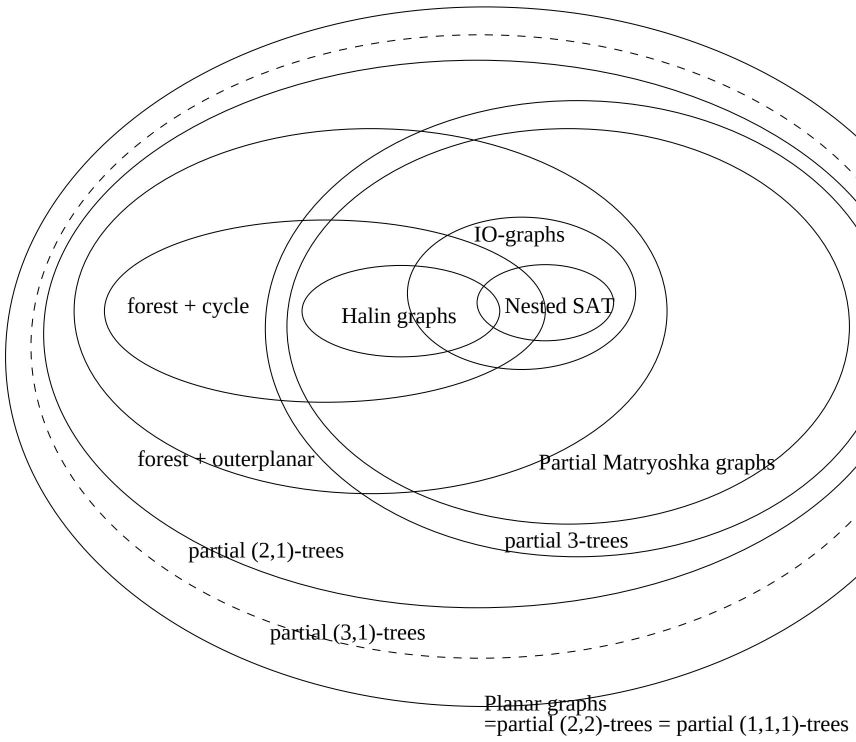


Figure 4.2: Planar Graph Class Hierarchy

Of course, more sophisticated results have been shown. For instance, it is known that planar graphs are partial $(2,2)$-trees [41, 73], and this result is also tight in terms of treewidth since planar graphs are not partial $(2,1)$-trees [42]. An earlier result by El-Mallah and Colbourn had shown that planar graphs are partial $(3,3)$-trees using IO-graphs to construct the decomposition [42]. We note that since outerplanar graphs have treewidth at most two, then all of these results are strongly related to the conjecture by Chartrand, Geller, and Hedetniemi that every planar graph can be decomposed into two outerplanar graphs [30]. Chartrand, Geller, and Hedetniemi had already shown that every planar graph had a vertex-decomposition into two outerplanar graphs when they made this conjecture, but it remains open today, despite claims to the opposite (e.g. [65][3]).

[3]Although I am unaware of the particular error in Heath's paper, websites such as

The Chartrand-Geller-Hedetniemi conjecture is trivial for planar graphs with a Hamiltonian cycle, since the cycle can be used to partition the graph's edges (inside and outside the cycle) into two outerplanar graphs. Since we only need to consider triangulated planar graphs, and since 4-connected planar graphs always have a Hamiltonian cycle (Theorem 1.1), then the only types of graph to consider are (non-Hamiltonian) 3-connected triangulated planar graphs. In any case, strengthening the result from two SP-graphs to two outerplanar graphs is rather difficult. Although outerplanar and SP-graphs both have at most $2n-3$ edges, the additional structural restrictions on outerplanar graphs is significant. In this thesis, we will investigate the different, but related, problem of finding partial $(k,1)$-tree decompositions.

4.2 Partial $(k,1)$-Tree Decompositions

4.2.1 Partial $(2,1)$-Trees and Partial $(3,1)$-Trees

From the infinite set of counter-examples given by El-Mallah and Colbourn [42], we know that planar graphs are not always partial $(2,1)$-trees. We will show that this set of graphs does not prove that planar graphs are not partial $(3,1)$-trees, but first we must recall some notation from page 34. Given triangulated planar graphs G and H with fixed planar embeddings, we use $G \leftarrow H$ to denote the graph obtained by substituting graph H into every face of graph G. Given this notation, El-Mallah and Colbourn's infinite set of graphs can be expressed as $\{H_G = (((G \leftarrow K_4) \leftarrow K_4) \leftarrow P_6)$, where G is any 5-connected triangulated planar graph$\}$. Recall from Theorem 1.12 and Figure 1.6 that K_4 is a forbidden minor of partial 2-trees and the octahedral graph P_6 is a forbidden minor of partial 3-trees. We shall now show that given a graph G that is a partial $(3,1)$-tree, H_G must also be a partial $(3,1)$-tree:

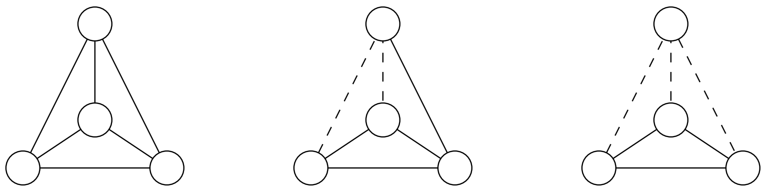


Figure 4.3: Extending G's Partial $(3,1)$-Tree Decomposition to $G \leftarrow K_4$

Lemma 4.2 *Let G be a triangulated planar partial $(3,1)$-tree, and let $H = (G \leftarrow K_4)$. Then H is a triangulated planar partial $(3,1)$-tree.*

http://www.emba.uvm.edu/~archdeac/problems/outplane.htm claim that the problem is unsolved. This claim is reinforced by the existence of later papers attempting to solve the conjecture, such as those by Kedlaya and Ding *et al.* [41, 73].

Proof: It is obvious that H will be planar and triangulated, so we need only argue that H is a partial $(3,1)$-tree. For each face into which K_4 is substituted, at most two of the edges on the boundary of this face can be in the forest of the partial $(3,1)$-tree decomposition for graph G (since all three edges would create a cycle). For each of these cases, choosing appropriate edges when extending the forest ensures that the new vertex in each face is simplicial in the subgraph forming the partial 3-tree and has degree at most three (see Figure 4.3, where dashed edges represent edges assigned to the forest). Thus, by Theorem 2.2, the result holds. □

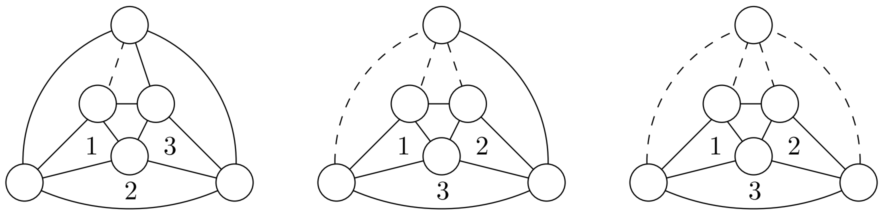


Figure 4.4: Extending G's Partial $(3,1)$-Tree Decomposition to $G \leftarrow P_6$

Lemma 4.3 *Let G be a triangulated planar partial $(3,1)$-tree, and let $H = (G \leftarrow P_6)$. Then H is a triangulated planar partial $(3,1)$-tree.*

Proof: Once again, it is obvious that H will be planar and triangulated, so we need only argue that H is a partial $(3,1)$-tree. We also know that at most two of the outer-face edges for P_6 will be in the forest. We shall select our forest edges for P_6 as shown in Figure 4.4 (where dashed edges represent forest edges). We can then remove the internal vertices of P_6 in the order shown (see Figure 4.4), by applying the almost simplicial rule to the first vertex, and the simplicial rule to remove the other two. Since every vertex is removed when it has degree at most three, then we can conclude from Theorems 2.2 and 2.3 that H is a partial $(3,1)$-tree. □

Theorem 4.1 *Let G be a triangulated planar partial $(3,1)$-tree. If we define graph H_G to be $(((G \leftarrow K_4) \leftarrow K_4) \leftarrow P_6)$, then H_G is also a partial $(3,1)$-tree.*

Proof: This result follows immediately from two applications of Lemma 4.2 and one application of Lemma 4.3. □

The icosahedral graph (the smallest 5-connected triangulated planar graph) can be decomposed into a forest and an outerplanar graph, as shown in Figure 4.5. Thus, by Theorem 1.7 the icosahedral graph is a partial $(2,1)$-tree, and hence a partial $(3,1)$-tree. Hence, we can conclude the following:

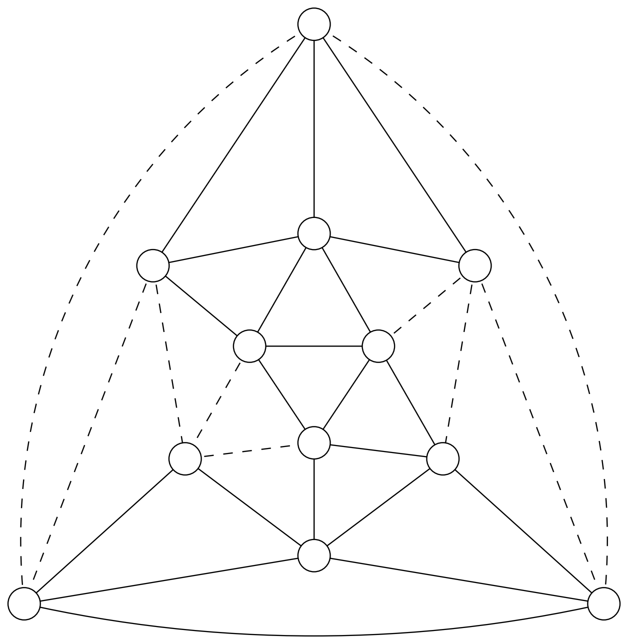


Figure 4.5: The Icosahedral Graph is a Partial $(2,1)$-Tree

Theorem 4.2 *Planar partial* $(2,1)$*-trees are a strict subset of planar partial* $(3,1)$*-trees.*

Proof: We have proven that the icosahedral graph is a partial $(3,1)$-tree, and thus by applying Theorem 4.1, we conclude that at least one of the (infinite set of) graphs given by El-Mallah and Colbourn [42] is a partial $(3,1)$-tree[4]. However, El-Mallah and Colbourn proved that these graphs are not partial $(2,1)$-trees. Thus, the result follows. $\square$

To summarize, we observe that planar graphs can be decomposed into three graphs of treewidth one, and into two graphs of treewidth two, but are not partial k-trees for any constant k. Thus the only remaining question is whether planar graphs are partial $(k,1)$-trees for any constant $k > 2$ (see Figure 4.2), which we will explore in the next section.

