4 Logarithms

Introduction

Every year roughly two or three questions are asked in CAT and other management entrance exams from logarithms. This topic can be mastered in a day or two since the fundamentals involved in this topic are easy to grasp. Proper understanding of the concepts involved in this topic would help students solve almost all the questions tested in various exams.

Definition

Every positive real number N can be expressed in an exponential form as

$N = a^x$

where 'a' is called the base and 'x' is called the exponent.

We can say that

x = logarithm of N to the base a and write it as $x = \log_a N$

$N = a^x \Leftrightarrow x = \log_a N$

(The above conversion from exponent to logarithm and vice versa is very important for solving questions.)

Thus, in general, we can say that,

If $A^B = C$, then it can be converted into logarithm as $B = \log_A C$.

For example, given $2^6 = 64$, we can write as $6 = \log_2 64$.

Or if given $\log_B A = C$, then it can be converted into exponential form as $A = B^C$.

For example, given $\log_3 81 = 4$, we can write it as $81 = 3^4$.

Some Basic Examples

Example 1:

If $\log_{81} 27 = x$, then find the value of x

Solution: 3/4

Given $\log_{81} 27 = x$

$\Rightarrow 27 = 81^x$

$\Rightarrow 3^3 = 3^{4x}$

$\Rightarrow 4x = 3$

$\Rightarrow x = 3/4$

Example 2:

If $\log_{\frac{1}{3}} 9\sqrt{3} = x$, then find the value of x.

Solution:

$$\left(\frac{-5}{2}\right)$$

$$\log_{\frac{1}{3}} 9\sqrt{3} = x$$

$$\Rightarrow \left(\frac{1}{3}\right)^x = 9\sqrt{3}$$

$$\Rightarrow 3^{-x} = 3^{\frac{5}{2}}$$

$$\Rightarrow x = \frac{-5}{2}$$

Example 3:

If $\log_{(x-2)}(x^2 - 5x + 7) = 0$, then find the value of x.

Solution: 3

Given that $\log_{(x-2)}(x^2 - 5x + 7) = 0$

$\Rightarrow x^2 - 5x + 7 = (x - 2)^0$

$\Rightarrow x^2 - 5x + 7 = 1$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 3)(x - 2) = 0$

$\Rightarrow x = 3, 2$

But at $x = 2$, the logarithm base will become zero. Hence, $x = 2$ is not acceptable.

Hence, $x = 3$ is the correct answer.

Example 4:

If $\log_4(x^2 - 9) = 2$, then find the value of x.

Solution: ±5

Given that $\log_4(x^2 - 9) = 2$

$\Rightarrow x^2 - 9 = 16$

$\Rightarrow x^2 = 25$

$\Rightarrow x = \pm 5$

Domain of Logarithm

For $\log_b a$ to be defined, the following conditions must be satisfied:

(i) $a > 0$ $a \to (0, \infty)$ (ii) $b > 0$ (iii) $b \neq 1$

For example, If $f(x) = \log_{x^2-5}(x^2 + 3x + 5)$, then find the range of x.

Case 1: $x^2 + 3x + 5 > 0 \Rightarrow x \in R$

Case 2: $x^2 - 5 > 0$

$\Rightarrow (x + \sqrt{5})(x - \sqrt{5}) > 0$

$\Rightarrow x \in \left(-\infty, -\sqrt{5}\right) \cup \left(\sqrt{5}, \infty\right)$

Case 3: $x^2 - 5 \neq 1$

$\Rightarrow x^2 \neq 6$

$\Rightarrow x \neq \pm\sqrt{6}$

Merging cases 1 to 3

$\Rightarrow x \in \left(-\infty, -\sqrt{5}\right) \cup \left(\sqrt{5}, \infty\right) - \left\{-\sqrt{6}, \sqrt{6}\right\}$

Properties of Logarithms

1. Logarithms of negative numbers are not defined.

For example, $\log_2(-10)$ = Not defined.

2. Logarithm of 1 to any positive base other than 1 is always equal to zero.

For example, $\log_2 1 = 0$, $\log_5 1 = 0$.

3. Logarithm of a number to the same base is equal to one.

For example,

a) $\log_{20} 20 = 1$

b) $\log_a a = 1$ if $a > 0$, $a \neq 1$

4. Product law

$\log_a mn = \log_a m + \log_a n$

where $m > 0$, $n > 0$, $a > 0$, and $a \neq 1$

Proof:

Let $\log_a m = x$ $a^x = m$...(i)

and $\log_a n = y$ $a^y = n$...(ii)

Multiplying equations (i) and (ii), we will get

$a^x \cdot a^y = mn$

$a^{x+y} = mn$

Now, after converting exponent to logarithm, we will get

$x + y = \log_a mn$

Put the value of x and y from equations (i) and (ii), we will get

$\log_a m + \log_a n = \log_a mn$

Example 5:

$\log_{10} 50 + \log_{10} \frac{1}{5} = ?$

Solution: 1

Given t

hat $\log_{10} 50 + \log_{10} \frac{1}{5}$

$\Rightarrow \log_{10}\left(50 \times \frac{1}{5}\right)$

$\Rightarrow \log_{10} 10 = 1$

Example 6:

If $\log_{20} 20x^2 + \log_{20} x = 2$, then find the value of x.

Solution: $\log_{20} 20x^2 + \log_{20} x = 2$

$\Rightarrow \log_{20} 20 + \log_{20} x^2 + \log_{20} x = 2$

$\Rightarrow 1 + \log_{20}(x^2 \times x) = 2$

$\Rightarrow \log_{20} x^3 = 1$

$\Rightarrow x^3 = 20$

$\Rightarrow x = 20^{\frac{1}{3}}$

5. Division law

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Where $m, n > 0$, $a > 0$, and $a \neq 1$

Proof

Let $\log_a m = x$ $a^x = m$ ...(i)

Similarly, $\log_a n = y$ $a^y = n$ ...(ii)

Dividing (i) and (ii)

$$\frac{a^x}{a^y} = \frac{m}{n}$$

$$a^{x-y} = \frac{m}{n}$$

On converting exponent into logarithm, we will get

$$x - y = \log_a \frac{m}{n}$$

Put the value of x and y from equations (i) and (ii)

$$\log_a m - \log_a n = \log_a \frac{m}{n}$$

Example 7:

If $\log_5 x - \log_5 2 = 2$, find the value of x.

Solution: $\log_5 x - \log_5 2 = 2$

$$\Rightarrow \log_5 \frac{x}{2} = 2$$

$$\Rightarrow \frac{x}{2} = 5^2$$

$$\Rightarrow x = 50$$

6. Exponential law

$\log_a mn = n \log_a m$ where $m > 0$, $a > 0$, $a \neq 1$, and $n \in R$

Proof:

Let $\log_a m = x$ $\quad\therefore a^x = m$

$\therefore (a^x)^n = mn$

$\therefore mn = a^{xn}$

Converting exponent into logarithm, we will get

$\log_a mn = x \times n$

$\therefore \log_a mn = (\log_a m) \times n$

$\therefore \log_a mn = n \log_a m$

For example, $\log_3 729 = \log_3 3^6 = 6 \log_3 3 = 6$

Example 8:

If $\log_2 x^2 + \log_2 x = 6$, then find the value of x.

Solution: 4

Given $\log_2 x^2 + \log_2 x = 6$

$\Rightarrow 2 \log_2 x + \log_2 x = 6$

$\Rightarrow 3 \log_2 x = 6$

$\Rightarrow \log_2 x = 2$

$\Rightarrow x = 4$

Example 9:

If $a^2 + 4b^2 = 12ab$, then $\log(a + 2b) = ?$

(A) $\frac{1}{2}(4\log 2 + \log a + \log b)$

(B) $\frac{1}{2}(3\log 2 + \log a + \log b)$

(C) $\frac{1}{2}(4\log 2 + \log a - \log b)$

(D) $\frac{1}{2}(3\log 2 + \log a - \log b)$

Solution: A

$a^2 + 4b^2 = 12ab$

Adding $4ab$ on both the sides to make it a perfect square

$a^2 + 4b^2 + 4ab = 16ab$

$(a + 2b)^2 = 16ab$

Taking log on both sides

$\log (a + 2b)^2 = \log 16ab$

$2\log (a + 2b) = \log 16 + \log a + \log b$

$2\log (a + 2b) = \log 2^4 + \log a + \log b$

$\therefore \log (a + 2b) = \frac{1}{2}[4\log 2 + \log a + \log b]$

7. Equality law

1. If $\log_a m = \log_a n$, then $m = n$ where $m, n > 0$ $a > 0, a \neq 1$	2. If $\log_a m = \log_b m$, then $a = b$ where $m, a, b > 0$ $a \neq 1, b \neq 1$

Example 10:

If $\log_6(x + 3) - \log_6 x = \log_6 36$, then find the value of x.

