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ON SIMPLE CHARACTERIZATIONS OF k-TREES*

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Abstract. k-trees are a special class of perfect elimination graphs which arise in the study of sparse linear systems. We present four simple characterizations of k-trees involving cliques, paths, and separators.

1. Introduction

k-trees are a special class of Michigan graphs,[1] that is, graphs $G = (X, E)$, where X is a nonempty finite set of vertices and E is a set of pairs of distinct vertices called edges. Recalling that a *clique* in G is a nonempty subset of vertices each distinct pair of which is an edge of G, k-trees are defined recursively as follows. A *k-tree* on k vertices is a graph whose vertex set is a clique on k vertices (k-clique); and given any k-tree $T_k(n)$ on n vertices, a k-tree on $n+1$ vertices is obtained when the $(n+1)$st vertex is made adjacent to each vertex of a k-clique in $T_k(n)$.

Let $T_k(n) = (X, E)$ be a k-tree on n vertices and let $X = \{x_i\}_{i=1}^{n}$, where x_i is the vertex added to the k-tree $T_k(i-1)$ to produce the k-tree $T_k(i)$, $i > k$, $\{x_i\}_{i=1}^{k}$ the "base" clique. Note then that k-trees are *perfect elimination graphs*, that is, graphs $G = (Y, E)$ for which there exists an ordering of the vertex set, say $Y = \{y_i\}_{i=1}^{n}$ ($|Y| = n$), such that in the vertex induced subgraph $G(Y - \{y_j\}_{j=1}^{i-1})$ the set

$$C_i = \{y_i\} \cup \operatorname{Adj}(y_i)$$

is a clique. That is, when $i = 1$, y_1 and its adjacent vertices in G are a

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[1] Generally our graph-theoretic terminology follows [1].

clique, while for $i > 1$, y_i and its adjacent vertices in the subgraph are a clique. For the k-tree $T_k(n) = (X, E)$ with x_i as above, the *perfect elimination ordering* $\{y_i\}_{i=1}^n$ is defined simply by $y_i = x_{n+1-i}$.

Let $G = (X, E)$ be a graph with $c(G)$ connected components. Recall that a separator S of G is a nonempty subset of X such that the vertex induced subgraph $G(X-S)$ has $c(G(X-S)) > c(G)$. For connected G, $G(X-S)$ has two or more connected components, say $C_i = (V_i, E_i)$. The subgraphs $G(V_i \cup S)$ then are the *leaves* of G with respect to S. Similarly, for $x, y \in X$ with $xy \notin E$ and x and y in the same component of G, an *x, y separator* S is a separator such that x and y are in distinct components of $G(X-S)$. Recall also that any minimal separator is a minimal x, y separator for some $x, y \in X$, but a minimal x, y separator need not be a minimal separator ([4, Fig.1, p. 193]).

Perfect elimination graphs and their role in the algebraic process of symmetric Gaussian elimination in sparse symmetric matrices has been discussed extensively in [4]. Here we apply a portion of the theory developed there to provide a simple characterization of k-trees.

Theorem 1.1. *A graph $G = (X, E)$ is a k-tree if and only if*
(i) *G is connected,*
(ii) *G has a k-clique but no $k+2$ clique,*
(iii) *every minimal x, y separator of G is a k-clique.*

The necessity of (iii) was essentially established in [4, p. 201], however, the presentation given below is clearer. Even in the case $k = 1$ (trees), the result does not appear to be well known, although in this case it follows easily from other characterizations of trees (see [1, Theorem 4.1, p. 32]). We have for trees the following:

Corollary 1.2. *A graph G is a tree iff G is connected, without triangles, and every minimal x, y separator is a single vertex.*

For a different approach to the characterization of 2-trees via a generalization of the notion of being "acyclic", see [2]. Our presentation deals only with cliques, paths and separators.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we will borrow some results about perfect elimination graphs as discussed in [4].

