Algorithm 884: A Simple Matlab Implementation of the Argyris Element

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In this work we propose a new algorithm to evaluate the basis functions of the Argyris finite element and their derivatives. The main novelty here is an efficient way to calculate the matrix which gives the change of coordinates between the bases of the Argyis element for the reference and for an arbitrary triangle. This matrix is factored as the product of two rectangular matrices with a strong block structure which makes their computation very easy. We show and comment on an implementation of this algorithm in Matlab. Two numerical experiments, an interpolation of a smooth function on a triangle and the finite-element solution of the Dirichlet problem for the biLaplacian, are presented in the last section to check the performance of our implementation.

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1. INTRODUCTION

The Argyris element [Argyris et al. 1968; Braess 2001; Brenner and Scott 2002; Ciarlet 1975] is possibly the best-known and most widely used of all 2D finite elements of class $\mathcal{C}^1$. Finite elements of class $\mathcal{C}^1$ are commonly employed for conforming the approximation of primal formulations of linear partial differential equations of order four, such as the Kirchhoff plate problem. They also are used in the community of D-spline approximation (see Arcangéli et al. [2004]), where exact interpolants cannot be computed and are approximated with finite elements of a smooth class.

Unlike the other popular $\mathcal{C}^1$ element, the Bogner-Fox-Schmidt rectangle [Bogner 1965] (that only works on grids of well-oriented rectangles), the Argyris element can be used in any triangulation. Its 21 degrees of freedom may seem disadvantageous at first, but the fact that the space involved is the full space of bivariate polynomials of degree up to 5 provides a very high order of convergence as the size of the triangulation decreases. However, the main drawback is the difficulty of implementation, a complication that is shared with most finite elements of Hermite type. The difficulties come from two different sources. The first is the fact that the use of normal vectors to define degrees of freedom is not respected by affine transformations, and mapping to a reference triangle is the most widely employed way of working with triangular finite elements of any type. Therefore, Lagrange bases (in the sense of bases of the space that cancel all degrees of freedom but one) are not naturally mapped to each other. In already classical work in plate and shell approximation by Bernadou (see Bernadou [1997]), an exact Lagrange basis is written in terms of barycentric coordinates for an arbitrary triangle. However, this kind of treatment gives rise to the second inherent problem: To make the basis relatively simple to use, Hermitian degrees of freedom on the vertices are written in terms of directional derivatives with respect to the sides of the triangle. Hence, the local degrees of freedom of a vertex of the triangulation are not shared by all triangles that have the vertex in common and, therefore, the global Lagrange basis of the finite-element space is not the local Lagrange basis when restricted to a triangle.

The Bell triangle [Bell 1969] and Clough-Tocher element [Clough and Tocher 1965] are two alternatives using fewer degrees of freedom. The Bell triangle is constructed from the Argyris element by removing the restriction on normal derivatives on the midpoint of the sides. This gives rise to a $\mathcal{C}^1$ finite element with 18 degrees of freedom. In turn, the Clough-Tocher element uses a $\mathcal{C}^1$ piecewise polynomial space of degree 3 with 12 degrees of freedom. The cost paid by these simplifications is that the corresponding finite-element spaces are not so simple. Actually, the spaces are suitable subspaces of the polynomials of degree 5 and 3 which depend on the triangle. Moreover, regarding the approximation properties, there is a loss in the order of convergence: 1 order for the Bell triangle and 2 for the Clough-Tocher element. Although the optimal order of convergence of the Argyris element is achieved only under rather strong smoothness conditions, which, for instance, rarely happens in partial differential problems, the use of Argyris elements can take better advantage from local regularity properties of the function to be approximated.

In this work we develop a simple implementation, especially adapted to Matlab (but easily modifiable to Fortran 90), to construct and evaluate the local basis functions of the Argyris element on any triangle. The degrees of freedom on vertices will not take into account the directions of the sides and, therefore, the global basis functions of the corresponding finite element space are constructed just by joining basis functions on adjacent triangles. The main novelty that permits such an easy implementation is noticing that the matrix transforming the basis functions from the reference triangle to a general triangle can be factored as the product of two simple rectangular matrices. The basis functions of the reference triangle can be computed with a symbolic algebra package: Here we show how to do it with the symbolic package of Matlab.

We finally check the code with two examples. In the first one, we interpolate a smooth function on successive (red) uniform refinements of an initial grid, and estimate the L^∞ error for the function and its derivatives up to order 2. The second experiment consists of the numerical solution of a clamped Kirchhoff plate with an exact polynomial solution. In this case, the L^2 and H^2 error are computed.