Solution:

$$\log_6(x + 3) - \log_6 x = \log_6 36$$

$$\Rightarrow \log_6\left(\frac{x+3}{x}\right) = \log_6 36$$

$$\Rightarrow \frac{x+3}{x} = 36$$

$$\Rightarrow x + 3 = 36x$$

$$\Rightarrow 35x = 3$$

$$\Rightarrow x = \frac{3}{35}$$

8. Base change property

$$\log_b a = \frac{\log_c a}{\log_c b}$$

where, $a > 0$, $b > 0$, $b \neq 1$, and $c > 0$, $c \neq 1$

Proof

Let $\log_b a = x$

$\Rightarrow a = b^x$

$\Rightarrow \log_c a = \log_c b^x$

$\Rightarrow \log_c a = x \log_c b$

$$\therefore x = \frac{\log_c a}{\log_c b}$$

$$\therefore \log_b a = \frac{\log_c a}{\log_c b}$$

Corollary 1:

$$\log_b a = \frac{1}{\log_a b}$$

For example, $\log_3 5 = \frac{1}{\log_5 3}$

Corollary 2:

$$\log_{b^k} a = \frac{1}{k}\log_b a$$

For example, $\log_{3^5} 5 = \frac{1}{5}\log_3 5$

Corollary 3:

$$\log_{b^n} a^m = \frac{m}{n}\log_b a$$

where, $n \neq 0$, $a > 0$, $b > 0$, $b \neq 1$

for example, $\log_{5^3} 7^2 = \frac{2}{3}\log_5 7$

Example 11: If $\log_3 x - 2\log_{\frac{1}{3}} x = 6$, then find the value of x.

Solution:

$$\log_3 x - 2\log_{\frac{1}{3}} x = 6$$

$$\Rightarrow \log_3 x - 2\log_3 \frac{1}{x} = 6$$

$$\Rightarrow \log_3 x - 2\log_3 x^{-1} = 6$$

$$\Rightarrow \log_3 x + 2\log_3 x = 6$$

$$\Rightarrow 3\log_3 x = 6$$

$$\Rightarrow \log_3 x = 2$$

$$\Rightarrow x = 3^2 = 9$$

Example 12:

$$\frac{1}{\log_{ab} abc} + \frac{1}{\log_{bc} abc} + \frac{1}{\log_{ac} abc} = ?$$

(where, a, b, $c > 0$ also, a, b, $c \neq 1$)

Solution:

$$\frac{1}{\log_{ab} abc} + \frac{1}{\log_{bc} abc} + \frac{1}{\log_{ac} abc}$$

$$\Rightarrow \log_{abc} ab + \log_{abc} bc + \log_{abc} ac$$

$$\Rightarrow \log_{abc}(ab \times bc \times ac) = \log_{abc}(abc)^2$$

$$\Rightarrow 2\log_{abc} abc = 2$$

9. $x = a^{\log_a x}$ where $a > 0$, $a \neq 1$, and $x > 0$

Proof

$\log_a x = \log_a x$

By converting logarithm into the exponent, we will get

$$x = a^{\log_a x}$$

for example, $5^{\log_5 7} = 7$

Example 13:

If $3^{\log_3 x-2} + 2^{\log_2 2x-3} = 5$, then find the value of x.

Solution:

$$3^{\log_3 x-2} + 2^{\log_2 2x-3} = 5$$

$$(x - 2) + (2x - 3) = 5$$

$$3x = 10$$

$$x = \frac{10}{3}$$

10. $a^{\log_c b} = b^{\log_c a}$ where $b > 0$, $a > 0$, $c > 0$, and $c \neq 1$

Proof

$$\text{L.H.S} = a^{\log_c b}$$

$$= a^{\log_c b \times \log_a a}$$

$$= a^{\log_a b \times \log_c a}$$

$$= \left(a^{\log_a b}\right)^{\log_c a}$$

$$= b^{\log_c a}$$

$$= \text{R.H.S}$$

Example 14:

$3^{\log_{81} 625} = ?$

Solution:

$$3^{\log_{81} 625}$$

$$= 625^{\log_{81} 3}$$

$$= 625^{\log_{3^4} 3}$$

$$= 625^{\frac{1}{4}\log_3 3}$$

$$= 625^{\frac{1}{4}}$$

$$= \left(5^4\right)^{\frac{1}{4}} = 5$$

Common Logarithms

These are the logarithms whose base is 10. If logarithms are given without any base, they can be taken as logarithms to the base 10.

For example, $\log 5 = \log_{10} 5$

The common logarithm of any positive number N can be expressed as:

Log N = characteristics + mantissa

where characteristic is always an integral part and mantissa is always a non-negative decimal part between 0 and 1.

For example, if $\log K = 2.0587$, then characteristic is two and mantissa is 0.0587.

The value of mantissa is generally given in the questions.

Characteristics of common logarithm

- Characteristics of the common logarithm of any number > 1 are positive, and it is one less than the number of digits in its integral part.

  For example, the characteristics of log 234 will be 2. (As 234 is a three-digit number and characteristics will be one less than the number of digits in 234.)

  Similarly,

  Characteristics of log 45678 = 4

  Characteristics of log 5 = 0

  Characteristics of log 45.56 = 1

  [As characteristics is one less than the integral part (45) of number.]

- Characteristics of the common logarithm of any number between 0 and 1 is negative, and its magnitude is one more than the number of consecutive zeros immediately after the decimal point.

  For example, the characteristics of log 0.007205 will be –3. (As we have two consecutive zeros immediately after the decimal.)

  Similarly,

  Characteristics of log 0.00089600004 = – 4.

Application of logarithm

Finding the number of digits in a^b.

Example 15:

Find the number of digits in 'n' where $n = 2^{12} \times 7^{10}$.

Given: $\log_{10}2 = 0.301$ and $\log_{10}7 = 0.8451$.

Solution:

$n = 2^{12} \times 7^{10}$

Taking log base 10 on both sides

$\log_{10}n$

$= \log_{10}(2^{12} \times 7^{10})$

$= \log_{10}2^{12} + \log_{10}7^{10}$

$= 12 \log_{10}2 + 10 \log_{10}7$

$= 12 \times 0.301 + 10 \times 0.8451$

$= 3.612 + 8.451 = 12.063$

So, $\log n = 12.063$ = characteristic (12) + mantissa (0.063).

Since, we know that the characteristic is one less than the number of digits.

Hence, the number of digits in $n = (12 + 1) = 13$.

Example 16:

Find the number of digits in 2^{2009}.

(Given that $\log_{10}2 = 0.301$).

Solution:

$n = 2^{2009}$

$\log_{10}n = \log_{10}2^{2009}$

$= 2009 \log_{10}2$

$= 2009 \times 0.301$

$= 604.709$ = characteristic (604) + mantissa (0.709).

Hence, the number of digits in 2^{2009}

$= 604 + 1 = 605$.

Graphical Representation of Logarithm

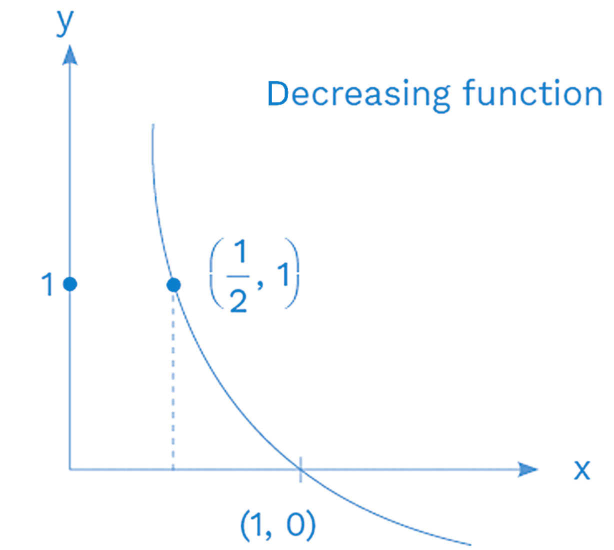
Case 1: If $y = \log_a x$ where $x > 0$, and $0 < a < 1$

