1 Ring Theory.

* R semi prime if $0 \neq I \not\Rightarrow I^n = 0 \quad \forall n \geq 1$.
R has DCC if every descending chain stabilises.

1908: Artin Wedderburn thm.
R DCC + semi prime $\Rightarrow R = S_1 \oplus \dots \oplus S_m$ where
S_i is a matrix ring over some division ring
$S_i = M_{n_i}(D_i)$

Thm: R unital ring, DCC $\Rightarrow$ ACC for right ideals.
so ACC is weaker condition of the two.

Q: Semiprime + ACC $\Rightarrow$?

Concept needed: Rings of quotients.

recall A comm integral domain R has a field of frac.
i.e. R subring of F and every elt of F is expressible
as ac^{-1} $a, c \in R$. and $c \neq 0$.

we need to have defns which apply to imp classes of ring – eg matricies

$c \in R$ is called regular if $\begin{matrix} xc = 0 \Rightarrow x = 0 \\ cx = 0 \Rightarrow x = 0 \end{matrix}$. $\forall x \in R$.

R is said to have a rt quotient ring Q if
(i) R is a subring of Q
(ii) regular elts of R are units in Q
(iii) every elt of Q is expressible as ac^{-1} $a, c \in R$, c regular

Similarly for left quotient ring, $c^{-1}a$.

in non-com ring?

Q: i) Does a rt quotient ring exist? always? sometimes?
ii) Can a quotient ring exist on one-side, but not the other?
iii) If quotient rings exist on both sides, do we have equality? or keep separate?

Suppose R has a rt quotient ring Q. Let $a, c \in R$. with c reg.

Consider $c^{-1}a \in Q$ (as $c^{-1}, a \in Q$).

$c^{-1}a = bd^{-1}$ as Q rt quotient ring

$\boxed{ad = cb}$

The Ore condition i.e. given a, c $\exists b, d$.

This condition must be true if rt quotient ring exists.

2 Ring Theory.

M_R right R-module.

S1Q7 unital R, M_R a right module. Then $M_R = M_1 \oplus M_2$
M_1 unital, $M_2 R = 0$.
For this reason, for unital R, all modules assumed to be unital.

$\{M_\lambda\}_{\lambda \in \Lambda}$ a collection of submodules of M_R.
we define $\sum_{\lambda\in\Lambda} M_\lambda$ to be collection of all finite sums

we consider right modules and write maps on the left.
This way R^o (op ring) will not show up.
(neither will anti-isomorphisms).

using this. $\theta\phi(m) = \theta(\phi(m))$.

R, S rings. $\theta : R \to S$ ring hm.
M, K mod $\phi : M \to K$ mod hm.
$Ker(\theta)$ is ideal of R $Im(\theta)$ is subring of S
$ker(\phi)$ submod of M $im(\phi)$ submod of K.

$\sum_{\lambda\in\Lambda} M_\lambda$ is direct if each elt of $\sum M_\lambda$ is uniquely expressible as $m_1 + \dots + m_k$ $m_i \in M_{\lambda_k}$
iff $M_\mu \cap \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq\mu}} M_\lambda = 0$. $\forall \mu \in \Lambda$.

The $\forall \mu \in \Lambda$ part is important check.

1.15 Zorn's / WOP / A o C

A non-empty set S is said to be partially ordered if $\exists$ binary relation $\leq$ on S which is defined for certain pairs of elements and satisfies

(i) $a \leq a$ (iden)??

(ii) $a \leq b$ $b \leq c \Rightarrow a \leq c$ (trans)

(iii) $a \leq b$, $b \leq a \Rightarrow a = b$.

Let S be a partially ordered set. A non-empty subset τ is said to be totally ordered if ~~for every pair, $a, b \in \tau$,~~ all elements are comparable.

Let S be a partially ordered set. An elt $x \in S$ is called maximal elt if $x \leq y$ with $y \in S$ $\Rightarrow x = y$.

Similarly for a minimal elt.

Let τ be totally ordered subset of p.o. set S. τ has upper bound in S if
$\exists x \in S : y \leq x \quad \forall y \in \tau$.

Zorns: If a poset S has the property that every ordered subset of S has an upper bound in S then S contains a maximal elt.

A nonempty set S is said to be well ordered if it is totally ordered and every non empty subset of S has a minimal elt.

Thm

The WOP: Any non-empty set can be well-ordered.

AoC: Given a class of non-empty sets, there exists a choice function i.e. a function which assigns to each of it's sets one of it's elements.

It can be shown, AoC, WoP and Zorns are all logically equivalent.

3 Ring Theory (late)

Zorn's Lemma (Axiom)

1.16 ~~E~~An application

$[M \lhd R]$

1) Defn let R be a non-zero ring and $M \underset{r}{\lhd} R$ such that $M \neq R$. Then M is said to be a maximal right ideal. [max'l ideal]

if $M' \underset{r}{\lhd} R$ $[M' \lhd R]$ with $M' \supsetneq M \Rightarrow M = R$.

2) Prop: let R be a ring with 1. and $I \underset{r}{\lhd} R$. $[I \lhd R]$ such that $I \neq R$. Then $\exists$ a max'l rt ideal [ideal] $M \underset{r}{\lhd} R$ st. $M \supseteq I$.

Ex: In $\mathbb{Z}$, $p\mathbb{Z}$ is max'l ideal for each prime. [check].

let $M \neq R$ be an ideal in $\mathbb{Z}$ (or any comm. integral domain).

$\mathbb{Z}/M$ field $\Leftrightarrow$ M max'l ideal.

Pf: Zorns: $I \underset{r}{\lhd} R$. let S be the set of all proper($\neq R$) rt ideal of R containing I. Then $S \neq \emptyset$ since $I \in S$. Partially ordered S by inclusion i.e. $\forall A, B \in S$ $A \leq B$ if $A \subseteq B$.

get caught up — Let $\{T_\alpha\}_{\alpha \in \Lambda}$ be a totally ordered subset

$T = \bigcup_{\alpha \in \Lambda} T_\alpha$ Then $T \underset{r}{\lhd} R$ (check).

because totally ordered.

∴ ~~by Zorns~~, each totally ordered set has an upper bound.

but does $T \in S$?

clearly $T \supseteq I$. moreover, $T \neq R$. since $T = R$
$\Rightarrow 1 \in T \Rightarrow 1 \in T_\alpha \quad \alpha \in A$
$\Rightarrow T_\alpha = R$ (Check! as proper right ideals)
$\Rightarrow T_\alpha \notin S$ ※.

$\therefore T \in S$.
Clearly $T \supseteq T_\alpha \ \forall \alpha \in A$.
so T is an upper bound for $\{T_\alpha\}_{\alpha \in \Lambda \cap S}$.

So Zorn's applies. Thus S contains a max'l elt.
M say.
M is a max'l rt ideal and $M \supseteq I$.
Similarly for $I \triangleleft R$.

Rmk: This is false if R does not have 1.
See R+M rmk 8.22.

Ex: Prove that every v.s. has a basis

3) A ring with 1 has a max'l rt ideal [ideal].

pf: Take $I = 0$ in the above.

Chapter 2: General properties of Rings.

2.1 The Jacobson Radical.

All rings are assumed unital in this section, unless stated.

2: Defn The intersection of all the max'l rt ideal of a ring R is called the <u>Jacobson Radical</u>. Usually denoted $J(R)$ or simply J.

Note: that by 1.16 (2) R has at least one max'l rt ideal.

rmk: At this stage, this is the right jacobson radical. but we'll show that the two are the same. (left J)

first, we shall show J is two sided.

2 Crucial lemma:
Let M be a maximal rt ideal of a ring R. and let $a \in R$. Define $K = \{r \in R : ar \in M\}$.
$K \triangleleft_r R$. if $a \in M$ $K = R$.
if $a \notin M$ K is also a max'l rt ideal of R.

pf check $K \triangleleft_r R$. (consider kernel argument?)
if $a \in M$ then $1 \in K$. so $K = R$.
ii) now assume $a \notin M$. Then $aR + M \triangleleft_r R$.
$aR + M \supsetneq M$.
$\therefore$ $aR + M = R$ as maximal.
def R-mod hm $\theta : R \to R/M$
by $\theta(r) = ar + M$. $\forall r \in R$.
check θ is R-hm. (as module).
by $aR + M = R$, θ is an onto map (surj) (epi)
$\mathrm{Ker}(\theta) = K$
$\therefore$ $R/M \cong R/K$

*?? $\therefore$ K is ~~a~~ max'l rt ideal. (as $R/M \cong R/K$ and M is max'l).

3) Thm $J(R) \triangleleft R$.
pf Clearly $J(R) \triangleleft_r R$

Now let $j \in J$ be $a \in R$ and suppose that $aj \notin J$.
Then by defn of J $\exists$ a max'l rt ideal M s.t.
$aj \notin M$.
Clearly $a \in M$. $K = \{r \in R : ar \in M\}$.

Then by 2: K is a max'l rt ideal but $j \notin K$.
Since $aj \notin M$. So $j \notin J$ ※.
Thus $aj \in J$ $\forall j \in J$ $a \in R$. Thus $J \trianglelefteq R$.

4 Defn: Let R be a ring, $x \in R$. We say
x is right-quasi-regular (rqr).
if $1 - x$ has right inverse.
i.e. $\exists y \in R : (1-x)y = 1$
A subset S of R is called rqr if every
elt of S is rqr. Left qr (lqr) defined analogously.
lqr + rqr = qr.

Ring Theory.

R unital ring. $J(R) = \bigcap$ max'l right ideal.
have shown $J \trianglelefteq R$.
$x \in R$ is right quasi-regular (rqr) if $\exists y \in R$ s.t.
$(1-x)y = 1$.

$M \trianglelefteq R$ M max'l. $a \notin M$ $a \in R \Rightarrow K = \{r \in R : ar \in M\}$ is
also maximal. $R/M \cong R/K$.

Lemma: Let I be a rqr right ideal of R.
Then $I \subseteq J$.

pf: Let M be a max'l rt ideal of R.
Suppose $I \not\subseteq M$. So $I + M \supsetneq M$ $\therefore I + M = R$.
$\therefore$ $1 = x + m$ for some $x, m \in I, M$.
$1 - x = m \in M$.
$*$ Now $\exists y \in R : my = 1$.
So $1 = (1-x)y \in M$.
$M = R$ $\divideontimes$.
$\therefore I \subseteq M$. hence $I \subseteq J$.

6 Lemma: Let R be a ring then $J(R)$ is a rqr ideal.
Pf: Let $j \in J$ suppose $1-j$ has no right inverse.
$\nexists y$: $(1-j)y = 1$.
then $(1-j)R \neq R$.
by 1.16.2 (Zorns) $\exists$ max'l rt ideal M s.t. $(1-j)R \subseteq M$
But J belongs to M by defn of $J(R)$.
So $1 = 1-j+j \in M \Rightarrow M = R$ $\divideontimes$.
thus $1-j$ has rt inverse $\forall j \in J$.
J is rqr. $\square$

Lemma 7 Let I be an ideal of R.
Then I rqr $\Leftrightarrow$ I lqr.

pf Suppose I rqr, not lqr. Let $x \in I$
Then $\exists a \in R$ s.t. $(1-x)(1-a) = 1$.
$\therefore xa = x + a$.
$a = x(a-1) \in I$ since $I \trianglelefteq R$.

$\therefore \exists t \in R : (1-a)(1-t) = 1$.
hence $1 - x = 1 - t$
$\therefore x = t$.
$\therefore x$ is lqr.
converse by symmetry

8 Thm The right Jacobson radical is a quasiregular ideal. and contains all the rqr right ideals of R.

9 Cor The Jacobson radical of a ring has left-right symm
i.e. left Jacobson radical $J_l \equiv$ rt Jacobson radical J_r.

Pf J_l is a quasi regular ideal by by Left hand version of thm 8. $\therefore J_l \subseteq J_r$ by 8.
but $J_r \subseteq J_l$ by symmetry.

10 Thm Let R be a ring with J. Then
$J(R/J) = 0$.

pf: The maxl rt ideals of R/J are precisely the maxl rt ideals of the form M/J wher M is a maxl rt ideal of R.

let R be comm ring with 1. Let $M \neq R$ be an ideal.
recall that M is max'l iff R/M is a field.

11 Examples : (i) $J(\mathbb{Z}) = 0$. $p\mathbb{Z}$ p prime are precisely the max'l ideals.

ii) $R = \{a/b : a, b \in \mathbb{Z} \ b \text{ odd}\}$. let $0 \neq I \subsetneq R$.

let $x/y \in I$ $x, y \in \mathbb{Z}$ If x odd $y/x \in R$.
so $1 = \frac{x}{y} \cdot \frac{y}{x} \in I$ then $I = R$ ※.
$\therefore$ x even.

$\therefore I \subseteq M = \left\{ \frac{2c}{2d+1} : c, d \in \mathbb{Z} \right\}$.

It shows that M, is an unique max'l ideal of R.
So $M = J$. $[R = \mathbb{Z}_{(2)}]$.

iii) Let S be a comm int domain with a field of fractions F.
(eg R as above and $\mathbb{Q}$).

Consider $R = \begin{bmatrix} S & F \\ 0 & F \end{bmatrix}$ (check it's a ring).
let $X = \begin{bmatrix} J(S) & F \\ 0 & 0 \end{bmatrix}$, check $X \lhd R$.
Claim $X = J(R)$

5

Ring Theory.

Computing the Jacobson radical in examples:

$R = \begin{bmatrix} S & F \\ 0 & F \end{bmatrix}$ S comm int dom, $F = \mathrm{FoF}(S)$.

$X = \begin{bmatrix} J(S) & F \\ 0 & 0 \end{bmatrix}$ $X \trianglelefteq R$ (check.

Claim $J(R) = X$.

$\begin{bmatrix} j & f \\ 0 & 0 \end{bmatrix} \in X$ where $j \in J(S)$ and $f \in F$.

Then $I - \begin{bmatrix} j & f \\ & \end{bmatrix} = \begin{bmatrix} 1-j & -f \\ & 1 \end{bmatrix}$.

$\exists y \in S : (1-j)y = 1$.

$$\begin{bmatrix} 1-j & -f \\ & 1 \end{bmatrix} \begin{bmatrix} y & fy \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ all elts are rqr $\therefore$ by (8) $X \subseteq J(R)$.

$$R \xrightarrow{\theta} S/J(S) \oplus F \text{ given by.}$$

$$\theta \begin{pmatrix} s & f_1 \\ & f_2 \end{pmatrix} = (s + J(S),\ f_2).$$

is hm of rings which shows that.

$$R/X \cong S/J(S) \oplus F \quad \text{as rings}$$

Now, for rings A and B we have

$J(A \oplus B) = J(A) \oplus J(B)$ (check!)

$$J(S/J(S) \oplus F) = J(S/J(S)) \oplus F = 0$$

$$\therefore J(R/X) = 0.$$

$\therefore$

but clearly, $J(R)/X$ is a rqr ideal in R/X.

$$J(R)/X \subseteq J(R/X) = 0.$$

$$\therefore \quad J(R) \subseteq X \quad \therefore \quad J(R) = X.$$

(check entire pf).

ex Show $\bigcap_{n=1}^{\infty} J^n \neq 0$. (Christine (?)'s example) (by this particular J).

in comm: $\bigcap J^n = 0$.

