6 Sequences and Series

Introduction

Every year around one or two questions are asked in CAT from this chapter. Rigorous practice and in-depth knowledge of concepts helps one to master this topic.

What is a sequence?

A group of numbers arranged in a definite order according to a given certain rule is called a sequence.

Terms of a sequence are written as $a_1, a_2, a_3, a_4, ..., a_n$, but when the terms are written as, $a_1 + a_2 + a_3 + a_4 + ...+ a_n$, it is known as series.

Before moving on, it is very important for one to know the usefulness of Arithmetic Progression, Geometric Progression, and Harmonic Progression.

Arithmetic Progression (A.P.)

An Arithmetic Progression (A.P.) is a sequence of numbers in which each term is obtained by adding a fixed number 'd' to the preceding term, except the first term. The fixed number 'd' is called as the common difference.

First terms of an A.P. is 'a' and common difference between any two consecutive terms is 'd'.

a	a + d	a + 2d,	....	'a + (n - 1)d'
↓	↓	↓		↓
T_1	T_2	T_3		T_n

Where n = (number of terms in a sequence), and $T_1, T_2, T_3..............T_n$ are the first, second, third .... and n^{th} term of the progression respectively.

$$T_n \boxed{= n^{th} \text{ term} = a + (n - 1)d}$$

Let us find the sum of first n terms of an A.P. (sum of n terms = S_n)

Sum of n terms can be written as:

$$S_n = T_1 + T_2 + T_3 + T_4 + T_5 + ...+ T_n$$

$$S_n = (a) + (a + d) + (a + 2d) + (a + 3d) +.........+ [a + (n - 1) \times d] \quad ...(i)$$

This can also be written as:

$$S_n = T_n + T_{(n-1)} + T_{(n-2)} +.........+ T_2 + T_1$$

$$S_n = (a + (n - 1) \times d) + (a + (n - 2) d) +.........+ (a + d) + (a) \quad ...(ii)$$

On adding equation (i) and (ii), we get,

$$2S_n = 2a + (n - 1) d + (2a + (n - 1)d) + (2a + (n - 1)d).........(n \text{ times})$$

$$2S_n = n \times (2a + (n - 1) \times d)$$

$$S_n = \frac{n}{2}(2a + (n - 1) \times d)$$

$$S_n = \frac{n}{2}[a + (a + (n - 1)d)]$$

$$S_n = \frac{n}{2}[a + T_n]$$

Arithmetic Mean:

The average of all the terms in an A.P. is called their Arithmetic Mean (A.M.)

$$\text{A.M.} = \frac{\text{Sum of n terms}}{n}$$

$$\frac{S_n}{n} = \frac{1}{2} \times (2a + (n - 1) \times d) = \left[\frac{a + a + (n - 1)d}{2}\right]$$

$$= \left[\frac{\text{First term} + \text{Last term}}{2}\right]$$

Rack Your Brain

Find the sum of the 14^{th} and 17^{th} terms of an arithmetic progression containing 30 terms, if the sum of its first and last terms is 61.

(A) 51 (B) 73

(C) 82 (D) 61

Note:

1. If an A.P. has an odd number of terms, then the average will be the middle term.
2. If an A.P. has an even number of terms, then the average will be the average of the middle two terms.
3. If any two terms which are equidistant from both ends (from the beginning and the end), then the average of an A.P. will be the average of these two terms.

   i.e., the average of the fourth term from the beginning and the fourth term from the end will be equal to the A.M. and so on.
4. Sum of the first n terms of an A.P can be written in terms of A.M.

   $[S_n = n \times \text{A.M.}]$
5. If three numbers are in A.P., then these numbers can be taken as (a – d), a and (a + d)

   Four numbers in A.P. can be taken as

   (a – 3d), (a – d), (a + d), and (a + 3d)

   Five numbers in A.P. can be taken as

   (a – 2d), (a – d), (a), (a + d), and (a + 2d)

Summations:

From the above learning, we can derive the formula for the sum of the first n natural numbers.

Here one can take an assumption that 'a' = 1 and 'd' = 1

$$S_n = n \times \text{A.M.}$$

$$S_n = n \times \frac{(1+n)}{2}$$

$$\boxed{S_n = \frac{n(n+1)}{2}}$$

$$\sum_1^n n = \frac{n(n+1)}{2}$$

Similarly, we can find the formulae for $\sum_1^n n^2$ and $\sum_1^n n^3$ and from an exam point of view, one must know about the end result.

$\sum_1^n n^2 = \frac{n(n+1)(2n+1)}{6}$ (Sum of squares of the first n natural numbers)

$\sum_1^n n^3 = \left(\sum_1^n n\right)^2 = \left[\frac{n(n+1)}{2}\right]^2$ (Sum of cubes of the first n natural numbers)

Let us understand some examples based on above learning.

Example 1:

Find the sum of the first 19 terms of an A.P., whose first term is 6 and common difference is $\frac{7}{3}$.

Solution: 513

$n = 19$

$a = 6$

$d = \frac{7}{3}$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_n = \frac{19}{2}\left(2 \times 6 + (19-1) \times \frac{7}{3}\right)$$

$$= \frac{19}{2}[12 + 42]$$

$$= 19 \times 27 = 513$$

Rack Your Brain

If the sum of the first 'n' terms of an Arithmetic Progression is a perfect square, and that is the 1st three digit perfect square, and the sum of the next 'n' terms of the same Arithmetic Progression is 3 times of that square, then the ratio of the first term and the common difference is $m:n$. Find the value of $m + n$.

(A) 2

(B) 3

(C) 4

(D) Cannot be determined

Example 2:

Find the sum of the first 18 even perfect squares.

Solution: 8436

First 18 even perfect squares = 4, 16, 36, 64..........1296

$$= 4\,(1^2 + 2^2 + 3^2 + 4^2 + 5^2 \text{..........} + 18^2)$$

$$= 4\left[\frac{18 \times 19 \times 37}{6}\right]$$

$$= 8436$$

Example 3:

Find the sum of the first 10 natural numbers which leaves a remainder of 2 when divided by 10.

Solution: 470

The first 10 natural numbers which leaves a remainder of 2 when divided by 10 are 2, 12, 22, 32, 42, 52, 62, 72, 82, and 92.

These 10 numbers are in A.P.

$a = 2$

$d = 12 - 2 = 10$

$$S_n = \frac{n}{2}(2a + (n-1)\ d)$$

$$S_{10} = \frac{10}{2}(2 \times 2 + (10-1) \times 10)$$

$\Rightarrow$ 5 (4 + 90)

$\Rightarrow$ 470 Ans.

Example 4:

If the sum of the first 7 terms of an A.P. is equal to the sum of the first 14 terms of the same A.P. Which term of this A.P. must necessarily be zero?

Solution: 11th term

(Traditional approach):

It is given that, $S_7 = S_{14}$

$$\frac{7}{2}(2a + 6d) = \frac{14}{2}(2a + 13d)$$

$(2a + 6d) = (4a + 26d)$

$2a + 20d = 0$

$a + 10d = 0$

Alternate Method: (Logical approach):

Sum of the first 7 terms = Sum of the first 14 terms.

$$(T_1 + T_2 + T_3 + \ldots + T_7) = (T_1 + T_2 + T_3 + \ldots + T_{14})$$

It implies that sum of the 8^{th} to 14^{th} term

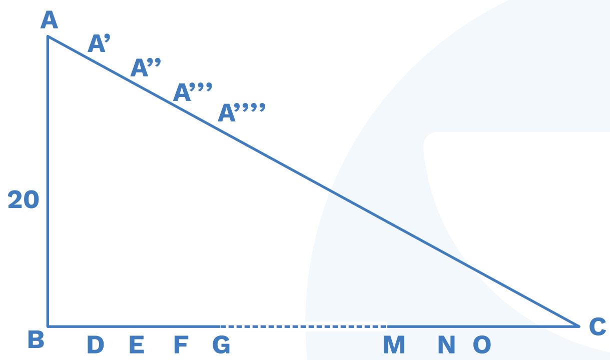
$(T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13} + T_{14})$ must be zero.

One can conclude that the average of those 7 terms must also be zero.

Average of 7 terms in A.P. is the 4^{th} term which means T_{11} must be equal to zero.

Example 5:

In the given figure, all the vertical lines are perpendicular to the base of the triangle, and the base of the triangle is divided into 12 equal parts. The length of the leftmost vertical line is 20. Find the sum of the length of all the vertical lines.



Solution: (130)

It is given that, AB = 20

Let BC = 12x

In ΔABC $\quad \text{TanC} = \dfrac{AB}{BC} \Rightarrow \dfrac{20}{12x}$ ...(1)

In $\Delta A'DC$ $\quad \text{TanC} = \dfrac{A'D}{DC} = \dfrac{A'D}{11x}$ ...(2)

From (1) and (2) we get,

$$\frac{20}{12x} = \frac{A'D}{11x};\ A'D = \frac{55}{3}$$

In $\Delta A''EC$ $\quad \text{TanC} = \dfrac{A''E}{EC} = \dfrac{A''E}{10x}$ ...(3)

From (1) and (3) we get,

$$\frac{AB}{BC} = \frac{A''E}{10x} \Rightarrow \frac{20}{12x} = \frac{A''E}{10x}$$

$$A''E = \frac{50}{3}$$

Here one can see a pattern in the length of the perpendicular.

$$20, \frac{55}{3}, \frac{50}{3}$$

$$a = 20 \text{ and } d = \frac{55}{3} - 20 = \left(-\frac{5}{3}\right)$$

$$T_{12} = a + 11d = 20 + 11 \times -\frac{5}{3} \Rightarrow \frac{5}{3}$$

$$S_{12} = \frac{12}{2}(a + T_n) = \frac{12}{2}\left(20 + \frac{5}{3}\right)$$

$$S_{12} = \frac{12}{2} \times \frac{65}{3} = 130$$

Geometric Progression (G.P.)

A Geometric Progression (G.P.) is a sequence of terms in which the ratio of any term to its preceding term is the same throughout. This ratio of any term to its preceding term is called as common ratio.

For example, 2, 6, 18, 54, 162, 486......... is a G.P.

This G.P. can be written as,

$2, 2 \times 3, 2 \times 3 \times 3, 2 \times 3 \times 3 \times 3, 2 \times 3 \times 3 \times 3 \times 3, ...$

(Here 3 is the common ratio, i.e., r = 3)

From this logic, we can say that 'a' which is the first term and 'r' which is the common ratio, then a, ar, ar^2, ar^3, ar^4 .... $a\, r^{(n-1)}$ are in G.P.

$$T_n \boxed{= a \times (r)^{(n-1)}}$$

Let's find the sum of the first n terms of a G.P. (Sum of n terms = S_n)

Sum of n terms can be written as,

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{(n-1)} \quad ...(1)$$

Mulitply 'S_n' by 'r' we get,

$$r \times S_n = ar + ar^2 + ar^3 + ... + ar^{(n-1)} + ar^n \ ...(2)$$

Subtract equation (1) from (2) we get

$$rS_n - S_n = ar + ar^2 + ar^3 \ldots ar^{n-2} + ar^{n-1} + ar^n$$

$- a - ar - ar^2 - ar^3 \ldots\ldots\ldots ar^{n-2} - ar^{n-1}$

$S_n (r - 1) = ar^n - a$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

Sometimes it is useful to write the formula for S_n as $S_n = a\frac{(1-r^n)}{(1-r)}$

One can use both the formulas as per one's convenience

If r > 1, then one can use $S_n = \frac{a(r^n - 1)}{(r-1)}$

If r < 1, then one can use $S_n = \frac{a(1-r^n)}{(1-r)}$

Geometric Mean:

The average of all the terms in a G.P. is called their Geometric Mean.

In other words, one can say that Geometric Mean of a Geometric Progression is the middle term, or else the geometric mean of the first and last term.

Formula for G.M. of two natural numbers a and b = $\sqrt{ab}$.

For example: The geometric mean of 5^5, 5^6, 5^7, 5^8, 5^9, 5^{10} and 5^{11} is 5^8

$$GM = \sqrt{5^5 \times 5^{11}} = \sqrt{5^{16}} = 5^8$$

Infinite Geometric Progression:

In a specific G.P. where $-1 < r < 1$, and the terms of G.P. becomes smaller and smaller as the number of terms keeps on increasing, such G.P. is known as infinite geometric progression.

The sum of a G.P. to infinite terms can be obtained from the formula

$$S_n = a\left(\frac{r^n - 1}{r-1}\right)$$

Here, $n = \infty$

$$S_\infty = \frac{a(r^\infty - 1)}{r-1} \Rightarrow \frac{a(-1)}{r-1}$$

(As $-1 < r < 1$, r to the power infinity will be equal to zero)

$$\left(S_\infty = \frac{a}{1-r}\right)$$

Example 6:

If the first term of a G.P. is $\frac{1}{12}$ and the common ratio is -3, then find the 13^{th} term of the G.P.

