2 Functions and Graphs

Introduction

The function is one of the most important topics of algebra in the quantitative aptitude section of the CAT exam. Types of function, compound function, greatest integer functions, domain and range, and graphs of some special functions all appear in this topic. One can generally expect one or two questions from functions.

Function

The function can be understood as an input–output machine that takes an input and produces an output. For example, at metro stations or shopping malls, we have seen automated vending machines, as shown below:

Through a pre-defined programme, if you press certain keys, you would get your desired object. In this case, the output (your desired object) is related to the input (the buttons pressed) through an algorithm. You must have seen for a particular input, they always produce the same output.

Functions: Let A and B be two non-empty sets. A function f from A to B is a correspondence that associates elements of set A to the elements of set B such that:

1. All elements of set A are associated with elements in set B.

2. An element of set A is associated with a unique element in set B.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the image of a under f and a is called the pre-image of b under f. The function f from A to B is denoted by f: A $\to$ B.

Let us look at some examples.

- If the selling price (SP) per unit of a quantity is constant, the revenue of a company (R), will be dependent on the number of units sold (n). This can be written mathematically as R = $f(n)$.
- If the distance (d) is constant, then the time taken (t) to cover that distance (d) will be dependent on the value of speed (v). This can be written mathematically as $t = f(v)$.
- The volume of a cube (V) is dependent on the side of the cube (a). This can be written mathematically as $V = f(a)$.
- Area of circle (A) = πr^2, where r is the radius. So, the area of the circle is dependent upon the value of the radius of the circle. We can write this mathematically as $A = f(r)$.

Evaluating Function

Example 1:

Given the function defined by $g(x) = \frac{3}{2}x - 1$, find the function values.

(A) $g(0)$ (B) $g(2)$

(C) $g(4)$ (D) $g(-2)$

Solution:

(A) put $x = 0$; $g(0) = \frac{3}{2}(0) - 1 = -1$

(B) put $x = 2$; $g(2) = \frac{3}{2}(2) - 1 = 2$

(C) put $x = 4$; $g(4) = \frac{3}{2}(4) - 1 = 6 - 1 = 5$

(D) put $x = -2$; $g(-2) = \frac{3}{2}(-2) - 1 = -3 - 1 = -4$

Domain of a Function

Consider a function defined by $y = f(x)$. The domain of f is the set of all x-values that when substituted into the function, produce a real number. The range of f is the set of all the y-values corresponding to the values of x in the domain.

To find the domain of a function defined by $y = f(x)$, keep these guidelines in mind:

Exclude values of x that make the denominator of a fraction zero.

Exclude values of x that make a negative value within a square root.

Example 2:

Find the domain of the following functions. Write the answers in interval notation.

(A) $f(x) = \dfrac{x+7}{2x-1}$ (B) $h(x) = \dfrac{x-4}{x^2+9}$

(C) $k(t) = \sqrt{t+4}$ (D) $g(t) = t^2 - 3t$

Solution: (A)

The function will be undefined when the denominator is zero, that is, when:

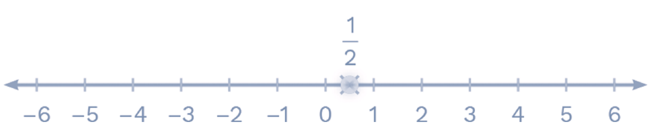
$2x - 1 = 0$

Or, $2x = 1$

Or, $x = \dfrac{1}{2}$,

So, the value of $x = \dfrac{1}{2}$ must be excluded from the domain.

Interval notation: $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$



(B) The quantity x^2 is greater than or equal to 0 for all real numbers x, and the number 9 is positive. Therefore, the sum $x^2 + 9$ must be positive for all real numbers x. So, the denominator of $h(x) = \dfrac{(x-4)}{(x^2+9)}$ will never be zero; so, the domain is the set of all the real numbers.

Interval notation: $(-\infty, \infty)$

(C) The function defined by $k(t) = \sqrt{t+4}$ will not be a real number when $t + 4$ is negative; hence, the domain is the set of all t-values that make the radicand greater than or equal to zero.

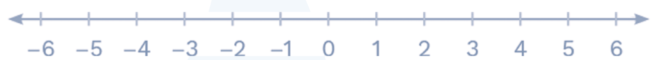
So, $t + 4 \geq 0$

Or, $t \geq -4$

Interval notation: $(-4, \infty)$

(D) The function defined by $g(t) = t^2 - 3t$ has no restrictions on its domain because any real number substituted for t will produce a real number. The domain is the set of real numbers.

Interval notation: $(-\infty, \infty)$



Example 3:

What is the domain of the function?

$$f(x) = \frac{1}{\sqrt{|x| - x}}$$

Solution: $x \in R-$

For domain: $|x| - x > 0$

$|x| > x$

This is only possible when $x \in R-$.

Example 4:

Find the domain of the function.

$$f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$$

Solution: $(-3, \infty) - (-1, -2)$

Here, $f(x) = \dfrac{\log_2(x+3)}{x^2+3x+2} = \dfrac{\log_2(x+3)}{(x+1)(x+2)}$ exist, if

numerator, $(x + 3) > 0 \Rightarrow x > -3$ ...(i)

And denominator, $(x + 1)(x + 2) \neq 0$

$\Rightarrow x \neq -1, -2$ ...(ii)

Thus, from equations (i) and (ii), we have:

The domain of $f(x) = (-3, \infty) - (-1, -2)$.

Example 5:

The domain of the function

$f\left(x\right)=\sqrt{x-x^2}+\sqrt{4+x}-\sqrt{4-x}$ is

(A) [-4, ∞) (B) [-4, 4]

(C) [0, 4] (D) [0, 1]

Solution: (D)

We know that, $f\left(x\right)=\sqrt{x-x^2}+\sqrt{4+x}-\sqrt{4-x}$

Clearly, $f(x)$ is defined if

$4 + x \geq 0 \Rightarrow x \geq -4$

$4 - x \geq 0 \Rightarrow x \leq 4$

$x(1 - x) \geq 0 \Rightarrow x \geq 0$ or $x \leq 1$

Therefore, domain of

$f = (-\infty, 4] \cap [-4, \infty] \cap [0, 1] = [0, 1]$

Hence, 4th option is correct.

Co-Domain and Range of a Function

If a function f is defined from a set of A to set B, then for f: A → B. Set A is called the domain of function f and set B *is* called the *co-domain* of function f. The set of all f-images of the elements of A is called the *range of function f*. In other words, we can say, domain = all possible values of x for which $f(x)$ exists.

Range = for all values of x, all possible values of $f(x)$ or the set of all the outputs is known as range.

Example 6:

Find the range of $\frac{1+x^2}{x^2}$.

(A) (0, 1) (B) (1, ∞)

(C) [0, 1] (D) [1, ∞)

Solution: (B)

Let $y = \frac{1+x^2}{x^2}$

$\Rightarrow x^2y = 1 + x^2 \quad \Rightarrow x^2(y - 1) = 1 \quad \Rightarrow x^2 = \frac{1}{y-1}$

Now, $x^2 \geq 0$

$\Rightarrow \frac{1}{y-1} \geq 0 \Rightarrow (y - 1) \geq 0 \quad \Rightarrow y > 1$ as y cannot be 1

Trick:

$$y=\frac{1+x^2}{x^2}=1+\frac{1}{x^2}$$

Now, since, $\frac{1}{x^2}$ is always > 0

$\Rightarrow \quad y > 1$

$\Rightarrow \quad y \in (1, \infty)$

Therefore, option (B) is correct.

Example 7:

Find the range of the function

$f\left(x\right)=\frac{x^2+11x+31}{x^2+11x+30}$, where $x \in$ R.

Solution: [1, 7/3)

Here, $f\left(x\right)=\frac{x^2+11x+31}{x^2+11x+30}$

$$f\left(x\right)=1+\frac{1}{x^2+11x+31}=1+\frac{1}{\left(x+\frac{11}{2}\right)^2+\frac{3}{4}}$$

Now, the maximum value of $f(x)$ is $\frac{7}{3}$ at $x = \frac{-11}{2}$

And the minimum value of $f(x)$ is 1 at $x = \infty$

Therefore, the range of $f(x)$ is [1, 7/3)

Note: For two sets A and B, the number of functions from A to B are $|B|^{|A|}$. For example, let A = {1, 2, 3, 4} and B = {p, q, r, s, t}. Therefore, the number of functions = $B^A = 5^4$ = 625 functions from A to B.

Types of Functions

One–one function (injective)

A function f: A → B is said to be a one–one function if different elements of A have different images in B.

Number of one–one functions from A to B having m and n elements respectively

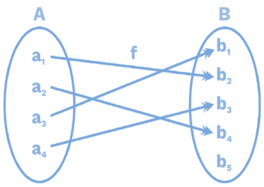
$$=\begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n = 0 \end{cases}$$

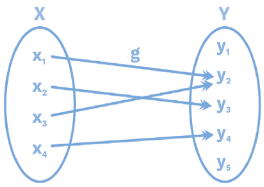
To check the injectivity of a function:

1. Take two arbitrary elements x_1 and x_2 in the domain of f.
2. Check whether $f(x_1) = f(x_2)$.

3. If $f(x_1) = f(x_2)$, which implies that $x_1 = x_2$, only then the function is a one–one function or injective function and otherwise, it is not.

Let f: A → B and g: X → Y be two functions, represented by the following diagram:

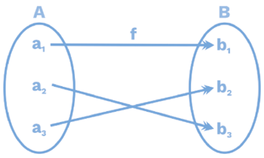


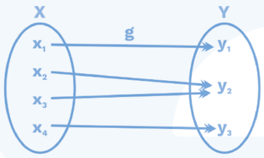


f: A → B is a one–one function. But g: X → Y is not a one–one function because two distinct elements x_1 and x_3 have the same image under function g.

Onto function (surjective)

Function f: A → B is said to be an onto function, if every element of B is the image of some element of A under f, i.e., for every element of $y \in$ B, there exists an element $x \in$ A such that $f(x) = y$, e.g., the following arrow diagram shows onto function.

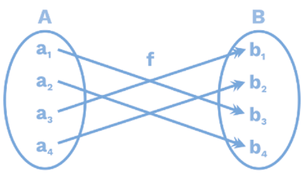




The number of onto functions: If A and B are two sets having m and n elements, respectively, such that $1 \le n \le m$, the number of onto functions from A to B is $\sum_{r=1}^{n}(-1)^{n-r} \times {}^{n}_{r}C \times r^{m}$.

One-one onto function (bijective)

A function f: A → B. is said to be a one-one and onto, if it is both one-one and onto.



Number of one-one onto functions: If A and B are finite sets and f: A → B is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijections from A to B is the total number of arrangements of n items taken all at a time, i.e., $n!$

Example 8:

The function f: R → R defined by $f(x) = (x - 3)(x - 4)(x - 5)$ is:

(A) One-one but not onto

(B) Onto but not one-one

(C) Both one-one and onto

(D) Neither one-one and onto

Solution: (B)

We have, $f(x) = (x - 3)(x - 4)(x - 5)$

Hence, $f(x)$ is not one-one

For each $y \in$ R, there exists $x \in$ R such that $f(x) = y$.

Therefore, f is onto.

Hence, f: R → R is onto but not one-one.

Therefore, option (B) is the correct answer.

Algebraic Functions

A function that contains a finite number of terms having different powers of the independent variable (x).

For example, $4x^2 - 3x^{1/2} + 7$, $\frac{x^2+1}{x^2-2}$ etc.

Polynomial Functions

A function $f(x)$ of the following form is known as a polynomial function:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n.$$

Where n is a whole number and $a_1, a_2, a_3, \cdots a_n \in R$.

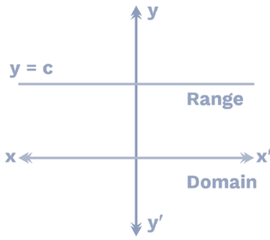
Domain: It is the set of all real numbers.

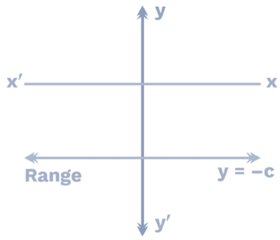
Range: It is also the set of all real numbers.

Constant Function

$f(x) = c$, $c \in$ R, where c is constant.

If c is a fixed real number, then a function $f(x)$ is given by $f(x) = c$, for all $c \in$ R.





Domain: It is the set of real numbers.

Range: It is a particular real number.

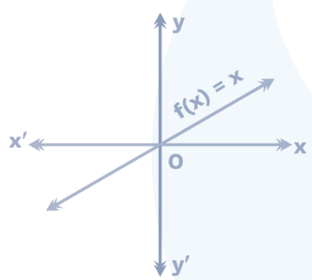
Identity Function

The function that associates each real number to itself is called the identity function and is usually denoted by *I*.

Thus, the function I: R $\rightarrow$ R defined by $I(x) = x$ for all $x \in$ R is called the identity function.

Domain: It is the set of real numbers.

Range: It is also the set of real numbers.

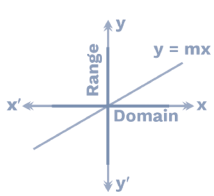


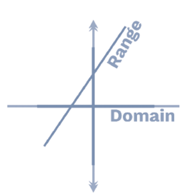
The domain and range of the identity function are both equal to R.

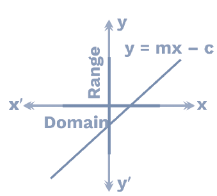
The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with the *x*-axis.

Linear Function

$y = mx + c$, (m, $c \in$ R is positive).







Domain: It is the set of real numbers.