Caution: R unital ring, M max'l ideal of R.
then $J(R) \subseteq M$. (prove!).
but $J(R)$ is <u>not</u> in general the intersection of all max'l ideals.
it's intersection of all <u>rt</u> max'l ideals.
and of all <u>left</u> max'l ideals.
Indeed $\exists$ a Noeth unital ring with unique non-zero proper ideals which is not it's Jacobson radical. ($J=0$).

2.1A: Jacobson Radical for rings without 1.

Suppose: R has 1, $a \in R$ is rqr.
$\exists b : (1-a)(1-b) = 1$.
$\cancel{1} + a + b = ab \quad + ab = a + b. \ (*)$
$(*)$ is indep of 1 so we define $a \in R$ to be rqr if $\exists b \in R$: $*$ is satisfied.
In general max'l rt ideals may not exist.
To overcome this.
$I \trianglelefteq_r R$ is called modular if $\exists e \in R$ s.t.

$r - er \in I \quad \forall r \in R$.

by Zorn's lemma: if R has a proper modular rt ideal I, then it has a max'l rt ideal $M \supseteq I$ (M modular too).

When R has 1, every rt ideal of R is modular.

2) Defn: $J(R) = \bigcap$ modular max'l rt ideals of R. if R has ~~...~~ any
$= R$ otherwise.

no pf given (ex!)
Standard properties follow.
employing methods we used previously.

2.2 Finitely generated module
1) Defn. let T be a subset of M_R. The "smallest" submodule of M containing T is called the submodule of M generated by T. i.e. it is the intersection of all submodules of M containing T. By convention we take 0 to be its submodule generated by the empty set. Of particular importance is the case when T consists of a single elt $a \in M$.

In general, the submodule gen by a is
$$\{ ar + \lambda a : r \in R \quad \lambda \in \mathbb{Z} \}.$$
- check this is submodule.

arb mod with a, must contain this one.
This equals aR if R has 1, M unital.

Def^n M_R is said to be f.g. (finitely generated) if it is the module generated by some finite ~~≠~~ subset.

If R has 1 and M is a unital f.g. module then $\exists\, a_1, \ldots, a_n \in M$ s.t. $M = a_1R + \ldots + a_nR$.
$a_1 \ldots a_n$ called generators.

2) Def^n: A module gen by single elt is called cyclic.
Thus an f.g. module is a finite sum of cyclic submodules.

(ii) Cyclic submodules of R_R are called principal rt. ideals.

6

Ring Theory.

Nakayama's Lemma: Let R be a ring with 1 and M_R a f.g. module $MJ = M \Rightarrow M = 0$: $J = J(R)$

pf: Let $MJ = M$ suppose $M \neq 0$.

$\exists\ m_1, \dots m_k \in M$: $M = m_1R + \dots + m_kR$.

Choose the generating set such k is least possible. we have

$$M = MJ = (m_1R + \dots + m_kR)J$$
$$\subseteq m_1J + \dots + m_kJ$$

Now $m_1 \in M$ so $\exists\ j_1 \dots j_k \in J$:

$$m_1 = m_1j_1 + \dots + m_kj_k \quad \text{so}$$
$$m_1(1-j_1) = m_2j_2 + \dots + m_kj_k \quad \text{case } k=1: \ [m_1(1-j_1) = 0]$$

Since j_1 is qr we have $m_1 = m_2j_2(1-j_1)^{-1} + \dots + m_kj_k(1-j_1)^{-1}$.

$\divideontimes$ k was least.

$[m_1 = 0 \Rightarrow \divideontimes\ k = 1]$.

Thus $M = 0$.

rmk its also true for rings without 1, adjusting proof with the equal condition for qr.

5: An application. Let R be a ring with 1 and M_R a f.g module

If $\left(M/MJ\right)_R$ is cyclic, then so is M_R.

pf: $\exists m \in M$: $\frac{M}{MJ} = \frac{mR+MJ}{MJ}$.

So we have $M = mR + MJ$.

This implies $\left[\frac{M}{mR}\right]J = \left[\frac{M}{mR}\right]$ where

$\frac{M}{mR}$ is viewed as a ~~A~~ R module.
(check!).

by Nakayamas lemma: $\left[\frac{M}{mR}\right] = 0$

which means $M \subseteq mR$.
but $mR \subseteq M$ $\therefore M = mR$.
In particular, J_R is f.g. J/J^2 is cyclic.

$\Rightarrow J$ principal.

2.3 Nil and Nilpotent subsets.

1) Defn: Let S be a non-empty subsets of a ring R
S is said to be <u>nil</u> if given any $s \in S$, $\exists$ an integer $k \geq 1$. (which depends on s) such that $s^k = 0$.
S is said to be nilpotent if $\exists k \geq 1$ s.t. $S^k = 0$ i.e. $s_1 \ldots s_k = 0$ $\forall s_i \in S$.

Ex: $R = \mathbb{Z}/4\mathbb{Z}$. $2\mathbb{Z}/4\mathbb{Z}$ is nilpotent.
If S consists of a single elt, then no difference between nil and nilpotent.
and usually the term nilpotent is used.
~~2. Prop: Let R be ring with 1.~~

2 Prop: let R be a ring with 1.
every nil one-sided ideal of R lies inside $J(R)$.

pf: let I be a nil rt ideal and $x \in I$. Then $x^k = 0$ for $k \geq 1$, we have $(1-x)(1+\dots+x^{k-1}) = 1$.
So I is rqr and so $x \in J$.

rmk: This is also true for rings without 1

3 lemma: Let R be a ring.
(i) if I and K are nilpotent rt ideals then so are $I+K$ and RI.
(ii) every nilpotent rt ideal lies inside a nilpotent ideal.

pf: (i) Suppose that $I^m = 0$ and $K^n = 0$ $m, n \geq 1$.
Consider $(I+K)^{m+n-1}$

Comm case $\left\{ \begin{array}{l} \text{by bin coeff } i+j = m+n-1 \\ \text{has no partition s.t. } i < m, j < n. \\ \therefore \text{ by bin thm: } (I+K)^{m+n-1} = 0. \end{array} \right.$

at least m I's in this term or at least n K's. general term $= x_1 \dots x_{m+n-1}$.
Since $IK \subseteq I$ $KI \subseteq K$ it follows $x_1 \dots x_{m+n-1} = 0$.
Thus $I+K$ is nilpotent.

$(RI)^m = (RI)\dots(RI)$ m times.
$\subseteq R(IRI\dots)I = R(IR)^{m-1}I \subseteq RI^m = 0$

(ii) Let I be nilpotent rt ideal. We have $I \subseteq I + RI$ and $I + RI$ nilpotent by (i).

4 Def'n: The sum of all nilpotent ideals of a ring R is called nilpotent radical of R (older terminology, older = wedderburn radical)

It is usually denoted by $N(R)$.
Note that $N(R) \subseteq J(R)$ always. (check)

It follows from (3) that $N(R) = \sum$ nilpotent right ideals.
(check) $= \sum$ " left "

(Clearly $N(R)$ is a nil ~~potent~~ ideal but need not be nilpotent itself.

Ex: Let n be a nilpotent elt of a comm ring. Show that (n) the ideal generated by n is nilpotent.

7

Ring Theory

$N(R) = \sum$ Nilpotent ideals (defns)
$= \sum$ " right ideals
$= \sum$ " left "

$N(R)$ is nil (check!) but need not be nilpotent

If $n \in R$ is nilpotent in a comm ring R, Then (n) the ideal generated by n is a nilpotent ideal.

5) ~~Zorn's Lemma~~ Zassenhaus Example.
Let F be a field, I the open interval $(0,1)$ and R a V.S. over F with basis $\{x_i : i \in I\}$
Define multiplication on R by extending the following product of basis elt $\quad x_i x_j = x_{i+j}$ if $i+j<1$
$= 0$ if $i+j \geq 1$.

Now every elt of R can be written uniquely in the form. $\sum_{i \in I} a_i x_i$ where $a_i \in F$ and $a_i = 0$ for all but finitely many i

Extend this product to arbitrary elts the usual way.
Check that R is **nil** but not nilpotent!

In fact $N(R) = R$.

6 Prop: Let R be a comm ring, then $N(R)$ equals the set of all nilpotent elts of R.

pf $n \in R$, n nilpotent $\Rightarrow (n)$ nilpotent

7 Example: The above is false in general in non-comm rings

Take $R = M_2(\mathbb{Q})$ 0 and R are the only ideals. (see 2.4.6).

So $N(R)=0$. but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = 0$. plenty of nilpotent elements.

Note also that $J(R)=0$. but the above elt is q.r.

2.4 Prime and Semi-Prime Ideals.

$p \in \mathbb{Z}$ prime if $p|ab \Rightarrow p|a$ or $p|b$. (i.e. $p\mathbb{Z}$ is prime)
generalised to:
R comm ring $P \triangleleft R$ P prime if $ab \in P$ $a,b \in R$
$\Rightarrow a \in P$ or $b \in P$

to non-comm rings:

1) Defn: P ideal of ring R is said to be a prime ideal if $AB \subseteq P$, $A,B \triangleleft R$ $A \subseteq P$ or $B \subseteq P$.

by convention, we exclude R from set of prime ideals.

Ex: R be a ring with 1. Show that a max'l ideal is prime.

2) Prop: The following are equivalent for an ideal $P \neq R$ of R
(i) P is prime ideal
(ii) $AB \subseteq P$, $A, B \trianglelefteq R \Rightarrow A \subseteq P$ or $B \subseteq P$
(iii) $CD \subseteq P$, $C, D \trianglelefteq_r R \Rightarrow C \subseteq P$ or $D \subseteq P$.
(iv) $aRb \subseteq P$ $a, b \in R \Rightarrow a \in P$ or $b \in P$.

pf: (ii) $\Rightarrow$ (i) trivial

(i) $\Rightarrow$ (iv) Let $aRb \subseteq P$ with $a, b \in R$.

$$\underbrace{(RaR)}_{A} \cdot \underbrace{(RbR)}_{B} \subseteq P.$$

$\Rightarrow$ $A \subseteq P$ or $B \subseteq P$.

WLOG: $RaR \subseteq P$.

Suppose that $RaR \subseteq P$.

Let $\langle a \rangle = \{\lambda a + ra + as + \sum_{\text{finite}} r_i a s_i : \lambda \in \mathbb{Z}, r, s, r_i, s_i \in R\}$

be the two sided ideal generated by a.

Check $\langle a \rangle \triangleleft R$.

* since $RaR \subseteq P$ we have $\langle a \rangle^3 \in P$.

hence $\langle a \rangle \subseteq P$ by assumption so $a \in P$.

(iv) $\Rightarrow$ (ii) let ~~A, B~~ $\subseteq P$ $A, B \triangleleft R$ suppose $A \not\subseteq P$

~~$a \in A$, $b \in B$ Then $aRb \subseteq P$~~

$\exists a \in A$ st. $a \notin P$.

$b \in B$ Then $aRb \subseteq P$ so by assumption $b \in P$ $\forall b \in B$.

$\therefore$ $B \subseteq P$.

Similarly, we can show (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). $\square$

3) Defn: R is called a <u>prime</u> ring if (0) is prime ideal of R.

A comm int domain is a prime ring.

4) Let $P \neq R$ be an ideal of a comm ring R.
Then P is a prime ideal iff $ab \in P \Rightarrow a \in P$ or $b \in P$
$a, b \in R$.

Let R be a <u>comm</u> ring w/ 1.
$P \triangleleft R$ $\quad P \neq R$. Then P is prime ideal
$\Leftrightarrow R/P$ is an int domain.

~~Let $R = M_2(\mathbb{Z})$,~~

5) Matrix Units

Let R be a ring with 1 and (a_{ij}) an $n \times n$ matrix with $a_{ij} \in R$.
Let E_{ij} be the matrix with 1 in the ij-th pos. and 0's elsewhere.

We have $(a_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij}$ uniquely.

$$E_{ij} E_{kl} = \begin{cases} E_{il} & \text{if } j = k \\ 0 & \text{else.} \end{cases}$$

8 RT

$P \lhd R$ $P \neq R$ P is a <u>prime ideal</u> if $AB \subseteq P$, $A, B \lhd R$
$\Rightarrow A \subseteq P$ or $B \subseteq P$.

R ring with 1. E_{ij} matrix units.

$$E_{ij} E_{kl} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwise} \end{cases} = \delta_{jk}$$

6 Thm: R ring w/ 1. Then.
(i) $I \lhd R \Rightarrow M_n(I) \lhd M_n(R)$ conversely, every ideal of $M_n(R)$ is of the form $M_n(I)$ for some $I \lhd R$.

if $R = M_2(\mathbb{Z})$ P prime in $\mathbb{Z}$.
we expect $M_2(P)$ to be prime, but how do we show it?

pf (i) trivial
(ii) Let $X \lhd M_n(R)$. We shall show that $\exists$ an ideal I of R : $X = M_n(I)$.

let $A = (a_{ij}) = \sum_i \sum_j a_{ij} E_{ij} \in X$

consider (fixed) α, β $\quad 1 \leq \alpha, \beta \leq n$.

we have $E_{1\alpha} \cdot (\sum_{i,j} a_{ij} E_{ij}) E_{\beta 1} \in X$ since $X \lhd M_n(R)$.

hence $a_{\alpha\beta} E_{11} \in X$ $\quad (*)$.

Thus, the matrix with $a_{\alpha\beta}$ in the 11-th position and 0 elsewhere belongs to X.

ll entries of all matrices* Now let I be set of all elts of R with occur in in the 1,1th entry of some matrix in X.
We shall show that $I \lhd R$ and $X = M_n(I)$.

Let $a,b \in I$. Easy to see that $a-b \in I$.
$a \in I$ $r \in R$.
let a be the $1,1$ th entry of $A = (a_{ij}) \in X$.
~~defn~~ $(a_{ij}) = \sum_{i,j} a_{ij} E_{ij}$

$$rE_{11}(A)(E_{11}) \in X \quad \therefore ra \in X$$
$$E_{11} A(rE_{11}) \in X \quad \therefore ar \in X.$$

$\therefore I \trianglelefteq R$.

Now let $C = (c_{ij}) = \sum_{ij} c_{ij} E_{ij} \in X$. $(c_{ij} \in R)$.

By (*), each $c_{ij} \in I$ So $C \in M_n(I)$
hence $X \subseteq M_n(I)$.

Finally let $D = (d_{ij}) = \sum d_{ij} E_{ij} \in M_n(I)$
by defn of I, $\forall i,j \in I^n$: $d_{ij} E_{11} \in X$.

So $E_{i1}(d_{ij} E_{11}) E_{1j} \in X$
hence $E_{ij} d_{ij} \in X$.

Since X is an ideal, we have $D = \sum d_{ij} E_{ij} \in X$.
$\therefore D \in X$
Thus $M_n(I) \subseteq X$ $\therefore M_n(I) = X$.