Solution: $\mathbf{\frac{(-3)^{12}}{12}}$

It is given that, $a = \frac{1}{12}$ and $r = -3$

$$T_n = a \times r^{(n-1)}$$

$$T_{13} = a \times (r)^{(13-1)}$$

$$= \frac{1}{12} \times (-3)^{12} = \frac{(-3)^{12}}{12}$$

Example 7:

Find three numbers in G.P. whose sum is 73 and product is 512.

Solution: (64, 8 and 1) or (1, 8 and 64)

Let the three numbers be $\left(\frac{a}{r}, a, ar\right)$

Product of terms = 512

$$\frac{a}{r} \times a \times ar = 512$$

$\therefore \quad a^3 = 512$

$$\boxed{a = 8}$$

$\Rightarrow \quad \frac{a}{r} + a + ar = 73$

$\Rightarrow \quad a + ar + ar^2 = 73r$

$\Rightarrow \quad 8 + 8r + 8r^2 = 73r$

$\Rightarrow$ $8r^2 - 65r + 8 = 0$

$\Rightarrow$ $8r^2 - 64r - r + 8 = 0$

$\Rightarrow$ $(8r - 1)(r - 8) = 0$

$\Rightarrow$ $\left(r = 8, \frac{1}{8}\right)$

Required G.P. when r = 8 is [1, 8, 64]

and when r = 1/8 is [64, 8, 1]

Example 8:

A baby plant grows 1.53 cm in its first week. Each week it grows by 3% more than it did in the week before. By how much does it grow in nine weeks, including the first week (approx)?

Solution: (15.54 cm)

It is given that, A plant grows 1.53 cm in its first week.

The growth in the first nine weeks are as follows:

$1.53,\ 1.53 \times 1.03,\ 1.53 \times (1.03)^2,\ 1.53 \times (1.03)^3 \ldots\ldots\ldots 1.53 \times (1.03)^8$

Total growth in first nine weeks is

$$S_9 = 1.53\left[\frac{(1.03)^8 - 1}{1.03 - 1}\right] = 15.54 \text{ cm}$$

Example 9:

The sum of 7 + 77 + 777 +.......... upto n terms is?

Solution: ($\frac{70}{81}(10^n - 1) - \frac{7n}{9}$)

It is given that,

$$S = (7 + 77 + 777 + \ldots\ldots\ldots n \text{ terms})$$

$$S = 7\,(1 + 11 + 111 + \ldots\ldots\ldots n \text{ terms})$$

$$S = \frac{7}{9}\left[\begin{array}{l}(10-1)+(100-1)+(1000-1)\\ +\ldots n \text{ terms}\end{array}\right]$$

$$S = \frac{7}{9}\left[\begin{array}{l}\underbrace{(10+10^2+10^3+\ldots\ldots\ldots+10^n)}_{\text{Forms a G.P.}}\\ -\,(1+1+1+\ldots\ldots n \text{ terms})\end{array}\right]$$

$$S = \frac{7}{9}\left[10\left(\frac{10^n - 1}{10 - 1}\right) - n \times 1\right]$$

$$S_n = \frac{7}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$$

$$S_n = \frac{70}{81}\left(10^n - 1\right) - \frac{7n}{9}$$

Hence, the required sum is $\frac{70}{81}\left(10^n - 1\right) - \frac{7n}{9}$

Example 10:

On the first day of its release, the movie NOISE 2 makes 300 crores. On each subsequent day, it makes one third as much as it did on the previous day. How much will NOISE 2 make by the end of its run?

Solution: (450 crores)

The first day the movie makes 300 crores.

On the second day = $300 \times \frac{1}{3} = 100$ crores

On the third day = $100 \times \frac{1}{3} = \frac{100}{3}$ crores

Required G.P. = 300, 100, $\frac{100}{3}$ ... till infinity

$$a = 300, \quad r = \frac{100}{300} = \frac{1}{3}$$

$$S_\infty = \frac{a}{1-r} \Rightarrow \frac{300}{1 - \frac{1}{3}} = 300 \times \frac{3}{2}$$

= 450 crores

Rack Your Brain

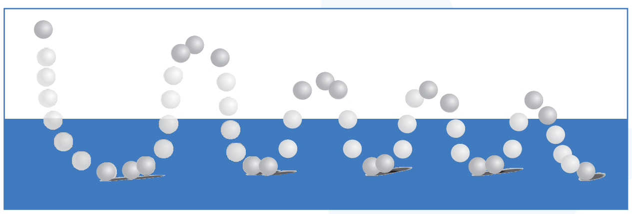
If the product of the twentieth and fifty third term of a geometric progression is 69, then what is the product of the first seventy-two terms of the progression?

(A) $(20)^{69}$ (B) $(53)^{69}$

(C) $(69)^{36}$ (D) $(69)^{53}$

Example 11:

A ball is dropped from a height of 100 feet. Every time it bounces to a height of $(3/4)^{th}$ of what it had achieved on the previous bounce. How far will it have travelled before it comes to rest?



Solution: (700 feet)

The distance that the ball travels the first time is 100.

Here the situation is slightly different as during the first downward journey the ball travels 100 feet but in consecutive bounces it moves up as well as down. In each bounce it will travel to a height of $(3/4)^{th}$ of that of the previous one.

Required distance will be

$$100 + 2 \times 75 + 2 \times \frac{225}{4} + \text{.........}$$

$$= 100 + 2 \times 75\left(1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots\right)$$

$$= 100 + \frac{150}{1 - \frac{3}{4}}$$

$$= 100 + 600 = 700 \text{ feet}$$

Harmonic Progression (H.P.)

If the reciprocal of the terms of a sequence are in Arithmetic Progression then the sequence is said to be a Harmonic Progression.

For Example:

$$\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \text{..........}$$

Reciprocal of terms are in A.P., so the given sequence is in Harmonic Progression.

If ‘a’, ‘b’ and ‘c’ are in Harmonic Progression, then b is said to be the harmonic mean of ‘a’ and ‘c’.

Arithmetic-Geometric Sequence (A.G.P.):

An arithmetic-geometric sequence is a type of a sequence in which each term is the product of the corresponding term of an A.P. and a G.P..

For Example:

Given A.P. = 1, 2, 3, 4, 5, 6,..........

Given G.P. = 2, 4, 8, 16, 32, 64,..........

Then the corresponding AGP would be

2, 8, 24, 64, 160,..........

Let’s understand some examples based on AGP:

Example 12:

Find the sum of infinite terms of the series $x, 3x^2, 5x^3, 7x^4, 9x^5$,..........

Solution: $\left(\frac{x}{(1-x)} + \frac{2x^2}{(1-x)^2}\right)$

Given AGP, $x, 3x^2, 5x^3, 7x^4, 9x^5$,..........

Sum of the series

$$(s) = x + 3x^2 + 5x^3 + 7x^4 + \text{.........} \quad \text{...(i)}$$

On multiplying equation (i) by x, we get

$$x \times s = x^2 + 3x^3 + 5x^4 + 7x^5 + 9x^6 \qquad ...(ii)$$

Subtracting equation (ii) from (i) we get,

$$S - xs = x + \underbrace{2x^2 + 2x^3 + 2x^4 + 2x^5 + \ldots\ldots\ldots}_{\text{Infinite G.P. is formed}}$$

$$S(1-x) = x + \frac{2x^2}{1-x}$$

$$S = \frac{x}{(1-x)} + \frac{2x^2}{(1-x)^2}$$

Example 13:

Find the sum of infinite terms of the series:

$$\frac{11}{13} + \frac{14}{169} + \frac{17}{2197} + \frac{20}{28561} + \ldots\ldots\ldots$$

Solution: $\left(\frac{15}{16}\right)$

Let $S = \frac{11}{13} + \frac{14}{169} + \frac{17}{2197} + \frac{20}{28561}$

This can be written as,

$$S = 11 \times \frac{1}{13} + 14 \times \frac{1}{(13)^2} +$$

$$17 \times \frac{1}{(13)^3} + 20 \times \frac{1}{(13)^4} + \cdots$$

Here, one can observe that the common ratio of the G.P. is $\left(\frac{1}{13}\right)$

Multiply S by 13 we get,

$$13S = 11 + \frac{14}{13} + \frac{17}{169} + \frac{20}{2197} + \ldots\ldots\ldots$$

(13S) – (S) we get,

$$= 11 + \frac{3}{13} + \frac{3}{169} + \frac{3}{2197} + \ldots\ldots\ldots$$

$$12S = 11 + 3 \times \left[\frac{1}{13} + \frac{1}{13^2} + \frac{1}{13^2} + \ldots\ldots\ldots\right]$$

$$12S = 11 + 3 \times \frac{1}{12}$$

$$12S = 11 + \frac{1}{4}$$

$$12S = \frac{45}{4}$$

$$\Rightarrow S = \frac{15}{16}$$

Rack Your Brain

If S is the sum, P is the product and R is the sum of the reciprocals of the n terms of a GP., then which of the following result is true?

(A) $\left[\frac{S}{R}\right]^{\frac{n}{2}} = P^2$ (B) $\left[\frac{S}{R}\right] = P^2$

(C) $\left[\frac{S}{R}\right]^{n} = P^2$ (D) $\left[\frac{S}{R}\right]^{2n} = P^2$

Example 14:

Find the sum series $\frac{3}{8} + \frac{7}{64} + \frac{11}{512} + \frac{15}{4096} + \cdots$ upto infinity

Solution: $\left(\frac{25}{49}\right)$

Let $S = \frac{3}{8} + \frac{7}{64} + \frac{11}{512} + \frac{15}{4096} + \ldots\ldots\ldots$

Multiply (S) by 8 $\left(\because \text{ Ratio of G.P is } \frac{1}{8}\right)$

$$8S = 3 + \frac{7}{8} + \frac{11}{64} + \frac{15}{512} + \ldots\ldots\ldots$$

$$8S - S = 3 + \frac{4}{8} + \frac{4}{64} + \frac{4}{512} + \ldots\ldots\ldots$$

$$7S = 3 + 4 \times \left[\frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^2} + \ldots\ldots\ldots\right]$$

$$7S = 3 + 4 \times \frac{1}{7}$$

$$7S = 3 + \frac{4}{7}$$

$\Rightarrow \quad S = \dfrac{25}{49}$

Previous Years' Question

A gentleman decided to treat a few children in the following manner. He gives half of his total stock of toffees and one extra to the first child, and then the half of the remaining stock along with one extra to the second and continues giving away in this fashion. His total stock exhausts after he takes care of 5 children. How many toffees were there in his stock initially?

Example 15:

Find the sum of first 20 terms of the series whose n^{th} term is given by n(n + 7).

Solution: 4,340

It is given that

$$T_n = n(n+7) \quad \Rightarrow \quad n^2 + 7n$$

$$S_n = \sum_{n=1}^{n} t_n$$

$$S_n = \sum_{n=1}^{n} n^2 + 7\sum_{n=1}^{n} n$$

Sum to the first n natural numbers

Sum to the square of first n natural numbers

$$S_n = \frac{n(n+1)(2n+1)}{6} + 7 \times \frac{n(n+1)}{2}$$

$$\left[\because \quad \sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{n=1}^{n} n = \frac{n(n+1)}{2}\right]$$

$$S_{20} = \frac{20(21)(41)}{6} + 7 \times \frac{20(21)}{2}$$

$$S_{20} = 70 \times 41 + 70 \times 21$$

$$S_{20} = 70(41 + 21)$$

$$S_{20} = 70 \times 62 \quad \Rightarrow \quad 4340$$

Example 16:

Find the sum to first 25 terms of the series,

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + \ldots\ldots..$$

Solution: 5,850

Given series,

$$\underset{T_1}{(1 \times 2)} + \underset{T_2}{(2 \times 3)} + \underset{T_3}{(3 \times 4)} + \underset{T_4}{(4 \times 5)} + \underset{T_5}{(5 \times 6)} + \ldots$$

$T_n = ?$

Here, T_1 start with 1, T_2 start with 2 and T_3 start with 3, so T_{25} must start with 25.

Gap between the product of two numbers is 1.

$$T_n = n \times (n+1) \quad \Rightarrow \quad n^2 + n$$

$$S_n = \sum_{n=1}^{n} (n^2 + n) \quad \Rightarrow \quad \sum_{n=1}^{n} n^2 + \sum_{n=1}^{n} n$$

$$S_n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$S_{25} = \frac{25(26)(51)}{6} + \frac{25 \times 26}{2}$$

$$S_{25} = 5850$$

Example 17:

Find the sum to first 25 terms of the infinite series, $(1 \times 3) + (2 \times 4) + (3 \times 5) + (4 \times 6) + \ldots\ldots..$ infinity.

