3 Inequalities and Modulus

Introduction

The concepts related to inequalities and modulus are repeatedly asked in CAT, XAT, and other management exams. One can expect one or two questions from this topic. One must consider the concept of inequalities as a tool to ease problem solving and score well in an exam.

Definition

When two mathematical quantities x and y are not equal, then either x is greater than y (denoted as $x > y$) or x is less than y (denoted as $x < y$). The relation 'x is not equal to y' is known as inequality or inequation. For example, $4x + 3 \geq 0$, $5x - 3 > 0$, $x^2 + 3x + 2 > 0$, and $x^4 + 7x^3 + 13x^2 + 19 > 0$ are all inequalities.

Sign Notation

'>' means 'greater than'.

'<' means 'less than'.

'≥' means 'greater than or equal to'.

'≤' means 'less than or equal to'.

Interval Notation

Understanding the interval notation is important because one can use this to express where a set of solutions begins and where it ends.

- (P, Q) means all the real numbers between P and Q, excluding P and Q. For example, if $x \in (2, 8)$, x can take all the real numbers between 2 and 8.
- [P, Q] means all the real numbers between P and Q, including P and Q. For example, if $x \in [2, 8]$, it means x can be 2 or 8 or can take all the real numbers between 2 and 8.
- [P, Q) means all the numbers between P and Q where P must be included, and Q must be excluded.
  For example, if $x \in [2, 8)$, it means x can be 2 or take all the real numbers between 2 and 8, but x cannot be 8.
- (P, Q] means all the numbers between P and Q where Q must be included, and P must be excluded.
  For example, if $x \in (2, 8]$, it means x can be 8 or can take all the real numbers between 2 and 8, but x cannot be 2.

Critical Points

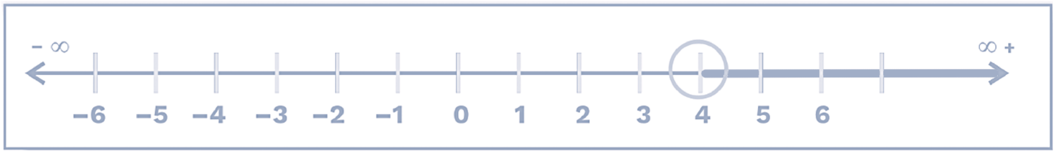
Those points at which given inequality becomes zero are known as critical points. For example, the critical points of the given inequality $(x - 4)(x - 2)(x + 1) > 0$ are (4, 2, and –1).

Finding the Solution Set of Inequalities

To find the solution for a set of inequalities, it is very useful to view the inequalities on the number line.

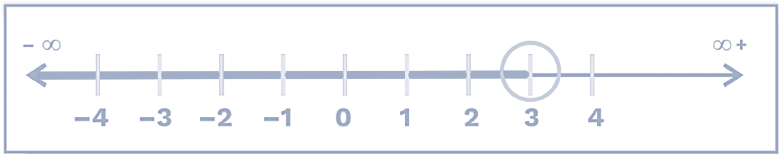
In the following figures, the thick line refers to the acceptable values. The empty circle over any value means that the value is not acceptable. The filled circle over any value means that the value is acceptable.

1. $x > 4$



The solution set of the given inequation is (4, ∞). An empty circle over 4 means $x = 4$ is not acceptable.

2. $x < 3$



The solution set of the given inequation is (–∞, 3). An empty circle over 3 means $x = 3$ is not acceptable.

3. $x \geq -2$

The solution set of the given inequation is [–2, ∞). A filled circle over –2 means $x = -2$ acceptable.

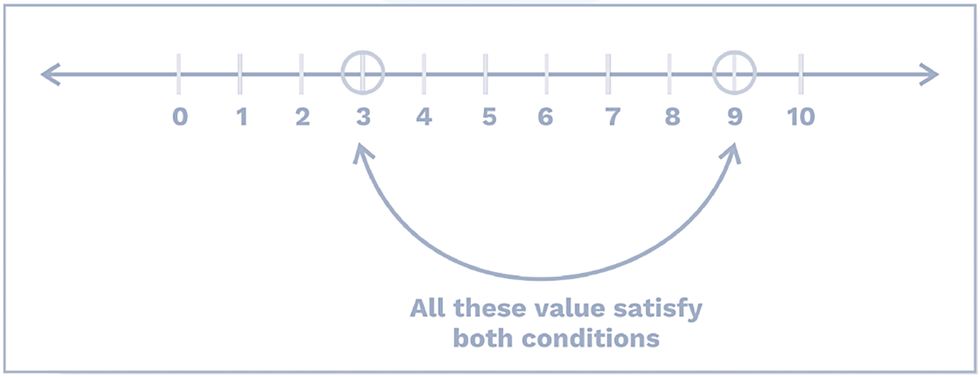
4. $x \leq 3$

The solution set of the given inequation is (–∞, 3]. A filled circle over 3 means $x = 3$ is acceptable.

Before moving on, let's understand the 'or' and 'and' situations.

Case of 'AND'

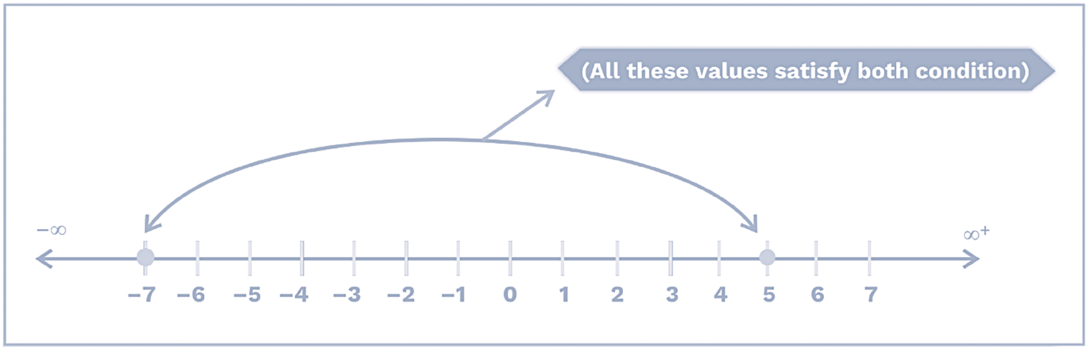
1. $x > 3$ and $x < 9$



Both the inequalities are combined into one single relation as ($3 < x < 9$).

The solution set of the given inequation will be (3, 9).

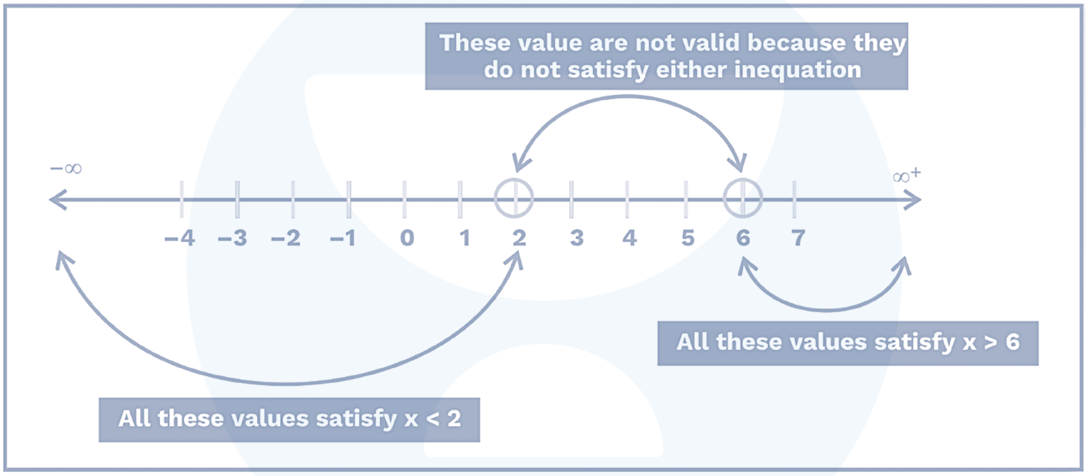
2. **$x \geq -7$ and $x \leq 5$**



Both the inequalities are combined into one single relation as $-7 \leq x \leq 5$ and the solution set of the given inequation will be $[-7, 5]$.

Case of 'OR'

1. **$x < 2$ or $x > 6$**



The solution set of the given inequation can be written as $(-\infty, 2) \cup (6, \infty)$. Here, the symbol '$\cup$' indicates union, i.e., OR.

Sign Changes in Inequalities

Rule 1

The same number added to (or subtracted from) both sides of an inequation will not change the sign of inequality. if $a > b$, then, adding 'P' to either sides or subtracting 'P' does not affect the inequality, i.e., the inequality remains the same for 'P' is positive or negative.

$\therefore$ $(a \pm P > b \pm P)$

As $6 > 4$, $6 + 2 > 4 + 2$, and $6 - 2 > 4 - 2$

Example 1:

Solve for x; $4x - 6 > 3x + 2$.

Solution:

When we transpose −6 to the RHS, we are adding 6 to both sides.

$\therefore$ $4x - 6 + 6 > 3x + 2 + 6$

$\therefore$ $4x > 3x + 8$

Here, adding 6 to both sides does not affect the inequality.

Similarly, transposing $3x$ on LHS simply means subtracting $3x$ from both sides.

$4x - 3x > 3x + 8 - 3x$

$x > 8$

Here, subtracting $3x$ from both sides does not affect the inequality.

Hence, the solution set of the given inequation is $(8, \infty)$.

Rule 2

There is no change in the sign of inequality when both sides of an inequation are multiplied or divided by the same positive real number.

The inequality sign is reversed only when both sides of an inequation are multiplied or divided by a negative number.

If $a > b$

1. When P is positive.

   $a \times P > b \times P$ (inequality remains the same)

   and $\frac{a}{P} > \frac{b}{P}$ (inequality remains the same)

2. When P is negative.

   $a \times P < b \times P$ (inequality sign reversed)

   $\frac{a}{P} < \frac{b}{P}$ (inequality sign reversed)

   Let's understand this with an example.

Example 2:

Solve for x; $\frac{4x-7}{x} > 3$ if $x \neq 0$

Solution:

It is given that, $\frac{4x-7}{x} > 3$

Here, the polarity of x (denominator) is unknown.

Two cases arise.

Case 1:

If 'x' is negative, i.e., $x < 0$, then we have $4x - 7 < 3x \Rightarrow x < 7$

The solution set of $x < 0$ and $x < 7$ is $(-\infty, 0)$.

Case 2:

If 'x' is positive, i.e., $x > 0$, then we have $4x - 7 > 3x \Rightarrow x > 7$.

The solution set of $x > 0$ and $x > 7$ is $(7, \infty)$.

The combined solution set of both cases are $(-\infty, 0) \cup (7, \infty)$.

Example 3:

Solve for x; $x^2 < 5x$.

Solution:

At any real value of x, LHS is always positive, but RHS may be positive or negative, depending upon the polarity of x.

Here, two cases arise.

Case 1:

If x is negative, i.e., $x < 0$ dividing both sides by $(-x)$ we get,

$$\frac{x^2}{-x} > \frac{5x}{-x}$$

$\Rightarrow x > 5$

The solution set of $x < 0$ and $x > 5$ is (ϕ) (null), i.e., no solution.

Case 2:

If 'x' is positive, i.e., $x > 0$ dividing both sides by $(+x)$ we get,

$$\frac{x^2}{x} < \frac{5x}{x}$$

$\Rightarrow x < 5$

The solution set of $x > 0$ and $x < 5$ is $(0, 5)$. This can be written as $0 < x < 5$.

Rule 3

Any term of an inequation can be taken to the other side without any change in the sign of the inequality.

Rule 4

In a given inequality, we cannot square or take the square root of both sides. Doing so, one will get the wrong solution set.

For example, if a and b are positive and assume that $a > b$; squaring both sides results in $a^2 > b^2$ (no sign change).

If 'a' and 'b' are negative, then squaring both sides results in $a^2 < b^2$ (inequality sign changed).

If 'a' is positive and 'b' is negative or 'a' is negative and 'b' is positive, then the sign of inequality changes based on the magnitude of 'a' and 'b'.

Let's understand some examples based on the above learning.

Example 4:

Solve for x; $3x - 6 \leq 0$.

Solution:

It is given that, $3x - 6 \leq 0$

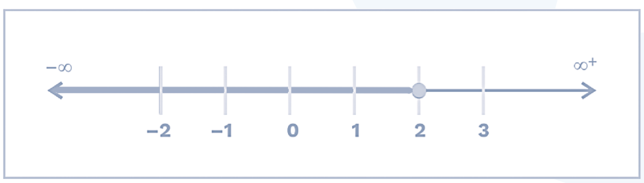
(Adding 6 on both sides) OR [transfer (-6) to RHS]

$\Rightarrow 3x \leq 6$

$\Rightarrow x \leq 2$

Hence, any real number less than or equal to 2 is a solution of the given inequation.

Graphically



The solution set is $(-\infty, 2]$.

Example 5:

Solve for x; $4x - 3 < 3x + 1$

When (i) x is a real number.

(ii) x is a natural number.

Solution:

It is given that, $4x - 3 < 3x + 1$

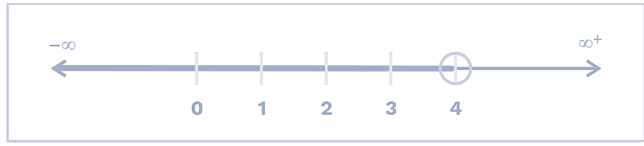
$\Rightarrow \quad 4x - 3x < 1 + 3$

$\Rightarrow \quad x < 4$

(i) If $x \in$ R, then

$x < 4$,

$\Rightarrow x \in (-\infty, 4)$



The solution set is $(-\infty, 4)$.

