Chapter 2

Preliminaries

We review some basic terminology and aspects of triangulated graphs. We define the concepts treewidth and pathwidth of a graph in terms of subgraphs of triangulated graphs. We then show the equivalence between the treewidth and pathwidth of a graph and the minimum width of a tree-decomposition and path-decomposition. We end this section with some notes on perfect graphs.

2.1 Triangulations

In this section we review some important aspects of triangulated graphs and of triangulations of graphs. The various names given to triangulated graphs, such as chordal graphs, rigid circuit graphs, perfect elimination graphs, monotone transitive graphs and so on, illustrate the importance of this class of graphs. We shall mainly use the term triangulated graphs. For a good overview of the different aspects of triangulated graphs the reader is referred to Golumbic's book [76].

Definition 2.1.1 *A graph is* triangulated *if it contains no chordless cycle of length greater than three.*

This is equivalent to saying that the graph does not contain an induced subgraph isomorphic to C_n (i.e., a cycle of length n) for $n > 3$. Notice that being triangulated is a hereditary property, i.e., if a graph G is triangulated then every induced subgraph of G also is triangulated.

There are many ways to characterize triangulated graphs. Characterizations have been appearing from 1960 [56] until very recently [11]. Although many of these characterizations are interesting and useful, it suffices for our purposes to list only some of them.

One of the most important tools that can be used when working with triangulated graphs is the concept of a perfect elimination scheme.

Definition 2.1.2 *Let* $G = (V, E)$ *be a graph. A* simplicial vertex *of* G *is a vertex of which the neighborhood induces a clique. An ordering of the vertices* $\sigma = [v_1, \ldots, v_n]$ *is called a* perfect elimination scheme *if for every* $1 \leq i \leq n$, v_i *is a simplicial vertex in* $G[v_i, \ldots, v_n]$.

Fulkerson and Gross, in a paper in 1965 [66], characterized triangulated graphs by means of a perfect elimination scheme.

Lemma 2.1.1 *A graph* G *is triangulated if and only if there exists a perfect elimination scheme for* G. *Moreover, if a graph is triangulated, any simplicial vertex can start a perfect elimination scheme for it.*

In 1975 Rose and Tarjan [148] gave a linear time algorithm for recognizing triangulated graphs, by finding a perfect elimination scheme if it exists. Many NP-hard problems are solvable in linear time when restricted to triangulated graphs like CLIQUE, INDEPENDENT SET, CLIQUE COVER etc. Lemma 2.1.1 has many useful consequences. For example, every triangulated graph has at least one simplicial vertex. In fact, Dirac showed that if a graph is not a clique, then it has at least two nonadjacent simplicial vertices. This can be used to obtain yet another characterization of triangulated graphs: a graph is triangulated if and only if every induced subgraph is either a clique or has two nonadjacent simplicial vertices. We conclude that if G is triangulated and C is the vertex set of a maximal clique in G, then there exists a perfect elimination scheme $\sigma = [v_1, \ldots, v_n]$ such that $C = \{v_{n-t+1}, \ldots, v_n\}$ (where $t = |C|$).

Notice that if x is a simplicial vertex in a triangulated graph, then $\{x\} \cup N(x)$ is a maximal clique, and in fact this is the only maximal clique containing x. It follows that there are at most n maximal cliques, with equality holding if and only if the graph has no edges. This was first pointed out in [66]. Notice that if $\sigma = [v_1, \ldots, v_n]$ is a perfect elimination scheme, then all maximal cliques are of the form $\{v_i\} \cup (N(v_i) \cap \{v_i, \ldots, v_n\})$.

The last immediate consequence of Lemma 2.1.1 that we mention here is that a triangulated graph is perfect (a graph is called perfect if the clique number is equal to the chromatic number for every induced subgraph; see section 2.3). This can be seen as follows. First, since every induced subgraph of a triangulated graph is again triangulated, we only have to show that the chromatic number $\chi(G)$ and clique number $\omega(G)$ are equal for any triangulated graph G. To prove it, let $\sigma = [v_1, \ldots, v_n]$ be a perfect elimination scheme. For $i = n, n-1, \ldots, 1$, color v_i with a color which is not used in $N(v_i) \cap \{v_{i+1}, \ldots, v_n\}$. This gives a correct coloring, and since each $\{v_i\} \cup (N(v_i) \cap \{v_{i+1}, \ldots, v_n\})$ is a clique, this coloring can be done using no more than $\omega(G)$ colors.

Another characterization of triangulated graphs was given by Dirac in 1961 [56], by means of minimal vertex separators.

Definition 2.1.3 *Given a graph* G *with vertex set* V, *a subset* $S \subset V$ *is called a* vertex separator *for nonadjacent vertices* a *and* b *in* $V \setminus S$ *if* a *and* b *are in different connected components of* $G[V \setminus S]$. *If* S *is a vertex separator for* a *and* b *but no proper subset of* S *separates* a *and* b *in this way, then* S *is called a* minimal vertex separator for a and b *or a* minimal a, b-separator. *A subset* $S \subseteq V$ *is called a minimal vertex separator if there exists a pair of nonadjacent vertices for which* S *is a minimal vertex separator.*

Since we deal only with vertex separators, we usually call a (minimal) vertex separator simply a (minimal) separator. We like to stress that one minimal vertex separator might well be properly contained in another minimal vertex separator.

Lemma 2.1.2 *A graph* G *is triangulated if and only if every minimal vertex separator induces a complete subgraph of* G.

The following lemma, which must have been rediscovered many times, appears as an exercise in [76].

Lemma 2.1.3 *Let* S *be a minimal vertex separator for nonadjacent vertices* a *and* b, *and let* C_a *and* C_b *be the connected components of* $G[V \setminus S]$ *containing* a *and* b, *respectively. Then every vertex of* S *has a neighbor in* C_a *and a neighbor in* C_b.

Proof. Assume $w \in S$ has no neighbor in C_a. We claim that $S' = S \setminus \{w\}$ is also an a, b-separator. Suppose there is a path between a and b in $G[V \setminus S']$. Then the path contains no vertex of S', hence the path passes through w. This leads to a contradiction. □

Especially useful is the fact that a triangulated graph has *balanced* separators. The following lemma is a generalization of probably the oldest separator theorem of which one version is due to C. Jordan in 1869 [90, 111]. For a similar result see e.g. [143].

