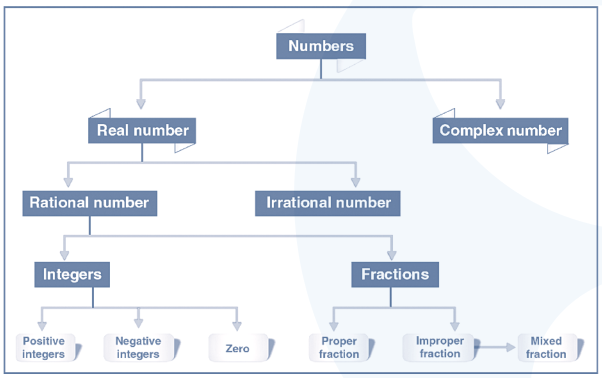
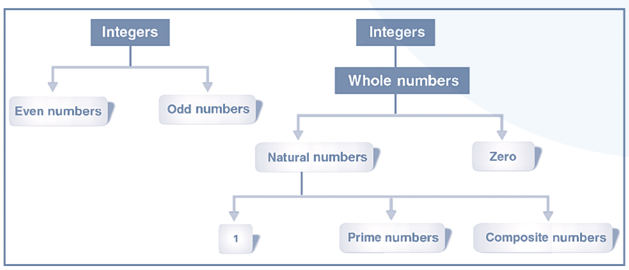
5 Numbers – Part 1

Introduction

It would not be an exaggeration to say that one cannot imagine having a stronghold in quantitative aptitude without knowing numbers. Numbers is one of the most important chapters for CAT. Almost three or four direct questions come from this chapter every year, but its applications are widely used to solve questions from other sections like Modern Maths, Geometry, and Algebra. One needs a very careful and rigorous practice to understand its fundamentals.

Classification of Numbers





Real Numbers

Numbers can be represented on the number line. There is a unique point on the number line for each real number.

Examples:

2, 3, 7.54, –3.47, $\sqrt{5}$, $\sqrt{23}$, etc.

Rational Numbers

Numbers which can be represented in p/q form (where p and q are integers and $q \neq 0$).

Examples:

2.5, 3/7, 11, –53, –207, 5.333, etc.

Irrational Numbers

Numbers that cannot be represented in p/q form. All decimals that are non-terminating and non-recurring come under the irrational category.

Examples:

$\sqrt{\text{Prime number}}$, $\sqrt[3]{5}$, etc.

Keynote

- Actual value of $\pi \neq \frac{22}{7}$. It is just approximated for calculation purpose.
- Any recurring decimal that form a fixed pattern is a rational number and hence can be expressed in $\frac{p}{q}$ form.

  Examples: 3.43333............
  0.55555............
  1.27777............
- Only non-terminating and non-recurring decimals are irrational numbers, rest in all cases it is rational.

Expressing Recurring Decimals to *p/q* Form

Example 1:
Convert K = 0.7777,... in p/q form.

Solution: $\boldsymbol{K = \frac{7}{9}}$

$K = 0.7777$ ...(i)
Multiply both side by 10 in equation (i)
$10K = 7.7777$ ...(ii)
Subtracting equation (i) from equation (ii), we get $9K = 7$

$$K = \frac{7}{9}$$

Example 2:

Convert $K = 1.233333 \ldots$ in p/q form:

Solution: $\boldsymbol{K = \frac{111}{90} = \frac{37}{30}}$

$K = 1.2333\ldots$

In the first step, we will take the non-recurring digits on the left side of the decimal.

$$10K = 12.3333 \quad \ldots(i)$$

Now, $100K = 123.3333 \quad \ldots(ii)$

Subtracting equation (i) from equation (ii), we get $90K = 111$

Therefore, $\boxed{K = \frac{111}{90} = \frac{37}{30}}$

Short Trick

Step 1: Represent the number in the form of bar.

Example: $1.573333\ldots = 1.57\bar{3}$

Step 2: For numerator, write the whole number as it is and subtract the numbers that are not getting repeated.

So, numerator of

$$1.57\bar{3} = 1573 - 157 = 1{,}416$$

Step 3: For denominator, write as many 9's as the number of digits repeating itself followed by 0's as many times as the number of digits not repeating itself immediately after the decimal point.

So, denominator of $1.\underbrace{57}_{\text{Two zeroes}}\underbrace{\bar{3}}_{\text{One 9}} = 900$

Hence, $1.57\bar{3} = \frac{1{,}416}{900} = \frac{118}{75}$

Example 3:

Convert $3.0535353\ldots$ in p/q form.

Solution: $\boldsymbol{\frac{3{,}023}{990}}$

Using trick: $3.0\overline{53} = \frac{3{,}053 - 30}{990} = \frac{3{,}023}{990}$

Example 4:

If $0.XYZXYZXYZ\ldots = \frac{23}{27}$, find $(X + Z - Y)$

Solution: 4

$0.XYZXYZXYZ = 0.\overline{XYZ} = \frac{XYZ}{999} = \frac{23}{27}$

$\Rightarrow \quad XYZ = \frac{23 \times 999}{27} = 851$

Hence, $X + Z - Y = 8 + 1 - 5 = 4$

Keynote

- 0 is neither positive nor negative integer.
- Mind here if in any question it is asked to take non-negative integers one need to consider zero as well as positive integers. Similarly, for non-positive integers one need to both zero and negative integers.

Example 5:

If $N = 0.\ a_1a_2a_1a_2a_1a_2\ldots$, then which of the following number should be multiplied with N so that it becomes an integer.

(A) 999 (B) 1,998

(C) 9,990 (D) 297

Solution: (D)

$N = 0.\ a_1a_2a_1a_2a_1a_2\ldots,$

Or, $N = 0.\ \overline{a_1a_2}$

Since the number is non-terminating but repeating.

Therefore, it can be expressed in the form of $\frac{p}{q}$

Hence, $N = \frac{a_1a_2}{99}$

To make N an integer, we need to multiply N by 99 leading to the formation of a multiple of 99.

Therefore, option (D) is the correct choice.

Integers

Integers are numbers that are not fractions. Integers are of three types:

1. Positive integers: 1, 2, 3, 4, 5, ...

2. Negative integers: –1, –2, –3, –4, ...

3. Zero

Fractions

A fraction represents the part or portion of a whole thing. It can be categorised in three categories:

1. Proper fraction: When the numerator is less than the denominator. For example, 2/3, 5/7, 11/15, etc.
2. Improper fraction: When the numerator is greater than the denominator. For example, 5/3, 9/7, 7/4, etc.
3. Mixed fraction: As the name suggests, it's a combination of natural number and a fractional part. For example, $2\frac{3}{5}$, $3\frac{1}{2}$, $1\frac{2}{7}$, etc.

Integers can be further classified into two important categories

1. Even numbers: Integers that are divisible by two or integers that are of the form 2*n* (where *n* is a whole number) are called as even numbers. For example, 0, 2, 4, 6, –10, 12, etc.
2. Odd numbers: Integers that are not divisible by two or integers that are of the form $(2n + 1)$ or $(2n - 1)$ (where *n* is a whole number) are called as odd numbers. For example, 1, 3, 5, 7, –3, –5, –13, etc.

Some Important Properties of Even and Odd Numbers

- Zero is an even number
- Even × any whole number = Even
- Odd × Odd = Odd
- Even + Even = Even
- Even + Odd = Odd
- Odd + Odd = Even
- $(\text{Even})^N$ = Even; where *N* is a natural number
- $(\text{Odd})^N$ = Odd; where *N* is a natural number

Example 6:

Given *a* and *b* are positive odd integers and *c* is a positive even integer. Which of the following is necessarily odd?

(A) $(a + b)^2 \times c$ (B) $a^2 + c^2 - b^2$

(C) $(a + b + c)(a - b)$ (D) $a^2(b + c)$

Solution: (D)

(A) $\underbrace{(a+b)^2}_{\text{(no need to check)}} \times \underset{\text{Even}}{\underset{\downarrow}{c}}$

(Since an even number multiplied by any number always results in an even number).

Any number × Even ⇒ Even

(B) $\underset{\text{Odd}}{\underset{\downarrow}{a^2}} + \underset{\text{Even}}{\underset{\downarrow}{c^2}} - \underset{\text{Odd}}{\underset{\downarrow}{b^2}}$

⇒ Odd + Even – Odd

⇒ Odd – Odd ⇒ Even

(C) $\underbrace{(a+b+c)}_{\text{(no need to check)}} \times (\underset{\text{Odd}}{\underset{\downarrow}{a}} - \underset{\text{Odd}}{\underset{\downarrow}{b}})$

Any number × Even ⇒ Even

(D) $\underset{\text{Odd}}{\underset{\downarrow}{a^2}}\ (\underset{\text{Odd}}{\underset{\downarrow}{b}} + \underset{\text{Even}}{\underset{\downarrow}{c}})$

⇒ Odd × (Odd + Even) ⇒ Odd × Odd ⇒ Odd

Hence, option (D) is correct.

Integers can also be classified as

1. Whole numbers: Whole numbers are non-negative integers. For example, 0, 1, 2, 3, 4, 5, ...
2. Natural numbers: Positive integers are natural numbers. These are also known as counting numbers. For example, 1, 2, 3, 4, 5, ...

Natural numbers can be classified into three categories

Prime number, composite number, and 1.

1. Prime Number: Natural numbers having exactly two distinct factors are called prime. For example, 2, 3, 5, 7, 11, ...

a) All prime numbers are only divisible by one or by itself.

b) Every prime number greater than 3 can be written in the form of $(6k + 1)$ or $(6k - 1)$ (where *k* is a natural number) but vice versa is not always true.

$13 = 6 \times 2 + 1 \rightarrow$ Prime

$29 = 6 \times 5 - 1 \rightarrow$ Prime

But, $6 \times 4 + 1 = 25 \rightarrow$ Not prime

c) If P is a prime number greater than 3, then $P^2 - 1$ is always divisible by 24.

Keynote

- 1 is neither prime nor composite.
- 2 is the only even prime number and it is also the smallest prime number.
- 3, 5, 7 is the only triplet of prime number which are at a difference of 2. No other triplet of prime numbers satisfies this condition.

2. Composite number: Natural numbers having at least three distinct factors are called composite. For example, 4, 6, 8, 9, 10, ...

How to check whether a number is prime or not?

One needs to check the divisibility of a given number by all prime numbers less than the square root of the given number.

For example: Let us check 259.

$\sqrt{259} < 17$

So, we will only check the divisibility of 259 by prime numbers less than 17.

$\because$ 259 is divisible by 7. Hence, it is not prime.

3. Co-prime numbers: Pair of numbers whose HCF is 1. For example, (2, 3), (7, 8), (11, 15) etc., are Co-prime pairs.

   It is not necessary for both the numbers to be prime in Co-prime pairs.

Divisibility Rules

The rule for 2^n or 5^n type numbers:

The last n digits of the number should be divisible by 2^n or 5^n

For example:

The rule for 4:

$4 = 2^2$; check only the last two digits of the number.

For example, 237,896 is divisible by 4 because 96 is divisible by 4.

$125 = 5^3$. Check if the last three digits of a number are divisible by 125.

Example 7:

Check if 1112131415...1920 is divisible by 16? If not, find the remainder.

Solution: 0

For the divisibility rule of 16, divide only the last four digits of the number.

Rem [1920/16] = 0

Hence, 1112131415...1920 is completely divisible by 16.

Example 8:

1457865XY is divisible by 8. What can be the maximum value of $(X + Y)$?

(A) 11 (B) 12

(C) 14 (D) 16

Solution: (C)

1457865XY is divisible by 8, so 5XY should be divisible by 8 (since $2^3 = 8$. So, we ought to investigate the last three digits of the given number).

Start putting the values as $X = 0$ and $Y = 0$; 500 is not divisible by 8.

Put $X = 0$ and $Y = 1$; 501 is not divisible by 8.

Put $X = 0$ and $Y = 2$; 502 is not divisible by 8.

Put $X = 0$ and $Y = 3$; 503 is not divisible by 8.

Put $X = 0$ and $Y = 4$; 504 is divisible by 8.

Next numbers divisible by 8 would be at difference of 8.

So, the possible values of XY are = {04, 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92}

The required maximum value of $(X + Y)$ is $(6 + 8) = 14$.