If we take $a = 1/2$, then we will get

$$y = \log_{\frac{1}{2}} x \Leftrightarrow x = \left(\frac{1}{2}\right)^y$$

at $x = 1, y = 0$

at $x = \frac{1}{2}, y = 1$

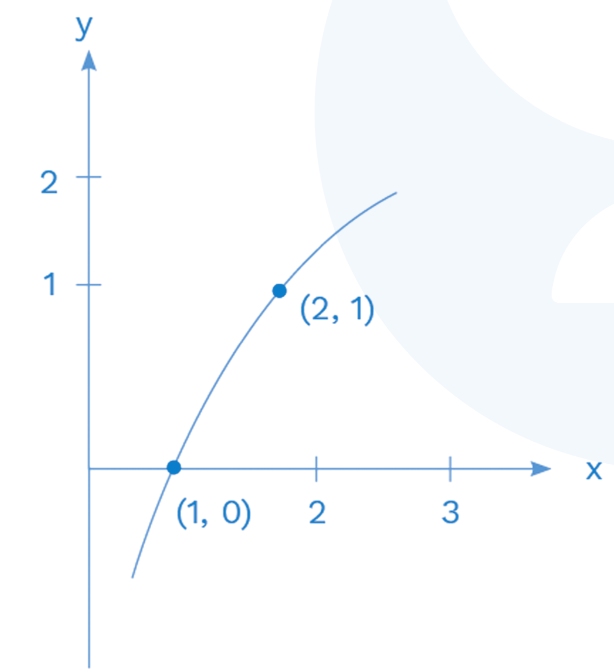


Case 2: If $y = \log_a x$ where $x > 0$, $a > 1$

If we take $a = 2$, then we will get $y = \log_2 x \Leftrightarrow x = 2^y$

at $x = 1$, $y = 0$

at $x = 2$, $y = 1$



Logarithmic Inequality

1. $\log_a x > \log_a y \Rightarrow x > y$ if $a > 1$

   $\log_a x > \log_a y \Rightarrow x < y$ if $0 < a < 1$

   For example, $\log_3 x > \log_3(x^2-5) \Rightarrow x > x^2 - 5$

   For example, $\log_{\frac{1}{7}}\left(x^2 - 8\right) > \log_{\frac{1}{7}} 3x$

   $\Rightarrow$ x2 – 8 < 3x

2. $\log_a b > c \Rightarrow b > a^c$ if $a > 1$

   $\log_a b > c \Rightarrow b < a^c$ if $0 < a < 1$

3. $\log_a b < c \Rightarrow b < a^c$ If $a > 1$

   $\log_a b < c \Rightarrow b > a^c$ If $o < a < 1$

4. If $a > 1$, $P > 1 \Rightarrow \log_a P > 0$

5. If $0 < a < 1$, $P > 1 \Rightarrow \log_a P < 0$

6. If $a > 1$, $0 < P < 1 \Rightarrow \log_a P < 0$

7. If $P > a > 1 \Rightarrow \log_a P > 1$

8. If $a > P > 1 \Rightarrow 0 < \log_a P < 1$

9. If $0 < P < a < 1 \Rightarrow \log_a P > 1$

10. If $0 < a < P < 1 \Rightarrow \log_a P < 1$

Example 17:

If $x = \log_3 7$, $y = \log_{17} 49$, then which of the following options is correct?

(A) $x < y$

(B) $x = y$

(C) $x > y$

(D) No relation

Solution: (C)

$x = \log_3 7$ and $y = \log_{17} 49$

$\therefore \ \frac{1}{x} = \log_7 3$ and $y = 2\log_{17} 7$

$\therefore \ \frac{1}{y} = \frac{1}{2}\log_7 17 = \log_7 \sqrt{17}$

$\sqrt{17}$ lies between 4 and 5

$\therefore \ \frac{1}{y} > \frac{1}{x}$

$\therefore \ x > y$

Example 18:

If $\log_{0.5} x - 2 < 0$, then x lies in the interval:

(A) (-3, -1) (B) (1, 3)

(C) $(3, \infty)$ (D) None of these

Solution: (C)

$\log_{0.5}(x-2) < 0$

$\therefore \ x - 2 > (0.5)^0$ (as base is less than 1

$\therefore$ the inequality is reversed).

$\therefore \ x - 2 > 1$

$\therefore \ x > 3$

$\therefore \ x \in (3, \infty)$

Practice Exercise – 1

Level of Difficulty – 1

1. If $2^{(\log_2 3)^x} = 3^{(\log_3 2)^x}$, then find the value of x.

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{-1}{2}$

(D) 2

2. If $\log_2 \log_3\left(\sqrt{k+3}+\sqrt{k}\right) = 1$, then find the value of k.

(A) $\frac{169}{9}$

(B) $\frac{169}{3}$

(C) $\frac{144}{5}$

(D) $\frac{144}{7}$

3. If $\log_m(mn) = y$, then $\log_n mn$.

(A) $\frac{1}{y}$

(B) $\frac{y}{y+1}$

(C) $\frac{y}{1-y}$

(D) $\frac{y}{y-1}$

4. If $\frac{\log P}{2a+b-3c} = \frac{\log Q}{2c-b} = \frac{\log R}{(c-2a)}$ then PQR is equal to:

(A) 0

(B) 1

(C) abc

(D) 2

5. If $\log_2 5 + \log_2(2x-1) = \log_2(3x-5) + 2$, then x is equal to:

(A) 12.5

(B) 7.5

(C) 8.5

(D) 8

Level of Difficulty – 2

6. The value of the expression $\sum \frac{1}{\log_i 100!}$ (where i = 2 to 100) is:

(A) 0.1

(B) 1

(C) 10

(D) 1,000

7. The number of solutions of $\log_4(x-1) = \log_2(x-3)$ is:

(A) 3

(B) 1

(C) 2

(D) 0

8. If $\log_{10}(2^x + x - 41) = x(1 - \log_{10}5)$, then find the value of x.

(A) 1

(B) $\frac{5}{2}$

(C) 9

(D) 41

9. The sum of two numbers P and Q is $\sqrt{13}$ and their difference is 3. The value of logQP is:

(A) 2

(B) 1

(C) $\frac{1}{2}$

(D) –1

10. If y is the smallest value of x satisfying the equation $2^x + \frac{15}{2^x} = 8$, then the value of $4y$ is equal to:

(A) 7

(B) 8

(C) 9

(D) 10

Level of Difficulty – 3

11. The domain of the function $f(x) = \log_7\{\log_3(\log_5(20x - x^2 - 91))\}$ is:

(A) (7, 13)
(B) (8, 12)
(C) (7, 12)
(D) (12, 13)

12. The least value of the expression $2\log_{10}x - \log_x 0.01$ for $x > 1$ is:
(A) 10
(B) 2
(C) −0.01
(D) None of these

13. Number of solutions possible for the following equation are:
$\log_{2x+3}(6x^2 + 23x + 21) + \log_{(3x+7)}(4x^2 + 12x + 9) = 4$
(A) 1
(B) 2
(C) 3
(D) 4

14. The equation $x^{\left(\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right)} = \sqrt{2}$ has
(A) At least one real solution
(B) Exactly three real solutions
(C) Complex roots
(D) No real solution

15. If 'm' and 'n' are real numbers such that $2\log(2n - 3m) = \log m + \log n$, then find the value of $\frac{m}{n}$.
(A) $\frac{4}{9}$
(B) 1
(C) $\frac{3}{5}$
(D) $\frac{5}{4}$

Solutions

1. (B)

$2^{(\log_2 3)^x} = 3^{(\log_3 2)^x}$

Taking log base 2 on both the sides,

$\Rightarrow (\log_2 3)^x \log_2 2 = (\log_3 2)^x \log_2 3$

$\Rightarrow (\log_2 3)^{x-1} = (\log_3 2)^x$

$\Rightarrow (\log_2 3)^{x-1} = \dfrac{1}{(\log_2 3)^x}$

$\Rightarrow (\log_2 3)^{2x-1} = (\log_2 3)^0$

$\therefore 2x - 1 = 0$

$\therefore x = \dfrac{1}{2}$

2. (A)

We know that if $\log_b a = k$ then $a = b^k$

$\log_2 \log_3 \left(\sqrt{k+3} + \sqrt{k}\right) = 1$

$\therefore \log_3 \left(\sqrt{k+3} + \sqrt{k}\right) = 2$

$\therefore \left(\sqrt{k+3} + \sqrt{k}\right) = 3^2$

$\therefore \left(\sqrt{k+3} + \sqrt{k}\right) = 9$

$\therefore \left(9 - \sqrt{k}\right) = \sqrt{k+3}$

Squaring both the sides

$\left(9 - \sqrt{k}\right)^2 = \left(\sqrt{k+3}\right)^2$

$81 - 18\sqrt{k} + k = k + 3$

$\therefore 18\sqrt{k} = 78$

$\therefore \sqrt{k} = \dfrac{13}{3}$

$\therefore k = \dfrac{169}{9}$

3. (D)

Given $\log_m(mn) = y$

$\therefore \dfrac{\log(mn)}{\log m} = y$

$\therefore \dfrac{\log m + \log n}{\log m} = y$

$\therefore 1 + \log_m n = y$

$\therefore \log_{mn} = y - 1$

We need to find

$\therefore \log_n mn$

$= \dfrac{\log(mn)}{\log(n)}$

$= \dfrac{\log m + \log n}{\log n}$

$= \log_n m + 1$

$= \left(\dfrac{1}{y-1} + 1\right) \quad \left[\begin{array}{l} \because \log_m n = y - 1 \\ \therefore \log_n m = \dfrac{1}{y-1} \end{array}\right]$