Ex let R be a ring with 1. P a prime ideal of R.
show $M_n(P)$ is a prime ideal of $M_n(R)$.

rmk: This does not hold for one-sided ideals.
$\begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 0 \end{bmatrix} \trianglelefteq_r M_2(\mathbb{Z})$ but
$\square$ not equal to $M_2(I)$ for any ideal of $\mathbb{Z}$.

ex: Let R be a ring w/ 1. and $I \triangleleft R$. Show that
$M_n(R/I) \cong \frac{M_n(R)}{M_n(I)}$

7 Cor: If R is a prime ring w/ 1. ~~Then~~ So is ring $M_n(R)$.

pf follows from the above

8 Defn: An <u>Integral domain</u> is a ring R s.t.
$ab = 0 \quad a, b \in R \Rightarrow a = 0$ or $b = 0$.

For us, an integral domain need <u>not</u> be comm.

An integral domain is trivially a prime ring.
A matrix ring over an int domain is a prime ring.
Thus $M_n(\mathbb{Z})$ is a typical example of a prime ring.

Defn 9: Let $I \neq R$ be an ideal of ring R. I is said to be a <u>semi-prime</u> ideal if $A^n \subseteq I$, $A \triangleleft R \Rightarrow A \subseteq I$.

R is called a <u>semi-prime ring</u> if 0 is a semi-prime ideal.

R semi prime ring $\Leftrightarrow$ R has no ~~zero~~ non-zero nilpotent ideal
$\Leftrightarrow$ R has no non-zero nilpotent rt ideal
$\Leftrightarrow$ " left
$\Leftrightarrow$ $N(R) = 0$

Clearly a prime ideal is semi-prime and more over an arbitrary intersection of prime ideals in a ring is semi-prime (but not nec. prime).
$2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is semi prime ideal of $\mathbb{Z}$.

which?
*

A result analogous to (2) can be proved for semi-prime ideals (Do this).

10) Prop: Let R be a ring. The intersection of all the prime ideals of R is a nil ideal.

pf: We shall show if $x \in R$ is not nilpotent then $\exists$ a prime ideal of R excluding it.
Let S be collection of all ideals of R with no power of x.
$S \neq \phi$ as $0 \in S$. (check that Zorn's lemma applies)
So S contains a max'l elt – P say claim P is prime.
If not $\exists$ ideals, A, B of R s.t. $AB \subseteq P$. but $A \nsubseteq P, B \nsubseteq P$. Then $A+P \supsetneq P$ $B+P \supsetneq P$
so by maximality $\exists$ m, n s.t.
$x^m \in A+P$, $x^n \in B+P$ by maximality of P.
then $x^{m+n} \in (A+P)(B+P) \subseteq P$. $\divideontimes$.
So P is prime. $\square$

9

Ring Theory

Have shown using Zorn's lemma $x \in R$, x not nilpotent
$\Rightarrow \exists$ a prime ideal P in R s.t. $x \notin P$.
hence $\bigcap_{P \text{ prime}} P$ is a nil ideal.

Ex: Show that Zassenhaus' Example cannot contain a prime ideal $P \neq R$ by defn.

~~(Assume ring has a prime ideal)~~
11) Cor: let X be intersection of all the prime ideals of a ring R. Then
(i) $N(R) \subseteq X$ always
(ii) $N(R) = X$ when ~~R~~ is commutative ring.

pf: (i) Easy
(ii) by 2.3.6 $N(R)$ consists of the nilpotent elts of R.
so by (10) $X \subseteq N(R)$

2.5 Completely Reducible Modules

1) Defn: A rt R-module M is said to be irreducible (or simple, but this word is not preferred).
if (i) $MR \neq 0$.
(ii) M contains no submodule other than 0 and M.

If R has 1, M unital then (i) is automatically true.

2) Examples: (i) If p is a prime then $\mathbb{Z}/p\mathbb{Z}$ is an irreducible $\mathbb{Z}$-module.
(ii) every ring with 1 has an irred submodule.

by 1.16.2 R contains a max'l rt ideal – M say. Then R/M is an irred rt R-module (check!)

(iii) Let V be a V.S. over a field F. Then any one dimensional subspace of V is an irreducible. F-module.

V has the following interesting property.
V is ~~the~~ a sum of irreducible submodules.
i.e. V is a sum of 1-dimn subspaces.
~~i.e. every vector space has~~
This sum is in fact direct
(i.e. every V.S. has a basis).

Over an arbitrary ring not all modules have this property.
e.g. consider $\mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module.
$2\mathbb{Z}/4\mathbb{Z}$ is only non-zero submodule.
So $\mathbb{Z}/4\mathbb{Z}$ is not expressible as a sum of irreducible submodules.

We investigate modules which do have this property.

Def'n 3) A module M_R is said to be completely reducible. (or semi-simple). if M is expressible as a sum of irred submodules.

Ex 4) Modules over a field, ie. vector spaces have this property. (every non-zero) is CR.

(ii) $\mathbb{Z}/6\mathbb{Z}$ is cR as a $\mathbb{Z}$-module.
$\mathbb{Z}/6\mathbb{Z} = \frac{2\mathbb{Z}}{6\mathbb{Z}} + \frac{3\mathbb{Z}}{6\mathbb{Z}}$. (Check) Clearly $2\mathbb{Z}/6\mathbb{Z}$, $3\mathbb{Z}/6\mathbb{Z}$ are irred.

4) Lemma: Let ~~each~~ M be a rt R-module s.t. $M = \sum_{\lambda \in \Lambda} M_\lambda$. where each M_λ is an irred submodule of M. let K be a submodule of M. Then $\exists$ a subset $\mathcal{M}$ s.t. sum $K + \sum_{\mu \in \mathcal{M}} M_\mu$ is a direct sum and

$$M = K \oplus \sum_{\mu \in \mathcal{M}} \oplus M_\mu$$

This says roughly for a v.s. and basis of subspace, it can be extended to a basis for whole v.s.

Pf sketch:
Consider S the set of submodules $K + \sum_{\alpha \in A} M_\alpha$ such that the sum $K + \sum_{\alpha \in A} M_\alpha$ is direct.

where A is a subset of Λ. Apply Zorn's lemma (after doing checks!!) to obtain a max'l elt in S, $X = K \oplus \sum_{\mu \in \mathcal{M}} \oplus M_\mu$ say

Claim $X = M$: choose $\lambda \in \Lambda$ we have either $X \cap M_\lambda = 0$ or $X \cap M_\lambda = M_\lambda$ since M_λ is irreducible.
The first possibility gives a direct sum,
$X \oplus M_\lambda$ which contradicts the maximality of X in S so $X \cap M_\lambda = M_\lambda$ and $M_\lambda \subseteq X$ $\forall \lambda \in \mathcal{M}$
hence $X = M$.

5) Dedekind Modular law: Let A, B, C be submodules of M_R s.t. $A \supseteq B$: Then

$$A \cap (B + C) = B + A \cap C.$$

pf elementary.

Thm 6) Let M be a non-zero rt R-module
Then ~~the~~ following are equivalent.
(i) M_R is CR
(ii) ~~and~~ M is a direct sum of irreducible submodules.
(iii) $mR = 0$ $m \in M \Rightarrow m = 0$ and every submodule of M is a direct summand of M.

pf (i) $\Rightarrow$ (ii) Take $K = 0$ in (4).

Ring Theory.

6) $M_R \neq 0$ TFAE:

(i) M is CR

(ii) M is direct sum of irred submodules

(iii) $mR = 0$ $m \in M \Rightarrow m = 0$ and every submodule M is direct summand of M.

pf i) $\Rightarrow$ (ii) Take $K = 0$ in (4).

(ii) $\Rightarrow$ (iii) Suppose that $mR = 0$ for some $m \in M$.
let $M = \bigoplus_{\lambda \in A} M_\lambda$ with M_λ irred.

Then $m = m_1 + \ldots + m_k$ for some $m_i \in M_{\lambda_i}$

Now $mr = 0$, $r \in R \Rightarrow m_1 r + \ldots m_k r = 0$
$\Rightarrow m_i r = 0$ for $i = 1, \ldots, k$
since the sum of M_λ's is direct.
Define $K_j = \{x \in M_{\lambda_j} : xR = 0\}$. is a submodule of M_{λ_j}. (check!).
So $K_j = 0$ or $K_j = M_{\lambda_j}$ since irred.
But $K_j \neq M_{\lambda_j}$ since $M_{\lambda_j} R \neq 0$ by defn of irred.
So $K_j = 0$ for $j = 1, \ldots, k$. hence $m_j = 0$. $\forall j$
hence $m = 0$

The second part follows from (4).

(iii) $\Rightarrow$ (i) Our first aim is to show that M has an irred submodule. Note that by the Dedekind Modular law, the hypothesis on M is inherited by every submodule of M. (check!)

Let $0 \neq y \in M$ let S be set of all submodules K of M. s.t. $y \notin K$.

S non-empty poset etc. ($0 \in S$).

Check Zorn's applies.

S contains a max'l elt. say A.

note $A \neq M$. since $y \notin A$.

by hypothesis $\exists$ a submodule $B \neq 0$ s.t. $M = A \oplus B$.

want to show B is irred:

$BR \neq 0$ by assumption.

Suppose B contains a submodule B_1 s.t. $0 \subsetneq B_1 \subsetneq B$

Then $\exists$ a submodule $B_2 \neq 0$ s.t. $B_1 \oplus B_2 = B$.

by Dedekind modular law. ~~let $y \in B_2$~~

now $y \in A \oplus B_1$, $A \oplus B_2$ by maximality of A in S.

$y \in (A \oplus B_1) \cap (A \oplus B_2)$

$y \in A$ (check)

$\times$

$\therefore$ B is irreducible.

Let K be the sum of all irred submodules of M.

If $K \neq M$ $\exists$ a submod $L \neq 0$ s.t.

$M = K \oplus L$. Since L must have one irred submodule, intersection cannot be $\emptyset$ $\therefore L = \{0\}$.

$\therefore K = M$, M is CR. $\square$

rmk: The first part of condition (ii) holds automatically when R has 1, and M is unital.

Question: What rings R are there $\ni$ R_R is cR?

7) Examples when ${}_RR$ and R_R are cR. ~~let $R = M_n(D)$ where D is a division~~

a) let $R = M_n(D)$ where D is a division ring.
here both R_R and ${}_RR$ are cR.

pf Let E_{ij} $1 \leq i,j \leq n$ be the matrix units of R.
Consider $I_j = E_{jj}R$. Then $I_j \trianglelefteq_r R$.
I_j is set of all matrices where all rows except possible the j^{th} are 0.

Claim: each I_j is an irred rt R-module.
Suppose that $0 \subsetneq X \subseteq I_j$ where $X \trianglelefteq_r R$.
Then X has a non-zero matrix $A = (\alpha_{\alpha\beta})$ say.
~~Why column? of 0?~~ A must have a non-zero entry since $A \in I_j$, must have $\alpha_{jk} \neq 0$ some k.
B = matrix with α_{jk}^{-1} in kj^{th} position. and 0's else. Then $AB = E_{jj}$

so $E_{jj} \in X$ since $A \in X$ $X \trianglelefteq_r R$.
Thus $E_{jj}R \subseteq X$ so $I_j = X$.

also $I_jR \neq 0$ as R has 1.

Now $R = I_1 \oplus \dots \oplus I_n$
Similarly, looking at columns, we can show
${}_RR = J_1 \oplus \dots \oplus J_n$ J being columns also irred.

b) let $R = R_1 \oplus \dots \oplus R_m$ a direct sum of rings
where $R_i = M_{n_i}(D_i)$. $n_i \in \mathbb{N}$, D_i div ring.
Again R_R, ${}_RR$ are cR

11 RT

Ex: R w/ 1, when is R_R CR?

a) $R = M_n(D)$ D division ring

b) $R = R_1 \oplus \dots \oplus R_m$ where $R_i = M_{n_i}(D_i)$ D_i div ring. n_i pos ints.

Pf b):

Since $R_i \triangleleft R$ each R_i can be considered as an R_i-module or an R-module. Further, the R-submodules and R_i-submodules coincide

(Note: $R_i R_j = R_i^2 \delta_{ij}$)

Now, by (a) each R_i is a sum of irreducible R_i-submodules. So R_i is sum of irred R-submods
So R is sum of irred R-submods $\square$

Goal: If R w/ 1 R_R CR then R is nec. of this form.
$\Rightarrow$ R_R CR $\Leftrightarrow$ ${}_RR$ CR.

§2.6 Idempotent Elements.

1) Def'n: elt $e \in R$ is idempotent if $e^2 = e$

2) Ex: (i) $0, 1$
(ii) in $\mathbb{Z}/6\mathbb{Z}$ $\bar{3}, \bar{4}$
(iii) in $M_2(\mathbb{Z})$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Lemma 3: $e \in R$ idempotent $R = eR \oplus K$ where $K = \{x - ex : x \in R\}$
$K \trianglelefteq R$.

pf: clearly eR, K rt ideals of R (check!)

Let $x \in R$, then $x = ex + x - ex \in eR \oplus K$.
$z \in eR \cap K$ then
$$z = ea = b - eb \quad \text{for some } a, b \in R.$$
$$ea = e^2a = eb - e^2b$$
$$= eb - eb$$
$$= 0$$
so $z = 0$. so $eR \cap K = \{0\}$.
sum is direct □

Cor 4: (Peirce Decomposition) R w/ 1. $e \in R$ idempotent.
$$R = eR \oplus (1-e)R$$

pf: $K = \{(1-e)x : x \in R\} = (1-e)R$.

11 addendum.

~~RT (ask ollie for whole?)~~

5 prop: Let R be a ring w/1. Suppose $R = I_1 \oplus \dots \oplus I_n$ a direct sum of rt ideals. Then we can write $1 = e_1 + \dots + e_n$ $e_j \in I_j$. e_j's have

- each e_j idempotent
- $e_i e_j = 0$ if $i \neq j$
- $I_j = e_j R$ $\forall j$
- $R = Re_1 \oplus \dots \oplus Re_n$ a direct sum of left ideals.

pf: (i) (ii): $e_j = 1 e_j = e_1 e_j + \dots + e_n e_j$

$$e_j - e_j^2 = e_1 e_j + \dots + e_{j-1} e_j + e_{j+1} e_j + \dots + e_n e_j$$

LHS $\in I_j$ RHS $\in (\sum_{s \neq j} I_s)$

$\therefore \in I_j \cap (\sum_{s \neq j} I_s) = \{0\}$.

hence e_j idempotent, $e_i e_j = \delta_{ij}$.

(since sum of I_j's is direct, we have $e_i e_j = 0$ $\forall i \neq j$.

(iii), (iv) exercise.

6 Examples

$R = M_n(\mathbb{Z})$ Take $e_j = [(\delta_{jj})]$. i.e. $\{ {1 \text{ if } j\text{'s th} \atop 0 \text{ else}}$.

$$R = e_1 R \oplus \dots \oplus e_n R = Re_1 \oplus \dots \oplus Re_n.$$

$e_j R =$ j'th row matrices.

$Re_j =$ jth column matricies.

7 defn: Let R be a ring. Define

$C(R) = \{x \in R : xr = rx \ \forall r \in R\}$.

$C(R)$ is called centre of R.

$C(R)$ subring of R. (check!)

It is in general not an ideal.

