1

Modular Forms

Upper half-plane: $\mathcal{H} = \{z \in \mathbb{C} : \text{im}(z) > 0\}$.

examples:

we can represent numbers and quadratic forms.

$r_k(n) = \{$# of ways to write n as sum of k squares allowing. 0, sign, ordering.$\}$

$r_2(1) = 4 \qquad r_2(100) = 12?$

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z} \quad \forall z \in \mathcal{H}. \text{ well-defined.}$$

let $q = e^{2\pi i z}$

$$\Theta(z) = \sum q^{n^2}$$

$$\Theta^2(z) = \sum q^{n^2} \sum q^{n^2}$$

$$= \sum r_2(n) q^n. \qquad * \text{ check}$$

$\vdots$

$$\Theta^k(z) = \sum_{n=-\infty}^{\infty} r_k(n) q^n.$$

$$\Theta^k(z+1) = \sum_{n=-\infty}^{\infty} r_k(n) e^{2\pi i (z+1) n}$$

$$= \sum_{n=-\infty}^{\infty} r_k(n) q^n.$$

$\therefore$ Θ^k is invariant to translation by 1.

$$\Theta^k\left(-\frac{1}{4z}\right) = \left(\frac{2z}{i}\right)^{k/2} \Theta^k(z)$$

Partition function:

$p(n)$ = # ways to write n as sum, not allowing 0, sign or ordering.

$$P(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

2 Modular Forms.

$GL_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc \neq 0 \}$

action.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$ $\quad z \in \mathbb{C}$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$

$GL_2^+(\mathbb{R}) \leq GL_2(\mathbb{R})$ $\quad$ action still works.

$\uparrow$ $ad-bc > 0$. $\quad$ from $\mathcal{H} \to \mathcal{H}$. from left.

$$\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \frac{ad-bc}{|cz+d|^2} \cdot \mathrm{Im}(z).$$

Some useful properties

$\gamma_1(\gamma_2 z) = (\gamma_1\gamma_2) z$.

$SL_2(\mathbb{R}) \leq GL_2^+(\mathbb{R})$ $\quad$ spec lin gp.

$\uparrow$ $ad-bc = 1$.

$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad$ proj spec lin gp.

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = z$.

Def The automorphic factor $j : SL_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$.

$$j(\gamma, z) := cz+d$$

Lemma: $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)\, j(\gamma_2, z)$.

pf: ① shabby calc

② write z as $\begin{pmatrix} z \\ 1 \end{pmatrix}$ $\quad \gamma z = \begin{pmatrix} a_1 z + b_1 \\ c_1 z + d_1 \end{pmatrix}$

$$= c_1 z + d_1 \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1 z \\ 1 \end{pmatrix} j(\gamma_1, z)$$

$$\gamma_2\begin{pmatrix}\gamma_1 z\\ 1\end{pmatrix} j(\gamma_1,z) = j(\gamma_1,z)\begin{pmatrix}\gamma_2\gamma_1 z\\ 1\end{pmatrix} j(\gamma_2,\gamma_1 z).$$

$\square$.

Def: ~~Anoth~~ weight k operator $k \geq 0 \in \mathbb{Z}$. $\gamma \in SL_2(\mathbb{R})$

$$(f|_k\gamma)(z) := j(\gamma,z)^{-k} f(\gamma z).$$

$$(f|_k\gamma_2\gamma_1)(z) = j(\gamma_2,\gamma_1 z)^{-k} j(\gamma_1,z)^{-k} f(\gamma_2\gamma_1 z).$$

$$(f|_k\gamma_1)z = j(\gamma_1,z)^{-k} f(\gamma_1 z).$$
$$((f|_k\gamma_2)|_k\gamma_1)z = j(\gamma_1,z)^{-k} j(\gamma_2,\gamma_1 z) f(\gamma_2\gamma_1 z)$$

$$\therefore \quad (f|_k\gamma_2\gamma_1)z = ((f|_k\gamma_2)|_k\gamma_1)z.$$

$$\star. \quad f|_k\gamma_2\gamma_1 = f|_k\gamma_2|_k\gamma_1$$

$$f : \mathcal{H} \to \mathbb{C}.$$

$$SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}).$$
$$Y(1) := SL_2(\mathbb{Z}) \backslash \mathcal{H}.$$
$$= \{\gamma\tau : \tau \in \mathcal{H}, \gamma \in SL_2(\mathbb{Z})\}.$$

$$\pi : \mathcal{H} \to SL_2(\mathbb{Z}) \backslash \mathcal{H} = Y(1).$$
$$\tau \mapsto SL_2(\mathbb{Z})\tau.$$

Give $Y(1)$ as a quotient topology.

~~open under π~~

a set in $Y(1)$ is open if its inverse under π is open in $\mathcal{H}$

well-defined inverse?

Q: is π an open mapping. <u>yes</u>.

$\mathcal{U} \subseteq \mathcal{H}$ open. $\pi(\mathcal{U})$ is open in $Y(1)$.

$\Updownarrow$

$\pi^{-1}\pi(\mathcal{U})$ is open in $\mathcal{H}$.

not just $\mathcal{U}$,

$\pi^{-1}\pi(\mathcal{U}) = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma \cdot \mathcal{U}$ arb union of open sets, so open.

$\tau = x + iy \in \mathcal{H}$.

$S_\tau = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \in SL_2(\mathbb{R})$

$S_\tau(i) = \tau$ action is transitive.

$\Rightarrow SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$.

Compute $\mathrm{Stab}(i) \leq SL_2(\mathbb{R})$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} i = i$.

i.e. $ai + b = (ci + d)i = di - c$.

$a = d \quad b = -c$.

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad a^2 + b^2 = 1$.

$= \left\{ \begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix} : \theta \in [0, 2\pi] \right\}$

$= SO_2(\mathbb{R})$ special orthogonal.

$= \mathrm{Stab}(i)$.

$SL_2(\mathbb{R}) \to \mathcal{H}$ $\gamma \mapsto \gamma \cdot i$ $SO_2(\mathbb{R}) \to i$.	$\exists$ a bijection $\frac{SL_2(\mathbb{R})}{SO_2(\mathbb{R})} \sim \mathcal{H}$. $\frac{\text{space}}{\text{stab}} \sim \text{orb}$ $S_\tau SO_2(\mathbb{R}) \Leftrightarrow \tau$

Lemma: let U_1, U_2 open sets in $\mathcal{H}$.

$$S = \{\gamma \in SL_2(\mathbb{Z}) : \gamma U_1 \cap \gamma U_2 \neq \phi\} \text{ is finite.}$$

pf: $S_{\gamma\tau} : i \to \gamma\tau \quad \gamma S_\tau : i \to \gamma\tau$

$\therefore S_{\gamma\tau}, \gamma S_\tau : i \to \gamma\tau.$

$$\frac{SL_2(\mathbb{R})}{SO_2(\mathbb{R})} \sim \mathcal{H}.$$

$$\therefore S_{\gamma\tau}, \gamma S_\tau \in S_{\gamma\tau} SO_2(\mathbb{R}).$$

$\tau_1, \tau_2 \in \mathcal{H}. \quad \gamma_* \tau_1 = \tau_2$

$$\Updownarrow$$

$$S_{\gamma\tau_1} SO_2(\mathbb{R}) = S_{\tau_2} SO_2(\mathbb{R}).$$

$$\gamma S_{\tau_1} SO_2(\mathbb{R}) = S_{\tau_2} SO_2(\mathbb{R}).$$

$$\therefore \gamma\tau_1 = \tau_2 \Leftrightarrow \gamma \in S_{\tau_2} SO_2(\mathbb{R}) S_{\tau_1}^{-1}$$

$\exists \gamma \in SL_2(\mathbb{Z})$
$\gamma U_1 \cap U_2 \Leftrightarrow \underbrace{SL_2(\mathbb{Z})}_{\text{discrete}} \cap \underbrace{S_{U_2} SO_2(\mathbb{R}) S_{U_1}^{-1}}_{\text{compact}}$

as $SO_2(\mathbb{R})$ is cmpt.

discrete $\cap$ compact = finite.

3 Modular Forms

$SL_2(\mathbb{Z}) \backslash \mathcal{H} = Y(1)$.

Goal: $Y(1)$ is Hausdorf.

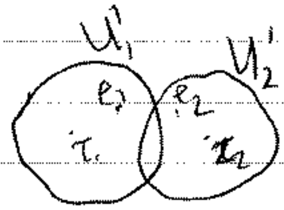
$SL_2(\mathbb{R}) \to \mathcal{H}$
$\gamma \mapsto \gamma i$

$SL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cong \mathcal{H}$
$\gamma \mapsto \gamma \cdot i$

$s_\tau SO_2(\mathbb{R}) \leftrightarrow \tau$.

Prop: Given $\tau_1, \tau_2 \in \mathcal{H}$. Then $\exists$ nbhds U_1 of τ_1, U_2 of τ_2 s.t. for all $\gamma \in SL_2(\mathbb{Z})$

- either $\gamma \tau_1 = \tau_2$
- $\gamma(U_1) \cap U_2 = \phi$.



pf: If $\exists \gamma \in SL_2(\mathbb{Z})$ s.t. $\gamma \tau_1 = \tau_2$ done.
Assume not,
choose U_1' of τ_1 and U_2' of τ_2
s.t. $\overline{U_1'}, \overline{U_2'} \subset \mathcal{H}$.
cmpt closure

choose $e_1 \in U_1'$ $e_2 \in U_2'$ s.t. $\gamma e_1 \neq e_2$ $\forall \gamma \in SL_2(\mathbb{Z})$.
$\mathcal{H} \cong SL_2(\mathbb{R}) / SO_2(\mathbb{R})$
$e_2 \mapsto S_{e_2} SO_2(\mathbb{R})$
$\gamma e_1 \mapsto S_{\gamma e_1} SO_2(\mathbb{R}) = \gamma S_{e_1} SO_2(\mathbb{R})$
$\gamma e_1 \neq e_2 \Rightarrow \gamma S_{e_1} SO_2(\mathbb{R}) \neq S_{e_2} SO_2(\mathbb{R})$
This might not hold $\forall e_1 \in U_1'$ and $e_2 \in U_2'$
assume a contradiction. $\exists$ some γ s.t. $\gamma e_1 = e_2$
$\gamma S_{e_1} SO_2(\mathbb{R}) = S_{e_2} SO_2(\mathbb{R})$.
$\gamma \in S_{e_2} SO_2(\mathbb{R}) S_{e_1}^{-1}$

$\gamma \in S_{\overline{U_2'}}\, SO_2(\mathbb{R}) S_{\overline{U_1'}}^{-1}$

This condition takes care of all γ and all pairs of pts $(e_1, e_2) \in (U_1' \times U_2')$ s.t. $\gamma e_1 = e_2$.

γ is lying in a compact set.

$\underbrace{\gamma \in SL_2(\mathbb{Z})}_{\text{disc}} \cap \underbrace{S_{\overline{U_2'}}\, SO_2(\mathbb{R}) S_{\overline{U_1'}}^{-1}}_{\text{compact}}$

The set $S = \{\gamma \in SL_2(\mathbb{Z}) : \gamma\tau_1 \neq \tau_2 \;\; \underset{\gamma e_1 = e_2}{\gamma(U_1') \cap U_2' \neq \emptyset}\}$

is finite.

Given $\gamma \in SL_2(\mathbb{Z})$ choose disjoint nbhds $U_{1,\gamma}$ of $\gamma\tau_1$
$U_{2,\gamma}$ of τ_2

$$U_1 = U' \cap \Big(\bigcap_{\gamma \in S} \gamma^{-1}(U_{1,\gamma})\Big)$$
$$U_2 = U_2' \cap \Big(\bigcap_{\gamma \in S} U_{2,\gamma}\Big)$$

open.

$\gamma(U_1) \cap U_2$

if $\gamma \notin S$ we are done.

if $\gamma \in S$ $\gamma^{-1}(U_{1,\gamma}) \supset U_1 \Rightarrow U_{1,\gamma} \supset \gamma(U_1)$
$U_{2,\gamma} \supset U_2 \qquad U_{2,\gamma} \supset U_2$

$U_{1,\gamma} \cap U_{2,\gamma} = \emptyset$
$\Rightarrow \gamma(U_1) \cap U_2 = \emptyset$.

$\pi : \mathcal{H} \to Y(1) = SL_2(\mathbb{Z}) \backslash \mathcal{H}$
τ_1, τ_2

Choose τ_1, τ_2 s.t. $\pi(\tau_1) \neq \pi(\tau_2)$ (i.e. $\nexists$ no $\gamma \in SL_2(\mathbb{Z})$ s.t. $\gamma\tau_1 = \tau_2$)

$U_1 \ni \tau_1$ s.t. $Y(U_1) \cap U_2 = \emptyset$
$U_2 \ni \tau_2$

$\Rightarrow \pi(U_1) \cap \pi(U_2) = \emptyset$

$\quad\;\; \pi(\tau_1) \in \quad \pi(\tau_2) \in$

$\Rightarrow Y(1)$ is Hausdorf.

Defn: Modular Form

Defn: A holomorphic fun $f : \mathcal{H} \to \mathbb{C}$ is called weakly modular of weight $k \in \mathbb{Z}^{\geq 0}$ for $SL_2(\mathbb{Z})$ if.

$$f|_k(\gamma)(z) = f(z) \quad \forall \gamma \in SL_2(\mathbb{Z}).$$

$$\Rightarrow f(\gamma z) = j(\gamma, z)^k f(z).$$

$\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad f(z) = (-1)^k f(z).$

$\therefore$ when k is odd, $f(z) = 0$.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$.

$f(z+n) = f(z) \quad n \geq 1.$

$\therefore$ weakly modular functions are $\mathbb{Z}$ periodic.

$D = \{q \in \mathbb{C} : |q| < 1\}$.

$D' = D^*$

$\exp : \mathcal{H} \mapsto D.$

$\quad \tau \mapsto e^{2\pi i \tau}$

$g : D' \mapsto \mathcal{H} \xmapsto{f} \mathbb{C}$

$g(q) = f\left(\frac{\log(q)}{2\pi i}\right)$

$q \mapsto q + 2\pi i n \quad f\left(\frac{\log(q)}{2\pi i} + n\right) = f\left(\frac{\log(q)}{2\pi i}\right).$

$g : D' \to \mathcal{H} \xrightarrow{f} \mathbb{C}$ $\quad g$ holomorphic

$\Downarrow$

Laurent expansion.

$$g(q) = \sum_{n=-\infty}^{\infty} a(n) q^n \quad \text{Laurent expansion.}$$

$$f(z) = \sum_{n=-\infty}^{\infty} a(n) e^{2\pi i n z}$$

Definition: f is meromorphic at ∞ (resp. holomorphic at ∞).

if $f(z) = \sum_{n \geq n_0} a(n) q^n$ - meromorphic at ∞.

If $f(z) = \sum_{n=0} a(n) q^n$ - holomorphic at ∞.

Defn: A holomorphic $f : \mathcal{H} \to \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}^{\geq 0}$ for $SL_2(\mathbb{Z})$ if

- $(f|_k \gamma)(z) = f(z) \quad \forall \gamma \in SL_2(\mathbb{Z})$
  $\Rightarrow f(\gamma z) = j(\gamma, z)^k f(z)$
- f is holomorphic at ∞
  $\Rightarrow f(z) = \sum_{n=0}^{\infty} a(n) q^n \qquad q = e^{2\pi i z}$.

Defn: Say a modular form vanishes at ∞.

if $f(\infty) = 0$

$f(\infty) = a(0) = 0.$

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n.$$

<u>Cusp forms</u>.

Denote space of modular forms of weight k for $SL_2(\mathbb{Z})$ by

$M_k(SL_2(\mathbb{Z}))$ - $\mathbb{C}$-v.s.

cusp forms - $S_k(SL_2(\mathbb{Z}))$

4 Modular forms

recall: holomorphic $f : \mathcal{H} \to \mathbb{C}$ is a modular form of wt k w/t $SL_2(\mathbb{Z})$ if

- $(f|_k \gamma) z = (cz+d)^{-k} f(\gamma z) = f(z)$
- f is holomorphic at $\infty \iff f(z) = \sum_{n=0}^{\infty} a(n) q^n$, $q = e^{2\pi i z}$

(1.4)

Example: Eisenstein series. $k \geq 3$

define $G_k(z) = \sum_{\substack{m,n \\ \in \mathbb{Z}^2}}' \left(\frac{1}{mz+n} \right)^k$

$G_k(z)$ is holomorphic on $\mathcal{H}$ (Ex)

$G_k(\gamma z) = (cz+d)^k \sum \left(\frac{1}{(ma+nc)z + (bm+dn)} \right)^k$

$(ma+nc \quad bm+dn) = (m, n) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= (cz+d)^k \sum \frac{1}{m'z+n'}$ $\qquad m' = am+nc$, $n' = bm+nd$.

$G_k(\gamma z)(cz+d)^{-k} = G_k(z)$.

* I know m', n' span $\mathbb{Z}^2$, but is this shown? — no, spans as γ invertible.

$q = e^{2\pi i z}$

lemma: $\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i - 2\pi i \sum q^m$

Pf: $\sin \pi z = \pi z \prod \left(1 - \frac{z^2}{n^2} \right)$

log + diff. wrt z.

$\pi \cot \pi z = \frac{1}{z} + \sum \frac{2z}{z^2 - n^2} = \frac{1}{z} + \sum \frac{1}{z-n} + \frac{1}{z+n}$.

$$\pi \cot \pi z = \pi \frac{\cot \pi z}{\sin \pi z} = \pi \frac{\left(\frac{e^{i\pi z}+e^{-\pi z i}}{2}\right)}{\left(\frac{e^{i\pi z}-e^{-\pi z}}{2i}\right)}$$

$$= \pi i \left(1 - 2\frac{e^{-i\pi z}}{e^{i\pi z} - e^{i\pi z}}\right)$$

$$= \pi i \left(1 - 2\frac{1}{1-e^{2\pi i z}}\right)$$

$$= \pi i \left(1 - 2\sum q^m\right) \qquad q = e^{2\pi i z}.$$

lemma: $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1} q^n.$ $(k \text{ even})$

pf $\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z-n} + \frac{1}{z+n} = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m.$ $\qquad q = e^{2\pi i z}$

$$-\frac{1}{z^2} + \left(\sum \frac{-1}{(z-n)^2} + \frac{-1}{(z+n)^2}\right) = -(2\pi i)^2 \sum_{m=0}^{\infty} m q^m$$

$$\sum_{n\in\mathbb{Z}} \frac{+1}{(z+n)^2} = (2\pi i)^2 \sum_{m=0}^{\infty} m q^m$$

pf for $k=2$.
diff more times for higher powers (ex).

Thm: $k \geq 4$ even.

$$G_k(z) = 2\zeta(k) E_k(z)$$

def

$$E_k(z) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$\sigma_m(n) = \sum_{d|m} d^m$ divisor function.

$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$ $\qquad B_k$ - bernoulli ~~integer~~ numbers.