Range: It is also the set of real numbers.

Quadratic Function

$y = f(x) = x^2$ or all functions in the form of $y = ax^2 + bx + c$ where a, b, $c \in R$, $a \neq 0$ will be known as a quadratic function.

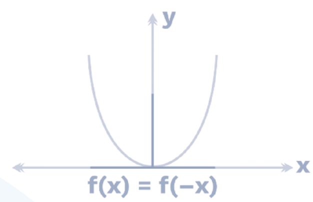
For, $f(x) = x^2$,

$f(x) = f(-x)$

$\therefore$ Graph of $f(x)$ is symmetric about the y-axis.

Domain: It is the set of real values.

Range: $R^+ \cup \{0\}$, i.e., non-negative real numbers.



Cubic Function

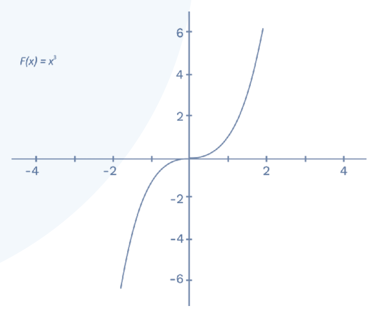
$y = f(x) = x^3$

$f(-x) = -f(x)$

$\therefore$ Graph of $f(x)$ is symmetric about the origin.

Domain: R (set of real numbers).

Range: R (set of real numbers).



Biquadratic Function

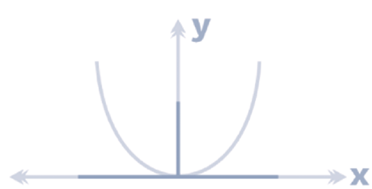
$y = f(x) = x^4$

$f(x) = f(-x)$

$\therefore$ Graph of $f(x)$ is symmetric about the y-axis.

Domain: R (set of real numbers).

Range: $R^+ \cup \{0\}$, i.e., set of non-negative real numbers.



Rational Function

f: A $\to$ R; $f\left(x\right)=\dfrac{P\left(x\right)}{Q\left(x\right)}$

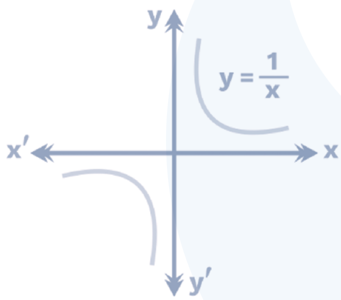
Here, $P(x)$ and $Q(x)$ are polynomial functions and A = $\{x$: $x \in$ R such that $Q(x) \neq 0\}$.

For example,

S. No.	Function	Domain
1.	$\dfrac{x^2+5x+8}{x^2-5x+6}$	R – {2, 3}
2.	$\dfrac{1}{x^n}; n \in N$	R – {0}

Graphs of important rational functions

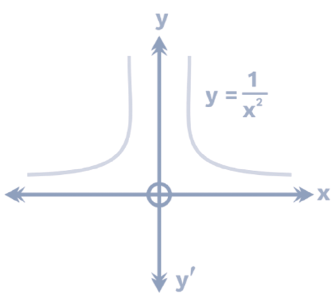
1. $y = f(x) = \dfrac{1}{x}$ $\Rightarrow$ $f(-x) = -f(x)$



Domain: R – {0}; set of non-zero rational numbers.

Range: R – {0}; set of non-zero rational numbers.

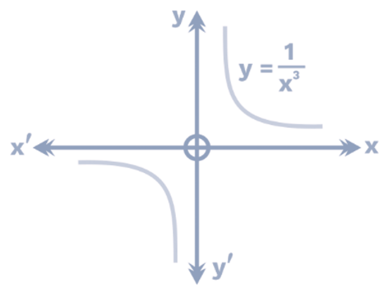
2. $y = f(x) = \dfrac{1}{x^2}$ $\Rightarrow f(x) = f(-x)$



Domain: R – {0}, i.e., set of non-zero rational numbers.

Range: R$^+$, i.e., set of positive rational numbers.

3. $y = f(x) = \dfrac{1}{x^3}$ $\Rightarrow$ $f(-x) = -f(x)$



Domain: R – {0}, i.e., set of non-zero rational numbers.

Range: R – {0}, i.e., set of non-zero rational numbers.

Irrational Functions

The algebraic functions containing one or more terms having non-integral rational powers of x are called irrational functions.

Example: $y = 4x^2+\sqrt{2x}$, then y is undefined if $x < 0$.

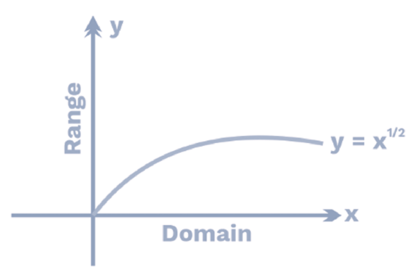
$y=\sqrt{x}$, then y is undefined if $x < 0$.

$y=\dfrac{5x^2+7x^2+3\sqrt{x}}{\sqrt{x+4}}$, then y is undefined if x < 0 and $\sqrt{x+4}\le 0$

These functions are not defined for $f(x) < 0$.

Graphs of important irrational functions

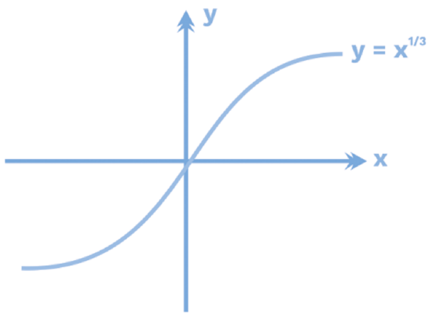
1. $y = f(x) = x^{1/2}$



Domain: R$^+$ U {0}, i.e., set of non-negative, real numbers.

Range: R$^+$ U {0}, i.e., set of non-negative real numbers.

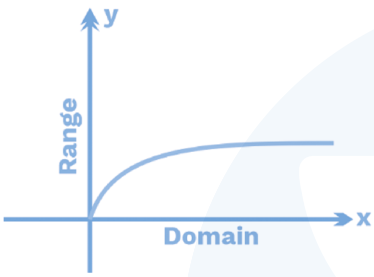
2. $y = f(x) = x^{1/3}$



Domain: R (set of real numbers).

Range: R (set of real numbers).

3. $y = f(x) = x^{1/4}$



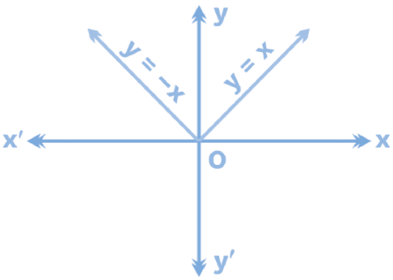
Domain: $R^+ \cup \{0\}$, i.e., set of non-negative real numbers.

Range: $R^+ \cup \{0\}$, i.e., set of non-negative real numbers.

Modulus Function

The function $f(x)$ defined by $f(x) = |x| = \begin{Bmatrix} x, \text{when } x \geq 0 \\ -x, \text{when } x < 0 \end{Bmatrix}$ is called the modulus function.

The graph of the modulus function is as shown in the figure, for $x \geq 0$, the graph coincides with the graph of the identity function, i.e., the line $y = x$, and for $x < 0$, it is coincident with the line $y = -x$.



Properties of modulus function

The modulus function has the following properties.

1. For any real number x, $\sqrt{x^2} = |x|$
2. If a, b are positive real numbers, then

   $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$

   $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$

   $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$

   $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or, $x > a$

   $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

   $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$
3. For real numbers x and y, we have

   $|x + y| = |x| + |y| \Leftrightarrow (x \geq 0$ and $y \geq 0)$ or $(x < 0$ and $y < 0)$

   $|x - y| = |x| - |y| \Leftrightarrow (x \geq 0, y \geq 0$ and $|x| \geq |y|)$ or, $(x \leq 0, y \leq 0$ and $|x| \geq |y|)$

   $|x \pm y| \leq |x| + |y|$

   $|x \pm y| \geq |x| - |y|$

Example 9:

What is the area of the following equation: $|x| + |y| = 5$?

(A) 100 square unit (B) 25 square unit

(C) 12.5 square unit (D) 50 square unit

Solution: (D)

We have, $|x| + |y| = 5$

We can write the four possible expressions from the above equation.

Case 1: $x + y = 5$

X	0	5
Y	5	0

Case 2: $x - y = 5$

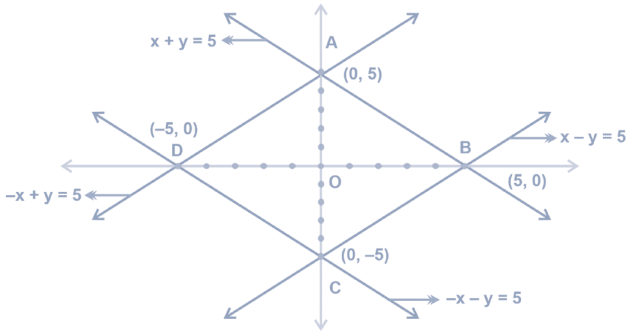
X	0	5
Y	–5	0

Case 3: $-x + y = 5$

X	0	–5
Y	5	0

Case 4: $-x - y = 5$

X	0	–5
Y	–5	0



Now, on right ΔAOB.

$$AB = \sqrt{(5)^2 + (5)^2}$$

$$AB = 5\sqrt{2}\text{units}$$

Hence, ABCD is a square of side-length $5\sqrt{2}$ units.

Therefore, area of ABCD = $(\text{side})^2 = \left(5\sqrt{2}\right)^2$

= 50 square unit.

Note: For the expression like $|x| + |y| = a$, the figure is always a square of the side length $a\sqrt{2}$ unit.

Hence, option (D) is the correct answer.

Example 10:

Find how many ordered pairs of values (x, y) satisfy $|x - 6| + |y - 4| \leq 4$, where x and y are positive integers.

Solution: 41

Given equation is $|x - 6| + |y - 4| \leq 4$

Consider it as $|k| + |l| \leq 4$

Now, if $k = 0$, $l = 0, 1, -1, 2, -2, 3, -3, 4, -4$ } 9 possibilities

if $k = 1, -1$, $l = 0, 1, -1, 2, -2, 3, -3$ } 14 possibilities

if $k = 2, -2$, $l = 0, 1, -1, 2, -2$ } 10 possibilities

if $k = 3, -3$, $l = 0, 1, -1$ } 6 possibilities

if $k = 4, -4$, $l = 0$ } 2 possibilities

Hence, total = 9 + 14 + 10 + 6 + 2 = 41.

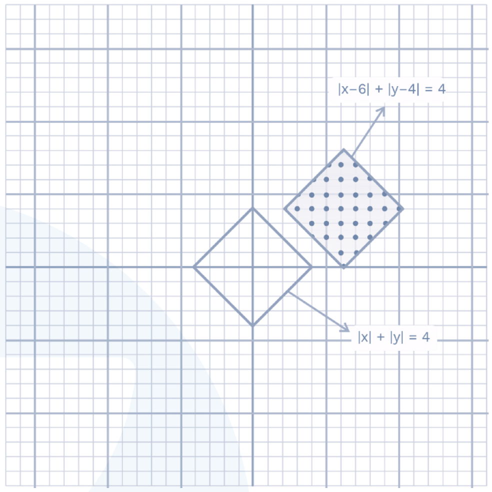
Graphical solution

If we observe: $|x - 6| + |y - 4| \leq 4$,

We can see that the shape will be the same as $|x| + |y| = 4$. The only difference is that $|x| + |y| = 4$ will be centred at (0, 0) while $|x - 6| + |y - 4| = 4$ will be centred at (6, 4).

This is the concept of shifting the origin.

You will get the same rhombus shape.

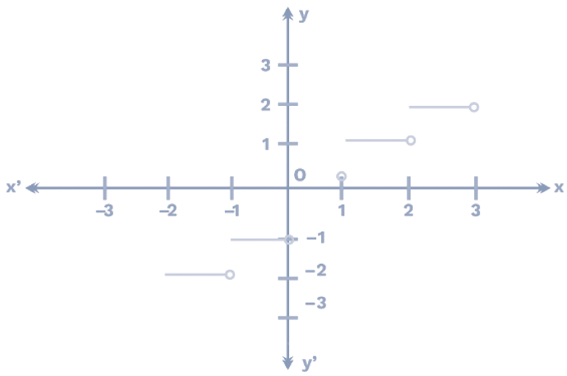


From the graph we can count 1 + 3 + 5 + 7 + 9 + 7 + 5 + 3 + 1 = 41 points.

Greatest Integer Function

The function f: R $\to$ R defined by $f(x) = [x]$ for all $x \in$ R is called the greatest integer function.

For any real number x, $[x]$ to denote the greatest integer less than or equal to x, i.e., $[n] = n$, if n is an integer and $[x] = n$, if $n \leq x < n + 1$, any real number x can be expressed as an integral part + fractional part, i.e., $I + f$, where, $0 \leq f < 1$ then $[x]$ gives an integral part of x.



Domain: R (set of real numbers).

Range: (Integral values) {–3, –2, –1, 0, 1, 2, 3} etc.