8) lemma: Let $I \triangleleft R$ with $I = eR = Rf$ $e = e^2$, $f = f^2$

Then

- $e = f$
- $e = 1$ of R.
- $e \in C(R)$.

pf: $e = e^2 \in I$ $\quad e = af \quad a \in R$.

$$= af^2$$
$$= ef$$

similarly $f = eb = e^2b = ef$

$$\therefore \underline{e = f}$$

ii) $x \in I \Rightarrow x = e\alpha = \beta e$ for some $\alpha, \beta \in R$

$ex = e\alpha = \beta e = x \quad \forall x \in R$

$xe = x$

iii) ex

Prop: R be a ring w/ 1. Suppose that $R = A_1 \oplus \dots \oplus A_k$. a direct sum of <u>ideals</u>.

Let $1 = e_1 + \dots + e_k$ w/ $e_j \in A_j$.

- $e_j \in C(R) \ \forall j$.
- $e_j^2 = e_j$
- $e_i e_j = \delta_{ij} e_j$
- $A_j = e_j R + R e_j$
- e_j is identity of the ring A_j.

pf check this follows from (5) and (8).

Ring Theory

Correction: proof of 8(ii) $ex = x = xe$ $\forall x \in I$ not $\forall x \in R$.

Peirce Decomposition: R w/ 1 $R = eR \oplus (1-e)R$.

2.7: Annihilators and minimal rt ideals:

Defn 1 Let S be a non-empty subset of M_R, we define the right ann of S to be $r(S) = \{r \in R : Sr = 0\}$.

When For $_RM$, the left ann $l(S) = \{r \in R : rS = 0\}$.
clearly $r(S) \trianglelefteq R$ and $l(S) \trianglelefteq R$.

In most applications S is a subset of R itself so we consider both $r(S), l(S)$

A rt ideal I is said to be an annihilator rt ideal or simply rt annihilator if $I = r(S)$ for some S subset of R.

Similarly for left ideals and left annihilators.

Defn 2 A non-zero rt ideal M of R is said to be a <u>minimal</u> rt ideal if $M \supsetneq M'$ $M' \trianglelefteq R \Rightarrow M' = 0$.

If R has 1, minimal rt ideals of R are precisely the irred submodules of R_R.

Lemma 3: Let M be a minimal rt ideal of a ring R. Then either (i) $M^2 = 0$ or $M = eR$ for some $e = e^2 \in M$.

pf: Suppose that $M^2 \neq 0$. then $\exists a \in M$ st $aM \neq 0$. now $aM \trianglelefteq R$, $aM \subseteq M$ since $a \in M$.
$\therefore$ $aM = M$ as $aM \neq 0$.

"it's v. easy to prove a large amount of false theorems".

Thus $\exists\, e \in M$ s.t. $a = ae$. In particular $e \neq 0$.

also $a = ae = ae^2 = \dots$

$a(e - e^2) = 0$

Thus $e - e^2 \in M \cap r(a)$.

$M \cap r(a) \trianglelefteq R$ $\quad M \cap r(a) \subseteq M$.

$\therefore M \cap r(a) = 0$ or M.

~~$e \in M$~~ if $= M \Rightarrow aM = 0$ $\cancel{\ }$.

$\therefore e - e^2 = 0$

$\therefore e$ idempotent.

So $eR \neq 0$. $eR \trianglelefteq R$

$eR \subseteq M$ since $e \in M$.

$\therefore eR = M$.

ex: $R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ Consider $M_1 = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{bmatrix}$

Both M_1 and M_2 are minimal rt ideals (check!).

Now, $M_1^2 = 0$, $M_2^2 = eR$ where $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$M_1 \not\cong M_2$ as rt R-modules (check!).

2.8 Homomorphisms of Irreducible Modules.

Prop 1 Let M, K be rt R-modules $\theta : M \to K$ a non-zero R-hm

(i) If M is irreducible, θ is a monomorphism.

(ii) If K is irred, θ is an epimorphism.

(iii) If K, M irred, θ is isomorphism.

pf: (i) $\ker(\theta)$ is a submodule of M

$\ker(\theta) \neq M$ as $\theta \neq 0$. $\therefore \ker(\theta) = 0$ $\therefore$ mono.

(ii) similar, (iii) follows from (i), (ii).

If $\theta : M \to M$ is an isomorphism, then θ^{-1} (set theoretic sense.) is the inverse map, which exists and is also an isomorphism (routine checking!)
Moreover $\theta\theta^{-1} = \theta^{-1}\theta = 1_M$ the identity map on M.

Cor 2: Schur's lemma: If M_R is irred, then $E_R(M)$ [(from H1)] is a div ring.

pf: by above every non-zero elt of $E_R(M)$ is an isomorphism

Defn 3: Let R be a ring. The <u>rt socle</u> $E(R)$ is defined to be

$E(R) :=$ the sum of all minimal rt ideals of R if R has any,
0 else.

$E'(R)$ is defined analogously left socle.

In general $E(R) \neq E'(R)$.

Prop 3: $E(R) \trianglelefteq R$
pf: trivial if $E = 0$ otherwise, let M be a minimal rt ideal of R and $x \in R$ Then the map
$\theta : x \mapsto xm, \forall m \in M$ shows that
$xM = 0$ or $xM \cong M$. and xM is also min rt ideal.

It follows $E \trianglelefteq_l R$. so $E \triangleleft R$ $\square$

Also define $E(M_R)$ = sum of all irred ~~##~~ submodules of M_R
if M has any
$= 0$ otherwise.

Chapter 3

3.1 Finiteness assumptions.

RT (Thanks to Ollie C.)

§3 Chain conditions.

Defn 3.1 Let S be a non-empty collection of submodules of M_R

(i) An elt $K \in S$ is max'l in S if $\nexists K' \in S$ s.t. $K' \supsetneq K$.
similarly for min'l.

(ii) M_R is said to have ACC for submodules in S if every increasing chain of submodules
$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \quad A_i \in S$$
has equal terms after finite number of steps.
i.e. $\exists N \geq 1$ s.t. $A_N = A_{N+1} = \ldots$

(iii) M_R is said to have maximum condition on submodules if every non-empty subcollection of submodules in S has a submodule max'l in that subcollection.

DCC, minimum condition defined similarly.

Prop 2: Let S be nonempty collection of submodules of M_R.

M has ACC [DCC] on submodules in S iff
M has max'm [min'm] condition on submodules in S.

pf ex $\square$

We shall be applying the above in particular, when S is the set of all rt annihilators of a ring.

The statement "M has ACC" means "M has ACC on set of all submodules of M". Similar for DCC.

Prop 3: TFAE for M_R:

(i) M has ACC

(ii) M_R has max'm cond.

(iii) Every submodule of M_R is f.g.

pf ex or R+M 5.3 $\square$

ex 4: $\mathbb{Z}_{\mathbb{Z}}$ has ACC.

Since every ideal of $\mathbb{Z}$ is principal.

rmk: ACC $\not\Rightarrow$ $\exists N$ s.t. all chains have stopped at n^{th} step, nor does DCC.

prop 5: K be a submodule of M_R, M has ACC [DCC] iff both K, M/K have ACC [DCC].

pf $\Rightarrow$ ✓

$\Leftarrow$ let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of submodules in M_R.

Consider $\underbrace{(M_i \cap K)_i}_{\substack{\text{submods of } K \\ \therefore \text{ stop}}}$ and $\underbrace{(M_i + K)_i}_{\substack{\text{submods of } M/K \text{ (after correspond-ance)} \\ \therefore \text{ stop}}}$.

$\therefore \exists k \geq 1$ s.t. first stops

$\therefore \exists l \geq 1$ s.t. second stops,

$n = \max \{k, l\}$

$$\begin{aligned} M_{n+i} &= M_{n+i} \cap (M_{n+i} + K) \\ &= M_{n+i} \cap (M_n + K) \\ &= M_n + (M_{n+i} \cap K) \\ &= M_n \end{aligned}$$

$\therefore$ M has ACC

similar for DCC $\square$

Previous prop has many useful consequences.

Cor 6: $M_1, \ldots, M_n$ submods of M_R. If M_i has ACC [DCC] then so do their sum.

pf: by induction on n

$n=1$ clear.

assume true for $n-1$ $L = M_1 + \ldots + M_{n-1}$

$$K/L = \frac{L+M_n}{L} \simeq \frac{M_n}{L \cap M_n} \quad (2^{nd} \text{ Isom})$$

K/L has ACC since $\frac{M_n}{L \cap M_n}$ is a factor module of M_n and M_n has ACC.

L has ACC by hypothesis

so K has ACC.

Similarly for DCC $\square$

Cor 7: Let R be ring w/ 1, ACC [DCC] on rt ideals, be a (unital) finitely gen'd module. Then M_R has ACC [DCC] on submodules.

pf Since M unital and f.g. $\exists\, m_1, \ldots, m_k \in M$ s.t.

$M = m_1R + \ldots + m_kR$

by (6) it is enough to show m_iR has ACC [DCC] on submodules.

TB Cont.

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Rwl 1. ACC [DCC] on rt ideals ~~$M_R = m_1R+\ldots$~~
$M_R = m_1R + \ldots + m_kR$
$\Rightarrow M_R$ has ACC [DCC] on submodules.

Pf ctd:
Enough to show that each $(m_iR)_R$ has ACC.
Let $\theta_i R \to m_iR$ be the map given by
$\theta_i(r) = m_ir \ \forall r \in R$ Then θ_i is an
R-hom from R_R onto m_iR.

So $m_iR \cong R_R/\ker(\theta)$ Since R_R has ACC on

submodules it follows that each m_iR has ACC
on submodules. Similarly for DCC. $\square$

rmk If R does not have 1, For ACC the result
is still true. For DCC, result is false!
(true as $\mathbb{Z}$ has ACC, but not DCC).
$m \in M$, $mR + m\mathbb{Z}$ is needed.

(or 8) If R has ACC [DCC] on rt ideals then so
does the ring $M_n(R)$.

Pf: Consider $M_n(R)$ as a rt R-module (see 1.8(2)(iv)).
Let T_{ij} = set of all matricies in $M_n(R)$ with
entries in $_{ij}$ and 0's elsewhere.

Then each T_{ij} is an R-submodule.
of $M_n(R)$. clearly (check!).

Clearly $T_{ij} \cong R_R$, each T_{ij} has ACC [DCC] on R-submodules. But trivially,

$$M_n(R) = \sum_{i,j} T_{ij}$$

So by (6) $M_n(R)$ has ACC [DCC] on R submodules. But clearly a rt ideal of $M_n(R)$ is clearly an R-submodule. (Not vice versa) So $M_n(R)$ has ACC [DCC] on rt ideals.

A module w/ ACC on submodules is called N'n module.
" w/ DCC " " A'n module.

A ring w/ ACC on rt ideals is called a rt N'n.
" w/ 1, DCC on " " rt A'n.
reason is DCC w/o 1 is "pathological" and weird.
Similarly, for lt N'n, lt A'n.

Let $R[x]$ be the ring of polys in x w/ coeffts from R. When multiplying two polys, we assume x commutes w/ elts in R. We quote w/o pf.

10 Hilberts basis thm.

If R is rt N'n ring w/ 1, then so is $R[x]$.

Pf see R+M 5.10.

3.2 Composition series.

Neither ACC nor DCC acting alone lead to an int n s.t. all chains stop after n steps. However, the two acting together, do give such n.

Def'n 1) A module M_R is said to have finite length if $\exists$ a chain of submodules $M = M_0 \supsetneqq \dots \supsetneqq M_k = 0$ (*) s.t. no submodule can be properly inserted between M_i and M_{i+1} $\forall i \in I_0^{k-1}$.

i.e. $M_i \geq K \geq M_{i+1} \Rightarrow M_i = K$ or $K = M_{i+1}$.

If R has 1 and M is unital, this just means M_i/M_{i+1} is irreducible.

Such a series is called a composition series for M_R. The factor module M_i/M_{i+1} are called factors of comp series. k is called the length of the series.

It is clear how ACC and DCC give finite length. but not immediately obvious the converse is true.

Let there be two comp series

$M = M_0 \supsetneqq \dots \supsetneqq M_s = 0$ and

$M = K_0 \supsetneqq \dots \supsetneqq K_t = 0$. they are equivalent if $s = t$ and $\exists$ a permutation π on $\{0, \dots, s-1\}$ s.t. $\frac{M_i}{M_{i+1}} \cong \frac{K_{\pi(i)}}{K_{\pi(i)+1}}$ for $i \in I_1^{s-1}$

2) Examples: $\mathbb{Z}/6\mathbb{Z} \supsetneqq 2\mathbb{Z}/6\mathbb{Z} \supsetneqq 0$ and $\mathbb{Z}/6\mathbb{Z} \supsetneqq 3\mathbb{Z}/6\mathbb{Z} \supsetneqq 0$.

equivalent. [R+M 6.10 (?)].

★ no submods other than $0, M \Rightarrow M$ field?

Lemma 3) M_R has comp series $\Leftrightarrow$ M_R has ACC and DCC on submodules.

pf: $\Leftarrow$ obvious approach works. :)
$\Rightarrow$ pf by induction on k, least length of a comp series for M. If $k=1$, M has no submodules other than $0, M$. So M has ACC, DCC.

★ Now assume true for $k-1$, let M have comp series size k. Then clearly M_1 has one length $k-1$. By induction M_1 has ACC, DCC. But of course M_0/M_1 has both ACC, DCC. So by 3.1(5) M has ACC, DCC. □

Lemma 4) Let M_R be a module which has a comp series, then any other series can be refined to a comp series by inserting extra terms as nec.

Ring Theory

$M_R = M_0 \supsetneq \dots \supsetneq M_k = 0$ is comp series if it cannot be lengthened

M_R has a comp series $\Leftrightarrow$ M_R has both ACC, DCC.

Lemma 4 If M has a comp series, then any series can be refined to a comp series.

pf: Let $M = A_0 \supsetneq \dots \supsetneq A_k = 0$ be a series of submodules of M. By (3) M_R has both ACC, DCC.

~~If you can~~, Choose a submodule B_1 with $A_0 \supseteq B_1 \supseteq A_1$ s.t. B_1 is minimal over A_1.

If $A_0 \neq B_1$ Choose B_2 s.t. $A_0 \supseteq B_2 \supsetneq B_1$ s.t. B_2 is minimal over B_1, by DCC, this stops.

Thus we obtain a chain between A_0, A_1 which cannot be lengthened. Repeat this between A_i, A_{i+1} to obtain a comp series. □

Thm 5 (Jordan-Hölder) Any two comp series of module of finite length are equivalent.

pf: For a module of finite length, let $\lambda(M)$ be the length of a shortest comp series. pf by induction on $\lambda(M)$.

If $\lambda(M) = 1$ then theorem holds trivially

Now assume that result holds for modules X_R s.t. $\lambda(X) \leq s-1$.

Now suppose M_R has a shortest comp series
$M = M_0 \supsetneqq \cdots \supsetneqq M_s = 0$ (i) Thus $\lambda(M) = s$.
Let $M = K_0 \supsetneqq \cdots \supsetneqq K_n = 0$ (ii) another comp series for M.