Lemma 2.1.4 *Let* $G = (V, E)$ *be a triangulated graph with* n *vertices. There is a clique* C, *such that every component of* $G[V \setminus C]$ *has at most* $\lceil \frac{1}{2}(n - |C|) \rceil$ *vertices.*

As mentioned earlier, there are many more characterizations of triangulated graphs. For example, a graph is triangulated if and only if every induced connected subgraph with $p \geq 2$ vertices contains at most $p - 1$ maximal cliques (see for example [15] page 84). Another interesting characterization appears in [11] and uses the so-called stability function of a graph. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. With each vertex we can associate a $0/1$-variable x_i. If x is some vector in $\{0, 1\}^n$ then the subgraph of

G induced by x is defined as the subgraph induced by exactly those vertices v_i for which the corresponding variable x_i is set to one. The stability function $\alpha_G : \{0,1\}^n \to \mathbf{N}$ is defined as follows. For each $x \in \{0,1\}^n$, $\alpha_G(x)$ is the stablity number (the maximum cardinality of an independent set) of the subgraph induced by x. It can be shown that this function can be expressed uniquely in the form $\sum_{t\in\Delta} a_t \prod_{i\in t} x_i$, where Δ is a collection of subsets of $\{1,\ldots,n\}$ and the a_t's are real coefficients. Now the following statement holds. A graph is triangulated if and only if the polynomial expression of the stability function has all its coefficients in $\{0,-1,1\}$.

We conclude with one more characterization, showing that triangulated graphs are a class of intersection graphs. This was shown for example in [172].

Lemma 2.1.5 *A graph* G *is triangulated if and only if* G *is the intersection graph of a family of subtrees of a tree.*

The kind of intersection graph referred to in the lemma is defined as follows. Given a family of subtrees of a tree, a graph is constructed in the following way. The vertices of the graph are the subtrees and two vertices are adjacent if the corresponding subtrees have at least one node in common. Notice that a family of subtrees of a tree satisfies the Helly property, which is the following (see for example [76]).

Definition 2.1.4 *A family* $\{S_i\}_{i\in I}$ *of sets satisfies the* Helly property *if for all* $J \subset I$*:* $S_i \cap S_j \neq \emptyset$ *for all* $i,j \in J$ *implies that* $\cap_{j\in J} S_j \neq \emptyset$.

Notice that this implies the following. Assume we have a set of connected subgraphs of a triangulated graph such that for every pair the subgraphs have at least one vertex in common. Then there is at least one vertex present in each of the subgraphs.

An important subclass of the triangulated graphs are the interval graphs (which were first mentioned by Hajös [80]).

Definition 2.1.5 *An* interval graph *is a graph for which one can associate with each vertex an interval on the real line such that two vertices are adjacent if and only if their corresponding intervals have a nonempty intersection.*

Definition 2.1.6 *An ordering* $(X_1,\ldots,X_t)$ *of the maximal cliques of a graph* G *is called an* interval ordering *if, for every vertex, the maximal cliques containing it occur consecutively in the ordering.*

An interval ordering of the maximal cliques of a graph is sometimes called a *consecutive clique arrangement*.

The following characterization of interval graphs was found by Gilmore and Hoffman in 1964 [72].

Lemma 2.1.6 *The following statements are equivalent.*

1. G *is an interval graph.*
2. G *is triangulated and its complement* $\overline{G}$ *is a comparability graph.*
3. *There is an interval ordering of the maximal cliques of* G.

A comparability graph is a graph which admits a transitive orientation of its edges [76].

Interval graphs have many practical applications in various fields like archeology, geology, criminology, genetics etc., see for example [46, 76]. For a much more extensive bibliographical list of applications we refer to page 90 of the overview paper of Duchet which appeared in [15].

As an interval graph is triangulated, the number of maximal cliques is bounded by the number of vertices. By Lemma 2.1.6 the maximal clique-versus-vertex incidence matrix has the consecutive ones property. In [35] Booth and Lueker give a fast algorithm to test for the consecutive ones property using PQ-trees, which implies the following result:

Lemma 2.1.7 *A graph* $G = (V, E)$ *can be tested for being an interval graph in* $O(n + e)$ *steps, where* n *is the number of vertices and* e *is the number of edges of* G.

Recently a simpler algorithm was discovered by Hsu [88] which does not use the maximal cliques and places the intervals directly. As with triangulated graphs, there are many more characterizations for interval graphs. For example, Lekkerkerker and Boland in 1962, found a list of graphs such that a graph is an interval graph if and only if it does not contain an induced subgraph isomorphic to a graph in this list [120]. Since the complements of interval graphs are comparability graphs, such a list is also easily obtained from the complete list of forbidden induced subgraphs for comparability graphs, which was found by Gallai [68] in 1967 (this list appears also in [15] on page 78). Lekkerkerker and Boland also proved another important characterization, using astroidal triples.

Definition 2.1.7 *Three vertices in a graph* G *are called an* astroidal triple *if any two of them is connected by a path which avoids the neighborhood of the third.*

For example, in a 3-sun (see figure 3.1 on page 30) the three vertices of the independent set are an astroidal triple.

Lemma 2.1.8 *A graph is an interval graph if and only if* G *is triangulated and does not contain an astroidal triple.*

A second important subclass of triangulated graphs is the class of k-trees. The first mention we found of 2-trees is in a paper of Harary and Palmer [83] and dates back to 1968 (in this paper they are introduced as being simple connected, acyclic 2-plexes). The related concept of a (acyclic) pure k-complex is defined in a paper of Harary in 1955 [81]. When restricted to 2-trees another related concept, called a two-terminal series-parallel network, can even be traced back to Macmahon in 1892 [123], who described a method to determine the number of distinct series-parallel two-terminal networks. This is mentioned in a paper of R. M. Foster [64] (see also [65]). Foster counts the number of different series-parallel networks without specification of terminals. (He mentions that in [163] the enumeration through $n = 7$ has been performed by Tellegen.)