4.2.2 Hamiltonian Planar Graphs

In this section we show that Hamiltonian planar graphs can be decomposed into a forest and graph of outerplanarity $O(\log n)$. As a consequence, we conclude that Hamiltonian

[4]Indeed, we have shown that all of these graphs are partial $(3,1)$-trees unless there exists a 5-connected triangulated planar graph that is not a partial $(3,1)$-tree.

planar graphs are partial $(k, 1)$-trees where k is $O(\log n)$. Whether or not this outerplanarity result can be extended to all planar graphs remains an open question at this time. It also remains to be seen whether planar graphs are partial $(k, 1)$-trees for some constant $k \geq 3$.

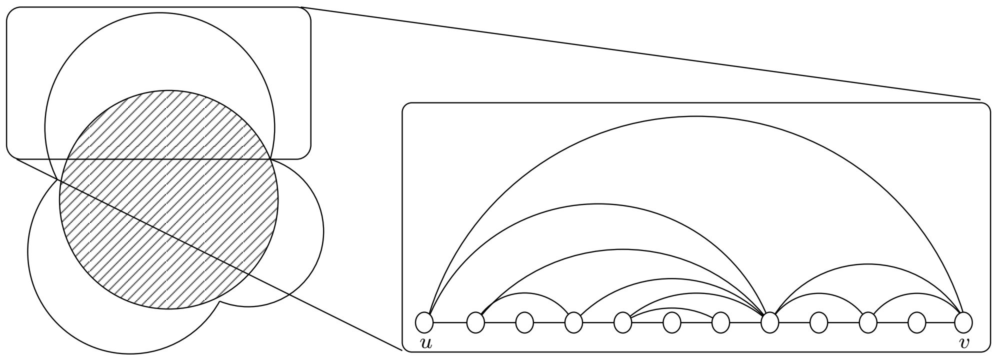


Figure 4.6: Outerplanar Sections of a Hamiltonian Graph

Since any decomposition of a graph into graphs of bounded treewidth must also hold for all its subgraphs (by Theorem 1.6), then without loss of generality we need only consider triangulated planar graphs. We begin this analysis by examining Hamiltonian triangulated planar graphs.

Let G denote our Hamiltonian triangulated planar graph, and let C denote its Hamiltonian cycle. Considering only edges on and outside C, we can envision this subgraph as being composed of three 2-connected outerplanar sections (see Figure 4.6)[5]. Note that the edges on and within the Hamiltonian cycle form an outerplanar graph, and hence a graph of treewidth at most two (by Theorem 1.7). In order to bound the outerplanarity of this graph minus a forest, we will only select forest edges from the outerplanar sections outside of the Hamiltonian cycle. The goal is to assign edges to the forest such that all remaining edges in these outerplanar sections add at most $O(\log n)$ outerplanarity to the graph inside of the Hamiltonian cycle.

We begin by introducing an algorithm that decomposes an outerplanar graph (minus one edge) into two forests with certain properties. The general idea of Algorithm 4 is to eliminate the edges that increase outerplanarity for the vertices furthest from the outer-face. In order to do this, we consider the dual tree of the outerplanar section minus the outer-face (recall Remark 1.7). As we shall see, the height of the dual tree relates to the outerplanarity of the graph, and hence at each stage we simply select the edges to the child subtree of greater height. An example of the decomposition produced by this algorithm, along with the dual tree guiding this decomposition, appears in Figure 4.7.

[5]Since we can select the outer-face of our planar embedding, we can guarantee that an edge on the Hamiltonian path will appear on the outer-face. In this manner we can reduce the number of outerplanar sections to two instead of three. However, this change does not simplify our proof or improve our results.

Algorithm 4: Decomposing a 2-Connected Outerplanar Graph Minus One Outer-Face Edge

```
Input: Internally-triangulated 2-connected outerplanar graph $G$ with a fixed
       planar embedding, and two vertices $u$ and $v$, adjacent on the outer-face
Output: A decomposition of $G - (u, v)$ into two forests such that neither forest
        has a path from $u$ to $v$

Let $T$ be the dual of $G$
Split the outer-face vertex into $n$ pairwise non-adjacent vertices, one per edge
incident to the outer-face
Delete from $T$ the ($n$th) outer-face vertex incident to the dual of edge $(u, v)$
Let $w$ denote the unique vertex in $G$ such that $u, v, w$ is an internal face of $G$
Root the tree $T$ at the face $u, v, w$
Starting at the root and traversing the internal (non-leaf) vertices of tree $T$, we do
the following at each vertex:
Let $p$ be the current vertex, $c_l$ its left child, $c_r$ its right child
if height of $c_r$'s subtree $c_r >$ height of the $c_l$'s subtree then
    Colour $(p, c_l)$ blue and $(p, c_r)$ red
else
    Colour $(p, c_l)$ red and $(p, c_r)$ blue
Transfer all colours from $T$'s edges to $G$'s edges
Return $R$ and $B$, the set of red and blue edges in $G$, respectively
```

In greater detail, since the dual of an outerplanar graph minus the outer-face is a tree, and since we split the outer-face into n non-adjacent vertices, then the T in Algorithm 4 is also a tree (where each leaf is one of the pairwise non-adjacent outer-face vertices). Also, every edge in the outerplanar section except for (u, v) has a unique dual edge in T, and thus the edge colours can be transferred uniquely from T to G. Since the outerplanar graph G is internally-triangulated, each vertex in the tree T has exactly two children and one parent except for the root and the $n - 1$ leaves (outer-face vertices). The root face is not adjacent to the outer-face (since the dual of edge (u, v) is deleted), and thus it has two children and no parent. Each of the $n - 1$ outer-face vertices is obviously only adjacent to its parent. Finally, we note that the order in which vertices are traversed in the tree makes no difference, as subtree height is not affected by the colouring of the edges.

We can now prove that the desired properties of Algorithm 4 do indeed hold:

Lemma 4.4 *Algorithm 4 assigns a unique colour to every edge in G except for (u, v).*

Proof: Every edge in G has a dual edge in T except for (u, v), and these edges are in one-to-one correspondence. Since each non-leaf vertex in T has exactly two children, and

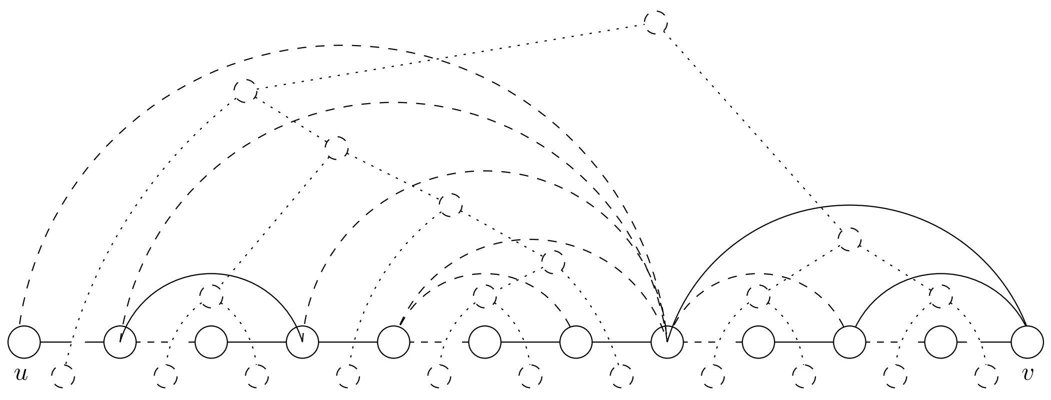


Figure 4.7: Dual Tree and Resulting Decomposition by Algorithm 4 of Figure 4.6

Algorithm 4 visits each non-leaf vertex once, then each edge to a child is coloured exactly once (when its parental vertex in T is visited). $\square$

Lemma 4.5 *Algorithm 4 returns a decomposition of $G - (u, v)$ such that there is no single-colour path from u to v.*

Proof: Proof by mathematical induction on n: If G has three vertices, then edge (u, v) does not appear in the decomposition, while the two remaining edges are assigned opposite colours. Hence the claim holds for $n \leq 3$. Assume the claim holds for all $n < k, k \geq 4$ and consider the case where $n = k$. There exists a unique vertex w such that u, v, w is an internal face of G. Hence w is a cut-vertex in $G - (u, v)$, and any path from u to v must pass through w. However, by our induction hypothesis, the only single-colour path from u to w is the edge (u, w), since the left child subtree is decomposed in the same manner as the entire tree. The same reasoning shows that the only single-colour path from w to v is the edge (w, v). However, since (u, w) and (w, v) are assigned different colours, then no single-colour path can exist from u to v. Thus the result holds for $n = k$. By the principle of mathematical induction, the statement is true for all n. $\square$

Lemma 4.6 *Algorithm 4 returns a decomposition of $G - (u, v)$ into two forests.*

Proof: Assume for contradiction that there is a cycle of some colour. Let $e = (x, y)$ be the top-most edge in the cycle according to heights in the tree T. Then there is a single-colour path from x to y in $G - (u, v) - e$. However, this implies that the subgraph rooted at edge e was coloured by Algorithm 4 such that it had a single-colour path between its two endpoints (not including the direct edge). However, this contradicts the results of Lemma 4.5. Thus no such cycle can exist. $\square$

Theorem 4.3 *Let G be an internally-triangulated 2-connected outerplanar graph with a fixed planar embedding, and let u, v be vertices adjacent on the outer-face of G. If we delete the edge (u, v), then the resulting graph can be decomposed into two forests, such that neither forest has a path from u to v.*

Proof: Follows immediately from Lemmas 4.5 and 4.6. □

Now we will show that using Algorithm 4 to decompose the three outerplanar sections of a Hamiltonian planar graph will result in a decomposition of Hamiltonian planar graphs into a forest and a graph of treewidth $O(\log n)$. The forest edges will be those marked as red, recalling that red edges were those to the subtree of greater height.

