Course Code: MAT 211

Course Title: Coordinate Geometry and Vector Analysis

Course Instructor

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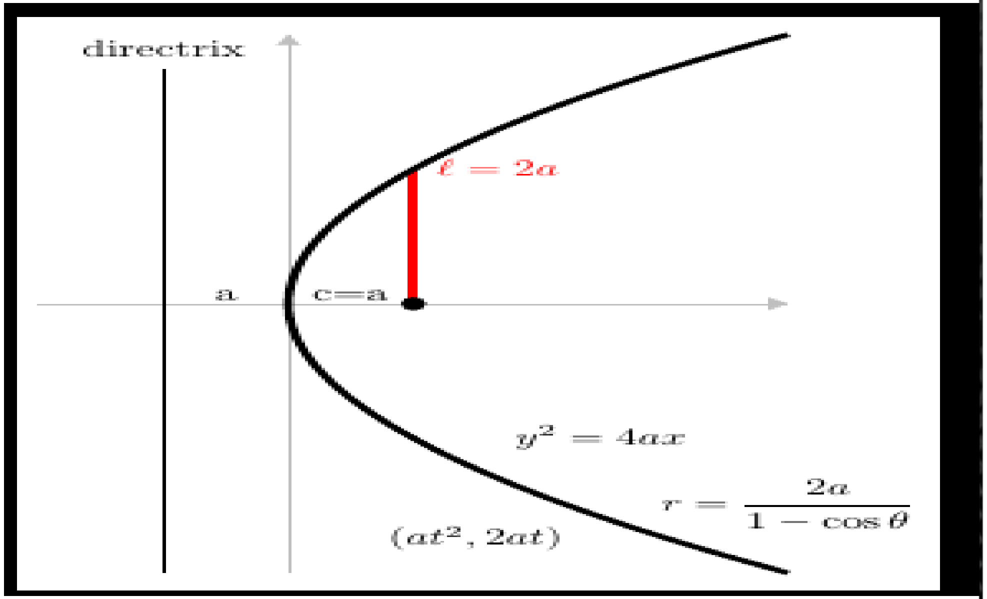
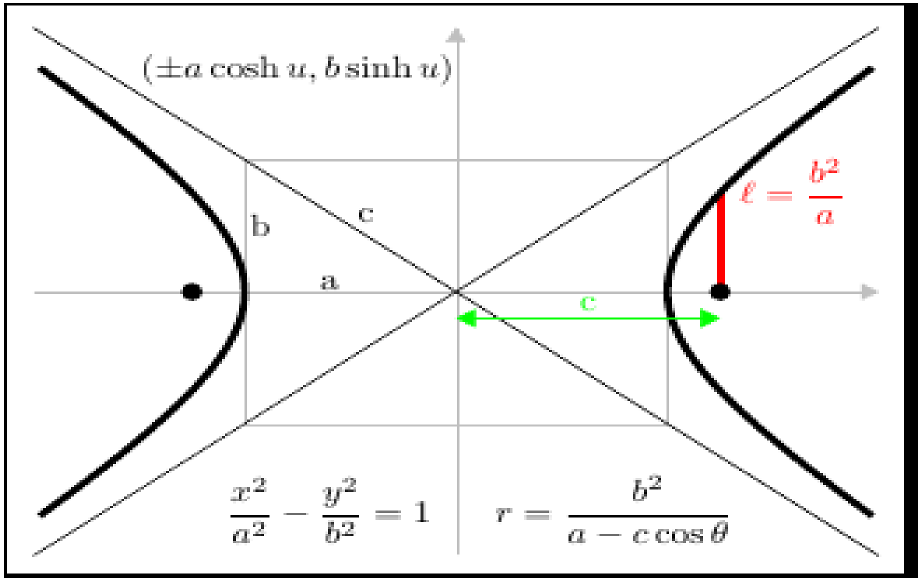
Standard forms of a parabola