In a paper from 1969 Beineke and Pippert enumerate labeled k-dimensional trees (after enumerating 2-dimensional trees in 1968) which they call k-trees for short [10].

Usually a clique with k vertices is also considered to be a k-tree. For convenience we prefer to define k-trees to have at least $k+1$ vertices. With this definition, the treewidth of a k-tree is k (the treewidth of a k-clique is $k-1$), and the clique number of a k-tree is $k+1$.

Definition 2.1.8 k-Trees *are defined recursively as follows: A clique with* $k+1$ *vertices is a* k*-tree; given a* k*-tree* T_n *with* n *vertices, a* k*-tree with* $n+1$ *vertices is constructed by taking* T_n *and creating a new vertex* x_{n+1} *which is made adjacent to a* k*-clique of* T_n *and nonadjacent to the* $n-k$ *other vertices of* T_n.

The first characterization shows that k-trees are indeed triangulated.

Lemma 2.1.9 *A graph* G *with* n *vertices is a* k*-tree if and only if* $n > k$ *and there exists a perfect elimination scheme* $\sigma = [v_1, \ldots, v_n]$ *such that for all* $i \leq n-k$, v_i *is adjacent to a* k*-clique in the subgraph* $G[v_i, \ldots, v_n]$.

Proof. Assume $G = (V, E)$ is a k-tree with n vertices. If $n = k+1$ then G is a $(k+1)$-clique and any ordering of the vertices yields a perfect elimination scheme satisfying the desired property. If $n > k+1$, by definition there is a vertex x such that $G[V \setminus \{x\}]$ is a k-tree with $n-1$ vertices, and x is adjacent to a k-clique. By induction we may assume that there is a perfect elimination scheme $\sigma' = [v_2, \ldots, v_n]$ for $G[V \setminus \{x\}]$ such that v_i is adjacent to a k-clique in $G[\{v_i, \ldots, v_n\}]$, for all $2 \leq i \leq n-k$. Define $v_1 = x$. Then $\sigma = [v_1, \ldots, v_n]$ is a perfect elimination scheme as mentioned for G.

Now assume $n > k$ and $\sigma = [v_1, \ldots, v_n]$ is a perfect elimination scheme such that for all $i \leq n-k$, v_i is adjacent to a k-clique in $G[v_i, \ldots, v_n]$. If $n = k+1$ it follows that the graph must be a $(k+1)$-clique and hence a k-tree. Assume $n > k+1$. Then $\sigma' = [v_2, \ldots, v_n]$ is a perfect elimination scheme for $G[V \setminus \{v_1\}]$ with the stated property guaranteeing by induction that $G[V \setminus \{v_1\}]$ is a k-tree. Since v_1 is adjacent to a k-clique it follows by definition of a k-tree that G is a k-tree. □

It follows that all maximal cliques in a k-tree have $k+1$ vertices. Lemma 2.1.9 makes it very easy to calculate the number of certain subgraphs in a k-tree. We give one example.

Lemma 2.1.10 *For* $1 \le t \le k+1$, *the number of* t*-cliques in a* k*-tree is*

$$\binom{k}{t} + (n-k)\binom{k}{t-1}$$

The following characterization of k-trees appears in [147].

Lemma 2.1.11 *A graph* G *is a* k*-tree if and only if*

1. G *is connected,*
2. $\omega(G) = k+1$,
3. *every minimal vertex separator of* G *is a* k*-clique.*

The following lemma shows that k-trees are triangulated graphs with clique number $k+1$ which have a maximum number of edges (see also [147]). In Lemma 2.1.13 we will show that, conversely, every triangulated graph with clique number $k+1$ and with a maximal number of edges is a k-tree.

Lemma 2.1.12 *A graph* G *with* n *vertices is a* k*-tree if and only if*

1. G *is triangulated,*
2. $\omega(G) = k+1$,
3. *the number of edges is at least* $nk - \frac{1}{2}k(k+1)$.

Proof. First assume G is a k-tree. By Lemma 2.1.9 G is triangulated and has clique number $k+1$. By Lemma 2.1.10 the number of edges in a k-tree is $nk - \frac{1}{2}k(k+1)$. Conversely, assume that a graph G satisfies the three conditions. Since G is triangulated there is a perfect elimination scheme $\sigma = [v_1, \ldots, v_n]$. Since $\omega(G) = k+1$, it follows that for each i, $|N(v_i) \cap \{v_i, \ldots, v_n\}| \le k$. Hence we find for the number of edges in G:

$$\sum_{i=1}^{n} |N(v_i) \cap \{v_i, \ldots, v_n\}| \le \sum_{i=1}^{n-k} k + \frac{1}{2}k(k-1) = nk - \frac{1}{2}k(k+1)$$

We can have equality only if for each $i \le n-k$ v_i has k neighbors in $\{v_i, \ldots, v_n\}$. By Lemma 2.1.9 G is a k-tree. □

Definition 2.1.9 *A* partial k-tree *is any subgraph of a* k*-tree.*

There exist linear time algorithms for many NP-complete problems when these problems are restricted to the class of partial k-trees for some *constant* k and when an embedding in a k-tree (i.e., a k-tree of which the graph is a subgraph) is given [3, 5, 7, 18]. There are also results stating that large *classes* of problems can be solved in linear time for the class of partial k-trees with k bounded by some constant [48, 49, 50].

In this book we shall primarily be concerned with the problem of finding nice embeddings of a graph in triangulated graphs and in interval graphs. We call such an embedding a triangulation of the graph.

Definition 2.1.10 *A triangulation of a graph* G *is a graph* H *with the same set of vertices such that* G *is a subgraph of* H *and such that* H *is triangulated. We say that* G *is* triangulated into H.

Clearly, every graph can be triangulated (into a clique). There are two problems which have drawn much attention because of the large number of applications. The first problem is to triangulate a graph such that the number of added edges is minimum and the second one is to triangulate a graph such that the clique number in the triangulated graph is minimum. The first one is called the MINIMUM FILL-IN problem and the second one is the TREEWIDTH problem. These problems are both NP-hard [4, 174]. Our concern, as one might guess by the title of this book, lies in finding embeddings which minimize the maximum clique.