Theorem 4.4 *Let G be a triangulated planar graph with Hamiltonian cycle C. Assume Algorithm 4 is used to decompose each of G's three outerplanar sections (see Figure 4.6), and returns the edges sets R_1 and B_1, R_2 and B_2, and R_3 and B_3, respectively. Then $G \setminus (R_1 \cup R_2 \cup R_3)$ has outerplanarity at most $\log_2(n) + 2$.*

Proof: We note that for any graph, the radius of its dual plus one is an upper-bound on the outerplanarity of the original graph (i.e. the face corresponding to the dual vertex of minimum eccentricity can be assigned as the outer-face). Thus, we need only consider a single outerplanar section on n vertices, and show that its dual graph has radius at most $\log_2 n + 1$. We do this by showing that the height of an outerplanar section's dual tree (omitting the outer-face of the outerplanar section) T is at most $\log_2 n$. Since the dual of edge deletion is edge contraction, and since the forest of red edges is being deleted from graph G, then essentially Algorithm 4 contracts the edges to the child subtree of greatest height for every non-leaf vertex in T. Thus, we need only show that in a rooted tree with ≤ 2 children per node, contracting the edge to the child subtree of greatest height for every non-leaf vertex results in a tree with at most $\log_2 n$ height. We prove this by induction:

A tree on one node has height zero, and this equals $\log_2 1$, so the statement holds for $n = 1$. Assume tree T has $n = k$ nodes, $k \geq 2$, and that the statement holds for all $n < k$. If the root has a single child, then the edge connecting the root to that child will be contracted, and so the height of tree T is the height of the subtree, which is at most $\log_2(k-1) < \log_2(k)$ by the induction hypothesis. Thus the statement holds in this case. If tree T has two children whose subtrees have different heights, then we contract the edge to the higher sibling, and the height of tree T will be the height of this child's subtree, which is at most $\log_2(k-2)$ by the induction hypothesis. Again, since $\log_2(k-2) < \log_2(k)$, the statement holds. Lastly, if the root of tree T has two child subtrees of equal height, the height of tree T will be one greater than the height of its subtrees, no matter which edge is contracted. We note that the child subtree with the fewest number of vertices has at most $\frac{k-1}{2}$ vertices, indicating that it has height at most

$\log_2(\frac{k-1}{2}) = \log_2(k-1) - 1$. Since the two child subtrees have equal height, it follows that they both have height at most $\log_2(k-1) - 1$. Thus the height of tree T (after contraction) will be at most $\log_2(k-1) < \log_2(k)$. Thus, the statement holds for all cases and, by the principle of mathematical induction, for all n.

Hence, we conclude that Algorithm 4 decomposes each outerplanar section such that its dual has radius at most $\log_2 n + 1$. In other words, each outerplanar section will have outerplanarity at most $\log_2 n + 2$, as desired. □

We note that the concept of a compressed tree with $O(\log n)$ height in Harel and Tarjan's paper [62] is quite similar, although not identical, to our contracted tree above. Similar ideas have also appeared in papers on set union [99] and algorithms relating to nearest common ancestor [1].

Now, having proved such a decomposition exists for planar graphs containing a Hamiltonian cycle, the question remains as to how efficiently we can find this decomposition. The dual of a planar graph can be computed in $O(n + m)$ time, and given the Hamiltonian cycle, we can identify the outerplanar sections in linear time. Thus we can also compute a dual tree for each outerplanar section in linear-time. For each of these (at most three) outerplanar sections, we can compute the height of each node in linear-time using a simple postorder traversal. Thus we can identify which dual edges to contract in linear-time, and this corresponds to identifying the decomposition of the original graph in linear-time. Hence, we can conclude the following:

Theorem 4.5 *Given a triangulated planar graph G and a Hamiltonian cycle C in G, we can decompose G into a partial $(k, 1)$-tree with $k \in O(\log n)$ in linear time.*

Finally, although finding a Hamiltonian cycle on planar graphs is $\mathcal{NP}$-complete on 3-connected graphs, a linear-time algorithm exists when we restrict ourselves to 4-connected planar graphs ([33, 50], recall Theorem 1.1). We can also triangulate a planar graph in linear time. Thus, we obtain the following result as well:

Theorem 4.6 *Given a planar 4-connected graph G, we can decompose G into a partial $(k, 1)$-tree with $k \in O(\log n)$ in linear time.*

The obvious next step is to see whether or not this result extends to all 3-connected triangulated planar graphs, as this would prove the existence of such a decomposition for all planar graphs. However, as we shall see, this result cannot easily be extended to such graphs.

If we have a 3-connected triangulated planar graph G, we can construct a tree of 4-connected components to represent this graph, with nodes representing 4-connected components and edges connecting 4-connected components with a 3-separating set in common

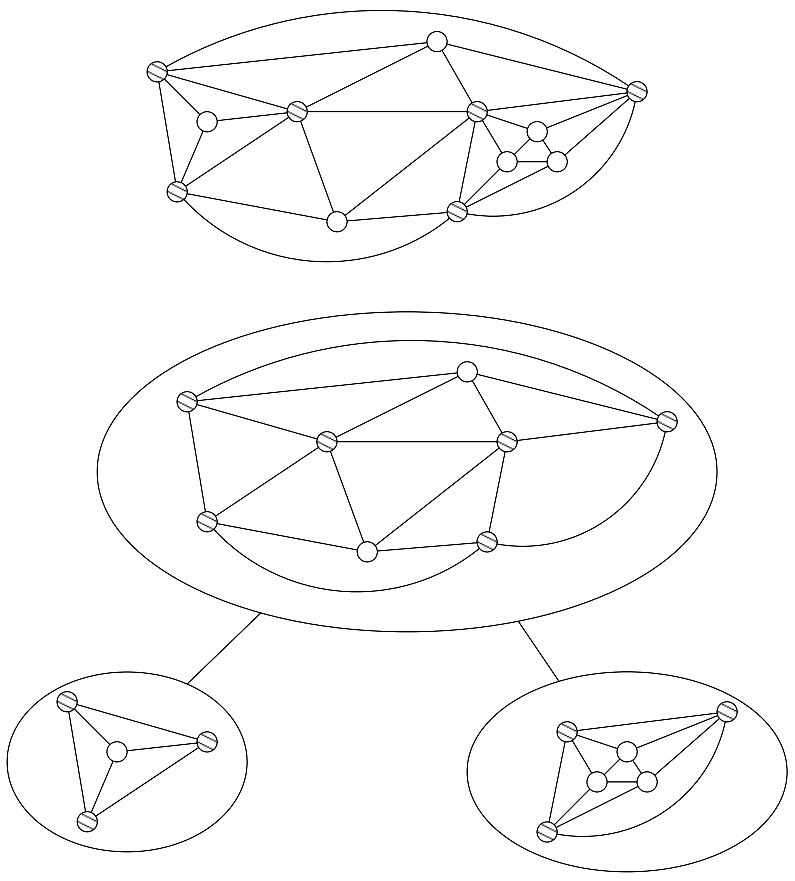


Figure 4.8: 3-Connected Graph and its Tree of 4-Connected Components

(see Figure 4.8). Using the decomposition method for Hamiltonian planar graphs we can derive a treewidth decomposition for each 4-connected component, but the problem is in combining these decompositions. Since graph G is triangulated, the 3-separating set must form a triangle (i.e. a cycle of length three), and thus one of the questions is whether or not the 4-connected components containing this triangle agree on its decomposition.

If we root this tree of 4-connected components, then we can have each parent component impose its decomposition onto its children. Since each 4-connected component will have to alter its bounded treewidth decomposition for a single parent, and since this alteration affects a constant amount of edges (since a single triangle is being imposed), then the increase in treewidth is at most a constant for each 4-connected component. In other words, we can identify a forest in G such that each 4-connected component within G is decomposed into a partial $(k, 1)$-tree for some $k \in O(\log n)$.

Unfortunately, combining the tree decompositions for each 4-connected component's partial k-tree is problematic. If zero or two of the triangle's edges are assigned to the partial 1-tree, then the separating triangle is a clique of size three or one, respectively. In this case, a simple clique sum combines the tree decompositions without increasing treewidth. However, if exactly one of the separating triangle's edges is assigned to the forest, then there is no guarantee that these tree decompositions can be merged without increasing treewidth. Furthermore, this problematic case cannot be avoided within every separating triangle without increasing the treewidth of these decompositions. Thus, our decomposition result is limited (for now) to 4-connected and Hamiltonian planar graphs.

4.2.3 Consequences

In conclusion, although the problem of decomposing planar graphs into a forest and partial k-tree for some constant k remains open, we give a decomposition for Hamiltonian planar graphs that bounds outerplanarity to $O(\log n)$. Consequently, we can solve most $\mathcal{NP}$-hard problems on the partial k-tree part of this decomposition in $O(2^k n) = O(n^2)$ time, which can help in the construction of approximation algorithms for this graph class, as we shall see in Chapter 5.

4.3 Restricted Decompositions of Bounded Treewidth

We now explore restricted decompositions of graphs into partial $(k, 1)$-trees. In particular, we add the restriction that the forest has maximum degree one. In other words, it is a matching (i.e. a set of pairwise non-adjacent edges, see page 3). We extend our current notation by denoting matchings as partial 1/2-trees.