For example, [5.2] = 5,

[10] = 10,

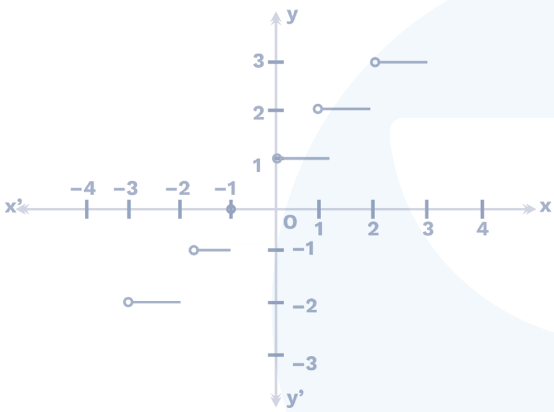
[–2.3] = –3,

[–9.9] = –10

Smallest Integer Function

The function f: R → R defined by $f(x)$ = $[x]$ for all $x \in$ R is called the smallest integer function or the ceiling function.

For example, [4.7] = 5, [–7.2] = –7, [5] = 5, [0.75] = 1, etc.



Domain: R (set of real numbers).

Range: (Integral values).

Transcendental Function

A function that is not algebraic and cannot be expressed in terms of a finite sequence of the algebraic operations of addition, subtraction, multiplication, division, and raising to a power is called a transcendental function, e.g., trigonometry, logarithmic, and exponential function.

Trigonometric Function

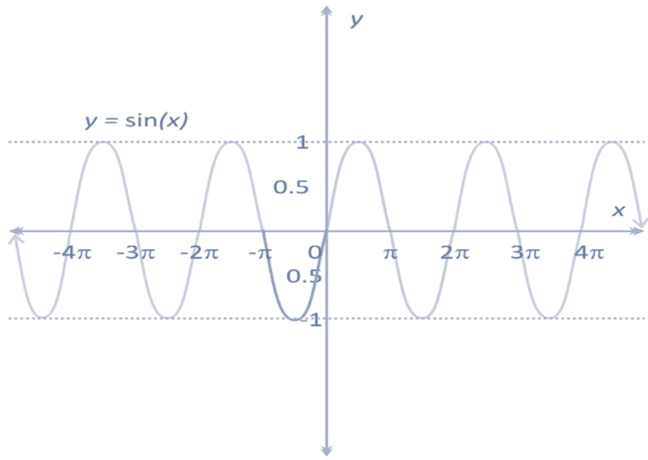
A function is said to be a trigonometric function if it involves sine, cosine, tangent co-tangent, secant, and cosecant.

$y = f(x) = \sin x$

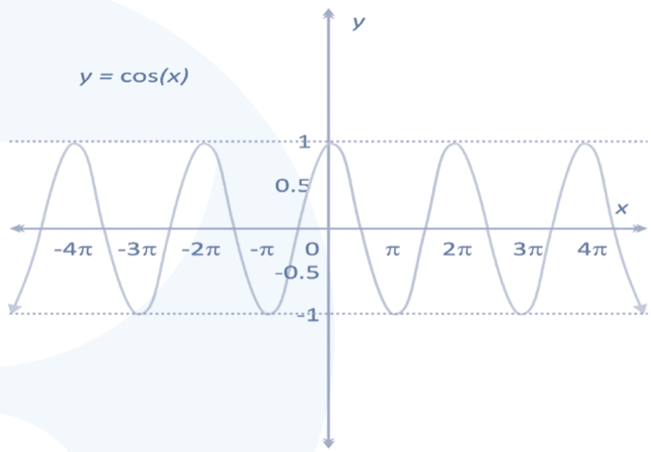
Domain: $x \in$ R

Range: $-1 \leq \sin x \leq 1$

sin (*x*):



cos (*x*):

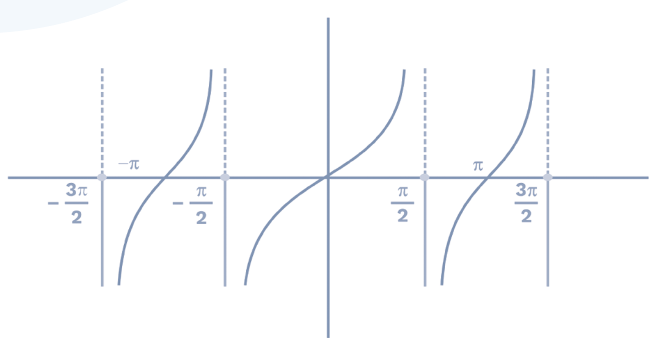


$y = \cos x$

Domain: $x \in$ R

Range: $-1 \leq \cos x \leq 1$

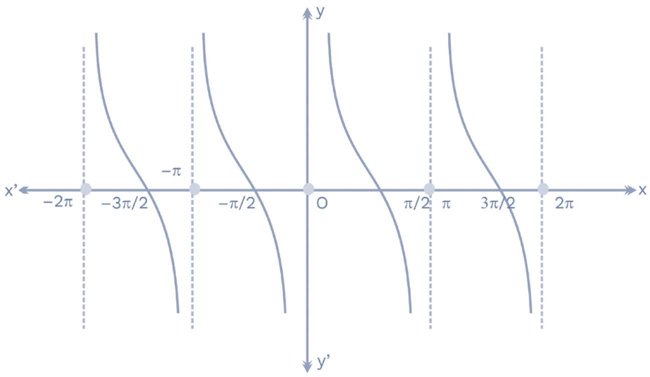
tan (*x*): $y = f(x) = \tan (x)$



Domain: $R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

Range: $-\infty < \tan x < \infty$.

cot (*x*): $y = f(x) = \cot x$



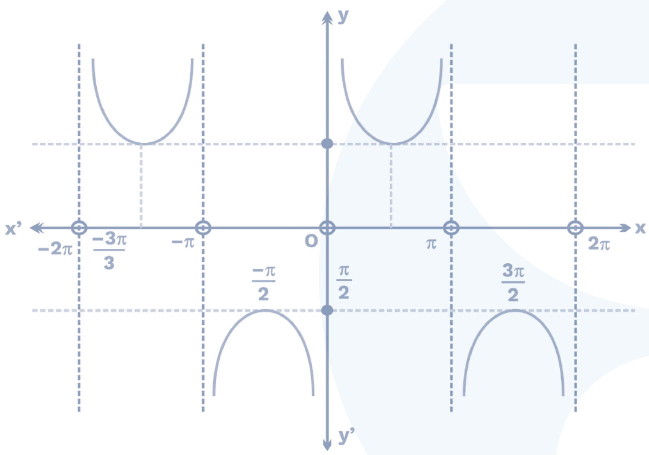
Domain: $x \in$ R − $\{n\pi: n \in$ I$\}$

Range: $-\infty < \cot x < \infty$.

cosec (x): $y = f(x) = \text{cosec } x$

Domain: R − $\{n\pi: n \in$ I$\}$

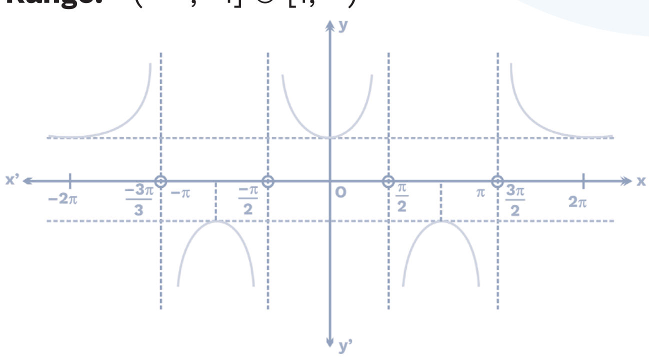
Range: $(-\infty, -1] \cup [1, \infty)$



sec (x): $y = f(x) = \sec x$

Domain: R − $\{(2n + 1); n \in I\}$

Range: $(-\infty, -1] \cup [1, \infty)$



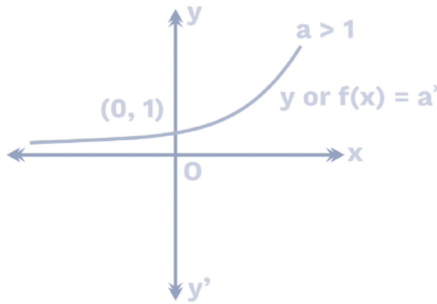
Exponential Function

Let $a \neq 1$ be a positive real number. Then f: R $\to (0, \infty)$ defined by $y = a^x$ is called exponential function.

1. $y = a^x$; $a > 1$

Domain: R

Range: R^+ or $(0, \infty)$



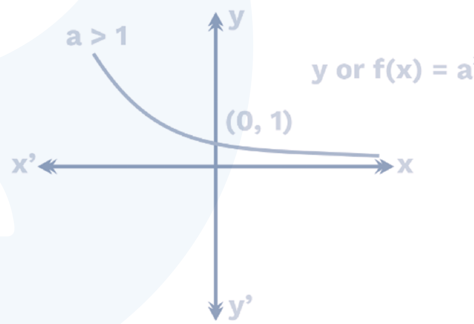
Keynote

$0 < a < 1$	Function is decreasing.
$a > 1$	Function is increasing.

2. $y = a^x$; $0 < a < 1$

Domain: R

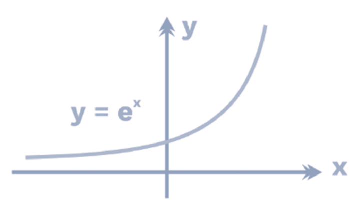
Range: R^+



3. a) $y = e^x$

Domain: R

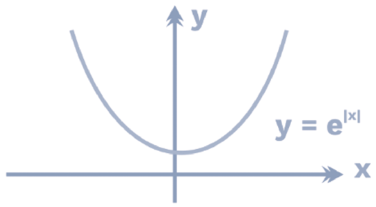
Range: R^+



b) $y = e^{|x|}$

Domain: R

Range: R^+



Keynote

- A one–one function is either increasing or decreasing across all values of a domain.
- A many-one function is a continuous function which has at least one local minimum or local maximum.

Logarithmic Function

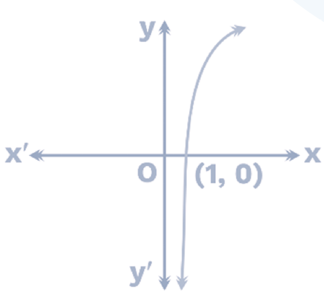
If $a > 0$ and $a \neq 1$, then the function defined by $f(x) = \log_a x$, $x > 0$ is called the logarithmic function.

Case 1: When $a > 1$, then

$$\log_a x = \begin{cases} < 0 \text{ for } 0 < x < 1 \\ = 0 \text{ for } x = 1 \\ > 0 \text{ for } x > 1 \end{cases}$$

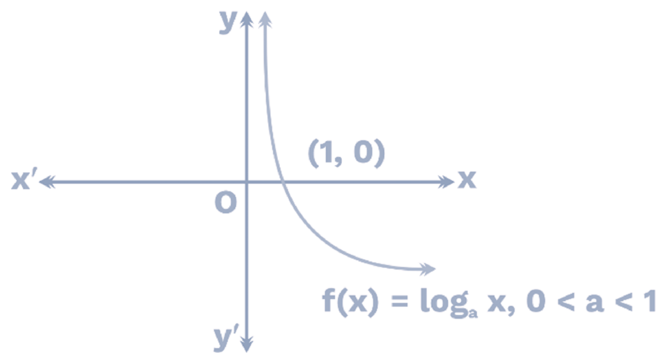
Domain: R^+

Range: R



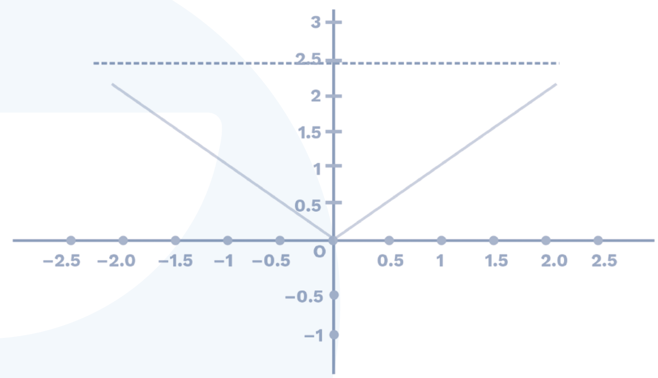
Case 2: When $0 < a < 1$

$$\text{We have } y = \log_a x = \begin{cases} > 0 \text{ for } 0 < x < 1 \\ = 0 \text{ for } x = 1 \\ < 0 \text{ for } x > 1 \end{cases}$$

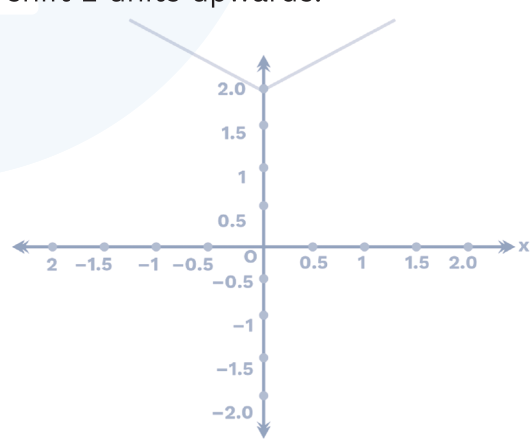


Transformation of Graphs

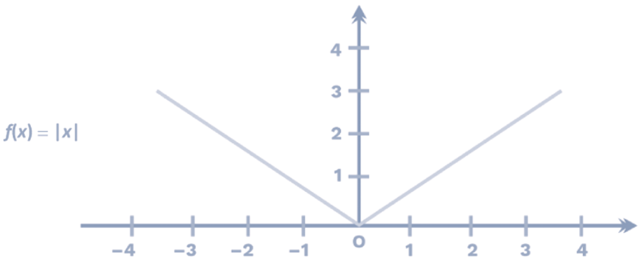
1. Transformation of function $f(x)$ to $f(x) + k$, k is a positive constant, and the graph shifts upward through k units.