Rule for 3

The sum of the digits of the number should be divisible by 3.

Example: 23,481 is divisible by 3 because 2 + 3 + 4 + 8 + 1 = 18 is divisible by 3.

Rule for 9

The sum of digits of the number should be divisible by 9.

Example: 365,184 is divisible by 9, because 3 + 6 + 5 + 1 + 8 + 4 = 27 is divisible by 9.

Digital sum

Digital sum is the sum of digits of the number till a single digit is obtained.

Example: Digital sum of 247 is 2 + 4 + 7 = 13 → 1 + 3 = 4

Remainder obtained by dividing any number by 9 equals the digital sum of that number. The digital sum of all the multiples of 9 is 9.

Hence, for the divisibility by 9, one should use the concept of digital sum to approach the problem easily.

Rule for 11

If the difference between the sum of digits at odd places and the sum of digits at even places is 0 or any multiple of 11 then the number is divisible by 11 (Assign alternate + and – sign from the right-hand side, add all + values and all – values and then take their difference).

Example: Check 1253478

+	–	+	–	+	–	+
1	2	5	3	4	7	8

= (8 + 4 + 5 + 1) - (7 + 3 + 2) = 18 - 12 = 6

which is not divisible by 11.

Hence, the given number is not divisible by 11.

Divisibility Rules for Composite Numbers

The divisibility rules of composite numbers are:

1. Select Co-prime factors of the composite number.
2. Product of these Co-prime factors should equal the composite number.

For example:

Divisibility rule of 6 → (2, 3); if a number is divisible by both 2 and 3, then it must be divisible by 6.

Divisibility rule of 21 → (3, 7); if a number is divisible by both 3 and 7, then it must be divisible by 21.

Divisibility rule of 72 → (8, 9); if a number is divisible by both 8 and 9, then it must be divisible by 72.

Example 9:

If a and b are two respective single digits of the five-digit number 843ab where a is not equal to b such that this number is divisible by 40, then $(a + b)$ can be?

(A) 1 (B) 2

(C) 3 (D) 5

Solution: (B)

For divisibility by 40, the number should be divisible by both 5 and 8.

For divisibility by 5, b can be 0 or 5 (since b is the last digit and for a number to be divisible by 5, the last digit is either 0 or 5). But, if we take b = 5 number becomes odd, and it would not be divisible by 8.

Hence, b = 0.

Now, the last three digits of 843ab, i.e., 3a0, should be divisible by 8. As for a number to be divisible by 8, we ought to take the last three digits of the number and divide it by 8.

Put, a = 2 → 320 is divisible by 8.

Next number would be 320 + 40 = 360.

So, there are two values of a = {2, 6}

Hence, $a + b$ can be either 2 + 0 = 2 or 6 + 0 = 6.

Option (B) is correct.

Common Divisibility Rules of 7, 11, and 13

7, 11, and 13 are the factors of 1,001, and they are Co-prime to each other.

1,001 = 7 × 11 × 13

Remainder [10^3/1,001] = –1

Also, Remainder [10^3/7] = –1,

Remainder [10^6/7] = 1

Remainder [10^9/7] = –1; and, so on.

The same result we get in the case of 11 and 13 also.

To check the divisibility of any large number by 7, 11, or 13 or to find the remainder, make pairs of 3 digits from the right-hand side and assign alternate + and – sign to the respective pairs. Add all + pairs and all – pairs and then take their difference. If this difference is divisible by 7 or 11 or by 13, then the original number would be divisible by 7 or 11 or by 13.

For example: Check the divisibility of 10,573,240,035 by 7.

$\Rightarrow \overset{-}{10} \mid \overset{+}{573} \mid \overset{-}{240} \mid \overset{+}{035}$ (Make pairs from the right side starting with "+")

$\Rightarrow$ (573 + 35) – (10 + 240)

$\Rightarrow$ 608 – 250 = 358

358 is not divisible by 7 so the given number is not divisible by 7.

Example 10:

Find the remainder when 151,152,153.......... 199,200 is divisible by 13.

Solution: 12

From 151 to 200, there are 50 three-digit numbers.

$\Rightarrow$ 

Take pair of two consecutive numbers from the right side and add them up.

$\Rightarrow$ + 1 + 1 + 1 + 1 .... up to 25 times. $\Rightarrow$ 25

$\text{Rem}\left[\frac{25}{13}\right] = 12$

Successive Division

In successive divisions, the quotient obtained in the first division acts as a dividend for the second division. The quotient obtained in the second division acts as a dividend for the third division and so on.

Let's consider a number N is successively divided by a, b, and c and the remainders obtained are r_1, r_2, and r_3, respectively.

$$a)\,N(q_1 \;\; \Rightarrow \;\; b)\,q_1(q_2 \;\; \Rightarrow \;\; c)\,q_2(q_3$$
$$r_1 \qquad\qquad r_2 \qquad\qquad r_3$$
$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$N = aq_1 + r_1 \quad q_1 = bq_2 + r_2 \quad q_2 = cq_3 + r_3$

Let $q_3 = 0 \Rightarrow q_2 = r_3$

$q_1 = bq_2 + r_2$

As $q_2 = r_3$ so, $q_1 = br_3 + r_2$

Also we know that

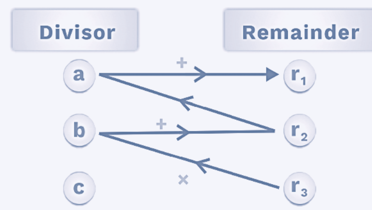
$N = aq_1 + r_1$ and $q_1 = br_3 + r_2$

So, $N = a(br_3 + r_2) + r_1$

or, $N = abr_3 + ar_2 + r_1$

$\Rightarrow$ This is the least such number

Short Trick

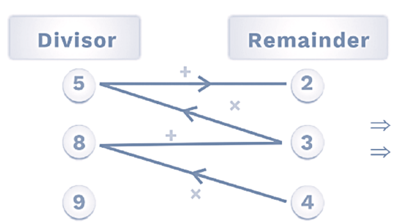


Create two columns of divisor and remainder. Start from last row of second column, go diagonally up and multiply (br_3) then add horizontally ($br_3 + r_2$) again go diagonally upward multiply $\{a(br_3 + r_2)\}$ then add horizontally $\{a(br_3 + r_2) + r_1\}$.

Example 11:

How many three-digit numbers exist which on successive division by 5, 8, and 9 leaves remainder of 2, 3, and 4, respectively?

Solution: 3



$\Rightarrow$ 35 × 5 + 2 = 177

$\Rightarrow$ 8 × 4 + 3 = 35

General form of the number is LCM (5, 8, 9) K + 177 = 360K + 177 , where K is a whole number

Put, K = 0, 1, 2, 3, ...

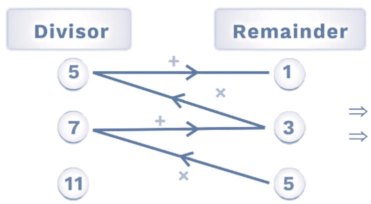
The desired three-digit numbers = 177, 537, and 897 for K = 0, 1, and 2, respectively.

Hence, three such numbers are possible.

Example 12:

A number is successively divided by 5, 7, and 11 and leaves the remainder 1, 3, and 5, respectively. Find the respective remainders when the order of division is reversed.

Solution: 2



$\Rightarrow$ **38 × 5 + 1 = 191**

$\Rightarrow$ **7 × 5 + 3 = 38**

The required number is LCM (5, 7, 11)K + 191 where *K* is a whole number = 385K + 191.

The least value of the number is obtained for *K* = 0, i.e., 191.

When the order of divisor is changed,

$$11\overline{)\,191\,}(17 \quad \begin{array}{r} -187 \\ \hline 4 \end{array} \quad \Rightarrow \quad 7\overline{)\,17\,}(2 \quad \begin{array}{r} -14 \\ \hline 3 \end{array} \quad \Rightarrow \quad 5\overline{)\,2\,}(0 \quad \begin{array}{r} -0 \\ \hline 2 \end{array}$$

Hence, the respective remainder when the number is successively divided by 11, 7, and 5 are 4, 3, and 2.

Factor Theory

Factors

Factors are the natural numbers less or equal to the given number that divides it completely.

Factors are the divisors of the given number.

Example: Factors of 12 $\rightarrow$ 1, 2, 3, 4, 6, 12

Factors of 56 $\rightarrow$ 1, 2, 4, 7, 8, 14, 28, 56.

Prime Factorisation

According to the fundamental theorem of arithmetic, every natural number (except 1) can be represented as the product of one or more prime numbers.

This process of breaking down any number to the product of prime numbers is known as prime factorisation.

Examples: $12 = 2 \times 2 \times 3 = 2^2 \times 3$

$60 = 6 \times 10 = 2^2 \times 3 \times 5$

In general, the prime factorisation of a number *N* is represented as:

$N = p^a \times q^b \times r^c$ (where *p*, *q*, and *r* are the prime numbers and *a*, *b*, and *c* are whole numbers).

Finding Number of Factors

$$36 = 6 \times 6 = \begin{array}{ccc} 2^2 & \times & 3^2 \\ \downarrow & & \downarrow \\ \begin{pmatrix} 2^0 \\ 2^1 \\ 2^2 \end{pmatrix} & & \begin{pmatrix} 3^0 \\ 3^1 \\ 3^2 \end{pmatrix} \\ \downarrow & & \downarrow \\ \text{3 ways} & & \text{3 ways} \end{array}$$

Total = 3 × 3 = 9 factors.

If $N = p^a \times q^b \times r^c$

Numbers of factors of *N*

$= (a + 1) \times (b + 1) \times (c + 1)$

Keynote

- Number of factors of any perfect square is always odd.
- Number of factors of square of any prime number is 3.

Number of Even and Odd Factors

Even Factors:

$1{,}800 = 18 \times 100 = 2 \times 3^2 \times 10^2 = 2 \times 3^2 \times 2^2 \times 5^2$

$$= \begin{array}{ccccc} 2^3 & \times & 3^2 & \times & 5^2 \\ \downarrow & & \downarrow & & \downarrow \\ \begin{pmatrix} 2^1 \\ 2^2 \\ 2^3 \end{pmatrix} & & \begin{pmatrix} 3^0 \\ 3^1 \\ 3^2 \end{pmatrix} & & \begin{pmatrix} 5^0 \\ 5^1 \\ 5^2 \end{pmatrix} \\ \underbrace{\qquad}_{3\ ways} & & \underbrace{\qquad}_{3\ ways} & & \underbrace{\qquad}_{3\ ways} \end{array}$$

Even factors = 3 × 3 × 3 = 27.

Note: One should not take 2° while calculating even factors because 2° = 1; any combination with other prime numbers will give an odd factor only.

Odd Factors:

$1,800 = 2^3 \times 3^2 \times 5^2$

2 is an even prime number, and any exponent of 2 (except 2^0) will also be an even number.

Hence, while calculating the number of odd factors, one should neglect 2 from the prime factorisation.

$$
\begin{array}{ccccccc}
1,800 = & 2^3 & \times & 3^2 & \times & 5^2 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \begin{pmatrix} 2^0 \end{pmatrix} & & \begin{pmatrix} 3^0 \\ 3^1 \\ 3^2 \end{pmatrix} & & \begin{pmatrix} 5^0 \\ 5^1 \\ 5^2 \end{pmatrix}
\end{array}
$$

The number of odd factors = $1 \times 3 \times 3 = 9$

Number of Perfect Squares and Perfect Cube Factors

For perfect square factors, only consider the multiple of 2 as power of a prime number.

For example:

$$
\begin{array}{cccccc}
N = & 2^5 & \times & 3^7 & \times & 7^3 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \begin{pmatrix} 2^0 \\ 2^2 \\ 2^4 \end{pmatrix} & & \begin{pmatrix} 3^0 \\ 3^2 \\ 3^4 \\ 3^6 \end{pmatrix} & & \begin{pmatrix} 7^0 \\ 7^2 \end{pmatrix} \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \text{3 ways} & & \text{4 ways} & & \text{2 ways}
\end{array}
$$

Number of perfect square factors of N

$= 3 \times 4 \times 2 = 24$

For a perfect cube, only consider the multiple of 3 as power of a prime numbers.