4.3.1 Cubic Graphs are Partial $(k, 1/2)$-trees

We begin by proving that cubic graphs can be decomposed into a matching and graph of treewidth at most two. We begin with the following well-known theorem:

Theorem 4.7 *Petersen's Theorem: Every cubic bridgeless graph has a perfect matching [86].*

Thus, for cubic bridgeless graphs, the problem is essentially solved:

Lemma 4.7 *A graph with maximum degree two has treewidth at most two.*

Proof: A connected graph with maximum degree two is either a path or a cycle. The former has treewidth one, while the latter has treewidth two. Thus the result follows for all such graphs. □

Theorem 4.8 *Every cubic bridgeless graph is a partial* $(2, 1/2)$*-tree.*

Proof: Let G be a cubic bridgeless graph. By Theorem 4.7, G has a perfect matching M. Since every vertex in G is covered by M, then $G - M$ is a graph of maximum degree two. The result follows from Lemma 4.7. □

In order to extend the result to all cubic graphs, we first make the following observations:

Lemma 4.8 *Let G be a graph containing one or more cycles (i.e. not a forest). Let H be the graph obtained by deleting all bridges in G. Then $tw(H) = tw(G)$.*

Proof: Adding an edge between two separate components cannot increase treewidth, unless neither component had an edge (since this is a clique sum). Thus, so long as some component of H contains an edge, the result will follow. However, this must be true since G had a cycle. □

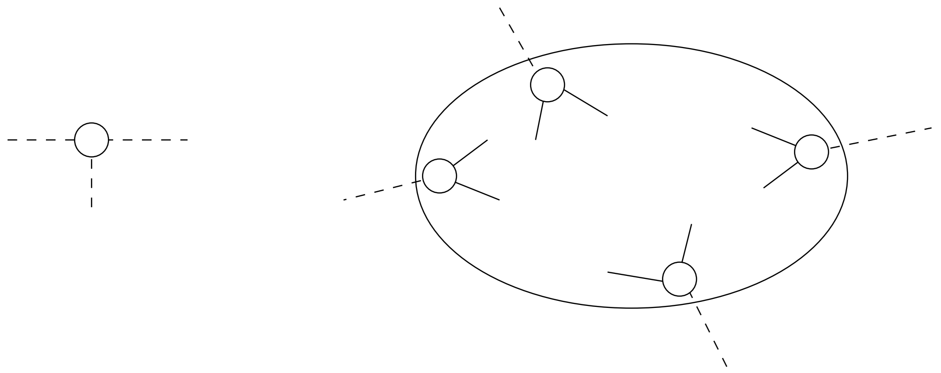


Figure 4.9: Components after Bridge Deletion

Thus our algorithm for decomposing cubic graphs into a matching and partial 2-tree begins with the deletion of all bridges. We note that any vertex incident to two bridges must be incident to three (if any incident edge is on a cycle, at least two of them must be). Thus the resulting components are either isolated vertices or components containing vertices of degree three and two (see Figure 4.9).

Isolated vertices obviously have treewidth less than two, so we only need consider components containing cycles. Let C be such a component, and let d represent the number of vertices of degree two in C. We shall now argue that for every $d > 0$, we can construct a matching that covers all vertices of degree three. Thus, by Lemma 4.7, such components are partial $(2, 1/2)$-trees.

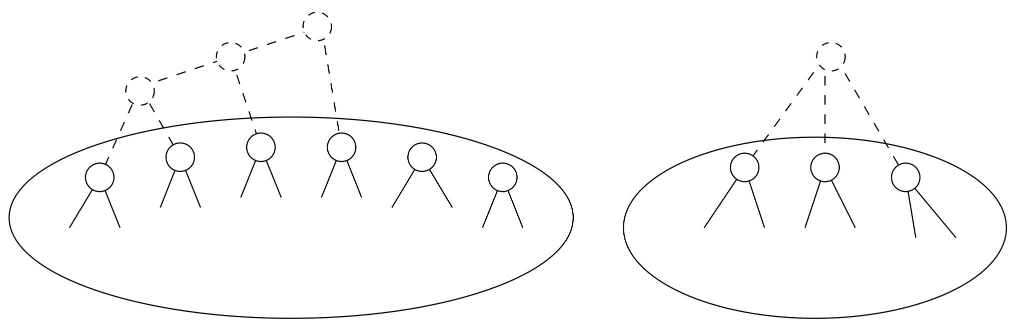


Figure 4.10: Components with Three or More Vertices of Degree Two

Lemma 4.9 *Let G be a connected graph with $d \geq 3$ vertices of degree two, and $n - d$ vertices of degree three. Then there exists a matching that covers all vertices of degree three.*

Proof: Since $d \geq 3$, we can add $d - 2$ new cubic vertices to graph G in order to create a cubic bridgeless graph H (see Figure 4.10). Having added such vertices and edges, we can find a perfect matching M_H for H, and then identify the subset of that matching contained within the original graph G (i.e. $M_G = \{e : e \in M_H \cap E(G)\}$). M_G must cover all vertices of degree three in G, since all vertices in G are covered by M_H, and no vertex of degree three in G is incident to any edges in $E(H) \setminus E(G)$. Hence we conclude that $G - M_G$ is a graph of maximum degree two. □

Lemma 4.10 *Let G be a connected graph with $d = 2$ vertices of degree two, and all other vertices of degree three. Then there exists a matching that covers all vertices of degree three.*

Proof: There exists a stronger version of Petersen's Theorem (Theorem 4.7) such that cubic bridgeless *multigraphs* (that is, graphs where E is a multi-set rather than a set) have perfect matchings (refer to [20] for details and applications). Thus, we can simply connect the two vertices of degree two with a new edge, and find a perfect matching M_H for this new graph H. Once again, we can obtain the subset of M_H on graph G, which we denote as M_G. Since only the two vertices of degree two may no longer be matched, then we conclude that $G - M_G$ has maximum degree two (see Figure 4.11). □

For this last proof we require the concept of a tree of 2-connected components. Recall from page 62 that a tree of k-connected components is formed by representing each k-connected component of the graph as a node, and connecting two nodes if and only if their respective components have some vertices in common (i.e. a separating set in the original graph of size at most $k - 1$). Thus, for a tree of 2-connected components, each

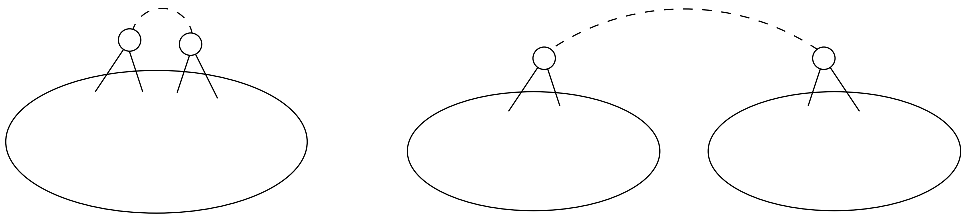


Figure 4.11: Components with One or Two Vertices of Degree Two

2-connected component of a graph would be represented by a node, and two nodes would be adjacent if and only if they shared a common cut-vertex. We can now proceed with the proof.

Lemma 4.11 *Let G be a connected graph with a single vertex v of degree two, and all other vertices of degree three. Then there exists a matching that covers all vertices of degree three.*

Proof: There exists a stronger version of Petersen's Theorem (Theorem 4.7) which states that cubic graphs have a perfect matching if their tree of 2-connected components forms a path [20]. Thus, if we create a copy of graph G, and connect v to its copy, we create a cubic graph H with a single bridge (see Figure 4.11). In other words, the tree of 2-connected components is K_2, which is certainly a path. Hence there exists a perfect matching M_H on this new graph H, and we can identify the subset M_G of this matching on graph G. Since only vertex v is incident to edges in $E(H) \setminus E(G)$, then it follows that every degree three vertex in graph G is covered by M_G. Hence, the result follows. □

We can now apply these Lemmas to conclude the following:

Theorem 4.9 *Let G be a graph of maximum degree three. Then G is a partial $(2, 1/2)$-tree.*

Proof: The case where G is cubic is handled by Theorem 4.8. Thus, without loss of generality we assume that G has one or more bridges. Using Lemma 4.8, we delete all bridges without affecting treewidth. Each resulting component is either an isolated vertex or a component with $d > 0$ vertices of degree two and all other vertices of degree three. In the second case, we can apply Lemmas 4.9, 4.10, and 4.11 accordingly in order to obtain a matching for each component that covers all vertices of degree three. It follows from Lemma 4.7 that each component is a partial $(2, 1/2)$-tree or an isolated vertex (i.e. of treewidth zero). Since every component is a partial $(2, 1/2)$-tree or isolated vertex, assigning the bridges of G to the partial 2-tree does not increase the treewidth, by Remark 1.9, since it is a clique sum of size one. Thus, we conclude that G must be a partial $(2, 1/2)$-tree. □

We note that if our partial $(2, 1/2)$-tree decomposition of a graph results in a partial 2-tree with more than one component, then edges should be transferred from the matching back to the partial 2-tree until it is connected (i.e. without increasing treewidth, by Remark 1.9). This reduces the number of edges in the matching, which will be beneficial for our algorithms, as we shall see in Chapter 5.

4.3.2 Maximum Degree Three is Tight

We now prove that planar graphs of higher degree are not partial $(k, 1/2)$-trees for any fixed k. We do so by constructing a planar graph that will contain the $k \times k$ rectangular grid graph as a minor no matter how one selects the matching. Since this grid graph has treewidth k and we can construct such a planar graph for any fixed k, the result immediately follows. Indeed, our result holds for planar graphs of maximum degree four, thus proving that maximum degree three is tight.

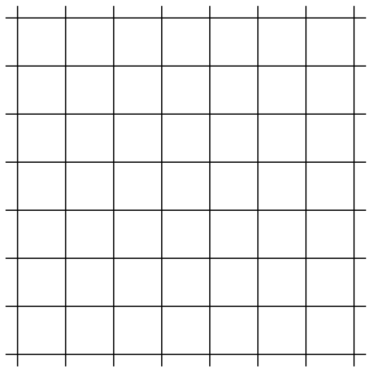


Figure 4.12: 8×8 Rectangular Grid Graph

Let D_k represent the graph constructed by replacing each edge in a $k \times k$ rectangular grid graph with two triangles (forming a diamond shape, see Figure 4.13). The resulting graph is certainly planar and, as we shall argue, must contain the original grid graph as a minor, no matter how one selects the matching.