Solution: 6,175

Given series,

$$\underset{T_1}{(1 \times 3)} + \underset{T_2}{(2 \times 4)} + \underset{T_3}{(3 \times 5)} + \underset{T_4}{(4 \times 6)} + \ldots\ldots.. \quad T_n = ?$$

Here, $T_1 = 1 \times 3$. Gap between 1 and 3 is 2.

T_2 = 2 × 4. Gap between 2 and 4 is 2.

According to the pattern,

T_{25} = 25 × 27 and gap between 25 and 27 is 2.

$$T(n) = n(n + 2)$$

$$S_n = \sum_{n=1}^{n} n(n+2) \Rightarrow \sum_{n=1}^{n} n^2 + 2n$$

$$S_n = \frac{n(n+1)(2n+1)}{6} + 2 \times \frac{[n(n+1)]}{2}$$

$$S_{25} = \frac{25 \times 26 \times 51}{6} + 25 \times 26$$

$$S_{25} = 25 \times 13 \times 17 + 25 \times 26$$

$$S_{25} = 25 \times 13\ [17 + 2]$$

$$S_{25} = 25 \times 13 \times 19$$

$$S_{25} = 6175.$$

Example 18:

Find the sum to the first 20 terms of the series (2 × 5) + (3 × 6) + (4 × 7) + (5 × 8) + (6 × 9) +........ ∞.

Solution: 4,000

Given series,

$$(2 \times 5) + (3 \times 6) + (4 \times 7) + (5 \times 8) + \ldots\ldots$$
$$\quad\downarrow\qquad\quad\ \downarrow\qquad\quad\ \downarrow\qquad\quad\ \downarrow$$
$$\quad t_1\qquad\quad\ t_2\qquad\quad\ t_3\qquad\quad\ t_4$$

Here, t_1 start with 2 and end with 5. Gap between 2 and 5 is 3.

t_2 start with 3 and end with 6. Gap between 3 and 6 is 3.

Now, one can conclude that t_n term start with (n + 1) and end with (n + 4). Gap between (n + 1) and (n + 4) is 3.

$$t_n = (n + 1)(n + 4)$$

$$S_n = \sum_{n=1}^{n} (n+1)(n+4) \Rightarrow \sum_{n=1}^{n} (n^2 + 5n + 4)$$

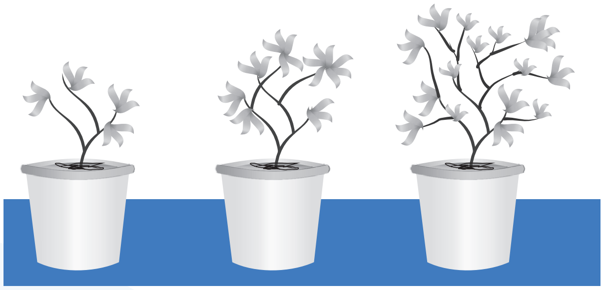
$$S_n = \sum_{n=1}^{n} n^2 + 5\sum_{n=1}^{n} n + 4\sum_{n=1}^{n} 1$$

$$S_n = \frac{n(n+1)(2n+1)}{6} + 5 \times \frac{n(n+1)}{2} + 4 \times n$$

$$S_{20} = \frac{20 \times 21 \times 41}{6} + 5 \times \frac{20(21)}{2} + 4 \times 20$$

$$S_n = 2870 + 1050 + 80$$

$$S_n = 4000$$



Series Based on Iterative Relation:

Iteration means repetition. So one can say that, it is a type of series in which some pattern appears. One don't need to find all the terms. One can reach to the desired solution by using the pattern.

Let's understand this concept with the help of some examples.

Example 19:

Given $t_{(n+2)} = t_{(n+1)} - t_{(n)}$ for n being any natural no ≥ 1.

If $t_1 = 12$ and $t_2 = 8$, then find t_{84}.

Solution: 4

It is given that, $t_{(n+2)} = t_{(n+1)} - t_{(n)}$

$$t_1 = 12$$

$$t_2 = 8$$

$$t_3 = t_2 - t_1 = 8 - 12 = -4$$

$$t_4 = t_3 - t_2 = (-4) - 8 = -12$$

$$t_5 = t_4 - t_3 = (-12) - (-4) = -8$$

$$t_6 = t_5 - t_4 = -8 + 12 = -4$$

$$t_7 = t_6 - t_5 = 4 - (-8) = 12$$

$$t_8 = t_7 - t_6 = 12 - 4 = 8$$

$t_9 = t_8 - t_7 = 8 - 12 = -4$

$t_{10} = t_9 - t_8 = -4 - 8 = -12$

$t_{11} = t_{10} - t_9 = -12 + 4 = -8$

$t_{12} = t_{11} - t_{10} = -8 + 12 = 4$

Now, one can see a pattern that values of terms repeats after a cycle of 6.

$a_1 = 12$

$a_2 = 8$

$a_3 = -4$

$a_4 = -12$

$a_5 = -8$

$a_6 = 4$

These terms are the negative of a_1, a_2 and a_3

$a_7 = 12$

$a_8 = 8$

$a_9 = -4$

These terms are the same as a_1, a_2 and a_3

$a_{10} = -12$

$a_{11} = -8$

$a_{12} = 4$

$a_1 = a_7 = a_{13}$........

$a_2 = a_8 = a_{14}$........

And so on

By using this pattern one can directly reach to t_{84}.

$T_6 = T_{12} = T_{18}$ .... $T_{84} = 4$

$\boxed{T_{84} = 4}$

Example 20:

Consider the sequence of numbers a_1, a_2, a_3, a_4,........ till infinity, where $a_1 = 81.33$, $a_2 = -19$ and $a_J = a_{J-1} - a_{J-2}$ for $J \geq 3$. What is the sum of the first 6002 terms of this sequence ?

(A) –100.33 (B) –30.00

(C) 62.33 (D) 119.33

Solution: (C)

Given relation: $a_J = a_{J-1} - a_{J-2}$

6002 is a big number. So one can directly conclude that it is a question based on iterated relation.

$a_1 = 81.33$

$a_2 = -19$

$a_3 = a_2 - a_1 = -100.33$

$a_4 = a_3 - a_2 = -81.33$

$a_5 = a_4 - a_3 = 19$

$a_6 = a_5 - a_4 = 100.33$

$a_7 = a_6 - a_5 = +81.33$

$a_8 = a_7 - a_6 = -19$

Values of terms repeats after a cycle of 6. One can see that every set of 6 terms will be the same. The sum of every set of 6 terms will also be the same and it will be = 0

So, the sum of first 6000 terms will be zero.

Sum of first 6002 terms are simply the sum of 6001th term and 6002th term.

= 6001th term + 6002th term

= 81.33 + (–19)

= 62.33

Series Based on Logarithm:

Previously CAT asked some Question based on logarithm series.

Let's understand this concept with the help of some examples.

Example 21:

What is the sum of 'n' terms in the following series:

$$\log m + \log\frac{m^2}{n^1} + \log\frac{m^3}{n^2} + \log\frac{m^4}{n^3} + \text{........} + \log\frac{m^n}{n^{n-1}}$$

(A) $\log\left[\frac{n^{(n-1)}}{m^{(n+2)}}\right]^{n/2}$ (B) $\log\left[\frac{m^m}{n^n}\right]^{n/2}$

(C) $\log\left[\frac{m^{(1-n)}}{n^{(1-m)}}\right]^{n/2}$ (D) $\log\left[\frac{m^{(n+1)}}{n^{(n-1)}}\right]^{n/2}$

Solution: (D)

$$\log m + \log\frac{m^2}{n} + \log\frac{m^3}{n^2} + \log\frac{m^4}{n^3}$$

$$+ \ldots + \log\frac{m^n}{n^{n-1}}$$

$= \log m + 2\log m - \log n + 3\log m - 2\log n \ldots\ldots\ldots n\log m - (n-1)\log n$

$= \log m\,(1 + 2 + 3 + 4 + \ldots\ldots. n) - \log n\,(1 + 2 + 3 + \ldots\ldots. + (n-1)]$

$$= \log m\left[\frac{n(n+1)}{2}\right] - \log n\left[\frac{(n-1)\times(n-1+1)}{2}\right]$$

$$= \log m\left[\frac{n(n+1)}{2}\right] - \log n\left[\frac{n(n-1)}{2}\right]$$

$$= \log\left[\frac{m^{(n+1)}}{n^{(n-1)}}\right]^{n/2}$$

Example 22:

Find the sum of the series

$\log x + \log\left(\frac{x}{2}\right)^2 + \log\left(\frac{x}{4}\right)^3 + \log\left(\frac{x}{8}\right)^4$ ... upto 10 terms.

(A) $55\log x - 338\log 2$

(B) $55\log x - 330\log 2$

(C) $55\log x - 300\log 2$

(D) $50\log x - 330\log 2$

Solution: (B)

It is given that,

$\log x + \log\left[\frac{x}{2}\right]^2 + \log\left[\frac{x}{4}\right]^3 + \log\left[\frac{x}{8}\right]^4$ ....10 terms

Expanded form of the given series.

$$= \log x + 2\log\left[\frac{x}{2}\right] + 3\log\left[\frac{x}{4}\right] + 4\log\left[\frac{x}{8}\right]$$

$$+ 5\left[\log\frac{x}{16}\right] + 6\left[\log\frac{x}{32}\right] + 7\left[\log\frac{x}{64}\right]$$

$$8\left[\log\frac{x}{128}\right] + 9\left[\log\frac{x}{256}\right] + 10\left[\log\frac{x}{512}\right]$$

$= \log x + 2\log x - 2\log 2 + 3\log x - 3\log 4 + 4\log x - 4\log 8 + 5\log x - 5\log 16 + 6\log x - 6\log 32 + 7\log x - 7\log 64 + 8\log x - 8\log 128 + 9\log x - 9\log 256 + 10\log x - 10\log 512.$

$= (1 + 2 + 3 + \ldots\ldots\ldots 10)\log x - [2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 + 90]\log 2$

$= 55\log x - 330\log 2$

Miscelleneous Series:

Example 23:

Find the 100th term of the sequence 3, 5, 9, 15, 23, 33, 45,..........

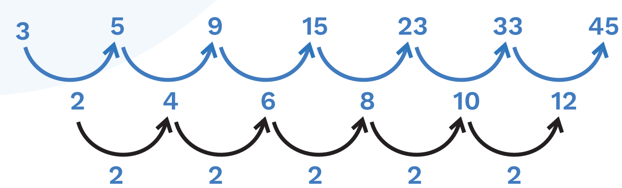
(A) 9,900 (B) 9,901

(C) 9,902 (D) 9,903

Solution: (D) 9,903

It is given that, 3, 5, 9, 15, 23, 33, 45,..........

While doing such type of Question one needs to identify the pattern of the sequence.



Here, one can see that the difference between successive terms of the series form on A.P. and the second level difference are constant.

So the nth term of such sequence will be a quadratic expression of the form $an^2 + bn + c$.

$T_1 = 3$

$$a \times (1)^2 + b \times (1) + c = 3$$

$$\Rightarrow \quad a + b + c = 3 \qquad ...(i)$$

$$T_2 = 5$$

$$a \times (2)^2 + b \times 2 + c = 5$$

$$\Rightarrow \quad 4a + 2b + c = 5 \qquad ...(ii)$$

$$T_3 = 9$$

$$a \times (3)^2 + b \times 3 + c = 9$$

$$\Rightarrow \quad 9a + 3b + c = 9 \qquad ...(iii)$$

Now, one have three equations with three variables on solving equation (i), (ii) and (iii) we get

$$a = 1, \ b = -1 \text{ and } c = 3$$

$$T_n = an^2 + bn + c$$

$$T_{100} = 1 \times (100)^2 + (-1) \times 100 + 3$$

$$T_{100} = 10000 - 100 + 3$$

$$= 9903$$

Example 24:

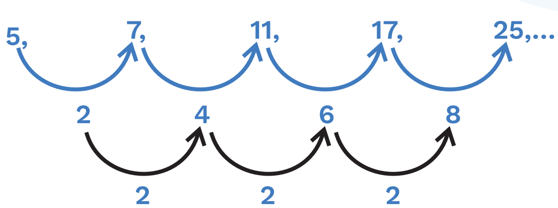
Find the sum of the first 97 terms of the sequence, 5, 7, 11, 17, 25,.......... .

(A) 3,00,677 (B) 3,04,677

(C) 3,05,690 (D) 3,05,688

Solution: (B)

Given Series, 5, 7, 11, 17, 25,..........