For example:

$$
\begin{array}{cccccc}
N = & 2^5 & \times & 3^7 & \times & 7^3 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \begin{pmatrix} 2^0 \\ 2^3 \end{pmatrix} & & \begin{pmatrix} 3^0 \\ 3^3 \\ 3^6 \end{pmatrix} & & \begin{pmatrix} 7^0 \\ 7^3 \end{pmatrix} \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \text{2 ways} & & \text{3 ways} & & \text{2 ways}
\end{array}
$$

Number of perfect cube factors of N

$= 2 \times 3 \times 2 = 12$

Finding Sum of the Factors

Consider a number $N = 2^3 \times 3^2 \times 5^4$

Sum of factors of N can be written as:

$(2^0 + 2^1 + 2^2 + 2^3) \times (3^0 + 3^1 + 3^2) \times (5^0 + 5^1 + 5^2 + 5^3 + 5^4)$

Now, the given expression is in G.P. and we know that:

Sum of G.P. $= \dfrac{a(r^n - 1)}{(r - 1)}$, $r > 1$

Where $a \rightarrow$ first term

$r \rightarrow$ common ratio

and $n \rightarrow$ number of terms

Therefore, $\dfrac{2^0(2^4 - 1)}{(2 - 1)} \times \dfrac{3^0(3^3 - 1)}{(3 - 1)} \times \dfrac{5^0(5^5 - 1)}{(5 - 1)}$

$= 15 \times 13 \times 781 = 152,295$

Note: If any number say N can be expressed as $N = a^p \times b^q \times c^r \times \ldots,$

where a, b, and c are prime factors and p, q, and r are whole numbers.

Sum of all factors (or divisors) of N

$= \dfrac{(a^{p+1} - 1)(b^{q+1} - 1)(c^{r+1} - 1)}{(a - 1)(b - 1)(c - 1)} \times \ldots\ldots$

Finding the Product of the Factors

Let's take a small number and understand the concept behind finding the product of factors of a natural number N.

20 = 1, 2, 4, 5, 10, 20

Product of factors = $1 \times 2 \times 4 \times 5 \times 10 \times 20$

$= (1 \times 20)(2 \times 10)(4 \times 5)$

$= 20 \times 20 \times 20 = 20^3$

Observe here, 20 has 6 factors, which can be paired in 3 ways.

So, it can be generalised that

Product of factors of $N = N^{\left(\frac{\text{no. of factors}}{2}\right)}$

In case of perfect square numbers, let $N = p^a \times q^b \times r^c$ where each of a, b, c is even.

There are $(a + 1)(b + 1)(c + 1)$ say d factors.

There are $(a + 1)(b + 1)(c + 1)$ say d factors.

We can form $(d - 1)/2$ pairs and we would be left with one lone factor, i.e., $\sqrt{N}$. The product of all these factors is $N^{(d-1)/2}$ x square root $(N) = N^{(d/2)}$.

Whether or not N is a perfect square, the product of all its factors is $N^{(d/2)}$ where d is the number of factors of N.

In how many ways a number can be written as the product of its two factors?

Let's see for 40

$40 \rightarrow 1, 2, 4, 5, 8, 10, 20, 40$ = Total 8 factors

$$\left.\begin{aligned}40 &= (1\times 40)\\ &= (2\times 20)\\ &= (4\times 10)\\ &= (5\times 8)\end{aligned}\right\}\text{4 ways}$$

So, it is obvious that if a number has n number of factors (and n is even), they can be paired in $n/2$ ways to get the product of it's two factors.

$$\text{Total no. of ways of writing any number as product of its two factors} = \frac{\text{Total number of factors}}{2}$$

This is applicable when total number of factors is even for any number (i.e., non-perfect square numbers).

When number of factors are odd i.e., for perfect square numbers?

$16 \rightarrow 1,\ 2,\ 4,\ 8,\ 16 \Rightarrow$ Total factors = 5

16 = There are 2 ways of writing 16 as the product of two distinct factors.

But, there is one more way = (4 × 4)

So, the whole ways of writing 16 the product of two factors are 3.

Therefore, one can generalise:

$$\text{Number of ways of writing any number as the product of two distinct factors} = \left(\frac{\text{Total number of factors} - 1}{2}\right)$$

$$\text{Total number of ways of writing any number as the product of two factors} = \left(\frac{\text{Total number of factors} + 1}{2}\right)$$

In how many ways a number can be written as the product of its two Co-prime factors?

Let's see for 48

$48 = 16 \times 3 = 2^4 \times 3^1$

Writing 48 as the product of its two Co-prime factors.

$$\left.\begin{aligned}48 &= (1,\ 2^4\times 3) &\Rightarrow&\ (1,\ 48)\\ &= (3,\ 2^4) &\Rightarrow&\ (3, 16)\end{aligned}\right\}\text{2 ways}$$

By permutation and combination concept, it can be shown that the number of ways of writing any number as the product it's two Co-prime factors = $2^{(p-1)}$

where, p is a number of prime factors.

Example: $120 = 12 \times 10 = 2^3 \times 3^1 \times 5^1$

So, $p = 3$ {∵ 3 prime numbers are used in prime factorisation}

Total ways of writing 120 as it's co-prime factors

$= 2^{3-1} = 2^2 = 4$

And, can manually verify also,

$120 = (1,\ 2^3 \times 3 \times 5),\ (3,\ 2^3 \times 5),\ (5,\ 2^3 \times 3),\ (2^3,\ 3 \times 5)$.

Finding the Number When Its Number of Factors Is Given

Let us understand it by an example:

If a number N has 10 factors.

$N = P_1^{a} \times P_2^{b} \times P_3^{c} \times \cdots$

So, $(a + 1)\ (b + 1)\ (c + 1)$ .... = 10

Now, see in how many ways 10 can be written as the product of two numbers, three numbers, and so on.

Accordingly, the value of a, b, c,............ can be determined.

$10 \rightarrow (1 \times 10) \rightarrow (a + 1)\ (b + 1) = 1 \times 10$

$\Rightarrow\ a = 0,\ b = 9$

$\rightarrow (2 \times 5) \rightarrow (a + 1)\ (b + 1) = 2 \times 5$

$\Rightarrow\ a = 1,\ b = 4$

Hence, $N = P_1^{0} \times P_2^{9}$ (where P_1 and P_2 can be any two distinct prime numbers)

Or,

$N = P_1^{1} \times P_2^{4}$

(where P_1 and P_2 can be any two distinct prime numbers)

Example 13:

If N^3 has 70 factors. Find the minimum value of N.

Solution: 72

Try to break down 70 as the product of two numbers, three numbers, etc.

$$70 \rightarrow 1 \times 70 \quad \Rightarrow N^3 = p^{69} \quad \Rightarrow N = p^{23}$$

$$\rightarrow 2 \times 35 \quad \Rightarrow N^3 = pq^{34}$$

(Not possible because exponents of prime numbers should be multiple of 3.)

$$70 \rightarrow 5 \times 14 \quad \Rightarrow N^3 = p^4q^{13} \text{ (Not possible)}$$

$$\rightarrow 7 \times 10 \quad \Rightarrow N^3 = p^6q^9 \quad \Rightarrow N = p^2q^3$$

$$\rightarrow 2 \times 5 \times 7 \Rightarrow N^3 = pq^4r^6 \text{ (Not possible)}$$

So, N can be either p^{23} or p^2q^3 type.

For minimum value of N.

Take $N = p^2q^3$

Put, $p = 3$ and $q = 2$ (Two least prime numbers)

Hence, $N_{min} = 3^2 \times 2^3 = 9 \times 8 = 72$.

Example 14:

If $N = 2^5 \times 3^4 \times 5^6 \times 7^2$. Find:

a) Number of perfect square factors of N.

b) Number of a perfect cube of factors of N.

c) Number of factors of N which are multiple of 100.

Solution: a) 72, b) 12, c) 300

a) Number of perfect squares factors of N

$$N = \underset{\underbrace{\begin{pmatrix}2^0\\2^2\\2^4\end{pmatrix}}_{3\text{ Ways}}}{\overset{\downarrow}{2^5}} \times \underset{\underbrace{\begin{pmatrix}3^0\\3^2\\3^4\end{pmatrix}}_{3\text{ Ways}}}{\overset{\downarrow}{3^4}} \times \underset{\underbrace{\begin{pmatrix}5^0\\5^2\\5^4\\5^6\end{pmatrix}}_{4\text{ Ways}}}{\overset{\downarrow}{5^6}} \times \underset{\underbrace{\begin{pmatrix}7^0\\7^2\end{pmatrix}}_{2\text{ Ways}}}{\overset{\downarrow}{7^2}}$$

Total perfect square factors $= 3 \times 3 \times 4 \times 2 = 72$

b) Number of perfect cube factors of N

$$N = \underset{\underbrace{\begin{pmatrix}2^0\\2^3\end{pmatrix}}_{2\text{ Ways}}}{\overset{\downarrow}{2^5}} \times \underset{\underbrace{\begin{pmatrix}3^0\\3^3\end{pmatrix}}_{2\text{ Ways}}}{\overset{\downarrow}{3^4}} \times \underset{\underbrace{\begin{pmatrix}5^0\\5^3\\5^6\end{pmatrix}}_{3\text{ Ways}}}{\overset{\downarrow}{5^6}} \times \underset{\underbrace{\begin{pmatrix}7^0\end{pmatrix}}_{1\text{ Way}}}{\overset{\downarrow}{7^2}}$$

Total perfect cube perfect $= 2 \times 2 \times 3 \times 1 = 12$

c) Factors which are multiple of 100

$N = 2^5 \times 3^4 \times 5^6 \times 7^2$

$\because \; 100 = 2^2 \times 5^2$

$N = 2^2 \times 5^2 \,(2^3 \times 3^4 \times 5^4 \times 7^2)$

So, number of factors which are multiple of 100

$= (3 + 1)(4 + 1)(4 + 1)(2 + 1) = 4 \times 5 \times 5 \times 3 = 300$

Example 15:

Find the number of solutions of $a \times b = 1980$, where a and b are natural numbers.

Solution: 36

$N = 1{,}980 = 198 \times 10 = 2^2 \times 3^2 \times 5 \times 11$

Total number of factors of N

$= (2 + 1)(2 + 1)(1 + 1)(1 + 1) = 3 \times 3 \times 2 \times 2 = 36$

Number of ways of writing N as product of its two factors

$$= \frac{\text{Total no. of factors}}{2} = \frac{36}{2} = 18$$

$\because$ Variables a and b are mentioned in the question, so one needs to find the ordered solution, i.e., (a, b) and (b, a) would be counted as 2 different solutions.

Therefore, $a \times b = 1980$ will have $= 2 \times 18 = 36$.

Example 16:

If all the factors of 5,880 are written in ascending order, from left to right, find the factor which occupies the 43rd from the left end.

Solution: 980

$N = 5{,}880 = 2^3 \times 3^1 \times 5^1 \times 7^2$

Number of factors of 5,880 = $4 \times 2 \times 2 \times 3$ = 48.

Let's write some of factors.
5,880 → 1, 2, 3, 4, 5, 6, 7,... 5,880

Notice here, first factor × 48th factor = 1 × 5,880 = 5,880

Similarly, second factor × 47th factor = 5,880 and so on. Hence, it can be concluded that the product of factors equidistant from both ends is equal to the number itself. Therefore,

⇒ Sixth factor × 43rd factor = 5,880

⇒ 6 × 43rd factor = 5,880

⇒ 43rd factor = $\frac{5{,}880}{6}$ = 980

HCF and LCM

Factors are numbers that divide the given number completely, and multiples are the numbers that are divisible by the given number.

For example, factors of 18 are 1, 2, 3, 6, 9, and 18.

Multiples of 18 are 18, 36, 54, 72............

Highest Common Factor (HCF)

It is the largest number that divides the given set of numbers completely.

Example: Find HCF of 15, 24, and 36.

Factors of 15 are → 1, 3, 5, 15.

Factors of 24 are → 1, 2, 3, 6, 8, 12, 24.

Factors of 36 are → 1, 2, 3, 4, 6, 9, 12, 18, 36.

So, the highest common factor of 15, 24, and 36 is 3.

Least Common Multiple (LCM)

The least number is completely divisible by the given set of numbers.

Example: Find LCM of 9, 18, and 24.