$$y^2 = 4ax$$



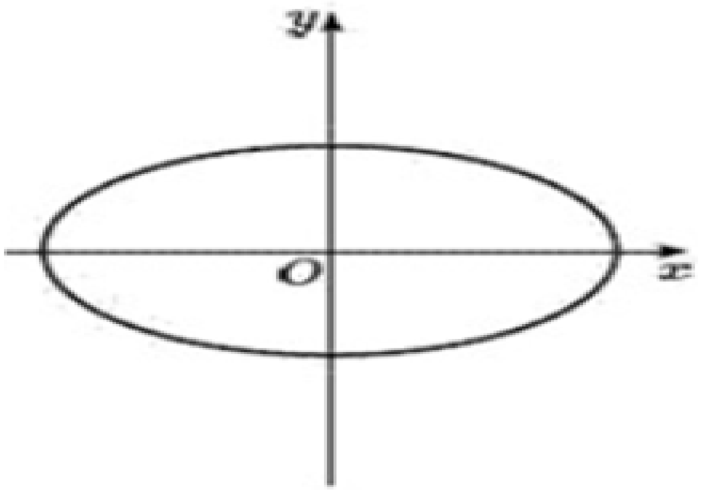


Standard forms of a hyperbola

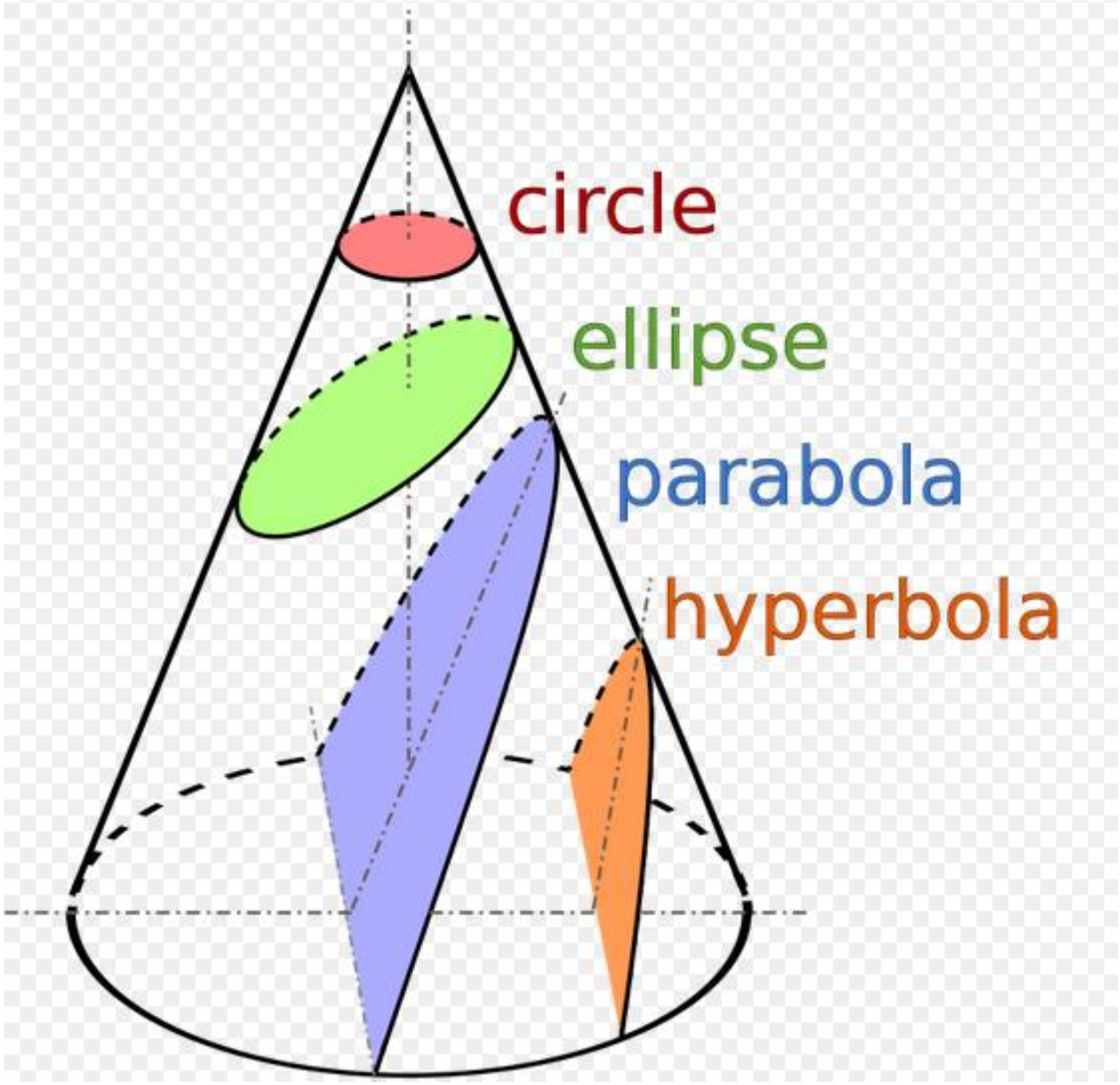
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