The difference between the successive terms of the series forms on A.P. and the second level difference in constant.

$$T_n = an^2 + bn + c$$

$$T_1 = a + b + c = 5 \qquad ...(i)$$

$$T_2 = 4a + 2b + c = 7 \qquad ...(ii)$$

$$T_3 = 9a + 3b + c = 11 \qquad ...(iii)$$

On solving equation (i), (ii) and (iii) we get

$$a = 1, \ b = -1 \text{ and } c = 5$$

$$T_n = 1 \times (n)^2 + (-1) \times n + 5$$

$$T_n = n^2 - n + 5$$

$$S_n = \Sigma(n^2 - n + 5)$$

$$S_n = \Sigma n^2 - \Sigma n + 5\ \Sigma_1$$

$$S_n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 5[1 \times n]$$

$$S_{97} = \frac{97(97+1)(2 \times 97+1)}{6} - \frac{97(97+1)}{2} + 5 \times 1 \times 97$$

$$S_{97} = \frac{97 \times 98 \times 195}{6} - \frac{97 \times 98}{2} + 485$$

$$S_{97} = 3,08,945 - 4753 + 485$$

$$\boxed{S_{97} = 3,04,677}$$

Example 25:

Find the sum of

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + ... + \sqrt{1+\frac{1}{15^2}+\frac{1}{16^2}}$$

(A) $\frac{240}{16}$ (B) $\frac{256}{16}$

(C) $\frac{255}{16}$ (D) $\frac{247}{16}$

Solution: (C)

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + ...$$

$$+ \sqrt{1+\frac{1}{15^2}+\frac{1}{16^2}}$$

$$= \sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} = \sqrt{2+\frac{1}{4}} = \sqrt{\frac{9}{4}} = \frac{3}{2} = \left(2-\frac{1}{2}\right)$$

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}$$

$$= \frac{3}{2} + \frac{7}{6} = \frac{16}{6} = \left(3-\frac{1}{3}\right)$$

Similarly,

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}=\left(4-\frac{1}{4}\right)$$

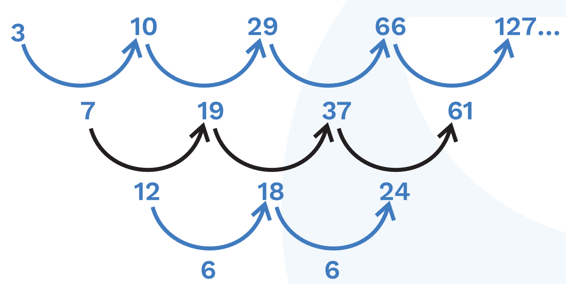
$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\ldots+\sqrt{1+\frac{1}{15^2}+\frac{1}{16^2}}$$

$$=16-\frac{1}{16}=\frac{256-1}{16}=\frac{255}{16}$$

Example 26:

Find the 37th term of the sequence 3, 10, 29, 66, 127,..........

Solution: 50655



Here, one can see that the difference between the second level terms is in A.P. and the third level difference is constant.

Here, the third level difference is constant, so it is a cubic sequence.

$$T_n = an^3 + bn^2 + cn + d$$

Here, one have 4 variables so one have four equation to solve the Question. But this is so time consuming here one can go for an alternate approach.

$3 = 1 + 2 = (1)^3 + 2$

$10 = 8 + 2 = (2)^3 + 2$

$29 = 27 + 2 = (3)^3 + 2$

So, nth term will be $(n^3 + 2)$.

$T_n = n^3 + 2$

$T_{37} = (37)^3 + 2$

$\boxed{T_{37} = 50655}$

Practice Exercise - 1

Level of Difficulty – 1

1. If f is a function satisfying $f(x + y) = f(x) \times f(y)$ for all $x, y \in N$ such that $f(1) = 5$ and $\sum_{x=1}^{n} f(x) = 155$, then find the value of n.

(A) 1

(B) 2

(C) 3

(D) 4

2. If x, 3x + 3 and 5x + 5 are the first three terms of a geometric progression whose all terms are negative, then the fourth term of the progression is?

3. x, 17, $3x - y^2 - 2$, and $3x + y^2 - 30$, are four consecutive terms of an increasing arithmetic sequence. The sum of these four numbers is divisible by?

(A) 2

(B) 3

(C) 5

(D) 7

4. The number of terms common to both the Arithmetic Progression 2, 5, 8, 11,........ 160 and 3, 5, 7, 9,........ 125 are?

(A) 17

(B) 19

(C) 21

(D) 23

5. The sum to 10^{th} term of the progression (90, 130, 170......) is equal to sum of the first N terms of another progression (220, 240, 260......). The value of N is?

(A) 6

(B) 9

(C) 10

(D) Cannot be determined

Level of Difficulty – 2

6. A pack contains 'P' cards numbered from 1 to P. Two consecutive cards are removed from the pack and sum of numbers on the remaining cards is 1224. If the smaller of number on the removed card is K. Find the minimum value of K–20. Given that K is a positive integer.

Write your answer here.

7. A WHO Research worker is working with two types of Covid-19 virus. One type of Covid-19 virus becomes 4 times in 10 minutes and other one become 5 times in the same time. If the total number of viruses after 30 minutes is 1259, then what was the total number of viruses at the beginning?

Write your answer here.

8. What is value of 10.11.12.13 + 11.12.13.14 +........+ 96.97.98.99?

(A) 1806869592

(B) 1806869594

(C) 1806869596

(D) 1806869598

9. The weights of 21 people are in Arithmetic Progression. The average weight of them is 32. If the heaviest person is of 62 kg, then find the weight of second lightest person.

(A) 2 kg

(B) 3 kg

(C) 5 kg

(D) 7 kg

10. Find the sum of first 291 terms of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4,........ .

(A) 4684

(B) 4242

(C) 4484

(D) 4648

Level of Difficulty – 3

11. Some identical balls are arranged in rows to form an equilateral triangle. The first row consist of one ball. The second row consist of two balls and so on. If 93 more identical balls are added to the total number of balls used in forming the equilateral triangle, then all these balls can be arranged in a square, where the number of balls used in the side of square is 3 less than the number of balls used in side of the equilateral triangle, then the number of balls used to form an equilateral triangle are?

Write your answer here.

12. $$S_n = \sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + \ldots + \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}$$

Find $S_3 + S_5 + S_7$

(A) $\frac{421}{24}$

(B) $\frac{419}{24}$

(C) $\frac{470}{24}$

(D) $\frac{413}{24}$

13. If the product of the 23rd and 70th of a Geometric Progression is 256, then what is the product of the first 92 terms of the same Geometric Progression?

(A) $(256)^{46}$

(B) $(256)^{42}$

(C) $(256)^{44}$

(D) $(256)^{92}$

14. Find the least value of n for which the sum $1 + 5 + 5^2 + 5^3 +$........ n terms is greater than 1601.

log 6405 = 3.806

log 5 = 0.698

15. In an increasing G.P., the sum of the first and the last term is 99, the product of the second term and the second the last term is 288 and the sum of all the terms is 189. How many terms are there in the progression?

(A) 5

(B) 6

(C) 7

(D) 8

Solution

1. (C)

Here, it is given that, $f(x + y) = f(x) \times f(y)$ for all $x, y \in N$

Substitute $y = 1$ then, $f(x + 1) = f(x) \times f(1)$

$= 5\ f(x) = \dfrac{f(x+1)}{f(x)} = 5$

The above shows that the ratio of two consecutive terms of f(x), when $x \in N$ is constant

Therefore, the sequence f(x) for natural n is G.P, with then $f(x) = 5^x$ for all $x \in N$

Now, $\sum_{x=1}^{n} f(x) = 155 \Rightarrow \sum_{i=1}^{n} 5^x = 155$

$(5 + 5^2 + 5^3 + \ldots\ldots\ldots + 5^n) = 155$

$5\left[\dfrac{5^n - 1}{5-1}\right] = 155$

$5^n = 125 \Rightarrow 5^n = (5)^3$

Hence, $n = 3$.

2. –24

x, $3x + 3$ and $5x + 3$ are the first three terms of a geometric progression,

Therefore, $(3x + 3)^2 = x\ (5x + 3)$

$9x^2 + 18x + 9 = 5x^2 + 3x$

$4x^2 + 15x + 9 = 0$

$4x^2 + 12x + 3x + 9 = 0$

$4x\ (x + 3) + 3\ (x + 3) = 0$

$(4x + 3)\ (x + 3) = 0$

i.e., $x = -\dfrac{3}{4}$ or $x = -3$

i.e., terms are

$-\dfrac{3}{4}, \dfrac{3}{4}, -\dfrac{3}{4}$ or $-3, -6, -12$

Thus, the possible geometric progression is –3, –6, –12.

Therefore, fourth term = –24.

3. (A)

Since x, 17, $3x - y^2 - 2$ and $3x + y^2 - 30$ are in A.P.

$\therefore\ 17 - x = 3x + y^2 - 30 - 3x + y^2 + 2$

$\Rightarrow\ x + 2y^2 = 45 \quad \ldots(i)$

Also, $17 - x = 3x - y^2 - 2 - 17$

$\Rightarrow\ 4x - y^2 = 36 \quad \ldots(ii)$

Solving equations (i) and (ii), we get

$x = 13$

Now, $x + 17 + 3x - y^2 - 2 + 3x + y^2 - 30$

$= 7x - 15 = 7\ (13) - 15$

$= 76$

Out of the given options, 76 is only divisible by 2.

4. (C)

Given Aps are, 2, 5, 8, 11, 14, 17, 20, 23,........ 160

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23,........ 125

Visible common terms are, 5, 11, 17, 23,........

$T_n = a + (n - 1) \times d$

$T_n = 5 + (n - 1) \times 6$

$T_n = 6n - 1$

By hit and trial

$T_{21} = 6 \times 21 - 1$

$T_{21} = 125$

125 exist in second AP.

Let's check it exist in first AP or net

$125 = 2 + (n - 1) \times 3$

$n - 1 = 41$

$n = 42$ (Yes)

Hence, 21 terms are common in both APs.

5. (B)

Sum up to 10 terms of the first series 90, 130, 170,.... is $\dfrac{10}{2}\ (2 \times 90 + 9 \times 40) = 2{,}700$.

The second series is 220, 240, 260,... Sum of the N terms of this series

$= \frac{N}{2}\{2 \times 220 + (N - 1)\,20\}$

$= \frac{N}{2}\{2 \times 220 + (N - 1) \times 20\} = 270 \times 10$

$= N^2 + 21N - 270 = 0$

$= (N + 30)(N - 9) = 0$

i.e., $N = 9$

6. 5

Let the removed cards be K and K + 1.

Sum of first 'P' cards is $\frac{P(P+1)}{2}$

According to the question,

$\Rightarrow \frac{P(P+1)}{2} - (K) - (K+1) = 1224$

$\Rightarrow \frac{P(P+1)}{2} - 2K - 1 = 1224$

$\Rightarrow \frac{P(P+1)}{2} - 2K = 1225$

$\Rightarrow K = \frac{P(P+1) - 2450}{4}$

for values of P, from 1 to 48, K will be negative and at P = 49, K will be = 0

At P = 50, K = (Positive integer).

$\Rightarrow K = \frac{50(51) - 2450}{4}$

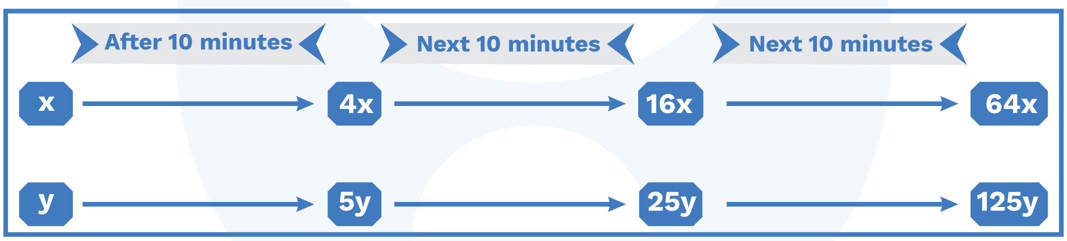
$\boxed{K = 25}$

Minimum value of $(K - 20) = 25 - 20 = 5$

7. 13

Let the Covid-19 virus of one type is 'x' in the beginning and of second type is 'y' in the beginning.

After 30 minute total number of virus = 1259.