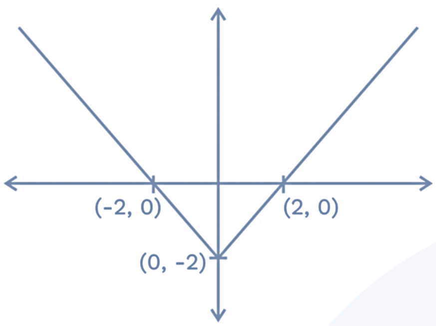
When $f(x) = |x| + 2$, then the graph will shift 2 units upwards.



2. $f(x) \to f(x) - k$; k is a positive constant.

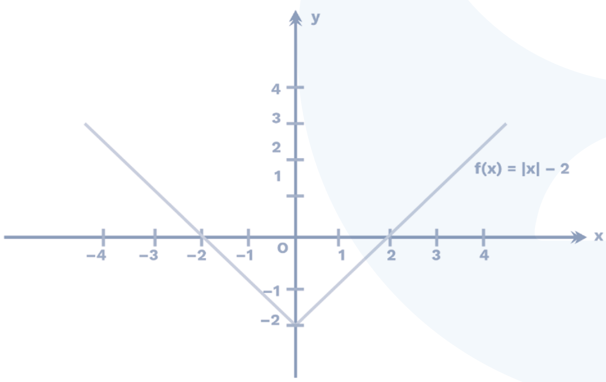


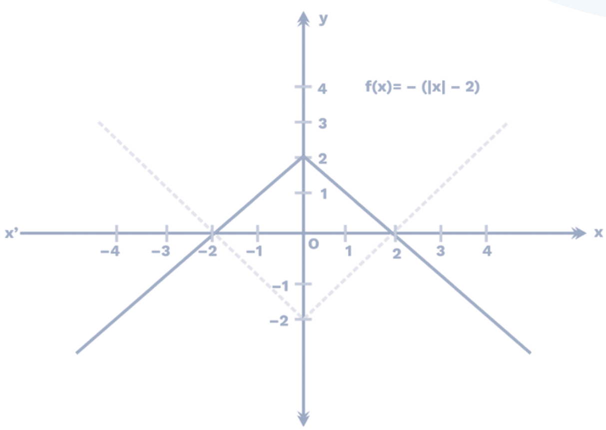
f(x) = |x| – 2



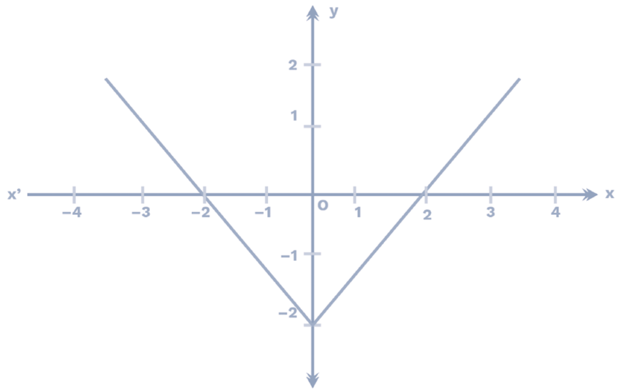
3. $f(x) \to -f(x)$

Turn the graph of $f(x)$ by 180° about the x-axis or take the mirror image of $f(x)$ on the x-axis.



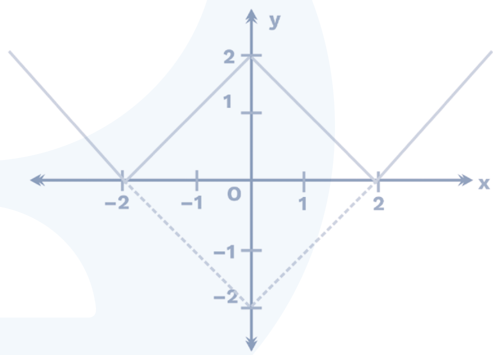


4. $f(x) = |f(x)|$, where $|f(x)| = \begin{cases} +f(x) \text{ is } f(x) \geq 0 \\ -f(x) \text{ if } f(x) < 0 \end{cases}$



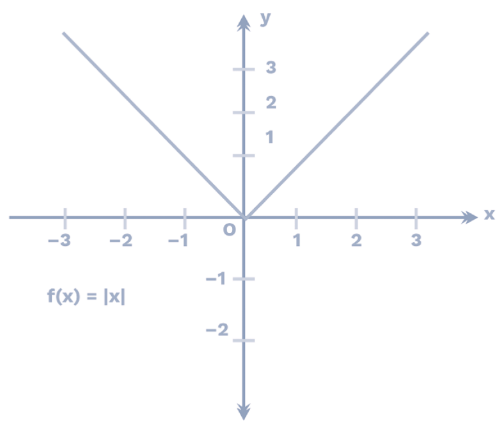
$f(x) = ||x| - 2|$

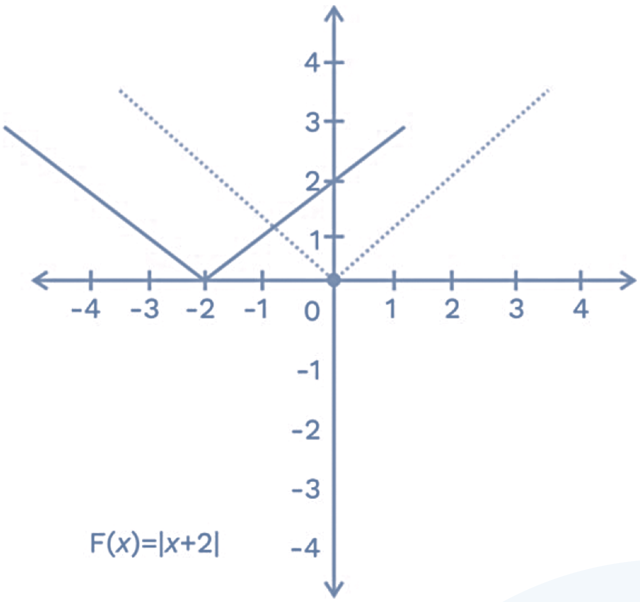
Turn the portion of the graph of $f(x)$ lying below the x-axis by 180° about the x-axis or take the mirror image (in x-axis) of the portion of the graph of $f(x)$ which lies below the x-axis.



5. $f(x) \to f(x + k)$

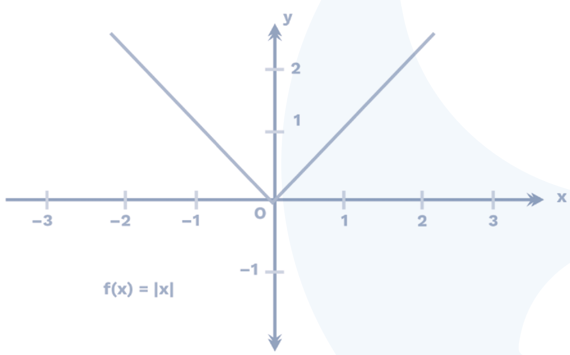
Shift the graph of $f(x)$ towards left by k.

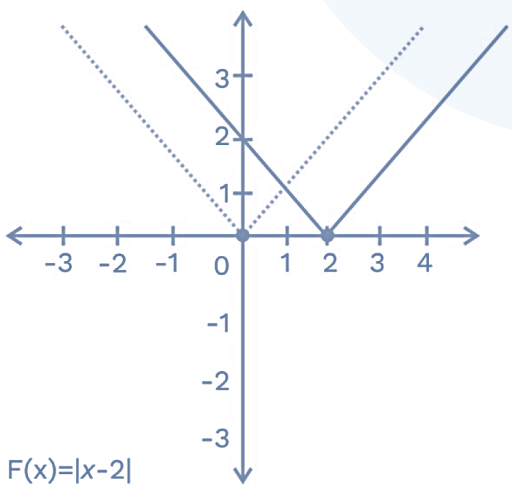




6. $f(x) \rightarrow f(x - k)$

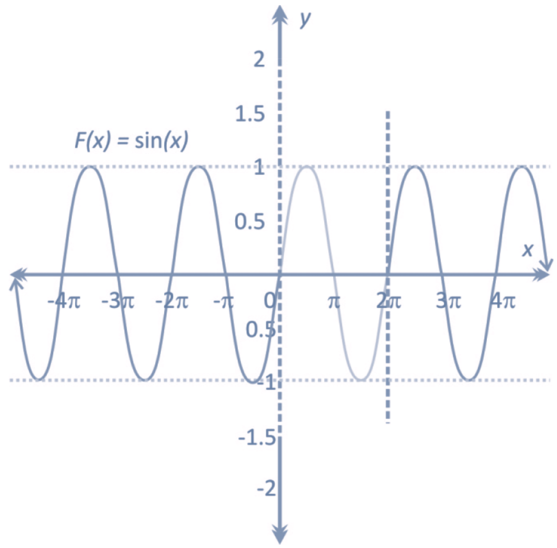
Shift the graph of $f(x)$ towards right by k.

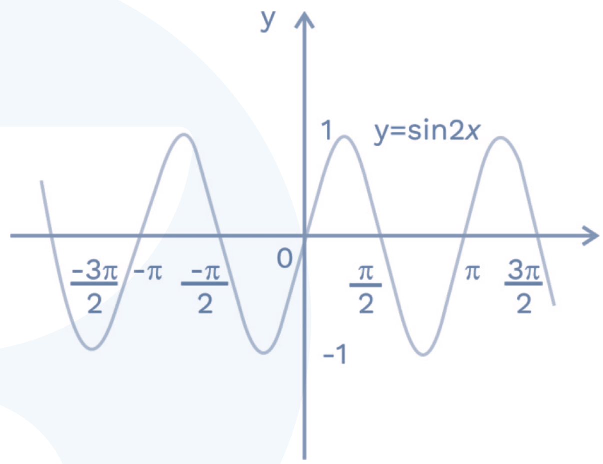




7. $f(x) \rightarrow f(kx)$; $(k > 1)$

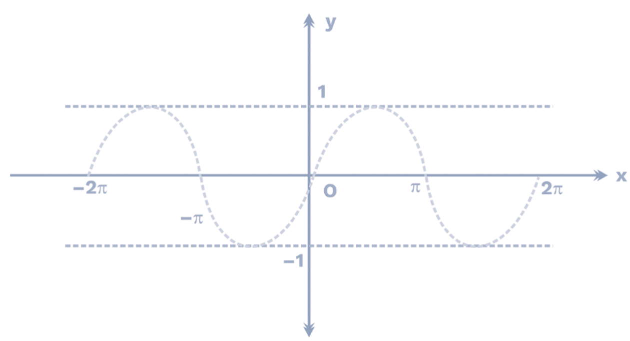
Shrink the graph of $f(x)$ k times along x-axis.



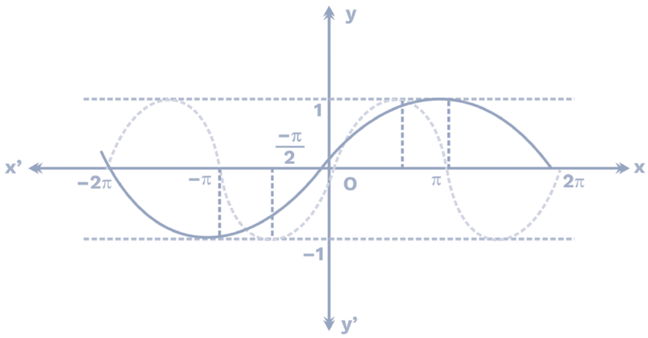


8. $f(x) \rightarrow f\left(\frac{x}{k}\right)$

Stretch the graph of $f(x)$ 'k' times along a-axis.



$f(x) = \sin x$



$f(x) = \sin(x/2)$

Example 11:

Let $f(x) = 2x^2 - 3x + 5$. If $g(x)$ represents $f(x)$ after it has been shifted to the left by four units, and then shifted down by 5 units, which of the following is equal to $g(x)$?

(A) $2x^2 - 19x + 52$ (B) $2x^2 - 19x + 47$

(C) $2x^2 + 13x + 20$ (D) $2x^2 + 13x + 25$

Solution: (C)

We are told that $g(x)$ is found by taking $f(n)$ and shifting it to the left by four and then down by five.

This means that we can represent $g(x)$ as follows:

$g(x) = f(x + 4) - 5$

Remember that the fusion $f(x + 4)$ represents $f(x)$ after it has been shifted to left by four, whereas $f(x - 4)$ represents $f(x)$ after being shifted to the right by four.

$f(x) = 2x^2 - 3x + 5$

Now, $g(x) = [2(x + 4)^2 - 3(x + 4) + 5] - 5$

$= 2x^2 + 16x + 32 - 3x - 12$

$g(x) = 2x^2 + 13x + 20$

Hence, option (C) is the correct answer.

Example 12:

If the point (8, 9) is reflected over the line $y = x$ and then over the x-axis, what is the result by coordinate?

(A) (8, –9) (B) (9, 8)

(C) (8, 9) (D) (9, –8)

Solution: (D)

A reflection of the line $y = x$ involves a switching of the coordinates to get us (9, 8).

A reflection over the x-axis involves a negation of the y-coordinate.

Thus, the result by points (9, –8).

Hence, option (D) is the correct answer.

Even and Odd Functions

If $f(x) = f(-x)$, the function will be an even function and $f(x) = -f(-x)$, the function will be an odd function.

Example: $f(x) = x^2 \sin x$

$f(-x) = -x^2 \sin x$

Here, $f(x) = -f(-x)$, so, it is an odd function.