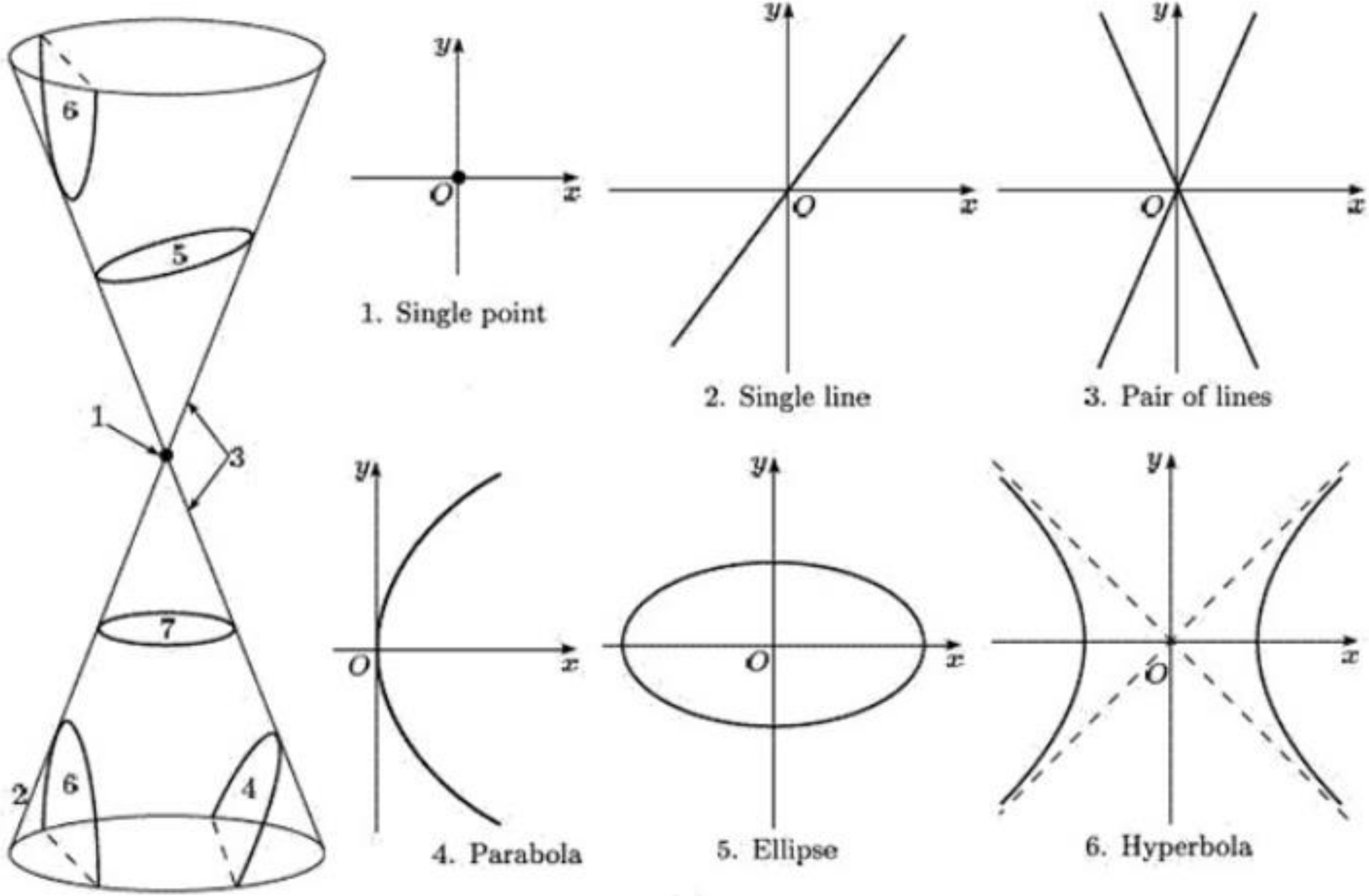
Standard forms of a Ellipse

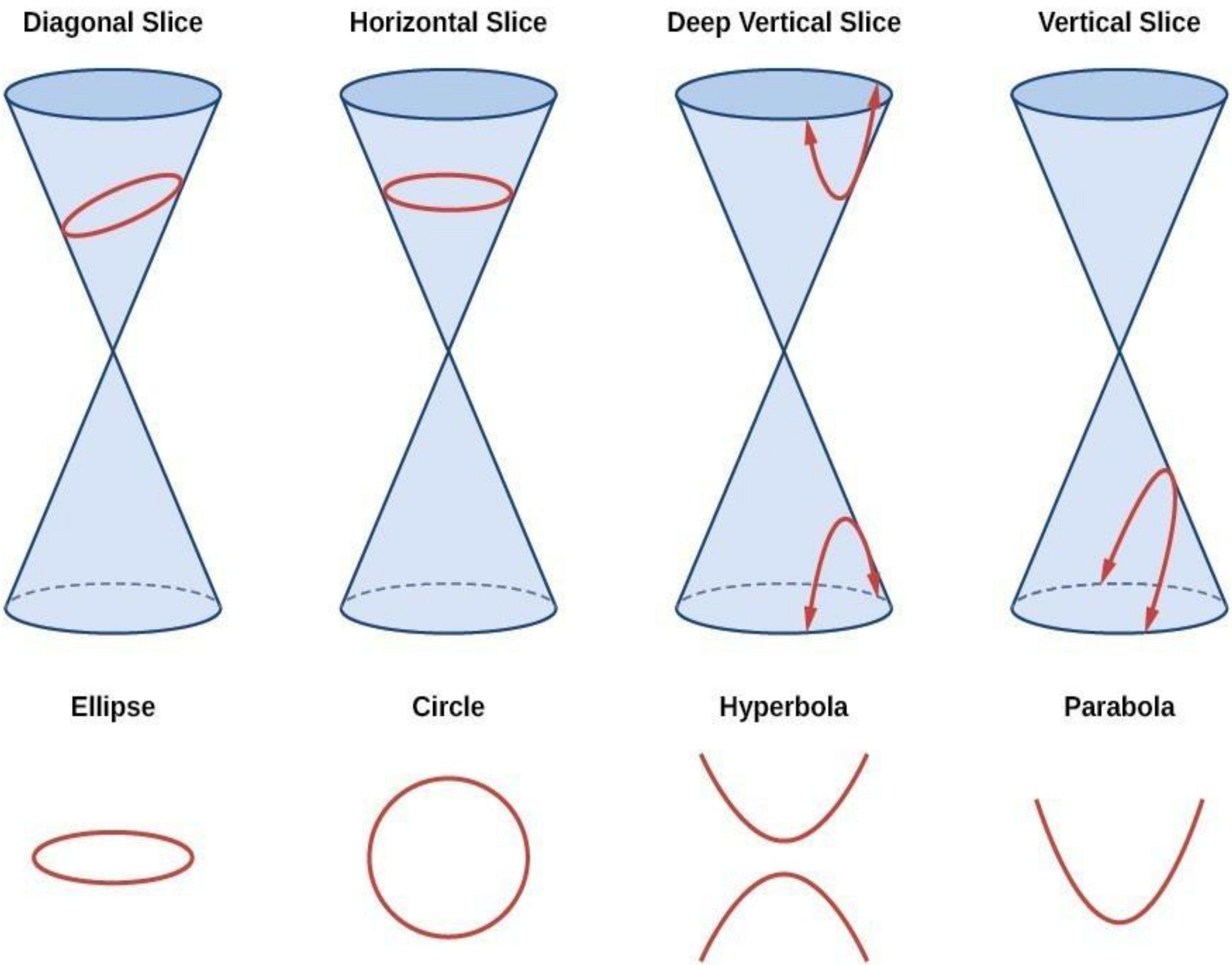
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Ellipse


circle
ellipse
parabola
hyperbola


1
3
2
4
5
6
6
7
1. Single point
2. Single line
3. Pair of lines
4. Parabola
5. Ellipse
6. Hyperbola
y
x
O


Diagonal Slice
Horizontal Slice
Deep Vertical Slice
Vertical Slice
Ellipse
Circle
Hyperbola
Parabola

General Equation of Second degree

The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent

1 (i) a pair of straight lines if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) Two parallel lines if $\Delta = 0, ab = h^2$

(iii) Two perpendicular lines if $\Delta = 0, a + b = 0$

2. a circle if $a = b, h = 0$

3. a parabola if $ab = h^2, \Delta \neq 0$

4. an ellipse if $ab - h^2 > 0, \Delta \neq 0$

5. a hyperbola if $ab - h^2 < 0, \Delta \neq 0$.