Remark. The fact that these problems are indeed different can be seen by the following example [24]. Let $\delta > 3$ be an integer. Let C be a cycle with $2\delta + 3$ vertices, I an independent set with δ vertices and K a clique with 2δ vertices. Consider the graph $G = C + (I \cup K)$. There are two possible triangulations with a minimal number of edges. In the first edges are added such that $I \cup K$ becomes a clique and such that C becomes triangulated. In this case the clique number becomes $3\delta + 3$ and the number of added edges is $\frac{1}{2}\delta(5\delta + 3)$. The second possible triangulation with a minimal number of edges, adds edges such that C becomes a clique. In this way the maximum clique has $4\delta + 3$ vertices and the number of added edges is $\delta(2\delta + 3)$. Hence the first triangulation minimizes the clique number and the second minimizes the number of edges.

Another result obtained in [24] is the following. An elimination scheme for a graph G is simply an ordering of the vertices $\sigma = [v_1, \ldots, v_n]$. Given an elimination ordering σ let $G' = G(\sigma)$ be the graph obtained from G by adding the minimum number of edges such that σ is a perfect elimination scheme for G'. Let a graph G have property $\mathcal{P}$ if for *every* elimination ordering σ the triangulation $G(\sigma)$ realizes the minimum fill-in. In [24] it is shown that the only graphs having property $\mathcal{P}$ are graph for which every connected component is either a clique, a cycle or a cocktailparty graph (i.e. the complement of $K_2 \cup K_2 \cup \ldots \cup K_2$). For these graphs every $G(\sigma)$ also minimizes the clique number. As far as we know, it is an open problem to characterize those graphs for which every triangulation with a minimal number of edges is a triangulation which minimizes the clique number.

Lemma 2.1.13 *If* G *is a triangulated graph with at least* $k+1$ *vertices and has clique number at most* $k+1$, *then* G *can be triangulated into a* k*-tree.*

Proof. Let $\sigma = [v_1, \ldots, v_n]$ be a perfect elimination scheme for G. We make an embedding of G in a k-tree recursively as follows. First we add edges such that the subgraph induced by $\{v_{n-k}, v_{n-k+1}, \ldots, v_n\}$ becomes a $k+1$-clique. For the induction step, assume the subgraph with vertices $\{v_{i+1}, \ldots, v_n\}$ has been triangulated into a k-tree T_{i+1}. In G vertex v_i is adjacent to a clique C with at most k vertices in $\{v_{i+1}, \ldots, v_n\}$. In the k-tree T_{i+1} C is contained in a k-clique C'. We make v_i adjacent to all vertices of C' and we obtain a k-tree T_i which is a triangulation of G. □

Lemma 2.1.14 *Every partial* k*-tree with at least* $k+1$ *vertices can be triangulated into a* k*-tree.*

Proof. Let the partial k-tree $G = (V, E)$ be a subgraph of the k-tree $H = (W, F)$. Then $H[V]$ is triangulated and has clique number at most $k+1$. By Lemma 2.1.13, $H[V]$ can be triangulated into a k-tree. Clearly this is a triangulation of G. □

We have seen that k-trees are triangulations with a maximal number of edges. We now show some properties of triangulations with a minimal number of edges.

Definition 2.1.11 *A* minimal triangulation H *of a graph* G *is a triangulation of* G *such that the following conditions are satisfied:*

1. *If* a *and* b *are nonadjacent vertices in* H *then every minimal* a, b*-separator of* H *is also a minimal* a, b*-separator in* G,
2. *If* S *is a minimal separator in* H *and* C *is the vertex set of a connected component of* $H[V \setminus S]$, *then* C *induces also a connected component in* $G[V \setminus S]$.

Theorem 2.1.1 *Let* H *be a triangulation of* G. *There exists a minimal triangulation* H' *of* G *such that* H' *is a subgraph of* H.

Proof. Suppose H has a minimal vertex separator W for nonadjacent vertices a and b, such that either W induces no minimal vertex separator for a and b in G, or the vertex sets of the connected components of $H[V \setminus W]$ are different from those of $G[V \setminus W]$. Let $S \subseteq W$ be a minimal a, b-separator in G. Let $C_1, \ldots, C_t$ be the connected components of $G[V \setminus S]$. Make a chordal graph H' as follows. For each $C_i \cup S$ take the chordal subgraph of H induced by these vertices. Since S is a clique in H, this gives a chordal subgraph H' of H. Notice that the vertex sets of the connected components of $H'[V \setminus S]$ are the same as those of $G[V \setminus S]$. We claim that the number of edges of H' is less than the number of edges of H which, by induction, proves the theorem. Clearly H' is a subgraph of H.

First assume that $S \neq W$, and let $x \in W \setminus S$. By Lemma 2.1.3, in H, x has a neighbor in the component containing a and a neighbor in the component containing b. Not both these edges can be present in H'.

Now assume $S = W$. Then, by assumption, the vertex sets of the connected components of $H[V \setminus W]$ are different from those of $H'[V \setminus W]$. Since H' is a subgraph of H, every connected component $H'[V \setminus W]$ is contained in some connected component of $H[V \setminus W]$. It follows that there must be a connected component in $H[V \setminus W]$ containing two different connected components of $H'[V \setminus W]$. This can only be the case if there is some edge between these components in $H[V \setminus W]$ (which is not there in $H'[V \setminus W]$). This proves the theorem. □

To illustrate that minimal triangulations are not very restrictive, notice that a clique is a minimal triangulation of G. We now show that we can restrict the triangulations to be considered somewhat more.

Definition 2.1.12 *Let Δ be the set of all minimal separators of a graph $G = (V, E)$. For a subset $\mathcal{C} \subseteq \Delta$ let $G_{\mathcal{C}}$ be the graph obtained from G by adding edges between vertices contained in the same set $C \in \mathcal{C}$. If the graph $G_{\mathcal{C}}$ is a minimal triangulation of G such that $\mathcal{C}$ is exactly the set of all minimal separators of $G_{\mathcal{C}}$, then $G_{\mathcal{C}}$ is called an* efficient *triangulation.*

Notice that for each $C \in \mathcal{C}$, the induced subgraph $G_{\mathcal{C}}[C]$ is a clique.