$\zeta(k) = \sum \frac{1}{n^s}$ $\qquad Re(s) > 1.$

$$\begin{cases} \zeta(k) = \sum \frac{1}{n^k} = -\frac{(2\pi i)^k}{2} \frac{B_k}{k!} & k>1. \\ \zeta(1-n) = -\frac{B_n}{n} & \forall n \geq 1. \end{cases}$$

pf $G_k = \sum' \frac{1}{(mz+n)^k}$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{n^k} + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$$

$$= 2\zeta(k) + 2\left(\sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}\right)$$

$$= 2\zeta(k) + 2\sum_{m=1}^{\infty} \frac{(-2\pi i)}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^{mn}.$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1} q^{mn} = \sum \sigma_{k-1}(n) q^n.$$

$\Rightarrow$ G_k is a mod form of wt k $SL_2(\mathbb{Z})$ $k \geq 4$ even.

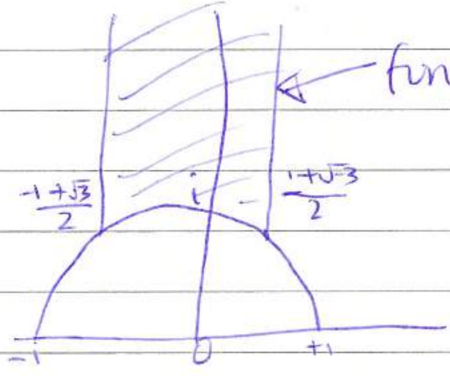
G_k is not a cusp form. □

Fundamental domain

Defn: G acts on $\mathcal{H}$. A fundamental domain for G $D_G \subset \mathcal{H}$ is a closed subset s.t. • D_G is closure of its interior.

• every pt in $\mathcal{H}$ is G-equivalent to a pt in D_G

• if $z, z' \in D_G$ s.t. $z' = \gamma z$ for some $\gamma \in G$ $\Rightarrow$ z, z' on boundary of D_G.



fundamental domain for $SL_2(\mathbb{Z})$.

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad Sz = \frac{-1}{z}$

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad Tz = z+1.$

$\gamma \in \langle S, T \rangle$

$\mathrm{Im}(\gamma z) = (cz+d)^{-2}\, \mathrm{Im}(z) \qquad ad-bc=1.$

$|cz+d|$ is bdd below away from 0

$\Updownarrow$

$\exists$ a minimal elt.

$\exists$ a $\gamma \in \langle S, T \rangle$: $\mathrm{Im}(\gamma z)$ is maximal.

$\forall \gamma' \in \langle S, T \rangle \qquad \mathrm{Im}(\gamma z) \geq \mathrm{Im}(\gamma' z).$

by applying T any times, we can obtain any strip of length 1 to be ar fundamental domain for $\langle T \rangle$.

choose ~~Re~~ $[-\frac{1}{2}, \frac{1}{2}]$.

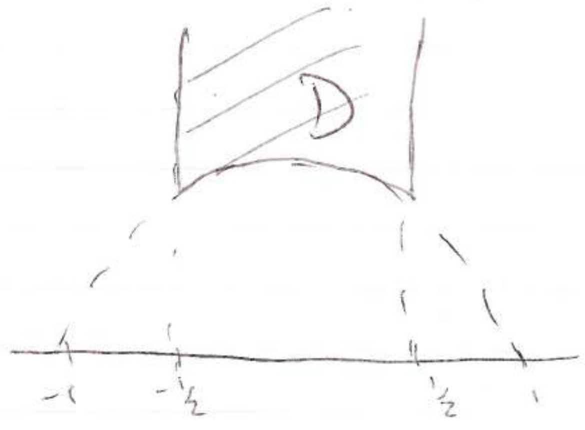
$\mathrm{Im}(\gamma z) \geq \mathrm{Im}(S\gamma z) = \mathrm{Im}\left(\frac{-1}{\gamma z}\right) = \frac{\mathrm{Im}(\gamma z)}{|\gamma z|^2}$

$\Rightarrow |\gamma z| \geq 1 \Rightarrow \gamma z \in D_{SL_2(\mathbb{Z})}$.

5

Modular Forms.

Fundamental domain for $SL_2(\mathbb{Z})$



Thm: This region above is a fundamental domain for $SL_2(\mathbb{Z})$.

i) D is closure of its interior ✓

ii) Every pt in $\mathcal{H}$ is $SL_2(\mathbb{Z})$-equivalent to a point in D.

iii)

ii): Choose $z \in \mathcal{H}$. $z = x+iy$. $\gamma \in SL_2(\mathbb{Z})$ $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

if $|cz+d| \leq 1$.

$(cx+d)^2 + c^2y^2 \leq 1$.

$c^2y^2 \leq 1$

$|c| \leq 1/y$ $\Rightarrow$ c has finite number of choices.

$-1 \leq (cx+d)^2 \leq 1$.

$-1-cx \leq d \leq 1-cx$ $\Rightarrow$ d has finite # of choices.

$(0,1)$ satisfies $|0 \cdot z+1| \leq 1$.

$\exists$ finite number of (c,d) : $0 \leq |cz+d| \leq 1$.

$\Rightarrow$ $|cz+d|$ attains a minimum.

$\mathrm{Im}(\gamma z) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$ $\rightarrow$ attains a maximum.

let γ be the element in $SL_2(\mathbb{Z})$ s.t. $\mathrm{Im}(\gamma z)$ is maximum.

$T^j z = z + j$ $\qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\mathrm{Re}(T^j z) = \mathrm{Re}(z) + j$

$\mathrm{Im}(T^j z) = \mathrm{Im}(z)$.

replacing γz by $T^j \gamma z$, we can assume that

$-\frac{1}{2} \le \mathrm{Re}(\gamma z) \le \frac{1}{2}$.

Assume $|\gamma z| < 1$. $\qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\mathrm{Im}(S\gamma z) = \dfrac{\mathrm{Im}(\gamma z)}{|\gamma z|^2} > \mathrm{Im}(\gamma z)$

※ assumed $\mathrm{Im}(\gamma z)$ is maximal.

c) If $\exists\ z, w \in D$ s.t. $w = \gamma z$ $\quad \gamma \in SL_2(\mathbb{Z})$.

then z, w on boundary.

pf: Assume $\mathrm{Im}(w) \ge \mathrm{Im}(z)$.

$\Rightarrow |cz+d| \le 1$.

$|\mathrm{Im}(cz+d)| = \cancel{c\,\mathrm{Im}(z)}\ |c|\ \mathrm{Im}(z)$.

$z \in D \Rightarrow \mathrm{Im}(z) \ge \sqrt{3}/2$.

$\Rightarrow |c| \le 2/\sqrt{3}$.

$\Rightarrow c = 0, 1, -1$. (c is int).

$c=0$: $|d| \le 1$ $\quad d = 0, 1, -1$.

$d \ne 0 \ \because \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \notin SL_2(\mathbb{Z})$.

$d = \pm 1$: $\begin{pmatrix} a & b \\ 0 & \pm 1 \end{pmatrix} \in SL_2(\mathbb{Z})$.

$\therefore a = \pm 1$.

$\therefore \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$.

$w = \gamma z = z + b \Rightarrow b = 1.$

$c = \pm 1$ similar, just more enumeration.

Valence formula.

- $f \in M_{k_1}$ $g \in M_{k_2}$ $f \cdot g \in M_{k_1 + k_2}$.

$$\Delta(z) := q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12}(SL_2(\mathbb{Z})).$$

Ramanujan's tau function.

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f - meromorphic. fn on some open set in $\mathcal{H}$.

$V_p(f)$ - order of f at $p \in U$.

$\rightarrow$ unique integer n s.t. $(z-p)^{-n} f(z)$ is holomorphic & non-vanishing at p.

If $n > 0$, $f(z)$ has a 0 at p of order n. $- f(z) = (z-p)^n g$

if $n < 0$, " pole " " $= \frac{g(z)}{(z-p)^n}$.

$$f(z) = \sum_{n \geq n_0} c_n (z-p)^n.$$

Defn: The residue of f at $p = c_{-1} \in \mathbb{C}$. $\mathrm{Res}_p(f) = c_{-1}$.

Lemma: $\mathrm{Res}_p\left(\frac{f'}{f}\right) = V_p(f)$.

$$V_p(f) = n \Rightarrow f(z) = (z-p)^n g(z).$$

$$f'(z) = n(z-p)^{n-1} g(z) + (z-p)^n g'(z).$$

$$f'/f = \frac{n}{z-p} + \frac{g'}{g}.$$

Thm: (Cauchy's Integral formula)

Let g be a holomorphic function $U \subseteq \mathbb{C}$ & let C be a contour in U

$$\int_C \frac{g(z)}{z-p}\, dz = 2\pi i\, g(p).$$

$C(p, r, \alpha)$ - arc centered at p of radius r & angle α.

$$\lim_{r\to 0} \int_{C(p,r,\alpha)} \frac{g(z)}{z-p}\, dz = \alpha i g(p)$$

Thm Let f be meromorphic fn. on $U \subseteq \mathbb{C}$ & let C be a contour in C not passing through its zeros or poles. $\displaystyle\int_C \frac{f'}{f}\, dz = 2\pi i \sum_{z \in \mathrm{Int}(C)} v_z(f)$.

Corollary: If f is meromorphic at p, then

$$\lim_{r\to 0} \int_{C(p,r,\alpha)} \frac{f'}{f}(z)\, dz = \alpha i v_p(f).$$

Ex: Suppose f is weakly modular of wt k.

$$v_p(f) = v_{\gamma p}(f)\ \forall \gamma \in SL_2(\mathbb{Z})$$

Thm: $$v_\infty(f) + \tfrac{1}{2} v_i(f) + \tfrac{1}{3} v_\rho(f) + \sum_{\tau \in SL_2(\mathbb{Z})\backslash \mathcal{H}'} v_\tau(f) = \frac{k}{12}.$$

$\mathcal{H}' = \mathcal{H} \setminus \{\infty, i, \rho\}$

$v_\infty(f) = n_0 \quad f = \sum_{n \geq n_0} a(n) q^n.$

$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}.$

$(G\backslash X)$???...

Thm:

$M_k = \{0\}$ if $k<0$ or $k=2$.
$S_k = \{0\}$ if $\forall k < 12$
$M_0 = \mathbb{C}$.
$S_{12} = \mathbb{C}\Delta$.

pf: plugging into valence, $k<0$ $\frac{k}{12}<0$ but LHS >0.

$k=2$. RHS $= \frac{1}{6}$ LHS $= a + \frac{b}{2} + \frac{c}{3} + d$.
no such combination exists.

If $f \in S_k \Rightarrow V_\infty(f) \geq 1$.
$\therefore$ RHS ≥ 1
$\therefore k \geq 12$.

If $f \in M_0$ and f not const $g = f(\infty) \in M_0$
$f - g \in S_0 = \{0\}$.
$\therefore 0 = f - g$
$\therefore f = g$.

RHS $= 1$. $\Delta \in S_{12}(SL_2(\mathbb{Z}))$.

$g(z) = \cancel{\frac{f(z)}{\Delta(z)}}$ $f(z) \in S_{12}(SL_2(\mathbb{Z}))$.
$= f(z) - \frac{f(i)}{\Delta(i)}\Delta(z)$.

g also $\in S_{12}$. $g(i) = 0 \Rightarrow V_i(g) \geq 1$.

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Modular Forms.

recall Valence formula. $f \in M_k(SL_2(\mathbb{Z}))$

$$V_\infty(f) + \frac{1}{2}V_i(f) + \frac{1}{3}V_\rho(f) + \sum_{\substack{\tau \in SL_2(\mathbb{Z})\backslash\mathcal{H}' \\ \mathcal{H}' = \mathcal{H}\setminus\{\infty, i, \rho\}}} V_\tau(f) = \frac{k}{12}$$

If $f = \sum_{n=0}^{\infty} a(n) q^n$

$V_\infty(f) = 0$ if $f \notin S_k$

$V_\infty(f) \geq 1$ if $f \in S_k$

$$\Delta(q) := q \prod_{n=1}^{\infty} (1-q^n)^{24} \in S_{12}(SL_2(\mathbb{Z}))$$

Thm: $S_{12} = \mathbb{C}\Delta$

$f \in S_{12}$ assume $f(z) \neq c\Delta$.

$V_\infty(f) \geq 1$ by Valence formula.

$\Rightarrow V_\infty(f) = 1$.

$g(z) := f(z) - \frac{f(i)}{\Delta(i)} \Delta(z) \quad \in S_{12}$

$V_\infty(g) \geq 1$.

$g(i) = 0. \Rightarrow V_i(g) \geq 1$. ✱ valence formula.

as $V_\infty(g) + \frac{1}{2}V_i(g) = 1$
$+ V_\rho(f) + \ldots$

$\Rightarrow g(z) = 0 \therefore f = c\Delta$.

$\therefore S_{12} = \mathbb{C}\Delta$.

Corollary: $S_{k+12} = \Delta M_k$.

pf: Let $f \in S_{k+12}$ $\quad g(z) = \frac{f(z)}{\Delta(z)}$

Makes sense $\because \Delta$ is holomphic on $\mathcal{H}$ and is non-vanishing ~~on~~

$V_\infty(g) = \underbrace{V_\infty(f)}_{\geq 1} - \underbrace{V_\infty(\Delta)}_{1} \Rightarrow V_\infty(g) \geq 0.$

$$g = \sum_{n \geq 0} a(n) q^n$$

$M_k = \mathbb{C} E_k \oplus S_k$: $\phi : M_k \to \mathbb{C}$

$\phi(f) := f(\infty)$

$\ker(\varphi) = S_k$ because cusp forms vanish at infinity

$\operatorname{Im}(\varphi) = E_k$

$\because \varphi(E_k) = E_k(\infty) = 1$

$G_k(z) = 2\zeta(k) E_k(z)$

$E_k(z) = 1 + \frac{2k}{B_k} \sum \sigma_{k-1}(n) q^n.$

$\varphi(cE_k) = c$

$M_k = \operatorname{Im}\varphi \oplus \ker\varphi$

$= \mathbb{C}E_k \oplus S_k$

$= \mathbb{C}E_k \oplus \Delta M_{k-12}.$

Cor: $\dim M_k = \lfloor \frac{k}{12} \rfloor$ if $k \equiv 2 \ (12)$

$\lfloor \frac{k}{12} \rfloor + 1 \quad k \not\equiv 2 \ (12)$

pf: by induction.

$\dim M_k = 1 + \dim M_{k-12}$

and then induce for k.

$S_k = 0 \ \forall k < 12$

E_4, E_6, E_8, E_{10} (only mod forms for $k < 12$).

Conc $\dim M_k < \infty$ and depends on k.

Cor $M_k = \langle E_4^a E_6^b : a \geq 0 \ b \geq 0 \quad 4a + 6b = k \rangle$.

$M_4 = \mathbb{C} E_4$
$M_6 = \mathbb{C} E_6$
$M_8 = \mathbb{C} E_8 = \mathbb{C} E_4^2$

$k \geq 8$, we again use induction.
$E_4^a E_6^b(\infty) = 1$ as $E_4^a(\infty) = 1$
$E_6(\infty) = 1$.

$g(z) = f(z) - f(\infty) E_4^a E_6^b \in S_k$

$g(z) = \Delta h(z) \quad h \in M_{k-12}$.
$\left(\sum_{a,b} E_4^a E_6^b\right)$

$\Delta(z) = \frac{1}{12^3}\left(E_4^3 - E_6^2\right)$

$g(z) = \frac{1}{12^3}\left(E_4^3 - E_6^2\right)\left(\sum_{ab} E_4^a E_6^b\right)$.

Thm: $f \in M_k$
$f = \sum_{n=0}^{\infty} a(n) q^n$

Suppose $a(j) = 0 \ \forall \ 0 \leq j \leq k/12$; then $f = 0$.

pf by valence $V_\infty(f) \geq k/12$ # $\therefore f = 0$.

- contradicts valence formula.

Cor: If $f, g \in M_k$: $a_f(n) = a_g(n) \quad \forall \ 0 \leq n \leq k/12$
$\Rightarrow f = g$.

Growth of Fourier coefficients

Thm: $a_n(E_k)$ grows as n^{k-1}

$\exists A, B > 0$

$$A n^{k-1} \leq |a_n(E_k)| \leq B n^{k-1}.$$

$$E_k = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

$$a_n = \frac{2k}{B_k} \sigma_{k-1}(n) = \frac{2k}{B_k} \sum_{d|n} d^{k-1}$$

$$\geq \frac{2k}{B_k} n^{k-1}$$

$$A = \frac{2k}{B_k}.$$

$$\frac{a_n}{n^{k-1}} = \frac{2k}{B_k} \sum_{d|n} \left(\frac{d}{n}\right)^{k-1}$$

$$= \frac{2k}{B_k} \sum_{d|n} d^{-(k-1)}$$

$$\leq \frac{2k}{B_k} \sum_{d=1}^{\infty} \frac{1}{d^{k-1}} = \cancel{\frac{2k}{B_k}} \zeta(k-1)$$

$$a_n(E_k) \leq \zeta(k-1) n^{k-1} \frac{2k}{B_k}.$$

$$\therefore B = \frac{2k}{B_k} \zeta(k-1).$$

Thm (Hecke) If $f = \sum a(n) q^n \in S_k$

$a_n(f) = O(n^{k/2})$

$|a_n| \leq M n^{k/2}$

$\phi : \mathcal{H} \to \mathbb{R}^{\geq 0}$

$z \mapsto |f(z)| y^{k/2}$

Invariant under $SL_2(\mathbb{Z})$.

as $y \to \infty$ $\phi \to 0$

$f(z) = \sum a_n q^n$. $q = e^{2\pi i z}$

$= e^{-2\pi y} e^{2\pi i n x}$

as $y \to \infty$ $q \to 0$.

☆ this page is not good v. distracted

∴ as a fn $|\Phi(z)|$ is bdd.

$$|f(z)| \leq M' y^{-k/2} \quad \forall z \in \mathcal{H}.$$

$$f(z) = a(1)q + a(2)q^2 + \ldots + q^n a(n) + \ldots$$

☆ $$\frac{f(z)}{q^{n+1}} = \frac{a(n)}{q} + a(n+1) + a(n+2)q + \ldots$$

by residue formula $a_n = \frac{1}{2\pi i} \int_{C_y} \frac{f(z)}{q^{n+1}} dz$

C_y: ~~circle~~ $e^{2\pi i(x+y)}$ $x \in [0,1]$

circle center clockwise.

$$q = e^{2\pi i(x+iy)}$$

$$dq = 2\pi i q \, dx.$$

$$|a_n| = \int_0^1 \frac{|f(z)|}{q^n} dx$$

$$|a_n| = \int_0^1 |f(z)|q^{-n} dx \leq \int_0^1 M' y^{-k/2}$$

$$|a_n(f)| \leq M'' y^{-k/2} e^{2\pi y n}$$

$$\forall z \in \mathcal{H}$$

taking $y = \frac{1}{n}$ $y \in \mathbb{R}^{\geq 0}$.

$$|a_n(f)| \leq M'' e^{2\pi} n^{k/2}$$

$$|a_n(f)| \leq M n^{k/2}$$

- take abs everywhere.

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Modular Forms.