Lemma 4.12 *Let k be any positive integer. Then D_{k+1} is not a partial $(k, 1/2)$-tree.*

Proof: For each cycle of length three (i.e. triangle) in D_{k+1}, at most one edge on the cycle can be in the matching. Hence, the vertices of a triangle remain in the same component after the deletion of any matching. The diamond shape (see Figure 4.13) consists of two triangles, and hence the two original endpoints (the two vertices not common to both triangles) are connected by a path using only diamond vertices and edges, even after the deletion of any matching[6]. By contracting this path (and deleting any unnecessary edges in the diamond shape), we obtain a single edge joining the two endpoints of the diamond shape. By doing this for every diamond in the graph, we obtain the $(k+1) \times (k+1)$ rectangular grid graph as a minor. In other words, deleting any matching from D_{k+1} results in a graph of treewidth $\geq k+1$ by Theorem 1.6. $\square$

Corollary 4.1 *Planar graphs are not partial $(k, 1/2)$-trees for any constant k.*

Proof: We need only show that D_k is planar for any integer $k > 0$. Since the rectangular grid graph is planar, and substituting the diamond shape for each edge does not make the graph non-planar, the result follows. $\square$

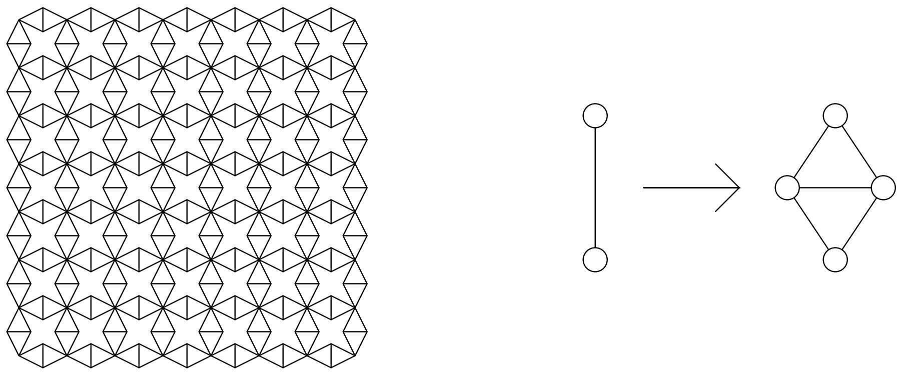


Figure 4.13: D_8 and the Diamond Formation

Since we know that planar graphs in general are not partial $(k, 1/2)$-trees yet cubic planar graphs are, the question arises as to where the cutoff is in terms of maximum degree. We shall now show that planar graphs with maximum degree ≥ 4 are not always decomposable into a matching and partial k-tree for any fixed k. In other words, the result on cubic graphs is tight.

Lemma 4.13 *Planar graphs with maximum degree at least four are not partial $(k, 1/2)$-trees for any constant $k > 0$.*

Proof: We employ the same argument as before, but we use a slightly different substitution. More specifically, each vertex is replaced by two triangles with a single vertex

[6]We can see here that a single triangle would have sufficed. However, we chose the diamond shape for symmetry.

in common, which we denote as an inverted diamond, and each edge is replaced with a diamond (see Figure 4.14). Note that each diamond is attached to a distinct vertex (not common to both triangles) on the inverted diamond. Once again, it is clear that this substitution does not make the graph non-planar, and the use of triangles ensures that the deletion of any matching cannot eliminate the path between two vertices that were adjacent in the original grid graph. Hence, once again we can construct a graph that contains the $(k+1)\times(k+1)$ rectangular grid graph as a minor (for any constant k) after the deletion of any matching. Finally, this modified substitution produces a graph with maximum degree four. Thus the result follows. □

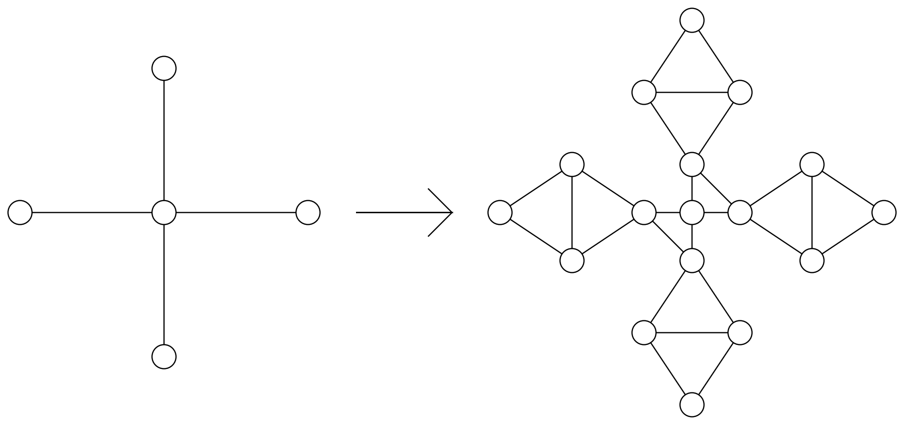


Figure 4.14: Modified Grid Graph of Maximum Degree Four

The purpose of using matchings was to bound the maximum degree of the forest. Although this technique worked on cubic graphs (planar and non-planar), it could not be extended to graphs of higher degree, even when restricted to planar ones. If we loosen this bound to two or some higher constant (e.g. if the forest is a set of vertex-disjoint paths instead of a matching), can we then decompose graphs of higher maximum degree (planar or otherwise)? Perhaps Eulerian paths could help construct appropriate decompositions for quartic graphs, but this remains an open question at this time. However, this bound on the maximum degree of the forest is desirable since it helps in the development of algorithms using these decompositions, as we shall see in Chapter 5.

4.4 Decompositions into a Bipartite Graph and Forest

It has been proven that planar graphs can be decomposed into two bipartite graphs [61, 59, 80]. One question that arose in our research is whether planar graphs can be decomposed into a bipartite graph and a forest (note that a forest is bipartite since it contains no (odd) cycles, by Remark 1.3). This question does not relate directly to our decompositions of bounded treewidth, since bipartite planar graphs can have unbounded treewidth (e.g. rectangular grid graphs). However, such a proof would be a strengthening of the Four Colour Theorem [59, 61, 80] and can also be used for approximation algorithms (see Chapter 5). Although the result was unpublished, Seymour answered this question in the negative in private discussions with Rick Mabry [81]. However, the result does hold for some types of planar graphs, which we explore below. We begin by exploring graphs of bounded treewidth, and we find that our first result does not require the graph to be planar:

Theorem 4.10 *Let G be a partial 3-tree. Then G can be decomposed into a bipartite graph and forest.*

Proof: Since G is a partial 3-tree, then it is the subgraph of a 3-tree, say H. Let $v_1, \ldots, v_n$ be the perfect elimination order of H. Then we can greedily 4-colour H using the perfect elimination order. Because the predecessors of a vertex form a clique, then each vertex's predecessors are assigned a unique colour. We can use this 4-vertex-colouring to assign the edges of H to three different sets (i.e. set A contains edges in $E(H)$ whose endpoints are colours 1 and 2 or colours 3 and 4, set B contains edges whose endpoints are colours 1 and 3 or 2 and 4, and set C contains all other edges in $E(H)$). We claim that $H[A]$ is a forest and $H[B \cup C]$ is a bipartite graph. The second half of this claim follows immediately from the definition of these edge sets since $H[B \cup C]$ does not contain any edges between vertices of colour 1 and 2, nor edges between vertices of colour 3 and 4. Thus we only need to prove the claim that $H[A]$ is a forest. Assume (for contradiction) that a cycle exists. Then choose the last vertex in the cycle according to the perfect elimination order. This vertex must have two predecessors of the same colour, contradicting the properties of our vertex colouring. Hence the assumption cannot hold, and $H[A]$ is a forest. Since H can be decomposed into a bipartite graph and forest, the same holds for its subgraph G. $\square$

Restricting the above result to planar partial 3-trees cannot make it false, thus we conclude the following:

Corollary 4.2 *Let G be a planar graph of treewidth at most three. Then G can be decomposed into a bipartite graph and a forest.*

For instance, applying the above method to the partial 3-tree from Figure 1.1, we obtain the perfect elimination ordering $v_{11}, v_5, v_3, v_7, v_{12}, v_{10}, v_1, v_2, v_4, v_9, v_8, v_6, v_{13}, v_{14}, v_{15}$, which results in the following 4-vertex-colouring:

$$(\{v_1, v_9, v_{11}, v_{13}, v_{15}\}, \{v_2, v_5, v_{10}, v_{14}\}, \{v_3, v_6, v_8\}, \{v_4, v_7, v_{12}\}).$$

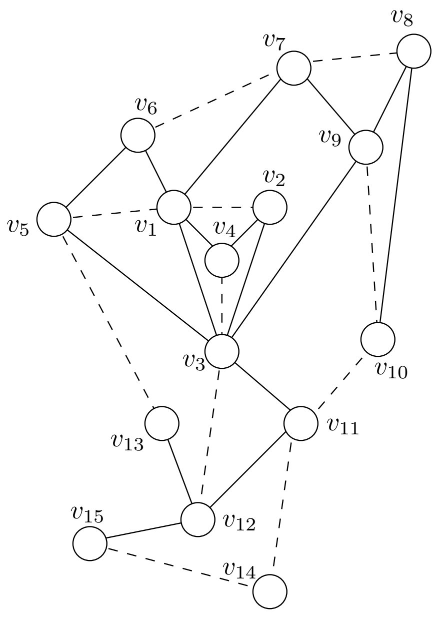


Figure 4.15: Partial 3-Tree Bipartite-Forest Decomposition

Thus we obtain a decomposition of this partial 3-tree into a bipartite graph and forest, as shown in Figure 4.15.

However, we wish to construct decompositions enabling efficient algorithms for $\mathcal{NP}$-hard problems and, as previously mentioned, such algorithms already exist for graphs of bounded treewidth. Thus Theorem 4.10 cannot be used to construct more efficient algorithms, although it does highlight the following piece of information:

Lemma 4.14 *A graph G can be decomposed into a bipartite graph and a forest if and only if it has a 4-vertex-colouring (using colours $1, 2, 3,$ and 4) such that there are no (even) cycles containing only colours 1 and 2 or only colours 3 and 4.*

Proof: $\Rightarrow$ Assume graph G is decomposed into a bipartite graph B and a forest F. Then we can 2-vertex-colour both B and F, using colours a, b and c, d respectively. We can then construct a 4-vertex-colouring for G using the cross-product of these two colourings (i.e. a vertex coloured with colour a in B and colour c in F is now assigned the colour (a, c), and so on). We note that if there are any 2-coloured cycles in G, they must be of even length. Furthermore, the colours (a, c) and (a, d) cannot form a 2-coloured cycle, since the alternation of colours implies that every edge in the cycle is in the forest F, which is a contradiction. Similarly, there cannot be any 2-coloured cycles on (b, c) and (b, d). Renaming colours $(a, c), (a, d), (b, c), (b, d)$ with colours $1, 2, 3, 4$ respectively results in the desired conclusion.