(ii) If $x \in$ N, then

$x \in$ N $\Rightarrow x = 1, 2, 3$

Example 6:

Solve for x; $\frac{4x+3}{5} + 7 \geq 5 + \frac{6x}{7}$

Solution:

It is given that, $\frac{4x+3}{5} + 7 \geq 5 + \frac{6x}{7}$

$\Rightarrow \frac{4x+3}{5} - \frac{6x}{7} \geq 5 - 7$

$\Rightarrow \frac{28x + 21 - 30x}{35} \geq -2$

$\Rightarrow -2x + 21 \geq -70$

$\Rightarrow -2x \geq -70 - 21$

$2x \leq 91$

$x \leq \frac{91}{2}$

The solution set of the given inequation is $\left(-\infty, \frac{91}{2}\right]$.

Example 7:

Find the range of real values of x satisfying the inequalities $3x + 2 \geq 5$ and $4x + 2 \leq 14$.

Solution:

It is given that,

$3x + 2 \geq 5$ and $4x + 2 \leq 14$

$3x + 2 \geq 5 \Rightarrow 3x \geq 3 \Rightarrow x \geq 1$

$4x + 2 \leq 14 \Rightarrow 4x \leq 12 \Rightarrow x \leq 3$

The required solution set of $x \geq 1$ and $x \leq 3$ is $[1, 3]$.

Wavy Curve Method

The wavy curve method also called the method of intervals, is used to solve inequalities involving a variable 'x'.

This method helps one to reach the solution set quickly.

To understand this concept, one should use the following algorithm:

1. Transpose all terms from RHS to LHS (i.e., RHS = 0).
2. Factorise the expression into as many linear factors as possible on LHS.
3. Make coefficient 'x' positive on LHS if it is not.
4. Identify the critical points, plot the critical points on the number line, and determine the regions.
5. Mark the right-most region positive and start moving from right to left and mark an alternatively negative and positive sign in other regions.
6. Select an appropriate region based on the sign of the inequation, i.e., if there is a '$\geq$' or '$>$' sign, then one should always take care of positive regions. If there is an '$\leq$' or '$<$' sign, one should always take care of negative regions.
7. Write these regions in the form of intervals to obtain the desired solution sets of the given inequation.

Let's understand the above learning with an example.

Example 8:

Solve the inequality for x; $x^2 - 11x + 24 \geq 0$.

Solution:

It is given that, $x^2 - 11x + 24 \geq 0$

Here, RHS is already equal to zero.

On factorising the given inequation, we get

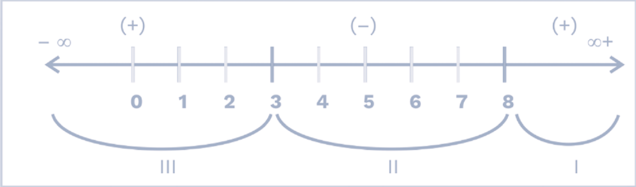
$x^2 - 11x + 24 \geq 0$

$x^2 - 8x - 3x + 24 \geq 0$

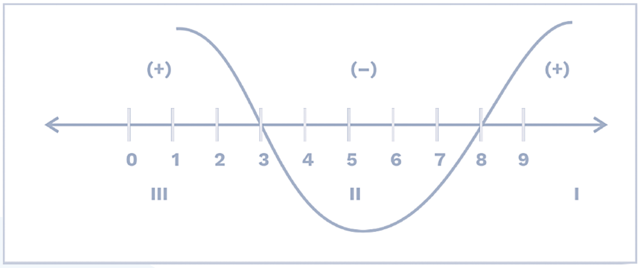
$\Rightarrow \quad x(x - 8) - 3(x - 8) \geq 0$

$\therefore \quad (x - 8)(x - 3) \geq 0$

Critical points of the given inequality are 8, 3.



Here, these critical points divide the number line into three regions, I, II, and III, where (I) is positive, (II) is negative, and (III) is positive.



Here, the sign of inequality is ($\geq$), so one should only take all the positive regions.

The required solution set of the given inequality will be $(-\infty, 3] \cup [8, \infty)$.

Let's understand some more examples:

Example 9:

Solve for x; $\dfrac{x-3}{x-2} \leq 3$

Solution:

It is given that, $\dfrac{x-3}{x-2} \leq 3$

$$\Rightarrow \frac{x-3}{x-2} - 3 \leq 0$$

$$\Rightarrow \frac{x-3-3x+6}{x-2} \leq 0$$

$$\Rightarrow \frac{-2x+3}{(x-2)} \leq 0$$

$$\Rightarrow \frac{2x-3}{x-2} \geq 0$$

Here, it is not known about the polarity of $(x - 2)$, so multiply and divide the LHS by $(x - 2)$, we get

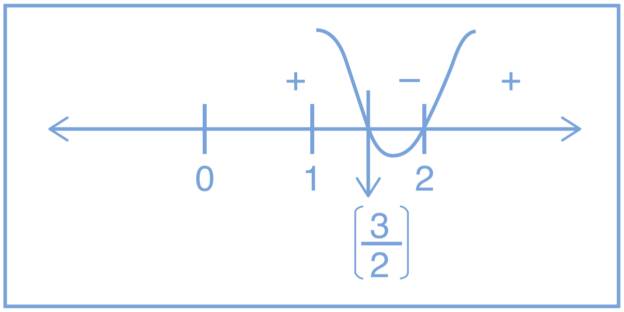
$$\Rightarrow \frac{(2x-3)}{(x-2)} \times \frac{(x-2)}{(x-2)} \geq 0$$

$$\Rightarrow \frac{(2x-3)(x-2)}{(x-2)^2} \geq 0$$

(As $(x-2)^2$ is always positive, multiplying both the sides by $(x-2)^2$ will not change the sign of inequality)

$\Rightarrow (2x-3)(x-2) \geq 0$

Required critical points $= \frac{3}{2}, 2 \;(x \neq 2)$



The required solution set of the given inequation $\left(-\infty\ \frac{3}{2}\right] \cup [2, \infty)$.

Example 10:

If $\frac{2x^2-4x+2}{x^2-3x-10} < 0$, then find the range of x.

Given that $x \in R$, $x \neq -2, 5$.

Solution:

It is given that, $\frac{2x^2-4x+2}{x^2-3x-10} < 0$

$\Rightarrow \frac{(x-1)(2x-2)}{(x-5)(x+2)} < 0$

Here, the polarity of $(x-5)(x+2)$ is not known, so multiply and divide the LHS by $(x-5)$ and $(x+2)$, we get,

$\Rightarrow \frac{(x-5)(x+2)(x-1)(2x-2)}{(x-5)^2(x+2)^2} < 0$

$\Rightarrow (x-5)(x+2)(x-1)(2x-2) < 0$

$\Rightarrow (x-5)(x+2)(x-1)(x-1) < 0$

$\Rightarrow (x-5)(x+2)(x-1)^2 < 0$

Now, $(x-1)^2$ is always positive

So, $(x-5)(x+2) < 0$

Critical points are $= -2$ and 5

Also at $x = 1$, the expression will be zero. So, that is not a part of the answer.

Hence, the required solution set of the given inequality is $(-2, 5) - \{1\}$

Example 11:

Solve for x; $(x-5)^2 \times (x-3) \times (x+2) > 0$.

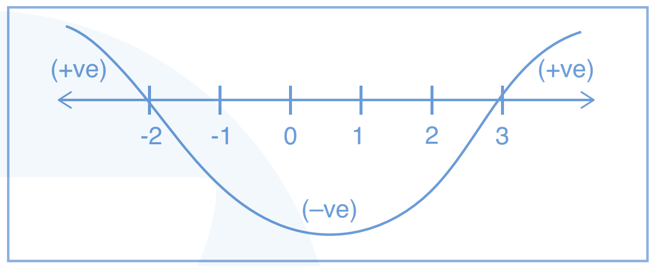
Solution:

It is given that $(x-5)^2 \times (x-3) \times (x+2) > 0$.

The polarity of $(x-5)^2$ will always be positive. To satisfy the given inequality $(x-3) \times (x+2)$ must be positive.

But at $x = 5$, the expression in the question becomes equal to zero, whereas the question requires the expression to be strictly greater than zero.

Critical points are $= (-2, 3)$.



The solution set of the given inequality is $(-\infty, -2) \cup (3, \infty) - \{5\}$.

Note: Whenever power is even, the entire base becomes irrelevant.

Example 12:

Solve for x; $(x-5)^3 \times (x-3) \times (x+2) < 0$.

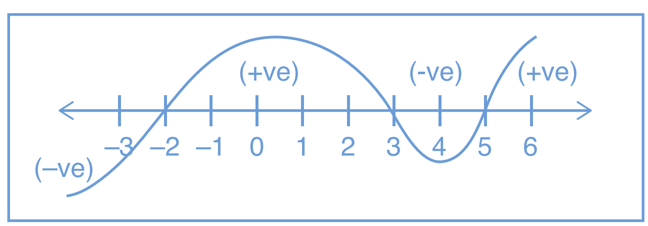
Solution:

It is given that, $(x-5)^3 \times (x-3) \times (x+2) < 0$.

Here, the polarity of $(x-5)^3$ is the same as that of $(x-5)$.

So, the given inequality can be considered as $(x-5) \times (x-3) \times (x+2) < 0$.

Critical points are $= (-2, 3, 5)$.



The solution set of the given inequality is $(-\infty, -2) \cup (3, 5)$.

Note: *The total power is irrelevant whenever the power is odd,* and the base becomes important.

Example 13:

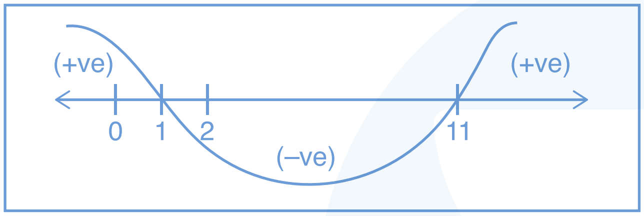
Solve for x; $(x-1)^5 \times (x+10)^6 \times (x-11)^7 \times (x+9)^8 < 0$.

Solution:

It is given that, $(x-1)^5 \times (x+10)^6 \times (x-11)^7 \times (x+9)^8 < 0$.

Here, $(x+10)^6$ and $(x+9)^8$ become irrelevant and given inequality can be written as $(x-1) \times (x-11) < 0$.

Critical points are = (1, 11).



The solution set of the given inequality is (1, 11).

Example 14:

Find the range of x for which

$$\frac{(x+7)^7 \times (x-11)^{10}}{(x-12)^{18} \times (x-7)^{11}} < 0.$$

Solution:

It is given that, $\frac{(x+7)^7 \times (x-11)^{10}}{(x-12)^{18} \times (x-7)^{11}} < 0$

Here, $(x-11)^{10}$ and $(x-12)^{18}$ become irrelevant, so ($x \neq 11$ and 12).

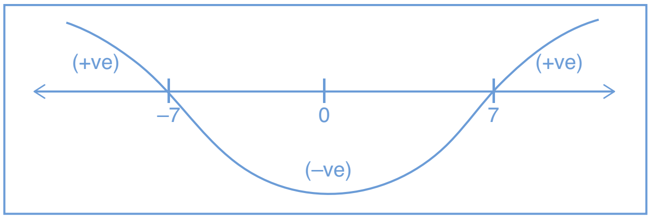
Given inequality can be written as $\frac{(x+7)}{(x-7)} < 0$

$$\Rightarrow \frac{(x+7)(x-7)}{(x-7)^2} < 0$$

$\because$ The polarity of $(x-7)$ is not known.

Now, $(x-7)^2$ becomes irrelevant, so $x \neq 7$.

Critical points are = (–7, 7).



The solution set of the given inequality is (–7, 7).

Absolute Value (Modulus)

The absolute value (modulus) of a real number 'x' is the non-negative value of x without regard to its sign. The absolute value of a real number 'x' is written as $|x|$ and read as 'modulus of x', or 'mod of x', or 'mod x'.

For any real number 'x', the absolute value is defined as:

$$|x| = \begin{Bmatrix} x & \text{if} & x \geq 0 \\ -x & \text{if} & x < 0 \end{Bmatrix}$$

Some Important Results

1. If 'a' is a positive real number, then

a) $|x| < a \quad \Rightarrow \quad -a < x < a$ i.e. $x \in (-a, a)$

b) $|x| \leq a \quad \Rightarrow \quad -a \leq x \leq a$ i.e. $x \in [-a, a]$

Proof: We know that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Case 1:

When $x \geq 0$

In this case $|x| = x$.

$\therefore \quad |x| < a \quad \Rightarrow \quad x < a$

Required solution set = $0 \leq x < a$ ...(i)

Case 2: When $x < 0$

In this case $|x| = -x$

$\therefore \quad |x| < a \quad \Rightarrow \quad -x < a$

$\therefore \quad x > -a$

Required solution set = $-a < x < 0$ ...(ii)

From equations (i) and (ii), we will get

$|x| < a \Rightarrow -a < x < a$

Similarly, we get the same result for $|x| \leq a$ $\Rightarrow -a \leq x \leq a$.