Congruent subgps of $SL_2(\mathbb{Z})$

$N \geq 1$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A \equiv I \bmod N \right\}$$

Defn: A subgp $\Gamma \leq SL_2(\mathbb{Z})$ is called a congruence subgp if $\exists$ some $N \geq 1$ s.t. $\Gamma(N) \leq \Gamma \leq SL_2(\mathbb{Z})$. level of Γ is minimum such N.

$\Gamma(1) = SL_2(\mathbb{Z})$ $\qquad \mathcal{H} \backslash SL_2(\mathbb{Z}) = Y(1)$

$\Gamma_1(N) = \{ SL_2(\mathbb{Z}) : A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \}$

$\Gamma_0(N) = \{ \quad " \quad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \}$

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N).$$

Ex:

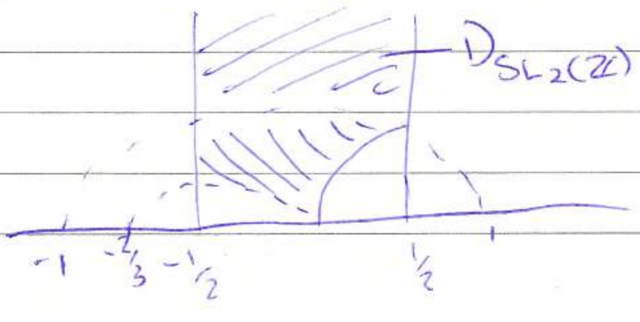
$\Gamma(N)$ is finite index in $SL_2(\mathbb{Z})$

Defn: A holomorphic $f : \mathcal{H} \to \mathbb{C}$ is called weakly modular of wt k for Γ if $f|_k\gamma(z) = j(\gamma, z)^{-k} f(\gamma z) = f(z)$ $\forall \gamma \in \Gamma$.

rem: $f(z+1) \neq f(z)$ if level $\neq 1$.
$f(z+N) = f(z)$ as $T^N \in \Gamma(N)$.

Fundamental domains for congruent subgps.

$\Gamma_0(2) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 2 \}$



/// + \\\ = $D_{\Gamma_0(2)}$.

$0 \in D_{\Gamma_0}(2)$
$0 \notin \mathcal{H}$.

$\Gamma \subset SL_2(\mathbb{Z})$

$\Gamma \backslash SL_2(\mathbb{Z}) = \bigcup_{h \in R} \Gamma h$ R-finite set

then $D_\Gamma = \bigcup_{h \in R} h D_{SL_2(\mathbb{Z})}$ - may not be connected.

pf: If $z \in \mathcal{H}$, $\exists g \in SL_2(\mathbb{Z})$ & $z_0 \in D_{SL_2(\mathbb{Z})}$

$z = g z_0$.

$\exists$ some $h \in R$ & $\gamma \in \Gamma$ s.t. $g = \gamma h$

$z_0' = h z_0 \in h D_{SL_2(\mathbb{Z})} \subseteq D_\Gamma$.

$z = \gamma z_0$

Now, we need to check boundary pts.

$\exists z, \gamma z \in D_\Gamma^\circ$ $\gamma \in \Gamma$. Then $\gamma = 1$.

pf: let $\varepsilon > 0$ small enough s.t. $B_\varepsilon(z) \subset \text{Int}(D_\Gamma) = D_\Gamma^\circ$

let $R' \subseteq R$ be finite set s.t. $B_\varepsilon(z)$ intersects $h D_{SL_2(\mathbb{Z})}$ $\forall h \in R'$.

$B_\varepsilon(z) \cap h D_{SL_2(\mathbb{Z})} \neq \emptyset \Leftrightarrow h \in R'$.

Consider $\gamma B_\varepsilon(z) = B_\varepsilon(\gamma z)$.

$\because \gamma z \in \text{Int}(D_\Gamma)$; $B_\varepsilon(\gamma z)$ should intersect with some $h D_{SL_2(\mathbb{Z})}$.

$\gamma B_\varepsilon(z) \cap h D_{SL_2(\mathbb{Z})} \neq \emptyset$.

$\Rightarrow B_\varepsilon(z) \cap \gamma^{-1} h D_{SL_2(\mathbb{Z})}$.

$\exists h_0 \in R'$ st. $\gamma^{-1} h = h_0$

$\Gamma \gamma^{-1} h = \Gamma h_0$

$\Gamma h = \Gamma h_0$ $(\because \gamma \in \Gamma)$.

$\therefore h = h_0 \therefore \gamma = 1$.

The 0 pt which lies in $D_{\Gamma_0(2)}$ but not $\mathcal{H}$.
is called a cusp.
∞ is only cusp for $SL_2(\mathbb{Z})$.

all cusps will be in $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$

$\gamma \cdot x = \frac{ax+b}{cx+d}$ $\gamma \in SL_2(\mathbb{Z})$ and $x \in \mathbb{P}^1(\mathbb{Q})$.

$\gamma\infty = \frac{a}{c}$ $\gamma x = \infty$ if $cx+d=0$.

★ $ax+b \neq 0$ always or is this needed?

Prop'n: The action of $SL_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is transitive and it induces a bijection.

$$SL_2(\mathbb{Z}) / SL_2(\mathbb{Z})_\infty \cong \mathbb{P}^1(\mathbb{Q})$$

$$SL_2(\mathbb{Z})_\infty = \mathrm{Stab}(\infty) = \langle \pm T \rangle.$$

pf $\mathrm{Orb}(\infty) = \mathbb{P}^1(\mathbb{Q})$ $\frac{a}{c} \in \mathbb{P}^1(\mathbb{Q})$.
$\Rightarrow (a,c) = 1$
$\exists\, b, d : ab - cd = 1$
$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
$M\infty = a/c$.

$\mathrm{Stab}(\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : M\infty = \infty \right\}$

i.e. $c = 0. \Rightarrow ab = 1$ $\therefore$ $\left\{ \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \right\} = \mathrm{Stab}(\infty)$
$= \langle T, -T \rangle$.

Def'n : The set of cusps of a congruent subgp Γ is the set of Γ-orbit of $\mathbb{P}^1(\mathbb{Q})$

$$\begin{aligned} \mathrm{cusps}(\Gamma) &= \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \\ &= \Gamma \backslash SL_2(\mathbb{Z}) / SL_2(\mathbb{Z})_\infty \\ &= \Gamma \backslash SL_2(\mathbb{Z}) / \langle -I, T \rangle \end{aligned}$$

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Modular Forms

$\mathrm{Cusps}(\Gamma) = \Gamma\backslash\mathbb{P}^1(\mathbb{Q}) = \Gamma\backslash {SL_2(\mathbb{Z})}/{SL_2(\mathbb{Z})_\infty}$

$p = [\frac{a}{c}] \in \mathrm{Cusps}(\Gamma)$, $\Gamma_p \leq \Gamma$

$\Gamma_p = \Gamma \cap \gamma_p SL_2(\mathbb{Z})_\infty \gamma_p^{-1}$ $\quad \gamma_p \in SL_2(\mathbb{Z})$; $\gamma_p(\infty) = p$.

lemma:

pf: $\gamma \in \Gamma_p \iff \gamma p = p$

$\gamma\gamma_p(\infty) = p = \gamma_p(\infty)$

$\gamma_p^{-1}\gamma\gamma_p(\infty) = \infty$

$\Rightarrow \gamma_p^{-1}\gamma\gamma_p \in SL_2(\mathbb{Z})_\infty$

$\Rightarrow \gamma \in \gamma_p SL_2(\mathbb{Z})_\infty \gamma_p^{-1} \cap \Gamma$.

Lemma: $H \leq SL_2(\mathbb{Z})_\infty$ of finite index h, then H is one of the following:

$h = [SL_2(\mathbb{Z})_\infty : \{\pm I\}H]$

$$H = \begin{cases} \langle T^h \rangle \\ \langle -I \cdot T^h \rangle \\ \langle T^h, -I \rangle \end{cases}$$

$H_p = \gamma_p^{-1}\Gamma\gamma_p \cap SL_2(\mathbb{Z})_\infty$.

$H_p = \gamma_p \Gamma_p \gamma_p^{-1}$ $\quad\square$

$\Gamma_0(p)$ p any prime. Calculate cusps for $\Gamma_0(p)$

$\begin{pmatrix} a & b \\ pc & d \end{pmatrix}$ $ad - pbc = 1$.

$\Gamma_0(p)\infty = \left\{\begin{bmatrix} a & b \\ pc & d\end{bmatrix}\infty\right\} = \left\{\frac{a}{pc}\right\} = \left\{\frac{r}{s} : r,s \in \mathbb{Z}, p \mid s\right\}$.

Clearly $\mathrm{Orb}(\infty) \subset \mathbb{P}^1(\mathbb{Q})$

$0 = 0/1 \notin \mathrm{Orb}(\infty)$.

$\mathrm{Orb}(0) = \{A0\} = \{\frac{b}{d}\}$
$= \{\frac{b}{d} : p \nmid d\}$

$\Rightarrow \mathbb{P}(\mathbb{Q}) = \mathrm{Orb}(\infty) \cup \mathrm{Orb}(0)$

$\Rightarrow \mathrm{Cusps}(\Gamma_0(p)) = \{0, \infty\}$.

Orbit Stab for ∞-gps:

lemma: Let G be a gp acting transitively on set X. and let H be finite index subgp. Then for any $x \in X$, the stabilizer of x in H has finite index in the stabilizer of x in G. for tomo.

$$\sum_{x \in X \backslash H} [G_x : H_x] = [G : H] \qquad H_x = H \cap G_x.$$

pf $x \in X$ $\quad G_x \hookrightarrow G \twoheadrightarrow G/H$
$G_x \to G/H$

Suppose $\exists g_1, g_2$ that are mapped to gH.

$g_1 H = g_2 H \Leftrightarrow g_2^{-1} g_1 \in H$.

$g_2^{-1} g_1 \in G_x$.

$\Rightarrow g_2^{-1} g_1 \in H_x$.

$\Rightarrow g_1 H_x = g_2 H_x$

$G_x/H_x \hookrightarrow G/H$

$\Leftrightarrow H_x$ is of finite index in G_x. $\square$

$H_p = \gamma_p^{-1} \Gamma \gamma_p \cap SL_2(\mathbb{Z})_\infty$

lemma: H_p doesn't depend on the choice of a class of cusp

$p = \left[\frac{a}{c}\right] \in \mathbb{P}(\mathbb{Q})$ $\quad p' = \left[\frac{a'}{c'}\right]$

s.t. $\exists \gamma \in \Gamma$ s.t. $\gamma\left[\frac{a}{c}\right] = \left[\frac{a'}{c'}\right]$

$H_p = H_{p'}$ $\quad \gamma_p(\infty) = \left(\frac{a}{c}\right)$

$\gamma\gamma_p(\infty) = \left(\frac{a'}{c'}\right)$. $\quad \gamma_{p'} = \gamma\gamma_p$

$$H_{p'} = (\gamma\gamma_p)^{-1} \Gamma \gamma\gamma_p \cap SL_2(\mathbb{Z})_\infty$$
$$= \gamma_p^{-1}\gamma^{-1} \Gamma\gamma\gamma_p \cap SL_2(\mathbb{Z})_\infty$$
$$= H_p.$$

Things we know so far:

* H_p - doesn't depend on p up to Γ-classes.
  - finite index in $SL_2(\mathbb{Z})_\infty$.

$h_\Gamma(p) = h$ - width of the cusp p.

$H_p = \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \longrightarrow p$ "irregular"

$= \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ $\longrightarrow p$ "regular".

$= (T^n, \pm I)$.

$X = \mathbb{P}^1(\mathbb{Q})$

$G = SL_2(\mathbb{Z})$

$H = \Gamma$

all requirements of lemma satisfied

~~$\sum_{x \in Cusps(\Gamma)} [SL_2(\mathbb{Z})]$~~

$$\sum_{p \in Cusps(\Gamma)} [SL_2(\mathbb{Z})_p : \Gamma_p] = [SL_2(\mathbb{Z}) : \Gamma]$$

$\#$

$$\sum_{p \in Cusps(\Gamma)} [SL_2(\mathbb{Z})_p : \{\pm I\}\Gamma_p] = [SL_2(\mathbb{Z}) : \{\pm I\}\Gamma]$$

$$\Downarrow$$

$$\boxed{\sum h_\Gamma(p) = [SL_2(\mathbb{Z}) : \{\pm I\}\Gamma]}$$

<u>Fourier expansions</u>

$\Gamma \subseteq SL_2(\mathbb{Z})$ - Congruent subgroup of level N.

$f : \mathcal{H} \to \mathbb{C}$ a holomorphic & weakly modular of wt k for Γ.

$\exists N > 0$ s.t. $\Gamma(N) \subseteq \Gamma \ni \langle \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \rangle$

Defn: The fan width of Γ is the ~~bes~~ least h st. $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \Leftrightarrow h \mid N$.

$q_h = q^{1/h} = e^{2\pi i z/h}$

$z \mapsto q_h(z)$ is h-periodic.

$f(z) = \sum_{n=-\infty}^{\infty} a(n) q_h^n \to$ Laurent expansion at ∞. (exactly the same).

Fourier expansion at other cusps

$s \in Cusps(\Gamma) \setminus \{\infty\}$.

$s = \alpha\infty$ for some $\alpha \in SL_2(\mathbb{Z})$

$f|_k \alpha(z) = j(\alpha, z)^{-k} f(\alpha z)$ - holomorphic in upper half plane.

- invariant under $\alpha^{-1}\Gamma\alpha$ (Ex).

is a cong subgp of level N.

$f|_k \alpha(z) = \sum_{n=-\infty}^{\infty} b(n) q_N^n$ (treat it at ∞ by Laurent expansion).

tomo new relation on $a(n), b(n)$.

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Modular Forms.

Orbit-Stab for infinite groups.

X - a set. G - acting transitively on X.

$H \leq G$ - subgps of finite index.

$[G:H] = \sum_{x \in X/H} [G_x : H_x]$ $H_x = G_x \cap H$

pf: Fix x_0 in X.

$\varphi : G/H \to X/H$

$Hg \to Hgx_0$ $\qquad$ φ is surjective $\because$ G acts transitively.

$\Rightarrow \forall x \in X.\ \exists g \in G$ st. $gx = x_0$.

Let $g_x \in G$ be the element s.t. $g_x x_0 = x$

* $T_{Hx} := \{ Hg \in G/H : Hgx_0 = Hx \}$ ← fi of φ for $x \in X \backslash H$.

Let $g' \in G$ s.t. $Hg = Hg'g_x$

$T_{Hx} = \{ Hg' \in G/H : Hg'g_x x_0 = Hx \}$

$= \{ Hg' \in G/H : Hg'x = Hx \}$ $\qquad$ as $Hg'x = Hx$ $\quad g'x = x$ mod H

$\cong G_x H / H \cong G_x / G_x \cap H$ (by Second Isom thm)

$\cong G_x / H_x$

$[G:H] = \sum_{x \in X/H} T_{Hx} = \sum_{x \in X/H} [G_x : H_x]$

recall: Γ - cong subgp of level N

$f : \mathcal{H} \to \mathbb{C}$

- f holomorphic on $\mathcal{H}$
- f is weakly modular of wt k for Γ
- f is holomphic at ∞

  $\Leftrightarrow f(z) = \sum_{n=0}^{\infty} a(n) q_N^n$

we defined Laurentz expansions at cusps P

$$f|_k \alpha (z) = \sum_{n=0}^{\infty} b(n) q_N^n$$

$$\alpha P = \infty.$$

Defn: A fn $f : \mathcal{H} \to \mathbb{C}$ is said to be a modular form of wt k for Γ if

- f is holomorphic on $\mathcal{H}$.
- $f|_k \gamma(z) = f(z) \quad \forall \gamma \in \Gamma$.
- $f|_k \alpha$ is holomorphic at ∞ $\forall \alpha \in SL_2(\mathbb{Z})$.

notation: $f \in M_k(\Gamma)$

f is a cusp form if $f|_k \alpha$ vanishes at ∞ $\forall \alpha \in SL_2(\mathbb{Z})$. (notation $f \in S_k(\Gamma)$)

if f is weakly modular for $SL_2(\mathbb{Z})$ its also for Γ as well.

can extend this:

$M_k(SL_2(\mathbb{Z})) \hookrightarrow M_k(\Gamma_0(N))$.

$f \in M_k(SL_2(\mathbb{Z}))$

$g(z) = f(Nz) \qquad \gamma \in \Gamma_0(N) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad N | c$

$\gamma' \in SL_2(\mathbb{Z}) = \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix}.$

$$
\begin{aligned}
g(\gamma z) &= f(N\gamma z) \\
&= f\left(N\left(\frac{az+b}{cz+d}\right)\right) \\
&= f\left(\frac{aNz+bN}{cz+d}\right) \\
&= f\left(\frac{a\,Nz+bN}{\frac{c}{N}Nz+d}\right) \\
&= f(\gamma' Nz) = \left(\tfrac{c}{N}(Nz)+d\right)^k f(Nz) \\
&= \left(\tfrac{c}{N}(Nz)+d\right)^k g(z)
\end{aligned}
$$

$\therefore\ g(\gamma z) = j(\gamma, z)^{+k} g(z)$

$f \mapsto g$ is an inclusion.

$[\sigma], [\tau] \in \mathrm{Cusps}(\Gamma)$ if $\sigma = \gamma\tau$ for some $\gamma \in \Gamma$.

$\sigma = \alpha\infty \quad \beta, \alpha \in SL_2(\mathbb{Z}) \quad \tau = \beta\infty$.

Proposition:

If $f|_k \alpha(z) = \sum_{n=-\infty}^{\infty} a(n) q_N^n$, then $f|_k \beta(z) = \sum_{n=-\infty}^{\infty} b(n) q_N^n$

where $b(n) = (\pm 1)^k e^{\frac{2\pi i n j}{N}} a(n) \quad j \in \mathbb{Z}$.

pf: $\alpha\infty = \gamma\beta\infty$

$\alpha^{-1}\gamma\beta(\infty) = \infty$.

$\alpha^{-1}\gamma_\beta \in$ ~~Stab~~ $SL_2(\mathbb{Z})_\infty = \langle \pm T \rangle$.

$\overset{\shortparallel}{\pm I}\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$

$\beta = \pm I \gamma^{-1} \alpha \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$

$$
\begin{aligned}
f|_k \beta(z) &= f|_k \pm I |_k \gamma^{-1} |_k \alpha |_k \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \\
&= (\pm 1)^k \sum a_n e^{\frac{2\pi i n z}{N}} e^{\frac{2\pi i j z}{N}}
\end{aligned}
$$

★

$\Leftrightarrow\ b(n) = (\pm 1)^k a_n e^{\frac{2\pi i n j}{N}}$.

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Mod Forms.