Multiples of 9 are → 9, 18, 27, 36, 45, 54, 63, 72, 81............

Multiples of 18 are → 18, 36, 54, 72, 90............

Multiples of 24 are → 24, 48, 72, 96............

So, the LCM of 9, 18, and 24 are 72.

Keynote

- HCF of given set of numbers is always less than or equal to the smallest number in the set.
- LCM of given set of numbers is always greater than or equal to the largest number in the set.
- HCF is the factor of LCM.

Finding HCF and LCM by factorisation method

Factorise all the given numbers. To find HCF, look for common prime numbers and take their highest common power. And to find LCM take all prime numbers used in prime factorisation and write their highest powers available.

Example 17:

Find HCF and LCM of 216, 252 and 306.

Solution: 18 and 25,704

Prime factorise all the numbers.

$216 \rightarrow 2^3 \times 3^3$

$252 \rightarrow 2^2 \times 3^2 \times 7$

$306 \rightarrow 2 \times 3^2 \times 17$

For HCF, it can be observed that prime numbers 2 and 3 are common in all, and the highest common power of 2 is 1, and the highest common power of 3 is 2.

HCF = $2^1 \times 3^2 = 18$

For LCM, write all the prime numbers used in the prime of factorisation with their highest power.

LCM = $2^3 \times 3^3 \times 7 \times 17$ = 25,704.

Example 18:

If the HCF of the first 40 natural numbers is N, then find the LCM of the first 46 natural numbers.

Solution:

HCF of the first 40 natural numbers is N, which means N includes all the prime numbers up to 40 with their highest power.

$\{1, 2, 3, 4, 5, \ldots\ldots\ldots\ldots 40\} = N$

Now, the numbers 41 to 46 are added to the list. So, one needs to include all new prime numbers added and look for any change in the highest power of already used prime numbers in N.

41	$\rightarrow$	Prime	$\rightarrow$	Add
42	$\rightarrow$	$2 \times 3 \times 7$	$\rightarrow$	It will not update the N
43	$\rightarrow$	Prime	$\rightarrow$	Add
44	$\rightarrow$	$2^2 \times 11$	$\rightarrow$	No update in N again
45	$\rightarrow$	$3^2 \times 5$	$\rightarrow$	Again, no update required
46	$\rightarrow$	2×23	$\rightarrow$	No update

Hence, LCM of $(1, 2, 3, 4, 5, \ldots\ldots\ldots\ldots 46)$

$= N \times 41 \times 43$

Example 19:

LCM of $(5^n \times 48, 250, 32 \times k)$ is $2^6 \times 5^5 \times 3^2$ find maximum of $(n + k)$.

Solution: 56,255

$5^n \times 48 \rightarrow 2^4 \times 3 \times 5^n$

$250 \rightarrow 2 \times 5^3$

$32K \rightarrow 2^5 \times K$

$\because$ LCM includes all the prime numbers with their highest power, so the maximum exponent of 5 that is possible is 5.

Hence, the maximum value of n can be 5 here.

To maximise K, it can be taken as $K = 2 \times 3^2 \times 5^5$ $= 56{,}250$

So, number $32K = 2^6 \times 3^2 \times 5^5$ (which is equal to LCM)

Therefore, maximum of $(n + k) = 5 + 56{,}250$ $= 56{,}255$

Finding HCF by minimum difference method

Find the minimum difference between the pair of numbers, and one of the factors of this difference would be the HCF of given numbers.

Example 20:

Find HCF of 276, 372, 588, and 708.

Solution: (12)

Take the minimum difference between any two numbers.

Minimum difference = 372 – 276 = 96

Now, start checking the factors of 96 that will divide the given numbers completely. Use the concepts of divisibility to figure out the value quickly.

Check the factors in decreasing order as one needs to find HCF here.

$96 \rightarrow 2^5 \times 3$; Factors	$\rightarrow$	96,	48,	32,	24,	16,	12
		×	×	×	×	×	√

So, 12 is the highest factor of 96, which divides all the numbers completely.

Hence, 12 is required for HCF.

Application Rules of HCF and LCM

- The largest number that will divide a, b, and c, is HCF (a, b, c).
- The largest number that will divide a, b, and c leaving the remainder of x, y, and z, respectively, is HCF $(a - x, b - y, c - z)$.
- The largest number that divides a, b, and c, leaving the same remainder in each case, is HCF $(a - b, b - c, c - a)$.
- The smallest number which is exactly divisible by a, b, and c is LCM (a, b, c).
- The smallest number, when divided by a, b, and c, leaves the same remainder r in each case is $[\text{LCM}(a, b, c) + r]$.
- The smallest number, when divided by a, b, and c, leaves the remainder p, q, and r, respectively, is $[\text{LCM}(a, b, c) - m]$, where $a - p = b - q = c - r = \text{m}$.

Example 21:

The least multiple of 8, which leaves a remainder of 4, when divided by 9, 15, 18 and 30 is ________ .

Solution: 184

The least number which is divisible by 9, 15, 18, and 30 is LCM of (9, 15, 18, 30) = 90.

The general form of all such numbers = $90K$.

As in the question, it says that number leaves a remainder of 4 on dividing by 9, 15, 18, and 30. So, required number = $90K + 4$.

Now, to figure out the least multiple of 8, put values of $K = 1, 2, 3,...$ and check.

Put $K = 1$, $90 \times 1 + 4 = 94 \rightarrow$ Not divisible by 8.

Put $K = 2$, $90 \times 2 + 4 = 184 \rightarrow$ Divisible by 8.

Hence, 184 is the desired number.

Example 22:

On dividing a certain number by 8, 10, and 12, the remainders obtained are 2, 4, and 6, respectively. Find the largest three-digit number satisfying this condition.

Solution: 954

Notice here, the difference between the divisors and their remainders is constant.

$\Rightarrow \quad 8 - 2 = 10 - 4 = 12 - 6 = 6$

General form of the number = LCM (8, 10, 12) $K - 6$

$= 120K - 6$

Largest three-digit number can be obtained at $K = 8$

$\Rightarrow \quad 120 \times 8 - 6 = 960 - 6 = 954$

Example 23:

For some industrial experiments, 40 L of mud water, 72 L of saline water, and 88 L of distilled water were ordered. The lab attendant wants to pack them in cans so that each can contains the same litres of water, and he can't mix any two in a can. What is the least number of can required?

Solution: 25

To keep the minimum number of cans, one needs to maximise each can's capacity. So, first out the maximum capacity of cans that can store all 40 L, 72 L, and 88 L of water completely. It is basically the HCF of (40, 72, 88) = 8.

Capacity of each can = 8 L

Number of cans required $= \dfrac{40}{8} + \dfrac{72}{8} + \dfrac{88}{8}$

$= 5 + 9 + 11 = 25$

Some General Rules of HCF and LCM

- If two numbers are given m and n, then $m \times n$ = HCF × LCM.
- If HCF of m and n is h; assume $m = ha$ and $n = hb$; where HCF $(a, b) = 1$.
- HCF of fractions = HCF of numerators/ LCM of denominators.
- LCM of fractions = LCM of numerators/ HCF of denominators.
- HCF of $(x^p - 1, x^q - 1) = x^{\text{HCF}(p,\, q)} - 1$.

Example 24:

The product of the two numbers is 6,480, and their HCF is 9. How many are such pairs of numbers possible?

Solution: 2

Let's take the numbers $9a$ and $9b$ where a and b are Co-prime, i.e., HCF $(a, b) = 1$

$\Rightarrow \quad 9a \times 9b = 6{,}480 \quad \Rightarrow ab = 80 \quad \Rightarrow \quad ab = 2^4 \times 5^1$

The number of ways of writing a number as the product of its two Co-prime factors = 2^{p-1} (where p is the number of prime numbers used in prime factorisation).

Here, $p = 2$

Co-prime pairs possible = $2^{2-1} = 2^1 = 2$

$a \times b$ = (1, 80) and (5, 16)

Hence, 2 pairs are possible.

Example 25:

The sum of the two numbers is 121, and their HCF is 11. How many are such pairs of numbers possible?

Solution: 5

Let's take the numbers as ha and hb where a and b are Co-prime numbers.

$\Rightarrow \quad ha + hb = 121 \qquad \Rightarrow \quad h(a + b) = 121$

$\Rightarrow \quad 11\,(a + b) = 121 \qquad \Rightarrow \quad a + b = 11$

$(a, b) \equiv (1, 10), (2, 9), (3, 8), (4, 7), (5, 6)$

So, total five pairs of numbers are possible.

Note: One can also use Euler number to find number of co-prime pairs. Number of ways of writing N as sum of two co-prime numbers

$= \frac{1}{2}(\textit{Euler number of } N)$

Example 26:

LCM of 2 natural numbers a and b where $a > b$ is 429. What is the maximum possible sum of the digits of b?

Solution: 12

$429 = 3 \times 11 \times 13$

So, a and b can have only these three prime numbers with their highest power as 1. So, different possibilities are:

$(a, b) \rightarrow (3 \times 11 \times 13, 1) = (429, 1)$

$\rightarrow (3 \times 11 \times 13, 3) = (429, 3)$

$\rightarrow (3 \times 11 \times 13, 11) = (429, 11)$

$\rightarrow (3 \times 11 \times 13, 13) = (429, 13)$

and, so on............

But to maximise sum of digits of b

$(a, b) \rightarrow (3 \times 11 \times 13, 3 \times 13) = (429, 39)$

Or,

$\rightarrow (11 \times 13, 3 \times 13) = (143, 39)$

Hence, maximum value of sum of digits of $b = 3 + 9 = 12$.

Factorials

Factorials are defined as:

$N! = N \times (N - 1) \times (N - 2) \times \cdots \times 3 \times 2 \times 1$

For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Note: $0! = 1$

Finding Highest Power of a Prime in any Factorial

Let's understand it by an example.

Example: Find highest power of 3 in 12!

$12! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12$

To find highest power of 3 take all multiples of 3 in 12! and neglect rest of the number.

$12! = 3 \times 6 \times 9 \times 12 \times K$

$= 3 \times (3 \times 2) \times (3 \times 3) \times (3 \times 4) \times K$

$= 3^5 \times K_1$

Hence, highest power of 3 in 12! is 5.

The question can also be asked to find the highest power of 3 that divides 12! completely. Both the questions have the same answer.

Alternate method

To find the highest power of prime P in any factorial N, divide N by P successively till the quotient obtained is less than P. And then add all the quotients.

Example: Find the highest power of 3 in 25!

$\left(\frac{25}{3}\right)_{\text{Quotient}} = 8$

$\left(\frac{8}{3}\right)_{\text{Quotient}} = 2$

Hence, highest power of 3 in 25! = 8 + 2 = 10.

Note: This method is applicable only for prime numbers.

Finding the Highest Power of Composite Numbers in any Factorial

To find out the highest power of composite numbers in a given factorial, first, factorise the composite number and find the highest powers of prime numbers available in factorisation. Accordingly, one can get composite number power as well.

Example 27:

What highest power of 15 divides 50! completely?

Solution: 12

$15 = 3 \times 5$

Power of 3 in 50!

$\frac{50}{3} = 16, \ \frac{16}{3} = 5, \ \frac{5}{3} = 1$

$= 16 + 5 + 1 = 22$

Power of 5 in 50!

$\frac{50}{5} = 10, \ \frac{10}{5} = 2$

$= 10 + 2 = 12$

So, $50! = 3^{22} \times 5^{12} \times K$

$= (3 \times 5)^{12} \times 3^{10} \times K$

$= 15^{12} \times K_1$

Therefore, the highest power of 15 that will divide 5! completely is 12.

Important observation

It can be observed from the above example that the highest power of larger prime numbers would always be less than the smaller prime numbers. As in the above example.

The highest power of 5 < the Highest power of 3.

Example 28:

For what maximum value of N, $\frac{150!}{63^N}$ is an integer?

Solution: 24

$63 = 3^2 \times 7$

Highest power of 3 in 150!

$$\frac{150}{3} = 50, \quad \frac{50}{3} = 16, \quad \frac{16}{3} = 5, \quad \frac{5}{3} = 1$$

$= 50 + 16 + 5 + 1 = 72$

Highest power of 7 in 150!

$$\frac{150}{7} = 21, \quad \frac{21}{7} = 3$$

$= 21 + 3 = 24$

$150! = 3^{72} \times 7^{24} \times K$

$= 3^{48} \times 7^{24} \times 3^{24} \times K$

$= (3^2 \times 7)^{24} \times 3^{24} \times K$

$150! = 63^{24} \times K_1$

So, maximum power of 63 that can be drawn from 150! is 24.