$= \left(\dfrac{y}{y-1}\right)$

4. (B)

Let $\dfrac{\log P}{2a+b-3c} = \dfrac{\log Q}{2c-b} = \dfrac{\log R}{(c-2a)} = k$

$\therefore \log P = k(2a + b - 3c)$ ...(i)

$\therefore \log Q = k(2c - b)$ ...(ii)

$\therefore \log R = K(c - 2a)$ ...(iii)

Adding equations (i) to (iii)

$(\log P + \log Q + \log R) = K[2a + b - 3c + 2c - b + c - 2a]$

$\therefore \log(PQR) = k(0)$

$\therefore \log(PQR) = 0$

$\therefore PQR = 1$

5. (B)

$\log_2 5 + \log_2(2x - 1) = \log_2(3x - 5) + 2$ (given)

$\therefore \dfrac{\log 5}{\log 2} + \dfrac{\log(2x-1)}{\log 2} = \dfrac{\log(3x-5)}{\log 2} + 2$

$\therefore \dfrac{\log 5(2x-1)}{\log 2} - \dfrac{\log(3x-5)}{\log 2} = 2$

$\therefore \log_2\left(\dfrac{5(2x-1)}{3x-5}\right) = 2$

$\therefore \dfrac{5(2x-1)}{3x-5} = 2^2$ (If $\log_b a = x$, then $a = b^x$)

$\therefore 5(2x - 1) = 4(3x - 5)$

$\therefore 10x - 5 = 12x - 20$

$\therefore 2x = 15$

$\therefore x = 7.5$

6. (B)

Given $\sum \dfrac{1}{\log_i 100!}$

Expanding the above expression

$= \dfrac{1}{\log_2 100!} + \dfrac{1}{\log_3 100!} + \dfrac{1}{\log_4 100!}$

$$+ \dots + \frac{1}{\log_{100} 100!}$$

$= \log_{100!}2 + \log_{100!}3 + \log_{100!}4 + \dots \log_{100!}100$
$= \log_{100!}(2 \times 3 \times 4 \times \dots 100)$
$= \log_{100!}100!$
$= 1$

7. **(B)**

$\log_4(x - 1) = \log_2(x - 3)$

$\Rightarrow \frac{1}{2}\log_2(x-1) = \log_2(x-3)$

$\Rightarrow \log_2(x-1)^{\frac{1}{2}} = \log_2(x-3)$

$\Rightarrow \sqrt{x-1} = x - 3$

$\Rightarrow (x - 1) = (x - 3)^2$
$\Rightarrow x - 1 = x^2 - 6x + 9$
$\Rightarrow x^2 - 7x + 10 = 0$
$\Rightarrow (x - 5)(x - 2) = 0$
$\Rightarrow x = 5, 2$

But $x = 2$ should be eliminated because log $(x - 3)$ becomes negative at $x = 2$.

$\therefore$ Only one solution is possible.

8. **(D)**

$\log_{10}(2^x + x - 41) = x(1 - \log_{10}5)$
$\Rightarrow \log_{10}(2^x + x - 41) = x(\log_{10}10 - \log_{10}5)$
$\Rightarrow \log_{10}(2^x + x - 41) = x\log_{10}2$
$\Rightarrow \log_{10}(2^x + x - 41) = \log_{10}2^x$
$\therefore 2^x + x - 41 = 2^x$
$\therefore x - 41 = 0$
$\therefore x = 41$

9. **(D)**

$P + Q = \sqrt{13}$
$P - Q = 3$

We know that

$(P + Q)^2 - (P - Q)^2 = 4PQ$

$\left(\sqrt{13}\right)^2 - (3)^2 = 4PQ$

$13 - 9 = 4PQ$

$\therefore PQ = 1 \therefore P = Q^{-1}$

Now, $\log_Q P = \log_Q Q^{-1} = -1$

10. **(C)**

$$2^x + \frac{15}{2^x} = 8$$

$\Rightarrow 2^{2x} - 8.2^x + 15 = 0$
$\Rightarrow (2^x - 3)(2^x - 5) = 0$
$\Rightarrow 2^x = 3$ or $2^x = 5$

Hence, smallest x is obtained by equating $2^x = 3$

(Taking log on both the sides)

$\Rightarrow x \log 2 = \log 3$
$\Rightarrow x = \log_2 3$

So, $y = \log_2 3$

Hence, $4^y = 2^{2y} = 2^{(2\log_2 3)} = 2^{(\log_2 9)} = 9$

11. **(B)**

Domain of any function $f(x)$ is the range of values of variable x for which the function is defined.

($\log_a b$ is defined, when $b > 0$, $a > 0$ and a is not equal to 1) $f(x) = \log_7\{\log_3(\log_5(20x - x^2 - 91))\}$

For $f(x)$ to be defined $\log_3\log_5(20x - x^2 - 91) > 0$

$\therefore \log_5(20x - x^2 - 91) > 1$
$\therefore 20x - x^2 - 91 > 5$
$\therefore 20x - x^2 - 96 > 0$
$\therefore x^2 - 20x + 96 < 0$
$\therefore (x - 8)(x - 12) < 0$
$\therefore x$ Î $(8, 12)$ or $8 < x < 12$

12. **(D)**

Let $y = 2\log_{10}x - \log_x 0.01$

where $x > 1$

$y = 2\log_{10}x - \log_x 10^{-2}$
$y = 2\log_{10}x + 2\log_x 10$

$$y = 2\log_{10} x + 2 \times \frac{1}{\log_{10} x}$$

$$y = 2\left(\log_{10} x + \frac{1}{\log_{10} x}\right)$$

We know that $m + \frac{1}{m} \geq 2$ for any positive real number m

$\therefore y = 2 \times (\geq 2)$
$\therefore y \geq 4$

Hence, the minimum value of $y = 4$.

13. **(A)**

$\log_{2x+3}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$

$\Rightarrow \log_{2x+3}(2x + 3)(3x + 7) = 4 - \log_{(3x+7)}(2x + 3)^2$

$\Rightarrow \log_{2x+3}(2x + 3) + \log_{2x+3}(3x + 7) = 4 - 2\log_{3x+7}(2x + 3)$

$\Rightarrow 1 + \log_{2x+3}(3x + 7) = 4 - 2\log_{3x+7}(2x + 3)$

$\Rightarrow \log_{2x+3}(3x + 7) + 2\log_{3x+7}(2x + 3) = 3$

$$\Rightarrow \log_{2x+3}(3x+7) + 2 \times \frac{1}{\log_{2x+3}(3x+7)} = 3$$

Let, $\log_{2x+3}(3x + 7) = m$

$$\therefore m + \frac{2}{m} = 3$$

$\Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow (m - 2)(m - 1) = 0$

$\therefore m = 2$ or $m = 1$

$\therefore \quad m = 2$ or $m = 1$

$\log_{2x+3} 3x + 7 = 2$	$\log_{2x+3} 3x + 7 = 1$
$3x + 7 = (2x + 3)^2$	$3x + 7 = 2x + 3$
$3x + 7 = 4x^2 + 12x + 9$	$x = -4$
$4x^2 + 9x + 2 = 0$	
$(4x + 1)(x + 2) = 0$	
$x = \frac{-1}{4}, -2$	

$x = -4, -2$ can't be possible because it makes $\log(2x + 3)$ negative.

$\therefore x = \frac{-1}{4}$ is the only solution.

14. (B)

$$x^{\left(\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right)} = \sqrt{2}$$

Let $\log_2 x = y$

$\therefore x = 2^y$

$$\therefore 2^{y\left[\frac{3}{4}y^2 + y - \frac{5}{4}\right]} = 2^{\frac{1}{2}}$$

$$\therefore y\left[\frac{3}{4}y^2 + y - \frac{5}{4}\right] = \frac{1}{2}$$

$$\therefore \frac{3y^3}{4} + \frac{4y^2}{4} - \frac{5y}{4} = \frac{1}{2}$$

$\therefore 3y^3 + 4y^2 - 5y - 2 = 0$

$\therefore 3y^2(y - 1) + 7y(y - 1) + 2(y - 1) = 0$

$\therefore (y - 1)(3y^2 + 7y + 2) = 0$

$\therefore (y - 1)(3y + 1)(y + 2) = 0$

$$\therefore y = 1, \frac{-1}{3}, -2$$

$$\therefore x = 2, 2^{\frac{-1}{3}}, 2^{-2}$$

Hence, the given equation has exactly three real solutions.