2. If a is a positive real number, then

a) $|x| > a \Rightarrow x < -a$ or $x > a$

b) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$

Proof:

Case 1:

When $x \geq 0$

$|x| = x$

$\therefore$ $|x| > a \Rightarrow x > a$ ($\because$ $a > 0$)

Required solution set $= x > a$ ...(i)

Case 2

When $x < 0$

$|x| = -x$

$\therefore$ $|x| > a \Rightarrow -x > a \Rightarrow x < -a$

Required solution set $= x < -a$ ...(ii)

From equations (i) and (ii), we get,

$|x| > a \Rightarrow (x < -a$ or $x > a)$

3. Let 'b' be a positive real number and 'a' be a fixed real number, then

a) $|x - a| < b \Leftrightarrow a - b < x < a + b$

i.e. $x \in (a - b, a + b)$

b) $|x - a| \leq b \Leftrightarrow a - b \leq x \leq a + b$

i.e. $x \in [a - b, a + b]$

c) $|x - a| > b \Leftrightarrow x < a - b$ or $x > a + b$

d) $|x - a| \geq b \Leftrightarrow x \leq a - b$ or $x \geq a + b$

Proof:

We have,

$|x - a| < b \Leftrightarrow -b < x - a < b$

$\Rightarrow a - b < x < a + b \rightarrow$ Proved (i)

Now

$|x - a| \leq b \Leftrightarrow -b \leq x - a \leq b$

$\Rightarrow a - b \leq x \leq a + b \rightarrow$ Proved (ii)

Now $|x - a| > b \Leftrightarrow x - a < -b$ or $x - a > b$

$\Rightarrow x < a - b$ or $x > a + b \rightarrow$ Proved (iii)

Now $|x - a| \geq b \Leftrightarrow x - a \leq -b$ or $x - a \geq b$

$\Leftrightarrow x \leq a - b$ or $x \geq a + b \rightarrow$ Proved (iv)

4. Let 'a' and 'b' be positive real numbers, then

a) $a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

b) $a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

c) $a \leq |x - c| \leq b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$

d) $a < |x - c| < b \Leftrightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$

Proof:

$a < |x| < b$

$|x| > a$ and $|x| < b$

$|x| > a \Rightarrow (x < -a$ or $x > a)$

$|x| < b \Rightarrow (-b < x < b)$

$\Rightarrow x \in (-b, -a) \cup (a, b)$ [Proved (i)]

Similarly, one can prove other results as well.

Properties of Modulus

For any real number x and y.

1. $|x| \geq 0$ (non-negativity)

2. $x = 0 \Leftrightarrow |x| = 0$ (positive-definiteness)

3. $|xy| = |x|\,|y|$ (multiplicativity)

4. $|x + y| \leq |x| + |y|$ (sub additivity)

5. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} (y \neq 0)$

6. $\big||x| - |y|\big| \leq |x - y|$

7. $|x|^2 = x^2$

Let's understand the above learning with the help of some examples.

Example 15:

Solve the inequality $|2x - 3| \leq \frac{1}{2}$.

Solution (Basic approach):

$$|2x - 3| \leq \frac{1}{2}$$

Case 1: $2x - 3 > 0$

$$\therefore \quad 2x - 3 \leq \frac{1}{2} \Rightarrow 2x \leq \frac{7}{2}$$

$$x \leq \frac{7}{4}$$

Case 2: $2x - 3 < 0$

$$\therefore\ -(2x-3) \le \frac{1}{2} \Rightarrow 2x-3 \ge -\frac{1}{2}$$

$$\Rightarrow 2x \ge \frac{5}{2} \Rightarrow x \ge \frac{5}{4}$$

$$x \in \left[\frac{5}{4}, \frac{7}{4}\right]$$

Time-saving approach

It is known that,

$$|x-a| \le b \Leftrightarrow a-b \le x \le a+b$$

$$\therefore\ |2x-3| \le \frac{1}{2} \Rightarrow 3-\frac{1}{2} \le 2x \le 3+\frac{1}{2}$$

$$\Rightarrow \frac{5}{2} \le 2x \le \frac{7}{2} \Rightarrow \frac{5}{2}\times\frac{1}{2} \le 2x\times\frac{1}{2} \le \frac{7}{2}\times\frac{1}{2}$$

$$\Rightarrow \frac{5}{4} \le x \le \frac{7}{4} \Rightarrow x \in \left[\frac{5}{4}, \frac{7}{4}\right]$$

The solution set of the given inequation is $\left[\frac{5}{4}, \frac{7}{4}\right]$

Example 16:

Solve the inequality $|x - 4| \ge 9$.

Solution:

It is known that,

$$|x-a| \ge b \Leftrightarrow x \le a-b \text{ or } x \ge a+b$$

$$\therefore\ |x-4| \ge 9 \Rightarrow x \le 4-9 \text{ or } x \ge 4+9$$

$$\Rightarrow x \le -5 \text{ or } x \ge 13$$

$$x \in (-\infty, -5] \text{ or } x \in [13, \infty)$$

The solution set of the given inequality $(-\infty, -5] \cup [13, \infty)$

Example 17:

Solve $\dfrac{|x|-2}{|x|-3} \ge 0\ \ x \in R,\ \ x \ne \pm 3$.

Solution:

It is given that $\dfrac{|x|-2}{|x|-3} \ge 0$

Let $|x| = y$

$$\Rightarrow \frac{y-2}{y-3} \ge 0$$

$$\Rightarrow \frac{(y-3)(y-2)}{(y-3)^2} \ge 0$$

$$\Rightarrow y \le 2 \text{ or } y > 3$$

$$\Rightarrow |x| \le 2 \text{ or } |x| > 3$$

$$\Rightarrow (-2 \le x \le 2) \text{ or } (x < -3 \text{ or } x > 3)$$

Combined range of x is

$$x \in [-2, 2] \cup (-\infty, -3) \cup (3, \infty)$$

The solution set of the given inequality is $(-\infty, -3) \cup [-2, 2] \cup (3, \infty)$.

Example 18:

Solve the inequation: $\left|\dfrac{3}{x-5}\right| > 1, x \ne 5$.

Solution:

It is given that,

$$\left|\frac{3}{x-5}\right| > 1, x \ne 5$$

$$\Rightarrow \frac{|3|}{|x-5|} > 1$$

$$\Rightarrow 3 > |x-5|$$

$$\Rightarrow 5-3 < x < 5+3$$

$$\Rightarrow 2 < x < 8 \Rightarrow x \in (2, 8) \text{ But } x \ne 5$$

The solution set of the given inequation $(2, 5) \cup (5, 8)$.

The solution set can also be written as $(2, 8) - \{5\}$.

Example 19:

Solve the inequation; $|x - 2| + |x - 3| \ge 5$.

Solution:

It is given that, $|x - 2| + |x - 3| \ge 5$

On the LHS side, both the terms contain modulus. By equating the expression within the modulus to zero, we get two critical points (2) and (3).

These points divide the number line into three regions, i.e., $(-\infty, 2)$, $[2, 3)$ $[3, \infty)$.

So, three cases arise.

Case 1:

When $-\infty < x < 2$

$|x - 2| = -(x - 2)$ and $|x - 3| = -(x - 3)$

$\therefore\ |x-2|+|x-3| \geq 5$

$\Rightarrow -(x-2)-(x-3) \geq 5$

$\Rightarrow -2x+5 \geq 5$

$\Rightarrow -2x \geq 0$

$x \leq 0$

Solution set is $(-\infty, 0]$. ...(i)

Case 2:

When $2 \leq x < 3$

In this case $|x-2| = (x-2)$ and $|x-3| = -(x-3)$

$\therefore\ |x-2|+|x-3| \geq 5$

$x-2-(x-3) \geq 5$

$1 \geq 5$ which is not possible.

Case 3:

When $x \geq 3$

$|x-2| = (x-2)$ and $|x-3| = (x-3)$

$\therefore\ |x-2|+|x-3| \geq 5$

$\Rightarrow x-2+x-3 \geq 5$

$\Rightarrow 2x-5 \geq 5$

$\Rightarrow 2x \geq 10$

$\Rightarrow x \geq 5$

Solution set is $[5, \infty)$. ...(ii)

On combining solution set (i) and (ii), we get $(-\infty, 0] \cup [5, \infty)$.

The solution set of the given inequation is $(-\infty, 0] \cup [5, \infty)$.

Example 20:

Solve the inequation $|x-2|+|x-3|-|x-5| \geq 9$.

Solution:

Critical points are 2, 3, and 5.

These points divide the number line into four regions $(-\infty, 2)$, $[2, 3)$, $[3, 5)$, $[5, \infty)$.

Four cases arise.

Case 1: When $-\infty < x < 2$

$|x-2| = -(x-2)$, $|x-3| = -(x-3)$ and $|x-5| = -(x-5)$

$\therefore\ |x-2|+|x-3|-|x-5| \geq 9$

$\Rightarrow -x+2-x+3+x-5 \geq 9$

$\Rightarrow -x \geq 9$

$\Rightarrow x \leq -9$

Solution set is $(-\infty, -9]$ ...(i)

Case 2: $2 \leq x < 3$

$|x-2| = (x-2)$, $|x-3| = -(x-3)$ and $|x-5| = -(x-5)$

$\therefore\ |x-2|+|x-3|-|x-5| \geq 9$

$\Rightarrow x-2-x+3+x-5 \geq 9$

$\Rightarrow x-4 \geq 9$

$x \geq 13$ (Not possible) ($\because$ Not in range)

Case 3: $3 \leq x < 5$

$|x-2| = (x-2)$, $|x-3| = (x-3)$, $|x-5| = -(x-5)$

$\therefore\ |x-2|+|x-3|-|x-5| \geq 9$

$\Rightarrow x-2+x-3+x-5 \geq 9$

$\Rightarrow 3x \geq 19$

$\Rightarrow x \geq \frac{19}{3}$ (Not possible) ($\because$ Not in range)

Case 4: $5 \leq x < \infty$

$|x-2| = (x-2)$, $|x-3| = (x-3)$ and $|x-5| = (x-5)$

$\therefore\ |x-2|+|x-3|-|x-5| \geq 9$

$x-2+x-3-x+5 \geq 9$

$x \geq 9$ (Possible)

Solution set is $[9, \infty)$. ...(ii)

On combining solution set (i) and (ii), we get $(-\infty, -9] \cup [9, \infty)$.

The solution set of the given inequation is $(-\infty, -9] \cup [9, \infty)$.

Example 21:

If $|x^2-9x+18| > -12$, then find the correct range of x.

(A) $[0, \infty)$

(B) $[-12, \infty)$

(C) All real values of x

(D) $[12, \infty)$

Solution: (C)

It is known that the modulus of something ≥ 0.

So, we can say that $|x^2 - 9x + 18|$ is always greater than -12.

Hence, all real values of 'x' satisfy the inequality.

Option (C) is correct.

Maximum/Minimum Value of Inequalities

There are three types of questions in which the maxima and minima of inequality are asked.

1. Maxima and minima of a polynomial inequation (simple inequality).
2. Minima of those inequalities, which involves modulus.
3. Maxima and minima based on AM, GM, HM inequality.

Let's understand some examples on type 1.

Example 22:

Suppose x and y are real numbers such that $y^2 \leq 11 - x^2 + 10x$. Which of the following is the least possible value of y?

(A) -5

(B) -6

(C) -9

(D) -8

Solution:

It is given that,

$y^2 \leq 11 - x^2 + 10x$

$\Rightarrow y^2 \leq 11 - x^2 + 10x + 25 - 25$

$\Rightarrow y^2 \leq 36 - (x^2 + 25 - 10x)$

$\Rightarrow y^2 \leq 36 - (x - 5)^2$

Minimum value of $(x - 5)^2$ is 0 when $x = 5$

$\Rightarrow y^2 \leq 36.$

So, the least possible value of $y = -6$.

Example 23:

What is the maximum value of $x^2 - 9x + 16$, if $\frac{1}{x-4} > \frac{1}{6}$ and x is a natural number?

(A) 16

(B) 9

(C) 11

(D) 15

Solution:

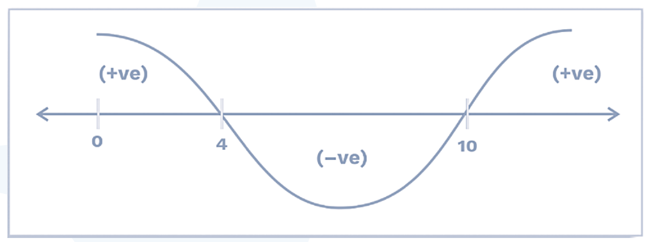
It is given that,

$$\frac{1}{x-4} > \frac{1}{6} \quad \Rightarrow \frac{1}{x-4} - \frac{1}{6} > 0$$

$$\Rightarrow \frac{6-x+4}{(x-4)\times 6} > 0 \quad \Rightarrow \frac{-x+10}{6(x-4)} > 0$$

$$\Rightarrow \frac{x-10}{6(x-4)} < 0 \quad \Rightarrow \frac{(x-10)(x-4)}{6(x-4)^2} < 0$$

Critical points are (10, 4).



So, $4 < x < 10$

So, $(x^2 - 9x + 16)$ max

when $x = 9$

$\Rightarrow (9^2 - 9 \times 9 + 16) = 16$

Example 24:

Find the maximum value of x such that $\sqrt{x} \geq 5x$.

(A) $\frac{1}{36}$

(B) $\frac{1}{4}$

(C) $\frac{1}{25}$

(D) $\frac{1}{9}$

Solution:

It is given that, $\sqrt{x} \geq 5x$

Let $x = y^2$ so $\sqrt{y^2} \geq 5y^2$

$\Rightarrow$ $y \geq 5y^2$ or $5y^2 - y \leq 0$

$y(5y - 1) \leq 0$

$\therefore$ $0 \leq y \leq \frac{1}{5}$ satisfies the above inequality.

Maximum value of y is $\frac{1}{5}$

Maximum value of x is $\left(\frac{1}{5}\right)^2 = \frac{1}{25}$

Example 25:

Given that $-7 \leq x \leq -1$ and $1 \leq y \leq 9$. What is the largest possible value of $\frac{x+y}{x}$?

(A) $\frac{1}{7}$

(B) $\frac{6}{7}$

(C) $-\frac{7}{9}$

(D) $-\frac{1}{9}$

Solution:

Here, $\left(\frac{x+y}{x}\right)$ can be written as $\left(1+\frac{y}{x}\right)$

Now x is negative, so to maximise $\left(1+\frac{y}{x}\right)$, $\frac{y}{x}$ must be minimised, which occurs when $|x|$ is largest and $|y|$ is smallest.