Valence formula for congruent subgps:

Γ - congruent subgp level N

$\bar{\Gamma} = \Gamma / \{\pm 1\}$ $\quad PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm 1\}$

Let f be a non-zero meromorphic on $\mathcal{H} \cup \{\infty\}$ weakly mod of wt k for Γ

$$\sum_{z \in \mathcal{H} \setminus \Gamma} \frac{V_z(f)}{|\bar{\Gamma}_z|} + \sum_{p \in \text{Cusps}(\Gamma)} V_p(f) = \frac{k}{12} [PSL_2(\mathbb{Z}) : \bar{\Gamma}]$$

Recall Orbit-Stabilizer for ∞ gps

X - set $\quad G$ - ∞ gp acting transitively on X

$H \leq G$ a finite index gp

$$[G : H] = \sum_{x \in X \setminus H} [G_x : H_x]$$

$$\phi : G/H \twoheadrightarrow X \setminus H$$
$$Hg \mapsto Hgx_0$$

T_{Hx} = fibre of $\varphi = \{Hg \in G/H : Hgx_0 = Hx\} \cong G_x / H_x$.

pf: $d = [PSL_2(\mathbb{Z}) : \bar{\Gamma}]$, write $M_1, \dots, M_d$ as repr. for $PSL_2(\mathbb{Z}) / \bar{\Gamma}$.

$F(z) = \prod f|_k M_j$ $\quad$ F - weakly modular for $PSL_2(\mathbb{Z})$ of wt kd. (ex)

- meromorphic at ∞.

$$V_\infty(F) + \tfrac{1}{2} V_i(F) + \tfrac{1}{3} V_\rho(F) + \sum_{w \in \mathcal{H}} V_w(F) = \frac{kd}{12} \quad (\text{weak } \underset{??}{2!})$$

"

$$V_\infty(F) + \sum_{z \in \mathcal{H} \setminus PSL_2(\mathbb{Z})} \frac{V_z(F)}{|PSL_2(\mathbb{Z})_z|} = \frac{kd}{12}$$

$$V_z(F) = V_z\left(\prod f|_k M_j\right)$$

$$= \sum V_z(f|_k M_j)$$

$$= \sum V_{M_j z}(f) \qquad w = M_j z$$

$$V_z(F) = \sum_{w \in (PSL_2(\mathbb{Z})/\bar{\Gamma})z} \frac{|PSL_2(\mathbb{Z})_w|}{|\bar{\Gamma}_w|} V_w(f) \qquad (1)$$

$PSL_2(\mathbb{Z})_w$ is independant of orbit $(PSL_2(\mathbb{Z})/\bar{\Gamma})z$

" $PSL_2(\mathbb{Z})_z$.

Divide (1) by $|PSL_2(\mathbb{Z})_z|$

$$\frac{V_z(F)}{|PSL_2(\mathbb{Z})_z|} = \sum_{w \in (PSL_2(\mathbb{Z})/\bar{\Gamma})z} \frac{V_w(f)}{|\bar{\Gamma}_w|}$$

$$= \sum_{z \in \mathcal{H}\backslash PSL_2(\mathbb{Z})} \sum_{w \in (PSL_2(\mathbb{Z})/\bar{\Gamma})z} \frac{V_w(f)}{|\bar{\Gamma}_w|}$$

$$= \sum_{w \in \mathcal{H}\backslash\bar{\Gamma}} \frac{V_w(f)}{|\bar{\Gamma}_w|}$$

back to

$$V_\infty(F) + \sum \frac{V_w(f)}{|\bar{\Gamma}_w|} = \ldots$$

$$V_\infty(F) = \sum_{p \in \text{cusps}(\bar{\Gamma})} V_p(f) \qquad \text{[Ex)}.$$

$$\text{Cusps}(\Gamma) = \Gamma\backslash SL_2(\mathbb{Z})/SL_2(\mathbb{Z})_\infty \qquad \text{(hint)}.$$

End of ch 2.

Lattices & Tori

Defn: A lattice is a free rank 2 $\mathbb{Z}$-module in $\mathbb{C}$ which contains an $\mathbb{R}$-basis for $\mathbb{C}$

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2; \quad \omega_1, \omega_2 \in \mathbb{C} \quad \omega_1 \neq \alpha\omega_2 \ \forall \alpha \in \mathbb{R}.$$

$\frac{\omega_1}{\omega_2} \in \mathcal{H}$ (can always do this).

Defn: A complete torus is the set $\mathbb{C} \backslash \Lambda = \{z + \Lambda \mid z \in \mathbb{C}\}$

- an abelian gp.
- Riemann surface of genus 1 (torus). (don't worry about this one).

$$\Lambda = \Lambda' \quad \langle \omega_1, \omega_2 \rangle = \Lambda$$
$$\Updownarrow \quad \langle \omega_1', \omega_2' \rangle = \Lambda'$$
$$\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix} = \gamma \begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} \text{ for some } \gamma \in SL_2(\mathbb{Z}).$$

Prop: Suppose $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ holomorphic $\exists\, m, b \in \mathbb{C}$ s.t.

- $\phi(z + \Lambda) = mz + b + \Lambda'$
- $m\Lambda \subseteq \Lambda'$ • ϕ is invertible $\Leftrightarrow m\Lambda = \Lambda'$

pf: $\mathbb{C}$ is the universal covering space for tori

ϕ lifts to a continuous map $\tilde{\phi}$:
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\phi}} & \mathbb{C} \\ \downarrow \pi & & \downarrow \pi' \\ \mathbb{C}/\Lambda & \xrightarrow{\phi} & \mathbb{C}/\Lambda' \end{array}$$
commutes.

$\lambda \in \Lambda$, $\quad f_\lambda(z) = \tilde{\phi}(z+\lambda) - \tilde{\phi}(z)$ — cts
— takes image in a discrete set

$\Rightarrow f_\lambda$ is constant.

$f_\lambda'(z) = 0 \Rightarrow \tilde{\phi}'(z+\lambda) = \tilde{\phi}'(z)$ — holomorphic, — bounded
— Λ-periodic

$\Rightarrow \phi'$ is constant by Liouville constant.

$\therefore\ \tilde{\varphi}(z) = mz + b$

$\varphi(z+\Lambda) + m\cancel{z} + b + \Lambda'$.

$\varphi(\Lambda) \subseteq \Lambda' \Rightarrow m\Lambda \subseteq \Lambda'$.

φ is gp hm. $\Leftrightarrow$ $\varphi(0) = 0 \Leftrightarrow b = 0$

$\varphi(z) = mz$.

$$\mathbb{C}/\Lambda \xrightarrow{[N]} \mathbb{C}/\Lambda$$

$$z+\Lambda \longrightarrow Nz+\Lambda$$

$\ker [N] = N$-torsion pts in Λ $\cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$

$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ $\qquad \Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$.

$\frac{\omega_1}{\omega_2} = \tau \in \mathcal{H}$

lemma: $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda_\tau$

pf: $\varphi(z+\Lambda) = \frac{1}{\omega_2} z$ or $\omega_2 z$ (one should work)

Lemma: $\mathbb{C}/\Lambda_\tau \cong \mathbb{C}_{\Lambda_{\tau'}} \Leftrightarrow \exists \gamma \in SL_2(\mathbb{Z})$ s.t. $\tau = \gamma\tau'$.

Eisenstein series attached to lattices

$k \geq 2$ - even $\quad \Lambda$ - a lattice

$$G_k(\Lambda) = \sum_{\omega\in\Lambda}' \omega^{-k}$$

$$G_k(\Lambda_\tau) = G_k(\tau)$$

$$\sum_{\omega\in\Lambda_\tau}'' \omega^{-k} = \sum_{m,n}' \frac{1}{(m+n\tau)^k}$$

$\omega = (m+n\tau)$

$$G_k(m\Lambda) = m^{-k} G_k(\Lambda)$$

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Modular Forms

Recall:

- $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is holomorphic

$\Updownarrow$

$\varphi(z+\Lambda) = mz + b + \Lambda'$

$m\Lambda \subseteq \Lambda'$

- φ is invertible $\Leftrightarrow m\Lambda = \Lambda'$
- φ is hm $\Leftrightarrow \varphi(z+\Lambda) = mz + \Lambda'$

Λ a lattice $k>2$ even

$G_k(\Lambda) = \sum_{\omega\in\Lambda}' \omega^{-k}$

$G_k(\Lambda_\tau) = G_k(\tau)$

$\uparrow$

Eisenstein series of wt k on $SL_2(\mathbb{Z})$.

$[N] : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$

$z \mapsto Nz \pmod \Lambda$.

Defn: A non-zero holomorphic hm between complex tori is called an isogeny.

$[N] \in \mathrm{End}(\mathbb{C}/\Lambda)$.

$\mathbb{Z} \subseteq \mathrm{End}(\mathbb{C}/\Lambda)$ when $\mathbb{Z} \subset \mathrm{End}(\mathbb{C}/\Lambda)$, we say Λ has CM (complex multiplication).

$\tau = \sqrt{d}$ $\quad d<0 \quad d = 2,3 \bmod 4$

$= \frac{1+\sqrt{d}}{2} \quad d = 1 \quad (4)$

$\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\tau = \mathcal{O}_K \quad K = \mathbb{Q}(\sqrt{d})$.

$\Lambda \subseteq \mathcal{O}$ - ideal

$m\Lambda \subseteq \Lambda \ \forall m \in \mathcal{O}$. $\Rightarrow$ multiplication by $m \in \mathrm{End}(\mathbb{C}/\Lambda)$.

$\therefore \mathcal{O} \subseteq \mathrm{End}(\mathbb{C}/\Lambda)$

(in fact $\mathcal{O} = \mathrm{End}(\mathbb{C}/\Lambda)$).

(see essay for more details)

Meromorphic fn on $\mathbb{C}/\Lambda$

$\mathbb{C}(\Lambda)$ - ring on meromorphic fn's on $\mathbb{C}/\Lambda$.

$\wp_\Lambda(z) = \frac{1}{z^2} + \sum'\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$ weirstrass $\wp$ fn.

- $\wp$ is even (pf: rearranging $\square$)
- $\wp$ is Λ periodic

pf: $\wp_\Lambda'(z) = -2\sum \frac{1}{(z-\omega)^3}$

$f_{\omega_1}(z) = \wp_\Lambda(z+\omega_1) - \wp_\Lambda(z)$ $\omega_1 \in \Lambda$

$f_{\omega_1}'(z) = \wp_\Lambda'(z+\omega_1) - \wp_\Lambda'(z) = 0$

$f_{\omega_1}(z) = C$.

$f_{\omega_1}(\frac{-\omega_1}{2}) = \wp(\frac{\omega_1}{2}) - \wp_\Lambda(-\frac{\omega_1}{2}) = 0$

$\Rightarrow f_{\omega_1}(z) = 0 \ \forall z$

$\therefore \wp_\Lambda(z+\omega) = \wp_\Lambda(z) \ \forall \omega \in \Lambda$

$\wp_\Lambda : \mathbb{C}/\Lambda \to \mathbb{C}$. $\therefore \wp_\Lambda \in \mathbb{C}(\Lambda)$.

also $\wp_\Lambda' \in \mathbb{C}(\Lambda)$.

rmk: $\mathbb{C}(\Lambda) = \langle \wp_\Lambda(z), \wp_\Lambda'(z) \rangle$.

prop: i) The Laurent expansion of $\wp_\Lambda$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{n\geq 2\\ \text{even}}} G_{n+2}(\Lambda)(n+1)z^n$$

$\forall z$ st. $0 < |z| < \inf\{|\omega| : \omega \in \Lambda\}$.

ii) $(\wp_\Lambda'(z))^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$

$g_2 = 60\, G_4$

$g_3 = 140\, G_6$.

iii) $E_\Lambda : Y^2 = 4X^3 - g_2 X - g_3$

$(\wp(z), \wp'(z))$ is a pt on $E_\Lambda(\mathbb{C})$.

★ $E_\Lambda : Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$ $\quad \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

$\omega_3 := \omega_1 + \omega_2$.

$\wp(\omega_i) = e_i$.

not doing pf - but should be in Marc's, as well as (i), (ii) in essay.

$\mathbb{C}/\Lambda \to E_\Lambda$

$z + \Lambda \to (\wp(z), \wp'(z))$ gives a bijection.

Uniformisation Thm

Let E be any elliptic curve over $\mathbb{C}$,

$E : Y^2 = 4X^3 - g_2 X - g_3$.

then $\exists\, \Lambda \subseteq \mathbb{C}$ st. $g_2(\Lambda) = g_2$

$g_3(\Lambda) = g_3$.

$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ an arbitrary lattice

$\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ $\quad \tau = \frac{\omega_1}{\omega_2} \in \mathcal{H}$. $\quad \mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda_\tau$.

Moduli space interpretation

$S := \{$isomorphism classes of Elliptic curves $/ \mathbb{C}\}$ $\quad [\mathbb{C}/\Lambda_\tau]$

$[\mathbb{C}/\Lambda_\tau] \mapsto SL_2(\mathbb{Z})\tau$ is a bijection.

$S \cong SL_2(\mathbb{Z}) \backslash \mathcal{H}$.

$f \in M_k(SL_2(\mathbb{Z}))$ $\quad F(\mathbb{C}/\Lambda_\tau) = f(\tau)$.

$\tau' = \tau + b \quad b \in \mathbb{Z}$.

$\Lambda_\tau = \Lambda_{\tau'}$

$F(\mathbb{C}/\Lambda_\tau) = f(\tau) = f(\tau+b) = f(\tau') = F(\mathbb{C}/\Lambda_{\tau'})$

So F is well defined.

$F(\mathbb{C}/m\Lambda_\tau) = m^{-k} F(\mathbb{C}/\Lambda_\tau)$.

Next time, we'll see for different subgps of $SL_2(\mathbb{Z})$.

12 Modular Forms (thanks to Ollie C.)

Moduli space interpretation of $\Gamma(1) = SL_2(\mathbb{Z})$, $\Gamma_0(N)$, $\Gamma_1(N)$.

$S(1) := S - \{$ set of isom classes of elliptic curves over $\mathbb{C}$

$$\begin{array}{ccc} S(1) & & \\ \downarrow \wr & & \\ \Gamma(1)/\mathcal{H} & & [\mathbb{C}/\Lambda_\tau] \mapsto \tau. \end{array}$$

Defn: An enhanced elliptic curve (over $\mathbb{C}$) for $\Gamma_0(N)$ is an ordered pair (E, C) where E is an elliptic curve over $\mathbb{C}$, C is a cyclic subgp of order N.

Recall: $\mathbb{C}/\Lambda \xrightarrow{[N]} \mathbb{C}/\Lambda$

$\mathrm{Ker}([N]) = E(N)$

An enhanced elliptic curve for $\Gamma_1(N)$ is ordered pair (E, P)
E - elliptic curve over $\mathbb{C}$
P - a point of exact order N.
i.e. $[N]P = 0$ $[n]P \neq 0$ $\forall n < N$.

$S_0(N) := \{ \text{Pairs } (E, C) \}/\sim$ [will prove $\simeq \Gamma_0(N)\backslash\mathcal{H}$]

$(E, C) \sim (E', C')$ if $\exists \varphi$ isom $\varphi : E \to E'$ s.t. $\varphi(C) = C'$.

$S_1(N) := \{ \text{Pairs } (E, P) \}/\sim$ [will prove $\simeq \Gamma_1(N)\backslash\mathcal{H}$]

$(E, P) \sim (E', P')$ if $\exists \varphi : E \to E'$ s.t. $\varphi(P) = P'$

Thm: i) Each class in $S_0(N)$ has a representative of the form:

$$\left(\mathbb{C}/\Lambda_{\tau'}, \left\langle \tfrac{1}{N} + \Lambda_{\tau'} \right\rangle\right) \text{ for some } \tau' \in \mathcal{H}.$$

Two pairs $\left(\mathbb{C}/\Lambda_{\tau}, \left\langle \frac{1}{N} + \Lambda_{\tau} \right\rangle\right)$ and $\left(\mathbb{C}/\Lambda_{\tau'}, \left\langle \frac{1}{N} + \Lambda_{\tau'} \right\rangle\right)$ are equivalent iff $\Gamma_0(N)\tau = \Gamma_0(N)\tau'$.

ii) Each class in $S_1(N)$ has a representative of form:

$$\left(\mathbb{C}/\Lambda_{\tau}, \tfrac{1}{N} + \Lambda_{\tau}\right) \text{ for some } \tau \in \mathcal{H}.$$

Two pairs (as above) are equivalent iff $\Gamma_1(N)\tau = \Gamma_1(N)\tau'$.

rmk: This gives the bijections:

$$S_0(N) \xrightarrow{\sim} \Gamma_0(N)\backslash\mathcal{H}$$
$$S_1(N) \xrightarrow{\sim} \Gamma_1(N)\backslash\mathcal{H}$$

pf: only (2) [(i) is same]

Take any (E,P) in $S_1(N)$. $E \xrightarrow{\sim} \mathbb{C}/\Lambda_{\tau'}$ for some $\tau' \in \mathcal{H}$.

take $E = \mathbb{C}/\Lambda_{\tau'}$

$$Q = \frac{c\tau'+d}{N} + \Lambda_{\tau'} \in E[N]$$
$$(c,d,N) = 1.$$

$\Rightarrow \exists\, a,b,s \in \mathbb{Z}$ st. $ad - bc - sN = 1$.

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will have det 1 (mod N).

So $\gamma \in SL_2(\mathbb{Z}/N)$ there is a surjective map:

$$SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/N)$$

So $\bar{\gamma} = \gamma \pmod N$

so $\bar{\gamma}$ lifts to $SL_2(\mathbb{Z})$

So we can choose γ s.t. it's in $SL_2(\mathbb{Z})$.

Let $\tau = \gamma\tau'$ $\quad m = c\tau' + d$.

$$(c\tau' + d)\tau = a\tau' + b$$

i.e. $m\tau = a\tau' + b$

consider $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$

$$\begin{aligned} m\Lambda_\tau &= m\tau\mathbb{Z} \oplus m\mathbb{Z} \\ &= (a\tau' + b)\mathbb{Z} \oplus (c\tau' + d)\mathbb{Z} \\ &= \tau'\mathbb{Z} \oplus \mathbb{Z} = \Lambda_{\tau'} \end{aligned}$$

$$m\Lambda_\tau = \Lambda_{\tau'}$$

$$\Rightarrow \mathbb{C}/\Lambda_\tau \cong \mathbb{C}/\Lambda_{\tau'}$$

$$m\left(\tfrac{1}{N} + \Lambda_\tau\right) = \frac{c\tau' + d}{N} + \Lambda_{\tau'} = Q$$

Suppose $\exists\, \tau, \tau' \in \mathcal{H}$ s.t. $\tau = A\tau'$ for $A \in \Gamma_1(N)$.

$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $\quad m := \gamma\tau' + d$ then:

$$m\Lambda_\tau = \Lambda_{\tau'}$$

$$m\left(\tfrac{1}{N} + \Lambda_\tau\right) = \frac{\gamma\tau' + d}{N} + \Lambda_{\tau'}$$

$A \in \Gamma_1(N)$

$(\gamma, \delta) \equiv (0, 1) \bmod N$.

So: $\quad = \frac{1}{N} + \Lambda_{\tau'}$

so $(\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau) \sim (\mathbb{C}/\Lambda_{\tau'}, \frac{1}{N} + \Lambda_{\tau'})$

$\Leftrightarrow \tau = A\tau' \quad A \in \Gamma_1(N)$.