15. (A)

$2\log(2n - 3m) = \log m + \log n$

$\Rightarrow \log(2n - 3m)^2 = \log mn$

$\Rightarrow (2n - 3m)^2 = mn$

$\Rightarrow 9m^2 - 12mn + 4n^2 = mn$Þ $9m^2 - 13mn + 4n^2 = 0$

$$\Rightarrow 9\left(\frac{m}{n}\right)^2 - 13\left(\frac{m}{n}\right) + 4 = 0 \text{ let } \frac{m}{n} = t$$

$\therefore 9t^2 - 13t + 4 = 0$

$\therefore (9t - 4)(t - 1) = 0$

$$\therefore t = \frac{4}{9} \text{ or } t = 1$$

$$\therefore \frac{m}{n} = \frac{4}{9} \text{ or } \frac{m}{n} = 1$$

But at $\frac{m}{n} = 1$, $\log(2n - 3m)$ becomes negative

$$\therefore \frac{m}{n} = \frac{4}{9}$$

Practice Exercise – 2

Level of Difficulty – 1

1. If $\log_7 \log_5 \log_2(x^2 + x - 40) = 0$, then what is the sum of all possible values of x?

(A) –1
(B) 1
(C) 17
(D) –17

2. Find the sum of the below-given series up to 30 terms.

$$\frac{1}{\log_3 x} + \frac{1}{\log_9 x} + \frac{1}{\log_{27} x} + \ldots\ldots\ldots\ldots + \text{ up}$$

to 30 terms is:

(A) $455 \log_3 x$
(B) $360 \log_x 3$
(C) $465 \log_x 3$
(D) $460 \log_x 3$

3. Find the value of

$$\left(\frac{1}{\log_3 255} + \frac{1}{\log_5 255} + \frac{1}{\log_{17} 255}\right).$$

(A) 0
(B) 1
(C) 17
(D) 51

4. If log6 = 0.778, then find the number of digits in 216^{10}

(A) 23
(B) 24
(C) 25
(D) 22

5. $\dfrac{3^{\log_2 125}}{27^{\log_2 5}} \times \dfrac{8\log_5 8}{3\log_5 256} = ?$

6. Find the sum of the following series:

$$\frac{\log a^2b^3}{\log c} + \frac{\log a^2b^3}{\log c^3} + \frac{\log a^2b^3}{\log c^9} + \frac{\log a^2b^3}{\log c^{27}} + \ldots$$

up to infinity

(A) $\dfrac{3}{2}\dfrac{\log a^2b^3}{\log c}$

(B) $\dfrac{5}{2}\dfrac{\log a^2b^3}{\log c}$

(C) $\dfrac{5}{3}\dfrac{\log a^2b^3}{\log c}$

(D) None of these

7. Suppose, $\log_4 a = \log_{16} b = K$, where a, b are positive numbers. If 'c' is the geometric mean of 'a' and 'b', then $\log_8 c$:

(A) $k/3$
(B) $K^{\frac{1}{2}}$
(C) $K^{\frac{1}{4}}$
(D) k

8. If $\log_{a^2+5a}\left(a^2+2a\right) = \log_{a^2+5a} 15$, then how many values '$a$' can take?

(A) 3
(B) 2
(C) 1
(D) 0

9. Find x, given that $\log_3 x = \log_3 7 \div \frac{1}{3}\log_4 7$.

10. $\log_{50625}\sqrt{3375} + \log_{49}\sqrt[3]{2401}$

(A) $\dfrac{41}{24}$

(B) $\dfrac{25}{24}$

(C) $\dfrac{3}{8}$

(D) $\dfrac{4}{9}$

Level of Difficulty – 2

11. If $\log_p x + \log_x p^6 = 5$ then find $\log_p x^3$.

(A) 2 or 3
(B) 2 or –3
(C) 6 or 9
(D) 6 or –9

12. If $4 - \log_{10}\sqrt{1+a} + 3\log_{10}\sqrt{1-a}$

$= \log_{10}\dfrac{1}{\sqrt{1-a^2}}$, then the value of 500a is:

(A) 198
(B) 99
(C) 395
(D) 495

13. Find the interval in which x lies if $\log_{0.25}(x + 2) - \log_{0.5}(x + 2) > 0$.
(A) $(-1, \infty)$
(B) $(1, \infty)$
(C) $(-1, 1)$
(D) None of these

14. If A, B, C, and D are positive quantities such that $A^2 = B^3 = C^5 = D^6 = E^{10}$, then find the correct range of $P = \log_{BD}(ACE)$.
(A) $1 < P < 1.5$
(B) $1.5 < P < 2$
(C) $2 < P < 2.5$
(D) $2.5 < P < 3$

15. $\log_3 5 \times \log_5 6 \times \log_6 7 \times \log_7 8 \ldots\ldots \times \log_{n+1}(n+2) \times \log_{n+2}(n + 3) = 5$, find n.
(A) 122
(B) 213
(C) 240
(D) 243

16. If $\log_2\left[5+\log_5\left\{6+\log_3(x+1)\right\}\right]-3=0$, then the value of $3x$ is ______.
(A) $3^{120} - 5$
(B) $3^{110} - 2$
(C) $3^{120} - 3$
(D) 3^{119}

17. If $\log_{216} 5 = a$ and $\log_5 7 = b$, what is the value of $\log_{42} 7$ in terms of a and b?
(A) $\dfrac{3ab}{3ab + 1}$
(B) $\dfrac{3a}{b + 1}$
(C) $\dfrac{1+a}{3b}$
(D) $\dfrac{ab}{3ab + 1}$

18. If $21 \log_k 5 + \log_5 k = 10$, find the product of all the possible values of k.
(A) 4^{-10}
(B) 10^{-4}
(C) 5^{10}
(D) 10^5

19. If x is a negative number such that $3^{x^2(\log_2 7)} = 7^{\log_3 2}$, then x is equal to:
(A) $\log_3\left(\dfrac{1}{2}\right)$
(B) $\log_2\left(\dfrac{1}{3}\right)$
(C) $\log_5\left(\dfrac{1}{3}\right)$
(D) $\log_3{}^2$

20. If x is an integer and $[\log_5(x^5 - x^2) - \log_{25}(x^6 + 1 - 2x^3)] = 6$, then find the sum of the digits of x.
(A) 7
(B) 8
(C) 13
(D) 5

Level of Difficulty – 3

21. Find the value of x which satisfies
$$log_5\left(\left(\frac{5}{2}\right)^x+2x-22\right)=x\left(2-log_5 10\right).$$
(A) 0
(B) 11
(C) 22
(D) $\left(\dfrac{5}{2}\right)^3$

22. If x is a positive number such that $3^{x^2 \log_2{}^5} = 5^{\log_3{}^2}$, then x is equal to:
(A) $\log_3{}^2$
(B) $\log_2{}^3$
(C) $\log_2{}^5$
(D) $\log_5{}^3$

23. If $\log_{27} p + \log_3 p = 4$, and y is the number of digits in 525, then find the value of $(p + y)/5$. Use $\log 2 = 0.301$.

24. If $x = 256!$ and y = product of the first 128 odd numbers, then find

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \ldots + \frac{1}{\log_{127} x} + \frac{1}{\log_{128} x}.$$

(A) $1 - \log_x y + 128 \log_x 2$

(B) $1 + \log_x y - 128 \log_x 2$

(C) $1 - \log_x y - 128 \log_x 2$

(D) $1 + \log_x y + 128 \log_x 2$

25. If $\log_a 30 = x, \log_a \left(\frac{5}{3}\right) = -y$, and $\log_2 a = \frac{1}{3}$, then find $\frac{\log_a 3}{\log_2 a}$.

(A) $\frac{7}{2}(y + 3 - x)$

(B) $\frac{3}{2}(x + y - 3)$

(C) $\frac{5}{2}(y - x + 3)$

(D) $x + y - 3$

26. Given that y is a positive integer, if $A^{(3-y)} \times B^{(7+y)} = A^{(5+y)} \times B^{(2-y)}$ and $\frac{\log A}{\log B} = \frac{25}{22}$, then y is:

27. If $\log 3$, $\log(3^x - 1)$ and $\log(3^x + 1)$ are in arithmetic progression, then x lies in which of the following intervals?

(A) (3,4)

(B) (2,3)

(C) (1,2)

(D) (0,1)

28. If $\log_{81} 3^{3^{33}} - \log_{81} 3^{3^{3^3}} = 3^{27} \times K$, then find the value of K.

29. If $\log_{25} 175 = x$ and $\log_{17} 25 = y$, then find the value of $\log_{60025} 2975$.

(A) $\frac{xy - 1}{y(4x + 3)}$

(B) $\frac{xy + 1}{y(4x - 3)}$

(C) $\frac{xy + 1}{y(4x + 3)}$

(D) None of these

30. Given that $\log_5 \log_{125} x + \log_{125} \log_5 x = 5$. If $x = 5^P$ satisfies the given equation, then find the value of P.