Problem1.

What does the equation $5x^2$-2xy+$5y^2$-8x-8y-8=0 represent?

Solution:

The given equation $5x^2$-2xy+$5y^2$-8x-8y-8=0 ....(1)

Comparing (1) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get a=5, h=-1,b=5,g=-4,f=-4,c=-8

$$\therefore \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

=-200-32-80-80+8

=-384

$$\therefore \Delta \neq 0$$

$$ab - h^2 = 5.5 - (-1)^2 = 25 - 1 = 24$$

$$ab - h^2 > 0.$$

Thus the given equation represents an ellipse.

Problem 2: Test the nature of the conic $x^2+2xy+y^2+2x-1=0$

Solution: The given equation is, $x^2+2xy+y^2+2x-1=0$ comparing the equation with the second degree general equation, $ax^2+2hxy+by^2+2gx+2fy+c=0$

We get, $a=1, h=1, b=1, g=1, f=0, c=-1$

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

$$=1.1.(-1)+2.0.1.1-1.(0)^2-1.(1)^2-(-1)(1)^2$$

$$=-1+0-0-1+1$$

$$= -1$$

Since $ab-h^2=1.1-(1)^2=1-1=0$

Since $\Delta \neq 0$ and $ab-h^2=0$, So the given equation represents a parabola.

Problem 3: Test the nature of the conic $x^2+4xy-2y^2+6x-12y-15=0$

Solution: The given equation is, $x^2+4xy-2y^2+6x-12y-15=0$

Comparing the equation with the second degree general equation,

$ax^2+2hxy+by^2+2gx+2fy+c=0$

We get, $a=1, h=2, b=-2, g=3, f=-6, c=-15$

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

$$=1.(-2).(-15)+2.(-6).3.2-1.(-6)^2-(-2).(3)^2-(-15)(2)^2$$

$$=30-72-36+18+60$$

$$=108\text{-}108$$

$$=0$$

Since $\Delta=0$, so the given equation represents a pair of straight line.

Problem 4:

Reduce the equation $5x^2 - 24xy - 5y^2 + 4x + 58y - 59 = 0$ standard form and what this represent?

Solution:

Let $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = 5x^2 - 24xy - 5y^2 + 4x + 58y - 59 = 0$

$$\frac{\delta f}{\delta x} = 10x - 24y + 4 = 0$$

$$=> 5x - 12y + 2 = 0 \ldots\ldots(1)$$

$$\frac{\delta f}{\delta y} = -24x - 10y + 58 = 0$$

$$=> 12x + 5y - 29 = 0 \ldots\ldots (2)$$

$$solving\ (1)\&(2)\ x = 2, y = 1. So\ the\ centre\ is\ (2,1)$$

$$New\ constant\ , C_1 = gx_1 + fy_1 + c$$

$$=2.2+29.1-59$$

$$=-26.$$

Now the equation of the conic referred to the center (2, 1) is

$$5x^2 - 24xy - 5y^2 = 26 \ldots\ldots\ldots\ldots (3)$$

When the xy term is removed by the rotation of axes, let the reduced equation be

$$a_1x^2 + b_1y^2 = 26 \ldots\ldots\ldots\ldots (4)$$

Then we have $a_1 + b_1 = a + b$

$$=5-5$$

$$=0.$$

And $a_1 b_1 = ab - h^2$

$=5.(-5)-12^2$

$=-169$

$(a_1-b_1)^2 = (a_1+b_1)^2 - 4a_1b_1$

$=0-4(-169)$

$(a_1-b_1)^2 = (a_1+b_1)^2 - 4a_1b_1$

$a_1 - b_1 = 26.$

Now by solving we get

$a_1 = 13, \qquad b_1 = -13$

Then (4) becomes $\therefore$ $13x^2$-$13y^2$=26.