[pf is same for (i), use subgp instead of m].

Defn: A cmplx valued fn F on $S(1), S_0(N), S_1(N)$ is called degree k homogeneous if

$$F(\mathbb{C}/m\Lambda) = m^{-k} F(\mathbb{C}/\Lambda) \qquad S(1)$$
$$F((\mathbb{C}/m\Lambda, mP)) = m^{-k} F((\mathbb{C}/\Lambda, P)) \qquad S_0(N)$$
$$F((\mathbb{C}/m\Lambda, mC)) = m^{-k} F((\mathbb{C}/\Lambda, C)) \qquad S_1(N)$$

$f : \mathcal{H} \to \mathbb{C}$.

$$f(\tau) = \begin{cases} F(\mathbb{C}/\Lambda_\tau) & S(1) \\ F(\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau) & S_1(N) \\ F(\mathbb{C}/\Lambda_\tau, \langle \frac{1}{N} + \Lambda_\tau \rangle) & S_0(N) \end{cases}$$

$$f(\gamma\tau) = \begin{cases} F(\mathbb{C}/\Lambda_{\gamma\tau}) & \gamma \in SL_2(\mathbb{Z}) \\ F(\mathbb{C}/\Lambda_{\gamma\tau}, \frac{1}{N} + \Lambda_{\gamma\tau}) & \gamma \in \Gamma_1(N) \\ F(\mathbb{C}/\Lambda_{\gamma\tau}, \langle \frac{1}{N} + \Lambda_{\gamma\tau} \rangle) & \gamma \in \Gamma_0(N) \end{cases}$$

$\gamma \in \Gamma_1(N), \quad f(\gamma_0 \tau) \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Lambda_{\gamma\tau} = \mathbb{Z} \oplus \mathbb{Z}\gamma\tau = \mathbb{Z} \oplus \mathbb{Z}\frac{a\tau+b}{c\tau+d} = \frac{\mathbb{Z}}{c\tau+d} \oplus \frac{\mathbb{Z}\tau}{c\tau+d}.$$

let $m = (c\tau+d)^{-1}$

$$\Lambda_{\gamma\tau} = m\Lambda_\tau$$

$$\frac{1}{N} + \Lambda_{\gamma\tau} = \frac{1}{N} + m\Lambda_\tau$$
$$= \frac{m(c\tau+d)}{N} + m\Lambda_\tau$$
$$= m\left(\frac{c\tau+d}{N} + \Lambda_\tau\right).$$

$(c,d) \equiv (0,1) \bmod N$

$= m \left(\frac{1}{N} + \Lambda_\tau \right)$

$$f(\gamma\tau) = F\left(\mathbb{C}/\Lambda_{\tau'},\ m\left(\tfrac{1}{N} + \Lambda_\tau\right)\right)$$

$$= m^{-k} F\left(\mathbb{C}/\Lambda_\tau,\ \tfrac{1}{N} + \Lambda_\tau \right)$$

$$= (c\tau + d)^k f(\tau)$$ ← weakly modular wt k.

Modular Forms.

Hecke Theory

- $M_k(\Gamma) \supseteq S_k(\Gamma)$ fin dim v.s. over $\mathbb{C}$.
- $M_k(\Gamma), S_k(\Gamma)$ going to be modules over a ring "Hecke algebra"

$\gamma \in GL_2^+(\mathbb{Q})$

$$f|_k\gamma(z) = \frac{\det(\gamma)^k}{(cz+d)^k} f(\gamma z)$$

$\Gamma \leq SL_2(\mathbb{Z})$ a congruence subgp $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$

$\Gamma' := \Gamma \cap \alpha^{-1}\Gamma\alpha$, $f \in M_k(\Gamma)$

$f|_k\alpha$ is invariant under wt k operation on $\alpha^{-1}\Gamma\alpha$. - why? $f|_k\alpha|_k\alpha^{-1}\gamma\alpha = f|_k\alpha^{-1}\alpha|_k\gamma|_k\alpha$
$= f|_k 1|_k\gamma|_k\alpha = f|_k\gamma|_k\alpha = f|_k\alpha$.

$f|_k\alpha$ is a modular form wt k on Γ'.

$$T_\alpha f := \sum_{\gamma \in \Gamma/\Gamma'} f|_k\alpha\gamma$$

Propn: if $f \in M_k(\Gamma)$, so is $T_\alpha f$.
if $f \in S_k(\Gamma)$, so is $T_\alpha f$.

$T_\alpha \in \mathrm{End}(M_k(\Gamma))$ which preserves $S_k(\Gamma)$.

$\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$

$\Gamma/\Gamma' \cong \alpha^{-1}\Gamma\alpha\Gamma/\alpha^{-1}\Gamma\alpha$ (2nd isom theorem)

Lemma: $\Gamma/\Gamma' \cong \Gamma\alpha\Gamma/\Gamma$

pf: $\phi : \Gamma \to \Gamma\alpha\Gamma/\Gamma$

$\gamma \mapsto \Gamma\alpha\gamma$ ϕ is surjective

$$\Gamma\alpha\gamma = \Gamma\alpha\gamma' \Leftrightarrow \gamma'\gamma^{-1} \in \alpha^{-1}\Gamma\alpha$$
$$\ker(\varphi) = \Gamma \cap \alpha^{-1}\Gamma\alpha = \Gamma'$$
$$\Rightarrow \Gamma/\Gamma' \cong \Gamma\alpha\Gamma/\Gamma. \qquad \square$$

upshot is $T_\alpha f = \sum_{\gamma \in \Gamma\alpha\Gamma/\Gamma} f|_k \gamma$.

<u>Hecke operators</u> $\Gamma_1(N)$

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod N$$ (check!) is well defined since d is coprime to N.

(all mod N) $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = 1 \bmod N$

$$\ker(\text{this map}) = \Gamma_1(N)$$
$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N)^*.$$

Given any d coprime to N, $\exists \alpha \in \Gamma_0(N)$ s.t. the lower rt entry of α is d.

Defn: The diamond operators acting on $M_k(\Gamma_1(N))$
$$\langle d \rangle f := T_\alpha f = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \quad w/ \ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

not sure: subtle point: $\alpha \in \Gamma_0$, $f \in M_k(\Gamma_1)$ $\therefore$ f is α invariant.

p - prime

$T_p \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

Defn: $T_p f = \frac{1}{p} T_\alpha f$.
$$\Gamma = \Gamma_1(N)$$
$$\Gamma' = \Gamma_1(N) \cap \alpha^{-1}\Gamma(\alpha)$$

Lemma: $\Gamma' = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \ \& \ p \mid b\}$

$$\Gamma/\Gamma' = \begin{cases} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad 0 \le b \le p-1 & \text{if } p \mid N \\ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad 0 \le b \le p-1 \ \cup \left\{\begin{pmatrix} ap & 1 \\ cN & 1 \end{pmatrix}\right\} & \text{if } p \nmid N \end{cases}$$

$a, c : ap - cN = 1$.

pf: $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$

$\alpha^{-1}\gamma\alpha = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix}$ $\qquad \Gamma' = \alpha^{-1}\Gamma\alpha \cap \Gamma$.

$\varphi : \Gamma_1(N) \to SL_2(\mathbb{Z}/p)$

$*$ ① $p \mid N$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$

$\mathrm{Im}(\varphi) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

$\varphi(\Gamma') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Gamma/\Gamma' \cong \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} / \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$\Gamma/\Gamma' \cong \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : 0 \le b \le p-1\right\}$.

$T_\alpha f = \sum_{\gamma \in \Gamma/\Gamma'} f|_k \gamma$

$T_\alpha f(z) = \sum_{0 \le b \le p-1} f|_k \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}(z)$

$= f(z+b)$ $\qquad$ on thursday, we'll see fourier expansions

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Modular Forms

Hecke Operators (Cont'd)

$\Gamma \leq SL_2(\mathbb{Z})$ - congruente subgp
$\alpha \in GL_2^+(\mathbb{Q})$
$\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$
$\Gamma/\Gamma' \cong \Gamma\alpha\Gamma/\Gamma$

$\therefore \Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha\gamma_j$ where $\Gamma = \bigcup_i \Gamma'\gamma_j$.

$$T_\alpha f = \sum_{\gamma \in \Gamma\alpha\Gamma/\Gamma} f|_k \gamma$$

$$= \sum_{\gamma \in \Gamma/\Gamma'} f|_k \alpha\gamma$$

$\Gamma = \Gamma_1(N)$ $\langle d \rangle$; T_p prime.

$T_p = \frac{1}{p} T_\alpha f$ $\quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in GL_2^+(\mathbb{Q})$

Lemma: $\Gamma' = \Gamma_1(N) \cap \alpha^{-1}\Gamma_1(N)\alpha$

$$= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \quad p \mid b \right\}$$

$$\Gamma/\Gamma' = \begin{cases} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad 0 \leq b \leq p-1 & p \mid N \\ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} ap & 1 \\ cN & 1 \end{pmatrix} & p \nmid N \end{cases}$$

$a, c \in \mathbb{Z}$ $ap - cN = 1$.

$$T_\alpha f(z) = \sum_{\gamma \in \Gamma/\Gamma'} f|_k \alpha\gamma(z)$$

when $p \mid N$: $\quad = \sum_{b=0}^{p-1} f|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}(z)$.

~~when $p \nmid N$~~ $\quad = \sum_{b=0}^{p-1} f|_k \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}(z)$.

$$= \frac{p^k}{p^k} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) = \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right).$$

When $p \nmid N$:

$$T_\alpha f(z) = \sum_{\gamma \in \Gamma / \Gamma'} f|_k \gamma(z)$$

$$= \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + f|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} ab & 1 \\ cN & 1 \end{pmatrix}(z)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} ap & 1 \\ cN & 1 \end{pmatrix} = \begin{pmatrix} a & 1 \\ cN & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ap & 1 \\ cNp & p \end{pmatrix}$$

$$= \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + f|_k \begin{pmatrix} a & 1 \\ cN & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}(z)$$

$$= \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + \langle p \rangle f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}(z)$$

$$= \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^k \langle p \rangle f(pz).$$

Notation: $f \in M_k(\Gamma_1(N))$

$a_n(f)$:- n^{th} Fourier co-efficient of the q expansion of f. at the cusp ∞.

Th'm: $a_n(T_p f) = a_{pn}(f) + p^{k-1} a_{\frac{n}{p}}(\langle p \rangle f) \quad \forall n \geq 0$

$a_{\frac{n}{p}}$ term included only when $p \nmid N$ and $p \mid n$.

pf: $T_p f(z) = \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^{k-1} \langle p \rangle f(pz)$

$$f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}.$$

$$T_p f(z) = \frac{1}{p} \sum_{b=0}^{p-1} \sum_{n=0}^{\infty} a_n(f) e^{\frac{2\pi i n (z+b)}{p}}$$

$$+ p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p \rangle f) e^{2\pi i p z} . \quad (1)$$

① Simplifies as

$$T_p f(z) = \frac{1}{p}\sum_{n=0}^{\infty}\left(\sum_{b=0}^{p-1} e^{\frac{2\pi i n b}{p}}\right) a_n(f) e^{\frac{2\pi i n z}{p}}$$

$$+ p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p\rangle f) e^{2\pi i p n z}$$

$$* \sum_{b=0}^{p-1} e^{\frac{2\pi i n b}{p}} = \begin{cases} p & \text{if } p \mid n \\ 0 & \text{if } p \nmid n \end{cases}$$

$$\Rightarrow T_p f(z) = \sum_{\substack{n \geq 0 \\ p \mid n}} a_n(f) e^{\frac{2\pi i n z}{p}}$$

$$+ p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p\rangle f) e^{2\pi i p z n}$$

$$= \sum_{n=0}^{\infty} a_{pn}(f) e^{2\pi i n z} + p^{k-1} \sum_{n=0}^{\infty} a_{\frac{n}{p}}(\langle p\rangle f) e^{2\pi i z n}$$

$$= \sum_{n=0}^{\infty} \left(a_{pn}(f) + p^{k-1} a_{\frac{n}{p}}(\langle p\rangle f)\right) e^{2\pi i n z}.$$

Hecke Algebra

$\langle d\rangle, T_p \in \mathrm{End}_{\mathbb{C}}(M_k(\Gamma_1(N)))$

Define The $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(M_k(\Gamma_1(N)))$ generated by $\langle d\rangle$; $d \in (\mathbb{Z}/N\mathbb{Z})^*$. and T_p p prime is called Hecke algebra & denoted $\mathbb{T}(M_k(\Gamma_1(N)))$.

similar for $\mathbb{T}(S_k(\Gamma_1(N)))$.

Th'm: $\mathbb{T}(M_k(\Gamma_1(N))$ is commutative.

pf: $\mathbb{T} = \langle \langle d \rangle, T_p \rangle$ it's sufficent to show:

①: $\langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle$

②: $\langle d \rangle T_p = T_p \langle d \rangle$

③: $T_p T_q = T_q T_p$.

①: $\langle d \rangle f = f|_k \alpha \qquad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$\langle d \rangle \langle e \rangle f = f|_k \alpha\alpha' \qquad \alpha' = \begin{pmatrix} a' & b' \\ c' & e \end{pmatrix} \in \Gamma_0(N)$.

$= f|_k \alpha'\alpha$ check $\alpha'\alpha$ have same

$= \langle e \rangle \langle d \rangle f$. lower rt entry $= ed\ (N)$.

(check!)

②: I have not prepared and is tomo.

Assume $\langle d \rangle T_p = T_p \langle d \rangle$.

③: $a_n(T_q f) = a_{qn}(f) + q^{k-1} a_{\frac{n}{q}}(\langle q \rangle f)$

$a_n(T_p T_q f) = a_{pn}(T_q(f)) + p^{k-1} a_{\frac{n}{p}}(\langle p \rangle T_q f)$

$= a_{qpn}(f) + q^{k-1} a_{\frac{pn}{q}}(\langle q \rangle f)$

$+ p^{k-1} a_{\frac{nq}{p}}(\langle p \rangle f) + (pq)^{k-1} a_{\frac{n}{pq}}(\langle p \rangle \langle q \rangle f)$

$= a_n(T_q T_p f)$ by symmetry of p, q.

$\Leftarrow \langle p \rangle T_q = T_q \langle p \rangle$.

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Mod Forms.

$\langle d\rangle T_p = T_p\langle d\rangle$.

pf: $\gamma\in SL_2(\mathbb{Z})$ s.t. $\gamma = \begin{pmatrix} * & * \\ 0 & d\end{pmatrix}$ mod N. $\gamma\in\Gamma_0(N)$.

$\Gamma = \Gamma_1(N)$ $\quad \Gamma \trianglelefteq \Gamma_0(N)$. $\Rightarrow \gamma\Gamma\gamma^{-1} = \Gamma$
$\Rightarrow \Gamma\gamma\Gamma = \Gamma\gamma$
$\langle d\rangle f = f|_k\gamma$

WTS: $\langle d\rangle^{-1} T_p \langle d\rangle = T_p$

$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}$ $\quad \Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j$
$\Gamma\alpha\Gamma = \bigcup_j \Gamma\gamma\beta_j\gamma^{-1}$ (WTS).

$$\begin{aligned}
&= \bigcup_j \Gamma(\gamma\beta_j\gamma^{-1})\\
&= \bigcup_j \gamma\Gamma\gamma^{-1}\gamma\beta_j\gamma^{-1}\\
&= \gamma\bigcup_j \cancel{\gamma}\,\Gamma\beta_j\gamma^{-1}\\
&= \gamma(\bigcup_j \Gamma\beta_j)\gamma^{-1}\\
&= \gamma\Gamma\alpha\Gamma\gamma^{-1}\\
&= \Gamma\gamma^{-1}\alpha\gamma\Gamma\\
&= \Gamma\alpha\Gamma
\end{aligned}$$

Sketch $\gamma = \hat{\gamma}\begin{pmatrix} 1 & 0 \\ 0 & d\end{pmatrix}$ $\quad \hat{\gamma}\in\Gamma$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & d\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}$.

This shows that $\mathbb{T}(M_k(\Gamma_1(N)))$ is a comm ring.

T_p $\quad p$ - prime $\quad \langle d\rangle$ $\quad d\in(\mathbb{Z}/N)^*$

Defn: • $\langle p\rangle = 0$ whenever $p|N$.
• $T_1 = Id$.
• $T_n T_m = T_m T_n = T_{mn}$ $\quad (m,n)=1$.
• $T_{p^r} = T_p T_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}}$ $\quad$ (when $p|N$ $\; T_{p^r} = (T_p)^r$.

Recall: $f \in M_k(\Gamma_1(N))$ $f(z) = \sum a_m(f) q^m$ at ∞.

$$a_m(T_p f) = a_{pm}(f) + p^{k-1} a_{\frac{m}{p}}(\langle p \rangle f)$$

$$T_n(f) = \sum a_m(T_n f) q^m$$

$$a_m(T_n f) = \sum_{d \mid (m,n)} d^{k-1} a_{\frac{mn}{d^2}}(\langle d \rangle f).$$

Defin: A Hecke Eigenform is a modular form, $f \in M_k(\Gamma_1(N))$ $f \neq 0$.
Which is an e-vector for $\mathbb{T}(M_k \Gamma_1(N))$. We say that it is normalised if $a_1(f) = 1$.

$$T_n f = \lambda_n f \qquad a_n(T_n f) = \lambda_n a_n(f).$$

$$a_1(T_n f) = \lambda_n a_1(f).$$

If f is normalised, we get $a_1(T_n f) = \lambda_1$.

$\ast$ if $a_1(f) = 0$ $a_1(T_n f) = a_n(f) = \lambda_n a_1(f) \Rightarrow a_n(f) = 0$ $\forall n$
$\Rightarrow f = 0$. so $a_1(f) \neq 0$.

f - e-form

$\frac{f}{a_1(f)}$ - normalised e-form.

$$T_{mn} = T_m T_n \qquad (m,n) = 1.$$

$f \in M_k(\Gamma_1(N))$ - normalised form.

$a_{mn}(f) = a_n(f) a_m(f)$.

$a_1(T_{mn} f) = a_{mn}(f)$

$a_1(T_m(T_n(f))) = a_1(T_m a_n f)) = a_m(f) a_n(f)$.

~~Similarly $a_{p^r}(f) = a_p(f)\, a_{p^{r-1}}(f) - p^{k-1}$.~~

if f normalised form, e-values are fourier coeffs.

$\mathbb{Q}(\{a_n(f)\}_{n\geq 0})$ = finite degree.

for E_k, coeffs = $\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$.

field associated to modular forms

$T_p : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$

$X^2 - a_p X + p^{k-1} = 0 \quad p \nmid N.$ char poly

$a_p = a_p(f)$.

Modular Forms

Γ cong subgp

$M_k(\Gamma)$ ~ modular forms
$\cup\!\!\mathsf{I}$
$S_k(\Gamma)$ ~ subspace

we'll define $\langle \cdot, \cdot \rangle_\Gamma : S_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$
"Peterson inner product".