Therefore, $N_{max} = 24$

Example 29:

For how many values of N, $N!$ is completely divisible by 5^7 but not by 7^5?

Solution: 4

For $N!$ to be a multiple of 5^7, N should be closer to 35.

(One needs to have a smart guess to approach such problems).

Any factorial from 30 to 34 will have the same number of 5s.

So, let's check for 30!

$$\frac{30}{5} = 6, \quad \frac{6}{5} = 1$$

So, 30! is a multiple of 5^7 and hence any factorial greater than 29 would be a multiple of 5^7.

Similarly, one can check (by smart guess) that the least factorial that has 7^5 is 35!

$$\frac{35}{7} = 5$$

So, any factorial greater than 34 would be the multiple of 7^5.

Accorting to the question, $N!$ should be a multiple of 5^7 but not 7^5.

Hence, N can be 30, 31, 32, and 34.

Therefore, 4 values of N can hold.

Finding the Number of Trailing Zeros in a Factorial

As $N!$ is the product of all the numbers starting from N to 1, and in any multiplication, zero can be only obtained when a pair of 5 and 2 is available.

The number of pairs of 5 and 2 available in any factorial will decide the number of trailing zeros.

∴ Factorial is a regular series of numbers. It can be witnessed that in any factorial

Number of 2s > Number of 5s

Or it can also be said

Power of 2 > Power of 5

But, to produce zero, a pair of 2 and 5 is required; hence, the power of 5 would be the deciding factor, i.e., the number of 5's available in factorial will give the number of trailing zeros in that factorial.

Example 30:

Determine the number of trailing zeros in 67!

Solution: 15

Just calculate the highest power of 5 available in 67!

$\frac{67}{5} = 13, \frac{13}{5} = 2$

$= 13 + 2 = 15$

$\therefore$ 5^{15} is available in 67!.

There are 15 trailing zeros in 67!

Example 31:

Find trailing zeros in 89! + 88!.

Solution: 21

$89! + 88! = 89 \times 88! + 88!$

$= 88!\,(89 + 1)$

$= 88! \times 90$

Trailing zeros in 88! is

$\frac{88}{5} = 17, \frac{17}{5} = 3$

$= 17 + 3 = 20$

So, trailing zeros in $89! \times 90$

$\underbrace{89!}_{20 \text{ zeros}} \times \underbrace{90}_{1 \text{ zero}}$

$= 20 + 1 = 21$

Skipping Zeros Concept in Factorials

Let's observe the behaviour of factorial from the trailing zero's point of view:

Factorials		Number of trailing zeros	
1!—4!	→	0	
5!—9!	→	1	
10!—14!	→	2	
15!—19!	→	3	
20!—24!	→	4	} Skipping
25!—29!	→	6	

It is clear that every multiple of 5' will contribute to 1 zero. Similarly, a multiple of 5^2 will contribute to 2 zeros.

Example: $24! \to 4$ trailing zeros $= 5^4K$

$25! = 25 \times 24!$

$= 5^2 \times 24!$

$= 5^2 \times 5^4K$

$= 5^6K \Rightarrow$ 6 trailing zeros

Hence, one cannot get 5 trailing zeros in any factorial.

This is the skipping zero concepts.

Again, $45! - 49! \to 10$ trailing zeros $= 5^{10}K$

$\left\{\because \frac{45}{5} = 9, \frac{9}{5} = 1\right\}$

$50! = 50 \times 49!$

$= 2 \times 5^2 \times 49!$

$= 2 \times 5^2 \times 5^{10}K = 5^{12}K_1$

$50! \to 12$ trailing zeros

$50! \to 54! \to 12$ trailing zeros.

So, there is no factorial which has 11 trailing zeros.

- Let's generalize this concept:

  If $N! \to P$ Trailing zeros

  then, $5^Q \times N! \to (P + Q)$ Trailing zeros

Example 32:

If m and n are two consecutive natural numbers, the difference between the number of trailing zeros of $n!$ and $m!$ is 3. Find the minimum value of $(m + n)$.

Solution: (249)

$m! \to P$ trailing zeros

$n! \to P + 3$ trailing zeros

It is possible when m! is multiplied by 5^3K, where K is not the multiple of 5.

So, $n! = 5^3K \times m!$

$n! = 125K \times m!$

To have minimum value of $m + n$, take $K = 1$

$n! = 125 \times m!$

Because, m and n are consecutive numbers $n = m + 1$

$\Rightarrow (m + 1)! = 125 \times m!$

$\Rightarrow (m + 1)\,m! = 125 \times m!$

$\Rightarrow m + 1 = 125$

$m = 124$

$n = 125$

Minimum of $(m + n) = 249$

Keynote

- Numbers which are equal to the sum of factorials of their digits are 1, 2, 145, and 40,585.
- Numbers for which $N!$ has N digits are 1, 22, 23, and 24.

Finding the Rightmost Non-Zero Digit in a Factorial

In any factorial, some 2s and 5s will produce zero, separate the pair of 2s and 5s that will produce trailing zeros and then find the unit digit of multiplication of the rest of the prime numbers. This would be the rightmost non-zero digit of the factorial.

For example:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= (2 \times 3) \times 5 \times (2 \times 2) \times 3 \times 2$$

$$= \underbrace{(2 \times 5)}_{\Downarrow} \times \underbrace{(3 \times 2 \times 2 \times 3 \times 2)}_{\Downarrow}$$

(This will produce Trailing zeros) (Unit digit of this would be right most non-zero digit)

$6! \Rightarrow$ Unit digit of $(3 \times 2 \times 2 \times 3 \times 2)$

$\Rightarrow 2 =$ Right most non-zero digit of $6!$

Let's try with some bigger number.

$15! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \times 14 \times 15$

$$= 2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13$$

$$= \underbrace{(2^8 \times 3^6 \times 7^2 \times 11 \times 13)}_{\Downarrow} \times \underbrace{(2^3 \times 5^3)}_{\Downarrow}$$

Unit digit of this would be right most non-zero digit Trailing zeros

Unit digit of $(2^8 \times 3^6 \times 7^2 \times 11 \times 13)$

$= (256 \times 729 \times 49 \times 11 \times 13)$

$= (6 \times 9 \times 9 \times 1 \times 3) = 8$

Hence, right most non-zero digit of $15!$ is 8.

Alternate method:

To find the rightmost non-zero digit of $N!$

$N = 5a + b$

$\Rightarrow$ Right most non-zero digit of $(2^a \times a! \times b!)$

Example 33:

Find the rightmost non-zero digit in $28!$.

Solution: 4

$28 = 5 \times 5 + 3$

Right most non-zero digit in $28!$

$=$ Right most non-zero digit of $(2^5 \times 5! \times 3!)$

$= (32 \times 120 \times 6) = (2 \times 2 \times 6) = 4$

- For any large value of N, to reduce the steps of the calculation, one can also use the following method.

  $N = 25a + b$

  Rightmost non-zero digit of $N!$ = Right most non-zero digit of $(4^a \times a! \times b!)$.

Example 34:

Find the right most non-zero digit in $81!$.

Solution: 8

$81 = 25 \times 3 + 6$

$=$ Right most non-zero digit of $(4^3 \times 3! \times 6!)$

$= (4 \times 6 \times 2) = 8$

Practice Exercise – 1

Level of Difficulty – 1

1. The highest power of 7 that completely divides 2200! is _______ .

2. A six-digit number of the form 'PQRPQR' will always be divisible by:
(A) 77
(B) 91
(C) 143
(D) All of the above

3. *A*, *B,* and *C* are defined as follows:

$A = (1.000002) \div [(1.000002)^2 + (4.000008)]$

$B = (2.000004) \div [(2.000004)^2 + (8.000016)]$

$C = (3.000003) \div [(3.000003)^2 + (9.000009)]$

Which of the following is true about the value of the above three expressions?
(A) *A* is the smallest
(B) *A* is twice of *C*
(C) *C* is the smallest
(D) *B* is the smallest

4. Find the product of all the factors of 5^{32}.
(A) 5^{1056}
(B) 5^{528}
(C) 5^{264}
(D) 5^{320}

5. *x*, *y*, and *z* are positive integers such that $x + y + z = 2021$.

Let $s = (-1)^x + (-1)^y + (-1)^z$. Find the number of possible values of *s*:
(A) 2
(B) 4
(C) 1
(D) 3

Level of Difficulty – 2

6. What is the number of ordered pairs of positive integers *a* and *b* that satisfy the condition ab = a + b + 16?
(A) 2
(B) 3
(C) 4
(D) 1

7. The product of four consecutive prime numbers is 4391633. Find the average of the first and the fourth prime number.

8. Highest power of 5 in 111 × 112 × 113 × ....998 × 999 is:
(A) 246
(B) 236
(C) 220
(D) Cannot be determined

9. How many natural numbers are there between 1 and 3149, which when divided by 3, 5, 7, and 9 gives remainders as 2, 4, 6, and 8, respectively?
(A) 9
(B) 10
(C) 11
(D) 12

10. *N*! is completely divisible by 11^{55}. What is the sum of the digits of the smallest such as number *N*?
(A) 10
(B) 11
(C) 12
(D) 13

Level of Difficulty – 3

11. If the LCM of two positive integers (N_1, N_2) where $N_1 < N_2$ is 30 times the HCF, which of the following is not a possible value of $\frac{N_2}{N_1}$?
(A) $7.\overline{7}$
(B) $3.\overline{3}$
(C) 1.2
(D) 7.5

12. How many values can natural number N take if $N!$ is a multiple of 2^{26} but not 3^{26}?

(A) 24
(B) 23
(C) 22
(D) 18

13. Find the number of pairs (x, y) which satisfy HCF (x, y) + LCM $(x, y) = 85$.

(A) 4
(B) 5
(C) 6
(D) 8

14. N is a 10-digit number and multiple of 24. If its digits are distinct, find the second largest value of N.

15. Among the first 100 natural numbers, how many numbers can be expressed as a difference between two perfect squares in at least one way?

(A) 25
(B) 40
(C) 50
(D) 75

Solutions

1. 364

The highest power of 7 completely divides 2200!

$= \frac{2200}{7} + \frac{2200}{7^2} + \frac{2200}{7^3}$

$= \frac{2200}{7} + \frac{2200}{49} + \frac{2200}{343}$

= 314 + 44 + 6

= 364

2. (D)

PQRPQR = PQR × 1000 + PQR
= PQR × (1000 + 1)
= PQR (1001)
= (PQR × 7 × 11 × 13)

So, PQRPQR is divisible by all of (77 = 11 × 7), (91 = 13 × 7) and (143 = 11 × 13).
Hence, option (D) is the correct answer.

3. (D)

Assume K = 1.00002

$\Rightarrow A = \frac{K}{K^2 + 4K} = \frac{1}{K+4} = \frac{1}{5.000002}$

Assume 2.00004 = L

$\Rightarrow B = \frac{L}{L^2 + 4L} = \frac{1}{L+4} = \frac{1}{6.000004}$

Similarly, $C = \frac{1}{6.000003}$

$\Rightarrow$ B is the smallest.

Hence, option (D) is the correct answer.

4. (B)

The factors of 5^{32} are $5^0, 5^1, 5^2, 5^3$ .... 5^{32}

$\therefore$ Product of all the factors

$= (5^0) \times (5^1) \times (5^2)$ .... $\times (5^{32})$

$= 5^{(0+1+2+3+\dots\dots\dots 32)}$

$= 5^{\left(\frac{32\times 33}{2}\right)} = 5^{16\times 33} = 5^{528}$

Hence, option (B) is the correct answer.

5. (A)

Since the given equation is $x + y + z = 2021$.
There are two possibilities to get $x + y + z$ = odd.
First one, if we take all the three numbers (x, y, z), all are odd in nature.