Since M_{s-1}, K_{n-1} have no proper submodules other than 0, we have either $M_{s-1} = K_{n-1}$ or else $M_{s-1} \cap K_{n-1} = 0$

Case 1: $M_{s-1} = K_{n-1}$. In this case, we have

$$\frac{M}{M_{s-1}} = \frac{M_0}{M_{s-1}} \supsetneqq \cdots \supsetneqq \frac{M_{s-2}}{M_{s-1}} \supsetneqq \frac{M_{s-1}}{M_{s-1}} = 0 \quad \text{(iii)}$$

$$\frac{M}{K_{n-1}} = \frac{M_0}{K_{n-1}} \supsetneqq \cdots \supsetneqq \frac{K_{n-1}}{K_{n-1}} = 0 \quad \text{(iv)}$$

These are composition series for $\frac{M}{M_{s-1}}$ and clearly $\lambda(\frac{M}{M_{s-1}}) = s-1$
by ind hypothesis $s-1 = n-1$ i.e. $s = n$.
and (iii) (iv) equivalent.

It follows that (i), (ii) equivalent.

Case 2: $M_{s-1} \cap K_{n-1} = 0$. So $M_{s-1} + K_{n-1}$ is direct sum, by (4) we can construct a comp series

$$M = Q_0 \supsetneqq \cdots \supsetneqq Q_{t-3} \supsetneqq M_{s-1} \oplus K_{n-1} \supsetneqq M_{s-1} \supsetneqq M_s = 0 \quad \text{(v)}$$

by ①: $s = t$ and (v) is equiv to (i).

$$M = Q_0 \supsetneqq \cdots \supsetneqq Q_{s-3} \supsetneqq M_{s-1} \oplus K_{n-1} \supsetneqq K_{n-1} \supsetneqq 0. \quad \text{(vi)}$$
(by ①: $s = n$ and (vi) is equiv to (ii).
is comp series equiv to (v) by inspection.

$\square$

3.3 Nil implies Nilpotent theorems.

$N(R) = \sum$ nilpotent ideals.
by Zassenhaus' example, we know $N(R)$ may not be nilpotent.

Let R be a ring, M minimal rt ideal of R. Then it is easy to check that $MJ = 0$ where $J = J(R)$ (check!). In particular $L(J) \neq 0$ $(*)$.

Thm 1) Let R be a ring w/ DCC on rt ideals. Then $J(R)$ is nilpotent.

pf: The chain $J \supseteq J^2 \supseteq J^3 \ldots$ stops by DCC.
so $\exists\, k$ s.t. $J^k = J^{k+1}$, in particular,
$$L(J^k) = L(J^{k+1})$$

Let $\bar{R}$ be the ring $R/L(J^k)$ and let bars denote images in $\bar{R}$. Clearly $\bar{J}$ is an rqr ideal in $\bar{R}$.
So $\bar{J} \subseteq J(\bar{R})$
Suppose $\bar{R}$ is non-zero ring. (for contra, it is 0).
Then by DCC $\bar{R}$ contains a minimal rt ideal.
so by comment above $(*)$, $\exists$ a non-zero elt $\bar{x} \in \bar{R}$ s.t. $\bar{x}\bar{J} = \bar{0}$. So in R, we have
$x J \subseteq L(J^k)$
so $x J^{k+1} = 0$ we have $x \in L(J^{k+1}) = L(J^k)$
Thus $\bar{x} = \bar{0}$ $\divideontimes$.
hence $\bar{R}$ must be zero ring. So $R \subseteq L(J^k)$
$J^{k+1} = 0$ ($J^k = 0$ if R has 1) and J is nilpotent. $\square$

Cor 2 [Hopkins] Nil one sided ideals are nilpotent in a ring w/ DCC on rt ideals.

pf They lie in $J(R)$.

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we showed R w/DCC on rt ideals $\Rightarrow J(R)$ is nilpotent

Cor [Hopkins] Nil one sided ideals are nilpotent on R w/ DCC on rt ideals. In particular $N(R)$ is nilpotent

pf $J(R)$ contains all nil one-sided ideals

Lemma3) (Utimi 1963) Let R be ring w/ ACC on rt annihilators. If R has non-zero nil one-sided ideal then R contains a non-zero nilpotent rt ideal.

pf: Suppose first R contains a non-zero nil left ideal A
Let $r(a)$ be max'l in the set $\{r(y) : 0 \neq y \in A\}$

claim $aRa = 0$: let $t \in R$ If $ta = 0$ then $ata = 0$
assume $ta \neq 0$. then $\exists$ int $k > 1$ s.t. $(ta)^k = 0$ and $(ta)^{k-1} \neq 0$, since $ta \in A$ which is nil.
So $ta \in r[(ta)^{k-1}]$ But clearly, $r[(ta)^{k-1}] \supseteq r(a)$
by maximality of $r(a)$, we must have
$r[(ta)^{k-1}] = r(a)$.

★ So $ta \in r(a)$ and thus $ata = 0$ always
This gives $(a)^3 = 0$ where (a) is rt ideal gen by $a \neq 0$.
Now suppose we have $0 \neq B \trianglelefteq R$ w/ B nil.
If $B^2 = 0$ then B is non-zero nilp rt ideal. Otherwise $\exists b \in B$ s.t. $Bb \neq 0$, Thus $Rb \neq 0$. Now Rb is a nil left ideal (check!).
Take $A = Rb$ in the first part to finish pf □

lemma 4) Let R be a ring w/ ACC on rt ideals.
Then R contains a unique max'l nilp ideal N and N contains all nilp one sided ideals of R.

pf Exercise

Thm 5) (Lentski). Let R be a ring w/ ACC on rt ideals
Then nil one sided ideals of R are nilp

pf: by (4) R has a unique max'l nilp ideal - N say.
Suppose that R has a nil one-sided ideal X s.t. $X \not\subseteq N$. Then $\frac{X+N}{N}$ is a non-zero nil one sided ideal of the rt Nh ring R/N By (3) R/N contains a non-zero nilp rt ideal. So R contains a non-zero nilp rt ideals which does not be inside N. ※ to (4)

Chapter 4: Semi simple A'n rings.

4.1 Idempotent Generators for Rt Ideals.

1) Def'n: A ring w/ no non zero nilpotent ideal and DCC on rt ideals is called a semi-simple A'n ring.

recall: M min rt ideal $\Rightarrow M^2 = 0$ or $M = eR$
$\therefore$ we don't allow M min rt ideals in defn.

Prop 2) Let R be a S.S. A'n ring $I \trianglelefteq R$. Then $I = eR$ for some $e = e^2 \in I$.

pf Trivial for $I=0$, so assume $I \neq 0$.

Claim. $\exists\ e=e^2 \in I$ s.t. $I \cap r(e)=0$.

by min condition, every non-zero rt ideal contains a minimal rt ideal and so by 2.7(3), contains a non-zero idempotent.

let $\mathcal{E}$ be set of all non-zero idempotents in I.

By above $\mathcal{E} \neq \emptyset$.

Suppose claim is false. Let $I \cap r(a)$ be minimal with the set $S=\{I \cap r(x) : x \in \mathcal{E}\}$

By assumption $I \cap r(a) \neq 0$. So by above $I \cap r(a)$ contains a non-zero idempotent $-b$ say. We have $b^2=b$ and $ab=0$.

Now consider $c=a+b-ba$ Then $c \in I$, as $a,b \in I$.

we have $ca=a \neq 0$. In particular $c \neq 0$.

$cb=(a+b-ba)b=b$. Hence

$c^2 = ca+cb-cba = a+b-ba = c$. $\therefore c \in \mathcal{E}$.

we shall next show, $I \cap r(c) \subsetneq I \cap r(a)$ — $(*)$

$t \in I \cap r(c) \Rightarrow ct=0 \Rightarrow act=0 \Rightarrow at=0$

$\Rightarrow t \in I \cap r(a)$.

Also, $b \in I \cap r(a)$ but $b \notin I \cap r(c)$. since $cb=b \neq 0$.

Thus $(*)$ is established. But $(*)$ contradicts the ~~num~~ minimality of $I \cap r(a)$ in S.

Thus our claim is proved & $\exists\ e \in \mathcal{E}$ s.t. $I \cap r(e)=0$.

pf: ~~Trivial if $I=0$, assume $I\neq 0$.~~
~~Claim. $\exists e = e^2 \in I$ s.t. $I \cap r(e) = 0$~~

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Correction: 3.3(i) pf has only J's no N

Prop 2) ~~pf~~ R s.s. A'n
$I \trianglelefteq R \Rightarrow I = eR$ for some $e = e^2 \in I$.

pf: continued: We have shown by contradiction that
$\exists e = e^2 \in I$ s.t. $I \cap r(e) = 0$
Now, for some $x \in I$
$$x - ex \in I \cap r(e) = 0$$
so $x = ex$ $\forall x \in I$. Thus $I \subseteq eR$
but $eR \subseteq I$ since $e \in I$ and $I \trianglelefteq R$
Thus $eR = I$ w/ $e = e^2 \in I$. □

Cor 3) Let R be a s.s. A'n ring and $A \trianglelefteq R$.
Then $\exists e = e^2 \in A$ s.t. $A = eR = Re$

pf: by (2) $A = eR$ for some $e = e^2 \in A$ since $A \trianglelefteq R$
let $K = \{x - xe : x \in A\}$ then $K \trianglelefteq_l R$.
we have $Ke = 0$ and so $KeR = 0$
hence $K^2 = 0$ since $K \subseteq eR = A$

Now $K = 0$ since R contains no non zero nilp left ideal so $x = xe$ $\forall x \in A$. Hence $A \subseteq Re$. But $Re \subseteq A$ as $e \in A$. and $A \trianglelefteq_l R$. Thus $A = eR = Re$ with $e = e^2$. □

Cor 4) A semi simple A'n ring has an identity.
pf: Take $A = R$ in (3). □

TFAE: for a ring R

Thm 5) (i) R is S.S. A'n
(ii) R has 1 and R_R is CR

pf: (i) $\Rightarrow$ (ii) by (4) R has 1.
Let $I \trianglelefteq_r R$ by (2) $I = eR$ for some $e = e^2 \in I$
So by peirce decomposition 2.6(4) I is a direct summand of R. (i.e. $\exists K$ s.t. $I \oplus K = R$).
Thus every submodule of R_R is a direct summand of R_R, So by 2.5.(6) R_R is CR.

(ii) $\Rightarrow$ (i) We have $R = \bigoplus_{\lambda \in \Lambda} I_\lambda$ where each I_λ is an irred submodule of R_R.
If R has 1, Λ is finite because
$1 = x_1 + \dots + x_n$ for some $x_i \in I_{\lambda_i}$ a finite sum.
in which case, for any $r \in R$, $r = 1r$
$r = x_1 r + \dots + x_r r \in I_{\lambda_1} \oplus \dots \oplus I_{\lambda_n}$.
So $R = I_{\lambda_1} \oplus \dots \oplus I_{\lambda_n}$.
$\therefore \Lambda$ is finite.
each I_{λ_i} has DCC trivially, so their direct sum does too (3.1(6)) $\therefore R_R$ has DCC on R-submodules $\therefore R$ has DCC on rt ~~modules~~ ideals

Now, let T be nilp rt ideal of R.
Then $R = T \oplus K$ for some $K \trianglelefteq_r R$ by 2.5(6).
We have $1 = t + k$ for some $t, k \in T, K$.
Now t is nilp so $t^m = 0$ for some $m \geq 1$
Thus $(1-k)^m = 0$
$1 - mk + \dots \pm k^m = 0$
$\therefore 1 \in K$. $(1 = mk + \dots \mp k^m)$.
So $K = R \therefore T = 0$.
$\therefore R$ is S.S. A'n. $\square$

Cor 6) Let R be S.S. Ar'n ring. Then $R = I_1 \oplus \ldots \oplus I_n$ where each I_j is a minimal rt ideal.

pf: This is shown in the above proof.

Cor 7) A direct sum of matrix rings over div rings is an S.S. Ar'n ring.

pf: we have shown in 2.5(7) that for such a ring R_R is CR.

4.2 Ideals in Semi-simple Ar'n rings

Prop 1) Let R be S.S. Ar'n ring. Then:
(i) Every ideal of R is generated by an idempotent which lies in $C(R)$ the centre of R.
(ii) There is a one to one correspondance between ideals of R and idempotents in $C(R)$

pf: (i) see 4(1)(3) and 2.6(8)(iii)
(ii) For $e = e^2 \in C(R)$ define $f(e) = eR \triangleleft R$
check that f is req bijection

Defn 2) An ideal I of R is said to be a min'l ideal if $I \neq 0$ and $I' \subsetneq I$ with $I' \triangleleft R \Rightarrow I' = 0$.

Thm: 3) Let R be a S.S. Ar'n ring. Then R has a finite # of min'l ideals. Their sum is direct and moreover it is R.

pf: Every non-zero ideal of R contains a min'l ideal, since R has DCC on rt. ideal. Let S_1 be a min'l ideal of R.

Then by (1) $S_1 = e_1 R = R e_1$ where $e_1^2 = e_1 \in C(R)$.

so $(1-e_1)^2 = 1-e_1 \in C(R)$ * why!?

and we have a direct sum of ideals

$$R = S_1 \oplus T_1 \text{ where } T_1 = (1-e_1)R = R(1-e_1).$$

(S_1 being two sided doesn't guarantee T_1 is).

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R semi-simple Ah $\Leftrightarrow$ R semi-prime + DCC on rt ideals.

$I \triangleleft_r R \Rightarrow I = eR$ w/ $e = e^2 \in I$

$R = I_0 \oplus \ldots \oplus I_n$ $\quad I_j$ minimal rt ideals.

$A \triangleleft R \Rightarrow A = eR = Re$ $\quad e^2 = e \in A$ $\quad e \in C(R)$

R has 1.

$R = S_1 \oplus \ldots \oplus S_m$ $\quad S_i$ minimal ideals.

and $S_1 \ldots S_m$ are only minimals ideals of R.

pf etd: $R = S_1 \oplus T_1$ with $T_1 = (1-e_1)R = R(1-e_1)$.

if $T_1 \neq 0$, T_1 contains a minimal ideal — S_2 say.

As above $R = S_2 \oplus K$ for some $K \triangleleft R$.

now $T_1 = T_1 \cap R = T_1 \cap (S_2 \oplus K)$

$= (T_1 \cap S_2) \oplus (T_1 \cap K)$ by modular law.

$= S_2 \oplus T_2$ $\qquad T_2 := T_1 \cap K$.

$R = S_1 \oplus S_2 \oplus T_2$

if $T_2 = 0$, we finish, else continue to $T_3, \ldots$

now $T_0 = R \supsetneq T_1 \supsetneq T_2 \supsetneq \ldots$

by DCC, must stop

can only stop when $T_m = 0$. A which point, we have $R = S_1 \oplus S_2 \oplus \ldots \oplus S_m$ direct sum of minimal ideals.

Now, let S be minimal ideal of R, $SR \neq 0$ as R has 1.

so $SS_j \neq 0$ for some j.

Now $SS_j \triangleleft R$. $\quad SS_j \subseteq S$, $SS_j \subseteq S_j$.

S, S_j minimal $\Rightarrow$ $S = SS_j = S_j$. $\quad \square$

§4.3 Simple A'n Rings

1) Defn: R is said to be a simple ring if 0 and R are only ideals of R.