Theorem 2.1.2 *Let H be a triangulation of a graph G. There exists an efficient triangulation $G_{\mathcal{C}}$ of G which is a subgraph of H.*

Proof. Take a minimal triangulation H' which is a subgraph of H such that the number of edges of H' is minimal (theorem 13.3.3). We claim that H' is efficient. Let $\mathcal{C}$ be the set of minimal vertex separators of H'. We prove that $G_{\mathcal{C}} = H'$.

Since every minimal separator in a triangulated graph is a clique, it follows that $G_{\mathcal{C}}$ is a subgraph of H'. Consider a pair of vertices a and b which are adjacent in H' but not adjacent in G. Remove the edge from the graph H'. Call the resulting graph H^*. Since the number of edges of H' is minimal, it follows that H^* has a chordless cycle. Clearly this cycle must have length four. Let $\{x, y, a, b\}$ be the vertices of this square. Then x and y are non adjacent in H'. But then a and b are contained in every minimal x, y-separator in H'. It follows that a and b are also adjacent in G_C. □

Corollary 2.1.1 *A triangulation of a graph with a minimal number of edges is efficient.*

Let $G = (V, E)$ be a graph and let C be a minimal vertex separator. Let $C_1, \ldots, C_t$ be the connected components of $G[V \setminus C]$. Denote by $\overline{C_i}$ $(i = 1, \ldots, t)$ the graph obtained as follows. Take the induced subgraph $G[C \cup C_i]$, and add all possible edges between vertices of C such that the subgraph induced by C is complete. The following lemma easily follows from Theorem 2.1.1 (a similar result appears in [4]).

Lemma 2.1.15 *A graph G with at least $k+2$ vertices is a partial k-tree if and only if there exists a minimal vertex separator with at most k vertices such that all graphs $\overline{C_i}$ are partial k-trees.*

Proof. Assume G is a partial k-tree. Let H be a minimal triangulation. Then $\omega(H) \leq k + 1$. Thus, because H has more than $k + 1$ vertices, there must exist nonadjacent vertices in H. Let C be a minimal vertex separator in H. Then this is also a minimal vertex separator in G. Since H is triangulated, C is a clique in H. Let $C_1, \ldots, C_t$ be the vertex sets of the connected components of $H[V \setminus C]$. These are also the vertex sets of the connected components of $G[V \setminus C]$. The graphs $H[C_i \cup C]$ are triangulations of $\overline{C_i}$, hence these are partial k-trees.

Conversely, let C be a minimal vertex separator and let $C_1, \ldots, C_t$ be the vertex sets of the connected components of $G[V \setminus C]$. Assume all $\overline{C_i}$ are partial k-trees. Make triangulations H_i of each of these graphs and identify the vertices of C. We claim that the result is a triangulation H of G. This can be seen for example as follows. Let the vertices of C be $c_1, \ldots, c_w$. For each triangulation H_i take a perfect elimination scheme $\sigma_i = [v^i_1, \ldots, v^i_{n_i}, c_1, \ldots, c_w]$. We can construct a perfect elimination scheme for H of the following form: $\sigma = [v^1_1, \ldots, v^1_{n_1}, \ldots, v^t_1, \ldots, v^t_{n_t}, c_1, \ldots, c_w]$. □

2.2 Treewidth and pathwidth

In this section we give a brief introduction to some important notions related to partial k-trees: treewidth, pathwidth, tree-decomposition and path-decomposition.

We define the treewidth of a graph in two ways and show the equivalence of the two definitions (see also [20, 117]). The first definition, which we have seen already in the previous section, is by means of partial k-trees and the second is by means of tree-decompositions.

Definition 2.2.1 *The* treewidth *of a graph* G *is the minimum value* k *for which* G *is a partial* k*-tree.*

Determining the treewidth or the pathwidth of a graph is NP-hard [4]. However, for *constant* k, graphs with treewidth $\leq k$ are recognizable in linear time, see chapter 13 and chapter 14.

Notice that a k-clique has treewidth $k-1$ (since it is a $(k-1)$-tree). The following result may serve as an example, and is needed in later chapters (see also, e.g., [32]).

Lemma 2.2.1 *The complete bipartite graph* $G = K(m, n)$ *has treewidth equal to* $\min(m, n)$.

Proof. First notice that in *any* triangulation of G at least one of the color classes is a clique. Otherwise there are nonadjacent vertices x and y in one colorclass and nonadjacent vertices p and q in the other colorclass, and $H[x, y, p, q]$ would be a chordless cycle. This shows that the treewidth is at least $\min(m, n)$.

For a bipartite graph $G = (X, Y, E)$ the graph $\mathrm{split}(G) = (X, Y, \hat{E})$ is defined as the graph with $\hat{E} = E \cup \{(x, x') \mid x, x' \in X\}$. A *split graph* is a graph for which there exists a partition of the vertex set into a clique and a stable set. Assume without loss of generality that $|X| \leq |Y|$. If G is complete bipartite then $\mathrm{split}(G)$ is a k-tree with $k = |X|$ consisting of a maximal clique with $|X| + 1$ vertices and a set of $|Y| - 1$ simplicial vertices. This proves that the treewidth is at most $\min(m, n)$. □

We now come to the second way to define the treewidth of a graph, namely by a concept called the tree-decomposition of a graph.

Definition 2.2.2 *A* tree-decomposition *of a graph* $G = (V, E)$ *is a pair* $D = (S, T)$ *with* $S = \{X_i \mid i \in I\}$ *a collection of subsets of vertices of* G *and* $T = (I, F)$ *a tree, with one node for each subset of* S*, such that the following three conditions are satisfied:*

1. $\bigcup_{i \in I} X_i = V$,
2. *for all edges* $(v, w) \in E$ *there is a subset* $X_i \in S$ *such that both* v *and* w *are contained in* X_i,
3. *for each vertex* x *the set of nodes* $\{i \mid x \in X_i\}$ *forms a subtree of* T.

Definition 2.2.3 *A* path-decomposition *of a graph* G *is a tree-decomposition* (S, T) *such that* T *is a path. A path-decomposition is also denoted as* $(X_1, X_2, \ldots, X_\ell)$.