Or
Two numbers will be even, and one number will be odd.
Therefore, the possible values of s if all the three numbers will be odd.
Then ‘s’ = $(-1)^{odd} + (-1)^{odd} + (-1)^{odd}$
Then = –1 – 1 –1 = –3
$'s' = -3$

Again, if two numbers are odd and one number is even then:
‘s’ $= (-1)^{odd} + (-1)^{odd} + (-1)^{even}$
= –1 – 1 + 1
$'s' = -1$

Hence only two values are possible for ‘s’.
Option (A) is the correct answer.

6. (A)

Since it is given in the question that ‘a’ and ‘b’ are positive integers.

$\therefore$ ab = a + b + 16
$\Rightarrow$ ab – a – b = 16
$\Rightarrow$ a(b – 1) – b + 1 = 16 + 1
$\Rightarrow$ a(b – 1) –1(b – 1) = 17
$\Rightarrow$ (a – 1) (b – 1) = 17

Here the product of two numbers is prime (17), which is possible only when one of the numbers is 1, and the other is 17.
So, two cases will be possible when a = 18 and b = 2 and a = 2 and b = 18
(a, b) = (2, 18) and (18, 2)
Hence only two ordered pairs of positive integers are possible that satisfy the given condition.
Hence, option (A) is the correct answer.

7. 47

Since prime numbers are consecutive, then for simplicity, we may take them equally.
Then, a^4 = 4391633
For approximation, we may take
4391633 ≈ 4000000
Then, a^2 ≈ 2000
Now, we know 40^2 = 1600 and 50^2 = 2500
Then, a ≈ 45

Now, check prime numbers near 45.
We have 37, 41, 43, 47, 53, 59
with the help of unit digits, we can easily conclude that prime numbers are: 41, 43, 47, 53.

Now, the required average $= \frac{41+53}{2} = 47$

Hence, 47 is the correct answer.

8. (C)

111 × 112 × 113 × ....998 × 999

$$= \frac{(1\times 2\times 3\times 4....110)\times(111\times 112\times 113\times 998\times 999)}{(1\times 2\times 3\times 4.........110)}$$

$$\frac{(1\times 2\times 3\times 4....110)\times(111\times 112\times 113\times 998\times 999)}{(1\times 2\times 3\times 4.........110)} = \frac{999!}{110!}$$

So, the highest power of 5 in 111 × 112 × 113 ....998 × 999 is the same as the highest power of 5 in $\frac{999!}{110!}$.

The highest power of 5 in 999!

$$= \left[\frac{999}{5}\right] + \left[\frac{999}{5^2}\right] + \left[\frac{999}{5^3}\right] + \left[\frac{999}{5^4}\right]$$

= 199 + 39 + 7 + 1 = 246

Where [x], represent greatest integer of x.

Similarly, the highest power of 5 in 110!

$$= \left[\frac{110}{5}\right] + \left[\frac{110}{5^2}\right] = 22 + 4 = 26$$

Hence, the highest power of 5 in $\frac{999!}{110!}$

$$= \frac{5^{246}}{5^{26}} = 5^{220}$$

Hence, option (C) is the correct answer.

9. (A)

Here, the difference between divisors and the remainders is fixed, and it is 1, so the number would be of the form LCM [(3, 5, 7, and 9) K] – 1 = 315K – 1, where K is a natural number.
Now for K = 1, the number would be 314
For K = 2, the number would be 629 and so on till K = 10, for which the number would be 3149.
Now we need to take numbers between 1 and 3149, so we can't take 3149.

So, for K = 9, the number = 315 × 9 – 1 = 2,835 – 1 = 2,834 would be the last number.

Hence, option (A) is the correct answer.

10. (C)

The number needs to be less than 11 × 55 = 605. The highest power of 11 in 605! = 60. The power of 11 is the smallest such number needs to be exactly 55. If we subtract 11x 5 = 55 from 605, we will get 550.
The highest power of 11 in 550! = 54. So, need to increase 550 by another 11, that is 561.

In 561! the highest power of 11 will be exactly 55 as per our requirement.

Hence, the minimum value of N is 561. Sum of the digits of the smallest value of N = 5 + 6 + 1 = 12.

Hence, option (A) is the correct answer.

11. (A)

Since LCM = 30 × HCF (Given)
Let HCF = m

$\therefore$ $N_1 = ma$ and $N_2 = mb$

Also, we know that:

$N_1 \times N_2$ = HCF × LCM

$ma \times mb = m \times 30 \times m$

$ab = 30$

Also, $a < b$ (Given in the question)

Therefore, $a \times b = 30$

$$\therefore \underbrace{(1, 30),\ (2, 15),\ (3, 10),\ (5, 6)}_{\text{All the 4 pairs are co-prime in nature.}}$$

For (a, b) = (1, 30) $\Rightarrow \frac{N_2}{N_1} = \frac{mb}{ma} = \frac{30}{1} = 30$

For (a, b) = (2, 15) $\Rightarrow \frac{N_2}{N_1} = \frac{15}{2} = 7.5$

For (a, b) = (3, 10) $\Rightarrow \frac{N_2}{N_1} = \frac{10}{3} = 3.\overline{3}$

For (a, b) = (5, 6) $\Rightarrow \frac{N_2}{N_1} = \frac{6}{5} = 1.2$

$\therefore$ $7.\overline{7}$, option (A) is not a possible value of $\frac{N_2}{N_1}$.

Hence, option (A) is the correct answer.

12. (A)

We first need to find, by trial and error method, the minimum value of N! which divides 2^{26}.

If $N = 10$ 10! is divisible by 2^8

$N = 20$ 20! is divisible by 2^{18}

$N = 25$ 25! is divisible by 2^{22}

$N = 30$ 30! is divisible by 2^{26}

Now $30! = 2^{26} \times 3^{14} \times$ ....

So 30! is the minimum number that divides 2^{26} but not 3^{26}.

So, 30! 31! 32! .... up to 53!.

There will be 24 possible numbers that divide 2^{26} but not 3^{26}.

As $53! = 2^{49} \times 3^{23} \times$ ......

whereas $54! = 2^{50} \times 3^{26} \times$ ......, which is divisible by both 2^{26} and $3^{26,}$ so not required.

Hence, option (A) is the correct answer.

13. (C)

Let $x = Kp$, $y = Kq$ where a and b are Co-prime.

HCF $(x, y) = K$

LCM $= (x, y) = Kpq$

According to question:

$K + Kpq = 85$

$\Rightarrow$ $K(1 + pq) = 85$

Possible option are 1×85 or 5×17

Case 1: When HCF = 1

$\Rightarrow$ $1\,(1 + pq) = 85$

$\Rightarrow$ $pq = 84$

There can be many combinations such that $pq = 84$. But we need to select where p and q are co-prime.

So, when HCF = 1, (p, q) can be (1, 84); (4, 21); (12, 7), and (28, 3).

Case 2: When HCF = 5

$\Rightarrow$ $5(1 + pq) = 85$

$\Rightarrow$ $1 + pq = 17$

$\Rightarrow$ $pq = 16$

Co-prime p and q can take values: (1, 16)

Case 3: When HCF = 17

$\Rightarrow$ $17\,(1 + pq) = 85$

$\Rightarrow$ $1 + pq = 5$

$\Rightarrow$ $pq = 4$

Possible values of p and q : (1, 4)

The total number of possible values of $(x, y) = 6$.

Hence, option (C) is the correct answer.

14. 9,87,65,41,320

As we know

A number divisible by 24 would be divisible by both 3 and 8.

Divisibility rule of 3: The sum of digits must be divisible by 3

Divisibility rule of 8: Last three digits must be divisible by 8

The largest number with all distinct digits would be 9,87,65,43,210, where the sum of its digits would be equal to 45 (always divisible by 3), so we must take care of the last three digits.

9,87,65,43,210

The last three digits can be arranged as 012, 021, 102, 120, 201, and 210. Only 120 is divisible by 8. So, the largest number would be 9,87,65,43,120.

For the second-largest number, we would play with 0, 1, and 3 (we should choose the lowest possible three digits to keep the Number N largest possible). The possibilities are 013, 031, 103, 130, 301, 310. None are divisible by 8.

Let's play with 0, 2, and 3 now. The possibilities are 023, 032, 203, 230, 302, and 320. Only 320 is divisible by 8. So, the second-largest number would be 9,87,65,41,320.

15. (D)

As we have to take the difference between two perfect squares, we can have three possibilities.

1. Both the perfect squares are of odd numbers.
2. Both the perfect squares are of even numbers.
3. One perfect square is of an odd number, and the other is of an even number.

Case 1: $(\text{odd number}_1)^2 - (\text{odd number}_2)^2$

(odd $number_1$ + odd $number_2$) (odd $number_1$ − odd $number_2$)
even number × even number
= multiple of 4, i.e., 4k ...(i)

Case 2: $(\text{even number}_1)^2 - (\text{even number}_2)^2$
(even $number_1$ + even $number_2$) (even $number_1$ − even $number_2$)
= even number × even number
= multiple of 4. i.e., 4l ...(ii)

Case 3: $(\text{odd number})^2 - (\text{even number})^2$
= (odd number − even number) (odd number + even number)
= Odd number × odd number
= odd number ...(iii)

From equations (i), (ii), and (iii), we can conclude that in the set of the first 100 natural numbers, all the even numbers which are not the multiple of 4 cannot be expressed as a difference between two perfect square numbers.
Or in other words:

In every set of four consecutive numbers, starting from 1, there will be three numbers which can be expressed as the difference between two perfect squares, or there will be only one number which cannot be expressed as the difference between two perfect square numbers.
Therefore, in the set of the first 100 natural numbers, numbers which cannot be expressed as the difference between two perfect squares $= \frac{1}{4} \times 100 = 25$.

Therefore, a number which can be expressed as the difference of two perfect squares = 100 − 25 = 75.
Hence, option (D) is correct.

Practice Exercise – 2

Level of Difficulty – 1

1. For how many prime numbers 'P' will '$37^P + 7$'be a multiple of P?
(A) 2
(B) 4
(C) 6
(D) 1

2. For $N = 2^{12} \times 3^8 \times 5^{10} \times 7^4$. Find the factors of N that are multiple of 360 but not of 420.
(A) 1,400
(B) 900
(C) 700
(D) 1,200

3. $\frac{3200!}{P^P}$ is completely divisible by P, where P is a prime number. For how many values of P is this possible?
(A) 16
(B) 8
(C) 15
(D) 11

4. $N = 2^{35} \times 3^{41}$. How many factors of N^2 are less than N but does not divide N?
(A) 2,936
(B) 1,435
(C) 1,570
(D) 1,511

5. Find the highest power of 96 in 72!.
(A) 34
(B) 17
(C) 13
(D) None of these

6. If $N = 3x^3 + 6x^2 + 5x + 36$, then for how many values of x, N is completely divisible by x?
(A) 6
(B) 8
(C) 9
(D) 12

7. If $15 < x < 2650$, then for how many values of x, $\frac{x}{5}$ is the square of a prime number?
(A) 10
(B) 9
(C) 7
(D) 6

8. If $N = 0.a1a2a3a2a3a2a3$ ....Find which of the following must be multiplied with N such that multiplication give an integral value.
(A) 900
(B) 1,287
(C) 1,800
(D) 2,970

9. How many natural numbers from 1 to 50 are there which have at least 2 and maximum 3 factors?
(A) 4
(B) 12
(C) 15
(D) 19

10. What can be the maximum value of HCF (x, y) where $x + y = 990$? x and y are distinct integers?
(A) 495
(B) 330
(C) 165
(D) 110

Level of Difficulty – 2

11. What is the sum of the even factors of 144 and the odd factors of 2,400?

12. Ravi was asked to add the numbers from 1 to n and report the Sum to his teacher. He gave the answer as 1020. The teacher noted that Ravi missed adding a few numbers between 1 and n, excluding 1 and n. If $n = 45$, what was the maximum number of numbers Ravi could have missed?