Or, x^2-y^2=2.Which is the reduced form.

This represents a hyperbola.

Problem 5: Reduce the equation $8x^2+4xy+5y^2-16x-14y+13=0$ to the standard form and what this represent?

Solution: The given equation is, $8x^2+4xy+5y^2-16x-14y+13=0$..........(1)

Comparing the equation with the second degree general equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$, we get

$a=8, h=2, b=5, g=-8, f=-7 \ and \ c=-13$

Now,

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

$$=8.5.13+2.(-7).(-8).2-8.(-7)^2-5.(-8)^2-13.(2)^2$$

$$=520+224-392-320-52$$

$$= -20$$

And $ab-h^2=8.5-(2)^2=36$

Since $\Delta \neq 0$ and $ab-h^2>0$. Therefore the given equation represents an ellipse.

$Let,$

$$F(x,y) = 8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0$$

$$\therefore \frac{\partial F}{\partial x} = 16x + 4y - 16 = 0$$

$$\therefore 4x + y - 4 = 0 \ldots\ldots(2)$$

$$and\ \frac{\partial F}{\partial y} = 4x + 10y - 14 = 0$$

$$\therefore 2x + 5y - 7 = 0 \ldots\ldots(3)$$

By cross multiplication from (2) and (3), we get,

$$\frac{x}{13} = \frac{y}{20} = \frac{1}{18}$$

$$\therefore x = \frac{13}{18}, y = \frac{10}{9}$$

Therefore, the center of the conic is,

$$(\alpha, \beta) = \left(\frac{13}{18}, \frac{10}{9}\right)$$

Shift the origin to the center of the conic, where the direction of axes remains unaltered, and the equation (1) takes the form:

$$8X^2 + 4XY + 5Y^2 + \lambda = 0........(4)$$

$$where\ \lambda = \frac{\Delta}{ab - h^2} = \frac{-20}{36} = -\frac{5}{9}$$

Putting the values of λ in (4) we get,

$$8X^2 + 4XY + 5Y^2 = \frac{5}{9}$$

When the XY term is removed by rotation of axes. Let the reduced equation be,

$$a_1X^2 + b_1Y^2 = \frac{5}{9}..............(5)$$

By the method of invariants, we have,

$$a_1 + b_1 = a + b = 8 + 5 = 13........(6)$$

$$and\ a_1b_1 = ab - h^2 = 36...........(7)$$

From (4) we get,

$$b_1 = 13 - a_1$$

Putting the value of b_1 in (7), we get

$$a_1(13 - a_1) = 36$$

$$or, 13a_1 - a_1^2 - 36 = 0$$

$$or, a_1^2 - 13a_1 + 36 = 0$$

$$or, (a_1 - 9)(a_1 - 4) = 0$$

$$\therefore a_1 = 9, 4$$

Putting the value of a_1 in (7) , we get

$b_1 = 4, 9$

Since λ is negative. So we choose $a_1 < b_1$.

$i.e\ a_1 = 4\ and\ b_1 = 9.$

Putting the values of a_1 and b_1 in (5) we get,

$$4X^2 + 9Y^2 = \frac{5}{9}$$

$$or, \frac{36}{5}X^2 + \frac{81}{5}Y^2 = 1$$

$$or, \frac{X^2}{\frac{5}{36}} + \frac{Y^2}{\frac{5}{81}} = 1$$

$$\therefore \frac{X^2}{\left(\frac{\sqrt{5}}{6}\right)^2} + \frac{Y^2}{\left(\frac{\sqrt{5}}{9}\right)^2} = 1$$

This is the slandered form of the ellipse.

Try yourself

Test the nature of the coins given by the following equations

i) $2x^2 - 4xy + 2y^2 + 6x - 2y - 12 = 0$

ii) $2x^2 + 4xy - y^2 - 4x - 10 = 0$

iii) $4x^2 + 9y^2 - 8x + 36y - 31 = 0$

iv) $x^2 - 2xy - 2y^2 + 6x - 12y - 20 = 0$

<u>* Equation of tangent and normal to the conic:</u>**

1. We know that the equation of the tangent to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at the point (x_1, y_1) is,

$$(y-y_1)=\frac{dy}{dx}(x-x_1)$$

2. We know that the equation of the tangent to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at the point (x_1, y_1) is,

$$(y-y_1)=\frac{-1}{\frac{dy}{dx}}(x-x_1)$$

<u>Problem 1:</u> Find the equation of the tangent and normal of the conic $2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$ at the point $(-2, -4)$