Take $x = -7$ and $y = 1$ we get $1+\frac{1}{(-7)}$

$\Rightarrow 1-\frac{1}{7} = \left(\frac{6}{7}\right)$

Type 2: Maxima and Minima of Inequalities that Involve Modulus

Let's understand this concept with the help of an example and try to generalise the concept.

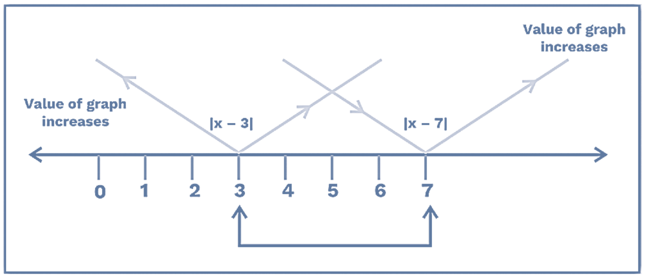
Example 26:

Find the minimum value of $|x - 3| + |x - 7|$.

Solution:

It is a combination of two mods

Critical points are (3, 7).



One can observe that between 3 and 7, one of the graphs increases while the other decreases, it implies that between 3 and 7, the value of $|x - 3| + |x - 7|$ is constant.

Let's check if it is constant or not

Put $x = 3$

$0 + 4 = 4$

Put $x = 4$

$1 + 3 = 4$

Put $x = 5$

$2 + 2 = 4$

Put $x = 6$

$3 + 1 = 4$

Put $x = 7$

$4 + 0 = 4$

Put $x = 4.5$

$1.5 + 2.5 = 4$

Here our value is constant, so we can say that the minimum value of $|x - 3| + |x - 7|$ is 4.

Note:

The minimum value of $|x - a| + |x - b|$ is between 'a' and 'b' where 'a' and 'b' are also included $[a < b]$. The minimum value is between $[a, b]$.

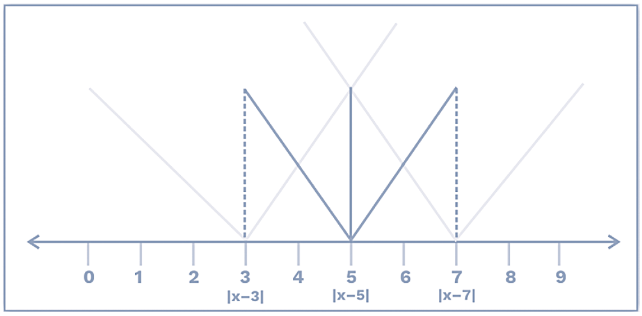
Example 27:

Find the minimum value of $|x - 3| + |x - 5| + |x - 7|$.

Solution:

It is a combination of three mods

Critical points are (3, 5, 7).



Here, one knows that the minimum value of $|x - 3| + |x - 7|$ lies between [3, 7].

If the value of $|x - 5|$ is zero, one can find the minimum value of $|x - 3| + |x - 5| + |x - 7|$.

The minimum value of $|x - 3| + |x - 5| + |x - 7|$ is minimum at $x = 5$.

$\therefore \quad |x - 3| + |x - 5| + |x - 7|$

$\Rightarrow \quad |5 - 3| + |5 - 5| + |5 - 7|$

$= 2 + 0 + 2 = 4$

Note:

Minimum value of $|x - a| + |x - b| + |x - c|$, where $a < b < c$ is at $x = b$.

1. In case of modulus, the minimum value can be found at the median of their critical points, where critical points are arranged in ascending or descending order.
   a) If the number of critical points is even then the minimum value is in between $\left[\left(\frac{N}{2}\right)^{th}\right]$ and $\left[\left(\frac{N}{2}+1\right)^{th}\right]$ critical point.
   (N = number of critical points).
   b) If the number of critical points is odd then the minimum value is at $\left(\frac{N+1}{2}\right)^{th}$ critical point.
   (N = number of critical points).

Example 28:

Find the minimum value of $|x - 1| + |x - 5| + |x + 7| + |x - 11| + |x - 13| + |x + 17|$.

Solution:

Critical points are (1, 5, −7, 11, 13, −17).

Ascending order of critical points = [−17, −7, 1, 5, 11, 13].

$N = 6$ (even)

$\left[\frac{N}{2}\right]^{th}$ and $\left[\frac{N}{2}+1\right]^{th} \Rightarrow (3^{rd}$ and $4^{th})$ critical points

Minimum value is in between [1, 5]

(i.e., Minimum value can be found at any value between 1 and 5, where 1, 5 is included).

Put $x = 5$

$\therefore \quad |x - 1| + |x - 5| + |x + 7| + |x - 11| + |x - 13| + |x + 17|$

$\Rightarrow \quad 4 + 0 + 12 + 6 + 8 + 22 = 52$

Thus, minimum value = 52.

Example 29:

Find the minimum value of $|x - 1| + |x - 11| + |x + 5| + |x| + |x - 17|$.

Solution:

Critical points = [1, 11, −5, 0, 17].

Ascending order of critical points = [−5, 0, 1, 11, 17].

N = 5 (odd) $\Rightarrow \left(\frac{N+1}{2}\right) = 3^{rd}$ critical point.

Minimum value at $x = 1$

$\therefore \quad |x - 1| + |x - 11| + |x + 5| + |x| + |x - 17|$

$\Rightarrow \quad 0 + 10 + 6 + 1 + 16 = 33$

Thus, minimum value = 33.

Example 30:

Find the minimum value of $32 + |5x + 8|$.

Solution:

Let $f(x) = 32 + |5x + 8|$

Given function is minimum when $|5x + 8| = 0$

$5x + 8 = 0$

$x = -\dfrac{8}{5}$

At $\left(x = -\dfrac{8}{5}\right)$ the given function becomes zero.

$\therefore\ 32 + \left|5 \times \left(-\dfrac{8}{5}\right) + 8\right| = 32 + 0$

Thus, minimum value = 32.

Example 31:

Find the maximum value of $f(x) = 15 - |7x + 3|$.

Solution:

It is known that, $|x| \geq 0$

When $|7x + 3| > 0$ the given function decreases.

One will get maximum value when $|7x + 3| = 0$.

$7x + 3 = 0$

$x = -\dfrac{3}{7}$

At $\left(x = -\dfrac{3}{7}\right)$ one will get the maximum value of the given function.

$\Rightarrow\ 15 - \left|7 \times \left(-\dfrac{3}{7}\right) + 3\right| = 15$

Thus, maximum value = 15 at $\left(x = -\dfrac{3}{7}\right)$.

Type 3

Maxima and minima are based on arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM).

Generally, two different types of questions are framed from this type.

1. **Type A:** The sum of the positive variable is given and the product of variables is asked (maximum product).
2. **Type B:** The product of variables is given and the minimum value of the sum of the variables is asked.

The following table shows the AM, GM, and HM for positive real numbers.

AM	GM	HM
$\dfrac{a+b}{2}$	$\sqrt{a \times b}$	$\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$
$\dfrac{a+b+c}{3}$	$\sqrt[3]{a \times b \times c}$	$\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$
$\dfrac{a+b+c+d}{4}$	$\sqrt[4]{a \times b \times c \times d}$	$\dfrac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}$
$\left(\dfrac{a_1+a_2+a_3+\ldots+a_n}{n}\right)$	$\left(\sqrt[n]{a_1 \times a_2 \times a_3 \ldots\ldots a_n}\right)$	$\left(\dfrac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots\ldots+\frac{1}{a_n}}\right)$

Note:

1. When observations are not same, then AM ≥ GM ≥ HM and $(GM)^2 = AM \times HM$.
2. When observations are same AM = GM = HM.
3. For any positive real number x, $\left(x + \frac{1}{x}\right) \geq 2$.
4. For $x \geq 1$, the following inequality holds true.

$$2 \leq \left(x + \frac{1}{x}\right)^x < 2.8 \left[x \to \infty\right]$$

Let's understand some examples based on type A.

Example 32:

If x, y, and z are positive real numbers and $x + y + z = 18$, find the maximum value of xyz.

Solution:

It is known that AM ≥ GM

$$\Rightarrow \frac{x+y+z}{3} \geq \sqrt[3]{x \times y \times z}$$

$$\Rightarrow \frac{18}{3} \geq \sqrt[3]{xyz} \Rightarrow (6)^3 \geq xyz.$$

$$\because \quad xyz \leq 216$$

Thus, the maximum value of xyz is 216.

Example 33:

If $x + 2y + 3z = 12$, then find the maximum value of xy^3z^2 (x, y, and z are positive real numbers).

Solution:

Here, AM includes $x + 2y + 3z$, but the GM will have $(x \times 2y \times 3z)$ and not xy^3z^2.

So, we will break the terms to achieve the required product xy^3z^2.

As you can see that we need y^3 (in GM), we will break the second term $2y$ into three equal parts, i.e., $\frac{2y}{3}$, $\frac{2y}{3}$, and $\frac{2y}{3}$.

Similarly, we need z^2 (in GM), so we will break the third term $3z$ into two equal parts, i.e., $\frac{3z}{2}$ and $\frac{3z}{2}$.

Thus, $x + 2y + 3z = 12$ or

$$x + \frac{2y}{3} + \frac{2y}{3} + \frac{2y}{3} + \frac{3z}{2} + \frac{3z}{2} = 12$$

Now the AM will include $(x + 2y + 3z)$ and GM will have (xy^3z^2).

$$\Rightarrow \frac{x + \frac{2y}{3} + \frac{2y}{3} + \frac{2y}{3} + \frac{3z}{2} + \frac{3z}{2}}{6} \geq \sqrt[6]{x \times \frac{2y}{3} \times \frac{2y}{3} \times \frac{2y}{3} \times \frac{3z}{2} \times \frac{3z}{2}}$$

$$\Rightarrow \frac{12}{6} \geq \sqrt[6]{x \times y^3 \times z^2 \times \frac{2}{3}}$$

$$2 \geq \sqrt[6]{xy^3z^2 \times \frac{2}{3}}$$

$$(2)^6 \times \frac{3}{2} \geq xy^3 z^2$$

$$96 \geq xy^3z^2$$

Hence, the maximum value of xy^3z^2 is 96.

Questions based on type B.

Example 34:

If $x \times y \times z = 512$, find the minimum value of $(x + y + z)$, where x, y and z are positive real numbers.

Solution:

Here,

$$\Rightarrow \frac{x+y+z}{3} \geq \sqrt[3]{x \times y \times z}$$

$$\Rightarrow \frac{x+y+z}{3} \geq \sqrt[3]{512}$$

$$\Rightarrow \frac{x+y+z}{3} \geq 8$$

$$\Rightarrow x + y + z \geq 3 \times 8$$

$$\Rightarrow x + y + z \geq 24$$

Hence, the minimum value of $(x + y + z)$ is 24.

Example 35:

If $x^3y^2 = 2^6 \times 3^6$, find the minimum possible value of $4x + 3y$ (x, y, and z are positive real numbers).

Solution:

$$x^3y^2 = 2^6 \times 3^6$$

$4x + 3y$

3 parts $\uparrow x^3$ $\quad$ 2 parts $\uparrow y^2$ $\Rightarrow \frac{4x}{3}, \frac{4x}{3}, \frac{4x}{3}, \frac{3y}{2}, \frac{3y}{2}$

Using AM $\geq$ GM

$$\Rightarrow \frac{\frac{4x}{3}+\frac{4x}{3}+\frac{4x}{3}+\frac{3y}{2}+\frac{3y}{2}}{5} \geq \sqrt[5]{\frac{4x}{3}\times\frac{4x}{3}\times\frac{4x}{3}\times\frac{3y}{2}\times\frac{3y}{2}}$$

$$\Rightarrow \frac{4x+3y}{5} \geq \sqrt[5]{\frac{16}{3}\times x^3y^2}$$

$$\Rightarrow \frac{4x+3y}{5} \geq \sqrt[5]{\frac{2^4\times2^6\times3^6}{3}}$$

$$\frac{4x+3y}{5} \geq \sqrt[5]{2^{10}\times3^5}$$

$$\frac{4x+3y}{5} \geq 2^2\times3^1$$

$$4x + 3y \geq 60$$

Hence, the minimum value of $4x + 3y$ is 60.

Questions based on $\left(2\left(x+\frac{1}{x}\right)^x < 2.8\right)$

Inequality ($x \to \infty$)

Example 36:

Find the minimum value of $\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{d}+\frac{d}{c}\right)$, where a, b, c, and d are positive real numbers.

Solution:

$\frac{a}{b}$ and $\frac{b}{a}$ is reciprocal to each other, $\frac{b}{c}$ and $\frac{c}{b}$ is reciprocal to each other, and $\frac{c}{d}$ and $\frac{d}{c}$ is reciprocal to each other.

$$\left(\frac{a}{b}+\frac{b}{a}\right) \geq 2$$

As if we take $\frac{a}{b} = P$, then $\frac{b}{a} = 1/P$ and we know that $\left(P+\frac{1}{P}\right) \geq 2$

Similarly, the value of $\left(\frac{b}{c}+\frac{c}{b}\right) \geq 2$ and $\left(\frac{c}{d}+\frac{d}{c}\right) \geq 2.$

$\therefore \left[\frac{a}{b}+\frac{b}{a}\right]\left[\frac{b}{c}+\frac{c}{b}\right]\left[\frac{c}{d}+\frac{d}{c}\right] = (\geq 2) \times (\geq 2) \times (\geq 2) = \geq 8$

$$\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{d}+\frac{d}{c}\right) \geq 8$$

Hence, the minimum value of $\left[\frac{a}{b}+\frac{b}{a}\right]\left[\frac{b}{c}+\frac{c}{b}\right]\left[\frac{c}{d}+\frac{d}{c}\right]$ is 8.