An alternative way to formulate the third condition is the following: for all $i, j, k \in I$: if j is on the path from i to k in T then $X_i \cap X_k \subset X_j$. Notice that if D is a tree-decomposition for a graph G and H is a subgraph of G with the same vertex set, then D is also a tree-decomposition for H. Also, if H is a subgraph of G and $D = (S, T)$ is a tree-decomposition for G, we can obtain a tree-decomposition D' for H by taking the restriction of every subset in S to the vertex set of H (the three conditions in Definition 2.2.2 are trivially satisfied).

Definition 2.2.4 *Let* $D = (S, T)$ *be a tree- or path-decomposition for a graph* G *and let* H *be any subgraph of* G. *The tree- or path-decomposition for* H *obtained from* D *by taking the restriction of every subset in* S *to the vertex set of* H *is called the* subdecomposition of D for H.

Definition 2.2.5 *The* width *of a tree-decomposition* $(\{X_i | i \in I\}, T = (I, F))$ *is* $\max_{i \in I}(|X_i| - 1)$.

To show the equivalence between the treewidth of a graph and the minimum width over all tree-decompositions, we make use of the following lemma [32], which is a direct consequence of the Helly property satisfied by a set of subtrees of a tree (for alternative proofs see [18, 150]).

Lemma 2.2.2 *Let* $(S = \{X_i \mid i \in I\}, T = (I, F))$ *be a tree-decomposition for* G. *For every clique* C *in* G *there exists a subset* $X_i \in S$ *such that* $C \subset X_i$.

Proof. Let C be a clique in G. For every pair of vertices x and y of C there exists a subset X_i containing both x and y. Also, for every vertex x of C the set of subsets X_i containing x forms a subtree of T. The lemma follows since a family of subtrees of a tree satisfies the Helly property (see Definition 2.1.4). □

Given a tree-decomposition D of a graph G of width k, we can triangulate G into a triangulated graph H with maximal clique number $k+1$ as follows.

Definition 2.2.6 *Let* $D = (S, T)$ *be a tree-decomposition for* $G = (V, E)$. *Define* $H = \mathcal{H}(G, D)$ *as the graph with the same vertex set as* G, *with two vertices in* H *adjacent if and only if they appear in some common subset* $X \in S$. *We call* H *the triangulation of* G *implied by* D.

Lemma 2.2.3 *Let* $D = (S, T)$ *be a tree-decomposition for* $G = (V, E)$ *of width* k. *Then* $H = \mathcal{H}(G, D)$ *is a triangulation of* G *and has clique number* $k+1$. *The tree-decomposition* D *is also a tree-decomposition for* H.

Proof. First we show that G is a subgraph of H. If (v, w) is an edge in G, then v and w appear in some common subset $X \in S$. By definition v and w are thus also adjacent in H. The fact that D is also a tree-decomposition for H follows immediately from Definition 2.2.2. By Lemma 2.2.2 the clique number of H is $k+1$.

It remains to show that H is triangulated. For each vertex $x \in V$, consider the subtree of T consisting of those nodes i for which $x \in X_i$. Two vertices x and y are adjacent in H if and only if the corresponding subtrees intersect. Hence, H is the intersection graph of a family of subtrees of a tree, and by Lemma 2.1.5 H is triangulated. □

An alternative way to prove Lemma 2.2.3 is to show that H has a perfect elimination scheme. This can be seen as follows. First, if T has only one node, H is a clique and thus H is triangulated. If T has more than one node, let i be a leaf and let j be the neighbor of i. Assume that for every vertex $x \in X_i$ there exists another subset X_k $(k \neq i)$ that also contains x. Then, by definition, x must also be contained in X_j and thus $X_i \subset X_j$. Now we can make a new tree-decomposition D' for H by removing X_i from the set of subsets and by removing i from the tree. Otherwise, let x be a vertex in X_i which is contained only in X_i. Then clearly all neighbors of x are contained in X_i and thus x is simplicial. In this way we obtain a perfect elimination scheme for H, showing that H is triangulated.

Corollary 2.2.1 *If* G *has a tree-decomposition of width* k, *then the treewidth of* G *is at most* k.

The next lemma shows the equivalence between the two concepts.

Lemma 2.2.4 *The treewidth of a graph* G *equals the minimum width over all tree-decompositions of* G.

Proof. Let the treewidth of G be k. We show how to construct a tree-decomposition of width k for G. Since the treewidth of G is k we know there exists a k-tree H such that G can be triangulated into H. First notice that a tree-decomposition for H is also a tree-decomposition for G, since G is a subgraph of H with the same set of vertices. Hence it is sufficient to show there exists a tree-decomposition for H of width k.

Since H is triangulated, it is the intersection graph of a family of subtrees of a tree $T = (I, F)$. For each node i in this tree, define a subset X_i consisting of those vertices for which the corresponding subtree contains i. Let $S =$

$\{X_i \mid i \in I\}$. We claim that $D = (S, T)$ is a tree-decomposition of width at most k. It is easy to check that D is indeed a tree-decomposition for H. Furthermore, each subset corresponds with a clique in H and hence has cardinality at most $k+1$. This shows that the width of D is at most k.

The converse is stated in Corollary 2.2.1. □

Lemma 2.2.5 *For any graph with* n *vertices and treewidth* k*, there exists a tree-decomposition* $D = (S, T)$ *with* $|S| \leq n - k$ *such that every subset* $X \in S$ *contains exactly* $k+1$ *vertices.*

Proof. Let $G = (V, E)$ be a graph with n vertices and treewidth k. Let H be a triangulation of G into a k-tree. We show there exists a tree-decomposition as claimed for H. If $n = k+1$, then we take a tree with one node and a corresponding subset containing all nodes. Otherwise let x be a simplicial vertex. By induction there exists a tree-decomposition for $H[V \setminus \{x\}]$ with $n - k - 1$ nodes and such that every subset in the tree-decomposition has $k+1$ elements. Since $N(x)$ is a clique, there is a subset X_i in this tree-decomposition such that $C \subseteq N(x)$. Take a new node $\imath'$ and make this adjacent to i. Let the corresponding subset be $X_{\imath'} = \{x\} \cup N(x)$. □

For a related result see also Lemma 7.2.3 on page 81.

We now show some related results for the path-decomposition of a graph. In the same manner in which triangulated graphs are related to tree-decompositions, interval graphs are related to path-decompositions.