$\Leftarrow$ Assume G has a 4-vertex-colouring such that there are no 2-coloured cycles on colours 1 and 2 and colours 3 and 4. Then we assign all $1-2$ and $3-4$ edges to set F, and all other edges to set B. F is clearly a forest by definition, and partitioning the vertices of G into those coloured 1 and 2 and those coloured 3 and 4 makes it apparent that B is bipartite. Hence G can be decomposed into a bipartite graph and a forest. $\square$

Based on this result, it seems likely that 3-colourable planar graphs can be decomposed into a bipartite graph plus forest. Indeed, to prove this result we need only introduce colour 4 to re-colour some vertices of colour 1 and 2 such that:

1. all cycles of colour 1 and 2 are eliminated, and
2. no cycles of colour 3 and 4 are introduced

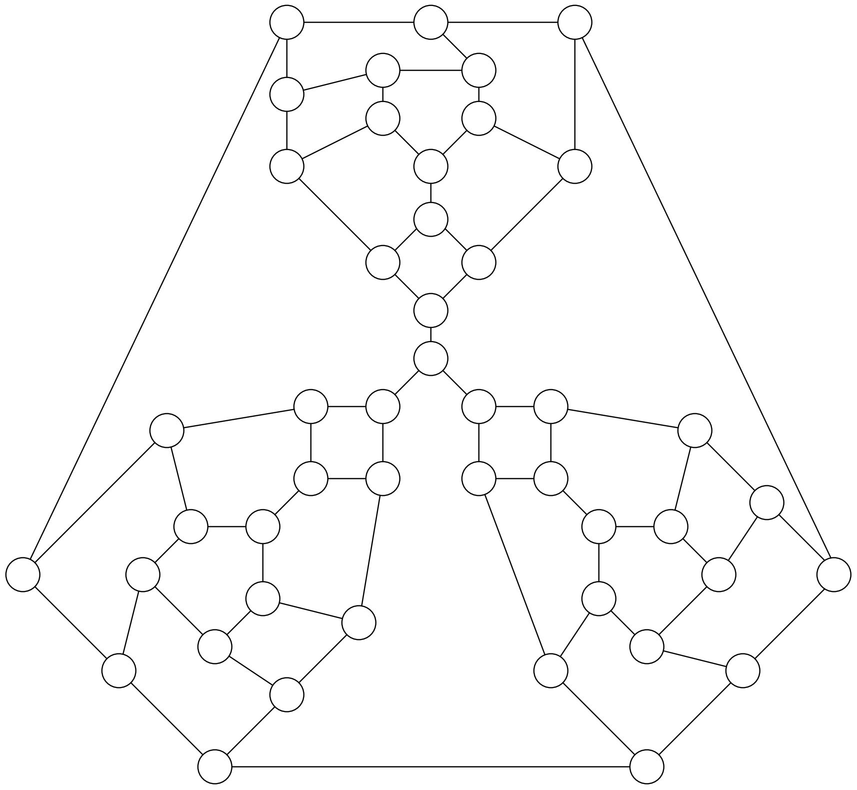


Figure 4.16: The Tutte Graph

We note that a consequence of Seymour's proof is that such a decomposition cannot exist for all Hamiltonian planar graphs, an example being the dual of the Tutte graph [81].

Other interesting questions include whether or not treewidth three is tight and, if the statement does not hold for all 3-colourable planar graphs, whether it holds for triangle-free planar graphs (a proper subset of 3-colourable planar graphs [56]).

Chapter 5

Algorithms using Bounded Treewidth Decompositions

In this chapter we discuss an application of bounded treewidth decompositions: namely, the development of approximation algorithms for $\mathcal{NP}$-hard problems on graph classes of unbounded treewidth. We provide an overview of the approximation classes and the status of certain graph-theoretic problems, and review the contributions of existing algorithms that use bounded treewidth decompositions. Finally, we introduce some basic algorithms that use the decompositions we have been developing.

5.1 Approximation Complexity Classes

The question as to whether $\mathcal{P} = \mathcal{NP}$ is still the most important open question facing computer scientists today. Given the extreme difficulty, and likely impossibility, of finding a polynomial-time algorithm for an $\mathcal{NP}$-hard problem, it makes sense to approach such problems with heuristics or approximation algorithms. In order to compare approximation methods, some proof of the closeness to optimality is necessary, and thus approximation complexity classes have developed [48, 68, 70].

We will concern ourselves with both the maximum error possible, as well as the expected error. Given that the optimal solution is of size k, we say an algorithm has an approximation factor $\leq \epsilon$ if the solution it returns is guaranteed to be $\leq (1+\epsilon)k$ (for minimization problems[1]). An *approximation scheme* is a class of algorithms $\mathcal{A}$ such that for any given $\epsilon > 0$, there exists a corresponding algorithm $\mathcal{A}_\epsilon$ that has an approximation factor of at most ϵ. If for any fixed ϵ the running time of $\mathcal{A}_\epsilon$ is polynomial in n,

[1]Maximization problems can be converted to minimization problems, and hence need not be considered separately.

the size of the input, then we say that the class of algorithms $\mathcal{A}$ is a *polynomial-time approximation scheme (PTAS)*. If we have the stronger condition that the running time of $\mathcal{A}_\epsilon$ is polynomial in n and $1/\epsilon$, then we say that the class of algorithms $\mathcal{A}$ is a *fully polynomial-time approximation scheme (FPTAS)*. Of course, not all optimization problems allow arbitrarily-close approximations. If an optimization problem allows approximation within a constant factor in polynomial-time, then it said to be in the *approximable (APX)* class.

Unless $\mathcal{P} = \mathcal{NP}$, it is known that there are problems in APX that are not in PTAS, and we call these algorithms APX-hard [68]. Similarly, there are PTAS-hard problems that do not permit an FPTAS. Thus we have the following hierarchy, with the inequalities holding if $\mathcal{P} \neq \mathcal{NP}$:

$$FPTAS \subset PTAS \subset APX.$$

5.2 Status of Independent Set and Vertex Cover

Recall from Chapter 1 that an independent set is a set of vertices no two of which are adjacent, and that a vertex cover is a set of vertices such that every edge in the graph is incident to some vertex in the cover. It is obvious that the empty set and $V(G)$ are an independent set and vertex cover, respectively, for all graphs G. However, the problem of finding the maximum independent set or minimum vertex cover of a graph is $\mathcal{NP}$-hard, even when restricted to cubic planar graphs [49]. These two problems are strongly related since the complement of an independent set is always a vertex cover (i.e. an edge cannot have both endpoints in the independent set, thus each edge is incident to a vertex not in the independent set, which satisfies the definition of a vertex cover), and thus finding the maximum independent set is equivalent to finding the minimum vertex cover.

The difficulty of approximating these problems depends on which graph class we consider[2]. For planar graphs, both problems are in PTAS [12]. However, both problems become APX-complete on graphs with bounded $\Delta(G) \geq 3$. That is, they can be approximated within a constant factor, but there does not exist any polynomial-time approximation scheme unless $\mathcal{P} = \mathcal{NP}$ [2, 17, 85].

The lower bounds on the approximation factors for independent set are 1.0005 for $\Delta(G) = 3$, 1.0018 for $\Delta(G) = 4$, and 1.0030 for $\Delta(G) = 5$ [18]. It is approximable within $\frac{\Delta(G)+3}{5}$ for small $\Delta(G)$, and within $O(\frac{\Delta(G) \log\log \Delta(G)}{\log \Delta(G)})$ for large $\Delta(G)$ [3, 17, 71]. In general, independent set is not approximable within $n^{1/2-\epsilon}, \epsilon > 0$ [64].

For vertex cover, the problem is approximable within 7/6 for $\Delta(G) = 3$, not approximable within 1.0029 for $\Delta(G) = 5$, and not approximable within 7/6 for large enough

[2]These results were mostly gathered from the online compendium found at http://www.nada.kth.se/∼viggo/wwwcompendium/wwwcompendium.html.

Table 5.1: Approximating Independent Set and Vertex Cover

Graph Class	Independent Set	Vertex Cover
Planar	$\in$ PTAS	$\in$ PTAS
$\Delta(G) = 3$	APX-com., $\epsilon \geq 0.0005$	APX-com., $\epsilon \leq 1/6$
$\Delta(G) = 4$	APX-com., $\epsilon \geq 0.0018$	APX-com.
$\Delta(G) = 5$	APX-com., $\epsilon \geq 0.003$	APX-com., $\epsilon \geq 0.0029$
Small $\Delta(G)$	APX-com., $\epsilon \leq \frac{\Delta(G)-2}{5}$	APX-com.
Large $\Delta(G)$	APX-com., $\epsilon \leq O(\frac{\Delta(G)\log\log\Delta(G)}{\log\Delta(G)})$	APX-com., $\epsilon \geq 1/6$
All graphs	$\notin$ APX	APX-com., $1/6 \leq \epsilon \leq 1 - \frac{\log\log n}{2\log n}$

$\Delta(G)$ [17, 18, 34, 63]. A simple and elegant greedy algorithm (select an arbitrary edge, add both endpoints to the cover, delete all incident edges, and repeat until no edges remain) is guaranteed to return a vertex cover no larger than twice the minimum possible size on all graphs. This has been tightened to $1 + \epsilon \leq 2 - \frac{\log\log n}{2\log n}$, where n is the number of vertices [14, 82]. We summarize these results in Table 5.1.