(A) 3
(B) 4
(C) 8
(D) 5

13. The sum of the squares of the two integers, X and Y, is less than the square of the sum of these two integers by 200. How many pairs of integers, X and Y, satisfy this condition?
(A) 6
(B) 8
(C) 10
(D) 12

14. If $x = 30^{12}$, how many factors of x will have at least one '0' at the end?

15. For how many natural numbers N below 150, $(n - 1)!$ will not be divisible by n?
(A) 45
(B) 35
(C) 40
(D) 38

16. $N = 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + 17 \times 17!$, then find the remainder of $19 \left.\dfrac{(N+7)}{19!}\right|_R$
(A) 1
(B) 7
(C) 19
(D) 114

17. Given that A, B, C, D... X, Y, and Z are 26 consecutive natural numbers such that

$(A + B + C + D + \cdots + X + Y + Z)^2 = A^3 + B^3 + C^3 + D^3 + \cdots X^3 + Y^3 + Z^3$. Find the number of zeros at the end of product $(A \times B \times C \times D \times \cdots \times Y \times Z)$.

18. There is a number X^2 that has 21 factors. Find the number of factors x can have.
(A) 12 or 8
(B) 11 or 8
(C) 10 or 6
(D) 12 or 6

19. The sum of two natural numbers and their LCM is 89. How many such pairs of numbers are there?
(A) 6
(B) 5
(C) 4
(D) 3

20. Find the number of factors of $2^6 \times 5^9 \times 12^3$ that are perfect squares.

Level of Difficulty – 3

21. A natural number N has k distinct prime factors. If the total number of factors of N is 72, then what is the product of all the possible values of k?

22. A positive integer is equal to the square of its number of factors. How many such integers are there?
(A) 3
(B) 4
(C) 2
(D) 5

23. N is a natural number which has four factors. If $10 \le N \le 65$, then how many values are possible for N?
(A) 12
(B) 15
(C) 13
(D) 19

24. How many values of A exist such that all among A, B, and $(B - A)$ are prime and both A and B are less than 100?
(A) 9
(B) 8
(C) 7
(D) 5

25. The product of all the factors of a positive integer P is equal to P^2. The Sum of all the factors of P excluding P is 20. P lies between ________.
(A) 0 and 20
(B) 20 and 40
(C) 40 and 60
(D) 60 and 80

26. What is the smallest natural number which when divided by 7 leaves a remainder of 5, when divided by 9 leaves a remainder of 6, and when divided by 5 leaves a remainder of 1?

27. What is the largest integer that always divide $x^5 - 5x^3 + 4x$; x is an integer?

(A) 240

(B) 120

(C) 480

(D) 160

28. If N = 10,800 then find the sum of thrice number of odd factors, twice number of even factors and thrice number of factors which are perfect square.

(A) 168

(B) 192

(C) 172

(D) 180

29. Find all the integers x for which $x^2 + 16x + 92$ is a perfect square.

(A) –2

(B) –14

(C) 8

(D) Both (A) and (B)

30. What is the highest power of 12 that will divide the product of the first 50 multiples of 12?

(A) 22

(B) 68

(C) 50

(D) 72

Solutions

1. (A)

$37^P + 7$ can be converted in $a^n - a$ form, i.e., $37^P - 37 + 44$

Here, $37^P - 37$ is always divisible by P

$\therefore$ For the whole term to be divisible by P.

The given P must be a factor of 44.

$44 = 4 \times 11$

or $2 \times 2 \times 11$,

Since P is a prime number.

$\therefore$ P can be 2 or 11.

Hence, P can take only two values.

Note: $a^n - a$ is always divisible by n if a is a prime number.

Hence, option (A) is the correct answer.

2. (C)

Factors of form 360 N, where N is a natural number.

$360\ N = 2^3 \times 3^2 \times 5^1\ [2^9 \times 3^6 \times 5^9 \times 7^4]$

Number of factors of $N = (9 + 1) \times (6 + 1) \times (9 + 1) \times (4 + 1)$
$= 10 \times 7 \times 10 \times 5 = 3{,}500$

Factors of form $420\ M = 2^2 \times 3 \times 5 \times 7\ [2^{10} \times 3^7 \times 5^9 \times 7^3]$

Common factors $= [2^9 \times 3^6 \times 5^9 \times 7^3]$

Number of common factors $= 10 \times 7 \times 10 \times 4 = 2{,}800$

Required number of factors $= 3{,}500 - 2{,}800 = 700$.

Hence, option (C) is the correct answer.

3. (A)

$$\frac{3200!}{P^P}$$

Here $P \le \sqrt{N}$, $P \le \sqrt{3{,}200}$, $\mathrm{P} \le 56$, $\mathrm{P} \approx 56$

Since P is a prime number. There are 16 prime numbers from 1 to 56.

$\therefore$ 16 is the required answer.

Hence, option (A) is the correct answer.

4. (B)

$N = 2^{35} \times 3^{41}$

Number of factors of $N = (35 + 1)(41 + 1) = 36 \times 42 = 1512$

$N^2 = 2^{70} \times 3^{82}$

Number of factors of $N^2 = (70 + 1)(82 + 1) = 71 \times 83 = 5{,}893$

For N^2 $\Rightarrow$ $\xleftarrow{2946} N \xrightarrow{2946}$

For N $\Rightarrow$ $\xleftarrow{1511} N$

Factors of N^2 that do not divide $N = 2{,}946 - 1{,}511 = 1{,}435$.

Hence, option (B) is the correct answer.

5. (D)

$72 = 8 \times 9 = 2^3 \times 3^2$

Highest power of 2 in $72! = 36 + 18 + 9 + 4 + 2 + 1 = 70$.

Highest power of 3 in $72! = 24 + 8 + 2 = 34$

So, $72! = 2^{70} \times 3^{34} \times 5^x \times 7^y \times \ldots$

$96^1 = 2^5 \times 3^1$

$96^K = 2^{5K} \times 3^K$

If we put $K = 34$, 3^{34} will be fully utilised, but we do not have 2^{170}. So, K can't be 34.

If we put $K = 14$, 2^{70} will be fully utilised, and 3^{14} will be utilised

$96! = (2^{70} \times 3^{14}) \times 3^{20} \times 5^x \times 7^y \times \ldots$

$96! = 96^{14} \times 3^{20} \times 5^x \times 7^y \times \ldots$

Hence, the highest power of 96 in $72! = 14$.

Hence, option (D) is the correct answer.

6. (C)

We have

$N = 3x^3 + 6x^2 + 5x + 36 \Rightarrow$

$$\frac{N}{x} = \frac{3x^3 + 6x^2 + 5x + 36}{x}$$

$$= 3x^2 + 6x + 5 + \frac{36}{x}$$

Now, for N to be completely divisible by x, x must be the factor of 36.

We can express '36' as:

$36 = 2^2 \times 3^2$

Now, total factors $= (2 + 1) \times (2 + 1) = 9$

Hence, option (C) is the correct answer.

7. (B)

$15 < x < 2{,}650$

$$\Rightarrow \quad 3 < \frac{x}{5} < 530$$

Since $\frac{x}{5}$ is the square of a prime number, the values which $\frac{x}{5}$ can take are 4, 9, 25, 49, 121, 169, 289, 361, and 529.

Thus for 9 values of x, $\frac{x}{5}$ will be the square of a prime number.
Hence, option (B) is the correct answer.

8. (D)

Here, $N = 0.\overline{a_1a_2a_3}$

There, we can express 'N' as:

$$N = \left(\frac{a_1 a_2 a_3 - a_1}{990}\right)$$

Now, from the given option, only option (e) is a multiple of 990.
Hence, multiplying 2,970 with given 'N' will get an integral value.

Hence, option (D) is correct.

9. (D)

As we know from the definition of prime numbers, that prime numbers always have exactly two factors.
Also, numbers in the form of $N = p^2$ (where p is prime)
i.e., All the squares of prime numbers will have exactly three factors.
Therefore, from 1 to 50, there are 15 prime numbers:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.
Also, from 1 to 50 there are 4 squares of prime numbers.
$2^2, 3^2, 5^2, 7^2$
Therefore, required answer = 15 + 4 = 19
Hence, option (D) is the correct answer.

10. (B)

$x + y = 990$
If x and can be equal then $x = 495$, $y = 495$ with HCF = 495 would have been the best possible solution. But x and y are given to be distinct positive integers.
Therefore, let us divide 990 into three parts for next best possible solution with maximum HCF
$\Rightarrow x = 330$, $y = 660$ is the best solution with HCF = 330.
Hence, option (B) is the correct answer.

11. 514

$144 = 2^4 \times 3^2$
$2{,}400 = 2^5 \times 3^1 \times 5^2$

To calculate the sum of even factors of 144 first we will calculate the sum of odd factors and then we will subtract it from the total sum of the factors.

$$\text{Sum of factors of } 144 = \left(\frac{2^5-1}{1}\right)\left(\frac{3^3-1}{2}\right)$$
$$= 31 \times 13 = 403$$

For odd factors, we will remove the powers which will make a factor even, thus will consider a sum of $2^0 \times 3^2$, and thus

$$\text{Sum of odd factors will be} \left(\frac{2^1-1}{1}\right)\left(\frac{3^3-1}{2}\right) = 13$$

Thus, the sum of even factors will be $403 - 13 = 390$.
For 2,400, we will consider $2^0 \times 3 \times 5^2$, thus the sum of odd factors will be

$$\left(\frac{2^1-1}{1}\right)\left(\frac{3^2-1}{2}\right)\left(\frac{5^3-1}{4}\right) = 124$$

Thus, required sum = 390 + 124 = 514.

12. (B)

Ravi missed numbers between 1 and n.
Thus, 1 and n were necessarily included.
Sum = 1 + 2 + 3 + ⋯ + 45 = 1,035
Ravi's sum = 1,020
Thus, 15 more is required. Since '1' wasn't missed, there could be maximum of four numbers 2, 3, 4, 6.
Hence, option (B) is the correct answer.

13. (C)

It has been given that $(x + y)^2 = x^2 + y^2 + 200$.
$$x^2 + y^2 + 2xy = x^2 + y^2 + 200$$
$$2xy = 200$$
$$xy = 100$$
The product of two numbers is 100.
100 can be written as $2^2 \times 5^2$.
Therefore, 100 will have $3 \times 3 = 9$ natural number factors.
But the pairs will be 5 as there will be four other sets of numbers other than 10×10, which will satisfy the condition.
We have to entertain the possibility that both x and y can be negative.
Therefore, $5 \times 2 = 10$ pairs of integers will satisfy the given condition.
Hence, option (C) is the right answer.

14. 1,872

Here, $x = 2^{12} \times 3^{12} \times 5^{12}$
In order to have a 0 at the end, the power of 2 and 5 should be at least 1 while the power of 3 can be anything.
So, the number of possibilities is $12 \times 12 \times 13 = 1{,}872$.

15. (B)

For the statement given in the question to be true, n must be a prime number

$(2 - 1)!$ is not divisible by 2
$(3 - 1)!$ is not divisible by 3
$(7 - 1)!$ is not divisible by 7
$(31 - 1)!$ is not divisible by 31

We see all the prime numbers satisfy this condition.
$\therefore$ Total numbers 'N' till 150 satisfying this condition will equal the number of primes till 150.
There are a total of 35 primes from 1 to 150 that are 2, 3, 5, 7, ... , 149.
$\therefore$ A total of 35 numbers satisfy the condition or $N = 35$.
Hence, option (B) is the correct answer.

16. (D)

$N = 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + 17 \times 17!$
$= (2 - 1)\,1! + (3 - 1)\,2! + (4 - 1)\,3! + \ldots + (18 - 1) \times 17!$
$= 2! - 1! + 3! - 2! + 4! - 3! + \ldots + 18! - 17! = 18! - 1$
$N = 18! - 1$
$19(N + 7) = 19! + 114$

$$19\left.\frac{(N+7)}{19!}\right|_R = \left.\frac{19! + 114}{19!}\right|_R = 114$$

$\therefore$ Required remainder = 114
Hence, option (D) is the correct answer.

17. 6

$(A + B + C + D + \ldots + X + Y + Z)^2 = A^3 + B^3 + C^3 + D^3 + \ldots + X^3 + Y^3 + Z^3$
This is true only when $A = 1$, so according $B = 2$, $C = 3$, and so on till $Z = 26$
which means
$(1 + 2 + 3 + 4 + \ldots + 24 + 25 + 26)^2 = \{(25 \times 26)/2\}^2$
$1^3 + 2^3 + 3^3 + 4^3 + \ldots + 24^3 + 25^3 + 26^3 = \{(25 \times 26)/2\}^2$
So, $(1 + 2 + 3 + 4 + \ldots + 24 + 25 + 26)^2 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + 24^3 + 25^3 + 26^3$
Now $(A \times B \times C \times D \times \ldots \times Y \times Z) = 1 \times 2 \times 3 \times 4$ ---------- $\times 25 \times 26 = 26!$
So, we need to find the number of zeros at the end of 26! which will be equal to the highest power of 5 in 26! = [26/5] + [26/25] = 5 + 1 = 6.