Example: $f(x) = x^2$ and $f(-x) = x^2$

$f(x) = f(-x)$

So, it is an even function.

Example 13:

Let $f(x)$ be a function satisfying $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y. If $f(300) = 4$, then what is the value of $f(400)$?

(A) $\frac{1}{3}$ (B) $\frac{4}{3}$

(C) $\frac{3}{4}$ (D) 3

Solution: (D)

$x = 300$ and $y = \frac{4}{3}$

$$f\left(300 \times \frac{4}{3}\right) = \frac{f(300)}{\frac{4}{3}}$$

$$f(400) = \frac{4}{\frac{4}{3}}$$

$f(400) = 3$

Hence, option (D) is the correct answer.

Example 14:

Let $f(x)$ denote the sum of the digits of the positive integers x. For example, $f(7) = 7$ and $f(135) = 1 + 3 + 5 = 9$. For how many two-digits values of x is $f[f(x)] = 3$?

Solution: 10

Let a and b be the digits of x.

$f(f(x)) = a + b = 3$

$f(x)$ can only be 3, 12, 21, or 30 and only 3 and 12 are possible to have two-digit sum.

If $f(x)$ sums to 3, then there are three different solutions:

12, 21, or 30.

If $f(x)$ sums to 12, then there are seven different solutions:

39, 48, 57, 66, 75, 84, or 93.

The total number of solutions is 3 + 7 = 10

Example 15:

If $f\left(x\right)=\dfrac{4^x}{4^x+2}$, then find the value of $f\left(\dfrac{1}{1999}\right)+f\left(\dfrac{2}{1999}\right)+\cdots+f\left(\dfrac{1998}{1999}\right)$

(A) 1998 (B) 1999 (C) 998 (D) 999

Solution: (D)

Notice that $f(x) + f(1 - x) = 1$

Hence, $f\left(\dfrac{1}{1999}\right)+f\left(\dfrac{1998}{1999}\right)=1$

$f\left(\dfrac{2}{1999}\right)+f\left(\dfrac{1997}{1999}\right)=1$, and so on.

Therefore, sum = $\dfrac{1998}{2} = 999$

Hence, option (D) is the correct answer.

Example 16:

If $f(x + y) = f(x) + f(y) + f(x) \times f(y)$ and $f(1) = 4$, then what is the value of $f(3)$?

Solution: 624

By putting $x = y = 1$ in the equation, we get:

$f(2) = f(1) + f(1) + f(1) \times f(1) \Rightarrow f(2) = 24$

$f(3) = f(2) + f(1) + f(2) \times f(1) \Rightarrow f(3) = 124$

Similarly, $f(4) = f(3) + f(1) + f(3) \times f(1)$

$= 124 + 4 + 124 \times 4 = 624$

Example 17:

A function $f(x)$ satisfies $f(1) = 3600$, and $f(1) + f(2) + \cdots + f(n) = n^2f(n)$, for all positive integers $n > 1$. What is the value of $f(9)$?

Solution: 80

We have, $f(1) + f(2) + \cdots + f(n) = n^2f(n)$ ...(i)

Put, $n = n + 1$, we get:

$f(1) + f(2) + f(3) + \cdots + f(n + 1) = (n + 1)^2 f(n + 1)$ ...(ii)

Subtracting equation (i) from (ii),

$f(n + 1) = (n + 1)^2 f(n + 1) - n^2f(n)$

$f(n + 1) = \dfrac{n}{n+2}f\left(n\right)$

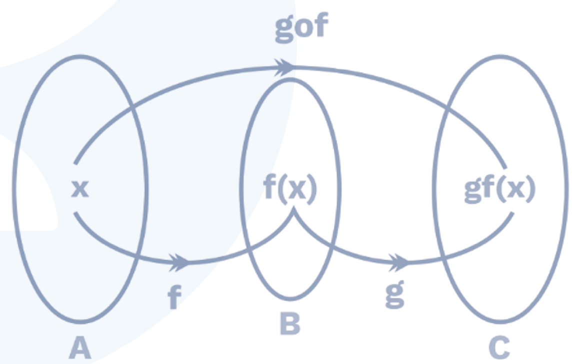
Therefore, $f(9)=\dfrac{8}{10}f\left(8\right)=\dfrac{8}{10}\times\dfrac{7}{9}\times f\left(7\right)=\ldots\ldots\ldots$

$=\dfrac{8}{10}\times\dfrac{7}{9}\times\dfrac{6}{8}\times\dfrac{5}{7}\times\dfrac{4}{6}\times\dfrac{3}{5}\times\dfrac{2}{4}\times\dfrac{1}{3}\times f\left(1\right)$

$= 80.$

Composition of Function

Let f: A → B and g: A → B be any two functions, the compositions of f and g, denoted by gof is defined as the function gof of A → C given by $gof(x) = g[f(x)]$ for all $x \in$ A.



Periodic Function

A function is said to be a periodic function if each value is repeated after a definite interval.

So, a function $f(x)$ will be periodic if a positive real number T exists such that, $f(x + T) = f(x)$ for all $x \in$ domain.

Here, the least positive value of T is called the period of the function.

Clearly, $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots$, e.g., $\sin x$, $\cos x$, $\tan x$ are periodic function with period 2π, 2π and π, respectively.

Example 18:

If $f(x + y) = f(x) + f(y) + f(xy)$ and $f(1) = 1$, then find the value of:

$f(2) + f(3) + \cdots + f(11)$.

Solution: 4,082

It is given that, $f(x + y) = f(x) + f(y) + f(xy)$

Put $x = y = 1$

$f(1 + 1) = f(1) + f(1) + f(1)$

$f(2) = 3f(1) = 3 \times 1 = 3$

Now, put $x = 2, y = 1$

$f(3) = f(2) + f(1) + f(2)$

$= 2f(2) + f(1)$

$= 2[3f(1)] + f(1)$

$= 7f(1) = 7 \times 1 = 7$

Putting, $x = 3, y = 1$

$f(4) = f(3) + f(1) + f(3)$

$= 2(f3) + f(1)$

$= 2(7f(1)) + f(1)$

$= 15\ f(1)$

$= 15 \times 1 = 15.$

Put $x = 4, y = 1$

$f(5) = 2f(4) + f(1)$

$= 2[15f(1)] + f(1)$

$= 31f(1)$

$= 31 \times 1 = 31$

Now we can find the pattern:

3,	7,	15,	31,...,
$2^2 - 1$,	$2^3 - 1$,	$2^4 - 1$,	$2^{11} - 1$,

Now required sum:

$f(2) + f(3) + \cdots +, f(11)$

$(2^2 - 1) + (2^3 - 1) + (2^4 - 1) + \cdots + (2^{11} - 1)$

$= (2^2 + 2^3 + \cdots + 2^{11}) - 10$

$= (2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{11}) - (2^0 + 2^1) - 10$

$= 2^{12} - 1 - 3 - 10$

$= 4{,}096 - 14$

$= 4{,}082.$

Practice Exercise – 1

Level of Difficulty – 1

1. If $f(x) = \dfrac{x+2}{x-2}$, $x \neq 2$, then find $f\left(f\left(f\left(f\left(f(3)\right)\right)\right)\right)$

(A) $\dfrac{15}{11}$

(B) $\dfrac{-37}{7}$

(C) 3

(D) 1

2. The domain of the function $f(x)$ = log $(5x - 6 - x^2)$ is:

(A) (5, 6)

(B) (2, 3)

(C) (2, ∞)

(D) None of these

3. If $f(x) = x^3 + 1$, and the value of the ex pression $f(2x) - 2f(x) = 4373$, then value of $\sqrt{x}$ is:

(A) 8

(B) 3

(C) 2

(D) 9

4. The domain of the function $f\left(x\right) = \sqrt{4x+4} + \sqrt{2x-10}$ is:

(A) [1, 5]

(B) [-1, 5]

(C) [5, ∞]

(D) [–1, ∞]

5. If $3x^2 - 2xy - y^2 + 4$, then $f[f(2, 3), f(-1, 1)]$ is equal to:

(A) -68

(B) 95

(C) 251

(D) 232

Level of Difficulty – 2

6. Let $f\left(n\right) = \left[\dfrac{1}{2} + \dfrac{n}{100}\right]$, where $[x]$ denotes an integral part of x. Then, the value of $\sum_{n=1}^{100} f\left(n\right)$ is:

(A) 50

(B) 51

(C) 1

(D) None of these

7. $f(x + y) = f(x)\, f(y)$ for all x, y, $f(4) = +5$. What is $f(-12)$?

(A) 125

(B) $\dfrac{1}{125}$

(C) 1728

(D) $\dfrac{1}{1728}$

8. Let $f(x)$ be a function satisfying $f(x)\, f(y) = f(xy)$ for real x, y. If $f(2) = 4$, then what is value of $f\left(\dfrac{1}{2}\right)$?

9. Let $f(x)$ be the greatest integer function and $g(x)$ be the modulus function, then what is the value of $\left(gof\right)\left(\dfrac{-5}{3}\right) + \left(fog\right)\left(\dfrac{-5}{3}\right)$?

(A) $\dfrac{3}{5}$

(B) $\dfrac{11}{3}$

(C) $\dfrac{2}{7}$

(D) 3

10. Let $f(x) = x^3 - 4x + p$, and $f(0)$ and $f(1)$ are of opposite signs, then which of the following is necessarily true?

(A) $-1 < p < 2$

(B) $0 < p < 3$

(C) $-2 < p < 1$

(D) $-3 < p < 0$

Level of Difficulty – 3

11. For a function $f(x)$, $f(x) + f(x - 1) = x^2$ and $f(20) = 40$. Find $f(40)$.

12. The operation $f(x)$ is defined by, (i) $f(1) = 2$, (ii) $f(x + y) = f(x)\, f(y)$, for all positive integers x and y, if $\sum_{x=1}^{n} f(x) = 1022$ then, n is:

(A) 8
(B) 9
(C) 10
(D) 11

13. The domain of the function $f(x) = \log_7(\log_3(\log_5(20x - x^2 - 91)))$ is:

(A) (7, 13)
(B) (8, 12)
(C) (7, 12)
(D) (12, 13)

14. Find $f(13)$ where $f(x + 2) = 2f(x) - f(x + 1)$ and $f(10) = 20$, $f(15) = 108$.

(A) 24
(B) 28
(C) 44
(D) Cannot be determined

15. Let $f(x) = \dfrac{(x^2 + 1)}{(x^2 - 1)}$; if $x \neq 1, -1$ and 1 if $x = 1, -1$.

Let $g(x) = \dfrac{x+1}{x-1}$; if $x \neq 1$ and 3, if $x = 1$. What are the minimum possible values of $\dfrac{f(x)}{g(x)}$?

(A) $\dfrac{1}{2}$
(B) –1
(C) $\dfrac{1}{3}$
(D) $\dfrac{1}{4}$

Solutions

1. (B)

Put $x = 2$; we have,

$$f\left(f\left(f\left(f\left(f(3)\right)\right)\right)\right) = f\left(f\left(f\left(f(5)\right)\right)\right)$$

$$= f\left(f\left(f\left(\frac{7}{3}\right)\right)\right)$$

$$= f\left(f(13)\right)$$

$$= f\left(\frac{15}{11}\right) = \frac{-37}{7}$$

Therefore, (B) option is correct.

2. (B)

$f(x) = \log(5x - 6 - x^2)$

The logarithmic function is defined only for positive values.

So, $5x - 6 - x^2 > 0$ or $x^2 - 5x + 6 < 0$

Roots of this equation are 2 and 3.

Put them on the number line.

Take the value from the internal 2–3 (say 2.5). It satisfies the inequality.

So, the value lies between 2 and 3.

So, the domain is $(2, 3) = 2 < x < 3$

(2 and 3 are not included)

Therefore, (B) option is correct.

3. (B)

We have, $f(2x) - 2f(x) = 4373$

$[(2x)^3 + 1] - 2[x^3 + 1] = 4373$

$6x^3 - 1 = 4373$

$$x = \sqrt[3]{\frac{4374}{6}}$$

$x = 9$

$\Rightarrow \sqrt{x} = 3$

Hence, the (B) option is correct.

4. (B)

For domain, $4x + 4 \geq 0$ and $2x - 10 \geq 0$

$x \geq -1$ and $x \geq 5$

Therefore, $x \in [-1, 5]$

Hence, (B) option is correct.

5. (B)

$f(x) = 3x^2 - 2xy - y^2 + 4$

Then, $f((f(2, 3), f(-1, 1))$

The value of $f(2, 3) = 3(2)^2 - 2 \times 2 \times 3 - 3^2 + 4$

$= 12 - 12 - 9 + 4 = -5$

Now, $f(-1, 1) = 3 \times (-1)^2 - 2 \times (-1) \times 1 - (1)^1 + 4$

$= 3 + 2 - 1 + 4 = 8$

Then, $f(f(2, 3), f(-1, 1)] = f[-5, 8] = 3 \times (-5)^2 - 2 \times (-5) \times 8 - 8^2 + 4$

$= 75 + 80 - 64 + 4 = 95$

Therefore, (B) option is correct.

6. (B)

$$f(n) = \left[\frac{1}{2} + \frac{n}{100}\right] = \left[\frac{50 + n}{100}\right]$$

For all the values of $n < 50$, $f(n) = 0$ and for all the $n \geq 50$, $f(n) = 1$.

Hence, 51 such values are there.

Hence, (B) option is correct.

7. (B)

Here, $f(0 + 4) = f(0)\, f(4)$

$5 = f(0)\, f(4)$

$f(0) = 1$

Now, $f(4 + (-4)) = f(4)\, f(-4)$

$f(0) = 5 \times f(-4)$

$$f(-4) = \frac{1}{5}$$

Therefore, $f(-4 - 4 - 4) = f(-4)\, f(-4)\, f(-4)$

$$f(-12) = \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{125}$$

Hence option (B) is the correct answer.

