PHYS 351
Topic 1:
Linear Systems

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Solving Diagonal Systems (1)

The system defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 6 \\ -15 \end{bmatrix}$$

is equivalent to

$$\begin{array}{rrrcr} x_1 & & & = & -1 \\ & 3x_2 & & = & 6 \\ & & 5x_3 & = & -15 \end{array}$$

The solution is

$$x_1 = -1 \qquad x_2 = \frac{6}{3} = 2 \qquad x_3 = \frac{-15}{5} = -3$$

Solving Triangular Systems (6)

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 9 \\ -1 \\ 8 \end{bmatrix}$$

is equivalent to

$$\begin{array}{rcrcrcr} -2x_1 & + & x_2 & + & 2x_3 & = & 9 \\ & & 3x_2 & + & -2x_3 & = & -1 \\ & & & & 4x_3 & = & 8 \end{array}$$

Solve in backward order (last equation is solved first)

$$x_3 = \frac{8}{4} = 2 \qquad x_2 = \frac{1}{3}(-1 + 2x_3) = \frac{3}{3} = 1$$

$$x_1 = \frac{1}{-2}(9 - x_2 - 2x_3) = \frac{4}{-2} = -2$$

Gaussian Elimination — Cartoon Version (2)

Eliminate elements under the pivot element in the first column. x' indicates a value that has been changed once.

$$
\begin{bmatrix} \boxed{x} & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}
\longrightarrow
\begin{bmatrix} \boxed{x} & x & x & x & x \\ 0 & x' & x' & x' & x' \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}
$$

$$
\longrightarrow
\begin{bmatrix} \boxed{x} & x & x & x & x \\ 0 & x' & x' & x' & x' \\ 0 & x' & x' & x' & x' \\ x & x & x & x & x \end{bmatrix}
$$

$$
\longrightarrow
\begin{bmatrix} \boxed{x} & x & x & x & x \\ 0 & x' & x' & x' & x' \\ 0 & x' & x' & x' & x' \\ 0 & x' & x' & x' & x' \end{bmatrix}
$$

Gaussian Elimination — Cartoon Version (3)

The pivot element is now the diagonal element in the second row. Eliminate elements under the pivot element in the second column. x'' indicates a value that has been changed twice.

$$
\begin{bmatrix}
x & x & x & x & x \\
0 & \boxed{x'} & x' & x' & x' \\
0 & x' & x' & x' & x' \\
0 & x' & x' & x' & x'
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
x & x & x & x & x \\
0 & \boxed{x'} & x' & x' & x' \\
0 & 0 & x'' & x'' & x'' \\
0 & x' & x' & x' & x'
\end{bmatrix}
$$

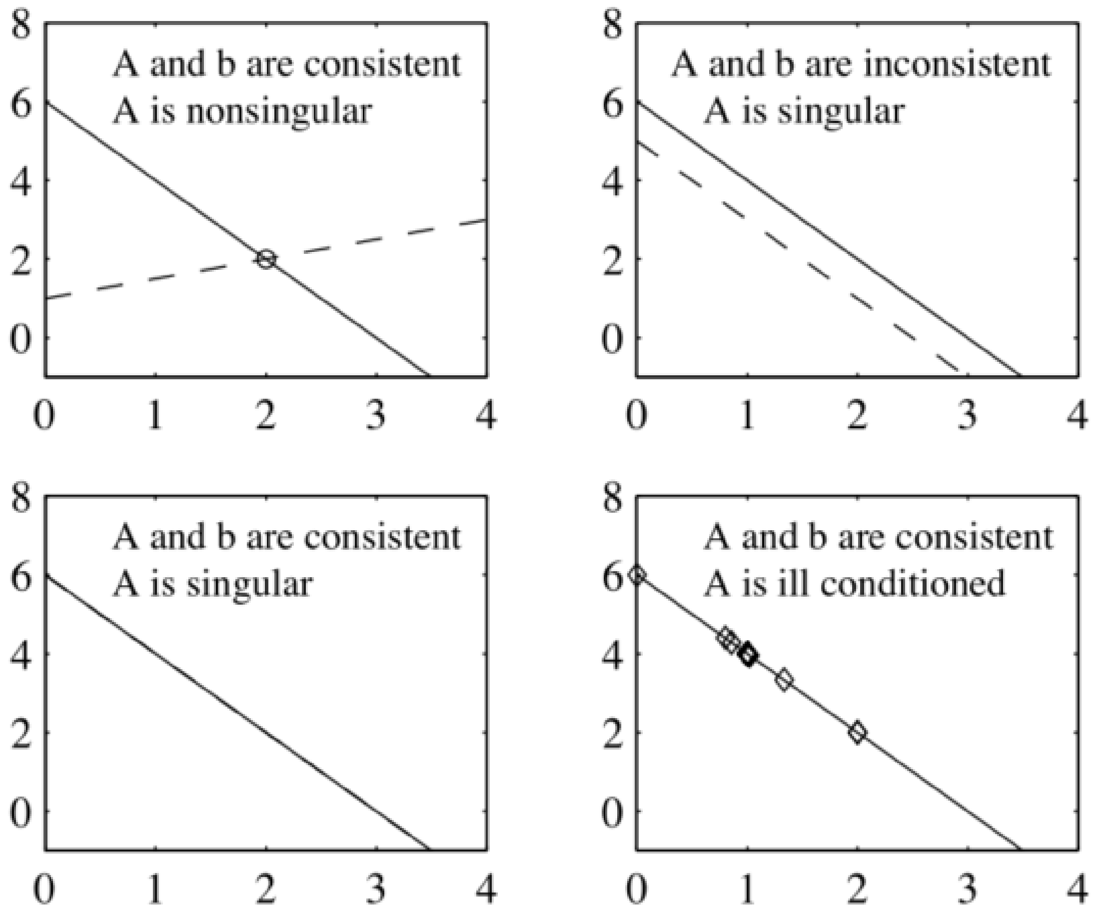
$$
\longrightarrow
\begin{bmatrix}
x & x & x & x & x \\
0 & \boxed{x'} & x' & x' & x' \\
0 & 0 & x'' & x'' & x'' \\
0 & 0 & x'' & x'' & x''
\end{bmatrix}
$$

Singularity of A

If an $n \times n$ matrix, A, is **singular** then

- the columns of A are linearly dependent
- the rows of A are linearly dependent
- $\mathrm{rank}(A) < n$
- $\det(A) = 0$
- A^{-1} does not exist
- a solution to $Ax = b$ may not exist
- If a solution to $Ax = b$ exists, it is not unique

Geometric Interpretation of Singularity (3)



Tiny fluctuations in numbers
cause a big change in solution

```python
A = np.array([[-0.5,1],[2,1]])
print(A)

b = np.array([1,6])
print(b)

np.linalg.solve(A,b)
```

```
[[-0.5  1. ]
 [ 2.   1. ]]
[1 6]
array([2., 2.])
```

numpy.linalg.solve

`linalg.solve(a, b)` [source]

Solve a linear matrix equation, or system of linear scalar equations.

Computes the “exact” solution, *x*, of the well-determined, i.e., full rank, linear matrix equation *ax* = *b*.

Parameters: **a** : *(..., M, M) array_like*

Coefficient matrix.

b : *{(..., M,), (..., M, K)}, array_like*

Ordinate or “dependent variable” values.

Returns: **x** : *{(..., M,), (..., M, K)} ndarray*

Solution to the system a x = b. Returned shape is identical to *b*.

Raises: **LinAlgError**

If *a* is singular or not square.