A comm. simple ring is a field.

Let R be a simple ring, consider $R^2 \triangleleft R$.
So $R^2 = R$ or $R^2 = 0$.

Suppose $R^2 = 0$. Then $xy = 0 \ \forall x, y \in R$.
So any additive subgp of R is an ideal of ~~of~~ R. So the additive ~~subgp~~ gp has no subgp other than 0 and R.

hence the additive gp of R must be cyclic gp of prime order. Thus the structure of R is completely determined.
$R = \{0, 1, \ldots, p-1\}$ where addition is mod p, and product of any two elts is 0.

Thus, when studying simple rings $R^2 = R$.
(ofc $R^2 = R$ when R has 1).
Thus $N(R) = 0$ in this case.

A simple ring with DCC on rt ideals will be called a simple A'n ring (right hand conditions for now)
Thus a simple A'n ring is S.S. A'n.

2 (lemma) Let R be S.S. A'n ring and $0 \neq I \triangleleft R$. Then I itself is a S.S. A'n ring. In particular, when I is a minimal ideal, I is simple A'n ring.

pf) consider I as a ring itself.

Claim: $K \trianglelefteq I \Rightarrow K \trianglelefteq R$.

by 4.1(3) we have $I = eR = Re$ with $e^2 = e \in I$.

now, $k \in K$ and $r \in R \Rightarrow kr = (ke)r$ since e is the identity of I and $k \in K \subseteq I$

So $kr = ker \in K$ since $k \in K$, $er \in I$ and $K \trianglelefteq I$.

So the claim is proved.

It follows that the ring I has DCC on its rt ideals and $N(I) = 0$.

If I is minimal, then by above it must be a simple A'n ring. (Note that I is a ring with identity e). □

Thm 3) Let R be a S.S. A'n ring. Then R is expressible as finite direct sum of simple A'n rings and this expression is unique

pf) by 4.2(3) and (2) above, we have $R = S_1 \oplus \cdots \oplus S_n$ where each S_i is simple A'n ring.

The uniqueness follows from the fact that the S_i are precisely the minimal ideals of R □

§4.4 A'n Wedderburn thm

Look up §1.14

Routine to check, if $A_R \cong B_R$ as modules, then $\mathcal{E}_R(A) \cong \mathcal{E}_R(B)$ as rings.

recall 1.14(1) For a ring w/ 1, $R \cong \mathcal{E}_R(R_R)$ as rings

c.f. R+M 7.6.

lemma 1 $\mathcal{E}_R(X^{(n)}) \cong M_n(\mathcal{E}_R(X))$ as rings.

pf see handout H3. $\square$

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Thm 2 Artin-Wedderburn.

R s.s. Art'n $\iff R = S_1 \oplus \ldots \oplus S_m$.

where $S_i \cong M_{n_i}(D_i)$ for integer n_i and division rings D_i

pf: "$\Rightarrow$" $R = I_1 \oplus \ldots \oplus I_m$, minimal r.t. ideal

$\mathcal{E}_R(R_R) \cong M_n \mathcal{E}_R(I_1)$

$R \cong \mathcal{E}_R(R_R)$

$R = S_1 \oplus \ldots \oplus S_m$ where S_i is simple Art'n (4.3(3))

$S_i = I_1 \oplus \ldots \oplus I_{n_i}$ direct sum of min rt ideals for same n_i 4.1(6)

But $I_j \cong I_k$ $\forall j, k$ by Ex5q2, so

$S_i \cong I_1 \oplus \ldots \oplus I_1$ (n_i-times).

also $\cong \mathcal{E}_{S_i}(S_{i S_i})$ (1.14(1))

$\cong M_{n_i}(\mathcal{E}_{S_i}(I_1))$ by (1)

$= M_{n_i}(D_i)$ $D_i = \mathcal{E}_{S_i}(I_1)$ is a division ring by Schur's Lemma 2.8(2)

"$\Leftarrow$" Done in 2.5(7) $\square$

Thm 3 A s.s. Art'n ring is left-right symmetric.

pf: rt hand conditions $\iff$ R direct sum of matrix rings over division rings $\iff$ left hand conditions. $\square$

rmk: Another pf by symmetry is Example sheet 6 Qu 5.

§4.5 Rt A'n rings are rt N'n.

Lemma 1) Let R be a rt A'n ring and $N = N(R)$. then R/N is s.s. A'n.

pf: by Hopkins 3.3(2) N is nilpotent, it follows that R/N is s.s. A'n. □

Thm 2: Let R be a s.s. A'n ring and M_R a non-zero unital module. Then M_R is CR.

pf: we have $R = I_1 \oplus \dots \oplus I_n$ a direct sum of minimal rt ideals 4.1(6).
Let $m \in M$ Then $m = m1 \subset mI_1 \oplus \dots mI_n$
Each mI_j is either 0 or irreducable (check)
Thus each $m \in M$ lies in a sum of irred submod.
Hence M is CR. □

If M_R is a submodule, $I \triangleleft R$, is M_R an R/I-module?
Yes! if $MI = 0$ $\quad m[x+I] = mx \ \forall m \in M_R \ x \in R$
but only well defined if $MI = 0$.

Recall §1.13 M rt R-module, $I \triangleleft R$ st. $MI = 0$ then M is also a rt R/I-module and R R-submodules of M coincide with R/I-submodules.

Thm 3: (Hopkins) A rt A'n ring is rt N'n.

pf: $N := N(R)$. by ① R/N is s.s. A'n
By 3.3(2) $\exists$ a smallest integer $k \geq 1$ s.t. $N^k = 0$
Consider the chain $R \supsetneq N \supsetneq N^2 \supsetneq N^3 \dots \supsetneq N^k = 0$.
let $N^0 = R$ for notation.

each $\frac{N^j}{N^{j+1}}$ is a unital rt R/N module. $j=0,\dots,k-1$.

by (2) $\frac{N^j}{N^{j+1}}$ is CR. (possibly infinite)
As N^j/N^{j+1} is Ain rt Module, it must be a finite direct sum of irred submodules (check)
So by 3.1(6) each N^j/N^{j+1} is a Nn rt module for $j=0,\dots,k-1$.
Thus, in particular $N^{k-1}, N^{k-2}/N^{k-1}$ are Nn rt modules so by 3.1(5) N^{k-2} has ACC as a rt R-module. Proceed in this way, R_R has ACC as req. □

Cor A finitely generated module over a rt Ain ring has a composistion series.

pf: M has both ACC, DCC.

R semiprime + DCC $\Rightarrow$ Artin-Wedderburn.
DCC $\Rightarrow$ ACC.
R semiprime + ACC $\Rightarrow$??

§5 Quotient rings.

5.1 Def'ns and elementary properties

Defn 1) An elt $c \in R$ is said to be rt regular if $r(c)=0$
left regular $l(c)=0$, regular $r(c)=l(c)=0$.

A ring R is called a quotient ring if it has 1 and every reg elt of Q is an unit in Q.
Thus a div ring is quotient ring.

pf) Let Q be a rt Asn ring and $c \in Q$ s.t. $r(c)=0$. Then c is a unit. In particular Q is a quotient ring

pf ex

RT

Defn 3) Q be a ring w/ 1. R subring of Q. The ring Q is said to be a rt quotient ring of R if i) every reg elt of R is a unit of Q.
ii) every elt of Q is expressible
ac^{-1} $a, c \in R$ c reg.

Last time:
Q is a quotient ring if Q has 1 and every reg elt is unit. A rt Atn ring is a quotient ring.

semi prime + ACC on rt ideals $\Rightarrow$???

A comm int domain D always has a field of fractions F. The right quotient ring is an attempt to generalise.

Note that if Q is a rt q. ring of R, then Q is a q. ring as defined in (1(ii)).

Examples 2) $\mathbb{Q}$ is a q ring (field) of $\mathbb{Z}$, $2\mathbb{Z}$, ...

and ~~$\mathbb{Z}_{p\mathbb{Z}}$~~ $\mathbb{Z}_{(p)} = \{\frac{a}{c} : a, c \in \mathbb{Z},\ p \nmid c\}$

ex: if $D \subseteq D_1 \subseteq F$ and F is q ring of D, then it is also for D_1.

A left quotient ring is defined analogously.

Defn 5) A ring is said to be a rt order in Q if Q is a rt q. ring for R.

$\frac{1}{2}, \frac{1}{5}, \frac{1}{3} \Rightarrow \frac{15}{30}, \frac{6}{30}, \frac{10}{30}$'s anologous result ↓

Lemma 6) Suppose that R has a rt q ring Q. Let $c_1, \ldots, c_n$ be regular elts. Then $\exists\, r_1, \ldots, r_n, c \in R$ w/ c regular. such that
$$c_i^{-1} = r_i c^{-1} \quad \text{for} \quad c = 1, \ldots n.$$

pf: by induction on n. for $n=1$, take $r_1 = c_1$, $c = c_1^2$.
assume true up to $n-1$
we have obtained $t_1, \ldots, t_{n-1}, d$ w/ d reg
s.t. $c_i^{-1} = t_i d^{-1}$ for $i = 1, \ldots, n-1$.
Consider $d^{-1} c_n$ since ~~q~~ Q is rt q ring for R
$$d^{-1} c_n = r r_n^{-1} \quad r, r_n \in R \quad r_n \text{ regular.}$$
so $c_n r_n = dr = c$ say, c is regular $\because c_n, r_n$ regular

$$c_i^{-1} = t_i d^{-1} = t_i (r c^{-1}) = r_i c^{-1}$$
where $r_i = t_i r \in R$. for $i \in 1, \ldots, n-1$.
also $c_n^{-1} = r_n c^{-1}$.
This completes the proof.

Prop 7) Let R be a ring w/ a rt q ring Q.
Then
(i) if $I \trianglelefteq R$, then $IQ \trianglelefteq Q$.
and every elt of IQ is expressible as $x c^{-1}$ with $x \in I$, c reg in R.
(ii) if $K \trianglelefteq Q$ then $K \cap R \trianglelefteq R$ and
$$(K \cap R) Q = K.$$

pf: (i) clearly $IQ \trianglelefteq Q$. A typical elt of IQ is
$$\alpha = t_1 q_1 + \ldots + t_k q_k \text{ with } t_i \in I, q_i \in Q.$$
$$= t_1 a_1 c_1^{-1} + \ldots + t_k a_k c_k^{-1} \quad a_i, c_i \in R \quad c_i \text{ reg}$$

by (6): $= t_1 a_1 r_1 c^{-1} + \ldots + t_k a_k r_k c^{-1} \quad r_i \in R \quad c \text{ reg.}$
$$= (t_1 a_1 r_1 + \ldots + t_k a_k r_k) c^{-1}$$

$t_i \in I,\ a_i, r_i \in R$ $\therefore$ $t_i a_i r_i \in I$

$\therefore$ $\alpha = x c^{-1}$ for some $x \in I$.

ii) Exercise.

Cor 8) Suppose that R has a rt q ring Q. Then R rt $N_n \Rightarrow Q$ rt N_n.

pf: Follows from (ii) above.

Lemma 9) Let R_1, R_2 be rings w/ rt q-rings Q_1, Q_2 resp. Suppose $R_1 \cong R_2$, then $Q_1 \cong Q_2$

pf: Let θ be the iso between R_1, R_2. A typical elt of Q_1 is ac^{-1} with $a, c \in R$ and c regular

define $\Theta : Q_1 \to Q_2$

$$\Theta(ac^{-1}) = \theta(a)[\theta(c)]^{-1}$$

Note that $\theta(c)$ has to be regular in R_2.

well defined: $ac^{-1} = bd^{-1}$ $a,b,c,d \in R$ $c,d \in \mathrm{Reg}$

$$ac^{-1}d = b$$

Now $c^{-1}d = ef^{-1}$ $e,f \in R$, f reg.

~~ae = bf~~ $df = ce$

and hence $\theta(d)\theta(f) = \theta(c)\theta(e)$.

$\theta(f), \theta(c)$ regular in R_2.

$$\theta(c)^{-1}\theta(d) = \theta(e)\theta(f)^{-1}$$

~~Similarly~~ $b = aef^{-1}$

so $bf = ae$

similarly $\theta(b)\theta(f) = \theta(a)\theta(e)$.

$$\theta(b) = \theta(a)\theta(c)^{-1}\theta(d)$$

$\theta(d)$ regular in R_2 $\therefore$

$$\theta(b)\theta(d)^{-1} = \theta(a)\theta(c)^{-1}$$

$\therefore$ well defined.

Similarly for the others.

Cor 10) The rt q ring is unique in the sense that if R has rt q rings Q_1 and Q_2 then the identity map on R can be extended to a ring iso between Q_1, Q_2.

pf: take θ as identity map on R, ~~Hence~~ ~~we~~ s

Hence, we speak of <u>the</u> rt q ring of R.

21.

RT: The Ore condition

review: ~~reason~~ R comm int domain, to construct a field of fractions, create pairs $[a,c]$. $c \neq 0$.

$$[a,c]+[b,d]=[ad+bc,\ cd].$$
$$[a,c][b,d]=[ab,\ cd].$$

but too many elements!

want $[1,2]\sim[2,4]$.

define $[a,c]\sim[\cancel{a}b,d] \Leftrightarrow ad=bc$.

Define $\frac{a}{c}$ = equiv class of $[a,c]$.

Defn 2) Let S be a non-empty subset of a ring R. S is mult. closed. if $s_1,s_2\in S \Rightarrow s_1s_2\in S$.

R ring w/ rt q ring Q. Let ~~...~~ $a,c\in R$, c reg.

(in Q) $c^{-1}a=a_1c_1^{-1}$ $\quad a_1,c_1\in R$ $\quad c_1$ reg.

(in R) $\boxed{ac_1=ca_1}$

Ore condition; "right multiplication of regulars"

Let S be as above. We say that R has the rt Ore condition wrt S if given $a\in R$, $s\in R$ $\exists a_1\in R, s_1\in S$ s.t. $as_1=sa_1$.

Ore's thm Let R be a ring with at least one regular elt.

R has rt q ring $\Leftrightarrow$ R has rt Ore condition w.r.t. its regular elts.

Pf later.

5.3: Integral Domain.

Def 1) A module M_R is said to be finite (Goldie) dimensional if it does not contain an infinite direct sum of submodules.

Ān, Nā modules are finite dimensional (check!)
R is called a rt finite dim'l ring if R_R is finite diml

Lemma 2) Let R be a ~~rt f.d.~~ ring and let $c \in R$ s.t. $r(c)=0$. Let $I \trianglelefteq R$ s.t. $cR \cap I = 0$. Then $I + cI + c^2I + \dots$ is a direct sum.

pf: check that the sum is direct.

recall that for us an 'int domain' need not be commutative. It is easy to see that if an int domain ~~D~~ has a rt q. ring ~~D~~, then D must be a division ring. (check).