$\therefore \quad 64x + 125y = 1259$

By hit and trial one get, x = 6 and y = 7

Number of virus at the beginning are

$\Rightarrow \quad 6 + 7 = 13$

8. (A)

$n(n + 1)(n + 2)(n + 3) = n(n + 1)(n + 2)(n + 3)\left(\frac{1}{5}\right)\{(n + 4) - (n - 1)\}$

$\frac{1}{5}[\{n(n + 1)(n + 2)(n + 3)(n + 4) - (n - 1)\,n(n + 1)(n + 2)(n + 3)\}]$

Hence,

$10.11.12.13 = \frac{1}{5}[10.11.12.13.14 - 9.10.11.12.13]$

$11.12.13.14 = \frac{1}{5}[11.12.13.14.15 - 10.11.12.13.14]$

$96.97.98.99 = \frac{1}{5}[100.99.98.97.96 - 99.98.97.96.95]$

Adding the terms given above, we get

$= 10.11.12.13 + 11.12.13.14 + \ldots\ldots + 96.97.98.99$

$= \frac{1}{5}[100.99.98.97.96 - 9.10.11.12.13]$

$= 1806869592.$

9. (C)

Number of peoples = 21

Average weight of 21 peoples = 32 kgs.

Sum of the weight of 21 peoples

$= 32 \times 21 = 672$ Kgs.

Let the peoples are $P_1, P_2, P_3 \ldots\ldots P_{21}$, where $P_{21} = 62$ Kgs.

Since, weights are in A.P.,

$$672 = \frac{n}{2}(P_1 + P_{21})$$

$$672 = \frac{21}{2}(P_1 + 62)$$

$$\Rightarrow \frac{1344}{21} = P_1 + 62$$

$\Rightarrow P_1 = 64 - 62$

$\Rightarrow P_1 = 2$

$P_{21} = a + 20d = 62$

$62 = 2 + 20 \times d$

$\boxed{d = 3}$

Weight of second lightest person

$= a + d = 2 + 3 = 5$ Kgs.

10. (A)

Given series, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4,........

Here, one can observe that n^{th} number occurs n^{th} times.

Total numbers written will be

$$\sum_1^n n = \frac{n(n+1)}{2}$$

Writing 23 times, bringing the total to 276.

Remaining numbers = 291 – 276

$\Rightarrow$ 15 numbers.

$\Rightarrow$ 1, 2, 2, 3, 3, 3, 4, 4, 4, 4 ,... 24 (15 times)

$\Rightarrow 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 23^2 + 24 \times 15$

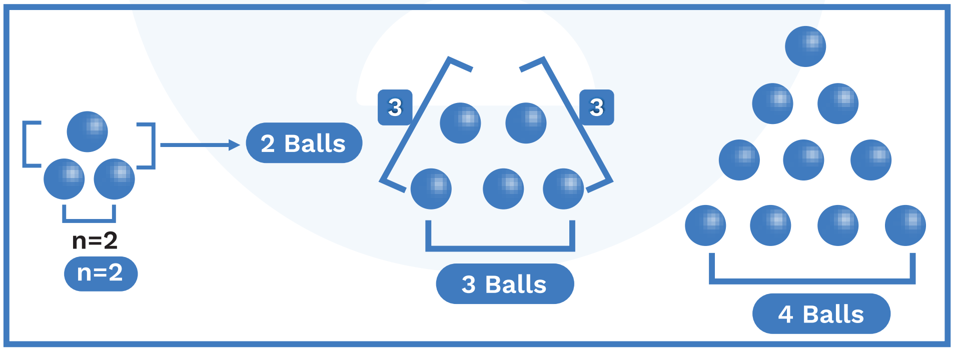
$$\Rightarrow \frac{n(n+1)(2n+1)}{6} + 24 \times 15$$

$$\Rightarrow \frac{23(24)(47)}{6} + 360$$

$\Rightarrow 4324 + 360 \Rightarrow 4684$

11. 231

Let's understand the pattern first.



One can see that number of balls at the bottom is simply side of an equilateral triangle. Let the side on an equilateral triangle is n. it implies that, the number of balls at the bottom is also 'n'.

Number of balls an equilateral triangle of side n contain,

$$\frac{n}{2}(2a + (n-1) \times d)$$

$$\Rightarrow \frac{n}{2}[2 + (n-1) \times 1] = \frac{n(n+1)}{2}$$

According to the question,

$$\Rightarrow \frac{n(n+1)}{2} + 93 = (n-3)^2$$

$$\Rightarrow \frac{n^2+n}{2} + 93 = n^2 + 9 - 6n$$

$\Rightarrow n^2 + n + 186 = 2n^2 + 18 - 12n$

$\Rightarrow n^2 - 13n - 168 = 0$

$\Rightarrow$ $n^2(n-21)+9(n-21)=0$

$\Rightarrow$ $(n-21)(n+8)=0$

$\Rightarrow$ $n=21$ and $n=-8$

[i.e., -8 is not possible]

$n=21$

Number of balls an equilateral triangle of side 21 contains

$$=\frac{n(n+1)}{2}=\frac{21\times 22}{2}=231 \text{ balls}$$

12. (B)

It is given that,

$$S_n=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots.+\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}$$

$$\Rightarrow (n+1)-\frac{1}{(n+1)}$$

$$\therefore\ S_n=(n+1)-\frac{1}{(n+1)}$$

$$S_3=\left(4-\frac{1}{4}\right),\ S_5=\left(6-\frac{1}{6}\right) \text{ and } S_7=\left(8-\frac{1}{8}\right)$$

$$S_3+S_5+S_7=\left(4-\frac{1}{4}\right)+\left(6-\frac{1}{6}\right)+\left(8-\frac{1}{8}\right)$$

$$\Rightarrow \frac{15}{4}+\frac{35}{6}+\frac{63}{8}$$

$$\Rightarrow \frac{419}{24}$$

13. (A)

Let the geometric progression be a, ar, ar^2, ar^4,........ ar^{91}

It is given that, $t_{23}\times t_{70}=256$

$\Rightarrow$ $a\times(r)^{22}\times(a\times r^{69})=256$

$\Rightarrow$ $a^2\times r^{91}=256$ ...(i)

Product of first 92 terms are,

$a\times ar\times ar^2\times ar^3\times\ldots\ldots\times ar^{91}$

$=a^{92}\times r^{(1+2+3+\ldots\ldots.91)}$

$=a^{92}\times r^{\frac{91\times 92}{2}}$

$=a^{92}\times r^{(91\times 46)}$

$=(a^2)^{46}\times(r^{91})^{46}$

$=(a^2\times r^{91})^{46}$ ...(ii)

From equation (i) and (ii) we get,

$\Rightarrow$ $(256)^{46}$

Required product $=(256)^{46}$

14. 6

Given series, $S_n=1+5+5^2+5^3+\ldots\ldots.+$ n terms

$$S_n=1\left[\frac{5^n-1}{5-1}\right]\ \Rightarrow\ \frac{5^n-1}{4}$$

$S_n>1601$

$$\therefore\ \frac{5^n-1}{4}>1601$$

$\Rightarrow$ $5^n-1>6404$

$\Rightarrow$ $5^n>6405$

Taking log on both sides

$\Rightarrow$ $n\log 5>\log 6405$

$$\Rightarrow\ n>\frac{\log 6405}{\log 5}$$

$$\Rightarrow\ n>\frac{3.806}{0.698}$$

$\Rightarrow$ $n>5.45$

Hence, the value of n is 6.

15. (B)

Let 'a' be the first term and 'r' be the common ratio of the G.P.

Given $a_1+a_n=99$ $\Rightarrow$ $a+ar^{n-1}=99$ ...(i)

Now, $a_2\times a_{n-1}=288$ $\Rightarrow$ $ar\times ar^{n-2}=288$

$$\therefore\ a^2r^{n-1}=288\ \Rightarrow\ ar^{n-1}=\frac{288}{a} \quad \ldots\text{(ii)}$$

Put the value of equation (ii) in equation (i) we get

$$a+\frac{288}{a}=99$$

$\Rightarrow$ $a^2-99a+288=0$

$\Rightarrow$ $(a-96)(a-3)=0$

$\therefore$ $a=96$ and $a=3$

As G.P. is increasing so $a=96$ is not possible.

Now, $S_n = 189$

$$\Rightarrow S_n = 3\left[\frac{r^n - 1}{r-1}\right] = 189$$

$$\Rightarrow 3\left[\frac{(r^{n-1}) \times r - 1}{r-1}\right] = 189$$

$$\Rightarrow \frac{(r^{n-1}) \times r - 1}{r-1} = 63 \qquad \text{...(iii)}$$

Put a = 3 in equation (i) we get

$3 + 3r^{n-1} = 99 \quad \Rightarrow r^{n-1} = 32$...(iv)

Put $r^{n-1} = 32$ in equation (iii) we get

$\Rightarrow \quad 32r - 1 = 63r - 63$

$31r = 62 \quad \Rightarrow r = 2$

Put r = 2 in equation (iv) we get

$(2)^{n-1} = 32 \quad \Rightarrow 2^{(n-1)} = (2)^5$

$n - 1 = 5 \quad \Rightarrow n = 6$

Hence, the number of terms in the given geometric progression is 6.

Alternate Solution:

$a_1 + a_n = 99$ and $a_2 \times a_{n-1} = 288$

As we know that the product of the first and the last term in a G.P. is equal to the product of the second term and second last term.

Sum of $a_1 + a_n = 99$

$a_1 \times a_n = 288$

By hit the trial we get a = 3 and $a_n = 96$

$a_1 \times a_n = 288$

$\Rightarrow \quad a \times ar^{n-1} = 288$

$\Rightarrow \quad 3 \times 3 \times r^{n-1} = 288$

$r^{n-1} = 32$

$\Rightarrow \quad r^{(n-1)} = (2)^5$

On comparing we get, r = 2 and n − 1 = 5

n = 6

Hence, the number of terms in the given geometric progression is 6.

Practice Exercise - 2

Level of Difficulty – 1

1. If 2, a, b, c, d, e, f, and 65 form an arithmetic progression, find out the value of 'e'.

(A) 48

(B) 47

(C) 41

(D) None of these

2. A child named Rambharosi, playing on the balcony of his multi-storied apartment, drops a ball from a height of 350 m. Each time the ball rebounds, it rises (4/5)th of the height it has fallen through. The total distance traveled by the ball before it comes to rest is which of the following?

(A) 2,530 m

(B) 2,800 m

(C) 3,150 m

(D) 3,500 m

3. What is the sum of the integers 54 through 196 inclusive?

(A) 28820

(B) 24535

(C) 20250

(D) 17875

4. Find the sum of the product of the corresponding terms of the sequence 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, and 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536

(A) $(4^{10} - 1)$

(B) $2(4^{10} - 1)$

(C) $2(6^{10} - 1)$

(D) $23(^{10} - 1)$

5. If p, q, r are in A.P and m, n, o are in G.P then $m^{q-r} \times n^{r-p} \times o^{p-q}$ is equal to:

(A) 0

(B) 1

(C) 2

(D) None of these

6. $1^2 - 2^2 + 3^2 - 4^2 + \ldots. - 99^2$ is equal to :

(A) 4350

(B) 4950

(C) 4582

(D) 5050

7. In a school, students were called for the flag hoisting ceremony on 15th August. After the ceremony, small boxes of sweets were distributed among the students. In each class, the student with roll number 1 got 1 box of sweets, student with roll number 2 got 2 boxes of sweets, student with roll number 3 got 3 boxes of sweets and so on. In class III, a total of 1200 boxes of sweets were distributed. By mistake, one of the students of class III got double the sweets that he was entitled to get. Identify the roll number of the student who got twice as many boxes of sweets as compared to his entitlement.

(A) 22

(B) 24

(C) 28

(D) 30

8. The sum of the 5^{th} and 18^{th} term of an Arithmetic Progression is equal to the sum of 8^{th}, 13^{th} and 19^{th} term of the same Arithmetic Progression. Then, which term of the same A.P should necessarily be equal to zero?

(A) 14^{th}

(B) 17^{th}

(C) 20^{th}

(D) 24^{th}

9. There are 5984 steel balls stacked in a pile with one ball on the top, 3 balls in the second layer, 6 balls in the third layer, 10 balls in the 4^{th} layer and so on. Find the total number of layers.

(A) 28

(B) 30

(C) 32

(D) 64

10. If 96 + 93 + 90 ------- till N terms = 1386, then how many values of N are possible?

(A) 0

(B) 1

(C) 2

(D) More than 2

Level of Difficulty – 2

11. Rashi dropped a bouncing ball from a 48 m tall building. Each time the ball touches the ground, it bounces back $\frac{2}{5}^{th}$ of the distance travelled before touching the ground and it goes on indefinitely. Calculate the total distance travelled by the ball before coming to rest.

(A)80 m

(B) 112 m

(C) 32 m

(D) 102 m

12. Find the sum of the first 50 terms of the series 2, 5, 10, 17, 26, 37, ...

Write your answer here.