We will also be considering the weighted generalizations of these problems, where the vertices are assigned positive real weights, and we must find the maximum weight independent set/minimum weight vertex cover. Thus we need to highlight some of the progress made for this variation as well. The PTAS that exists for planar graphs also applies to the weighted versions of both these problems [12]. As well, the approximation within a factor of $2 - \frac{\log\log n}{2\log n}$ for vertex cover in general also holds for weighted vertex cover [14, 82]. As for independent set, the weighted version can be approximated within a factor of 3/2 on graphs with $\Delta(G) = 3$, and within $\frac{\Delta(G)+2}{3}$ for $\Delta(G) > 3$ [58, 67].

Since the weighted versions of both these problems have a PTAS for planar graphs, we shall focus our attention on producing approximation algorithms for graphs of bounded degree. However, before proceeding with our own algorithms, we shall now review some existing algorithms that use bounded treewidth decompositions to perform tasks efficiently.

5.3 Existing Algorithms using Bounded Treewidth Decompositions

Recalling that arboricity is essentially a decomposition into partial 1-trees, Chiba and Nishizeki have developed a subgraph-listing algorithm whose efficiency relates to bounded treewidth decompositions [32]. A similar use is found in Eppstein's paper on listing maximal complete bipartite subgraphs [44]. A more obvious application of bounded treewidth decompositions is Baker's development of a PTAS for problems on planar graphs, such as independent set and vertex cover [12]. Baker's method is to decompose a planar graph into n/k overlapping k-outerplanar graphs. Since k-outerplanar graphs have treewidth at most $3k-1$ (recall Theorem 1.7), then most $\mathcal{NP}$-hard problems can be solved on these subgraphs. It can easily be shown that the best answer among these n/k solutions is at most a factor of $1/k$ away from optimal. Eppstein has also developed algorithms using bounded treewidth decompositions, and generalized Baker's approach to decompositions of graphs with locally-bounded treewidth [45, 46]. Further improvements were recently made by Demaine *et al.* [38].

5.4 Approximation Algorithms for Partial $(k, 1)$-Trees

5.4.1 General Approach

Our algorithmic approach for partial $(k, 1)$-trees will be to use the partial k-tree decomposition to solve the $\mathcal{NP}$-hard problem exactly on the subgraph of treewidth k in linear time, and then to use the partial 1-tree to adjust this solution appropriately. We note that this approach will also work on graphs that have a decomposition into a bipartite graph and forest, subject to the restriction that the $\mathcal{NP}$-hard problem can be solved efficiently on bipartite graphs (e.g. independent set [76]). However, we shall confine our attention to algorithm design for decompositions of bounded treewidth.

5.4.2 Basic Approximation Algorithms

Assuming that we have a partial $(k, 1)$-tree decomposition for some fixed k, then we can apply the following basic algorithm:

Algorithm 5: Approximating Independent Set on Partial $(k, 1)$-Trees

```
Input: A partial $(k,1)$-tree decomposition for graph $G$, for some fixed constant $k$
Output: An independent set for $G$

Compute the maximum independent set $I$ for the partial $k$-tree
Take the partial 1-tree, delete all edges that do not have both endpoints in $I$, and
define the resulting graph to be $F$
Compute the minimum vertex cover $C$ of $F$
Return $R = I \setminus C$
```

The maximum independent set and minimum vertex cover can be computed for a graph of treewidth x in $O(2^x n)$ time [90]. This is equivalent to linear time if x is a fixed constant (i.e. k or 1). Thus steps 1 and 3 take linear time. Step 2 iterates through all edges in the forest, and hence also takes $O(n)$ time. Finally, step 4 can be implemented in linear time. Thus this is a linear-time algorithm.

However, this is an approximation algorithm, so we must also examine how accurately we approximate the maximum independent set. To do so, we must first review some well-known graph properties:

Theorem 5.1 *Turán's Theorem: A graph on n vertices and m edges has an independent set of size at least $\lceil \frac{n^2}{2m+n} \rceil$. [100].*

Since we are considering regular graphs, the following corollary will be important to our analysis:

Corollary 5.1 *Every graph with n vertices and average vertex degree k contains an independent set of size $\geq \lceil \frac{n}{k+1} \rceil$.*

Thus, cubic graphs have an independent set of size at least $\lceil n/4 \rceil$. Bipartite graphs—including forests, by Remark 1.3—must have a maximum independent set of size at least $n/2$, since we can simply pick the larger half of the bipartition. We note that this second observation also holds for the weighted version of maximum independent set.

Returning to Algorithm 5, we see that step 1 computes the maximum independent set for the partial k-tree, but the next three steps correct for the fact that the edges in the partial 1-tree were not considered. Thus the initial set I is clearly an upper-bound on the maximum independent set for G, and the minimum vertex cover is computed on a forest whose number of vertices is at most $|I|$. Since the minimum vertex cover of a forest on n vertices is at most $n/2$ vertices, then we delete at most $|I|/2$ vertices. Hence, the independent set R that we return is at least $1/2$ of the optimal size. This analysis also holds for the weighted version of maximum independent set, since set I will still be the maximum for the partial k-tree, and the partial 1-tree will select a vertex cover of at most half the weight (i.e. the smaller weighted half of the bipartition is an upper-bound on C).

A similar algorithm allows us to approximate vertex cover (directly) on partial $(k, 1)$-trees:

Algorithm 6: Approximating Vertex Cover on Partial $(k, 1)$-Trees

```
Input: A partial $(k, 1)$-tree decomposition for graph $G$, for some fixed constant $k$
Output: A vertex cover for $G$

Compute the minimum vertex cover $C_1$ for the partial $k$-tree
Take the partial 1-tree, delete all edges that are incident to a vertex in $C_1$, and
define the resulting graph to be $F$
Compute the minimum vertex cover $C_2$ of $F$
Return $R = C_1 \cup C_2$
```

By the same reasoning, this algorithm also takes linear time, so we only need to consider the approximation factor. The set C_1 is optimal for the partial k-tree, and hence a lower bound on the minimum vertex cover size for graph G. Since no vertex in C_1 can be incident to an edge in F, then in the worst-case F will be a tree on $n - |C_1|$ vertices. Thus $|C_2|$ is at most $\frac{n-|C_1|}{2}$. So we conclude that R is at most $|C_1| + \frac{n-|C_1|}{2} = n/2 + |C_1|/2$ vertices. In other words, our approximation factor is no worse than $\frac{n+|C_1|}{2|C_1|}$. Although this does not give us a constant factor for all planar graphs, a simple edge-counting argument for regular graphs proves that a vertex cover includes at least half of the vertices, giving us an approximation factor of at most $3/2$.

A better way of analyzing Algorithm 6 is to notice that a vertex cover on the partial k-tree and a vertex cover on the partial 1-tree are both lower-bounds on the size of a minimum vertex cover. Since $|C_1|$ corresponds to this first bound, and $|C_2|$ is no larger than this second bound, then we conclude that $|C_1| + |C_2|$ is at most twice the optimal size. In other words, our approximation factor is never worse than 2.

We can combine these results for independent set and vertex cover, since Algorithms 5 and 6 are essentially equivalent. That is, finding the maximum independent set and minimum vertex cover for the partial k-tree will return complementary vertex sets. In both cases, F is defined to be the subset of partial 1-tree edges not covered by the vertex cover, or equivalently, incident to two independent set vertices. Lastly, in both cases we identify the minimum vertex cover of F and transfer these vertices from the independent set to the vertex cover. Thus, our approximation bounds on Algorithm 6 also apply to Algorithm 5, and vice-versa. Thus from now on we shall only discuss Algorithm 5.

Finally, although Algorithm 5 claims that k must be a fixed constant, the algorithm remains polynomial-time when $k \in O(a \log(n) + b)$ for constants a and b, since $O(2^k n) = O(2^b n^{a+1}) \in \mathcal{P}$. Thus we obtain the following Theorem:

Theorem 5.2 *Assume $\mathcal{G}$ is a graph class that can be decomposed into partial $(k,1)$-trees, where $k \in O(a\log(n)+b)$ for some fixed constants a and b. Let us also say that Algorithm 5 is performed on graph $G \in \mathcal{G}$, and returns the solution (I, C), where I is an independent set and C is its complementary vertex cover. Then we can conclude that the optimal solution (weighted or unweighted) (I^*, C^*) has the property that $|I^*| - |I| = |C| - |C^*| = \Delta$ and that $\Delta \leq min\{|I|, |C|/2\}$. Furthermore, this approximate solution can be computed in linear time when k is a constant independent of n (i.e. $a = 0$), and polynomial time otherwise.*

Looking at Table 5.1 and the results for the weighted versions of these problems, we can see that our algorithm does not improve on the best-known results for graphs with maximum degree three. However, if regular graphs of higher degree are also partial $(k,1)$-trees for $k \in O(\log n)$, then our approach will offer a better approximation for these graph classes.

Finally, we note that by reducing the number of edges in the forest F, we can automatically improve our approximation factor for both of these algorithms, since the vertex cover of a graph can never be larger for a subgraph (i.e. the adjustment factor can only decrease by reducing the partial 1-tree). Thus, there are significant benefits to constraining the treewidth decompositions in ways that decrease the number of edges in the partial 1-tree. For instance, using a matching when possible, transferring as many edges as possible from the forest to the partial k-tree, and even using a non-minimal k for the partial $(k, 1)$-tree decomposition, thereby increasing the number of edges that can be transferred to the partial k-tree.