<u>Surface integrals</u>

Let $V \subseteq \mathbb{C}$, A 2-form on V is an expression

$$\omega = f(z, \bar{z}) \quad dz \wedge d\bar{z}$$

write $z = x + iy$,

$$dz \wedge d\bar{z} = d(x+iy) \wedge d(x-iy)$$
$$= -2i dx \wedge dy$$

$$\int_V \omega = \int_V f(z, \bar{z}) \, dz \wedge d\bar{z}$$

$$= \iint -2i \, f(x+iy, x-iy) \, dx \, dy$$

let $\alpha \in GL_2^+(\mathbb{Q})$, we have the standard action:
$z \mapsto \alpha z$

$$\mathrm{Im}(\alpha z) = \frac{\det(\alpha)}{|cz+d|^2} \mathrm{Im}(z)$$

$$d(\alpha z) = \frac{\det(\alpha)}{|cz+d|^2} dz \qquad \overline{d(\alpha z)} = \frac{\det(\alpha)}{|cz+d|^2} d\bar{z}$$

so $d(\alpha z) \wedge \overline{d(\alpha z)} = \dfrac{\det(\alpha)}{|cz+d|^4} dz \wedge d\bar{z}$

$\Rightarrow \dfrac{dz \wedge dz}{\mathrm{Im}(z)^2} \leftarrow$ Invariant under α.

def $d\mu(z) := \frac{-1}{2i} \frac{dz \wedge d\bar{z}}{\operatorname{Im}(z)^2}$

$= \frac{dx \wedge dy}{y^2} \qquad z = x+iy$

$d\mu$ is invariant under $z \mapsto \alpha z$

def: The covolume of $SL_2(\mathbb{Z})$ is the quantity:

$$\operatorname{covol}(SL_2(\mathbb{Z})) = \int_{D(SL_2(\mathbb{Z}))} z \, d\mu \quad \left(= \frac{\pi}{3}\right)$$

Cor: If $\varphi : D_{SL_2(\mathbb{Z})} \to \mathbb{C}$ is bdd then

$\int_{D_{SL_2(\mathbb{Z})}} \varphi(z) \, d\mu(z)$ is well defined cmpx numbe

Integral over $X(\Gamma) = \mathcal{H}/\Gamma$

$D_\Gamma = \bigcup_j \alpha_j D_{SL_2(\mathbb{Z})}, \quad \{\alpha_j\} = SL_2(\mathbb{Z})/\pm \mathrm{Id}\, \Gamma$

if φ is Γ invariant fn, $\int_{X(\Gamma)} \varphi(\tau) \, d\mu(\tau)$

$$= \sum_j \int_{\alpha_j D_{SL_2(\mathbb{Z})}} \varphi(\tau) \, d\mu(\tau)$$
$$= \text{"} \int_{D_{SL_2(\mathbb{Z})}} \varphi(\alpha_j \tau) \, d\mu(\tau)$$

$$\operatorname{covol}(\Gamma) = \int_{X(\Gamma)} d\mu(\tau)$$
$$= \sum_j \underbrace{\int_{D_{SL_2(\mathbb{Z})}} d\mu(\tau)}_{\operatorname{covol}(SL_2(\mathbb{Z}))}$$

$$= [PSL_2(\mathbb{Z}) : \bar{\Gamma}] \cdot \frac{\pi}{3}$$

let $f, g \in S_k(\Gamma)$, $\varphi(\tau) = f(\tau) \overline{g(\tau)} \operatorname{Im}(\tau)^k$

$\boxed{\text{ex: } \varphi \text{ is } \Gamma \text{ invariant}}$

Furthermore, $\forall \alpha \in SL_2(\mathbb{Z})$; $\varphi(\alpha\tau)$ is bdd in $D_{SL_2(\mathbb{Z})}$

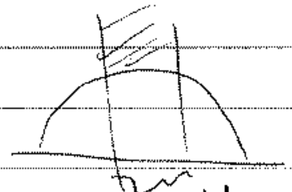
pf: $\varphi(\alpha\tau) = f|_k \alpha \, \overline{g|_k \alpha} \quad \mathrm{Im}(\tau)^k$

$\sim O(q_n) O(\overline{q_n}) y^k \quad (z = x+iy)$

$q_n = e^{2\pi i (x+iy)/n}$ n-form width of the cusp at α.

So $\varphi(\alpha\tau) \sim O(|q_n|^2) y^k$

fund dom



It suffices to check $\varphi(\alpha\tau) \to 0$ as $y \to \infty$; $\tau = x+iy$

As $y \to \infty$, $q_n = e^{2\pi i x/n} e^{-y} \to 0$

$\Rightarrow \varphi(\alpha\tau)$ is bdd on $D_{SL_2(\mathbb{Z})}$

$$\frac{1}{\mathrm{covol}(\Gamma)} \int_{X(\Gamma)} \underbrace{f(\tau)\overline{g(\tau)} \, \mathrm{Im}(\tau)^k}_{} \, d\mu(\tau)$$

$\Rightarrow$ from above, this is well defined as a cplx number.

This is the petersson inner product

$$\langle f, g \rangle_\Gamma := \frac{1}{\mathrm{covol}(\Gamma)} \int_{X(\Gamma)} f(\tau)\overline{g(\tau)} \, \mathrm{Im}(\tau)^k \, d\mu(\tau)$$

Prop : ① $\langle a_1 f_1 + a_2 f_2, g \rangle_\Gamma = a_1 \langle f_1, g \rangle_\Gamma + a_2 \langle f_2, g \rangle_\Gamma$
where $a_i \in \mathbb{C}$ $f_i, g \in S_k(\Gamma)$

② $\langle g, f \rangle_\Gamma = \overline{\langle f, g \rangle_\Gamma}$ (Hermitian)

③ $\langle f, f \rangle \geq 0$, $\langle f, f \rangle = 0 \iff f = 0$.

we can extend,

$$\langle \cdot, \cdot \rangle : M_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$$

we cannot extend to $M_k(\Gamma) \times M_k(\Gamma) \to \mathbb{C}$.

Def: The Eisenstein space for Γ is the orthogonal compliment of $S_k(\Gamma)$ under $\langle , \rangle_\Gamma$

$$\mathcal{E}_k(\Gamma) := \{ f \in M_k(\Gamma) : \langle f, g \rangle_\Gamma = 0 \ \forall g \in S_k(\Gamma) \}$$

= Adendum to last lecture

$$f = \sum_{m=0}^{\infty} a_m(f) q^m \in M_k(\Gamma_1(N))$$
$$a_m(T_n f) = \sum_{d \mid (m,n)} d^{k-1} a_{\frac{mn}{d^2}}(\langle d \rangle f)$$

$m=1$: $\quad a_1(T_n f) = a_n(f)$

If f is a Hecke eigenform,
$$T_n f = \lambda_n f \qquad \forall n \geq 1$$
$$a_n(f) = a_1(T_n f)$$
$$= \lambda_n a_1(f)$$
If $a_1(f) = 0$ then $a_n(f) = 0 \ \forall n$.

we normalise f by dividing by $a_1(f)$.
When f is a normalised eigenform,
$$a_n(f) = \lambda_n a_1(f)$$
$$\Rightarrow a_n(f) = \lambda_n$$

So e-values for T_n are just $a_n(f)$.

$\langle d \rangle$ on $f \in M_k(\Gamma_1(N))$, $d \in (\mathbb{Z}/N)^\times$

$\langle d \rangle$ on $f \in M_k(\Gamma_1(N))$, $d \in (\mathbb{Z}/N)^*$

f normalised eigen form

Defn ~~The~~
$\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}$
$\chi(d) :=$ e-value of $\langle d \rangle f$.

χ is a Dirichlet character
$\rightsquigarrow$ check! (because $\langle d \rangle \langle e \rangle = \langle de \rangle$)

Defn: Space of modular forms on $\Gamma_1(N)$ of wt k and char χ to be the subspace of
$M_k(\Gamma_1(N))$
$\langle d \rangle f = \chi(d) f$.

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \in \{(\mathbb{Z}/N)^\times \to \mathbb{C}\}} M_k(\Gamma_1(N), \chi)$$

In particular if $f \in M_k(\Gamma_1(N), \chi)$ normalised e-form,
$$\langle d \rangle f = \chi(d).$$

$$a_m(T_n f) = \sum_{d \mid (m,n)} d^{k-1} \chi(d)\, a_{\frac{mn}{d^2}}(f)$$

$$a_{p^r}(f) = a_p(f)\, a_{p^{r-1}}(f) - p^{k-1}\chi(p)\, a_{p^{r-2}}.$$

Mod Forms

recall: we defined the Petersson inner product
$\langle \cdot , \cdot \rangle_\Gamma : M_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$.

Defn: if V is a v.s. with $\langle , \rangle$. $T : V \to V$.
$T^+ :=$ adj of T. $\langle Tf, g \rangle = \langle f, T^+ g \rangle$ $f, g \in V$.

$\alpha \in GL_2^+(\mathbb{Q})$ $\alpha^* = (\det \alpha)\alpha^{-1}$
$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\alpha^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Prop: Γ-congruence subgps, $k \geq 0 \in \mathbb{Z}$, $f, g \in M_k(\Gamma)$
s.t. at least one of f, g in $S_k(\Gamma)$
$\langle T_\alpha f, g \rangle_\Gamma = \langle f, T_{\alpha^*} g \rangle$ $T_\alpha^+ = T_{\alpha^*}$

$\Gamma_\alpha = \Gamma \cap \alpha \Gamma \alpha$
$T_\alpha f(z) = \sum_{\gamma \in \Gamma / \Gamma_\alpha} f|_k \alpha\gamma (z)$
$= \sum_{\gamma \in \Gamma/\Gamma_\alpha} \frac{(\det \alpha\gamma)^k}{j(\alpha\gamma, z)^k} f(\alpha\gamma \cdot z)$.

$\langle T_\alpha f, g \rangle_\Gamma = \int_{z \in X(\Gamma)} \sum_{\gamma \in \Gamma/\Gamma_\alpha} \frac{(\det \alpha\gamma)^k}{j(\alpha\gamma, z)^k} f(\alpha\gamma z) \overline{g(z)} \operatorname{Im}(z)^k d\mu(z)$.

$\det(\alpha\gamma) = \det(\alpha)$
$j(\alpha\gamma, z) = j(\alpha, \gamma z) j(\gamma, z)$
$g(z) = j(\gamma, z)^{-k} g(\gamma z)$ $(g \in M_k(\Gamma))$
$\operatorname{Im}(z)^k = |j(\gamma, z)|^{2k} \operatorname{Im}(\gamma z)^k$

Change of variable $z \mapsto \gamma z$, $z \in X(\Gamma) = \mathcal{H}/\Gamma$
$\gamma \in \Gamma/\Gamma_\alpha \to \gamma z \in X(\Gamma_\alpha)$.

$\langle T_\alpha f, g \rangle_\Gamma = \int_{z \in X(\Gamma_\alpha)} (\det(\alpha))^k j(\alpha, z)^k f(\alpha z) \overline{g(z)} \operatorname{Im}(z)^k d\mu(z)$.
$= \det(\alpha)^k \int$ " "

Now consider $\langle f, T_\alpha^* g\rangle$, ~~$\langle f, \det(\alpha) T_{\alpha^{-1}} g\rangle$ $\det\alpha$~~.

$$\langle f, T_\alpha^* g\rangle = \langle f, T_{\det\alpha\,\alpha^{-1}} g\rangle$$
$$= \langle f, \det(\alpha)^{+k} T_{\alpha^{-1}} g\rangle$$
$$= \det(\alpha)^k \overline{\langle f, T_{\alpha^{-1}} g\rangle}$$
$$= \det(\alpha)^k \langle T_{\alpha^{-1}} g, f\rangle$$
$$= (\det\alpha)^k (\det\alpha^{-1})^k \int_{z\in X(\Gamma_{\alpha^{-1}})}$$

$$= (\det\alpha)^k (\det\alpha^{-1})^k \int_{z\in X(\Gamma_{\alpha^{-1}})} \overline{j(\alpha^{-1}, z)^{-k} g(\alpha^{-1} z)\, \overline{f(z)}\, \mathrm{Im}(z)^k} \, d\mu(z)$$

$$\int_{z\in X(\Gamma_{\alpha^{-1}})} \overline{j(\alpha^{-1}, z)^{-k} g(\alpha^{-1} z)}\, f(z)\, \mathrm{Im}(z)^k \, d\mu(z)$$

$$g(\alpha^{-1} z) = j(\alpha^{-1}, z)^k g(z)$$

if $w = \alpha^{-1} z$

$z \in X\Gamma_{\alpha^{-1}}$ ~~$\Rightarrow \Gamma_w$~~ $\Rightarrow w \in X(\Gamma_w)$.

$$j(\alpha^{-1}, \alpha w) = j(\alpha^{-1}, z) = j(\alpha, w)^{-1}$$
$$\mathrm{Im}(z) = \mathrm{Im}(\alpha w) = \frac{(\det\alpha)^k}{|j(\alpha, w)|^{2k}} \mathrm{Im}(w)^k.$$

$$\langle f, T_\alpha^* g\rangle = \int_{z\in X(\Gamma_{\alpha^{-1}})} \overline{j(\alpha^{-1} z)^k g(\alpha^{-1} z)},$$

$$= \int_{w\in X(\Gamma_\alpha)} (\det\alpha)^k j(\alpha, w)^{-k} f(\alpha w)\, g(w)\, \mathrm{Im}(w)^k \, d\mu(w)$$

$$= \langle T_\alpha f, g\rangle.$$

$$T_\alpha^* = T^{+}_\alpha.$$

Cor: $f \in \mathcal{E}_k(\Gamma)$ $g \in S_k(\Gamma)$. T_α preserves $\mathcal{E}_k(\Gamma)$

$\langle T_\alpha f, g\rangle = \langle f, T_\alpha^* g\rangle = 0$ $\quad T_\alpha^* g \in S_k(\Gamma)$

$\langle T_\alpha f, g\rangle = 0 \Rightarrow T_\alpha f \in \mathcal{E}_k(\Gamma)$.

Prop $\Gamma = \Gamma_1(N) \quad \langle d\rangle^+ = \langle d\rangle^{-1}$

$T_m^+ = \langle m\rangle^{-1} T_m \quad \forall\ (m,n)=1.$

pf $\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \quad \alpha^* = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix} \quad \langle d\rangle^+ = \text{acting by } \alpha^*.$

$ad - bc = 1$

$ad \equiv 1 \pmod N$ as $N \mid bc$.

$\therefore\ a \equiv d^{-1} \bmod N.$

$\langle d\rangle^+ = \langle a\rangle = \langle d^{-1}\rangle = \langle d\rangle^{-1}$

$(\langle d\rangle\langle d^{-1}\rangle = \langle d^{-1}d\rangle = I_d$

p prime $p \nmid N$

$T_p = \frac{1}{p} T_{\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}} \qquad \alpha^* = \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}$

$T_p^+ = \langle p\rangle^{-1} T_p \qquad (p,N)=1$ so we can choose

$d \equiv 1 \quad (N)$

$d \equiv 0 \quad (p) \qquad$ by CRT.

$(d,N)=1 \Rightarrow \exists a,b \in \mathbb{Z}$ s.t. $\begin{pmatrix} a & b \\ N & d\end{pmatrix} \in \Gamma_1(N).$

$T_{\begin{pmatrix} a & b \\ N & d\end{pmatrix}} f = f$

$$\begin{pmatrix} a & b \\ N & d\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} ap & b \\ Np & d\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\begin{pmatrix} ap & b \\ N & d/p\end{pmatrix} \in \Gamma_0^1(N).$$

$$T_{\alpha^*} = T_{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}} = T_{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}} I = T_{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}} T_{\begin{pmatrix} a & b \\ N & d\end{pmatrix}} = T_{\begin{pmatrix} a & b \\ N & d\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}}$$

$$= T_{\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}} T_{\begin{pmatrix} ap & b \\ N & d/p\end{pmatrix}}$$

$$= \langle d/p\rangle\, T_{\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}}.$$

$d \equiv 1 \ (N) \qquad \frac{d}{p} \equiv p^{-1}\ (N). \qquad T_{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}} = \langle p\rangle^{-1} T_{\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}}$

$T_p^+ = \langle p\rangle^{-1} T_p.$

$m = p^r$. Proof by induction.

Assume $T_{p^{r-1}}^{+} = \langle p^{r-1}\rangle^{-1} T_{p^{r-1}}$ (also for $< r$).

$T_{p^r} = T_p T_{p^{r-1}} - p^{r-1}\langle p\rangle T_{p^{r-2}}$

$= \cancel{\langle p^{r-1}\rangle^{-1}}$ take adjoint on both sides.

$T_{p^r}^{+} = T_{p^{r-1}}^{+} T_p^{+} - p^{k-1} T_{p^{r-2}}^{+} \langle p\rangle^{+}$

$= \langle p^{r-1}\rangle^{-1} T_{p^{r-1}} \langle p\rangle^{-1} T_p - p^{k-1}\langle p^{r-2}\rangle^{-1} T_{p^{r-2}} \langle p\rangle^{-1}$.

$= \langle p^r\rangle^{-1}\left[T_{p^{r-1}} T_p - p^{r-1}\langle p\rangle T_{p^{r-2}}\right]$

$= \langle p^r\rangle^{-1}\left[T_{p^r}\right]$. ∎

Finally if (m,n) coprime $T_{mn}^{+} = T_n^{+} T_m^{+}$

$= \langle n\rangle^{-1} T_n \langle m\rangle^{-1} T_m$

$= \langle nm\rangle^{-1} T_{nm}$

$= \langle mn\rangle^{-1} T_{mn}$ true if $(m,n,N) = 1$

This proves that $T_m^{+} = \langle m\rangle^{-1} T_m \; \forall \, (m,N) = 1$.

Mod Forms

New forms and old forms [Atkin-Lehner-Li-Theory]

Def'n: $T: V \to V$ is called normal if it commutes with its adjoint T^*.

Cor: $M_k(\Gamma_1(N))$ T_n & $\langle n \rangle$ are normal $(n,N)=1$

Spectral Thorem: V - fin dim $\mathbb{C}$-v.s. and has an Inner product. T - normal operator on V. V has an orthogonal basis of e-vectors w.r.t T.