Example 37:

Find the minimum value of $\frac{(x+6)(x+11)}{x+2}$ $(x > 0)$.

Solution:

It is given that, $\frac{(x+6)(x+11)}{x+2}$; $(x > 0)$

Let $x + 2 = a$

$$\Rightarrow \frac{(x+2+4)(x+2+9)}{a} \Rightarrow \frac{(a+4)(a+9)}{a}$$

$$\Rightarrow \frac{a^2+9a+4a+36}{a} \Rightarrow \frac{a^2+36+13a}{a}$$

$$\Rightarrow a+\frac{36}{a}+13$$

Minimum value of $a+\frac{36}{a}$

$$\frac{a+\frac{36}{a}}{2} \geq \sqrt{a\times\frac{36}{a}}$$

$$a+\frac{36}{a} \geq 12$$

The minimum value of $a+\frac{36}{a} = 12$

$\therefore$ Minimum value of $a+\frac{36}{a}+13 = 12+13 = 25$

Hence, minimum value of $\frac{(x+6)(x+11)}{x+2}$ is 25.

Alternate Method (Formula-Based):

Minimum value of $\dfrac{(x+a)\,(x+b)}{(x+c)} = a - c + b - c + 2\sqrt{(a-c)\,(b-c)}$

$$\frac{(x+6)(x+11)}{(x+2)} \Rightarrow a = 6,\ b = 11, \text{ and } c = 2$$

Minimum value of $\dfrac{(x+6)(x+11)}{(x+2)} =$

$4 + 9 + 2\sqrt{4 \times 9} = 13 + 2\sqrt{36} = 13 + 12 = 25.$

Example 38:

Find the minimum value of $\dfrac{(7a^2+a+7)(11b^2+b+11)(9c^2+c+9)}{19abc}$ (a, b, and $c > 0$).

Solution:

It is given that,

$\dfrac{(7a^2+a+7)(11b^2+b+11)(9c^2+c+9)}{19abc}$;

(a, b, and $c > 0$)

$$\Rightarrow \frac{1}{19}\left[\left(\frac{7a^2+a+7}{a}\right)\left(\frac{11b^2+b+11}{b}\right)\left(\frac{9c^2+c+9}{c}\right)\right]$$

$$\Rightarrow \frac{1}{19}\left[7\left(a+\frac{1}{a}\right)+1\right]\left[11\left(b+\frac{1}{b}\right)+1\right]\left[9\left(c+\frac{1}{c}\right)+1\right]$$

The minimum value of $\left(a+\dfrac{1}{a}\right)$, $\left(b+\dfrac{1}{b}\right)$ and $\left(c+\dfrac{1}{c}\right)$ is 2.

$= \dfrac{1}{19}\left[(7\times 2+1)(11\times 2+1)(9\times 2+1)\right]$

$= \dfrac{1}{19}[15\times 23\times 19]$

$= 15 \times 23$

$= 345$

Hence, the minimum value of $\dfrac{(7a^2+a+7)\,(11b^2+b+11)\,(9c^2+c+9)}{19abc}$ is 345.

Cauchy-Schwarz Inequality

$\{(x_1)^2 + (x_2)^2 + (x_3)^2 + \ldots + (x_n)^2\}\,\{(y_1)^2 + (y_2)^2 + (y_3)^2 + \ldots + (y_n)^2\} \geq (x_1y_1 + x_2y_2 + x_3y_3 + \ldots\ x_ny_n)^2$

($x_1, x_2, x_3, \ldots\ldots, x_n \in R$)

($y_1, y_2, y_3, \ldots\ldots, y_n \in R$)

Let's understand the above learning with the help of some examples.

Example 39:

If $a^2 + b^2 + c^2 + d^2 = 500$, then find the maximum value of $(3a + 4b + 5c + 6d)$ where a, b, c, and d are real numbers.

(A) $10\sqrt{430}$

(B) $5\sqrt{420}$

(C) $10\sqrt{420}$

(D) $15\sqrt{430}$

Solution:

Here, $a^2 + b^2 + c^2 + d^2 = 500$

$$\begin{array}{ccccccccccc} 3 & a & + & 4 & b & + & 5 & c & + & 6 & d \\ \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \\ x_1 & y_1 & & x_2 & y_2 & & x_3 & y_3 & & x_4 & y_4 \end{array}$$

According to inequality,

$[(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2]\,[(y_1)^2 + (y_2)^2 + (y_3)^2 + (y_4)^2] \geq (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2$

$[(3)^2 + (4)^2 + (5)^2 + (6)^2]\,[(a^2 + b^2 + c^2 + d^2)] \geq (3a + 4b + 5c + 6d)^2$

$(9 + 16 + 25 + 36)\,(a^2 + b^2 + c^2 + d^2) \geq (3a + 4b + 5c + 6d)^2$

$(86)(500) \geq (3a + 4b + 5c + 6d)^2$

$\sqrt{86 \times 500} \geq (3a + 4b + 5c + 6d)$

$(3a + 4b + 5c + 6d)_{Max} = \left(10\sqrt{430}\right)$

Example 40:

If $a^2 + b^2 + c^2 = 108$, then find the maximum value of $(a + b + c)$ where a, b, and $c \in R$.

(A) 9

(B) 12

(C) 18

(D) 21

Solution:

Here, $a^2 + b^2 + c^2 = 108$

$$\begin{array}{cccccccc} 1 & a & + & 1 & b & + & 1 & c \\ \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \\ x_1 & y_1 & & x_2 & y_2 & & x_3 & y_3 \end{array}$$

$[(x_1)^2 + (x_2)^2 + (x_3)^2]\,[(y_1)^2 + (y_2)^2 + (y_3)^2] \geq (x_1y_1 + x_2y_2 + x_3y_3)^2$

$(1^2 + 1^2 + 1^2)\,(a^2 + b^2 + c^2) \geq (a + b + c)^2$

$(3 \times 108) \geq (a + b + c)^2$

$324 \geq (a + b + c)^2$

$18 \geq (a + b + c)$

Hence, the maximum value of $(a + b + c)$ is 18.

Practice Exercise – 1

Level of Difficulty – 1

1. The set of values of x which satisfy $5x + 7 < 3x + 7$ and $\dfrac{6x+2}{x-1} < 4$ is:

(A) $(-2, -3)$

(B) $(-\infty, 1) \cup (2, 3)$

(C) $(-3, 0)$

(D) $(1, 3)$

2. How many solutions are possible for the inequality $|x - 2| + |x - 7| < 4$?

(A) 2

(B) 7

(C) 0

(D) Infinite many solutions

3. What values of x satisfy $x^{2/5} + x^{1/5} - 2 \le 0$ (where $x \in$ real number)?

(A) $-32 \le x \le 1$

(B) $-8 \le x \le 1$

(C) $1 \le x < 8$

(D) $-32 < x \le 1$

4. Find the minimum value of $9^x + 9^{1-x}$, where $x \in R$.

5. Find the minimum value of the $|4x + 24| + |2x - 8|$.

Level of Difficulty – 2

6. Find the sum of all the integral possible solution of inequality: $(x^2 + 7x - 30) \times (x^2 - 5x - 14) < 0$.

(A) –21

(B) –27

(C) 21

(D) 27

7. Find the minimum value of the given expression: $|x - 1| + |x - 2| + |x - 3| + \ldots\ldots + |x - 97|$.

8. $|x - 5| + |x + 4| = a$.

The given expression will have no solution for how many (positive) integral values of a?

(A) 10

(B) 12

(C) 8

(D) 4

9. If a, b, and c are integers such that $|a| < |b| < |c| < 50$ and $a + b + c = 25$, then what is the maximum possible value of $a \times b \times c$?

(A) 7,056

(B) 7,007

(C) 7,248

(D) 7,247

10. The following relation holds true for how many integral values of n: $8^{28} < n^{21} < 5^{42}$?

(A) 5

(B) 6

(C) 7

(D) 8

Level of Difficulty – 3

11. Find the range of x where $||x - 6| - 2| > 4$.

(A) $(-\infty, 0) \cup (14, \infty)$

(B) $(-\infty, 0) \cup (12, \infty)$

(C) $(-\infty, 0) \cup (10, \infty)$

(D) $(-\infty, 0) \cup (8, \infty)$

12. If $x \in (a, b)$ satisfies the inequality $\frac{x-3}{x^2+3x+2} \geq 1$, then the largest possible value of $(b - a)$ is:

(A) 3

(B) 1

(C) 2

(D) Cannot be determined

13. Consider the expression

$$\frac{(a^2+a+1)(b^2+b+1)(c^2+c+1)(d^2+d+1)(e^2+e+1)}{abcde}$$

where a, b, c, d, and e are positive real numbers. The minimum value of the expression is:

(A) 32

(B) 243

(C) 100

(D) 10

14. If a, b, and c are all the distinct positive integers less than 1,000, such that $|a - b| + |b - c| - |c - a| = 0$, then find the difference between maximum and minimum value of b.

(A) 996

(B) 997

(C) 998

(D) 995

15. If $x + |y| = 10$, $y + |x| = 14$, then how many pairs of x, y satisfy these two equations?

(A) 2

(B) 3

(C) 1

(D) 0

Solutions

1. (C)

$5x + 7 < 3x + 7$

$\Rightarrow 2x < 0$

$\Rightarrow x < 0$ ...(i)

This implies that x is negative.

Now, $\dfrac{6x+2}{x-1} < 4$

$\Rightarrow 6x + 2 > 4(x - 1)$ (assuming x is negative)

$\Rightarrow 2x > -6$

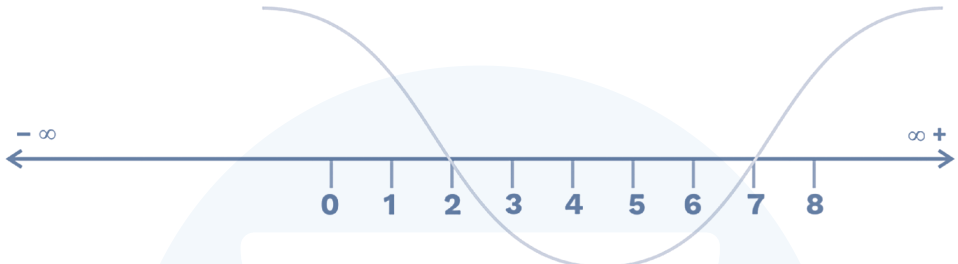
$\Rightarrow x > -3$ ...(ii)

Combining (i) and (ii) we get $-3 < x < 0$

2. (C)

It is given that $|x - 2| + |x - 7| < 4$.

Critical points are (2, 7).



Now, the minimum value of $|x - 2| + |x - 7|$ will be at any value x in the range (2, 7).

Let's take the value of x in the range (2, 7) to calculate the minimum value.

Let's take $x = 3$, then the minimum value will be $|3 - 2| + |3 - 7| = 1 + 4 = 5$.

Hence, no solution exists for $|x - 2| + |x - 7| < 4$.

3. (A)

It is given that

$x^{2/5} + x^{1/5} - 2 \leq 0$

$\Rightarrow x^{2/5} + 2x^{1/5} - x^{1/5} - 2 \leq 0$

$\Rightarrow x^{1/5}(x^{1/5} + 2) - 1(x^{1/5} + 2) \leq 0$

$\Rightarrow (x^{1/5} - 1)(x^{1/5} + 2) \leq 0$

$\Rightarrow -2 \leq (x)^{1/5} \leq 1$

$\Rightarrow -32 \leq x \leq 1$

4. 6

For two positive real number A and B

$\text{AM} \geq \text{GM} \longrightarrow \dfrac{\text{A}+\text{B}}{2} > (\text{A} \times \text{B})^{1/2}$

$\Rightarrow \dfrac{9^x + 9^{1-x}}{2} \geq (9^x \times 9^{1-x})^{1/2}$

$\Rightarrow \dfrac{9^x + 9^{1-x}}{2} \geq (9)^{1/2}$

$\Rightarrow \dfrac{9^x + 9^{1-x}}{2} \geq 3$

$\Rightarrow 9^x + 9^{1-x} \geq 6$

Hence, the minimum value is 6.

5. 20

It is given that,

$|4x + 24| + |2x - 8|$

This can be written as

$\Rightarrow |4(x + 6)| + |2(x - 4)|$

$\Rightarrow 4|x + 6| + 2|x - 4|$

$\Rightarrow |x + 6| + |x + 6| + |x + 6| + |x + 6| + |x - 4| + |x - 4|$

$\Rightarrow$ Critical points are $(-6, -6, -6, -6, 4, 4)$.

Six critical points (even)

Ascending order of critical points

$(-6, -6, -6, -6, 4, 4)$

Minimum value at $x = -6$

$\Rightarrow |4 \times (-6) + 24| + |2 \times (-6) - 8| = 20$

6. (B)

It is given that,

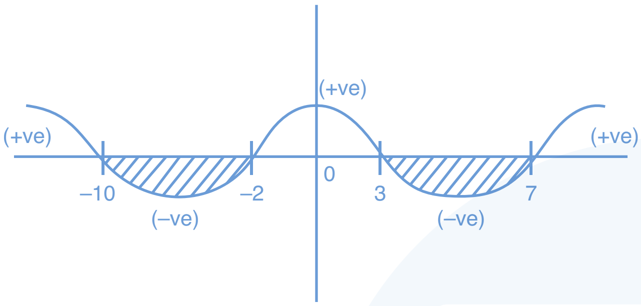
$(x^2 + 7x - 30) \times (x^2 - 5x - 14) < 0$

$\Rightarrow (x^2 + 10x - 3x - 30) \times (x^2 - 7x + 2x - 14) < 0$

$\Rightarrow [x(x + 10) - 3(x + 10)]\,[x(x - 7) + 2(x - 7)] < 0$

$\Rightarrow [(x + 10)(x - 3)] \times [(x - 7) \times (x + 2)] < 0$

Critical points are (–10, 3, 7, –2).