13. Find the sum

$$100+\frac{121}{9}+\frac{144}{81}+\frac{169}{729}+\frac{196}{9801}+\text{.....upto } \infty$$

(A) $\frac{29665}{256}$

(B) $\frac{29565}{128}$

(C) $\frac{29665}{128}$

(D) $\frac{29565}{256}$

14. Find the value of ‘n’, given that

$$\frac{1^3+3^3+5^3+----(2n-1)^3}{2^3+4^3+6^3+----+(2n)^3}=\frac{337}{392}$$

15. If

$$\frac{1}{2}+\frac{1}{2+4}+\frac{1}{2+4+6}\dots\frac{1}{2+4+6\dots398}=\frac{P}{Q},$$

then find the sum of the last 2 digits of P^Q.

16. If $S_N = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 \dots (4N-1)^2 - (4N)^2 + (4N+1)^2$, then which of the following options is correct ?

(A) $S_{10} = 1681$

(B) $S_{20} = 3321$

(C) $S_{30} = 7350$

(D) $S_5 = 241$

17. If the first four terms of an AP are m, m + 2n, 3m + n, and 30 respectively, find the value of the 2020th term of the progression.

(A) 14344

(B) 16532

(C) 15130

(D) 16158

18. If

$$K = \frac{1}{21} + \frac{1}{77} + \frac{1}{16} + \frac{1}{285} ---------- \infty,$$

then find the value of 84K.

19. 1,000 apples are distributed among the maximum possible number of children such that each child gets a different number of apples. What is the maximum number of apples received by a child?

20. If $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots\ldots + \text{to } \infty = \frac{\pi^2}{8}$, then find the value of $\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \ldots\ldots +$ to ∞ :

(A) $\frac{7}{32}\pi^2$

(B) $\frac{1}{8}\pi^2$

(C) $\frac{7}{64}\pi^2$

(D) $\frac{11}{64}\pi^2$

Level of Difficulty – 3

21. Find the sum of 20 terms of the following series: ---------------

(A) 355/60036

(B) 355/60032

(C) 354/60036

(D) 365/60032

22. Three distinct numbers x, 2y and 3z form a GP in that order and the numbers x + 2y, 2y + 3z and 3z + x form an A.P in that order. Find the common ratio of the GP

(A) 1

(B) 2

(C) –2

(D) Either (A) or (C)

23. Find the value ofX_{75} when $X_1 = -1$, $X_2 = 1$ and $X_{m+1} = X_{m-1} + m$

(A) 1400

(B) 1205

(C) 1206

(D) 1405

24. The value of the sum 6 × 8 + 9 × 11 + 12 × 14 + ...+ 93 × 95 is ?

25. Given an equilateral triangle T_1 with each side measuring 32 cm, a second triangle T_2 is formed by joining the midpoints of the sides of T_1. Then a 3rd triangle is formed by joining the midpoints of the sides of T_2. If this process of forming a triangle is continued, the sum of the areas (in cm²) of infinitely many such triangles T_1, T_2, T_3 .... will be :

(A) $\frac{1024}{3}$

(B) $1024\sqrt{3}$

(C) $\frac{1024}{\sqrt{3}}$

(D) None of these

26. Given a series of letters:

a,b,b,b,b,c,c,c,c,c,c,c,c,c,d,d,d,d,d,d,d,d,d,d,d,d,d,d,d,d,.......................

Which of the following letter will be at 3000th position?

(A) R

(B) S

(C) T

(D) U

27. Find the sum of 'n' terms of the series:

$$\log P + \log \frac{P^3}{q} + \log \frac{P^5}{q^2} + \log \frac{P^7}{q^3} + \ldots.$$
$$\ldots\ldots + n^{th} \text{ term}$$

(A) $\frac{n}{2} \log \frac{P^n}{q^{n-1}}$

(B) $\frac{n}{2}\log\frac{P^{2n}}{q^{n-1}}$

(C) $\frac{n}{4}\log\frac{P^{n}}{q^{n}}$

(D) $\frac{n}{2}\log\frac{q^{n-1}}{P^{n}}$

28. Find the sum of the terms common to the sequences {3, 7 , 11, 15, ..., 399} and {2, 9, 16, 23, ..., 702}.

(A) 2100

(B) 2870

(C) 3200

(D) 3400

29. If the sum of the first ‘n’ terms of an Arithmetic Progression is a square, and that of the 1st three-digit is a square, the sum of the next ‹n› terms of the Arithmetic Progression is 3 times that of a square, then the ratio of the first term and the common difference is m:n. Find the value of m + n.

(A) 2

(B) 3

(C) 4

(D) CND

30. What maximum power of 7 divides z! where z is the sum of the digits of (8 + 88 + 888 + ... 12 terms) ?

(A) 9

(B) 10

(C) 11

(D) 12

Solution

1. **(B)**

Let the first term and common difference of the given AP be ‘2’ and ‘D’, respectively.

Then, a + (n – 1)D = 65

2 + (8 – 1)D = 65

7D = 63

D = 9

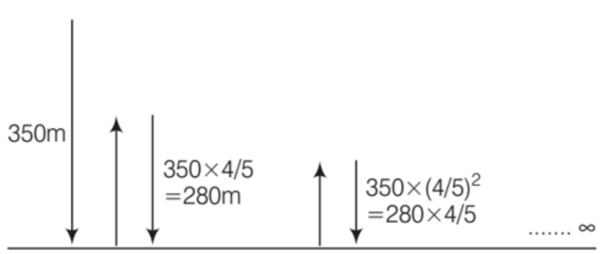
Now, 6th term = a + 5D

e = 2 + 5 × 9 = 47

Hence option (B) is the correct answer.

2. **(C)**

As per the diagram,



Total distance covered

$$= 350 + \left[2\times350\times\frac{4}{5} + 2\times350\times\left(\frac{4}{5}\right)^2 + 2\times350\times\left(\frac{4}{5}\right)^3 + \ldots\infty\right]$$

$$= 350 + 2\times350\left[\frac{4}{5} + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \ldots\ldots\infty\right]$$

Here, we can clearly see a decreasing G.P

Total distance covered

$$= 350 + 2 \times 350 \times \left(\frac{\frac{4}{5}}{1-\frac{4}{5}}\right)$$

= 350 + 2 × 350 × 4 = 3,150 m

Total distance covered by the ball is 3150 m.

3. (D)

Sum of integers from 54 to 196 = (sum of integers from 1 to 196) – (sum of integers from 1 to 53)

Sum of integers from 54 to 196

$= \frac{196 \times 197}{2} - \frac{53 \times 54}{2}$

= 19306 – 1431= 17,875

4. (B)

As we know, if a, ar, ar^2 .... and A, AR, AR^2......... are two geometric sequences, then the sequence having terms as the product of corresponding terms of the two sequences is also a geometric sequence with first term aA and common ratio rR.

In our first geometric sequence, the first term is 2 and the common ratio is 2 and in our second geometric sequence, the first term is 3 and the common ratio is 2. Therefore, the sequence is formed by multiplying the corresponding terms of the sequence in a G.P with first term a = 2×3 = 6 and common ratio r = 2 × 2 = 4

Required sum = $\frac{a(r^n - 1)}{r-1}$ where n = 10

$= \frac{6(4^{10} - 1)}{4-1}$

$= 2\ (4^{10} - 1)$

Hence, the required sum is $2\ (4^{10} - 1)$.

5. (B)

Here p, q, r are in A.P then

2q = p + r ...(1)

m, n, o are in G.P then $n^2 = m \times o$ ...(2)

Now,

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q}$

$= m^{q-r} \times (\sqrt{mo})^{r-p} \times o^{p-q}$

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q}$

$= m^{q-r} \times m^{\frac{r-p}{2}} \times o^{\frac{r-p}{2}} \times o^{p-q}$

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q} = m^{q-r+\frac{r-p}{2}} \times o^{\frac{r-p}{2}+p-q}$

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q} = m^{\frac{2q-(p+r)}{2}} \times o^{\frac{r+p-2q}{2}}$

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q} = m^{\frac{2q-2q}{2}} \times o^{\frac{2q-2q}{2}}$

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q} = m^0 \times 0^0$

$\Rightarrow m^{q-r} \times n^{r-p} \times o^{p-q} = 1$

Hence option (B) is the correct answer.

6. (B)

$1^2 - 2^2 + 3^2 - 4^2 + ... - 99^2$

$= (1^2 + 3^2 + 5^2 + 7^2 + ... + 99^2) - (2^2 + 4^2 + 6^2 + ...98^2)$

Now, $(1^2 + 3^2 + 5^2 + 7^2 + ... 99^2)$

$= (1^2 + 2^2 + 3^2 + 4^2 ......99^2) - (2^2 + 4^2 + ... 98^2)$

Therefore,$1^2 - 2^2 + 3^2 - 4^2 + ... - 100^2$

$= 1^2 + 2^2 + 3^2...... + 99^2) - 2\ (2^2 + 4^2 + 6^2 + ... + 98^2)$

$= \frac{(99 \times 100 \times 199)}{6} - 2^3(1^2 + 2^2 + 3^2 .... + 49^2)$

$= \frac{(99 \times 100 \times 199)}{6} - 8 \times \frac{49 \times 50 \times 99}{6}$

$= 99 \times \frac{50}{6} \times (398 - 392)$

$= 99 \times \frac{50}{6} \times 6 = 4950$

Hence option (B) is the correct answer.

7. (B)

Let N be the number of students and the student who gets double the sweets has roll number R.

Then,(1 + 2 + 3 + ... + N) + R

= 1200 ...(i)

$\because\ 1 + 2 + 3 + ... + N = \frac{N(N+1)}{2}$

[The value of N for which $\frac{N(N+1)}{2} < 1200$]

For N = 49, $\frac{N(N+1)}{2} = \frac{49 \times 50}{2} = 1225$

ForN = 48, $\frac{N(N+1)}{2} = \frac{48 \times 49}{2} = 1176$ ...(ii)

From Equations. (i) and (ii),

1176 + R = 1,200

So, the roll number of the student is 24.

8. (B)

$T_5 + T_{18} = T_8 + T_{13} + T_{19}$

A + 4D + A + 17D = A + 7D + A + 12D + A + 18D

2A + 21D = 3A + 37D

A + 16D = 0

Now in A.P A + 16D = T_{17}

Hence, option (B) is the correct answer.

9. (C)

If we observe the pattern, then the nth layer contains $\frac{n(n+1)}{2}$ balls

$$\sum \frac{n(n+1)}{2} = 5984$$

$$\Rightarrow \frac{1}{2}\sum (n^2 + n) = 5984$$

$$\Rightarrow \frac{1}{2}\left(\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)\right) = 5984$$

$$\Rightarrow \frac{n(n+1)}{4}\left[\frac{2n+1}{3} + 1\right] = 5984$$

$$\Rightarrow \frac{n(n+1)}{4} \times \frac{2(n+2)}{3} = 5984$$

$\Rightarrow$ n(n + 1) (n + 2) = 35904

$\Rightarrow$ n = 32

10. (C)

Given series 96 + 93 + 90 ------- till N terms is an AP having N terms with first term a = 96 and common difference d = – 3. Also given that the sum of these N terms = 1386

Sum of first N terms in an AP = N/2 [2a + (N – 1) d] = 1386

N/2 [2 x 96 + (N – 1) x3] = 1386

N/2 [195 – 3N] = 1386

N/2 [65 – N] = 462

N [65 – N] = 924

Solving which we will get N = 21 or N = 44, both of which are correct

Thus, 2 values of N are possible

Hence, option (C) is the correct answer.

11. (B)

Height of the building = 48 m.

Let us divide the distance into two parts.

Downwards and upwards:

D_1 = Distance Downward

$$= 48 + \frac{2}{5}(48) + \left(\frac{2}{5}\right)^2 (48) + ------$$

D_2 = Distance Upward

$$= \frac{2}{5}(48) + \left(\frac{2}{5}\right)^2 48 + --------$$

$$\Rightarrow D_1 = 48\left[1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + ------\right]$$

= (sum to G.P. with r = $\frac{2}{5}$

$$\Rightarrow D_1 = 48 \times \frac{5}{3} = 80$$

Now, $D_2 = \frac{2}{5}(48)\left[1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + -----\right]$

$$= \frac{2}{5} \times 48 \times \frac{5}{3} = 32$$

Total distance = 80 + 32 = 112 m

Hence option (B) is the correct answer.