8. 1/4

Put $x = 1$ and $y = 2$, we get

$f(1) \times f(2) = f(2)$

$\Rightarrow f(1) = 1$

Now, put $x = \frac{1}{2}$ and $y = 2$, we get

$$f\left(\frac{1}{2}\right) \times f(2) = f\left(\frac{1}{2} \times 2\right)$$

$$f\left(\frac{1}{2}\right) \times f(2) = f(1)$$

$$\Rightarrow f\left(\frac{1}{2}\right) = \frac{f(1)}{f(2)} \quad \Rightarrow f\left(\frac{1}{2}\right) = \frac{1}{4}$$

9. (D)

If it is given that $f(x) = [x]$ and $g(x) = |x|$

Then, $gof\left(\frac{-5}{3}\right) = gof\,[-1.66] = g(-2) = 2$

$$fog\left(\frac{-5}{3}\right) = fog\left[-1.66\right] = f\left(1.66\right) = 1$$

Then, $gof\left(\frac{-5}{3}\right) + fog\left(\frac{-5}{3}\right) = 2 + 1 = 3.$

Hence, (D) option is correct.

10. (B)

$f(0) = p$

$f(1) = 1 - 4 + p = p - 3$

Since they are of opposite sign, $p(p - 3) < 0$

$\Rightarrow \quad 0 < p < 3$

Hence, (B) option is correct.

11. 650

Here, $f(x) = x^2 - f(x - 1)$

$$\begin{aligned} f(40) &= 40^2 - f(39) \\ &= 40^2 - [39^2 - f(38)] \\ &= 40^2 - 39^2 + f(38) \\ &= 40^2 - 39^2 + 38^2 - f(37) \\ &= 40^2 - 39^2 + 38^2 - 37^2 + f(36) \\ &= 40^2 - 39^2 + 38^2 - 37^2 + 36^2 + 35^2 + \cdots + f(20) \\ &= (40 + 39)(40 - 39) + (38 + 37) + (38 - 37) + (36 + 35)(36 - 35) + \cdots + (22 + 21)(22 - 21) + f(20) \\ &= [79 + 75 + 71 + \cdots + 43] + f(20) \\ &= [43 + 47 + \cdots + 71 + 75 + 79) + f(20) \end{aligned}$$

$f(40) = s + f(20)$ ...(i)

For s: $a_n = 79; \quad a = 43; \quad d = 4$

Since we know that

$a_n = a + (n - 1)\, d$

$79 = 43 + (n - 1)\, 4$

$\frac{36}{4} = n - 1 \qquad$ or $\quad n = 10$

Also, a sum of n terms, $S = \frac{n}{2}\,[a + a_n] = \frac{10}{2}\,[43 + 79] = 5 \times 122 = 610$

Taking equation (i)

$f(40) = 610 + f(20)$

$f(40) = 610 + 40$

$f(40) = 650$

12. (B)

Since $f(x + y) = f(x) \times f(y)$ and $f(1) = 2$

Putting $x = y = 1$

Now, $f(1 + 1) = f(1) \times f(1)$

$\Rightarrow \quad f(2) = 2 \times 2 = 4$

If $x = 2, y = 1$

$\Rightarrow \quad f(3) = f(2) \times f(1) = 4 \times 2 = 8$

Similarly, $f(4) = f(3)\, f(1) = 8 \times 2 = 16$

The pattern followed: $f(n) = 2^n$

Now, $\sum_{x=1}^{n} f\left(x\right) = 1022 = f(1) + f(2) + f(3) + \cdots + f(n) = 1022$

$\Rightarrow 2^1 + 2^2 + 2^3 + \cdots + 2^n = 1022$

The series is a GP with first term, $a = 2$ and a common ratio $r = 2$

$\Rightarrow$ Sum of GP $= \frac{a\left(r^n - 1\right)}{\left(r - 1\right)}$

Therefore, $\frac{2\left(2^n - 1\right)}{\left(2 - 1\right)} = 1022$

$2^n - 1 = \frac{1022}{2} = 511$

$2^n = 512$

$2^n = 2^9$

On comparing both the sides, we get

or, $n = 9$

Hence, (B) option is correct.

13. (B)

A logarithm function of the form $\log_a b$ is true,

if $b > 0$ and $c = a^b$ is written as:

$\log_a c = b$

Since $f(x) = \log_7 (\log_3 (\log_5 (20x - x^2 - 91)))$

$\Rightarrow \quad \log_3 (\log_5 (20x - x^2 - 91)) > 0$

$\Rightarrow \quad \log_5 (20x - x^2 - 91) > 3^0$

$\Rightarrow \quad \log_5 (20x - x^2 - 91) > 1$

$\Rightarrow \quad 20x - x^2 - 91 > 5^1$

$\Rightarrow \quad x^2 - 20x + 96 < 0$

$\Rightarrow \quad (x - 8)(x - 12) < 0$

$\Rightarrow \quad 8 < x < 12$

Therefore, domain of $f(x) = (8, 12)$.

Hence, (B) option is correct.

14. (C)

$(x + 2) = 2f(x) - f(x + 1)$

Let $f(11) = K$

$\Rightarrow \quad f(12) = 2f(10) - f(11) = 40 - K$

Also, $f(13) = 2f(11) - f(12) = 2K - 40 + K$

$\Rightarrow \quad f(13) = 3K - 40$

$f(14) = 2f(12) - f(13)$

$= 80 - 2K - 3K + 40 = 120 - 5K$

Now, $f(15) = 2f(13) - f(14) = 6K - 80 - 120 + 5K$

$\Rightarrow \quad 108 = 11K - 200$

$\Rightarrow \quad 11K = 308$

$\Rightarrow \quad K = 28$

Therefore, $f(13) = 3K - 40 = 84 - 40 = 44$.

15. (D)

$$\frac{f(x)}{g(x)} = \frac{(x^2+1)}{(x^2-1)} \times \frac{(x1)}{(x+1)} = \frac{(x^2+1)}{(x+1)^2}$$

This function is greater than 0.

Let $y = \dfrac{(x^2+1)}{(x+1)^2}$

$\Rightarrow \quad x^2(y - 1) + 2yx + (y - 1) = 0$ which is quadratic in x.

Discriminant should be greater than 0

$4y^2 - 4(y - 1)^2 \geq 0$

$\Rightarrow \quad y \geq \dfrac{1}{2}$

When $x = 1$, then $\dfrac{f(x)}{g(x)} = \dfrac{1}{3}$

Therefore, either the value should be greater than $\dfrac{1}{2}$ or equal to $\dfrac{1}{3}$.

Hence, (D) option is correct.

Practice Exercise – 2

Level of Difficulty – 1

1. If $f\left(x\right)=\dfrac{5x+2}{3x-5}$ and $g(x) = 16x - 15$, then the value of $g(f(f(5)))$ is:
(A) 33
(B) 49
(C) 65
(D) 81

2. If $f(x) = px + q$ and $f[f[f(x)]] = 64x + 147$, the value of $2p + 3q$ is:
(A) 29
(B) 35
(C) 24
(D) 31

3. For a positive number x, $[x]$ = the largest integer less than x and $\{x\}$ = the smallest integer greater than x, what is the value of $[7.2] + \{8.3\} - [4] - [6] + \{8.7\} - \{9.5\} - [8.1]$?
(A) −3
(B) −7
(C) −5
(D) −1

4. Let $f(x)$ be a function satisfying $f(x)\ f(y) = f(xy)$ for all real values of x and y. If $f(3) = 6$, then what is the value of $f\left(\dfrac{1}{3}\right)$?
(A) 1
(B) $\dfrac{1}{6}$
(C) $\dfrac{1}{2}$
(D) 6

5. For a positive integer a, $f(a)$ is such that $f(a + K) = f(a \times K)$, where K is an integer and $f(1) = 6$. If the value of $f(1{,}007) = K$, the value of K will be.
(A) 6
(B) 5
(C) 4
(D) 3

6. A function $g(x)$ is an even function, if $g(-x) = g(x)$, where $x \in R$.
A function $g(x)$ is an odd function, If $g(x) = -\ g(x)$, where $x \in R$.
If $g(x) = 4x^2 - 5x$, then $g(x) + g(-x)$ is:
(A) Always odd
(B) Always even
(C) Always equal to 1
(D) Neither odd nor even

7. A function is defined as $f(x) = \sin x\ (\sin x - 4) + 3$ for all the real values of x. Find the maximum value of $f(x)$.

8. $f(x)f(y) = f(x) + f(y) + f(xy) - 2$ where x and y are positive real numbers. If $f(3) = 10$ and $f(2) = 5$, find $f(12)$.

9. Given that $f(x) = 1$, $f(2x) = 4f(x) + 5$ and $f(x + 2) = f(x) + 3x + 5$ for all real values of x. What is the value of $f(20)$?

10. Find the range of value of x for which $[x] + [2x] + [3x] = 8$, where x is a real number and $[x]$ is the greatest integer less than or equal to x.
(A) $1 \le x \le \dfrac{4}{3}$
(B) $x < \dfrac{5}{3}$
(C) $\dfrac{3}{2} \le x < \dfrac{5}{3}$
(D) None of these

Level of Difficulty – 2

11. If $f\left(x\right)=\dfrac{1}{x}$ and $\left(x\right)=x-\dfrac{1}{x}$, then which of the following is true?
(A) $f\left(g\left(x\right)\right)\cdot g\left(f\left(x\right)\right)=1$
(B) $f\left(g\left(x\right)\right)\cdot g\left(f\left(x\right)\right)=-1$
(C) $\dfrac{f\left(g\left(x\right)\right)}{g\left(f\left(x\right)\right)}=1$
(D) $\dfrac{f\left(g\left(x\right)\right)}{g\left(f\left(x\right)\right)}=-1$

12. For how many integral values of x the function $f(x)=\dfrac{\sqrt{16-x^2}}{\log_{x+2}\left(\left|\dfrac{x}{5}\right|\right)}$ is defined?

13. $f(x)$ is function such that $f(a \times b) = \dfrac{f(a)}{f(b)}$, $(a < b)$ for all positive real values of a and b. Find $f(0.125)$, given that $f(8) = 0.5$.
(A) 0.125
(B) 0.25
(C) 0.50
(D) None of these

14. If $f(x)=\dfrac{1-x^2}{1+x^2}$, then find the value of $f\left(\dfrac{1-x^2}{1+x^2}\right)$.
(A) x^2
(B) $\dfrac{x^2+2x}{3+x}$
(C) $\dfrac{2}{x^2+\dfrac{1}{x^2}}$
(D) $\dfrac{2x^2}{x^2+1}$

15. The range of the function $g(a) = |a| - a$ is:
(A) $(-\infty, \infty)$
(B) $[-2, 0]$
(C) $[0, 1]$
(D) $[0, \infty)$

16. Find x, when $f(x) = 3x - 1$ and $f(f(f(2x))) = 365$.
(A) 5
(B) 7
(C) 9
(D) None of these

17. Let $g(x)$ be a function such that $g(x + 1) + g(x - 1) = g(x)$ for every real x. Then, for what value of p is the relation $g(x + p) = g(x)$ necessarily true for every real x?
(A) 3
(B) 4
(C) 2
(D) 6

18. Which of the following function is an even function?
(A) $\log\left(\dfrac{1+x^5}{1-x^5}\right)$
(B) $\dfrac{\left(1+3^x\right)^2}{3^x}$
(C) $e^{x^{12}}$
(D) All of these

19. Let $f(x)$ be a fourth-degree polynomial, the coefficient of whose highest power is 3. If $f(2) = 4$, $f(3) = 9$, $f(4) = 16$ and $f(5) = 25$, find $f(x)$.
(A) $3x^4 + 14x^3 + 72x^2 + 154x + 120$
(B) $3x^4 + 42x^3 + 214x^2 + 462x + 360$
(C) $3x^4 - 14x^3 + 72x^2 - 154x + 120$
(D) $3x^4 - 42x^3 + 214x^2 - 462x + 360$

20. The function $f(x)$ denotes a parabola which cuts the x-axis at the points $x = -15$ and $x = -8$. If $f(a) = f(b)$, then find the value of $a + b$, if $b > a$.
(A) 0
(B) −7
(C) 23
(D) −23
(E) −11.5

Level of Difficulty – 3

21. $f(x)$ is a function such that $f(2) = f(5) = f(8) = 1$ and $f(11) = 163$. Find $f(3)$.

22. If $f(a^2 - a) = a^4 - 2a^3 + 5a^2 - 4a - 5$, then find the approximate value of $\dfrac{f(f(10))}{f(10)}$.
(A) 139
(B) 143
(C) 150
(D) 172

23. Find $f(13)$ where $f(x + 2) = 2f(x) - f(x + 1)$ where $f(10) = 20$, $f(15) = 108$.
(A) 24
(B) 28
(C) 44
(D) Cannot be determined

24. Find the sum $\left[\sqrt[3]{1}\right]+\left[\sqrt[3]{2}\right]+\left[\sqrt[3]{3}\right]+\left[\sqrt[3]{4}\right]+\cdots+\left[\sqrt[3]{150}\right]$

Note: $[a] \rightarrow$ Represents the greatest integer less than or equal to a.