Solution set = $(-10, -2) \cup (3, 7)$

Possible integral solution = (–9, –8, –7, –6, –5, –4, –3, 4, 5, 6)

Required sum = (–9) + (–8) + (–7) + (–6) + (–5) + (–4) + (–3) + 4 + 5 + 6

$\Rightarrow$ (–27)

7. 2,352

It is given that,

$|x - 1| + |x - 2| + |x - 3| + |x - 4| + \dots\dots\dots + |x - 95| + |x - 96| + |x - 97|$.

Critical points = (1, 2, 3, ...97)

N = 97 (odd)

If we arrange all the critical points, then the median value of critical points will be the point, where there will be a minimum value of a given expression.

At x = 49 (median), given expression takes minimum value,

$\therefore |49 - 1| + |49 - 2| + \dots\dots\; |49 - 49| + |49 - 50| + |49 - 51| + \dots\dots\; |49 - 97|$

$\therefore \underbrace{48 + 47 + 46 \dots\dots\dots 1}_{} + 0 + \underbrace{1 + 2 + 3 + \dots\dots 48}_{}$

$\therefore \dfrac{48 \times 49}{2} + 0 + \dfrac{48 \times 49}{2}$

$\Rightarrow$ 1,176 + 1,176 = 2,352

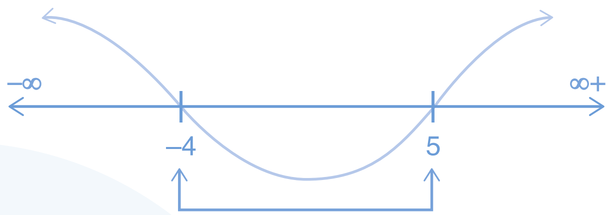
Hence, the minimum value of the given expression is 2,352.

8. (C)

It is given that, $|x - 5| + |x + 4| = a$

Note: In such types of questions, one must find the minimum value of the expression first.

Critical points are (–4, 5).



The minimum value can be found in between –4 and 5.

Put $x = 0$

$|-5| + |4| = a$

$a = 9$

Below 9, the given expression has no solution.

At a = 1, 2, 3, 4, 5, 6, 7, 8 given expression has no solution.

Hence, the given expression has no solution at 8 positive integral values of a.

9. (B)

It is given that, $|a| < |b| < |c| < 50$ and $a + b + c = 25$.

One need to maximise $a \times b \times c$ so two cases arise.

a) Two are negative and one positive

b) All three positive

Two negative and one positive results in maximum value of abc.

Maximum value of c is 49.

$\therefore \quad a + b + c = 25$

$\quad\;\; a + b = -24$

Here $|a| < |b|$ so to maximise the product one must assume $b = -13$ and $a = -11$.

Required product = $a \times b \times c = -11 \times -13 \times 49 = 7{,}007$.

10. (D)

It is given that, $8^{28} < n^{21} < 5^{42}$

$\Rightarrow \quad (8^4)^7 < (n^3)^7 < (5^6)^7$

$\Rightarrow \quad (8)^4 < (n)^3 < (5)^6$

$\Rightarrow \quad (2)^{\frac{4}{3}\times 3} < n < (5)^{\frac{6}{3}}$

$\Rightarrow \quad 16 < n < 25$

Thus, the value which n take is (17, 18, 19, 20, 21, 22, 23, 24).

Hence, n can take 8 values.

11. (B)

It is given that, $||x-6|-2|>4$

Here two cases arise.

Case 1:

$|x - 6| - 2 > 4$

$|x - 6| > 6$

$x - 6 > 6 \quad$ or $\quad x - 6 < -6$

$x > 12 \quad$ or $\quad x < -6 + 6$

$x > 12 \quad$ or $\quad x < 0$

Required set of solution: $(-\infty, 0) \cup (12, \infty)$

Case 2:

$|x - 6| - 2 < -4$

$|x - 6| - 2 < -4$

$|x - 6| < -4 + 2$

$|x - 6| < -2$

Mod of something is always greater than or equal to zero.

So, case 2 is not possible.

The range of x is $(-\infty, 0) \cup (12, \infty)$.

12. (B)

It is given that, $\dfrac{x-3}{x^2+3x+2} \geq 1$

$$\frac{x-3}{x^2+3x+2} - 1 \geq 0$$

$$\Rightarrow \frac{x-3-x^2-3x-2}{(x+1)(x+2)} \geq 0$$

$$\Rightarrow \frac{-x^2-2x-5}{(x+1)(x+2)} \geq 0$$

$$\Rightarrow \frac{x^2+2x+5}{(x+1)(x+2)} \leq 0$$

$$\Rightarrow \frac{x^2+2x+1+4}{(x+1)(x+2)} \leq 0$$

$$\Rightarrow \frac{(x+1)^2+4}{(x+1)(x+2)} \leq 0$$

Here, one can conclude that the numerator can't be negative.

[$\because$ $(x + 1)^2$ is always positive]

So, $(x + 1)(x + 2) < 0$

$x \in (-2, -1)$

$a = -2$ and $b = -1$

Largest possible of $b - a = -1 - (-2) = -1 + 2 = 1$

13. (B)

It is given that

$$\frac{(a^2+a+1)(b^2+b+1)(c^2+c+1)(d^2+d+1)(e^2+e+1)}{abcde}$$

By AM-GM relation, one concludes that,

$$\Rightarrow \frac{a^2+a+1}{3} \geq \sqrt[3]{a^2 \times a \times 1}$$

$$\Rightarrow \frac{a^2+a+1}{3} \geq a \Rightarrow \frac{a^2+a+1}{a} \geq 3$$

A similar relation for b, c, d, and e can be established

$$\frac{(a^2+a+1)}{a} \times \frac{(b^2+b+1)}{b} \times \frac{(c^2+c+1)}{c} \times \frac{(d^2+d+1)}{d} \times \frac{(c^2+c+1)}{e} \geq (3)^5$$

$$\Rightarrow \frac{(a^2+a+1)}{a} \times \frac{(b^2+b+1)}{b} \times \frac{(c^2+c+1)}{c} \times \frac{(d^2+d+1)}{d} \times \frac{(c^2+c+1)}{e} \geq 243$$

Hence, the minimum value of the expression can be 243.

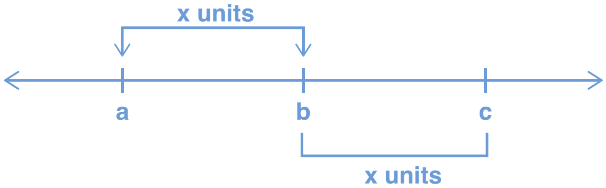
14. (A)

It is given that,

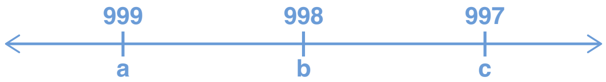
$|a - b| + |b - c| - |c - a| = 0$

$\Rightarrow \quad |a - b| + |b - c| = |c - a|$

From the above expression, one can conclude that a, b, and c are collinear points.

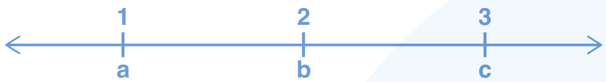


For the maximum value of b



Above inequality satisfy.

For the minimum value of b



Above inequality satisfy.

$b_{(max)} - b_{(min)} = 998 - 2 = 996$

15. (C)

It is given that, $x + |y| = 10$ and $y + |x| = 14$

$x + |y| = 10$

Two cases arise

a) if $y > 0$, then $x + y = 10$ ...(i)

b) If $y < 0$, then $x - y = 10$ ...(ii)

$y + |x| = 14$

Two cases arise

a) If $x > 0$, then $y + x = 14$ ...(iii)

b) If $x < 0$, then $y - x = 14$ ...(iv)

On solving equations (i) and (iv) we get

$y = 12$ and $x = -2$ ($x < 0$ and $y > 0$: condition satisfied).

On solving equations (ii) and (iii) we get

$x = 12$ and $y = 2$ ($x > 0$ and $y < 0$: condition not satisfied).

Solving equations (ii), (iv), (i), and (iii) is not worthy at all.

($\therefore$ Both results give no solution).

Only one pair of (x, y) satisfies these two equations.

Practice Exercise – 2

Level of Difficulty – 1

1. If $\left|\frac{45-x}{3}\right| < 5$, then find the correct range of x.

(A) $10 < x < 20$

(B) $30 < x < 60$

(C) $10 < x < 40$

(D) $x \geq 50$

2. The geometric mean proportion between $30+\sqrt{200}$ and $54-\sqrt{648}$ is:

(A) $6\sqrt{2}$

(B) $4\sqrt{5}$

(C) $6\sqrt{35}$

(D) $5\sqrt{6}$

3. Find the number of real number solutions for the equation $a^2 + |a| - 56 = 0$.

(A) 4

(B) 3

(C) 2

(D) 1

4. For how many integral values of x does the inequality $x^2 - 16x + 28 < 0$ would be true?

(A) 12

(B) 8

(C) 7

(D) 11

5. Find the set of values of x for which $(x + 2)(x - 7) > -20$.

(A) $x > 12$ or $x < 3$

(B) $x < 2$ or $x > 3$

(C) $x = 2$ or $x = 3$

(D) None of these

6. How many integer solutions of the form of (P, Q) are there, which satisfy the equation $|P| + |Q| = 7$?

(A) 12

(B) 14

(C) 24

(D) 28

7. If $x < -4$ and $y > 3$, then which of the following is definitely true for real values of x and y?

(A) $x + 3y < -1$

(B) $2x + 5y > 0$

(C) $3x - 2y < 0$

(D) $5x + 2y > 0$

8. Which of the following is the largest integral value of x satisfying the inequality $(|x| - 4)(x - 3) < 0$?

(A) -7

(B) -2

(C) -5

(D) -6

9. How many ordered triplets of positive integers k, l, m exist, such that $\frac{k}{l} + \frac{l}{m} + \frac{m}{k} = 2$?

(A) 7

(B) 5

(C) 3

(D) None of these

10. If $|x - 5| \leq 10$ and $|3y - 6| \leq 12$, then find the maximum possible value of $|x| - |y|$.

(A) 12

(B) 13

(C) 15

(D) 9

Level of Difficulty – 2

11. How many integer values of x satisfy $(x - 7)(x - 8) - 4(3x - 21) \leq 0$?

(A) 11

(B) 13

(C) 14

(D) 15

12. For what values of k is the expression $x^2 - 5kx + 6k^2 + 9 > 0$?

(A) $(-\infty, -6)$

(B) $(6, \infty)$

(C) $(-6, 6)$

(D) $(-\infty, -6) \cup (6, \infty)$

13. If P, Q, R, S are distinct natural numbers whose sum is 70, then find the maximum value of their product.

14. Which of the following solutions will satisfy the equation given below?

$|x^2 + 3x + 1| + 2x + 5 = 0$

(A) $\left(-2 \text{ and } \frac{-1+\sqrt{17}}{2}\right)$

(B) $\left(-3 \text{ and } \frac{-1+\sqrt{17}}{2}\right)$

(C) $\left(-3 \text{ and } \frac{-1-\sqrt{17}}{2}\right)$

(D) $\left(-2 \text{ and } \frac{-1-\sqrt{17}}{2}\right)$

15. The solution set for the inequation $(2x^2 + 3x + 1)^x < 1$ is:

(A) $(-\infty, -1)$

(B) $(-\infty, -1/2)$

(C) $(-\infty, 0)$

(D) $(-\infty, 1)$

16. If $|p + q| = |p| + |q|$, $|R + S| < |R| + |S|$, and $|T + U| < |T| + |U|$ and P, Q, R, S, T, U are real numbers, then the value of $P \times Q \times R \times S \times T \times U$ is:

(A) Positive

(B) Negative

(C) Zero

(D) Cannot be determined

17. What is the maximum value of the expression $(5x - 6)(15 - 5x)$?

(A) 40.25

(B) 20.25

(C) 60.15

(D) 15.25

18. Find the sum of all the possible values of x, which satisfies the equation

$|x + 5| + |x - 6| = 14.$

(A) 0.5

(B) 2

(C) 1

(D) -2

19. $|x-5| \leq 13$ and $|3y-12| \leq 18$. Find the minimum possible value of $|x| - |y|$.

(A) -10

(B) -15

(C) -18

(D) -23

20. How many integral values of x satisfy both the given inequalities: $|x - 2| \leq 4$ and $|x - 1| \geq 2$?

Level of Difficulty – 3

21. How many integers will satisfy the given inequality: $(x - 2)(x + 3)(x - 4) \leq (x + 2)(x - 3)(x + 4)$?

22. For how many values of x the given equation $\left|\left||x-5|+4\right|-3\right| = 0$ holds true?

(A) 1

(B) 2

(C) 3

(D) None of these

23. If $4P + 12Q + 3R = 32$, where P, Q, and R are positive real numbers, what is the maximum possible value of $P^2Q^3R^3$?

24. For $-3 \le x \le -1$, $-4 \le y \le 4$, $-6 \le z \le -2$, and $u = \frac{xy}{z}$ find the range of values u can take.

(A) $-6 \le \frac{xy}{z} \le 18$

(B) $-12 \le \frac{xy}{z} \le 12$

(C) $-24 \le \frac{xy}{z} \le 12$

(D) $-6 \le \frac{xy}{z} \le 6$

25. If P is a real number greater than 4, then the minimum value of the expression $P - 5 + \frac{9}{P-4}$ is.

26. If a, b, c, d are positive real numbers and $(abc + abd + acd + bcd) = 864$, what is the maximum value of $a \times b \times c \times d$?