12. 42,975

Let S_N = 2 + 5 + 10 + 17 + 26 ... T_N ...(1)

S_N = 2 + 5 + 10 + 17 ... T_{N-1} + T_N ...(2)

(1) – (2)

0 = 2 + (3 + 5 + 7 + 9 ... (N – 1) terms) – T_N

T_N = 2 + (3 + 5 + 7 + 9 ... (N – 1) terms)

$$T_N = 2 + \frac{N-1}{2}\{2 \times 3 + (N-2)(2)\}$$

$\mathbf{T_N = 2 + (N-1)(N+1) = N^2 + 1}$

$T_1 = (1)^2 + 1$, $T_2 = (2)^2 + 1$ .... and so, on till $T_{50} = (50)^2 + 1$

$S_{50} = T^1 + T^2 + T^3$ .... T_{50}

$S_{50} = (1^2 + 2^2 + 3^2 \ldots 50^2) + (1 + 1 + \ldots 50$ times)

$$S_{50} = \frac{50 \times 51 \times 101}{6} + 50$$

= 42925 + 50 = 42975

13. (D)

We have,

$$S = 100 + \frac{121}{9} + \frac{144}{81} + \frac{169}{729} + \frac{196}{9801} + \ldots.. \quad \ldots Eq\ (1)$$

Multiplying by '9' on both sides:

$$9S = 900 + 121 + \frac{144}{9} + \frac{169}{81} + \frac{196}{729} + \ldots.. \quad \ldots Eq\ (2)$$

Now subtract equation (1) from equation (2)

$$8S = 921 + \frac{23}{9} + \frac{25}{81} + \frac{27}{729} + \frac{29}{9801} + \ldots.. \quad \ldots Eq\ (3)$$

Multiplying by '9' on both sides:

$$72S = 8289 + 23 + \frac{25}{9} + \frac{27}{81} + \frac{29}{729} + \ldots.. \quad \ldots Eq\ (4)$$

Now subtract equation (3) from equation (4)

$$64S = 7391 + \frac{2}{9} + \frac{2}{81} + \frac{2}{729} + \ldots..$$

$$64S = 7391 + 2\left(\frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \ldots\right)$$

$$64S = 7391 + 2 + \left[\frac{\left(\frac{1}{9}\right)}{1 - \left(\frac{1}{9}\right)}\right]$$

$$64S = 7391 + 2 \times \frac{1}{9} \times \frac{9}{8}$$

$$S = \frac{29565}{64 \times 4}$$

$$S = \frac{29565}{256}$$

Hence, option (D) is correct.

14. 13

$$\frac{1^3 + 3^3 + 5^3 + ---- (2n-1)^3}{2^3 + 4^3 + 6^3 + ---- + (2n)^3} = \frac{337}{392}$$

We know that if $\frac{a}{b} = \frac{c}{d}$

Then $\frac{a+b}{b} = \frac{c+d}{d}$

(property of components)

Therefore,

$$\frac{1^3 + 2^3 + 3^3 + 4^3 + ---- (2n)^3}{2^3(1^3 + 2^3 + 3^3 + ---- + n^3)} = \frac{729}{392}$$

$$\frac{(2n)^2(2n+1)^2}{8n^2(n+1)^2} = \frac{729}{392} = \frac{(27)^2}{2(14)^2}$$

$$\frac{(2n+1)^2}{(n+1)^2} = \frac{(27)^2}{(14)^2}$$

$$\Rightarrow \frac{(2n+1)}{(n+1)} = \frac{27}{14}$$

$$\Rightarrow n = 13$$

Hence '13' is the correct answer.

15. 1

$$\frac{1}{2} + \frac{1}{2+4} + \frac{1}{2+4+6} \ldots\ldots.. \frac{1}{2+4+6 \ldots\ldots..398}$$

$$= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} \ldots \frac{1}{199 \times 200}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \ldots \left(\frac{1}{199} - \frac{1}{200}\right)$$

$$= 1 - \frac{1}{200} = \frac{199}{200} = \frac{P}{Q}$$

Now last 2 digits of $199^{200} = 01$

We must know that for any odd number 'N' ending in 1, 3, 7, or 9, the last 2 digits of $(N)^{20k} = 01$, Where k is a natural number.

Last 2 digits of $(199)^{20} = (199)^{20 \times 10} = 01$

Sum of the last 2 digits = 0 + 1 = 1

16. (B)

$S_N = (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + (5 - 6)(5 + 6) \ldots\ldots\ldots.. (4N - 1 - 4N)(4N - 1 + 4N) + (4N + 1)^2$

$= [-(1+2)-(3+4)-(5+6) \ldots (4N-1+4N)] + (4N-1)^2$

$= -(1+2+3+4+5+6+\ldots+4N-1+4N) + (4N+1)^2$

$= -\left(\frac{4N(4N+1)}{2}\right) + (4N+1)^2$

$= -2N(4N+1) + (4N+1)(4N+1)$

$= (4N+1)(-2N++4N+1)$

$= (4N+1)(2N+1) = 8N^2 + 6N + 1$

$\mathbf{S_N = 8N^2 + 6N + 1}$

$S_{10} = 8 \times 10^2 + 6 \times 10 + 1 = 861$

$S_5 = 8 \times 5^2 + 6 \times 5 + 1 = 231$

$S_{20} = 8 \times 20^2 + 6 \times 20 + 1 = 3321$

$S_{30} = 8 \times 30^2 + 6 \times 30 + 1 = 7{,}381$

Hence, option (B) is the correct answer.

17. (D)

It is given that m,m + 2n, 3m +n, and 30 are in AP.

$\therefore$ m + (3m+n) = 2(m+2n) or 2m = 3n

or m : n=3 : 2 i.e., m = 3x and n = 2x

Thus, the terms are 3x, 7x, 11x and 15x

So 15x = 30, given or x = 2

which implies m = 6,n = 4 and common difference(d) = 4x = 8

$\therefore$ The 2020th term of the AP = 6 + (2019 × 8) = 16,158

Hence option (D) is the correct answer.

18. (7)

$$K = \frac{1}{21} + \frac{1}{77} + \frac{1}{165} + \frac{1}{285} - - \infty$$

$$= \frac{1}{3\times7} + \frac{1}{7\times11} + \frac{1}{11\times15} + \frac{1}{15\times19} - - \infty$$

$$= \frac{1}{4}\left[\frac{4}{3\times7} + \frac{4}{7\times11} + \frac{4}{11\times15} + \frac{4}{15\times19} - - \infty\right]$$

$$= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{7} + \frac{1}{7} - \frac{1}{11} + \frac{1}{11} - \frac{1}{15} + \frac{1}{15} - \frac{1}{19} - - \infty\right]$$

Now, in the above series except for the first and the last terms all the other terms will cancel out each other, and also the last term will be approximately zero, so only the first term will remain.

$$= \frac{1}{4}\left[\frac{1}{3}\right] = \frac{1}{12}'$$

Hence, 84K = 84 × $\frac{1}{12}$ =7

19. 54

Let different children get a different number of apples, also let different children get 1, 2, 3, ... n apples.

Then we have:(1 + 2 + 3 + 4 ....+ n) apples

$$\therefore \quad \frac{n(n+1)}{2} < 1000 \text{ and}$$

$$\frac{(n+1)(n+2)}{2} > 1000$$

If we take n = 44

Then

$$\frac{44\times45}{2} < 1000 \quad \text{and} \quad \frac{45\times46}{2} > 1000$$

If the first 43 children get

1, 2, 3, 4 ... 43 apples.

$\therefore$ Total apples =(43 x 44)/2 = 946

Therefore, the 44th child will get = 1000 – 946 = 54 apples.

Thus, 54 apples, is the maximum possible number of apples which is obtained by a child.

20. (C)

We have : $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots.. + \text{to } \infty = \frac{\pi^2}{8}$

$$\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \ldots\ldots\ldots$$

$$= \left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \ldots\ldots + \text{to } \infty\right)$$

$$\left(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \ldots.\right) = \frac{\pi^2}{8} - \frac{1}{8}\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots.\right)$$

$$= \frac{\pi^2}{8} - \frac{1}{8} \cdot \frac{\pi^2}{8}$$

$$= \frac{\pi^2}{8}\left(1 - \frac{1}{8}\right) = \frac{7}{64}\pi^2$$

Thus, the required option (C) $\frac{7}{64}\pi^2$ is correct.

21. (B)

$\frac{1}{4\times7}-\frac{1}{7\times10}$ $\frac{10-4}{4\times7\times10}=\frac{6}{4\times7\times10}$	$\frac{1}{7\times10}-\frac{1}{10\times13}$ $\frac{13-7}{7\times10\times13}=\frac{6}{7\times10\times13}$

So, $\frac{6}{4\times7\times10}$ can be written as

$\frac{1}{4\times7}-\frac{1}{7\times10}$

$\frac{6}{7\times10\times13}$ can be written as $\frac{1}{7\times10}-\frac{1}{10\times13}$

First number (in the denominator) of each term 4, 7, 10, 13, 16 are in A.P. with N^{th} term

$= A + (N - 1) D = 4 + (N - 1) 3 = (3N + 1)$.

N^{th} term of the series

$$=\frac{1}{(3N+1)(3N+4)(3N+7)}$$

Also,

$$\frac{6}{(3N+1)(3N+4)(3N+7)}$$

$$=\frac{1}{(3N+1)(3N+4)}-\frac{1}{(3N+4)(3N+7)}.$$

$$\frac{1}{4\times7\times10}+\frac{1}{7\times10\times13}+\frac{1}{10\times13\times16}\ldots\ldots$$

$$\ldots\ldots\frac{1}{(3N+1)(3N+4)(3N+7)}$$

$$=\frac{1}{6}\left[\frac{1}{4\times1}-\frac{1}{7\times10}+\frac{1}{7\times10}-\frac{1}{10\times13}+\frac{1}{10\times13}-\frac{1}{13\times16}\cdots\frac{1}{(3N+1)(3N+4)}-\frac{1}{(3N+4)(3N+7)}\right]$$

$$=\frac{1}{6}\left[\frac{6}{4\times7\times10}+\frac{6}{7\times10\times13}+\frac{6}{10\times13\times16}\cdots\ldots\frac{6}{(3N+1)(3N+4)(3N+7)}\right]$$

$$\frac{1}{6}\left[\frac{1}{4\times7}-\frac{1}{(3N+4)(3N+7)}\right]$$

For the sum of 20 terms, Put n = 20

$$\frac{1}{6}\left[\frac{1}{4\times7}-\frac{1}{64\times67}\right]=\frac{355}{60032}$$

Hence, option (B) is the correct answer.

22. (D)

x, 2y and 3z from a G.P

(common ratio) ...(1)

Also, x + 2y, 2y + 3z and 3z + x are in A.P

$\Rightarrow$ 2(2y + 3z) = 3z + x + x + 2y

$\Rightarrow$ 2y + 3z = 2x ...(2)

Put the value of 2y = xR and

$3z=\frac{4y^2}{x}$ in (2)

$\Rightarrow xR+\frac{4y^2}{x}=2x$

$\Rightarrow R+\frac{4y^2}{x^2}=2$

$\Rightarrow R + R^2 = 2$

$\Rightarrow R^2 + R - 2 = 0$

$(R + 2)(R - 1) = 0$

$R = -2$ or 1

Hence, option (D) is the correct answer.

23. (D)

$X_1 = -1$ and $X_2 = 1$ (given)

$X_{m+1} = X_{m-1} + m$ (given) ...(1)

Put m = 2 in(1) :

$X_3 = X_1 + 2$

$X_3 = -1 + 2$ ...(2)

Put m = 3 in(1) :

$X_4 = X_2 + 3$

$X_4 = 1 + 3$ ...(3)

Put m = 4 in(1) :

$X_5 = X_3 + 4$

$X_5 = -1 + 2 + 4$

Put m = 5 in(1) :

$X_6 = X_4 + 5$

$X_6 = 1 + 3 + 5$ (using 3)

Put m = 6 in(1) :

$X_7 = X_5 + 6$

$X_7 = -1 + 2 + 4 + 6$

from above values, we can find the pattern such that

X_K (K is odd),

$X_K = -1 + (2 + 4 + \ldots\ldots + K-1)$

So, $X_{75} = -1 + (2 + 4 + 6 + \ldots\ldots + 74)$

$= -1 + 2\,(1 + 2 + 3 + \ldots\ldots + 37)$

$= -1 + (37 \times 38) = 1405$

Hence, option (D) is the correct answer.

24. 96,270

Let $P = 6 \times 8 + 9 \times 11 + 12 \times 14 + \ldots + 93 \times 95$

$\therefore$ n^{th} term of the series can be written as:

$T_n = (3n + 3)\,(3n + 5)$

Since the last term is given so that we can find the value of n.