5.5 Sample Performance

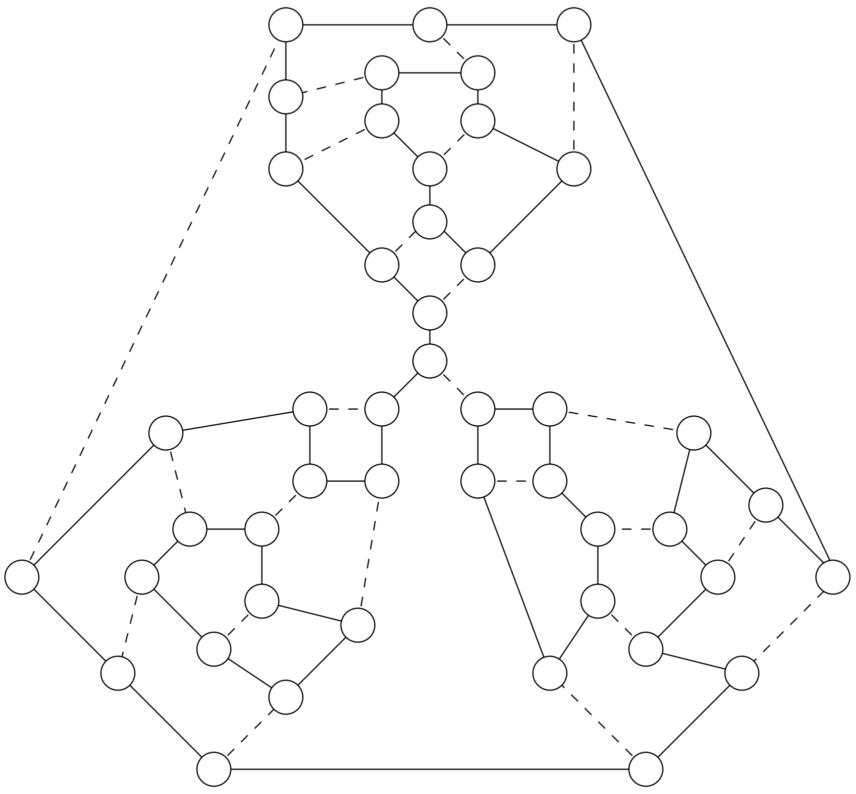


Figure 5.1: A Perfect Matching for the Tutte Graph

We will now show how Algorithm 5 performs on the Tutte graph. We begin by finding a perfect matching for this cubic graph in order to decompose it into a partial $(2, 1/2)$-tree (see Figure 5.1).

In order to reduce the number of adjustments that will be required, we should transfer some of the matching edges back into the partial 2-tree. In particular, bridges and edges covered by the tree decomposition should not be kept in the matching. Adding back such edges (seven of them) in the Tutte graph, we get the partial 2-tree and its maximum independent set, as shown in Figure 5.2.

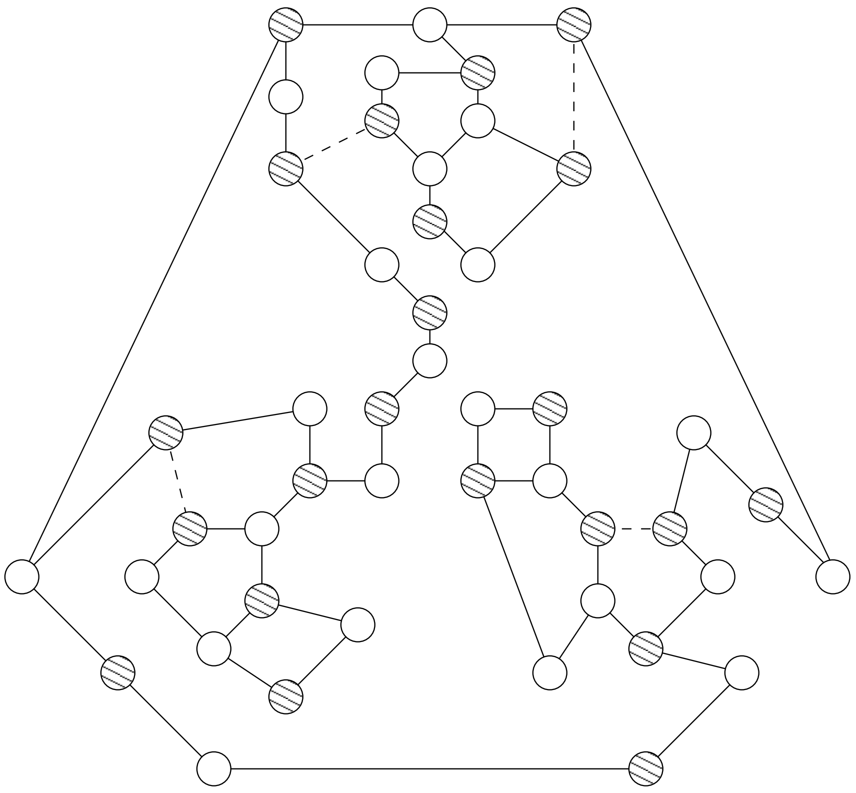


Figure 5.2: A Maximum Independent Set for the Tutte Graph's Partial 2-Tree

Since the partial 2-tree has a maximum independent set of size 22, it follows that the Tutte graph cannot have an independent set of size greater than 22. Furthermore, four of the matching edges create a conflict with the above independent set (see the dashed edges in Figure 5.2), so our algorithm returns a maximum independent set of size 18.

Thus we can conclude that our algorithm returns a maximum independent set that is, at worst, 18/22 of the optimum solution (an error margin of approximately 22%). However, it can be verified that the Tutte graph actually has a maximum independent set of size 19, and thus our algorithm's error is actually $< 6\%$ for this example. However, it should be noted that the solution for the partial 2-tree is not unique, and other answers can result in a smaller independent set (e.g. of size 17, for instance).

Chapter 6

Conclusion

In summary, we used the concept of treewidth to simplify and extend the results by Knuth, and Kratochvíl and Křivánek relating to nested satisfiability. This observation demonstrated that a restricted form of graph construction was associated with treewidth at most three, and that the tree decomposition could easily be constructed from this structure.

We then identified that this class could be extended even further to include more planar partial 3-trees, thereby creating a new class of nested graphs, which we named partial Matryoshka graphs. We showed that partial Matryoshka graphs were a strict subset of planar partial 3-trees whose standardized tree decomposition had at most two children per node (i.e. binary planar partial 3-trees), and that the induced graphs of such tree decompositions were always Hamiltonian. We also demonstrated that partial Matryoshka graphs, although not closed under minors, contained many planar graph classes of treewidth at most three (besides Nested SAT graphs) including outerplanar graphs, Halin graphs, and IO-graphs.

We determined that the structure of partial Matryoshka graphs allowed for a simple vertex decomposition into a forest and an outerplanar graph, as well as an edge decomposition into a forest and an SP-graph. Tighter decompositions were identified for both Halin and IO-graphs. Our observations directed us towards other mathematical results relating to decompositions of bounded treewidth, such as the research surrounding the conjecture by Chartrand, Geller, and Hedetniemi, as well as the concept of arboricity and the identification of partial $(p+q)$-trees as partial (p,q)-trees for all p, q. Hence, we focused on planar graphs and summarized the known relationships between graphs of bounded treewidth and graphs with decompositions of bounded treewidth.

Having summarized the known relationships for planar graphs, we then attempted to answer the open question as to whether or not planar graphs are partial $(k,1)$-trees for some constant k. We developed an efficient algorithm which produced such a decomposition for 4-connected planar graphs with $k \in O(\log n)$, and demonstrated obstructions to generalizing this approach to 3-connected planar graphs. We also examined decompositions of bounded treewidth for graphs of bounded degree and determined that cubic graphs were partial $(2, 1/2)$-trees, and that this decomposition could be constructed efficiently. We also demonstrated that graphs of bounded degree greater than three were not partial $(k, 1)$-trees for any constant k, not even when restricted to planar graphs or for any $k \in O(\log n)$. Finally, we explored other constrained decompositions for planar graphs, and showed that such decompositions (bipartite graph plus forest) were always possible for partial 3-trees.

Our final area of study was the potential use of partial $(k, 1)$-tree graph decompositions for the development of approximation algorithms. We developed such an algorithm for the complementary problems of weighted maximum independent set and weighted minimum vertex cover, and determined that our algorithm was guaranteed to produce a solution within an approximation ratio of at most two for both of these problems. This algorithm also produces upper and lower bounds on the optimal solution during its computation, thereby giving us a tighter range and ratio for many problem instances.

6.1 Future Research

This thesis produced many interesting open questions. With regards to partial Matryoshka graphs, the most important is to develop a polynomial-time algorithm for the recognition of such graphs, as well as an efficient algorithm for producing a Matryoshka drawing. In order to develop such an algorithm, a clearer knowledge of the characteristics of partial Matryoshka graphs is required, such as their relation to SP-graphs, and a clearer identification of their differences with binary planar partial 3-trees. It may also be interesting to examine the duals of partial Matryoshka graphs.

With regards to graph decompositions of bounded treewidth, the most natural continuation of our research is to determine which graph classes (of unbounded treewidth) are partial (p, q)-trees. In particular, the decomposition of planar graphs into partial $(k, 1)$-trees for minimal k would complete this exploration for planar graphs. In order to solve this problem, there are several possible approaches which can be considered. Firstly, we can use 2-outerplanar graphs in a manner similar to El-Mallah and Colbourn's use of IO-graphs [42]. Although this technique fails when cut-vertices appear at certain levels of outerplanarity, it does prove that a subset of planar graphs are partial $(4, 1)$-trees. Another approach likely to work is to identify forests that eliminate all $k \times k$ grid graph minors from planar graphs for some fixed k. This obtains a constant bound on the treewidth of the remaining graph since K_5 is a forbidden minor of planar graphs, although the constant is far too large to be of any practical use (see page 270 of Diestel's book for details [39]). Finally, Hamiltonian cycles and dual graphs seem to help in the construction of such decompositions, so a natural question is whether or not a non-Hamiltonian planar triangulated graph always has a Hamiltonian dual. Given the high likelihood of a cubic planar bridgeless graph being Hamiltonian [93], this conjecture may seem likely, though similar conjectures have failed before [98, 103]. In any case, examining this problem in the dual graph seems to offer some interesting possibilities.

Lastly, improvements should be made regarding our approximation algorithms for partial $(k, 1)$-trees. Although we proved that our algorithms have an approximation ratio of at most two, we have not shown that this bound is tight. Also, it would be interesting to investigate how constraints on the forest affect the performance of such algorithms. For instance, cubic graphs are both partial $(1, 1)$-trees (i.e. have arboricity two by the Nash-Williams formula) and partial $(2, 1/2)$-trees. Although our analysis does not distinguish between these two decompositions, improved performance would be expected as more edges are shifted to the partial k-tree half of the decomposition. Thus, a more detailed analysis of the current approximation algorithms, as well as an investigation of potential improvements (i.e. making adjustments for the forest while computing for the partial k-tree) is certainly warranted. However, to make such approximation algorithms more worthwhile, we need to identify a wider range of graph classes as partial $(k, 1)$-trees, both for constant k and any $k \in O(\log n)$.

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