27. What is the least and greatest possible value of $\frac{|a+b|}{|a|+|b|} + \frac{|b+c|}{|b|+|c|} + \frac{|c+a|}{|c|+|a|}$?

(A) 0 and 3

(B) 0 and 1

(c) 1 and 3

(D) 1 and 2

28. What are the values that a can take, if $\left|\frac{a-6}{a+5}\right| > 1$?

(A) $a < 3$, except $a = -5$

(B) $a < \frac{1}{4}$, except $a = -5$

(C) $a < \frac{1}{2}$, except $a = -5$

(D) $a < 2$, except $a = -5$

29. In a right-angle triangle, P and b are the perpendicular sides, and h is the hypotenuse. Find the minimum value of $\left\{\sqrt{2}\left(\frac{h}{P} + \frac{h}{b}\right)\right\}$

30. If $4x + 3|y| + y = 8$ and $x + |x| + 4y = 2$, then the value of $x \times 5y$ is:

(A) 2

(B) −3

(C) −3.6

(D) −7.5

Solutions

1. (B)

Since the given expression is:

$\left|\frac{45-x}{3}\right| < 5$

$|45 - x| < 15$

$-15 < (45 - x) < 15$

$-15 - 45 < -45 + 45 - x < 15 - 45$

$-60 < -x < -30$

$30 < x < 60$

Hence, option (B) is the correct answer.

2. (C)

The geometric mean proportion of two numbers a and b is given by $\sqrt{a \times b}$.

The given two numbers are $30 + \sqrt{200}$ and $54 - \sqrt{648}$, which are also equal to $10(3 + \sqrt{2})$ and $18(3 - \sqrt{2})$.

Hence, the geometric mean proportion

$= \sqrt{10(3+\sqrt{2}) \times 18\,(3-\sqrt{2})} =$

$\sqrt{180 \times (3^2 - (\sqrt{2})^2)}$

$= \sqrt{1260} = 6 \times \sqrt{35}$

Hence, option (C) is the correct answer.

3. (C)

Since the given equation is $a^2 + |a| - 56 = 0$

We can write this equation in this format also

$|a|^2 + |a| - 56 = 0$

or $|a|^2 + 8|a| - 7|a| - 56 = 0$

$|a|\,(|a| + 8) - 7\,(|a| + 8) = 0$

$(|a| - 7)\,(|a| + 8) = 0$

$|a| = 7$ or $|a| = -8$

But the value of $|a|$ cannot be negative.

Thus $|a| = 7$

$\therefore$ $a = 7$ or $a = -7$

Hence, option (C) is the correct answer.

4. (D)

Since the given equation is

$x^2 - 16x + 28 < 0$

$x^2 - 14x - 2x + 28 < 0$

$x(x - 14) - 2(x - 14) < 0$

$(x - 2)\,(x - 14) < 0$

$\therefore$ Integral values are [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Therefore, total integral values of x are 11.

Hence, option (D) is the correct answer.

5. (B)

$(x + 2)(x - 7) > -20$

$\Rightarrow$ $x^2 - 5x - 14 > -20$

$\Rightarrow$ $x^2 - 5x + 6 > 0$

$\Rightarrow$ $(x - 3)\,(x - 2) > 0$

$\Rightarrow$ $x < 2$ or $x > 3$ is the range in which $(x + 2)(x - 7) > -20$

Hence, option (B) is the correct answer.

6. (C)

$|P| + |Q| = 7$

$P = \pm 1$ $Q = \pm 6$ $\rightarrow$ 4 solutions

$P = \pm 2$ $Q = \pm 5$ $\rightarrow$ 4 solutions

$P = \pm 3$ $Q = \pm 4$ $\rightarrow$ 4 solutions

$P = \pm 4$ $Q = \pm 3$ $\rightarrow$ 4 solutions

$P = \pm 5$ $Q = \pm 2$ $\rightarrow$ 4 solutions

$P = \pm 6$ $Q = \pm 1$ $\rightarrow$ 4 solutions

Hence, a total of 24 ordered pairs of non-zero integer solutions are these.

Hence, option (C) is the correct answer.

7. (C)

When $x < -4$ and $y > 3$

If we check option A:

If $x = -5$, and $y = 4$

Then $x + 3y = -5 + 3 \times 4 = -5 + 12 = 7$

which is not less than -1.

Again, if we check for option B:

If $x = -5$ and $y = 4$

Then, $2x + 5y = 2 \times (-5) + 5 \times 4 = -10 + 20$

but if we put $x = -20$ and $y = 4$

then, $2x + 5y = 2 \times (-20) + 5 \times 4 = -40 + 20 = -20$ which does not satisfy.

Option (B) do not satisfies the condition for real values of x and y.

Now for option C:

If we take $x = -5$ and $y = 4$.

Then, $3x - 2y = 3 \times (-5) - 2 \times 4 = -15 - 8 = -23$.

which is less than zero, hence satisfy the condition

Also, you can see as x is negative and y is positive, so final values of $3x - 2y$ will always be less than zero.

For option D:

If we take $x = -5$ and $y = 4$

Then $5x + 2y = 5 \times (-5) + 2 \times 4$

$= -25 + 8 = -17$

which is less than zero.

Hence, option (C) is the correct answer.

8. (C)

Given: $(|x| - 4)(x - 3) < 0$

putting $|x| - 4 = 0$

$\Rightarrow |x| = 4$ or $\Rightarrow x = \pm 4$

And putting $x - 3 = 0 \Rightarrow x = 3$

Therefore, at $x = -4$, 3 and 4, LHS = 0

Range of values that will satisfy the above inequality will be $= x \in (-\infty, -4) \cup (3, 4)$.

Hence, the largest integer value of x satisfying the inequality $(|x| - 4)$ $(x - 3) < 0$ is -5.

Hence, option (C) is the correct answer.

9. (D)

Since we know that the arithmetic mean of two or more than two positive numbers is greater than or equal to their geometric mean.

AM > GM

$$\therefore \frac{\frac{k}{l} + \frac{l}{m} + \frac{m}{k}}{3} \geq \sqrt[3]{\frac{k}{l} \times \frac{l}{m} \times \frac{m}{k}}$$

$$\frac{\frac{k}{l} + \frac{l}{m} + \frac{m}{k}}{3} \geq 1$$

$$= \frac{k}{l} + \frac{l}{m} + \frac{m}{k} \geq 3$$

Hence, no such triplets of positive integers exist whose sum equals to 2.

Hence, option (D) is the correct answer.

10. (C)

Since the given equation is:

$|x - 5| \leq 10$ and $|3y - 6| \leq 12$

$\therefore x - 5 \leq 10$ or $x - 5 \geq -10$

$\therefore -10 \leq x - 5 \leq 10$

$-5 \leq x \leq 15$ ...(i)

Also, for the equation $|3y - 6| \leq 12$

$-12 \leq 3y - 6 \leq 12$

$-6 \leq 3y \leq 18$

$-2 \leq y \leq 6$ ...(ii)

Since we have to find the maximum value of $|x| - |y|$.

Therefore, $|x|$ should be the maximum and $|y|$ should be the minimum.

$\therefore |x| = 15$ and $|y| = 0$

Thus, the maximum value of $|x| - |y|$ $= 15 - 0 = 15$.

Hence, option (C) is the correct answer.

Level of Difficulty – 2

11. (C)

Given equation is $(x - 7)(x - 8) - 4(3x - 21) \leq 0$

$\Rightarrow x^2 - 15x + 56 - 12x + 84 \leq 0$

$\Rightarrow x^2 - 27x + 140 \leq 0$

$x^2 - 7x - 20x + 140 \leq 0$

$(x - 7)(x - 20) \leq 0$

Therefore, x can take any integer value from 7 to 20 (7 and 20 included) so as to satisfy the given inequality.

Total number of values = 14 values $(7 \leq x \leq 20)$.

Hence, option (C) is the correct answer.

12. (C)

It is given that,

$x^2 - 5kx + 6k^2 + 9 > 0$

If $b^2 - 4ac < 0$ then the given expression is always positive

$\Rightarrow b^2 - 4ac < 0$

$\Rightarrow (5k)^2 - 4 \times 1 \times (6k^2 + 9) < 0$

$\Rightarrow 25k^2 - 24k^2 - 36 < 0$

$\Rightarrow k^2 < 36$

$\Rightarrow |k| < 6$

$-6 < k < 6$

$k \in (-6, 6)$

Hence, option (C) is the correct answer.

13. 93,024

Since we know that

$$\text{AM} \geq \text{GM}$$

$$\therefore \frac{P+Q+R+S}{4} \geq (P \times Q \times R \times S)^{\frac{1}{4}}$$

$$\frac{70}{4} \geq (P \times Q \times R \times S)^{\frac{1}{4}}$$

$$17.5 \geq (P \times Q \times R \times S)^{\frac{1}{4}}$$

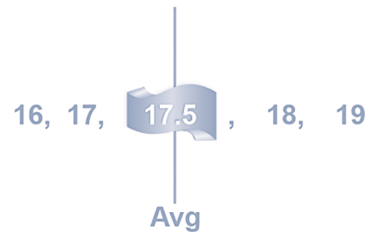
$\therefore (P \times Q \times R \times S)^{\frac{1}{4}} \leq 17.5$

$P \times Q \times R \times S \leq (17.5)^4$

If we take P = 17.5, Q = 17.5, R = 17.5, S = 17.5, then we will get maximum value of their product.

But in the question, it is given that P, Q, R, S are distinct natural numbers.

Therefore, we take all the four numbers nearby 17.5.



$\therefore$ P = 16, Q = 17, R = 18, and S = 19, if we take then we can find the maximum value of their product also.

Hence, $16 \times 17 \times 18 \times 19 = 93{,}024$.

14. (C)

Let $x^2 + 3x + 1 \geq 0$ ...(i)

Then, $x^2 + 3x + 1 + 2x + 5 = 0$

$x^2 + 5x + 6 = 0$

$x^2 + 3x + 2x + 6 = 0$

$x(x + 3) + 2(x + 3) = 0$

$\Rightarrow (x + 2)(x + 3) = 0$

$x = -2, x = -3$

$x = -2$ does not satisfy the inequation (i), but $x = -3$ does.

Now, if $x^2 + 3x + 1 < 0$ ...(ii)

Then, $-(x^2 + 3x + 1) + 2x + 5 = 0$

$$-x^2 - 3x - 1 + 2x + 5 = 0$$

$$-x^2 - x + 4 = 0$$

$$x^2 + x - 4 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 1 \times (-4)}}{2 \times 1}$$

$$x = \frac{-1 \pm \sqrt{17}}{2}$$

$$\therefore x = \frac{-1 + \sqrt{17}}{2} \text{ and } x = \frac{-1 - \sqrt{17}}{2}$$

But only $x = \dfrac{-1 - \sqrt{17}}{2}$ satisfies equation (i).

Hence, the solution set for the x is $\left(-3 \text{ and } \dfrac{-1 - \sqrt{17}}{2}\right)$.

Hence, option (C) is the correct answer.

15. (C)

It is given that

$(2x^2 + 3x + 1)^x < 1$

$\Rightarrow (2x^2 + 2x + x + 1)^x < 1$

$\Rightarrow [2x(x + 1) + 1(x + 1)]^x < 1$

$\Rightarrow [(x + 1)(2x + 1)]^x < 1$

Critical points = (–1, –1/2)

As one can conclude, the given expression becomes undefined at (–1) and (–1/2).

Also, one can conclude that when x becomes positive, LHS becomes greater than 1, so x must be negative.

A possible set of solutions = $(-\infty, -1)$.

16. (A)

$|p + q| = |p| + |q|$, then p and q will be of the same sign, which means

$p \times q = +\text{ve}$

Also, $|R + S| < |R| + |S|$

It will be possible only when both have different signs.

Then, $R \times S = -\text{ve}$

In the similar way $T \times U = -\text{ve}$

Thus $\underset{(+)}{(p \times q)} \times \underset{(-)}{(R \times S)} \times \underset{(-)}{(T \times U)} \Rightarrow +\text{ve}$

Hence, option (C) is the correct answer.

17. (B)

$(5x - 6) \times (15 - 5x)$

If we check sum of the above expression $5x - 6 + 15 - 5x = 9$ (constant).

Since the sum of the above expression is constant.

Therefore, its multiplication will be maximum when both expressions are equal to each other.

$5x - 6 = 15 - 5x$

$10x = 21$

$x = \frac{21}{10} = 2.1$

Hence, the above expression will attain maximum value when $x = 2.1$.

Now, we have to find maximum value of the expression

$= (5x - 6)(15 - 5x)$

$= [(5 \times 2.1 - 6) \times (15 - 5 \times 2.1)]$

$= 4.5 \times 4.5 = 20.25$.

Hence, option (B) is the correct answer.

18. (C)

Case 1:

$x > 6$

$|x + 5| + |x - 6| = 14$

$(x + 5) + (x - 6) = 14$

$2x - 1 = 14$

$2x = 14 + 1$

$x = 7.5$ ...(i)

Case 2:

$x < -5$

$|x + 5| + |x - 6| = 14$

$-(x + 5) - (x - 6) = 14$

$-x - 5 - x + 6 = 14$

$-2x = 13$

$x = -\frac{13}{2} = -6.5$ ...(ii)

Case 3:

$-5 < x < 6$

$|x + 5| + |x - 6| = 14$

$(x + 5) - (x - 6) = 14$

$-1 = 14$, which is not possible.

Hence, the sum of all possible values of x that satisfies the given equation $= 7.5 - 6.5 = 1$.

Hence, option (C) is the correct answer.

19. (A)

We have:

$|3y - 12| \le 18$

$-18 \le (3y - 12) \le 18$

$-6 \le 3y \le 30$

$-2 \le y \le 10$ ...(i)

$| x - 5 | \leq 13$

$- 13 \leq x - 5 \leq 13$

$- 8 \leq x \leq 18$ ...(ii)

For, minimum value of $|x| - |y|$, $|y|$ must be maximum and $|x|$ must be minimum.