Proposition 2.1 ([4, p. 196]). *Let $G = (X, E)$ be connected with a separator S which is a clique (separation clique) and leaves L_i, $1 \leq i \leq n$. If S_0 is a separator of some L_i, then S_0 is a separator of G. Furthermore, if S_0 is a minimal x, y separator of L_i, then S_0 is a minimal x, y separator of G.*

Proposition 2.2 ([4, p. 194]). *A graph $G = (X, E)$ is a perfect elimination graph if and only if every minimal x, y separator is a clique.*

We now begin the proof of Theorem 1.1. Let $G = (X, E)$ be a graph with $|X| = n$; for any fixed $k \leq n$, we proceed by induction on n. When $n = k$ or $n = k + 1$, the equivalence of the k-tree definition and (i)–(iii) is immediate since X of G must then be a clique. Assuming the equivalence for graphs with $k + 1 \leq |X| \leq n-1$, we consider a graph with $|X| = n$.

Necessity. G is a k-tree on n vertices; let x_n be the vertex added to the k-tree on $n-1$ vertices in the recursive definition of G. Hence $G(X- x_n)$ is a k-tree on $n-1$ vertices. G is connected, and G contains a k-clique but no $k+2$ clique since this is true for $G(X-x_n)$ and $|\mathrm{Adj}(x_n)| = k$.

It remains to show that every minimal x, y separator S of G is a k-clique. Certainly, $x_n \cup \mathrm{Adj}(x_n)$ must be in the same leaf of G with respect to S. If $S = \mathrm{Adj}(x_n)$, S is a k-clique. Otherwise, (since $n \geq k + 2$) S is a minimal x, y separator of $G(X-x_n)$, or, if $x = x_n$, S is a minimal a, y separator of $G(X-x_n)$ for some $a \in \mathrm{Adj}(x_n)$.

Sufficiency. Let $G = (X, E)$ with $|X| = n$ satisfy (i)–(iii) of Theorem 1.1. Then G is a perfect elimination graph (Proposition 2.2) and has a vertex, say x, such that $\{x\} \cup \mathrm{Adj}(x)$ is a clique. Certainly, $|\mathrm{Adj}(x)| \leq k$ since there are not $k+2$ cliques in G. Furthermore, since $|X| \geq k+2$ and G is connected, $\mathrm{Adj}(x)$ is a separator, hence $|\mathrm{Adj}(x)| \geq k$.

So $|\mathrm{Adj}(x)| = k$ and we finish by showing that $G(X-x)$ is a k-tree; then G is a k-tree by definition. But certainly $G(X-x)$ satisfies (i) and (ii); (iii) follows by applying Proposition 2.1 since $G(X-x)$ is a leaf of G with respect to $\mathrm{Adj}(x)$. By the induction hypothesis, $G(X-x)$ is a k-tree.

Applying Theorem 1.1 and Proposition 2.1, we have immediately:

Corollary 2.3. *Let $G = (X, E)$ be a k-tree with separation clique S. Then each leaf of G with respect to S is a k-tree. In particular, for* $|X| \geq k+1$, *if* $\{x\} \cup \mathrm{Adj}(x)$ *is a clique of G, then* $G(X-x)$ *is a k-tree.*

Suppose S is a minimal x, y separator of a k-tree; then $|S| = k$. If S were not a minimal separator, it must contain properly a minimal separator, say S_0, which is a minimal u, v separator for some $u, v \in X$. But then $k = |S_0| \leq k-1$ so S itself must be a minimal separator. Hence we have:

Corollary 2.4. *For a k-tree, every minimal x, y separator is a minimal separator.*

3. Other characterizations

In this section we consider some related results about k-trees. Recall that a graph $G = (X, E)$ is a tree iff G is connected and $|E| = |X| - 1$. For a k-tree,

$$|E| = \tfrac{1}{2} k(k-1) + (|X| - k)k = k|X| - \tfrac{1}{2}k(k+1)\,. \tag{3.1}$$

Two characterizations involving (3.1) are presented below.