Definition 2.2.7 *A* k-path *is a* k*-tree which is an interval graph. A* partial k-path *is a subgraph of a* k*-path. The* pathwidth *of a graph* G *is the minimum value* k *for which* G *is a partial* k*-path.*

Analogous to Lemma 2.1.13 and Lemma 2.1.14 we can obtain the following results.

Lemma 2.2.6 *An interval graph* G *with at least* $k+1$ *vertices and with* $\omega(G) \leq k+1$ *can be triangulated into a* k*-path.*

Proof. Let $G = (V, E)$ be an interval graph with $n \geq k+1$ vertices and with $\omega(G) \leq k+1$. Let $(X_1, \ldots, X_t)$ be an interval ordering of the maximal cliques of G. We prove there exists a triangulation H of G into a k-path such that there is an interval ordering $(Y_1, \ldots, Y_{n-k})$ of the maximal cliques of H, with $X_1 \subseteq Y_1$. This can be seen as follows. If G has $k+1$ vertices then we can take for H a clique with all vertices and let $Y_1 = V$. Otherwise G can not be a clique and there must exist a vertex $x \in X_1 \setminus X_2$. Then x is a simplical vertex in G and $G[V \setminus \{x\}]$ is an interval graph with at least $k+1$ vertices and with clique number at most $k+1$. There are two cases to consider.

First assume that $(X_1 \setminus \{x\}, X_2, \dots, X_t)$ is an interval ordering of the maximal cliques of $G[V \setminus \{x\}]$. By induction there is a k-path H', which is a triangulation of $G[V \setminus \{x\}]$, and there is an interval ordering $(Y_1, \dots, Y_{n-k-1})$ with $X_1 \setminus \{x\} \subseteq Y_1$. Notice there must be an element $z \in Y_1 \setminus X_1$ (otherwise X_1 contains more than $k+1$ vertices). Define $Y_0 = \{x\} \cup (Y_1 \setminus \{z\})$. Let H be the graph obtained from H' by making x adjacent to all vertices of $Y_0 \setminus \{x\}$. Then H is a k-tree, and $(Y_0, Y_1, \dots, Y_{n-k-1})$ is an interval ordering of the maximal cliques of H (hence H is a k-path) with $X_1 \subseteq Y_0$.

Now assume $X_1 \setminus \{x\}$ is not a maximal clique in $G[V \setminus \{x\}]$. Then $X_1 \setminus \{x\} \subset X_2$. Clearly $(X_2, \dots, X_t)$ is an interval ordering of the maximal cliques of $G[V \setminus \{x\}]$. Let H' be a triangulation of $G[V \setminus \{x\}]$ into a k-path, and let $(Y_2, \dots, Y_{n-k-1})$ be an interval ordering of the maximal cliques of H with $X_2 \subseteq Y_2$. Let $z \in Y_2 \setminus X_1$ (this vertex must exist since $X_2 \setminus X_1 \neq \emptyset$). Let $Y_1 = \{x\} \cup (Y_2 \setminus \{z\})$. Let H be the graph obtained from H' by making x adjacent to all vertices of $Y_1 \setminus \{x\}$. Then H is a k-tree and $(Y_1, \dots, Y_{n-k})$ is an interval ordering of the maximal cliques of H with $X_1 \subseteq Y_1$. □

Lemma 2.2.7 *Every partial* k*-path with at least* $k+1$ *vertices can be triangulated into a* k*-path.*

Proof. Let $G = (V, E)$ be a partial k-path with at least $k+1$ vertices. There is a k-path H such that G is a subgraph of H. Notice that $H[V]$ is an interval graph with maximal clique number at most $k+1$, such that G is a subgraph of $H[V]$. By Lemma 2.2.6, $H[V]$ can be triangulated into a k-path. □

Lemma 2.2.8 *The pathwidth of a graph* G *equals the minimum width over all path-decompositions of* G.

Proof. Assume the pathwidth of G is k. Hence G is the subgraph of an interval graph H which is simultaneously a k-tree. Since H is a k-tree all maximal cliques of H have number $k+1$. We show there exists a path-decomposition for H of width k. By Lemma 2.1.6 there is an interval ordering $(X_1, X_2, \dots, X_\ell)$ of the maximal cliques of H. It is easy to check that $(X_1, X_2, \dots, X_\ell)$ is a path-decomposition for H of width k.

Conversely let $D = (X_1, X_2, \dots, X_\ell)$ be a path-decomposition for G of width k. Let H be the triangulation of G implied by D. Each maximal clique of H is contained in some subset X_i and each subset contains at most one maximal clique. Hence there exists an ordering of the maximal cliques of H such that for every vertex the maximal cliques containing it are consecutive. Hence H is an interval graph with clique number $k+1$. It follows that H can be triangulated into a k-path. □

We show that the treewidth and pathwidth of a graph differ at most by a factor $\log n$. We need the following lemma. For a slightly more general result see also [143]. See also Lemma 2.1.4.

Lemma 2.2.9 *Every* k*-tree* $G = (V, E)$ *contains a clique* C *with* $k+1$ *vertices such that every connected component of* $G[V \setminus C]$ *has at most* $\frac{1}{2}(n-k)$ *vertices.*

Proof. Consider the following algorithm. Start with any $k+1$-clique S_0. Assume there is a connected component C in $G[V \setminus S_0]$ which has more than $\frac{1}{2}(n-k)$ vertices. Notice that the other components together have less than $\frac{1}{2}(n-k)-1$ vertices. There exists a vertex x in C which has k neighbors in S_0. Let $y \in S_0 \setminus N(x)$. Define $S_1 = \{x\} \cup (N(x) \cap S_0)$. Notice that S_1 also has $k+1$ vertices. The algorithm continues with S_1.