$(3n + 5) = 95$

$3n = 95 - 5$

$$\boxed{n = \frac{90}{3} = 30}$$

$$\therefore \sum_{n=1}^{n=30} (3n+3)(3n+5)$$

$$= \sum_{n=1}^{n=30} (9n^2 + 15n + 9n + 15)$$

$$= \sum_{n=1}^{n=30} (9n^2 + 24n + 15)$$

$$= 9 \times \sum_{n=1}^{n=30} n^2 + 24 \times \sum_{n=1}^{n=30} n + 15$$

$$= 9 \times \frac{n(n+1)(2n+1)}{6} + 24 \times \frac{n(n+1)}{2} + 15$$

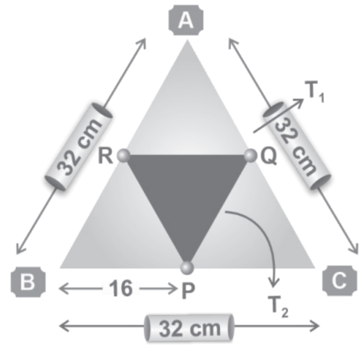
$$= 9 \times \frac{30 \times 31 \times 61}{6} + 24 \times \frac{30 \times 31}{2} + 15$$

$= 85,095 + 11,160 + 15 = 96,270$

Hence, the sum of the above series is 96, 270.

25. (C)

$$\therefore \text{ Area of the equilateral } \Delta T_1 = \frac{\sqrt{3}}{4} \times (32)^2$$



Since we know that, if we join the midpoints of an equilateral triangle then another equilateral triangle will form but the area will become $\frac{1}{4}$th the original triangle.

$\therefore$ Area of the equilateral Δ_s

If we again join the midpoints of the $\Delta\, T_2$ then we will get another equilateral $\Delta\, T_3$ whose area becomes 1/4th the area of $\Delta\, T_2$.

Therefore, $T_1 + T_2 + T_3 + \ldots \infty$

$$= \frac{\sqrt{3}}{4}a^2 + \frac{\frac{\sqrt{3}}{4}a^2}{4} + \frac{\frac{\sqrt{3}}{4}a^2}{16} + \ldots \infty$$

or we can say that ...

$$T_1 + \frac{T_1}{4} + \frac{T_1}{4^2} + \frac{T_1}{4^3} + \frac{T_1}{4^4} + \ldots \infty$$

Since the above series is in GP.

$$\therefore \text{ Common ratio } (r) = \frac{\frac{T_1}{4}}{T_1} = \frac{1}{4}$$

Also, the sum of infinite terms

$$= \frac{a}{1-r} \longrightarrow \text{First term}$$

$$= \frac{T_1}{1-\frac{1}{4}} = \frac{4T_1}{3} = \frac{4}{3} \times \frac{\sqrt{3}}{4}(32)^2$$

$$= \frac{1}{\sqrt{3}}(32)^2 = \frac{1024}{\sqrt{3}} = \frac{1024\sqrt{3}}{3}$$

Hence, the sum of the areas of infinitely many such triangles

$(T_1, T_2, T_3, \ldots\ldots \infty) = \frac{1024}{\sqrt{3}}$,

Hence, Option (C) is the correct answer.

26. (D)

Here letter 'a' occurs 1-time

letter 'b' occurs 4-time

letter 'c' occurs 9-time

Here nth letter will be written n^2 time, where 'n' represents the position of the letter in alphabetical series.

Therefore, $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 \leq 3000$

$$\frac{n(n+1)(2n+1)}{6} \leq 3000$$

By hit and trial (considering given options) we will get:

When, n = 20,

$$\frac{\{n(n+1)\ (2n+1)\}}{6} = \frac{17220}{6} = 2870$$

So, 2870th letter would be T and then letter U will start and U will come $21^2 = 441$ times

So, up to 2870 + 441 = 3311, letter U will come

Hence, Letter U will come at 3000th position.

Hence, option (D) is the correct answer

27. (B)

Since the given series is:

We can observe that the series follows the pattern:

$P, P^3, P^5 \ldots. P^{2n-1}$

Also, $1, q, q^2 \ldots. q^{n-1}$

Thus,

$$\log(P) + \log\left(\frac{P^3}{q}\right) + \ldots + \log\left(\frac{P^{2n-1}}{q^{n-1}}\right)$$

$$= \log\left[P \times \frac{P^3}{q} \times \frac{P^5}{q^2} \times \frac{P^7}{q^3} \cdots \times \frac{P^{2n-1}}{q^{n-1}}\right]$$

$$= \log\left[\frac{P^{(1+3+5+7+\ldots.+(2n-1))}}{q^{(1+2+3+\ldots+(n-1))}}\right]$$

$$= \log\left[\frac{P^{n^2}}{q^{\frac{(n-1)\ (n)}{2}}}\right] = \log\left[\frac{P^{2n}}{q^{n-1}}\right]^{n/2} = \frac{n}{2}\log\left[\frac{P^{2n}}{q^{n-1}}\right]$$

Hence, option (B) is the correct answer.

28. (B)

The given sequences are

{3, 7, 11, 15, ..., 399} ...(1)

and {2, 9, 16, 23, ..., 702} ...(2)

Let the m^{th} term of the 1^{st} sequence be equal to the n^{th} term of the 2^{nd} sequence,

i.e.,3 + (m − 1)4 = 2 + (n − 1)7

$\Rightarrow$ 7n − 4m = 4

$\Rightarrow \quad n = \frac{4(m+1)}{7}$

$\Rightarrow$ n = 4 when m = 6

Thus 6^{th} term of the 1^{st} sequence i.e., 3 + (5×4) = 23 which is equal to 4^{th} term of the second sequence i.e., 2 + (3×7) = 23

Now the sequence with the common terms with common difference 28 (i.e., LCM of the common difference of 1^{st} and 2^{nd} sequence 4 and 7 respectively) is :

{23, 51, 79, ...t_k };

where t_k= 23 + (k − 1)28 = 28k − 5

Thus we have,

28k − 5 ≤ 399 (since 399 < 702, from the last term of 1^{st} and 2^{nd} AP's)

$\Rightarrow \quad k \geq \frac{404}{28}$

$\Rightarrow$ k ≤ 14.42

$\therefore$ The number of common terms would be 14.

Hence, the sum of these common terms

$$= \frac{14}{2}[2(23) + (14 - 1)28] = 2870.$$

Hence, option (B) is the correct answer

29. (B)

Let the nth term of the A.P. be a_n and common difference be d.

a_1 = First term.a_2 = Second term.

$a_{n+1} = a_1 + (n + 1 - 1)d$

$= a_1 + nda_{n+2} = a_1 + (n + 2 - 1)d$

$a_1 + (n+1)d = (a_1 + d) + nd$

$= a_2 + nd$

.

.

.

$a_{2n} = a + (2n - 1)d = (a + (n - 1)d) + nd$

$= a_n + nd$

Sum of first n terms = $a_1 + a_2 + a_3 + \ldots\ldots + a_n = 100$ (Given)

Sum of next n terms=$a_{n+1} + a_{n+2} \ldots\ldots + a_{2n} = 300$

The difference of next 'n' terms and first 'n' terms is 300 – 100 = 200

$= a_{n+1} + a_{n+2} \ldots\ldots + a_{2n} - a_1 + a_2 + a_3 + \ldots\ldots a_n = 200$

$= (a_1 + nd) + (a_2 + nd) \ldots\ldots (a_n + nd) - (a_1 + a_2 + a_3 + \ldots\ldots + a_n) = 200$

n (nd=) 200

Hence, $n^2d = 200$.

Also, the sum of the first n terms

$$\frac{n}{2}(2a + (n-1)d = 100$$

$$an + \frac{n^2d}{2} - \frac{nd}{2} = 100$$

$$an + \frac{200}{2} - \frac{nd}{2} = 100$$

$$an = \frac{nd}{2}$$

$$\frac{a}{d} = \frac{1}{2} = \frac{m}{n}$$

m + n = 3

Hence option (B) is the correct answer.

30. (B)

Let z = (8 + 88 + 888 + ... 12 terms)

z = 8(1 + 11 + 111 + ... 12 terms)

$$z = \frac{8}{9}(9 + 99 + 999 + \ldots\ldots 12 \text{ terms})$$

$$z = \frac{8}{9}(10 - 1 + 10^2 - 1 + 10^3 - 1 + \ldots\ldots 12 \text{ terms})$$

$$z = \frac{8}{9}[10 + 10^2 + \ldots\ldots + 10^{12} - 12]$$

$$z = \frac{8}{9}\left[\frac{10 \times (10^{12} - 1)}{10 - 1} - 12\right]$$

On solving further, we get,

z = 987654320976

Sum of digits of z = 66

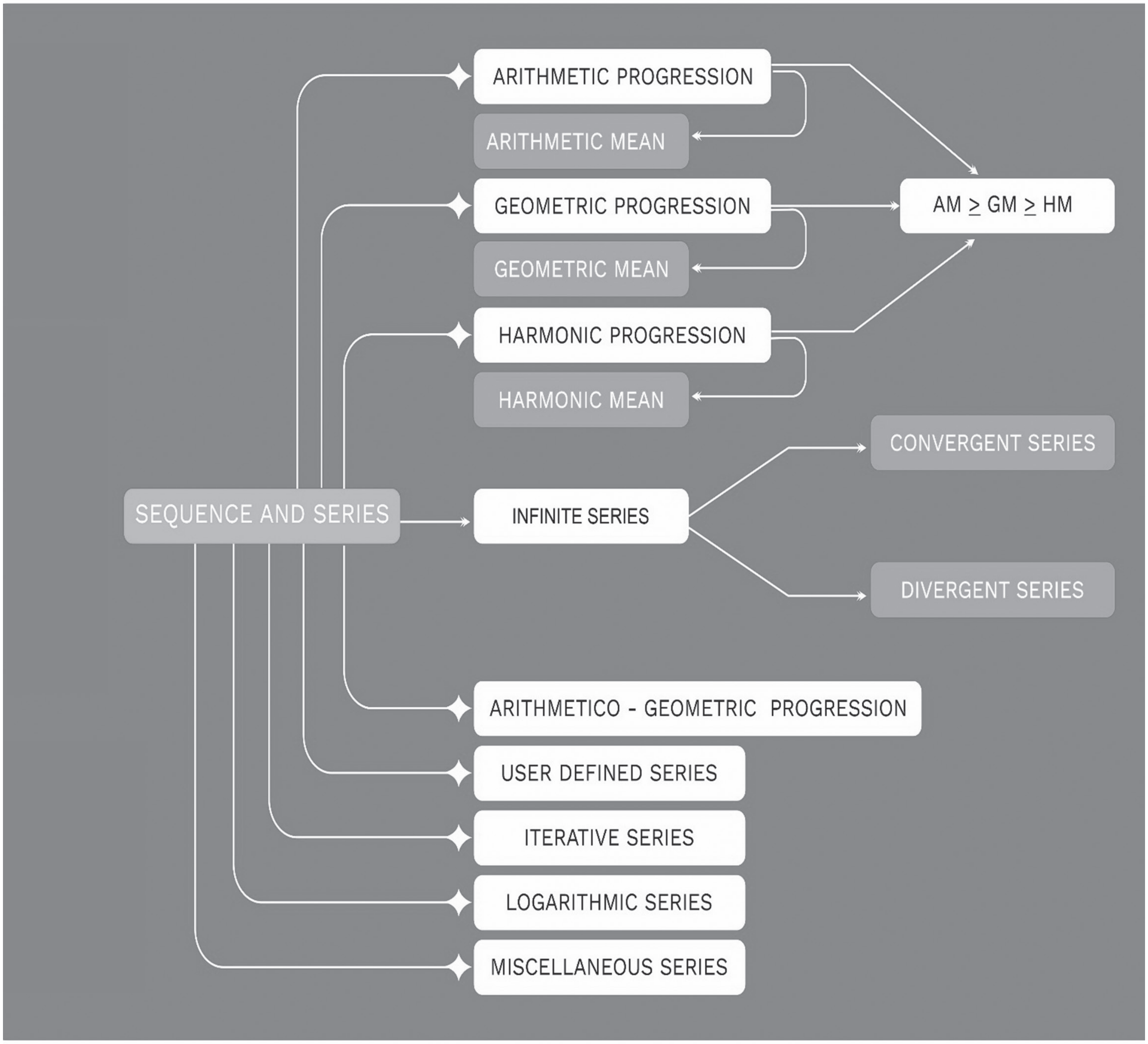
So, the maximum power of 7 in 66! is to be found.

Maximum power = $[66/7] + [66/7^2]$

= 9 + 1 = 10

Hence option (B) is the correct answer.

Mind Map



ARITHMETIC PROGRESSION
ARITHMETIC MEAN
GEOMETRIC PROGRESSION
GEOMETRIC MEAN
HARMONIC PROGRESSION
HARMONIC MEAN
AM ≥ GM ≥ HM
CONVERGENT SERIES
SEQUENCE AND SERIES
INFINITE SERIES
DIVERGENT SERIES
ARITHMETICO - GEOMETRIC PROGRESSION
USER DEFINED SERIES
ITERATIVE SERIES
LOGARITHMIC SERIES
MISCELLANEOUS SERIES