<u>Solution:</u>

Given equation is $2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$

Differentiating with respect to x

$$\frac{d}{dx}\left(2x^2 - 3xy + y^2 - 5x + 4y + 6\right) = 0$$

$$or, 4x - 3(\mathrm{x}\frac{dy}{dx} + \mathrm{y}) + 2\mathrm{y}\frac{dy}{dx} - 5 + 4\frac{dy}{dx} = 0$$

$$or, \frac{dy}{dx}(2y - 3x + 4) = 3y - 4x + 5$$

$$or, \frac{dy}{dx} = \frac{(3y - 4x + 5)}{(2y - 3x + 4)}$$

$$At\ po\mathrm{int}\ (-2, -4)\frac{dy}{dx} = \frac{-12 + 8 + 5}{-8 + 6 + 4} = \frac{1}{2}$$

$The\ equation\ of\ the\ tangent\ to\ the\ conic\ at\ the\ point\ (-2,-4)\ \text{is}$

$$y+4=\frac{1}{2}(x+2)$$

$$\text{or}, 2\text{y}+8=\text{x}+2$$

$$or, x+2-2y-8=0$$

$$or, x-2y-6=0(\text{Ans})$$

$The\ equation\ of\ the\ tangent\ to\ the\ conic\ at\ the\ point\ (-2,-4)\ \text{is}$

$$y+4=\frac{-1}{\frac{1}{2}}(x+2)$$

$$\text{or}, \text{y}+4=-2\text{x}-4$$

$$\text{or}, \text{y}+2\text{x}+8=0$$

Problem 2: Find the equation of the tangent and normal of the conic $x^2 - xy + y^2 - x + y + 1 = 0$ at the point $(1, -3)$.

Solutions: The equation of the tangent to the conic at $(-2, -4)$ **is**

Given that $x^2 - xy + y^2 - x + y + 1 = 0$

Differentiating with respect to x we get

$$\frac{d}{dx}\left(x^2 - xy + y^2 - x + y + 1\right) = 0$$

$$or,\ 2x - (\mathrm{x}\frac{dy}{dx} + \mathrm{y}) + 2\,\mathrm{y}\frac{dy}{dx} - 1 + \frac{dy}{dx} = 0$$

$$or,\ \frac{dy}{dx}(2y - x + 1) = y - 2x + 1$$

$$or,\ \frac{dy}{dx} = \frac{(y - 2x + 1)}{(2y - x + 1)}$$

At point $(1,-3)$ $\frac{dy}{dx}=\frac{-3-2+1}{-6-1+1}=\frac{2}{3}$

The equation of the tangent to the conic at the point $(1,-3)$is

$$y+3=\frac{2}{3}(x-1)$$

$$\text{or},3y+9=2x-2$$

$$\text{or},2x-3y-11=0(\text{Ans})$$

The equation of the tangent to the conic at the point $(1,-3)$is

$$y+3=\frac{-1}{\frac{2}{3}}(x-1)$$

$$\text{or},2y+6=-3x+3$$

$$\text{or},2y+3x+3=0(\text{Ans})$$

Problem 3: Find the equation of the tangent and normal of the conic $2x^2+xy+5x+7y+5=0$ at the point (-2,-1).