(A) $5^{\frac{3}{4}[5 + \log_5 3]}$

(B) $5^{\frac{1}{4}[5 + \log_5 3]}$

(C) $5^{\frac{5}{4}[5 + \log_5 3]}$

(D) None of these

Solutions

1. (A)

We know $\log_B A = C \rightarrow A = B^C$

Apply the same logic in this question

$\log_7 \log_5 \log_2 (x^2 + x - 40) = 0$

$\log_5 \log_2 (x^2 + x - 40) = 7^0 = 1$

$\log_2 (x^2 + x - 40) = 5^1 = 5$

$(x^2 + x - 40) = 2^5 = 32 \quad x^2 + x - 72 = 0$

$x^2 + 9x - 8x - 72 = 0$

$x(x + 9) - 8(x + 9) = 0$

$(x + 9)(x - 8) = 0$

$x = -9$ or 8

So, sum of all possible values of $x = -9 + 8 = -1$.

2. (C)

$$\frac{1}{\log_3 x} + \frac{1}{\log_9 x} + \frac{1}{\log_{27} x} + \ldots\ldots\ldots\ldots +$$ up to 30 terms

$= \log_x 3 + \log_x 9 + \log_x 27 + \ldots\ldots\ldots\ldots\ldots\ldots +$ up to 30 terms

$= 1.\ \log_x 3 + 2\ \log_x 3 + 3\log_x 3 + \ldots\ldots\ldots\ldots +$ up to 30 terms

$= [1 + 2 + 3 + \ldots\ldots\ldots\ldots + 30] \times \log_x 3$

$$= \frac{30 \times (30+1)}{2} \times \log_x 3$$

Note:

$$\left[1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2}\right]$$

$$= \frac{30 \times 31}{2} \times \log_x 3 = 465 \log_x 3.$$

Hence, option (C) is the correct answer.

3. (B)

We know that $\log_a b = \dfrac{1}{\log_b a}$

$$\Rightarrow \left(\frac{1}{\log_3 255} + \frac{1}{\log_5 255} + \frac{1}{\log_{17} 255}\right)$$

$$= (\log_{255} 3 + \log_{255} 5 + \log_{255} 17)$$

$= \log_{255}(3 \times 5 \times 17) = \log_{255} 255 = 1$ (as $\log_a a = 1$).

4. (B)

Let $y = 216^{10}$

Taking log in both sides, we get:

$\log y = \log 216^{10}$.

If $\log y$ = abcde.., then the number of digits in y will be equal to, $y = a + 1$

$\Rightarrow \log y = 10 \log 216$

$\Rightarrow \log y = 10 \log 6^3$

$\Rightarrow \log y = 30 \log 6$

$\Rightarrow \log y = 30 \times 0.778 = 23.34$

Thus, the number of digits = 23 + 1 = 24.

Hence, option (B) is the correct answer.

5. 1

$$\frac{3^{\log_2 125}}{27^{\log_2 5}} \times \frac{8 \log_5 8}{3 \log_5 256}$$

$$= \frac{3^{\log_2 5^3}}{27^{\log_2 5}} \times \frac{8\ \log_5 2^3}{3\ \log_5 2^8}$$

$$\Rightarrow \frac{3^{3\log_2 5}}{27^{\log_2 5}} \times \frac{24 \log_5 2}{24 \log_5 2}$$

$$\Rightarrow \frac{27^{\log_2 5}}{27^{\log_2 5}} \times 1$$

$= 1 \times 1 = 1$

6. (A)

$$\frac{\log a^2b^3}{\log c} + \frac{\log a^2b^3}{\log c^3} + \frac{\log a^2b^3}{\log c^9} + \ldots\ldots.$$

$$= \frac{\log a^2b^3}{\log c} + \frac{\log a^2b^3}{3\log c} + \frac{\log a^2b^3}{9\log c} + \ldots\ldots.$$

$$= \frac{\log a^2b^3}{\log c}\left[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots.\right]$$

$= \dfrac{\log a^2b^3}{\log c}\left(\dfrac{1}{1 - 1/3}\right)$ (sum to infinite terms in G.P.)

$$= \frac{3}{2}\frac{\log a^2b^3}{\log c}$$

7. (D)

$\log_4 a = K$ and $\log_{16} b = K$ (is given)

$\therefore\ 4^k = a$ and $16^k = b$

Since 'c' is the geometric mean of 'a' and 'b'

$\therefore\ C = \sqrt{a \times b}$

Therefore, $\log_8 C = \log_8 \sqrt{a \times b}$

$= \log_8 \sqrt{4^k \times 16^k}$

$= \log_8(2^K \times 4^K)$

$= \log_8 8^k = k \log_8 8 = k$

Hence, option (D) is the correct answer.

8. (C)

The given equation is:

$\log_{a^2+5a}\left(a^2+2a\right) = \log_{a^2+5a} 15$

$\therefore\ a^2 + 2a = 15$

$a^2 + 2a - 15 = 0$

$a^2 + 5a - 3a - 15 = 0$

$a(a + 5) - 3(a + 5) = 0$

$(a - 3)(a + 5) = 0$

$a = 3, a = -5$

If $a = -5$, then $a^2 + 5a = (-5)^2 + 5 \times (-5) = 0$ which is not possible (as base of a logarithm cannot be zero).

If $a = 3$, then $a^2 + 5a = 32 + 5 \times 3 = 9 + 15 = 24$.

This is possible.

Therefore, 'a' can take only one value.

9. 64

$\text{Log}_3 x = \log_3 7 \div \frac{1}{3}\log_4 7$

$\Rightarrow \log_3 7 \div \log_3 x = \frac{1}{3}\log_4 7$

$\Rightarrow \log_x 7 = \frac{1}{3}\log_4 7$ ...(i)

Let $\frac{1}{3}\log_4 7 = k$ ...(ii)

$\Rightarrow \log_4 7 = 3k$

$\Rightarrow 4^{3k} = 7$ ...(iii)

From equations (i) and (ii):

$\log_x 7 = \frac{1}{3}\log_4 7 = k$

$\Rightarrow \log_x 7 = k$

$\Rightarrow x^k = 7$ ...(iv)

Using equation (iii):

$x^k = 4^{3k} = (4^3)^k$

$\Rightarrow x = 4^3 = 64$

10. (B)

We have: $\log_{50625}\sqrt{3375} + \log_{49}\sqrt[3]{2401}$

$= \log_{15^4}\sqrt{15^3} + \log_{7^2}\sqrt[3]{7^4}$

$= \frac{1}{4}\times\log_{15} 15^{3\times\frac{1}{2}} + \frac{1}{2}\times\log_7 7^{4\times\frac{1}{3}}$

$= \frac{1}{4}\times\frac{3}{2}\times\log_{15} 15 + \frac{1}{2}\times\frac{4}{3}\times\log_7 7$

$= \frac{3}{8} + \frac{4}{6} = \frac{9+16}{24} = \frac{25}{24}$

Thus, the required option (B) is correct.

11. (C)

$\log_p x + \log_x p^6 = 5$

$\log_p x + \frac{6}{\log_p x} = 5$

Let $\log_p x = z$

$\Rightarrow z + \frac{6}{z} = 5$

$Z^2 - 5z + 6 = 0$

$Z^2 - 3z - 2z + 6 = 0$

$Z(z - 3) - 2(z - 3) = 0$

$z = 2, 3$

Therefore, $\log_p x = 2$ or 3

$\log_p x^3 = 3\log_p x$

When $\log_p x = 2, \log_p x^3 = 3\times 2 = 6$

When $\log_p x = 3, \log_p x^3 = 3\times 3 = 9$

So, $\log_p x^3 = 6$ or 9.

Hence, option (C) is the correct answer.

12. (D)

The given equation is

$4 - \log_{10}\sqrt{1+a} + 3\log_{10}\sqrt{1-a}$

$= \log_{10}\left(\sqrt{1-a^2}\right)^{-1}$

$= -1\left[\log_{10}\left(\sqrt{1+a}\times\sqrt{1-a}\right)\right]$

$4 - \log_{10}\sqrt{1+a} + 3\log_{10}\sqrt{1-a}$

$= -\log_{10}\sqrt{1+a} - \log_{10}\sqrt{1-a}$

$4 + 4\log_{10}\sqrt{1-a} = 0$

$4\log_{10}\sqrt{1-a} = -4$

$\log_{10}\sqrt{1-a} = -1$

$\sqrt{1-a} = 10^{-1}$

$\sqrt{1-a} = \frac{1}{10}$

$1 - a = \frac{1}{100}$

$100 - 100a = 1$

$-100a = -99$

$a = \dfrac{99}{100}$

Therefore, the value of $500a$ is =

$500a = 500 \times \dfrac{99}{100} = 5 \times 99 = 495$

Hence, option (D) is the correct answer.