Cor: $S_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous e-forms $\forall\, T_n, \langle n \rangle$, $(n,N)=1$.

d-prime $V_d : M_k(\Gamma_1(M)) \to M_k(\Gamma_1(Md))$

$$f(\tau) \mapsto f(d\tau) := d^{1-k} f|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

$V_d T_n = T_n V_d$ whenever $d \nmid n$

Suppose $M, N \in \mathbb{Z}_{>0}$ $M|N$

$$S_k(\Gamma_1(M)) \hookrightarrow S_k(\Gamma_1(N)).$$

$$V_d : S_k(\Gamma_1(M)) \hookrightarrow S_k(\Gamma_1(Md))$$

$$d \mid M/N.$$

Def'n: Space of old forms is a subspace of $S_k(\Gamma_1(N))$ that are in the image of V_d.

$$S_k(\Gamma_1(N))^{old} := \operatorname{span}_{\mathbb{C}} \langle V_d(S_k(\Gamma_1(M))) : d \mid M/N \quad M \neq N \rangle$$

$$\langle\, , \rangle : S_k(\Gamma_1(N)) \times S_k(\Gamma_1(N)) \to \mathbb{C}.$$

$$S_k(\Gamma_1(N))^{new} := (S_k(\Gamma_1(N))^{old})^{\perp}$$

$$= \{ f \in S_k(\Gamma_1(N)) : \langle f, g \rangle = 0 \ \forall g \in S_k(\Gamma_1(N))^{old} \}.$$

Theorem: $S_k(\Gamma_1(N))^{old}$ & $S_k(\Gamma_1(N))^{new}$ are stable under the action of all Hecke operators

It's enough to prove it on old, ~~as new is just the adjoint~~

T – diamond operator or T_p (p prime) or their adjoints

l-prime $l | N$.

$$S_k(\Gamma_1(N))^{l\text{-old}} := i S_k(\Gamma_1(N/l)) + V_l(S_k(\Gamma_1(N/l)))$$

$S_k(\Gamma_1(N/l)) \to S_k(\Gamma_1(N))$ $\quad V_l : S_k(\Gamma_1(N/l)) \to S_k(\Gamma_1(N))$

$i : f \mapsto f$

Note $S_k(\Gamma_1(N))^{old} = \bigoplus_{l|N} S_k(\Gamma_1(N))^{l\text{-old}}$.

Let $f \in S_k(\Gamma_1(N/l))$ $T(if)$ & $T(V_l f)$ are in $S_k(\Gamma_1(N))^{l\text{-old}}$

case 1: Diamond operators:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ this defines the $\langle d \rangle$ on both $S_k(\Gamma_1(N))$ & $S_k(\Gamma_1(N/l))$

$\langle d \rangle$ preserves $iS_k(\Gamma_1(N/l))$

$$\langle d \rangle if \in i(S_k(\Gamma_1(N/l))) \in S_k(\Gamma_1(N))^{l\text{-old}}.$$

$$V_l(\langle d \rangle f) = \langle d \rangle(V_l f)$$

$$\underbrace{\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}}_{V_l}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\langle d \rangle} = \underbrace{\begin{pmatrix} a & bl \\ c/l & d \end{pmatrix}}_{\langle d \rangle}\cdot\underbrace{\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}}_{V_l}$$

c/l also int.

$l | N$ and $N | c$.

N/l divides c/l.

$\begin{pmatrix} a & bL \\ c/L & d \end{pmatrix}$ acts as $\langle d \rangle$ on $S_k(\Gamma_1(N(L)))$.

Case 2: T_L Case 3 T_p $p \mid N$ 4 T_p $p \nmid N$

Case 5 T_n^+. These cases will take time and be proved later. (Tuesday).

Definition: $f \in S_k(\Gamma_1(N))^{new}$ is called a new form if it is
- normalised ($\Leftrightarrow a_1(f) = 1$)
- ~~eigenvector~~ for all the Hecke algebra ~~operators~~ e-form.

Strong Multiplicity One Theorem

1) $S_k(\Gamma_1(N))^{new}$ has a basis of new forms

2) If $f \in S_k(\Gamma_1(N))^{new}$ is an e-vector $\forall \{T_q\}$ $q \nmid N$.
q-prime.
then f is a scalar multiple of a new form.

3 $f \in S_k(\Gamma_1(M))^{new}$ $g \in S_k(\Gamma_1(N))^{new}$
$a_q(f) = a_q(g)$ $\forall$ but finitely many primes q; $M = N$ and $f = g$

$\mathcal{E}_k(\Gamma)$

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Mod Forms

recap L-prime $L | N$.

$$S_k(\Gamma_1(N))^{L\text{-old}} = iS_k(\Gamma_1(N/L)) + V_L S_k(\Gamma_1(N/L))$$

$$i : f(z) \mapsto f(z)$$
$$V_L : f(z) \mapsto f(Lz)$$

Thm: $S_k(\Gamma_1(N))^{L\text{-old}}$ is stable under the diamond operators, the T_n operators & ~~and~~ their adjoints

(i) Diamond Operators & Adjoints (Done)

(ii) T_p $p \nmid N$

(iii) T_p $p | N$ $p \neq L$

(iv) T_L

(v) T_p^{+} for $p | N$.

(ii) T_p preserves $S_k(\Gamma_1(N))^{L\text{-old}}$ $\forall p \nmid N$.

$S_k(\Gamma_1(N))$, $S_k(\Gamma_1(N/L))$

The formula for T_p is identical on both spaces

$f \in S_k(\Gamma_1(N/L))$

$$T_p(if) = i\,T_p(f)$$

$$\overset{\cap}{S_k(\Gamma_1(N))^{L\text{-old}}} \qquad \overset{\cap}{iS_k(\Gamma_1(N/L))} \subseteq S_k(\Gamma_1(N))^{L\text{-old}}$$

$T_p(V_\ell f) = T_p f(\ell z)$ easy to check.
$= V_\ell(T_p f)$

(iii) $p \mid N$, $p \neq \ell$

Same pf as above, but w/ a different formula for T_p.

(iv)' $\ell \| N$ ℓ exactly divides N.

$S_k(\Gamma_1(N))$ $\quad S_k(\Gamma_1(N/\ell))$
$\leftarrow \ell \nmid (N/\ell)$ (*??)

$f \in S_k(\Gamma_1(N/\ell))$

$T_\ell(if) = \sum_{n=1}^{\infty} a_{n\ell}(f) q^n$
$T_\ell(V_\ell f) = if$ (check)!

$f \in S_k(\Gamma_1(N/\ell))$, $\ell \nmid (N/\ell)$

$T_\ell f = \sum_{n=1}^{\infty} a_{n\ell}(f) q^n + \ell^{k-1} \sum_{n=1}^{\infty} a_{n/\ell}(\langle \ell \rangle f) q^n$

$= T_\ell(if) + \ell^{k-1} \sum_{n=1}^{\infty} a_{n\ell}(\langle \ell \rangle f) q^{n\ell}$

we have: $a_n(\langle \ell \rangle f) q^{n\ell} = V_\ell(\langle \ell \rangle f)$.

$T_\ell f = T_\ell(if) + \ell^{k-1} V_\ell(\langle \ell \rangle f)$.
$T_\ell(if) = \underset{S_k(\Gamma_1(N))^{\ell\text{-old}}}{\underset{\cap}{i(T_\ell f)}} - \ell^{k-1} \underset{S_k(\Gamma_1(N))^{\ell\text{-old}}}{\underset{\cap}{V_k(\langle \ell \rangle f)}}$.

(iv)'' $l^2 | N$

$S_k(\Gamma_1(N)) \qquad S_k(\Gamma_1(N/l))$

T_l acts by the same formula,

$T_l(if) = \sum a_{nl}(f) q^n$

$f \in S_k(\Gamma_1(N/l))$

$T_l(if) = i(T_l(f)) \in S_k(\Gamma_1(N))^{l\text{-old}}$

$T_l(V_l(f)) = if \in S_k(\Gamma_1(N))^{l\text{-old}}$

$p \nmid N$, $T_p^+ = \langle p \rangle^{-1} T_p$ - this formula doesn't hold when $p | N$.

$\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \quad \omega_N = T_{\alpha_N}$ - Atkin-Lehner Operator

$\alpha_N \underset{\substack{\cap \\ GL_2^+(\mathbb{Q})}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \alpha_N^{-1} = \begin{pmatrix} d & -c/N \\ -bN & a \end{pmatrix}$

α_n normalises $\Gamma_1(N)$

$\omega_N = f|_k \alpha_N(z)$

$\omega_N(f) = z^{-k} f\left(\frac{-1}{Nz}\right)$

$T_p^+ = \frac{1}{p} T_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}$

$= \frac{1}{p} T_{\alpha_N \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \alpha_N^{-1}}$ (by above)

$= \frac{1}{p} T_{\alpha_N} T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} T_{\alpha_N^{-1}}$

$= \omega_N^{-1} T_p \omega_N$

So it is enough to show ω_N preserves space of old forms $S_k(\Gamma_1(N))^{l\text{-old}}$

$f \in S_k(\Gamma_1(N/L))$

$$\omega_N(if)(z) = z^{-k}\, if\left(\frac{-1}{Nz}\right)$$
$$= L^k (zL)^{-k}\, if\left(\frac{-1}{\frac{N}{L}(Lz)}\right)$$

$$= L^k\, \omega_{N/L} f(Lz)$$
$$= L^k\, \omega_{N/L} V_L(f)$$
$$= L^k\, V_L(\omega_{N/L} f)$$
$$\cap$$
$$S_k(\Gamma_1(N))^{L\text{-old}}$$

$$\omega_N(N_L f)(z) = z^{-k}\, V_L\left(f\left(\frac{-1}{Nz}\right)\right)$$
$$= z^{-k} f(-1/Nz)$$
$$= z^{-k} f\left(-1/(\tfrac{N}{L})z\right)$$

$$= i\left(\omega_{N/L} f(z)\right) \in S_k(\Gamma_1(N))^{L\text{-old}}$$

Eisenstien Series for congruence subgps

$G_k(z) = \sum' \frac{1}{(mz+n)^2}$ $\quad SL_2(\mathbb{Z}) = \Gamma(1)$.

$P_\infty := SL_2(\mathbb{Z})_\infty = \{\pm T^n \; n \in \mathbb{Z}\}$

$P_\infty^+ = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ $\; b \in \mathbb{Z}$.

Lemma: $SL_2(\mathbb{Z})/P_\infty^+ \simeq \{\{c,d\} \in \mathbb{Z}^2 : (c,d) = 1\}$

given c,d s.t. $(c,d)=1$, can find a,b s.t.
$ad - bc = 1$ so surjective.
every sol'n $xd - cy = 1$ of form
$x = a + tc$
$y = b + td$.

$$\begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$

$$= \underset{\substack{\cap \\ P_\infty}}{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}} \underset{\substack{\cap \\ SL_2(\mathbb{Z})}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \qquad \square$$

So now we may we may rewrite the sum for Eisenstein series.

Prop $G_k(z) = \zeta(k) \sum_{\gamma \in SL_2(\mathbb{Z})/\Gamma_\infty^+} j(\gamma, z)^{-k}$

pf $(m,n) \in \mathbb{Z}^2/\{0,0\}$.

$(m,n) = (gc, dg)$ $\quad g = (m,n)$. $\quad g > 0$. $\quad (c,d) = 1$.

$$G_k(z) = \sum{}' \frac{1}{(mz+n)^k} = \sum_{\substack{(c,d)=1 \\ g=1}}^{\infty} \frac{1}{(gcz+gd)^k}$$

$$= \sum_{g=1}^{\infty} \frac{1}{g^k} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^n}$$

$$= \zeta(k) \sum_{\gamma \in SL_2(\mathbb{Z})/\Gamma_\infty^+} j(\gamma, z)^{-k}$$

Let Γ be arb. cong subgp.

$\Gamma_\infty^+ = \Gamma \cap P_\infty^+$.

~~we~~

Def The Eisenstein series of wt k and ~~lev~~ level Γ at cusp ∞ is

$$G_{k,\Gamma,\infty}(z) = \sum_{\gamma \in \Gamma / \Gamma_\infty^+} j(\gamma, z)^{-k}$$

we will show:

$$G_{k,\Gamma,\infty} \in \mathcal{E}(\Gamma)$$

$$\langle f, G_{k,\Gamma,\infty} \rangle_\Gamma = 0 \quad \forall f \in S_k(\Gamma)$$

then look at $G_{k,\Gamma_1(N),\infty}$.

Modular Forms

$$G_{k,\Gamma,\infty}(z) := \sum_{\gamma \in \Gamma/\Gamma_\infty^+} j(\gamma, z)^{-k} \qquad \Gamma_\infty^+ := \Gamma \cap P_\infty^+$$

$$P_\infty^+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}$$

rmk: $j(h\gamma, z) = j(\gamma, z)$ whenever $h \in \Gamma_\infty^+$ so well defined.

$$\Gamma/\Gamma_\infty^+ \hookrightarrow SL_2(\mathbb{Z})/P_\infty^+$$

prop: If either:

① k even

② k odd, $-Id \notin \Gamma$ and inf is reguler cusps of Γ.

then $G_{k,\Gamma,\infty} \in M_k(\Gamma)$, $G_{k,\Gamma,\infty}(\infty) \neq 0$

$G_{k,\Gamma,\infty}(s) = 0 \quad \forall s \in Cusps(\Gamma) \setminus \{\infty\}$.

③ k odd, $-Id \in \Gamma$ or ∞ is irregular cusp;

$G_{k,\Gamma,\infty} = 0$.

pf: most should be obvious, we just need to check calculations in ②:

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $j(\gamma, z)^{-k} = (cz+d)^{-k}$

$$\lim_{Im(z) \to \infty} (cz+d)^{-k} = \begin{cases} d^{-k} & c = 0 \\ 0 & c \neq 0. \end{cases}$$

$$c = 0 \iff \gamma \in \Gamma_\infty.$$

$$\lim_{Im(z) \to \infty} G_{k,\Gamma,\infty}(z) = \lim_{Im(z) \to \infty} \sum_{\gamma \in \Gamma/\Gamma_\infty^+} (cz+d)^{-k}$$

$$= \begin{cases} 1 & \Gamma_\infty = \Gamma_\infty^+ \\ 1 + (-1)^k & [\Gamma_\infty : \Gamma_\infty^+] = 2 \end{cases}$$

$$= \begin{cases} 1 & [\Gamma_\infty : \Gamma_\infty^+] = 1 \\ 2 & [\Gamma_\infty : \Gamma_\infty^+] = 2,\ k \text{ even} \\ 0 & [\Gamma_\infty : \Gamma_\infty^+] = 2\ \ k \text{ odd} \end{cases}$$

0 case can't happen

$\Rightarrow G_{k,\Gamma,\infty}(\infty) = \{1, 2\}$.

$s \in \mathrm{Cusps}(\Gamma)/\{\infty\}$ $\quad \gamma_s \in SL_2(\mathbb{Z})$ s.t. $\gamma_s \infty = s$.

$$G_{k,\Gamma,\infty}(s) = G_{k,\Gamma,\infty}|_k \gamma_s(\infty)$$

$$= \sum_{\gamma \in \Gamma/\Gamma_\infty^+} j(\gamma, \gamma_s z)^{-k} j(\gamma_s, z)^{-k}$$

$$= \sum_{\gamma \in \Gamma/\Gamma_\infty^+} j(\gamma\gamma_s, z)^{-k}$$

$$= 0.$$

Def'n: $G_{k,\Gamma,s} := G_{k,\gamma_s^{-1}\Gamma\gamma_s,\infty}|_k \gamma_s^{-1}$

$$= \sum_{\gamma \in \Gamma/\Gamma_s^+} j(\gamma_s^{-1}\gamma, z)$$

$$\Gamma_s^+ := \{\gamma \in \Gamma : \gamma_s^{-1}\gamma\gamma_s = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}.$$
i.e. $\in \Gamma_\infty^+$.

Prop $G_{k,\Gamma,s} \in \mathcal{E}_k(\Gamma)$ [i.e. $\langle f, G_{k,\Gamma,s}\rangle_\Gamma = 0\ \forall f \in S_k(\Gamma)$].

pf: Enough to show $s = \infty$.

$$\langle f, G_{k,\Gamma,\infty}\rangle_\Gamma = \int_{z \in X_\Gamma} \sum_{\gamma \in \Gamma/\Gamma_\infty^+} f(z)\overline{j(\gamma, z)^{-k}} \rightarrow$$

$\leftarrow \mathrm{Im}(z)^k \, d\mu(z)$

$$= \sum_{\gamma \in \Gamma / \Gamma_\infty^+} \int_{z \in X_\Gamma} f(z) j(\gamma, z)^{-k} \mathrm{Im}(z)^k \, d\mu(z).$$

$\omega = \gamma z$

$$\langle f, G_{k,\Gamma,\infty} \rangle_\Gamma = \sum_{\gamma \in \Gamma/\Gamma_\infty^+} \int_{\omega \in \gamma \mathcal{H}/\Gamma} f(\omega) \mathrm{Im}(\omega)^k \, d\mu(\omega)$$

$$f(\omega) = j(\gamma, z)^k f(z)$$
$$\mathrm{Im}(\omega) = |j(\gamma, z)|^{-2} \mathrm{Im}(z). \qquad \omega = x + iy$$

$$\sum_{\gamma \in \Gamma/\Gamma_\infty^+} \int_{\omega \in \gamma \mathcal{H}/\Gamma} f(\omega) y^k \frac{dx\,dy}{y^2}$$

$$\int_{\omega \in \mathcal{H}/\Gamma_\infty^+} f(\omega) y^{k-2} \, dy\, dx.$$

f is a cusp form $\therefore f = \sum_{n=1}^{\infty} a(n) q_h^n$

$h =$ width of ∞.

$$\int_{\omega \in \mathcal{H}/\Gamma_\infty^+} \left(\sum_{n=1}^{\infty} a_n e^{\frac{2\pi i n \omega}{h}} \right) y^{k-2} \, dx\, dy$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_{x=0}^{h} e^{\frac{2\pi i n x}{h}} \, dx}_{= 0} \int_{y=0}^{\infty} e^{-\frac{2\pi n y}{h}} y^{k-2} \, dy$$

$(n \geq 1)$.

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MF

Eisenstein series for $\Gamma_1(N)$

$M, N \in \mathbb{Z}_{>0} \quad M|N$

A Dirichlet character $\chi: \mathbb{Z} \to \mathbb{C}$ χ is defined modulo M can be lifted to a Dirichlet character $\chi^{(N)}$ modulo N

$$\chi^N(m) = \begin{cases} \chi(m) & (m,N)=1 \\ 0 & (m,N)>1 \end{cases}$$

The conductor of a dirichlet character χ mod N is the smallest divisor of M of N of which χ is a lift. If conductor is N itself, then χ is primitive.

For χ_1, χ_2

$$\sigma_{k-1}^{\chi_1,\chi_2}(n) := \sum_{d|n} \chi_1\left(\frac{n}{d}\right)\chi_2(d)\,d^{k-1}.$$

~~Defn~~

Thm: χ_1, χ_2 be two primitive Dirichlet characters modulo N_1, N_2 respectively.

$\chi = \chi_1 \cdot \chi_2$. – character modulo N_1N_2 need not be primitive.