18. (B)

X^2 have 21 factors.
We know that any number can be expressed as the product of prime factors $= p^a q^b r^c$.
Number of factors of that number $= (a + b)(b + 1)(c + 1)$.
Now, $21 = 1 \times 21$ or 3×7 (there is no other case)
If $X^2 = p^{20}$ or $p^2 q^6$ then $X = p^{10}$ or $p^1 q^3$
Therefore, number of factors of X will be $(10 + 1)$ or $(2) \times (4) = 11$ or 8.
Hence, option (B) is the correct answer.

19. (B)

Let two numbers be $H \times A$ and $H \times B$, where H is the HCF of two numbers and A, B Relatively Co-prime.
According to the question:

$H \times A + H \times B + \text{LCM}\,(H \times A \text{ and } H \times B) = 89$
$H \times A + H \times B + H \times A \times B = 89$
$H\,(A + B + AB) = 89$
89 is a prime number the only possibility is
$H = 1$ and $A + B + AB = 89$
$A + B + AB = 89$
$A + B\,(1 + A) = 89$
$1 + A + B\,(1 + A) = 1 + 89 = 90$
$(1 + A)(1 + B) = 90$
$\Rightarrow (A + 1)(B + 1) = 90$

$(A + 1)(B + 1) = 90$	A	B
1×90	0	89
2×45	1	44
3×30	2	29
5×18	4	17
6×15	5	14
9×10	8	9

Except
(0, 89), all other five pairs are Co-prime.
Hence, 5 such pairs are possible.
Hence, option (B) is the correct answer.

20. 70

$2^6 \times 5^9 \times 12^3 = 2^6 \times 5^9 \times (2 \times 2 \times 3)^3 = 2^6 \times 5^9 \times 2^6 \times 3^3 = 2^{12} \times 3^3 \times 5^9$

Now, for a number to be a perfect square, the power of its prime factor should be even.

For powers of 2, only 0, 2, 4, 6, 8, 10, and 12 will give a perfect square.

For powers of 3, only 0, 2 will give a perfect square.

For powers of 5, only 0, 2, 4, 6, and 8 will give a perfect square.

For perfect square factors, we need to select any one number out of 2^0, 2^2, 2^4, 2^6, 2^8, 2^{10}, and 2^{12}, any one number from 3^0, 3^2, and any one number from 5^0, 5^2, 5^4, 5^6, and 5^8.

This can be done in $7 \times 2 \times 5 = 70$ ways

Hence, the total number of square factors = 70.

21. 120

Let $N = a^p \times b^q \times c^r \times \ldots$, where a, b, c,... are prime numbers and p, q, r, ... are natural numbers.

$\therefore \quad (p + 1) \times (q + 1) \times (r + 1) \times \ldots = 72 = 2 \times 2 \times 3 \times 3$

The number of prime factors of N can be 1, 2, 3, 4, or 5.

For example, if the number of prime factors of N is 1, then N will be of the form a^{71}.

Similarly, the number of prime factors of N can be 2, 3, 4, or 5.

Hence, the product = $1 \times 2 \times 3 \times 4 \times 5 = 120$.

22. (C)

One such number is 1, which has no factor other than itself.

If the number has only one prime factor, i.e., it is of the form p^a where p is a prime number, and a s a natural number, then according to the question:

$(a + 1)^2 = p^a$.

This is possible only if $a = 2$ and $p = 3$. So the number is 9.

If the number has two prime factors, it would be of the type $p^a \times q^b$, where p and q are two distinct prime numbers. Then according to the question:

$(a + 1)^2 (b + 1)^2 = p^a \times q^b$

This is possible only if p and q are both 3. Since they are different, this is not a valid case. So, there would be no such case with two or more prime factors.

So, there are only two such integers, 1 and 9.

Hence, option (C) is the correct answer.

23. (D)

Since N has 4 factors, N must be of the form P_1^3 or $P_2 \times P_3$ where P_1, P_2, P_3 are prime numbers.

If N takes the form P_1^3 then P_1 can only be 3.

If N takes the form $P_2 \times P_3$ then the possibilities are:

Nine possible values where 2 is the smaller factor:

2×5, 2×7, 2×11, 2×13, 2×17, 2×19, 2×23, 2×29, 2×31.

Six possible values where 3 is the smaller factor:

3×5, 3×7, 3×1, 3×13, 3×17, 3×19

Three possible values where 5 is the smaller factor.

5×7, 5×11, 5×13.

So, N can take $1 + 9 + 6 + 3 = 19$ values in all.

Hence, option (D) is the correct answer.

24. (A)

All prime numbers except 2 are odd. Let us assume that both A and B are odd.

$\Rightarrow$ $B - A$ must be even. Since only even prime number is 2, $B - A$ must be 2.

For, such cases, the possible values of B and A are

(5, 3), (7, 5), (13, 11), (19, 17), (29, 31), (43, 41), (61, 59) (73, 71).

Other than this, A can also take a value of 2. In that case, B can take values (5, 7, 13, 19, 31, 41, 63, 73). In all these cases, $B - A$ will also be prime.

So, the values that A can assume are 2, 3, 5, 11, 17, 29, 41, 59, and 71.

Thus, a total of 9 values.

Hence, option (A) is the correct answer.

25. (B)

The product of all the factors of a positive integer P is $P^{\{\phi(P)/2\}}$, where $^{\phi}(P)$ is the number of factors of P.

Given that, product of all the factors of P is P^2.

$\therefore \quad \frac{\phi(P)}{2} = 2$

$\Rightarrow \quad ^{\phi}(P) = 4$, i.e., P has four factors.

P is either the cube of a prime number or the product of two distinct primes.

Suppose P is the cube of the prime number p.

Sum of all the factors of P, excluding P $= 1 + p + p^2 = 20$.

Since $p + p^2$ is always even, $1 + p + p^2$ is always odd.

It cannot be 20.

So, P is the product of two primes.

Let these be p_1 and p_2.

Let $p_1 < p_2$.

The factors of $P$are 1, p_1, p_2, p_1p_2.

$1 + p_1 + p_1 = 20 \quad \Rightarrow \quad p_1 + p_1 = 19$

$p_1 + p_2$ is odd.

$\therefore$ One of p_1 and p_2 is an odd prime and the other must be 2

($\because$ 2 is the only even prime)

$p_1 < p_2$.

$\therefore \quad p_1 = 2$ and $p_1 = 17$.

$P = p_1p_2 = 34$.

Hence, option (B) is the correct answer.

26. 96

This is a problem of the type LCM model 3.

7	9	5	—	divisors
5	6	1	—	remainders

Let us first take

7	9	—	divisors
5	6	—	the remainder

The number is of the form $9k + 6$

$7)\ \frac{9k + 6}{5}($

So, $\frac{9k + 6 - 5}{7}$ leaves no remainder.

$\frac{9k + 1}{7}$ Leaves no remainder.

$k = 3$ (by trial and error)

So, the smallest number is $9k + 6 = 9 \times 3 + 6 = 33$

General form is $63k + 33$.

This, when divided by 5, leaves a remainder of 1.

$33 + 63K_1 = 5k_2 + 1$

$63k_1 + 32 = 5k_2$

$k_1 = 1$ and $k_2 = 19$

So, the smallest number is $33 + 63 = 96$.

27. (B)

Here, we use the hit-and-trial method and try putting values of x as 3, 2, 1, 0, –1, –2, –3.

Putting value of $x = 0$.

$f(0) = (0)^5 - 5(0)^3 + 4(0) = 0$

Hence, we can say x is a factor of $f(x)$

x is a factor of $f(x)$

$x\,(x^4 - 5x^2 + 4)$

Putting, $x = 1$

We get, $f(1) = 0$

$\Rightarrow \quad x(x - 1)\,(x^3 + x^2 - 4x - 4)$

Putting, $x = 2$ and so on

$f(x) = (x - 2)\,(x - 1) \times (x + 1)\,(x + 2)$

Here, we see that $f(x)$ is the product of 5 continuous numbers.

$\therefore \quad f(x)$ must be a multiple of 5! or 120

Note: The product of n continuous numbers is always divisible by n!

Hence, option (B) is the correct answer.

28. (A)

$N = 10{,}800 = 2^4 \times 3^3 \times 5^2 = a^p b^q c^r$

$\Rightarrow$ Total number of factors $= (p + 1)\,(q + 1)\,(r + 1)$

$= (4 + 1)\,(3 + 1)\,(2 + 1)$

$= 5 \times 4 \times 3 = 60$

For odd factors, we only consider factors other than 2 and power of 2, i.e., $3^3 \times 5^2$

$\Rightarrow$ Number of odd factors $= (3 + 1)\,(2 + 1) = 4 \times 3 = 12$

For even factors, we consider multiple of 2, i.e., $2 \times (2^3 \times 3^3 \times 5^2)$

$\Rightarrow$ Number of even factors $= (3 + 1)\,(3 + 1)\,(2 + 1) = 4 \times 4 \times 3 = 48$

For perfect square factors, we consider even powers of prime numbers

$\Rightarrow$ Number of perfect square factors $= [2^0, 2^2, 2^4]\,[3^0, 3^2]\,[5^0, 5^2]$

$= 3 \times 2 \times 2 = 12$

Now, 3 × odd factors + 2 × even factors + 3 × perfect square factors

$= 3 \times 12 + 2 \times 48 + 3 \times 12 = 168$

Hence, option (A) is the correct answer.

29. (D)

$x^2 + 16x + 92 = k^2$

$x^2 + 16x + 64 + 28 = k^2$

$(x + 8)^2 + 28 = k^2$

$k^2 - (x + 8)^2 = 28$

$\underbrace{(k + x + 8)}_{a}\ \underbrace{(k - x - 8)}_{b} = 28$

a	b	
28	1 ×	Not possible (x must be integer)
14	2	Possible
7	4 ×	Not possible (x must be integer)
2	14	Possible

Case 1:

$k + x + 8 = 14$ and $k - x - 8 = 2$

Adding both, $2k = 16$ or $k = 8$

Now, put this in any of the equation

We get $x = -2$

Case 2:

$k + x + 8 = 2$ and $k - x - 8 = 14$

Solving both, we get $k = 8$

Putting this value of k in any of the equations, we get $x = -14$

$\therefore \quad x = -2$ or -14

Hence, option (D) is the correct answer.

30. (D)

The first 50 multiples of 12 are (12 × 1), (12 × 2), (12 × 3),............, (12 × 50)

Product of first, so multiples of 12 is

$\Rightarrow \quad 12^{50} \times (1 \times 2 \times 3 \times 4 \times \ldots \times 50)$

$\Rightarrow \quad 12^{50} \times 50!$

For checking the power of 12 in 50! We must check powers of 2 and 3.

(1) Checking power of 2.

$$\left|\frac{50!}{2}\right| = 25 \quad \left|\frac{50!}{2^4}\right| = 3 \quad \left|\frac{50!}{2^2}\right| = 12 \quad \left|\frac{50!}{2^5}\right| = 1 \quad \left|\frac{50!}{2^3}\right| = 6$$

Total power of 2 = 25 + 12 + 6 + 3 + 1 = 47

$50! = 2^{47} \times a$ (where a is the other factor)

(2) Checking power of 3

$$\left|\frac{50!}{3}\right| = 16 \qquad \left|\frac{50!}{3^2}\right| = 5 \quad \left|\frac{50!}{3^3}\right| = 1$$

Total power of 3 = 16 + 5 + 1 = 22

Here, power of 3 is the limiting factor: $12 = 2^2 \times 3$.

Highest power of 12 that divides 50!

$\Rightarrow (2^{47} \times 3^{22} \times b) \quad \Rightarrow \quad (2^2 \times 3)^{22} \times 2^3 \times b \quad \Rightarrow \quad 12^{22}$

$\therefore$ Highest power of 12 that will divide first 50 multiples of 12

$= 12^{(50+22)} = 12^{72}$.

Hence, option (D) is the correct answer.