13. (A)

Given that $\log_{0.25}(x + 2) - \log_{0.5}(x + 2) > 0$

$\Rightarrow \log_{(0.5)^2}(x+2) - \log_{0.5}(x + 2) > 0$

$\Rightarrow \dfrac{1}{2}\log_{0.5}(x+2) - \log_{0.5}(x+2) > 0$

(Using property $\log_{a^2} x = \dfrac{1}{2}\log_a x$)

$\Rightarrow -\dfrac{1}{2}\log_{0.5}(x+2) > 0$

or $\log_{0.5}(x + 2) < 0$ (removing negative sign).

Now $\log 1 = 0$;

$\Rightarrow \log_{0.5}(x + 2) < \log 1$

When base is less than 1, inequality is reversed.

Therefore, $x + 2 > 1$

$\Rightarrow x > -1$

$\Rightarrow x$ lies in the interval $(-1, \infty)$.

14. (B)

Let's assume $A^2 = B^3 = C^5 = D^6 = E^{10} = K^{\text{LCM (2, 3, 5, 6, and 10)}}$

$A^2 = B^3 = C^5 = D^6 = E^{10} = K^{30}$

$\Rightarrow A = K^{15}$

$B = K^{10}$

$C = K^6$

$D = K^5$

$E = K^3$

Now $P = \log_{BD}(ACE) = \dfrac{\log ACE}{BD}$

$P = \dfrac{\log(K^{15} \times K^6 \times K^3)}{\log(K^{10} \times K^5)}$

$P = \dfrac{\log(K^{24})}{\log(K^{15})} = \dfrac{24\log K}{15\log K}$

$P = \dfrac{24}{15} = 1.6$

So, P lies between 1.5 and 2; hence, option (B) is the correct answer.

15. (C)

We know that $\log_a b = \dfrac{\log_c b}{\log_c a}$

Given $\log_3 5 \times \log_5 6 \times \log_6 7 \ldots \times \log_{n+1}(n + 2) \times \log_{n+2}(n + 3) = 5$.

$\Rightarrow \log_3 5 \times \dfrac{\log_3 6}{\log_3 5} \times \dfrac{\log_3 7}{\log_3 6} \ldots\ldots \times \dfrac{\log_3(n+2)}{\log_3(n+1)}$

$\times \dfrac{\log_3(n+3)}{\log_3(n+2)} = 5$

$\Rightarrow \log_3(n+3) = 5$

$\Rightarrow n + 3 = 3^5 = 243$

$\Rightarrow n = 240$

16. (C)

Since the given equation is:

$\log_2\left[5 + \log_5\left\{6 + \log_3(x+1)\right\}\right] - 3 = 0$

$\log_2\left[5 + \log_5\left\{6 + \log_3(x+1)\right\}\right] = 3$

$\left[5 + \log_5\left\{6 + \log 3(x+1)\right\}\right] = 2^3$

$\log_5\left\{6 + \log_3(x+1)\right\} = 3$

$6 + \log_3(x + 1) = 5^3$ $\log_3(x + 1) = 125 - 6$

$\log_3(x + 1) = 119$

$x + 1 = 3^{119}$

$x = 3^{119} - 1$

$\therefore 3x = 3(3^{119} - 1)$

$= 3^{120} - 3$

Hence, option (C) is the correct answer.

17. (A)

We have:

$\log_{216} 5 = a \Rightarrow \dfrac{\log 5}{\log 216} = a$

$\Rightarrow \dfrac{\log 5}{3\log 6} = a$

$\Rightarrow \log 6 = \dfrac{\log 5}{3a}$ ... (i)

Also: $\log_5 7 = b \Rightarrow \dfrac{\log 7}{\log 5} = b$

$\Rightarrow \log 7 = b \log 5$ ... (ii)

$\therefore \log_{42} 7 = \dfrac{\log 7}{\log 42} = \dfrac{\log 7}{\log 6 + \log 7}$

Using equations (i) and (ii), we get:

$\dfrac{b\log 5}{\dfrac{\log 5}{3a} + b\log 5} = \dfrac{3ab}{3ab + 1}$

Hence, option (A) is the correct answer.

18. (C)

$21 \log_k 5 + \log_5 k = 10$ ...(i)

Let $\log_5 k = x$, then $\log_k 5 = \frac{1}{x}$

Putting these values is equation (i), we will get

$$21\left(\frac{1}{x}\right) + x = 10$$

$21 + x^2 = 10x$

$x^2 - 10x + 21 = 0$

$x^2 - 7x - 3x - 21 = 0$

$(x - 7)(x - 3) = 0$

$x = 7, 3$

Now, when $\log_5 k = 7$, then $k = 5^7$ and when $\log_5 k = 3$, then $k = 5^3$.

Required product $= 5^7 \times 5^3 = 5^{10}$.

Hence, option (C) is the correct answer.

19. (A)

Let $\log_2 7 = a$ and $\log_3 2 = b$

$$\therefore 3^{x^2 a} = 7^b$$

$$3^{x^2} = 7^{\frac{b}{a}}$$

$$\log_3 7^{\frac{b}{a}} = x^2$$

$$x^2 = \frac{b}{a}\log_3 7$$

Put the values of a and b in the above equation

$$x^2 = \frac{\log_3 2}{\log_2 7} \times \log_3 7$$

$$= \frac{\log 2}{\log 3} \times \frac{\log 2}{\log 7} \times \frac{\log 7}{\log 3}$$

$$x^2 = \left[\frac{\log 2}{\log 3}\right]^2$$

$$x = \frac{\log 2}{\log 3}$$

Since x is a negative number.

$$x = -\frac{\log 2}{\log 3}$$

$$x = \log_3 2^{-1}$$

$$x = \log_3\left(\frac{1}{2}\right)$$

Hence, option (A) is the correct answer.

20. (B)

$\log_5 (x^5 - x^2) - \log_{25} (x^6 + 1 - 2x^3) = 6$

$\log_5 \{x^2 (x^3 - 1)\} - \log_{25} (x^3 - 1)^2 = 6$

$\log_5 \{x^2 (x^3 - 1)\} - \log_5 (x^3 - 1) = 6$

$$\log_5\left(\frac{x^2(x^3-1)}{(x^3-1)}\right) = 6 \quad \log_5 (x^2) = 6$$

$2 \log_5 x = 6 \Rightarrow \log_5 x = 3 \Rightarrow x = 5^3 = 125$

21. (B)

$$x(2 - \log_5 10) = x(2\log_5 5 - \log_5 10),$$

(because $\log_5 5 = 1$)

$$= x(\log_5 5^2 - \log_5 10) = x\left(\log_5 \frac{25}{10}\right)$$

$$= x\left(\log_5 \frac{5}{2}\right) = \log_5\left(\frac{5}{2}\right)^x$$

$$\text{So, } x(2 - \log_5 10) = \log_5\left(\frac{5}{2}\right)^x$$

$$\text{Given } \log_5\left(\left(\frac{5}{2}\right)^x + 2x - 22\right) = x(2 - \log_5 10)$$

$$\Rightarrow \log_5\left(\left(\frac{5}{2}\right)^x + 2x - 22\right) = \log_5\left(\frac{5}{2}\right)^x$$

$$\Rightarrow \left(\frac{5}{2}\right)^x + 2x - 22 = \left(\frac{5}{2}\right)^x$$

$\Rightarrow 2x - 22 = 0$ or $x = 11$

22. (A)

The given equation is:

$$3^{x^2 \log_2 5} = 5^{\log_3 2}$$

Let $\log_2 5 = a$

And $\log_3 2 = b$

Therefore, $3^{x^2 \times a} = 5^b$

$$3^{x^2} = 5^{\frac{b}{a}}$$

Taking log on both the sides:

$$\log 3^{x^2} = \log 5^{b/a}$$

$$x^2 = \frac{b}{a}\frac{\log 5}{\log 3}$$

$$x^2 = \frac{\log_3 2}{\log_2 5} \times \frac{\log 5}{\log 3}$$

$$\therefore x^2 = \frac{\log 2}{\log 3}\times\frac{\log 2}{\log 5}\times\frac{\log 5}{\log 3}$$

$$x^2 = \left(\frac{\log 2}{\log 3}\right)^2$$

$$x = \frac{\log 2}{\log 3}$$

$$x = \log_3{}^2$$

Hence, option (A) is the correct answer.

23. 9

$\log_{27}$ p + $\log_3$ p = 4

$$\Rightarrow \frac{\log p}{\log 27} + \frac{\log p}{\log 3} = 4.$$

$$\Rightarrow \frac{\log p}{3 \log 3} + \frac{\log p}{\log 3} = 4.$$

$$\Rightarrow \frac{\log p + 3 \log p}{3 \log 3}$$

= 4 ⇔ 4 log p = 12 log 3

⇔ log p = 3 log 3 ⇔ log p = log (3^3) = log 27 ⇔ p = 27.

Now,

Since log 5^{25} ⇒ 25 log (10/2)

⇒ 25 [log 10 – log 2] ⇒ 25 (1 – 0.3010)

⇒ 25 × 0.699 ⇒ 17.5

Characteristic = 17. Hence, the number of digits in 5^{25} is 18.