Thm 3) Let R be an integral domain. Then R has a rt q div ring
$\iff R_R$ is finite dimensional.

pf: $\Rightarrow$ Let I, K be non-zero rt ideals of R. Let $a \neq 0, a \in I$ and $0 \neq c \in K$. By the rt Ore condition, $\exists\, a_1, c_1 \in R$ w/ $c_1 \neq 0$ s.t.
$$ac_1 = ca_1.$$
Thus $0 \neq ac_1 = ca_1 \in I \cap K$.
Hence R_R cannot contain an infinite direct sum

$\Leftarrow$ Let $a, c \in R$ w/ $c \neq 0$ if $a=0$ then we have.
$ac = ca$.
assume $a \neq 0$. Then $aR \neq 0$. by 2.
$aR \cap cR \neq 0$.
hence $\exists a_1, c_1 \in R$ s.t. c_1 regular ($c_1 \neq 0$).
s.t. $ac_1 = ca_1$
so by Ore's thm R has a rt q ring which must be a division ring.

rmk: (i) A comm int domain satisfies the above condition.
(ii) The above theorem is a special case of ~~Gol~~ Goldies thm.

Recall if K is a field, then the comm ring $K[x]$ is a PID. This was proved using Euclidean Algorithm on degree of poly.

Example 4) (G. Higman). There exists an integral domain which has a quotient ring on the left. but not on the rt.
Let F be a field, with monomorphism $F \to F$.
$a \mapsto \bar{a}$ $(a \in F)$ which is not an automorphism.
(see RT Lecture). ~~{0 map?}~~

Let $\bar{F} = \{\bar{a} : a \in F\}$. Then $\bar{F} \neq F$.
Taking x to be an indeterminant over F,
Let $R = F[x]$ be the ring of polys
$a_0 + \ldots + a_k x^k$ $k \geq 0$ $a \in F$ where the mult is defined by $xa = \bar{a}x$ $(a \in F)$ and the distribution laws.

It can be checked that

i) R is an int domain (degree arg)

ii) R has the Euclidean algorithm so that every left ideal of R is principal.
(The argument doesn't work on the rt).

By (3) R has a <u>left q. div ring</u>.

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RT: continuing the example.

R does not have the rt Ore condition w.r.t. non-zero elts.

consider $x+a, x^2 \in R$ with $a \in F \setminus \bar{F}$

suppose $\exists$ polynomials f, g s.t.

$$(x+a)f(x) = x^2 g(x)$$

$$(x+a)(b_0 + \dots + b_k x^k) = x^2(c_0 + \dots + c_{k-1}x^{k-1})$$

$$b_i, c_j \in F.$$

so: $ab_0 + (\bar{b_0} + ab_1)x + \dots + (\bar{b_{k-1}} + ab_k)x^k + \bar{b_k}x^{k+1}$

$$= \bar{\bar{c_0}}x^2 + \bar{\bar{c_1}}x^3 + \dots + \bar{\bar{c_{k-1}}}x^{k+1}.$$

so $\bar{b_k} = \bar{\bar{c_{k-1}}}$ $\quad b_k = \bar{c_{k-1}} \in \bar{F}$.

so now, $\bar{b_{k-1}} + ab_k = \bar{\bar{c_{k-2}}}$

This will give $a \in \bar{F}$ unless $b_k = 0$.

Hence $b_{k-1} = \bar{c_{k-2}}$

and so $b_{k-1} = \bar{c_{k-2}} \in \bar{F}$

⋮

and continue like this and force $b_i = 0 \ \forall i$.

so $f(x) = g(x) = 0$.

So R does not have rt ore condition.

rmk: (i) $R \oplus R^{op}$ will have q. ring on neither side

(ii) Infact Mal'cer has ~~produced~~ constructed an int domain which is not embeddable in any division ring.

prop 3) If R has a left q ring Q' and a rt q ring Q, then Q is also a left q ring and so $Q' \cong Q$.

pf: Consider arbitary elt $ac^{-1} \in Q$ $a, c \in R$ c reg. since R has the left ore condition, we have $c_1 a = a_1 c$ $a_1, c_1 \in R$ c_1-reg.

so $ac^{-1} = c_1^{-1} a_1 \in Q$.
thus Q is a left q. ring of R.
by 5.I(10) $Q \cong Q'$. $\square$

§6 Goldies Thms.

6.1. The singular submodule.

Defn1) A submodule E of M is said to be <u>essential</u> in M if $E \cap K \neq 0$ whenever K is a non-zero submodule of M.

Every non-zero ideal of a comm int domain is essential.

(key lemma) 2) Let E be an essential submodule of M_R. Let $a \in M$ and define:
$$F = \{ r \in R : ar \in E \}.$$
Then F is also an essential rt ideal.

pf: clear that $F \trianglelefteq R$.
let $0 \neq I \trianglelefteq R$.
If $aI = 0$ then $I \subseteq F \cap I$.

so $F \cap I \neq 0$, now assume $aI \neq 0$. Then aI is a non-zero submodule of M.

Hence $aI \cap E \neq 0$.

so $\exists\, x \in E$, $t \in I$ s.t. $0 \neq x = at$.

hence $0 \neq t \in F \cap I$ and thus.

$F \cap I \neq 0$. $\square$

Prop 3: Let M_R be a rt module.

define:

$Z(M) = \{m \in M : mE = 0$ for some E essential rt ideal of $R\}$.

(E depends on m).

Then $Z(M)$ is a submodule of M.

pf: check that $m_1, m_2 \in Z(M)$.

$\Rightarrow m_1 - m_2 \in Z(m)$ easy.

Let $m \in Z(M)$, $a \in R$ $\exists\, E$ essential rt ideal s.t. $mE = 0$.

define $F = \{v \in R : av \in E\}$

by (2) applied to R_R & E, F is essential rt ideal in R.

now $maF \subseteq mE = 0$.

hence $ma \in Z(M)$.

thus $Z(M)$ is a submodule of M $\square$

Defn 4) $Z(M)$ defined above is called the <u>singular submodule</u> of M.

$Z(R_R)$ is clearly an ideal. It is called the rt. singular ideal of R.

$Z'(R)$ the left singular ideal defined analogously.

In general $Z(R) \neq Z'(R)$.

lemma 5) Let R be a ring w/ ACC on rt annihilators.
then (i) $Z(R)$ is nil ideal
(ii) if R semi-prime, $Z(R) = 0$.

pf: let $z \in Z = Z(R)$.
claim: $\exists n \geq 1$ s.t. $z^n R \cap r(z^n) = 0$.

pf: $r(z) \subseteq r(z^2) \subseteq \dots$ stops so
$\exists n \geq 1$ s.t. $r(z^n) = r(z^{n+1}) = \dots = r(z^{2n})$
let $y \in z^n R \cap r(z^n)$.
Then $y = z^n t$ for some $t \in R$.
and $z^n y = 0$.
$z^{2n} t = 0$ and $t \in r(z^{2n}) = r(z^n)$.
hence $y = z^n t = 0$
Q.E.D in claim.

<u>proof Hbc..</u>

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RT

~~Connection~~: rt singular ideal $Z(R)=\{z\in R:$
$r(z)$ is essential in $R\}$.

Lemma: R has ACC on rt annihilators, then:
- Z is nil ideal.
- R semi-prime $\Rightarrow Z=0$.

pf i) ctd.
for $z\in Z$ then $\exists n\geq 1$ s.t. $z^nR\cap r(z^n)=0$
but $z^n\in Z$ since $Z\trianglelefteq R$. Hence $r(z^n)$ is essential.
It follows that $z^nR=0$ so $z^{n+1}=0$
and Z is a nil ideal.

(ii) Follows from Utumi's lemma 3.3(3). □

6) Lemma: Let R be a rt f.d. ring
and $c\in R$ s.t. $r(c)=0$.
Then cR is an essential rt ideal.

pf: Since R_R is a ~~rt~~ f.d. $cR\cap I\neq 0$ for any non-zero rt ideal by 5.3(2). □

Defn R is called a rt Goldie ring if R_R is f.d. and R has ACC on rt ~~ideals~~ annihilators

A comm int domain is trivially a Goldie ring.
A rt Nñ ring is a ~~[illegible]~~ Goldie ring.

lemma 8) Let R be a semi-prime rt Goldie ring and $c \in R$. Then $r(c) = 0 \Rightarrow l(c) = 0$.

pf: by (6) cR is essential rt ideal.
by 5ii $Z(R) = 0$. It follows that $l(c) = 0$.
(as killed by essential rt ideal).

rmk: $l(c) = 0 \not\Rightarrow r(c) = 0$ in the above ring.
~~6.2: Goldies~~

6.2 Goldies thms

1) <u>key prop</u>: every essential rt ideal of a semi-prime rt Goldie ring contains a regular elt.

pf: Let E be an essential rt ideal of R.
Then by 3.3(3), E is not nil.
Choose $x_1 \in E$ s.t. $r(x_1)$ is max'l in the sense $\{r(x) \mid 0 \neq x \in E,\ x \text{ is not nilpotent}\}$.
Then $r(x_1) = r(x_1^2)$.

If $E \cap r(x_1) = 0$, then $r(x_1) = 0$. So by 6.1 (6) $x_1 \in E$ is regular elt.

~~If $E \cap r(x_1) = \phi$, then by ... as above~~
If $E \cap r(x_1) = \phi$, then as above $E \cap (r(x_1))$ is not nil. Chose $x_2 \in r(x_1) \cap E$. s.t. $r(x_2)$ is max'l in $\{r_1(x) : 0 = x \in E_{\text{nongy}}$

$r(x_2) \neq r(x_2^2)$.

claim: $v: r(x_1+x_2) = r(x_1) \cap r(x_2)$ clearly
$r(x_1) \cap r(x_2) \subseteq r(x_1+x_2)$.
and $(x_1+x_2)y = 0 \quad y \in R$
$\Rightarrow x_1 y = -x_2 y$.

$\Rightarrow x_1^2 y = -x_1 x_2 y = 0$
$\Rightarrow x_1^2 y = -x_1 x_2 y = 0$
$\Rightarrow y \in r(x_1^2) = r(x_1)$.

$=$

$(x_1+x_2)y = 0 \quad y = R \Rightarrow x_1 y = -x_2 y$

$\Rightarrow x_1^2 y = 2 r_1 r_2$

$= \Rightarrow x_1 y = x_2 y = 0$, proving the claim"
The same argument holds, and shows that
$\therefore x_1 R + x_2 R -$

If $r(x_1+x_2) \neq 0$, then $E \cap r(x_1) \cap r(x_2) \neq 0$.
Now choose $x_3 \in E \cap r(x_1) \cap r(x_2)$. and
~~and $r(x_1)$~~ check that the process
repeats once again. $r(x_1+x_2+x_3) =$
$r(x_1) \cap r(x_2) \cap r(x_3)$
is a direct sum.

Since R_R is a f.d. the procedure
cannot contribute indefinitely
Thus we obtain f.d. The procedure
cannot continue indefinitely
$\therefore$ we obtain $c = x_1 + \ldots + x_n$ s.t. $c \in E$ $r(c) = 0$
by 6.16 c is a reg elt.

lemma 2: If K is a nilpotent ideal, then $l(K)$ is essential as a rt

pf: Let $0 \neq I \subsetneq R$ if $IK = 0$, then $0 \neq I \subseteq I \cap l(K)$

otherwise $\exists n > 1$: $IK^n = 0$ but $IK^{n-1} \neq 0$
Then $0 = IK^{n-1} \subseteq I \cap l(K)$.

$l(K)$ is essential as a rt module

lemma 3: R ring w/ 1 : Q ring Q
- E essential rt of $R \Rightarrow$ EQ essential rt ideal of Q_R.
- essential rt ideal of R. $\Rightarrow E(Q)$ essem,

pf ③ (i) essential rt ideal of $R \Rightarrow EQ$ as essar. ?

(ii) semi-prime + δ

(exercise as semi-prime + ACC

Goldie 1960 map R has a semisimple

Thm 4 An rt q map

$\Leftrightarrow$ R is a ~~sem~~ semiprime rt Goldie ring

pf: $\Leftarrow$ Let $a, c \in R$ w/ c reg

By 6.1(i), cR is essential

in R. Let $F = \{r \in R : ar \in cR\}$.

Then ~~F~~ $F \triangleleft R$.

and by 6.1(2) F_R is also essential.

So by (1) F contains a regular elt.

– c_1 say.

So $a_1 = a_i$

So $ac_1 = ca_1$ for some ~~b~~ $a_1 \in R$.

Thus by ~~Ore~~ Ore's ~~condition~~ th'm,

R has a ~~rt~~ rt q ring – Q say.

RT: Goldie's thm pf.

recall: R semi prime rt Goldie ring (i.e. R_R fin dim + ACC on rt annihilators).

Then every essential rt ideal of R contains a regular elt. $c \in R$ $r(c)=0 \Rightarrow cR$ is essential. and $l(c)=0$.

Since cR is essential, so is $F = \{r \in R : ar \in cR\}$.

So F contains a reg. elt $- c_1$ say.

thus $\boxed{ac_1 = ca_1}$ for some $a_1 \in R$ D = rt ore cdn.

Goldie's thm: ring R has a s.s. Ar'n rt quot. ring $\Leftrightarrow$ R is a semi-prime rt Goldie ring.

pf $\Leftarrow$ ctd. by Ore's thm R has a rt q. ring Q say. Let G be an essential rt ideal of Q.

by (3) $G \cap R$ is an essential rt ideal of R.
by (1) $G \cap R$ contains a regular elt.
So G contains a unit of Q.
Hence $G = Q$. Thus every rt ideal of Q is a direct summand. of Q using Q7 HW2.
Also Q has 1.

Thus Q is s.s. Ar'n.

"$\Rightarrow$" Let E be an essential rt ideal of R, by (3) EQ is an essential rt ideal of Q. But Q is s.s. Ar'n so $EQ = Q$.

So $1 \in EQ$. Hence $1 = xc^{-1}$ for some $x \in E$ and $c \in R$ regular. so $x = c \in E$.

$\therefore$ every essential rt ideal of R contains a reg elt.

Let K be a nilpotent ideal of R. By (2) $l(K)$ is essential rt ideal of R.
and so contains a reg elt.
Thus $K=0$ and R is semi-prime.
It is easy to see that direct sums in R extend to direct sums in Q.
Thus R_R is finite dim'l.
Finally for any annihilator rt ideal.
$r_R(T)$ (some $T \subseteq R$).
we have $r_R(T) = r_Q(T) \cap R$. — (*)
Now Q has ACC on rt ideals (why?)
~~(simplest: every ideal~~
Using (*) R has ACC on rt annihilators. □

This proof is also due to Goldie 1967.

Q: Semi-prime + DCC ⇒ Q s.s. A'n
Direct sum of matrix rings over division rings.
⇓
(R Semi-prime) + ACC ⇒ R has a rt ~~eq~~ q. ring Q s.t. above applies.

6.3 The prime case.

lemma 1) Let R be a ring w/ rt q ring Q. Suppose that Q is rt N'n. Then $A \triangleleft R \Rightarrow AQ = Q$.

pf: let c be a reg. elt of R. Consider chain
$$AQ \subseteq c^{-1}AQ \subseteq c^{-2}AQ \subseteq \ldots \subseteq Q$$
It's an ascending chain ~~of~~ of rt ideals in Q.
Q is rt N'n ∴ it must stop. $\exists k \geq 1$ s.t.
$$c^{-k}AQ = c^{-k-1}AQ = \ldots$$

So $AQ = c^{-1}AQ$. (mult by c^k).