To show that this algorithm terminates, we prove that in each step of the algorithm the number of vertices in the largest component decreases. Notice that $G[V \setminus S_1]$ has two types of components. One type consists only of vertices of $C \setminus \{x\}$. If the largest component of $G[V \setminus S_1]$ is among these, the number of vertices has clearly decreased. The other type of components consists only of vertices of $\{y\} \cup V \setminus (C \cup S_0)$. By the remark above, the total number of vertices in this set is less than $\frac{1}{2}(n-k)$. As the largest component of $G[V \setminus S_0]$ has more than this number of vertices, this shows that the number of vertices in the largest component of $G[V \setminus S_1]$ is at least one less than the number of vertices in the largest component of $G[V \setminus S_0]$. This shows that the algorithm terminates. □

Corollary 2.2.2 *Let* G *be a graph with* n *vertices and treewidth* k*. There exists a set* S *with* $k+1$ *vertices such that every connected component of* $G[V \setminus S]$ *has at most* $\frac{1}{2}(n-k)$ *vertices.*

Lemma 2.2.10 *Let* $G = (V, E)$ *be a graph with* n *vertices and treewidth* $k \geq 1$*. Then the pathwidth of* G *is at most* $(k+1)\log n - 1$*.*

Proof. Lemma 2.2.9 shows there is a clique X with $k+1$ vertices such that every connected component of $G[V \setminus X]$ has at most $\frac{1}{2}(n-k)$ vertices.

If $n = k+1$ the upperbound on the pathwidth clearly holds. We proceed by induction. Let $n > k+1$, and let X be a balanced separator as mentioned above. Let $C_1, \ldots, C_t$ be the connected components of $G[V \setminus X]$. By induction there are path-decompositions P_i for the induced subgraphs $G[C_i]$ with pathwidth $\leq (k+1)\log|C_i| - 1$. Add X to every subset of every path-decomposition, and let P_i' $(i = 1, \ldots, t)$ be the new path-decompositions so obtained. Let P be the concatenation of the P_i''s, i.e., the path-decomposition obtained by putting all P_i''s after each other: $P = P_1' +\!\!+ \ldots +\!\!+ P_t'$. Clearly, the width of P is at most $k + (k+1)\log\frac{n-k}{2} \leq (k+1)\log n - 1$. □

2.3 Perfect graphs

Perfect graphs were introduced by Claude Berge around 1960 (see [13]). A graph $G = (V, E)$ is called *perfect* if the following two conditions are both satisfied: First the *clique number* and the *chromatic number* must be equal for all induced subgraphs, (i.e. $\omega(G[A]) = \chi(G[A])$ for all $A \subseteq V$) and second, the *stability number* must equal the *clique cover number* for all induced subgraphs of G (i.e. $\alpha(G[A]) = k(G[A])$ for all $A \subseteq V$). Notice that it is quite a natural question to ask which graphs are perfect, since all graphs satisfy $\omega(G) \leq \chi(G)$ and $\alpha(G) \leq k(G)$. Notice also that the two conditions are dual in the sense that a graph satisfies the first condition if and only if its complement satisfies the second. The remarkable fact that a graph satisfies the first equality if and only if it satisfies the second equality, was conjectured by Berge [12] and proven by Lovász [121]. This has become known as the *perfect graph theorem*. Lovász also proved that the two conditions are equivalent to a third condition, namely: $\omega(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V$.

A graph is called *minimal imperfect* (or p*-critical*) if it is not perfect but every proper induced subgraph of it is. The *strong perfect graph conjecture* made by Berge (see [14]) states that the only minimal imperfect graphs are the chordless odd cycles of length at least five and their complements. This is equivalent to saying that a graph G is perfect if and only if in G and in $\overline{G}$ every chordless odd cycle of length at least five has a chord. The chordless odd cycles of length at least five and their complements are often referred to as the odd holes and the odd antiholes, respectively. Until now, the strong perfect graph conjecture is unsettled.

It is interesting to notice that for every constant k there is a polynomial time algorithm to test whether a given partial k-tree satisfies the strong perfect graph conjecture. This can be seen as follows. Let G be a partial k-tree. We may assume that a tree-decomposition of G of width bounded by some constant is given. Using standard techniques it can be tested in linear time whether or not the graph is perfect (for example, it can be stated in monadic second order logic [49] or, it can be tested using dynamic programming). Also, testing whether the graph has an odd hole can be done in linear time using one of these techniques. The only thing left to test is whether the graph has an odd antihole. Now notice that an antihole $\overline{C_{2t+1}}$ has a clique of number at least t. Since a partial k-tree can only have cliques of size at most $k+1$ it follows that we can restrict the search for odd antiholes to those with at most $2k+3$ vertices. It follows that we can perform this test also in linear time. In chapter 14 we show a linear time algorithm to obtain an optimal tree-decomposition for the graph. This proves that for each k there is a linear time algorithm to test whether a given partial k-tree satisfies the strong perfect graph conjecture. Using results of [48] we can even obtain a much stronger result; for each k there is an algorithm to decide whether or not for *every* partial k-tree the strong perfect graph conjecture holds. We do not claim that this is a very practical algorithm. This is even more so since for partial 3-trees the strong perfect graph conjecture is already known to hold. This can be seen as follows. Notice that every graph with at most three vertices is either bipartite or a clique. Consider a partial k-tree for $k \leq 3$ which is minimal imperfect. Then there is a vertex u with at most three neighbors. Hence the neighborhood of this vertex induces a bipartite graph or a clique (which is multipartite). In [87] it is shown that every minimal imperfect graph which has a vertex of which the neighborhood induces a bipartite or multipartite graph, must be an odd hole or an odd antihole.

Since the discovery of perfect graphs in 1960, much research has been devoted to special classes of perfect graphs. Among the most well-known classes of perfect graphs are the comparability graphs and the triangulated graphs. The class of triangulated graphs contains graph classes such as interval graphs, split graphs, k-trees, and indifference graphs. The class of comparability graphs contains complements of interval graphs, permutation graphs, threshold graphs and P_4-free graphs (or cographs). Much work has been done to characterize these graph classes and to find relationships between them. Interest has only increased since Lovász settled the perfect graph conjecture. From an algorithmic point of view, perfect graphs have become of great interest since Grötschel, Lovász and Schrijver discovered polynomial time algorithms for NP-hard problems like CLIQUE, STABLE SET and CHROMATIC NUMBER for perfect graphs [15]. Special classes of perfect graphs have proven their importance by the large number of applications (see for example [76, 15, 141] for applications in general and [46] for an overview of applications of interval graphs).