So, y = 18

Now, (p + y)/5 = (27 + 18)/5 = 9.

24. (C)

x=1 × 2 × 3 × ... × 255 × 256

y = 1 × 3 × 5 × ... × 253 × 255

$$\frac{x}{y} = 2\times 4\times 6\times\ldots\times 254\times 256 = 2^{128}(128!)$$

Now,

$$\frac{1}{\log_2 x}+\frac{1}{\log_3 x}+\frac{1}{\log_4 x}+\ldots+\frac{1}{\log_{127} x}$$

$$+\frac{1}{\log_{128} x}$$

$$= \log_x 2 + \log_x 3 + \log_x 4 + \ldots$$

$$+ \log_x 127 + \log_x 128$$

$$= \log_x(1\times 2\times 3\times 4\ldots\times 127\times 128)$$

$$= \log_x\left(\frac{x}{y}\times 2^{-128}\right)$$

$$= \log_x x - \log_x y + \log_x 2^{-128}$$

$$= 1 - \log_x y - 128\log_x 2$$

25. (B)

Since $\log_a 30 = x$

$\log_a(2 \times 3 \times 5) = x$

$\log_a 2 + \log_a 3 + \log_a 5 = x$....(i)

Again, $\log_a\left(\frac{5}{3}\right) \Rightarrow \log_a 5 - \log_a 3 = -y$

$\log_a 3 - \log_a 5 = y$....(ii)

$$\log_2 a = \frac{1}{3} \Rightarrow \frac{\log a}{\log 2} = \frac{1}{3}$$

$$\frac{\log 2}{\log a} = 3$$

$\log_a 2 = 3$ ...(iii)

put the value of $\log_a 2 = 3$ in equation (i)

$\log_a 2 + \log_a 3 + \log_a 5 = x$

$3 + \log_a 3 + \log_a 5 = x$

$\log_a 3 + \log_a 5 = x - 3$ ...(iv)

Now, solve equations (ii) and (iv)

$$\begin{array}{lcl} \log_a 3 + & \log_a 5 = & x - 3 \\ \log_a 3 - & \log_a 5 = & y \\ + & + & + \\ \hline 2\log_a 3 & & = x + y - 3 \end{array}$$

$$\log_a 3 = \frac{x + y - 3}{2}$$

Now we have to find:

$$\frac{\log_a 3}{\log_2 a} = \frac{\frac{x+y-3}{2}}{\frac{1}{3}} = \frac{3(x+y-3)}{2}$$

$$\frac{\log_a 3}{\log_2 a} = \frac{3}{2}(x + y - 3)$$

Hence, option (B) is the correct answer.

26. 10

Given $A^{(3-y)} \times B^{(7+y)} = A^{(5+y)} \times B^{(2-y)}$

Take log both sides

$$\log\left[A^{(3-y)}\times B^{(7+y)}\right] = \log\left[A^{(5+y)}\times B^{(2-y)}\right]$$

$$\log A^{(3-y)} + \log B^{(7+y)} = \log A^{(5+y)} + \log B^{(2-y)}$$

$(3-y)\log A+(7+y)\log B$

$=(5+y)\log A+(2-y)\log B$

$\log B(7+y-2+y)=\log A(5+y-3+y)$

$\frac{\log A}{\log B}=\frac{2y+5}{2y+2}$ ......(i)

Also, $\frac{\log A}{\log B}=\frac{25}{22}$ ......(ii)

From equations (i) and (ii)

$\frac{2y+5}{2y+2}=\frac{25}{22}\Rightarrow y=10$

27. (C)

Given: log3, log(3^x – 1) and log(3^x + 1) are in arithmetic progression.

Since log3, log(3^x – 1) and log(3^x + 1) are in arithmetic progression

2.log(3^x – 1) = log3 + log(3^x + 1)

⇒ log$(3^x - 1)^2$ = log{3(3^x + 1)}

⇒ $(3^x - 1)^2 = 3(3^x + 1)$

⇒ $3^{2x} - 2.3^x + 1 = 3.3^x + 3$

Putting $3^x = y$

$y^2 - 5y - 2 = 0$

Solving the above equation, $y = (\sqrt{33} + 5)/2$, {neglecting $y \neq (-\sqrt{33} + 5)/2$}.

$y = (\sqrt{33} + 5)/2$ = {(*a* value between 5 and 6) + 5}/2 = *a* value between 5 and 6

⇒ $3^x = (\sqrt{33} + 5)/2$ = *a* value between 5 and 6, which lies between 3^1 and 3^2.

Hence, *x* belongs to (1, 2).

28. 182

$\log_{81}3^{3^{33}}-\log_{81}3^{3^{3^3}}=3^{27}\times K$

$=\log_{81}3^{3^{33}}-\log_{81}3^{3^{27}}=3^{27}\times K$

$=\log_{81}\left(\frac{3^{3^{33}}}{3^{3^{27}}}\right)=3^{27}\times K$

$=\frac{\log 3^{(3^{33}-3^{27})}}{\log 81}=3^{27}\times K$

$=\frac{\log 3^{(3^{27}\times 728)}}{\log 81}=3^{27}\times K$

$=\frac{3^{27}\times 728\times\log 3}{4\log 3}=3^{27}\times K$

$182=K$

29. (B)

We have:

$\log_{25}175=x$

$\Rightarrow\log_{25}(25\times 7)=x$

$\Rightarrow\log_{25}25+\log_{25}7=x$

$\Rightarrow\log_{25}7=x-1$ .... (i)

Also:

$\log_{17}25=y$

$\Rightarrow\log_{25}17=\frac{1}{y}$ .... (ii)

$\therefore\log_{60025}2975=\frac{\log_{25}2975}{\log_{25}60025}$

$=\frac{\log_{25}(25\cdot 7\cdot 17)}{\log_{25}(25\cdot 2401)}$

$=\frac{\log_{25}25+\log_{25}7+\log_{25}17}{\log_{25}25+\log_{25}2401}$

$=\frac{1+\log_{25}7+\log_{25}17}{1+4\log_{25}7}$

Using equations (i) and (ii), we get:

$=\frac{1+x-1+\frac{1}{y}}{1+4(x-1)}=\frac{x+\frac{1}{y}}{1+4x-4}$

$=\frac{xy+1}{y(4x-3)}$

30. (A)

$\log_5\log_{125}x+\log_{125}\log_5 x=5$

$\log_5\left(\frac{1}{3}\log_5 x\right)+\frac{1}{3}\log_5\log_5 x=5$

Let $\log_5 x = m$

$\therefore\log_5\left(\frac{m}{3}\right)+\frac{1}{3}\log_5 m=5$

$\log_5 m-\log_5 3+\frac{1}{3}\log_5 m=5$

$\frac{4}{3}\log_5 m-\log_5 3=5$

$\frac{4}{3}\log_5 m=5+\log_5 3$

$\log_5 m=\frac{3}{4}[5+\log_5 3]$

Again, put the value of $m=\log_5 x$

$\therefore \log_5 \log_5 x = \frac{3}{4}[5 + \log_5 3]$

$\log_5 \log_5 5^P = \frac{3}{4}[5 + \log_5 3]$

$\log_5 (P \log_5 5) = \frac{3}{4}[5 + \log_5 3]$

$\log_5 P = \frac{3}{4}[5 + \log_5 3]$

$P = 5^{\frac{3}{4}[5 + \log_5 3]}$

Hence, option (A) is the correct answer.

Mind Map

LOGARITHM

Application of logarithm

Finding the number of digits in a^b

If $n = a^b$

and $\log_{10} n = x.****$

Then number of digits in $a^b = x + 1$

Types of Logarithm

Natural logarithm where base is 'e'

Common logarithm where base is 10

Definition

$N = a^x \Leftrightarrow x = \log_a N$

Domain of Logarithm

$\log_a N$ is valid if $N > 0,\ a > 0,\ a \neq 1$

Properties of Logarithm

1. $\log_a mn = \log_a m + \log_a n$
2. $\log_a \frac{m}{n} = \log_a m + \log_a n$
3. $\log_a m^n = n \log_a m$
4. $\log_a 1 = 0$
5. $\log_a a = 1$
6. $\log_b a = \frac{\log_c a}{\log_c b}$
7. $\log_b a = \frac{1}{\log_a b}$
8. $a^{\log_a x} = x$
9. $a^{\log_c b} = b^{\log_c a}$

Logarithmic inequalities

$\log_a x > \log_a y \Rightarrow x > y$ If $a > 1$

$x < y$ If $0 < a < 1$

Note : If base is greater than 1, then the inequality remains the same and if base is positive but less than 1 then the sign of inequality reverses.

$\log_b a > c \Rightarrow a > b^c$ if $b > 1$

$\Rightarrow a < b^c$ if $0 < b < 1$