(A) 430
(B) 130
(C) 510
(D) 530

25. If $f(x)=\log\left(\frac{1+x}{1-x}\right)$ and $g(x)=\frac{3x+x^3}{1+3x^2}$, then find the value of $fog(x)$

(A) $3f(x)$

(B) $-f(x)$

(C) $(-f(x))^2$

(D) $-3f(x)$

26. Consider the function $f(x) = (x + 5)(x + 7)(x + 9) \ldots (x + 97)(x + 99)$. The number of integers x for which $f(x) < 0$ is:

(A) 24
(B) 23
(C) 46
(D) 48

27. If $f(x) + f(1 - x) = 8$, find the value of $f\left(\frac{1}{100}\right)$

$+ f\left(\frac{2}{100}\right) + f\left(\frac{3}{100}\right) + \ldots f\left(\frac{98}{100}\right) + f\left(\frac{99}{100}\right)$

.

(A) 400
(B) 392
(C) 396
(D) 404

28. For two sets A and B, $n(A) = 6$ and $n(B) = 5$. Find the number of functions that are not onto that can be defined from A to B.

29. A function $f(x)$ exists for all the real values of x such that $f(3 - x) = f(3 + x)$ and $f(10 - x) = f(10 + x)$. If $f(3) = 0$, then what is the minimum number of roots for $f(x) = 0$, if x lies between -300 and 300 (both inclusive)?

30. $\log_{\frac{1}{4}}\left(x^2-2x-24\right) > \log_{\frac{1}{4}} 11$. The range of x is:

(A) (-5, 7)
(B) (-4, 6)
(C) $(-5, -4) \cup (6, 7)$
(D) $(-\infty, -4) \cup (6, \infty)$

Solutions

1. **(C)**

We have, $f(x) = \dfrac{5x+2}{3x-5}$

$\therefore f(5) = \dfrac{5(5)+2}{3(5)-5} = \dfrac{27}{10}$

$$f(f(5)) = \dfrac{5\left(\dfrac{27}{10}\right)+2}{3\left(\dfrac{27}{10}\right)-5} = \dfrac{155}{31} = 5$$

$\therefore g(f(f(5))) = 16(5) - 15 = 65$

Thus, option (C) is correct.

2. **(A)**

$f(x) = px + q$

$f[f(x)] = f[px + q]$

$= p(px + q) + q$

$= p^2x + pq + q$

$f[f(f(x))] = f(p^2x + pq + q)$

$= p[p^2x + pq + q] + q$

$= p^3x + p^2q + pq + q$

Given, this is equal to $64x + 147$

$\Rightarrow$ $p^3 = 64$ and $p^2q + pq + q = 147$

$p = 4$ and $q(p^2 + p + 1) = 147$

$\Rightarrow$ $q = \dfrac{147}{21} = 7$

$\therefore$ $2p + 3q = 2(4) + 3(7)$

$= 8 + 21 = 29.$

Hence option (A) is the correct answer.

3. **(D)**

[7.2] + {8.3} – [4] – [6] + {8.7} – {9.5} – [8.1]?

= 7 + 9 – 3 – 5 + 9 – 10 –8

= – 1

Hence, option (D) is the correct answer.

4. **(B)**

We have $f(xy) = f(x)f(y)$

Also, $f(3) = 6$

The above expression can be written as:

$$f\left(9\times\frac{1}{3}\right) = f(9)\times f\left(\frac{1}{3}\right)$$

$$f(3) = f(3\times 3)\times f\left(\frac{1}{3}\right)$$

$$f(3) = f(3)\times f(3)\times f\left(\frac{1}{3}\right)$$

$$\frac{1}{f(3)} = f\left(\frac{1}{3}\right)$$

Therefore, $f\left(\dfrac{1}{3}\right) = \dfrac{1}{6}$

Hence option (B) is the correct answer.

5. **(A)**

Let $a = 1$ and $K = 0$

Then $f(a + 0) = f(1 \times 0)$

$f(1) = f(0)$

Since $f(1) = 6$ is given in the question.

$\therefore$ $f(1) = f(0) = 6$

If we consider $a = 1{,}007$ and $K = 0$

Then, $f(1{,}007 + 0) = f(1{,}007 \times 0)$

$f(1{,}007) = f(0) = 6$

$f(1{,}007) = 6 = K$

Therefore, it is confined that this is a constant function for all values of a.

Hence option (A) is the correct answer.

6. **(B)**

Since, the given function is $g(x) = 4x^2 - 5x$.

If we check for $g(x) + g(-x) = 4x^2 - 5x + 4(-x)^2 - 5(-x) = 8x^2$

where $8x^2$ is always even.

Hence, (B) is the correct answer.

7. **8**

The expression $f(x)$ can be written as:

$f(x) = \sin^2 x - 4\sin x + 3 = \sin^2 x - 4\sin x + 4 - 4 + 3$

$= (\sin x - 2)^2 - 1$

To maximize the value of $f(x)$, the value of $\sin x$ will be -1.

Hence, the maximum value of $f(x) = (-1 - 2)^2 - 1 = 8$.

8. **148**

$f(x)f(y) = f(x) + f(y) + f(xy) - 2$

$f(3)f(2) = f(3) + f(2) + f(6) - 2$

$10(5) = 10 + 5 + f(6) - 2$

$\Rightarrow$ $f(6) = 50 - 15 + 2 = 37$
Put $x = 6$, $y = 2$
$f(6)f(2) = f(6) + f(2) + f(12) - 2$
$f(12) = 37(5) - 37 - 5 + 2 = 145$.

9. 393

$f(1) = 1$
Put $x = 1$ in $f(x + 2) = f(x) + 3x + 5$
$\Rightarrow f(1 + 2) = f(1) + 3 + 5$ or $f(3) = 1 + 3 + 5 = 9$
Put $x = 3$ in $f(x + 2) = f(x) + 3x + 5$
$\Rightarrow f(3 + 2) = f(3) + 3 \times 3 + 5$
or $f(5) = 9 + 9 + 5 = 23$
Put $x = 5$ in $f(2x) = 4f(x) + 5$
$f(2x) = 4f(x) + 5$
or $f(10) = 4f(5) + 5 = 97$
Put $x = 10$ in $f(2x) = 4f(x) + 5$
$f(20) = 4\,f(10) + 5 = 4 \times 97 + 5 = 393$.

10. (C)

By observing we can find that $x > 1$ and $x < 2$.
Else the RHS $\neq + 8$.
So, the combinations are $[x] = 1$, $[2x] = 2$ or 3, and $[3x] = 4$ or 5.
The combinations that give RHS = 8 are 1 + 2 + 5 or 1 + 3 + 4.
For any value of x, the case of '1 + 2 + 5' is not possible. Hence, it has to be the case of '1 + 3 + 4'. Which will occur when, $x \geq \frac{3}{2}$ and $x < \frac{5}{3}$.

Hence the solution is $\frac{3}{2} \leq x < \frac{5}{3}$.

11. (B)

We have, $f(x) = \frac{1}{x}$ and

$$g(x) = x - \frac{1}{x} \quad f(g(x)) = f\left(x - \frac{1}{x}\right) = \frac{1}{x - \frac{1}{x}}$$

$$= \frac{x}{x^2 - 1}$$

And $g(f(x)) = g\left(\frac{1}{x}\right)$

$$= \frac{1}{x} - \frac{1}{\frac{1}{x}} = \frac{1}{x} - x = \frac{1 - x^2}{x} = -\left(\frac{x^2 - 1}{x}\right)$$

$$\therefore f(g(x)) \cdot g(f(x)) = \frac{x}{x^2 - 1} \times -\left(\frac{x^2 - 1}{x}\right) = -1$$

Thus, the required option (B) is the correct answer.

12. 4

We have, $f(x) = \dfrac{\sqrt{16 - x^2}}{\log_{x+2}\left(\left|\frac{x}{5}\right|\right)}$

We know that the base of a log cannot be 1. The value inside the square should be positive. Also, the argument of the log should be positive.

The argument $\left|\frac{x}{5}\right|$ is always positive.
So, $x \neq 0$

Also, $\left|\frac{x}{5}\right|$ it cannot be one as the denominator would become 0.
Hence, $x \neq 5$ and $x \neq -5$
Also, $16 - x^2 \geq 0 \Rightarrow -4 \leq x \leq 4$
Now, $x + 2 > 0 \Rightarrow x > -2$
Also, $x + 2 \neq 1 \Rightarrow x \neq -1$
Hence, $x \in (-2, -1) \cup (-1, 0) \cup (0, 4]$
The integral values of x are 1, 2, 3, 4.
Hence, there are four integral values of x for which $f(x)$ is defined.

13. (A)

$$f(a \times b) = \frac{f(a)}{f(b)}; a < b$$

Now $f(8) = 0.5$ (given)

$$\Rightarrow f(1 \times 8) = \frac{f(1)}{f(8)}; (1 < 8)$$

$$\Rightarrow f(8) = \frac{f(1)}{f(8)}$$

$$\Rightarrow f(1) = f(8).f(8)$$

$f(1) = 0.5 \times 0.5 = 0.25$
Now, for $f(1/8)$:

Also, $f\left(\frac{1}{8} \times 8\right) = \dfrac{f\left(\frac{1}{8}\right)}{f(8)}; \left(\frac{1}{8} < 8\right)$

$$\Rightarrow f(1) = \frac{f\left(\frac{1}{8}\right)}{f(8)}$$

$$\Rightarrow f\left(\frac{1}{8}\right) = f(1) \times f(8) = 0.25 \times 0.5 = 0.125$$

14. (C)

Since $f(x) = \frac{1-x^2}{1+x^2}$

Then, $f\left(\frac{1-x^2}{1+x^2}\right) = \frac{1-\left(\frac{1-x^2}{1+x^2}\right)^2}{1+\left(\frac{1-x^2}{1+x^2}\right)^2}$

$$= \frac{1-\frac{1+x^4-2x^2}{1+x^4+2x^2}}{1+\frac{1+x^4-2x^2}{1+x^4+2x^2}}$$

$$= \frac{\frac{1+x^4+2x^2-1-x^4+2x^2}{\left(1+x^4+2x^2\right)}}{\frac{1+x^4+2x^2+1+x^4-2x^2}{\left(1+x^4+2x^2\right)}}$$

$$= \frac{4x^2}{2x^4+2}$$

$$= \frac{2x^2}{x^4+1}$$

$$= \frac{2}{\frac{x^4}{x^2}+\frac{1}{x^2}}$$

So, $f\left(\frac{1-x^2}{1+x^2}\right) = \frac{2}{x^2+\frac{1}{x^2}}$

Hence option (C) is the correct answer.

15. (D)

When $a < 0$

Then, $g(a) = -a - a = -2a$

So, $g(a)$ will be a positive number. Therefore, all the positive numbers come in this range.

Also, if $a \geq 0$ then $g(a) = 0$

$\therefore$ The range of the function is $[0, \infty)$.

16. (B)

$f(x) = 3x - 1$

$\Rightarrow f(2x) = 3(2x) - 1 = 6x - 1$

Now, $f(f(2x)) = f(6x - 1) = 3(6x - 1) - 1 = 18x - 3 - 1 = 18x - 4$

For $f(f(f(2x)))$: $f(f(f(2x))) = f(18x - 4) = 3(18x - 4) - 1 = 54x - 12 - 1 = 54x - 13$

$\Rightarrow 54x - 13 = 365 \Rightarrow 54x = 378 \Rightarrow x = 7$

Hence, option (B) is the correct answer.

17. (D)

Given that $g(x + 1) + g(x - 1) = g(x)$

Putting $(x + 1)$ at the place of x gives us the following:

$g(x + 2) + g(x) = g(x + 1)$

Adding these two equations, we get

$g(x + 2) + g(x - 1) = 0$

$\Rightarrow g(x + 3) + g(x) = 0$

$\Rightarrow g(x + 4) + g(x + 1) = 0$

$\Rightarrow g(x + 5) + g(x + 2) = 0$

$\Rightarrow g(x + 6) + g(x + 3) = 0$

$\Rightarrow g(x + 6) - g(x) = 0.$

Hence, $g(x + 6) = g(x)$.

So, $p = 6$

Hence, option (D) is the correct answer.

18. (C)

If $f(-x) = f(x)$, then it is an even function.

After verifying the options, we can see option (C) is an even function.

(A) $f(x) = \log\left(\frac{1+x^5}{1-x^5}\right) \Rightarrow$

$$f(-x) = \log\left(\frac{1+(-x)^5}{1-(-x)^5}\right) = \log\left(\frac{1-x^5}{1+x^5}\right) \neq f(x)$$

(B) $f(x) = \frac{\left(1+3^x\right)^2}{3^x} \Rightarrow$

$$f(-x) = \frac{\left(1+3^{-x}\right)^2}{3^{-x}} \neq f(x)$$

(C) $f(x) = e^{x^{12}} \Rightarrow f(-x) = e^{(-x)^{12}} = e^{x^{12}} = f(x)$, is an even function.

19. (D)

Given $f(2) = 4$, $f(3) = 9$, $f(4) = 16$ and $f(5) = 25$

Thus $f(x) = x^2$

But given $f(x)$ is a fourth-degree polynomial with the coefficient of highest power as 3.

So, $f(x) = 3(x - 2)(x - 3)(x - 4)(x - 5) + x^2$

(as putting $x = 2, 3, 4$ or 5,
$f(x) = 0 + x^2 \Rightarrow f(x) = x^2$)
$= 3(x^4 - 14x^3 + 71x^2 - 154x + 120) + x^2$
Hence, $f(x) = 3x^4 - 42x^3 + 214x^2 - 462x + 360$.