Proposition 3.1. *Let $G = (X, E)$ be a graph with* $|X| \geq k$ *satisfying* (ii) *of Theorem* 1.1 *and*

(iv) *every minimal x, y separator is a clique.*

Then $|E| \leq k\,|X| - \frac{1}{2}k(k+1)$ *with equality holding iff G is a k-tree.*

Proof. We note that there exists a perfect elimination ordering, say $X = \{x_i\}_{i=1}^{n}$ ($|X| = n$), by Proposition 2.2. Furthermore, we may assume without loss of generality by [4, Corollary 4, p. 198] that the k-clique C guaranteed by (ii) is ordered last; i.e., $C = \{x_i\}_{i=m+1}^{n}$ with $m = |X| - k$. Thus $\{x_i\} \cup \mathrm{Adj}(x_i)$ in $G(X - \{x_j\}_{j=1}^{i-1})$ is a clique, $1 \leq i \leq m$, and by (ii), $|\mathrm{Adj}(x_i)| \leq k$. Now such adjacency sets for $1 \leq i \leq m$ in their respective induced subgraphs, count exactly all edges of E except for those $G(C)$. Hence

$$|E| \leq \tfrac{1}{2}k(k-1) + (|X| - k)k = k|X| - \tfrac{1}{2}k(k+1)\,.$$

Clearly the inequality is strict unless $|\mathrm{Adj}(x_i)| = k$, $1 \leq i \leq m$, in which case, by definition, G is a k-tree.

The following result is an immediate corollary since necessity is clear.

Theorem 3.2. *$G = (X, E)$ is a k-tree if and only if* (3.1), *and* (ii) *and* (iv) *are satisfied.*

Proposition 3.3. *Let $G = (X, E)$ be a graph with $|X| \geq k$ satisfying* (i) *and* (iii) *of Theorem* 1.1. *Then* $|E| \geq k\,|X| - \tfrac{1}{2}k(k+1)$.

Proof. We sketch the inductive proof, letting G be a graph with $|X| \geq k+1$. Let x be a vertex such that $\{x\} \cup \mathrm{Adj}(x)$ is a clique (existence by Proposition 2.2). Then, since G is connected, either $X = \{x\} \cup \mathrm{Adj}(x)$ and the inequality is satisfied, or $\mathrm{Adj}(x)$ is a separation clique with $|\mathrm{Adj}(x)| \geq k$ by (iii). Using induction on $G(X-x) = (X', E')$, we have

$$|E'| \geq k\,|X'| - \tfrac{1}{2}k(k+1)\,.$$

Adding $|\mathrm{Adj}(x)|$ on the left and k on the right gives

$$|E| \geq k\,|X| - \tfrac{1}{2}k(k+1).$$

Theorem 3.4. *$G = (X, E)$ is a k-tree if and only if* (i), (3.1) *and* (iii) *are satisfied.*

Proof. Sufficiency is by induction and follows by observing that (3.1) implies that $|\mathrm{Adj}(x)| = k$ (where $\{x\} \cup \mathrm{Adj}(x)$ is a clique) and the inequality of Proposition 3.3 for $G(X-x)$ is an equality. Hence by induction on $|X|$, $G(X-x)$ is a k-tree, implying G is a k-tree.

With a little help from Menger's theorem [1, p. 47], we have:

Theorem 3.5. *A graph $G = (X, E)$ is a k-tree iff* (ii), (iv) *and*

(v) *for all distinct nonadjacent pairs $x, y \in X$, there exist exactly k vertex-disjoint (except for x and y) x, y paths;*

are satisfied.

Proof. Sufficiency is proved by induction of $|X|$, the cases $|X| = k$ and $|X| = k+1$ being clear. For $|X| \geq k+2$, let $\{x\} \cup \mathrm{Adj}(x)$ be the clique in G guaranteed by Proposition 2.2. Since G is connected by (v), we have that (ii) and (v) imply $|\mathrm{Adj}(x)| = k$.

Clearly (ii) holds in $G(X-x)$; (iv) holds by Proposition 2.1. Finally, given any nonadjacent u, v in $G(X-x)$, the k disjoint u, v paths in G imply k disjoint u, v paths in $G(X-x)$ since any u, v path in G containing x also contains two vertices of the clique $\mathrm{Adj}(x)$. Hence (v) holds in $G(X-x)$, there being no more than k disjoint u, v paths in $G(X-x)$. Thus by induction, $G(X-x)$ is a k-tree as is G.

We need only show necessity of (v) which follows from (iii) by Menger's theorem.

As a final remark we note that (iv) above may be replaced by any of several known equivalent conditions. See, for example, [4, p. 194] and [3].

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