Let $k \geq 3$ be s.t. $\chi(-1) = (-1)^k$.

$$\delta(\chi_1) = \begin{cases} 1 & \text{if } \chi_1 = 1^{(N_1)} \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{1}: \mathbb{Z} \to \mathbb{C}.$$

trivial character.

$$\mathbb{1}^{(N_1)}(m) = \begin{cases} 1 & (m,N)=1 \\ 0 & \text{else} \end{cases}.$$

$$L(\chi_2, s) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s}$$

$$= \prod_{p\text{-prime}} \frac{1}{1-\chi_2(p)p^{-s}} \qquad Re(s)>1.$$

$$E_k^{\chi_1\chi_2}(z) = \delta(\chi_1) L(\chi_2, 1-k)$$

$$+2 \sum_{n=1}^{\infty} \sigma_{k-1}^{\chi_1,\chi_2}(n)\, q^n$$

$$\in \mathcal{E}_k(\Gamma_1(N_1 N_2)).$$

<u>L-function of modular forms</u>

$$f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N))$$

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad s \in \mathbb{C}.$$

Prop: If $f \in S_k(\Gamma_1(N))$, then $L(f,s)$ converges absolutely $\forall s \in \mathbb{C}$ s.t. $Re(s) > \frac{k}{2}+1$

If $f \in M_k(\Gamma_1(N))$, $f \notin S_k(\Gamma_1(N))$, then $L(f,s)$ converges absolutely $\forall s \in \mathbb{C}$ s.t. $Re(s) > k$.

pf: $|a_n| \leq M n^{r(k)}$ $\qquad r(k) = \frac{k}{2}$ if $f \in S_k$

$\qquad = k-1$ if $f \notin S_k$.

(did this for $k=1$, but pf holds $\forall k$).

If $Re(s) > r(k)+1$

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leq M \sum n^{r(k)-Re(s)} < \infty.$$

Exercise $f \in M_k(\Gamma_0(N), \chi)$.

i.e. $f \in M_k(\Gamma_1(N))$, $\langle d \rangle f = \chi(d) f$ } can be lifted to $f \in M_k(\Gamma_0(N), \chi)$.

and $f = \sum a_n q^n$. Then f is a normalised e-form iff the L function admits an Euler product.

$$L(f,s) = \prod_{p \in \text{primes}} \frac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1-2s}}$$

analogous to showing ~~equiv to showing~~ zeta. $\Rightarrow \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$

L-function of Eisenstein series for $SL_2(\mathbb{Z})$.

$$E_k = \frac{G_k}{2\zeta(k)} = 1 +$$

$T_p E_k = \sigma_{k-1}(p) E_k$ holds.

$$\overline{E_k} = \frac{E_k}{a_1(E_k)} \xrightarrow{\text{consequences}} a_p(\overline{E_k}) = \sigma_{k-1}(p)$$

$\overline{E_k} \in M_k(SL_2(\mathbb{Z}))$

$$= 1 + p^{k-1}.$$

$$L(\overline{E_k}, s) = \prod_{p \text{ prime}} \frac{1}{1 - (1+p^{k-1})p^{-s} + p^{k-1-2s}}$$

$$= \prod_{p \text{ prime}} \frac{1}{(1-p^{-s})(1-p^{k-1-s})}.$$

$$= \zeta(s)\zeta(s-k+1).$$

$$\overline{E}_k^{\chi_1,\chi_2} = \frac{1}{2} E_k^{\chi_1,\chi_2} \in M_k(\Gamma_1(N_1 N_2))$$

$$L(\overline{E}_k^{\chi_1,\chi_2}, s) = L(\chi_1, s) L(\chi_2, s-k+1).$$

$$L(\chi, s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s} \qquad s \in \mathbb{C}.$$

L22

MF.

L-function of a cusp form

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N))$

recall that

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges for } Re(s) > \frac{k}{2}+1$$

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t} \to \text{Gamma fn}$$

$\Gamma(n+1) = n! \quad \forall n \geq 1.$

Defn: The completed L-function for f:

$$\Lambda(f,s) = (2\pi)^{-s} \Gamma(s) L(f,s)$$

Prop: $\Lambda(f,s) = \int_0^\infty f(it) t^s \frac{dt}{t}$ → Mellin transform of f.

pf. $$\Lambda(f,s) = \left(\frac{1}{2\pi}\right)^s \int_0^\infty t^s e^{-t} \frac{dt}{t} \sum \frac{a_n}{n^s}$$

$$= \left(\frac{1}{2\pi}\right)^s \sum_{n=1}^{\infty} a_n \int_0^\infty \left(\frac{t}{n}\right)^s e^{-t} \frac{dt}{t}$$

Make change of var $t \mapsto \frac{t}{2\pi n}$

$$\Lambda(f,s) = \sum_{n=1}^{\infty} a_n \int_0^\infty t^s e^{-2\pi n t} \frac{dt}{t}$$

$$= \int_0^\infty \sum_{n=1}^{\infty} a_n e^{-2\pi n t} t^s \frac{dt}{t}$$

$$= \int_0^\infty f(it)\, t^s \frac{dt}{t} \qquad \square$$

Interlude

$$W_N : S_k(\Gamma_1(N)) \to S_k(\Gamma_1(N))$$
$$f \mapsto i^k N^{1-\frac{k}{2}} f|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

$$\omega_N f(z) = i^k N^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right)$$

$$\omega^2_N f(z) = i^{2k} N^{-k} z^{-k} f\left(\frac{-1}{N\left(\frac{-1}{Nz}\right)}\right)$$

$$= f(z)$$

$$\langle \omega_N f, g \rangle_\Gamma = \langle f, \omega_N g \rangle_\Gamma$$

$$\omega_N^2 = \mathrm{Id}.$$

W_N has e-value ± 1.
$$S_k(\Gamma_1(N)) = S_k(\Gamma_1(N))^+ \oplus S_k(\Gamma_1(N))^-$$
$$S_k(\Gamma_1(N))^{\pm} = \{ f \in S_k(\Gamma_1(N)) : \omega_N f = \pm f \}.$$

$$\Lambda_N(s) := N^{sk} \Lambda(f, s)$$

Thm: Suppose $f \in S_k(\Gamma_1(N))^{\pm}$; then $\Lambda(f,s)$ extends to the entire cmplx plane.
$$\Lambda_N(s) = \pm \Lambda_N(k-s).$$

pf $\Lambda_N(s) = N^{s/2} \int_0^\infty f(it) t^s \frac{dt}{t}$

$$t \mapsto \frac{t}{\sqrt{N}}$$

$$= \int_0^\infty f\left(\frac{it}{\sqrt{n}}\right) t^s \frac{dt}{t}$$
$$= \int_0^1 f\left(\frac{it}{\sqrt{n}}\right) t^s \frac{dt}{t} + \underbrace{\int_1^\infty \text{''}}_{\text{converges as } t \to \infty}$$

$$\to f\left(\frac{it}{\sqrt{N}}\right) \sim O\left(e^{-2\pi i \sqrt{N}}\right)$$

$$\int_0^1 f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}$$

$$= \int_0^1 w_N f\left(\frac{i}{\sqrt{N}t}\right) \cancel{t}^{s-k} \frac{dt}{t}$$

$$\left[\begin{array}{l} w_N f\left(\frac{i}{\sqrt{N}t}\right) = t^k f\left(\frac{it}{\sqrt{N}}\right) \\ t \mapsto t^{-1} \end{array}\right]$$

$$= \int_1^\infty w_N f\left(\frac{it}{\sqrt{N}}\right) t^{k-s} \frac{dt}{t}$$

$$= \underbrace{\pm \int_1^\infty f\left(\frac{it}{\sqrt{N}}\right) t^{k-s} \frac{dt}{t}}$$ converges.

$$\Lambda_N(s) = \int_0^1 f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t} + \int_1^\infty f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}$$

$$= \int_1^\infty \pm f\left(\frac{it}{\sqrt{N}}\right) t^{k-s} \frac{dt}{t} + f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}$$

$$= \pm \Lambda_N(k-s).$$

Suppose $f \in S_k(\Gamma_1(N))^-$

$\Lambda_N(s) = -\Lambda_N(k-s) \quad \forall s \in \mathbb{C}.$

$s = k/2.$

$0 = \Lambda_N\left(\frac{k}{2}\right) = -\Lambda_N\left(\frac{k}{2}\right) \quad \therefore \Lambda_N(k/2) = 0.$

$\sim$ as it also takes on Gamma values (?).

$L(f, k/2) = 0.$

MA4H9 Modular Forms Revision

Disclaimer: Not in "right" order

§1: Modular Forms for $SL_2(\mathbb{Z})$

A Group theoretic description of $\mathcal{H}$.

Def: $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ upper half plane
$GL_2(\mathbb{R}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0\}$ general linear
$GL_2^+(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) : ad - bc > 0\}$ positive general linear
$SL_2(\mathbb{R}) = \{A \in GL_2^+(\mathbb{R}) : ad - bc = 1\}$ special linear g
$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ projective spec lin

rmk: Notice matricies act on $\mathcal{H}$ from left $[GL_2^+, SL_2, PSL_2]$
$(gh)z = g(hz)$ $\quad gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$.

lem: action of $SL_2(\mathbb{R})$ is transitive [i.e. $\forall z \in \mathcal{H}, \exists g, x \in SL_2, \mathcal{H}$ s.t. $gx = z$.]

Pf: let $\tau = x + iy \in \mathcal{H}$, $s_\tau := \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$. $S_\tau i = \tau$ □

lem: $\mathrm{stab}(i) = SO_2(\mathbb{R}) = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi]\}$.

pf: direct calc $g \in \mathrm{stab}(i)$, $g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ w/ $a^2 + b^2 = 1$ $\therefore SO_2$ □

rmk: by orb-stab, there is a bijection from $\mathcal{H}$ to $SL_2(\mathbb{R}) / SO_2(\mathbb{R})$
$\tau \mapsto S_\tau SO_2(\mathbb{R})$ $\quad$ inv: $gSO_2(\mathbb{R}) \mapsto g \cdot i$.

Def: $Y(1) := SL_2(\mathbb{Z}) \backslash \mathcal{H} = \{\gamma\tau : \tau \in \mathcal{H}, \gamma \in SL_2(\mathbb{Z})\}$
$\pi : \mathcal{H} \to Y(1)$ $\quad \tau \mapsto SL_2(\mathbb{Z})\tau$
$Y(1)$ is a quotient topology, with a set being open if the inverse image under π is open in $\mathcal{H}$.

rmk: π is an open mapping (takes open to open)
$\pi^{-1}\pi(U) = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma \cdot U$ arb. union of open sets.

lem U_1, U_2 open sets in $\mathcal{H}$, then
$S = \{\gamma \in SL_2(\mathbb{Z}) : \gamma U_1 \cap \gamma U_2 \neq \emptyset\}$ is finite

pf
- $S_{\gamma\tau} : i \to \gamma\tau$, $\gamma S_\tau : i \to \gamma\tau$ $\therefore$ by $\mathcal{H} \leftrightarrow SL_2(\mathbb{R})/SO_2(\mathbb{R})$
  $S_{\gamma\tau} SO_2(\mathbb{R}) = \gamma S_\tau SO_2(\mathbb{R})$.
- $\alpha, \beta \in \mathcal{H}$ $\quad \gamma\alpha = \beta \Leftrightarrow S_{\gamma\alpha} SO_2(\mathbb{R}) = S_\beta SO_2(\mathbb{R})$
  $\Leftrightarrow \gamma S_\alpha SO_2(\mathbb{R}) = S_\beta SO_2(\mathbb{R})$
  $\Leftrightarrow \gamma \in S_\beta SO_2(\mathbb{R}) S_\alpha^{-1}$

★
- $\therefore S \subseteq \{\gamma \in SL_2(\mathbb{Z}) : \gamma \overline{U_1} \cap \overline{U_2} \neq \emptyset\}$
  $= \underset{\text{discrete}}{SL_2(\mathbb{Z})} \cap \underset{\text{cmpt as } SO_2(\mathbb{R}) \text{ is.}}{S_{\overline{U_2}} SO_2(\mathbb{R}) S_{\overline{U_1}}^{-1}}$
- $=$ finite □

prop [action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ is proper discts] $\forall \alpha, \beta \in \mathcal{H}$ $\exists$ nbhds U of α, V of β s.t. $\forall \gamma \in SL_2(\mathbb{Z})$
- $\gamma\alpha = \beta$ or
- $\gamma(U) \cap V = \emptyset$.

pf
- Assume not first case, and $\gamma(U) \cap V \neq \emptyset$
- replace U with U_1, V with $\gamma^{-1} V_1$ where $U_1 \cap V_1 = \emptyset$, $\gamma\alpha \in V_1$, $\beta \in V_2$
- Since set of γ (s.t. above) is finite, process terminates.

Cor $Y(1)$ is Hausdorff.

pf
- pick τ_1, τ_2, U_1, U_2 s.t. $\pi(\tau_1) \neq \pi(\tau_2)$, U_i nbhd of τ_i.
- $\forall \gamma \in SL_2(\mathbb{Z})$, $\gamma\tau_1 \neq \tau_2$
  $\therefore \gamma U_1 \cap U_2 = \emptyset$ $\quad \therefore \pi(U_1) \cap \pi(U_2) = \emptyset$
- need to show $\pi(U_i)$ open set.
  done as π is open mapping □

Basic definitions of modular forms.

def The automorphy factor is the fn
$j : SL_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C} \quad j(\gamma, z) = cz + d \quad [\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}]$

lem cocycle relation $j(gh, z) = j(g, hz)\, j(h, z)$

def weight-k slash operator $k \in \mathbb{Z}_{\geq 0},\ \gamma \in SL_2(\mathbb{R})$
$$f|_k\gamma\, z = j(\gamma, z)^{-k} f(\gamma z)$$
note: $(f|_k \gamma_1\gamma_2)\, z = ((f|_k\gamma_1)|_k\gamma_2)\, z$

def fn $f : \mathcal{H} \to \mathbb{C}$ is a weakly modular fn of wt $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ if • holomorphic • $f|_k\gamma = f \ \forall \gamma \in SL_2(\mathbb{Z})$.

rmk note $f(z+1) = f(z)$ ∴ Fourier expansion exists
$\exp : \mathcal{H} \to \{0 < |q| < 1\}$. $z \mapsto q = e^{2\pi i z}$
$g(q) = f\left(\frac{\log q}{2\pi i}\right)$ choosing any branch of log.
$g(q) = \sum_{n=-\infty}^{\infty} a(n) q^n$
$\therefore f(z) = \sum_{n=-\infty}^{\infty} a(n) e^{2\pi i n z}$

def f meromorphic at infinity (resp. holomorphic) if
$f(z) = \sum_{n \geq n_0} a(n) q^n$ (resp. $n_0 = 0$)
f vanishes at ∞ if $n_0 = 1 \Leftrightarrow a(0) = 0 \Leftrightarrow f(\infty) = 0$.

def fn $f : \mathcal{H} \to \mathbb{C}$ is modular form of wt k for $SL_2(\mathbb{Z})$ if
- f holomorphic
- $f|_k\gamma = f \ \forall \gamma \in SL_2(\mathbb{Z})$
- f holomorphic at ∞

a cusp form is a modular form which vanishes at infinity.
$M_k = M_k(SL_2(\mathbb{Z}))$ space of mod forms wt k for $SL_2(\mathbb{Z})$
$S_k = S_k(SL_2(\mathbb{Z}))$ space of cusp forms wt k for $SL_2(\mathbb{Z})$

Eisenstein series

Def $k \geq 3$, $G_k(z) := \sum'_{m,n\in\mathbb{Z}^2} \left(\frac{1}{mz+n}\right)^k$ $\left[\sum' = \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0\neq n}}\right]$

- is holomorphic on $\mathcal{H}$ [ex]

Lem G_k is a modular form. $(k\geq 3)$

pf
- holomorphic ✓
- $G_k(\gamma z) = (cz+d)^k G_k(z)$ [easy calc]
- holormphic at ∞.

(later as cor, we'll find fourier series)

def $\sigma_m(n) = \sum_{d|n} d^m$ (divisor fn)

$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$ B_k = bernoulli numbers

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $Re(s)>1$ Riemann zeta fn.

$\cancel{= -\frac{(2\pi i)^s}{2}\frac{B_s}{k}}$

$\zeta(k) = -\frac{(2\pi i)^k}{2}\frac{B_k}{k!}$ $k\geq 2$, $\zeta(1-n) = -\frac{B_n}{n}$ $\forall n\geq 1$.

Lem $\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right) = \pi\cot\pi z = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m$.

pf
- $\sin\pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$
- log + diff wrt z, use $\tan(\pi z) = \frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}$

$\pi\cot(\pi z) = \pi i\left(1 - 2\frac{1}{1-e^{2\pi i z}}\right)$

- $q = e^{2\pi i z}$ and □

lem $\sum_{d\in\mathbb{Z}} \frac{1}{(z+d)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}q^n$ (k even ≥ 2)

pf $\frac{1}{z} + \sum_{d=1}^{\infty}\left(\frac{1}{z-d}+\frac{1}{z+d}\right) = \pi i - 2\pi i \sum_{d=0}^{\infty} q^d$

- diff wrt z, reorder

$\sum_{d\in\mathbb{Z}} \frac{1}{(z+d)^2} = (2\pi i)^2 \sum_{d=1}^{\infty} d q^d$ (diff more for others). □

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Thm $k \geq 4$ even, $G_k(z) = 2\zeta(k) E_k(z)$ where $\in \mathbb{Q}[[q]]$

pf
$$G_k = \sum{}' \frac{1}{(mz+n)^k}$$
$$= \sum_{n\in\mathbb{Z}} \frac{1}{n^k} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$$
$$= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$$
$$= 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!} \underbrace{\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1} q^{md}}_{=\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n} \qquad \square$$

$\therefore$ G_k is mod form wt k for $SL_2(\mathbb{Z})$ $k \geq 4$ even

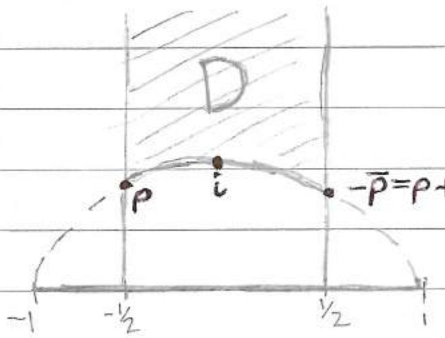
Fundamental domains

Def G acts on $\mathcal{H}$. A fundamental domain for G is a closed subset $D_G \subset \mathcal{H}$ s.t.

- $D_G = \overline{(D_G^{\circ})}$ [closure of its interior]
- every pt in $\mathcal{H}$ is G-equivalent to a pt in D_G
- if $z, z' \in D_G$ s.t. $z' = \gamma z$ for some $\gamma \in G$, $\Rightarrow z, z'$ on boundary

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $Sz = \frac{-1}{z}$ $Tz = z+1$

em



$SL_2(\mathbb{Z}) = \langle S, T \rangle$
$D = D_{SL_2(\mathbb{Z})}$

pf
- $\gamma \in \langle S, T\rangle$, $\gamma z \in D$ by construction
  - $\mathrm{Im}(\gamma z) = |cz+d|^{-2}\,\mathrm{Im}(z)$
    $|cz+d|$ bdd below away from $0 \Leftrightarrow \exists \gamma \in \langle S, T\rangle$ s.t. $\mathrm{Im}(\gamma z)$ max.
  - apply T to be in $[-\frac{1}{2}, \frac{1}{2}]$.
  - $\mathrm{Im}(\gamma z) \geq \mathrm{Im}(S\gamma z) \Rightarrow |\gamma z| \geq 1$ $\therefore \gamma z \in D$.
- D is closure of interior
  - true from metric space knowledge
- $\gamma \in SL_2(\mathbb{Z})$ $\gamma z \in D$ by const also.