Line 1 Clearly $AQ \subseteq Q$.

It follows $AQ \trianglelefteq Q$. Thus $AQ \triangleleft Q$. □

Thm 2: (Goldie 1958) The ring R has a simple A'n rt q. ring $\iff$ R is a prime rt Goldie ring

pf. $\Leftarrow$ by 6.2(4) R has a s.s. A'n rt q. ring Q.
Let $AB = 0$ w/ $A, B \triangleleft Q$.
then $(A \cap R)(B \cap R) = 0$ in R.
So $(A \cap R) = 0$ or $(B \cap R) = 0$.
Hence $A = (A \cap R)Q = 0$ or $B = (B \cap R)Q = 0$
Thus Q is a prime ring.
prime + s.s $\Rightarrow$ simple. or you can use.
by Q1HW6: Q is simple A'n.

$\Rightarrow$ by 6.2(4). R is a semiprime rt Goldie ring.
Let $AB = 0$ where $A, B \triangleleft R$. $B \neq 0$.
by (1) $BQ \triangleleft Q$. so $BQ = Q$ since Q simple.
Hence $0 = AB = ABQ = AQ$ Thus $A = 0$.
and R is a prime ring. □

Chapter 7: On the Jacobson problem (conjecture)

Assume all rings in this chapter have 1.

7.1: The commutative case:
- Primary Decomposition.

Defn 1) Let $I \triangleleft R$, then I is said to be <u>meet-irreducible</u> if $I = A \cap B \quad A, B \triangleleft R$.
$\Rightarrow I = A$ or $I = B$.

RT

recall: • Every essential rt ideal of a semi-prime rt Goldie ring contains a regular elt.

• An ideal I is meet-irreducible if $I = A \cap B$, $A, B \triangleleft R$
$\Rightarrow I = A$ or $I = B$

prop 2) Let R be a ring w/ ACC on ideals. Then every ideal of R is expressible as a finite intersection of meet-irred ideals

pf Suppose not. By ACC $\exists$ a max'l counterexample $I \triangleleft R$
Then in particular, I is not meet irreducible. So $\exists$ ideals A, B in R s.t. $I = A \cap B$ w/ $A \supsetneq I$, $B \supsetneq I$.

Since I is max'l counterexample, both A and B are finite intersections of meet irreducible ideals. Hence so is I ※. Hence no such counterexample exists. □

|| Assume all rings are commutable for this section.

Defn 3) An ideal Q of R is said to be primary if.
~~$a b \in$~~ $ab \in Q$ $a, b \in R \Rightarrow a \in Q$ or $b^n \in Q$ for some $n \geq 1$.

• Clearly, prime ideal is primary.
• If $R = F[x]$, F a field then $x^2 R$ is primary.

• R is called a primary ring if 0 is a primary ideal.
R is said to have a primary decomposition if every ideal of R is a finite intersection if primary ideals.
Let $Q \triangleleft R$, Note Q is primary
$\Leftrightarrow R/Q$ is primary ring.

RT (ctd).

Thm 4 (Noether) Every N'n ring has a primary decomp.

pf: by (2) it is enough to show that a meet-irred ideal is primary.
WLOG. assume that 0 is meet-irred.
let $ab=0$ $a,b\in R$. Then by Fitting's Lemma (in 6.1.5)
$\exists n\geq 1$ s.t. $b^nR \cap r(b^n)=0$.

Since 0 is meet. irred. either $b^nR=0$ or $\operatorname{ann}(b^n)=0$.
$\therefore$ $b^n=0$ or $a=0$.
so 0 is a primary ideal. $\square$

Defn 5: Let Q be a primary ideal. Let P/Q be a nilpotent ~~redu~~ radical of R/Q. Then P is called the <u>radical</u> of Q and Q is said to be P-primary. We denote the radical by $\sqrt{Q}$. or $\sqrt[4]{Q}$.

Recall that in a comm ~~case~~ ring, $N(R)$ = set of all nil elts of R, it is easy to see that a f.g. nil ideal in a comm ring is nilp.

Prop 6 Let Q be a primary ideal and $P=\sqrt{Q}$. Then
i) P is prime
ii) R N'n, then $P^k\subseteq Q$ for some $k\geq 1$.

Pf Let $ab\in P$ with $a,b\in R$. Then $(ab)^n\in Q$ for some $n\in\mathbb{N}_{\geq 1}$. so $a^nb^n\in Q$.
If $a\notin P$, then $a^n\notin Q$. so $(b^n)^s\in Q$ for some $s\geq 1$. by defn of primary. Thus $b\in P$. $\therefore$ P prime ideal.

*? ii) P/Q is a f.g. nil ideal of R/Q and hence is nilpotent. □

Thm 7) Let R be a comm N'n ring. Then $\bigcap_{n=1}^{\infty} J^n = 0$, $J = J(R)$.

pf Let $X = \bigcap_{n=1}^{\infty} J^n$. We can use Nakayama's Lemma.

Let $XJ = Q_1 \cap \dots \cap Q_n$ with $P_i = \sqrt{Q_i}$ be a primary decomp. by 6 (ii) $\exists k \geq 1$ s.t. $P_i^{k_i} \subseteq Q_i$.
Now that $X \not\subseteq Q_i$. Then $J \subseteq P_i$ since Q_i is P_i primary.
But $X \subseteq J^{k_i}$ so $X \subseteq Q_i$ for all i, in any case.
Thus $X \subseteq XJ$. So $X = XJ$.
Now $X = 0$ by Nakayama's Lemma. □

Given this, Jacobson wondered if $\bigcap_{n=1}^{\infty} J^n = 0$, also in non-comm (rt) N'n rings.
Not true for rt N'n rings: Herstein's example (1965).
~~ESH~~ HW5Q5.
It is still open for N'n rings.
We can define a topology on R where the powers of J are the open sets. This topology is Hausdorff $\iff \cap_{n=1}^{\infty} J^n = 0$.

Def'n 8) A comm ring R is said to be local if $J(R)$ is it's unique max'l ideal.
Let R be a local ring. It's easy to see that $u \in R$ is a unit $\iff u \notin J$.

Thm 9: Let R be a comm Nn ring s.t. $J = aR$ for some $a \in J(R)$. Then R is either a principal ideal domain (PID) or an An principal ideal ring.

pf We shall show that every proper non-zero ideal of R is a power of J. We have $J^m = a^m R$ $\forall m \geq 1$.

RT

recall: ~~F~~For problems to do w/ q rings of matrix rings use:
Let R be a comm int dom and $A \in M_n(R)$.
Then $A(\text{adj } A) = (\text{adj} A)A = |A| I_n$.

For a comm. Nh ring $\bigcap_{n=1}^{\infty} J^n = 0$.
For non-comm Nh rings?

Thm 9: R comm Nh local ~~ring~~ ~~ideal~~. $J = aR$ for some $a \in J$.
Then R is either a PID or an Ah principal ideal ring.

pf we shall show that every proper non-zero ideal of R is a power of J.
Firstly $J^m = a^m R$ for all $m \geq 1$.
Let $I \triangleleft R$ $R \supsetneq I \neq 0$. Then ~~...~~ $I \subseteq J$.
and since $\bigcap_{n=1}^{\infty} J^n = 0$, $\exists k \geq 1$ s.t. $I \subseteq J^k$ but $I \nsubseteq J^{k+1}$. choose $x \in I$ s.t. $x \notin J^{k+1}$. Then $x = a^k t$ for some $t \in R$ w/ $t \notin J$.

So t is a unit of R. Hence $a^k = xt^{-1} \in I$.
so $J^k \subseteq I \subseteq J^k$ $\therefore I = J^k = a^k R$ and I is principal.

2 cases:

① Suppose J is not nilpotent. Let a, b be non-zero, non-units. in R. Then $aRbR = J^\alpha J^\beta = J^{\alpha+\beta} \neq 0$ for some $\alpha, \beta \geq 1$. Hence R is an int domain.
Thus R is a PID.

② Suppose J is nilpotent. Let $J^s = 0$.
Then $R, J, J^2, \ldots, J^{s-1}$ ~~are~~, $J^s = 0$ are the only ideals ~~of~~ of R. So R is Ah. $\square$

10) $\boxed{\text{Noether's example}}$ $R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$. R A'n.

~~Pxt~~ 0 is meet-irred ideal, but not primary in any useful way.

7.2 Rings of Krull dim 1.

Defn: 1) The Gabriel-Rentschler Krull dimension of M_R is defined as follows.

Krull dimesion: K dim

$K\dim M = -1$ if $M = 0$

$K\dim M = 0$ if $M \neq 0$ A'n.

$K\dim M = \alpha$ (α an ordinal) if $K\dim M \not< \alpha$ and for all chains $M_1 \geq M_2 \geq \ldots$ of submodules $\exists n \geq 1$ s.t. $K\dim(M_i/M_{i+1}) < \alpha$ for all $i \geq n$.

ex: $K\dim \mathbb{Z} = 1$.

$K\dim k[x] = 1$ k field.

in fact, $K\dim R = 1$ for any R comm. PID.

Lemma 2) Let R be a ring s.t. $K\dim R_R \leq 1$. and let $c \in R$. s.t. $r(c) = 0$. then R/cR is an A'n R-module.

pf: consider $R \geq cR \geq c^2R \geq \ldots$

Since $K\dim R \leq 1$ $\exists n \geq 1$ s.t. $c^nR/c^{n+1}R$ is A'n.

but $R/cR \cong c^nR/c^{n+1}R$ since $r(c) = 0$ (check).

Thus $(R/cR)_R$ is A'n.

Lemma 3): Let M_R be A'n, N'n. (i.e. let M_R be a module w/ a comp series). Then $MJ^n = 0$ for some $n \geq 1$.

Let $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_n = 0$ be a comp series.
Then each M_i/M_{i+1} is irreducible. $i = 0, \dots, n-1$.

Now $\left(M_i/M_{i+1}\right) J = 0$ (see 3.3)
It follows that $MJ^n = 0$

Thm 4) Let R be a prime N'n ring w/ $\text{Kdim } R \leq 1$.
Then $\bigcap_{n=1}^{\infty} J^n = 0$.

pf: If $E(R)$ = sum of all min'l rt ideals, socle $\neq 0$.
then R is simple A'n [HW8, Q2].
In this case $J = 0$.
So now assume that $E(R) = 0$.

Then $\bigcap F = 0$ where F runs over all essential rt ideals of R [Q1 HW5].
Now F contains a reg elt, c say [6.2(i)]
By (2) R/cR is A'n. So R/F is A'n.
So by (3) $\exists\, n_F \geq 1$ s.t. $J^{n_F} = RJ^{n_k} \subseteq F$

Hence $\bigcap_{n=1}^{\infty} J^n \subseteq \bigcap_{\substack{F \text{ ess.} \\ \text{rt. ideals}}} F = 0$. $\square$

<u>Thm of Lenagan</u>: R N'n $K \dim R \leq 1$
$\Rightarrow \bigcap_{n=1}^{\infty} J^n = 0$.

R N'n $K \dim R \leq 2$ <u>still open</u>.

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RT

(no lec friday).

Krull dim: Just learn the finite case.
we only used it for $n=1$.

Exam will be same style as last years.

5.2 The Ore Condition.

Lemma 2) Let S be a mult closed subset of R
Suppose that S consists of reg elts of R
and that R has the rt Ore conditn wrt S.
Let $(x,c),(y,d)$ and $(r,s) \in R\times S$ s.t.
$cr = ds$ and $xr = ys$.
Then $ca = db \Rightarrow ya = yb$ $\forall a,b \in R$.

pf. Since R has the rt Ore cond'n. wrt S
$\exists (\lambda,\mu) \in R\times S$ s.t. $b\mu = s\lambda$.
Now $ca\mu = db\mu = ds\lambda = cr\lambda$
so $a\mu = r\lambda$ since $r(c)=0$.
Since $xa\mu = xr\lambda = ys\lambda = yb\mu$.
so $xa = yb$ since $l(\mu)=0$. $\square$

rmk: observe that we need both l, r regularity of S.

Thm 3) (Ore 1931) Let R be a ring w/ at least one reg elt.
Let S be the set of all reg elts of R.
R has a rt q ring $\Leftrightarrow$ R has rt Ore condn wrt S.
Q.

pf: Let $a, c \in R$ w/ c regular.
Then $c^{-1}a \in Q$. So $\exists a_1, c_1 \in R$ w/ c_1 regular
s.t. $c^{-1}a = a_1c_1^{-1}$ by defn of a rt q ring.
so $ac_1 = ca_1$

(last page in note pad sorry for handwriting).

$\Leftarrow$ Define an equiv reln on $R \times S$ as follows
$(x,c) \sim (y,d) \iff \exists\, (r,s) \in R \times S$ s.t. $cr = ds$ $\quad xr = ys$
To check $\sim$ is equiv reln:
Clearly $(x,c) \sim (x,c)$. [reflexivity]?

symm
Suppose $(x,c) \sim (y,d)$. By rt Ore: $\exists\, r_1, s_1 \in R \times S$.
$dr_1 = cs_1$ (i)
since $(x,c) \sim (y,d)$. $\exists\, (r,s) \in R \times S$ s.t. $cr = ds$, $xr = ys$ (ii)
by (2), (i), (ii) $\Rightarrow$ $yr_1 = xs_1$ (iii).
thus $(y,d) \sim (x,c)$ by (i) and (iii).

trans-itivit-y
Now suppose $(x,c) \sim (y,d) \sim (z,e)$.
by rt Ore condition: $\exists\, (r,s)$ w/ $cr = es$.
$\exists\, (r_2, s_2)$ in $R \times S$ s.t. $dr_2 = (es)s_2$.
by (2), this gives $yr_2 = xrs_2$. Again, ~~this gives~~ (2) and $dr = ess_2$ $\Rightarrow yr_2 = zss_2$
so $xrs_2 = yr_2 = zss_2$ $\therefore xr = zs$. since s_2 is regular. $\therefore$ transitive

Denote equiv class of (x,c) under $\sim$ by x/c.
$Q = \{x/c : (x,c) \in R \times S\}$. define operations so Q is rt q ring of R.
$x/c + y/d = \frac{xr + ys}{ds}$ w/ (r,s): $cr = ds$... we must check $+$ is well defined
$(x,c) \sim (x',c')$, $(y,d) \sim (y',d')$. By rt ore cdn: $\exists\, r', s'$: $cr' = d's'$. $\exists\, \rho, \sigma$ s.t.
$ds\rho = d's'\sigma$. by (2) $ys\rho = y's'\sigma$. Now $cr\rho = ds\rho = d's'\sigma = c'r'\sigma$
by (2) $xr\rho = x'r'\sigma$. $\therefore (xr + ys)\rho = xr\rho + ys\rho = x'r'\sigma + y's'\sigma = (x'r' + y's')\sigma$.
$\therefore +$ is well defined. Proceeding w/ similar techniques, it can be shown that
$(Q,+)$ is Abelian gp. Next given $x/c, y/d \in Q$ define $\frac{x\lambda}{d\mu}$ w/ (λ, μ) s.t. $y\mu = c\lambda$.
$\therefore$ can be shown $(Q,+,\cdot)$ is ring.

pf of ore's thm is not examinable.