20. (D)

As –15 and –8 are the roots of a parabolic (quadratic) function, we can write the function as $f(x) = (x + 8)(x + 15)$

$\Rightarrow f(x) = x^2 + 23x + 120$

From the question, $f(a) = f(b)$

$\Rightarrow a^2 + 23a + 120 = b^2 + 23b + 120$

$\Rightarrow a^2 - b^2 = 23(b - a)$

$\Rightarrow (a + b)(a - b) = 23(b - a)$

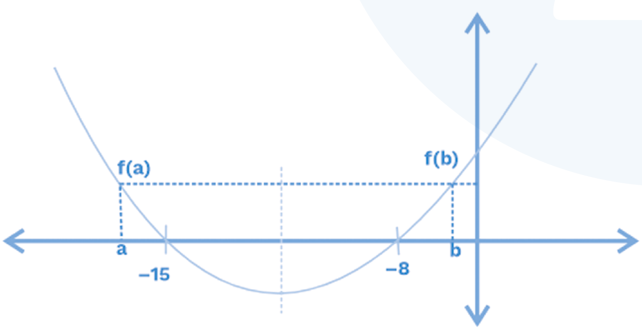
As $a \neq b$

$\Rightarrow a + b = -23$.

Alternate approach

From the question, we can understand that the function is quadratic and $x = -15$ and $x = -8$ are its roots.

From the following figure, we can comprehend that 'a' is as far from –15 as 'b' is from –8. This can be generalised for any parabola or quadratic function.



It is mentioned in the question that $b > a$, so we can comprehend that $b > -8 > -15 > a$.

So, $-15 - a = b - (-8)$

Or, $a + b = -23$.

21. 11

We have:

$f(2) = f(5) = f(8) = 1$ and $f(11) = 163$

Therefore, $f(x)$ can be given by:

$f(x) = (x - 2)(x - 5)(x - 8) + 1$
Now, $f(3) = (3 - 2)(3 - 5)(3 - 8) + 1$
$f(3) = (1) \times (-2) \times (-5) + 1$
$f(3) = 11$
Hence, 11 is the correct answer.

22. (A)

$f(a^2 - a) = a^4 - 2a^3 + 5a^2 - 4a - 5$
$= \underbrace{a^4 - 2a^3 + a^2} + 4a^2 - 4a - 5$
$= (a^2 - a)^2 + 4(a^2 - a) - 5$
Let $a^2 - a = x$
$f(x) = x^2 + 4x - 5$
$= x^2 + 5x - x - 5$
$= x(x + 5) - 1(x + 5)$
$f(x) = (x - 1)(x + 5)$
$\therefore f(f(x)) = (x^2 + 4x - 5 - 1) \times (x^2 + 4x - 5 + 5)$
$= (x^2 + 4x - 6) \times (x^2 + 4x)$
Also, let $a^2 - a = y$
Then, $f(y) = y^2 + 4y - 5$

Then, $\dfrac{f(f(x))}{f(y)} = \dfrac{(x^2 + 4x - 6) \times (x^2 + 4x)}{y^2 + 4y - 5}$

$\therefore \dfrac{f(f(10))}{f(10)} = \dfrac{(10^2 + 4 \times 10 - 6) \times (10^2 + 4 \times 10)}{(10^2 + 4 \times 10 - 5)}$

$= \dfrac{134 \times 140}{135} = 138.9629 \approx 139$

Hence, option (A) is correct answer.

23. (C)

$f(x + 2) = 2f(x) - f(x + 1)$
Let $f(11) = K$
$\Rightarrow f(12) = 2f(10) - f(11) = 40 - K$
Also, $f(13) = 2f(11) - f(12) = 2K - 40 + K$
$\Rightarrow f(13) = 3K - 40$
$f(14) = 2f(12) - f(13) = 80 - 2K - 3K + 40$
$= 120 - 5K$
Now, $f(15) = 2f(13) - f(14) = 6K - 80 - 120 + 5K$
$\Rightarrow 108 = 11K - 200$
$\Rightarrow 11K = 308$
$\Rightarrow K = 28$
Therefore, $f(13) = 3K - 40 = 84 - 40 = 44$.

24. (D)

Here, from $\left[\sqrt[3]{1}\right]$ to $\left[\sqrt[3]{7}\right]$ of each term value is 1, total sum = $1 \times 7 = 7$

From $\left[\sqrt[3]{8}\right]$ to $\left[\sqrt[3]{26}\right]$, value of each term is 2, total sum = 2 × 19 = 38

From $\left[\sqrt[3]{27}\right]$ to $\left[\sqrt[3]{63}\right]$, value of each term is 3, total sum = 3 × 37 = 111

From $\left[\sqrt[3]{64}\right]$ to $\left[\sqrt[3]{124}\right]$, value of each term is 4, total sum = 4 × 61 = 244

From $\left[\sqrt[3]{125}\right]$ to $\left[\sqrt[3]{150}\right]$, value of each term is 5, total sum = 5 × 26 = 130

Therefore, required sum = 7 + 38 + 111 + 244 + 130 = 530.

25. (A)

We have: $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and

$$g(x) = \frac{3x+x^3}{1+3x^2}$$

$$\therefore\ fog(x) = f\left[g(x)\right] = f\left(\frac{3x+x^3}{1+3x^2}\right)$$

$$= \log\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right) = \log\left(\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}\right)$$

$$= \log\left(\frac{(1+x)^3}{(1-x)^3}\right) = \log\left(\frac{1+x}{1-x}\right)^3$$

$$= 3\log\left(\frac{1+x}{1-x}\right)$$

$$= 3f(x)$$

26. (A)

Total terms in $f(x)$ = 48

Product → $(x + 5)(x + 7)(x + 9) \ldots (x + 97)(x + 99) < 0$.

When

Positive Terms	Negative Terms
47	1 when x = -6
45	3 when x = -10
. . .	. . .
1	47 when x = -98

So total valid cases will be = 24.

27. (C)

$f(x) + f(1 - x) = 8$

$$f\left(\frac{1}{100}\right) + f\left(1-\frac{1}{100}\right) = 8$$

$$f\left(\frac{1}{100}\right) + f\left(\frac{99}{100}\right) = 8$$

Similarly, $f\left(\frac{2}{100}\right) + f\left(\frac{98}{100}\right) = 9$ and so on.

There are 99 terms, out of which we can make 49 pairs of value 8 each.

One middle term $f\left(\frac{50}{100}\right)$ will be left.

$$f\left(\frac{50}{100}\right) = f\left(\frac{1}{2}\right)$$

So total value will be = 49 × 8 + $f\left(\frac{1}{2}\right)$

Now $f(x) + f(1 - x) = 8$.

Put $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) + f\left(1-\frac{1}{2}\right) = 8$$

$$2f\left(\frac{1}{2}\right) = 8$$

$$f\left(\frac{1}{2}\right) = 4$$

Hence, total value = 49 × 8 + $f\left(\frac{1}{2}\right)$

= 49 × 8 + 4

= 392 + 4 = 396.

28. 13,825

When $n(A) = 6$, and $n(B) = 5$

The total number of functions from A to B is given by $n(B)^{n(A)} = 5^6$

The number of onto functions from A to B is given by

$$n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \cdots$$

Here, n = 5, m = 6

$= 5^6 - {}^5C_1(4)^6 + {}^5C_2(3)^6 - {}^5C_3(2)^6 + {}^5C_4(1)^6$

$= 5^6 - 5(4{,}096) + 10(729) - 10(64) + 5$

$= 5^6 - 13{,}825$

$\therefore$ The required number of functions = $5^6 - \{5^6 - 13{,}825\} = 13{,}825$.

29. 43

We have, $f(3 - x) = f(3 + x)$ ....(i)

So, the function is symmetric about 3.

$f(10 - x) = f(10 + x)$ ....(ii)

Hence, it is also symmetric about 10.

Now, if $f(3) = 0$, then from 2, $f(10 - 7) = f(10 + 7) \Rightarrow f(3) = f(17) = 0$

Using 1 again, $f(17) = f(3 - (14)) = f(3 - 14) = f(-11) = 0$

All the roots with such as $f(-11)$, $f(3)$, $f(17)$ are 0.

$f(-25)$, $f(-11)$, $f(3)$, $f(17)$, $f(31)$ are equal to 0.

So the roots are a series of arithmetic progression with common difference = 14

nth term = $a + (n - 1)d$

$\Rightarrow$ $300 > 3 + (n - 1) \times 14$

$n - 1 < \dfrac{297}{14}$

$n < 22.22$

$n = 22$

$\Rightarrow$ $-300 < 3 + (n - 1)(-14)$

$= n < 22.6$

So, $n = 22$

Since 3 has been counted twice, hence minimum number of roots

$= 22 + 22 - 1 = 43$.

30. (C)

$\left(\dfrac{1}{4}\right)^x$ decrease as x increases ...(i)

Let $x^2 - 2x - 24 = \left(\dfrac{1}{4}\right)^a$ and $11 = \left(\dfrac{1}{4}\right)^b$

Given $a > b$. $\therefore$ (1) $\Rightarrow x^2 - 2x - 24 < 11$

$\therefore$ $x^2 - 2x - 35 < 0 \Rightarrow (x - 7)(x + 5) < 0$

$\therefore$ $-5 < x < 7$ ...(ii)

For $\log_{\frac{1}{4}}\left(x^2 - 2x - 24\right)$ to be defined,

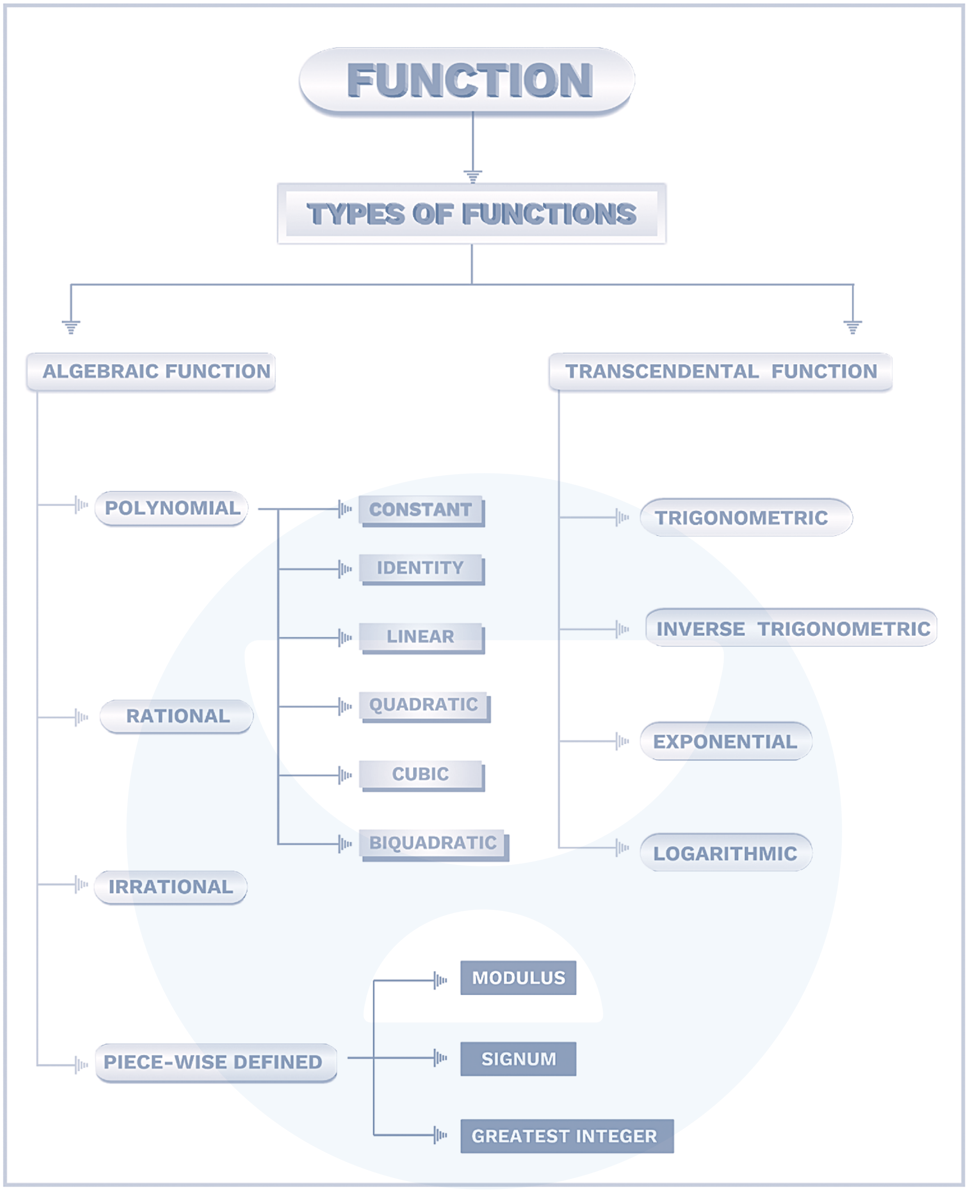
$x^2 - 2x - 24$ must be positive

i.e. $(x - 6)(x + 4)$ must be positive

$\therefore$ $x < -4$ or $x > 6$ ...(iii)

From equations (ii) and (iii), the range of x is $(-5, -4) \cup (6, 7)$.

Mind Map


FUNCTION
TYPES OF FUNCTIONS
ALGEBRAIC FUNCTION
TRANSCENDENTAL FUNCTION
POLYNOMIAL
CONSTANT
IDENTITY
LINEAR
QUADRATIC
CUBIC
BIQUADRATIC
RATIONAL
IRRATIONAL
PIECE-WISE DEFINED
MODULUS
SIGNUM
GREATEST INTEGER
TRIGONOMETRIC
INVERSE TRIGONOMETRIC
EXPONENTIAL
LOGARITHMIC