Now, from equations (i) and (ii),

maximum value of $|y| = 10$

minimum value of $|x| = 0$

Therefore, $|x| - |y| = 0 - 10 = -10$.

Hence, option (A) is correct.

20. 6

Given: $|x - 2| \leq 4$ ...(i)

and

$|x - 1| \geq 2$ ...(ii)

Using eqation (i)

$|x - 2| \leq 4$

$-4 \leq x - 2 \leq 4$

$\Rightarrow -4 + 2 \leq x \leq 4 + 2$

$\Rightarrow -2 \leq x \leq 6$ ...(iii)

Using equation (ii)

$|x - 1| \geq 2$

$\Rightarrow x - 1 \leq -2$ or $x - 1 \geq 2$

$\Rightarrow x \leq -2 + 1$ or $x \geq 2 + 1$

$\Rightarrow x \leq -1$ or $x \geq 3$ ....(iv)

From equations (iii) and (iv)

$-2 \leq x \leq -1$ and $3 \leq x \leq 6$

So, solution set for x is [–2,–1] U [3,6].

Hence, 6 integral values of x, i.e., –2, –1, 3, 4, 5, and 6 are there, which satisfy both the given inequalities.

Level of Difficulty – 3

21. 5

The given equation is:

$(x - 2)(x + 3)(x - 4) \geq (x + 2)(x - 3)(x + 4)$

$(x^2 + 3x - 2x - 6)(x - 4) \geq (x^2 - 3x + 2x - 6)(x + 4)$

$X^3 + 3x^2 - 2x^2 - 6x - 4x^2 - 12x + 8x + 24$
$\geq x^3 - 3x^2 + 2x^2 - 6x + 4x^2 - 12x + 8x - 24$

$- 3x^2 - 10x + 24 \geq 3x^2 - 10x - 24$

$6x^2 \leq 48$

$x^2 \leq 8$

Therefore, –2, –1, 0, 1, and 2 are five integers only that satisfy the given inequality.

22. (D)

From the question:

$|||x - 5| + 4| - 3| = 0$

$\Rightarrow ||x - 5| + 4| = 3$

$\Rightarrow |x - 5| + 4 = +3$ or -3

As the value of modulus can never be negative, we can eliminate –3.

$\Rightarrow |x - 5| = -1$

Again, as the value of modulus can never be negative, no solution is possible.

Hence, option (D) is correct.

23. 256

$$\frac{2P + 2P + 4Q + 4Q + 4Q + R + R + R}{8} \geq \sqrt[8]{(2P)^2(4Q)^3(R^3)}$$

$$\Rightarrow \frac{32}{8} \geq (256 \times P^2 \times Q^3 \times R^3)^{1/8}$$

$$\Rightarrow 4 \geq 2(P^2Q^3R^3)^{1/8}$$

$$\Rightarrow 2 \geq (P^2Q^3R^3)^{1/8}$$

$$\Rightarrow (P^2Q^3R^3)^{1/8} \leq 2$$

$$\Rightarrow P^2Q^3R^3 \leq 2^8$$

$$\Rightarrow P^2Q^3R^3 \leq 256$$

Hence, the maximum possible value of $P^2Q^3R^3 = 256$.

24. (D)

The range of values of xy is $-12 \leq xy \leq 12$

(For any value of $-3 \leq x \leq -1$, $-4 \leq y \leq 4$).

Now, $\frac{xy}{z}$ the range will be

$$\left(\frac{xy}{z}\right) = \left(\frac{-12}{-2}\right) = 6; \quad \frac{xy}{z} = \frac{12}{-2} = -6$$

So, the possible range is

$$-6 \le \frac{xy}{z} \le 6$$

Hence, option (D) is the correct answer.

25. 5

$$P - 5 + \frac{9}{P-4} = P - 4 + \frac{9}{P-4} - 1$$

Since, AM > GM

$$\therefore \frac{P - 4 + \frac{9}{P-4}}{2} \ge \sqrt{(P-4) \times \frac{9}{(P-4)}}$$

$$P - 4 + \frac{9}{P-4} \ge 3 \times 2$$

$$(P-4) + \frac{9}{P-4} \ge 6$$

$\therefore$ minimum value of $(P-4) + \frac{9}{P-4} = 6$

Therefore, the minimum value of

$$p - 5 + \frac{9}{p-4} = (P-4) + \frac{9}{P-4} - 1 = 6 - 1 = 5.$$

26. 1,296

We should know that, for positive real numbers:

Geometric mean ≥ Harmonic mean

Here, we should consider: GM ≥ HM

Therefore, $(abcd)^{\frac{1}{4}} \ge \dfrac{4}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}$

$$(abcd)^{\frac{1}{4}} \ge \frac{4abcd}{bcd + acd + abd + abc}$$

Let $abcd = k$, then

$$k^{\frac{1}{4}} \ge \frac{4k}{864}$$

$$k^{\frac{1}{4}} \ge \frac{k}{216}$$

$$216 \ge \frac{k}{k^{\frac{1}{4}}}$$

$$216 \ge k^{\frac{3}{4}}$$

$$(216)^{\frac{4}{3}} \ge k$$

$$(6^3)^{\frac{4}{3}} \ge k$$

$$1,296 \ge k$$

Or $\quad 1,296 \ge abcd$

Here the maximum value of $abcd$ is 1,296.

Hence, 1,296 is the correct answer.

27. (C)

Since all the numerators and denominators are in modulus, the least possible value will be more than 0 as all three of a, b, and c cannot be zero simultaneously

We know that

$|a + b| \le |a| + |b|$, then $\dfrac{|a+b|}{|a| + |b|} \le 1$.

This applies to all terms.

Thus, the maximum value can be 3. Also, it is not possible for all terms to be zero.

To make two terms zero

$a = b = -c$

$$\Rightarrow \frac{|2a|}{2\,|a|} + \frac{|0|}{|b| + |c|} + \frac{|0|}{|c| + |a|} = 1$$

So the least possible value will be 1 and the greatest value 3.

Hence, option (C) is the correct answer.

28. (C)

The given equation is $\left|\dfrac{a-6}{a+5}\right| > 1$

$$\Rightarrow \frac{a-6}{a+5} > 1 \quad \text{or} \quad \frac{a-6}{a+5} < -1$$

$$\Rightarrow \frac{a-6}{a+5} - 1 > 0 \quad \text{or} \quad \frac{a-6}{a+5} + 1 < 0$$

$$\Rightarrow -\frac{11}{a+5} > 0 \quad \text{or} \quad \frac{2a-1}{a+5} < 0$$

$$\Rightarrow \frac{11}{a+5} < 0 \quad \text{or} \quad \frac{2a-1}{a+5} < 0$$

$\Rightarrow -5 < a < \frac{1}{2}$

$\Rightarrow a < -5$

If we combine both the equation:

$a < -5$ and $(-5) < a < \left(\frac{1}{2}\right)$

Then we will get

$a < \frac{1}{2}$ and $a \neq -5$

Hence, option (C) is the correct answer.

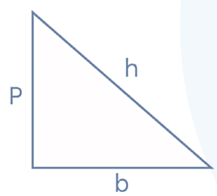
29. 4

Since we know that AM ≥ GM

$$\therefore \frac{\frac{h}{P}+\frac{h}{b}}{2} \geq \sqrt{\frac{h}{P} \times \frac{h}{b}}$$

$$\frac{h}{p}+\frac{h}{b} \geq 2\sqrt{\frac{h^2}{Pb}}$$

$$\frac{h}{p}+\frac{h}{b} \geq 2\sqrt{\frac{P^2+b^2}{Pb}}$$



$\left[h^2 = P^2 + b^2\right]$

$$\frac{h}{p}+\frac{h}{b} \geq 2\sqrt{\frac{P}{b}+\frac{b}{P}}$$

$$\frac{h}{p}+\frac{h}{b} \geq 2\sqrt{2}$$

Because $\left[X + \frac{1}{X} \geq 2\right]$

Therefore, $\frac{h}{P}+\frac{h}{b} \geq 2\sqrt{2}$

So, the minimum value of the expression $\left(\frac{h}{P}+\frac{h}{b}\right)$ is $2\sqrt{2}$.

Hence, the minimum value of

$$\left\{\sqrt{2}\left(\frac{h}{P}+\frac{h}{b}\right)\right\} = \sqrt{2} \times 2\sqrt{2} = 4.$$

30. (C)

In this question we have to check for all the regions:

Region 1 : $x > 0$ and $y > 0$

Region 2 : $x > 0$ and $y < 0$

Region 3 : $x < 0$ and $y > 0$

Region 4 : $x < 0$ and $y < 0$

If we find the answer from any of the regions, there is no need to go to other regions.

Therefore, region 1: $x > 0$ and $y > 0$

Then, $4x + 3y + y = 8$ and $x + x + 4y = 2$

$x + y = 2$ and $x + 2y = 1$

$$\begin{array}{r} x + y = 2 \\ x + 2y = 1 \\ - \quad - \quad - \\ \hline -y = 1 \end{array}$$

$\Rightarrow y = -1$

Since in region 1: $y > 0$

Thus, the region 1 is not satisfying the above condition.

Now, we have check for region 2: $x > 0$ and $y < 0$

$\therefore \quad 4x - 3y + y = 8$ and $x + x + 4y = 2$

$4x - 2y = 8$

$2x - y = 4$ and $x + 2y = 1$

Solving which we will get $x = 1.8$ and $y = -0.4$.

This satisfies both the conditions.

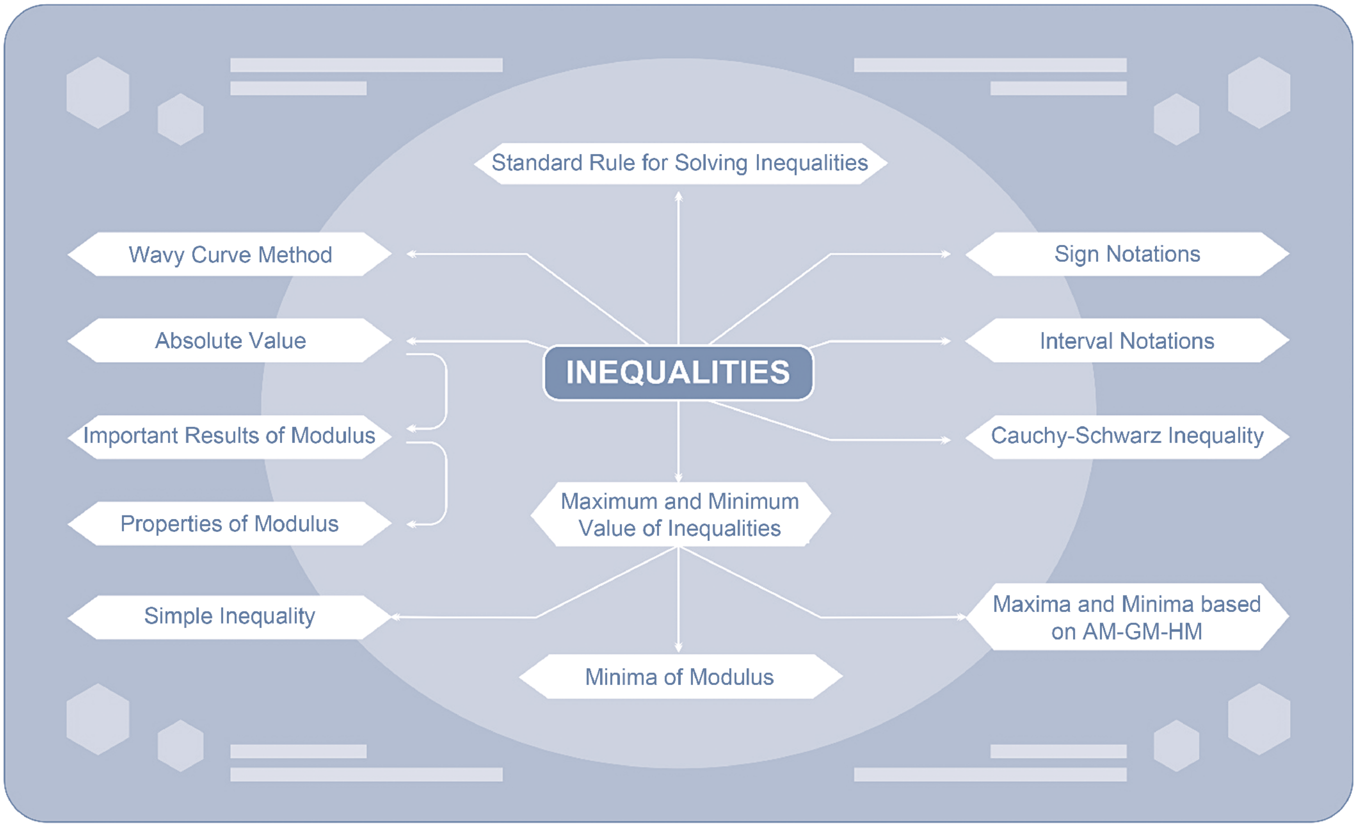
Therefore $x = 1.8$ and $y = -0.4$ is the correct point.

Therefore, the value of $x \times 5y = 1.8 \times 5 \times (-0.4) = -3.6$

Region 3 and region 4 will also not satisfy the conditions.

Hence, option (C) is the correct answer.

Mind Map



Standard Rule for Solving Inequalities
Wavy Curve Method
Sign Notations
Absolute Value
INEQUALITIES
Interval Notations
Important Results of Modulus
Cauchy-Schwarz Inequality
Properties of Modulus
Maximum and Minimum Value of Inequalities
Simple Inequality
Maxima and Minima based on AM-GM-HM
Minima of Modulus