Solution: Given $2x^2+xy+5x+7y+5=0$

Differentiating W.r.t x we get

$$\frac{d}{dx}\left(2x^2+xy+5x+7y+5\right)=0$$

$$or,\ 4x+x\frac{dy}{dx}+y+5+7\frac{dy}{dx}=0$$

$$or,\ \frac{dy}{dx}(x+7)=-4x-y-5$$

$$or,\ \frac{dy}{dx}=\frac{-4x-y-5}{x+7}$$

At point (-2,-1) $\dfrac{dy}{dx}=\dfrac{8+1-5}{-1+7}=\dfrac{2}{3}$

The equation of the tangent to the conic at the point (-2,-1)is

$$y+1=\frac{2}{3}(x+2)$$

$$or, 3y+3=2x+4$$

$$or, 2x-3y+1=0(Ans)$$

The equation of the tangent to the conic at the point (-2,-1)is

$$y+1=\frac{-1}{\frac{2}{3}}(x+2)$$

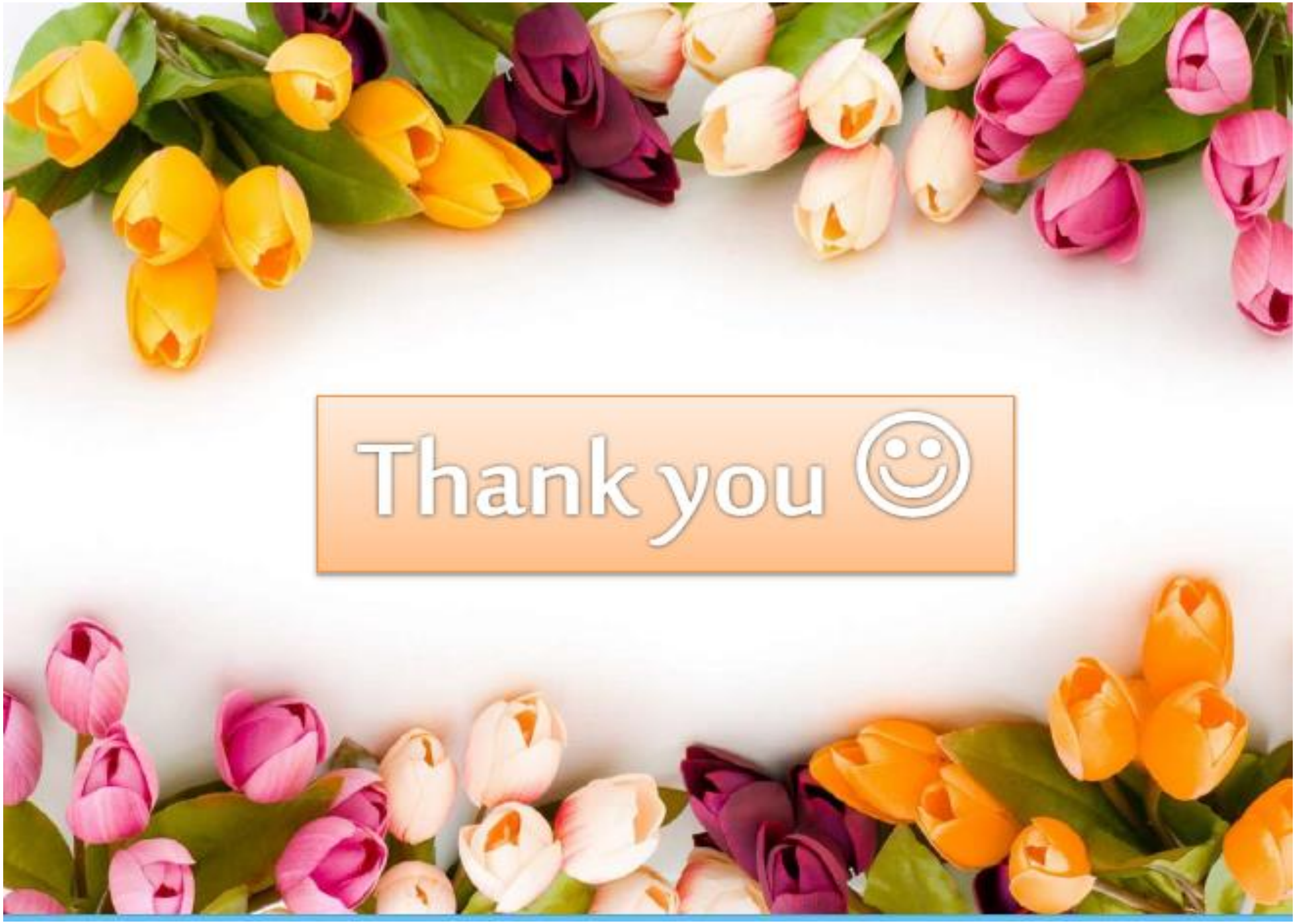
$$or, 2y+2=-3x-6$$

$$or, 2y+3x+8=0(Ans)$$

Try yourself

1. Find the equation of the tangent and normal of the conic $2x^2+2y^2-4x+8y+6=0$ at the point $(1,2)$.

2. Find the equation of the tangent and normal of the conic $x^2+y^2-x+y+6=0$ at the point $(3,4)$.


Thank you ☺