2. THE ARGYRIS ELEMENT

Consider the reference triangle $\widehat{K}$ with vertices on $\widehat{\mathbf{x}}_1 = (0, 0)$, $\widehat{\mathbf{x}}_2 = (1, 0)$ and $\widehat{\mathbf{x}}_3 = (0, 1)$. Given three nonaligned points of the plane $\mathbf{x}_\alpha = (x_\alpha, y_\alpha)$ $(\alpha = 1, 2, 3)$, we consider an affine transformation that maps $\widehat{K}$ bijectively onto K, the triangle formed by these points, $F : \widehat{K} \to K$

$$F(\widehat{\mathbf{x}}) = B\widehat{\mathbf{x}} + \mathbf{b} := \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}. \tag{1}$$

(We follow the convention of writing $\mathbf{x}$ to refer to points on the triangle K and $\widehat{\mathbf{x}}$ for points on $\widehat{K}$).

The vectors joining the vertices will be numbered in the following way (see Figure 1).

$$\mathbf{v}_1 = \mathbf{x}_2 - \mathbf{x}_1, \qquad \mathbf{v}_2 = \mathbf{x}_3 - \mathbf{x}_1, \qquad \mathbf{v}_3 = \mathbf{x}_3 - \mathbf{x}_2 \tag{2}$$

Also, $\mathbf{n}_\alpha$ denotes the unit normal vector to the corresponding side, obtained (after normalization) by rotating the vector $\mathbf{v}_\alpha$ by an angle of $\pi/2$ in the positive direction. Finally $\mathbf{m}_\alpha$ will be the midpoint of the side, respecting the same numbering as described before. We note that any other numbering of sides (some are more common in part of the finite element literature) amounts to a simple reordering of what follows. Also, taking the normals pointing outwards amounts only to changing some signs. We will stick to these notations, since some of the forthcoming computations will become simpler.

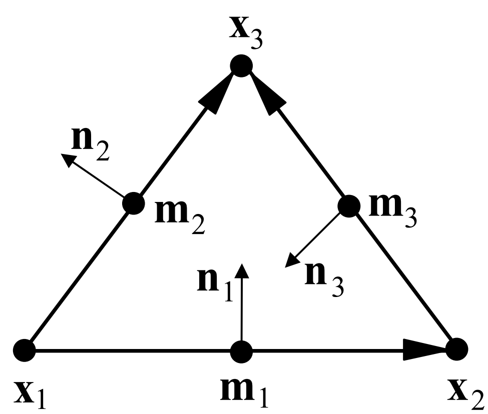

Fig. 1. Geometrical elements of an arbitrary triangle.

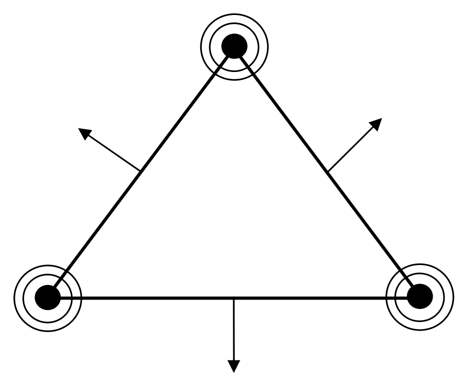

Fig. 2. The usual graphical representation of the degrees of freedom of the Argyris triangle.

We consider the functionals associated to the vertices of K (see Figure 2) as

$$\begin{aligned}\mathcal{L}^0_\alpha(\phi) &:= \phi(\mathbf{x}_\alpha),\\ \mathcal{L}^\circ_\alpha(\phi) &:= \partial_\circ\phi(\mathbf{x}_\alpha), \quad \circ \in \{x, y\},\\ \mathcal{L}^\circ_\alpha(\phi) &:= \partial^2_\circ\phi(\mathbf{x}_\alpha), \quad \circ \in \{xx, xy, yy\},\end{aligned}$$

for $\alpha \in \{1, 2, 3\}$, and the functionals associated to the sides

$$\mathcal{L}^n_\alpha(\phi) := \nabla_{\mathbf{x}}\phi(\mathbf{m}_\alpha) \cdot \mathbf{n}_\alpha, \qquad \alpha \in \{1, 2, 3\}.$$

We list these functionals as $\mathcal{L}_j$ for $j = 1, \dots, 21$ in the following order.

$$\begin{aligned}&\mathcal{L}^0_1, \mathcal{L}^0_2, \mathcal{L}^0_3,\\ &\mathcal{L}^x_1, \mathcal{L}^y_1, \mathcal{L}^x_2, \mathcal{L}^y_2, \mathcal{L}^x_3, \mathcal{L}^y_3\\ &\mathcal{L}^{xx}_1, \mathcal{L}^{xy}_1, \mathcal{L}^{yy}_1, \mathcal{L}^{xx}_2, \mathcal{L}^{xy}_2, \mathcal{L}^{yy}_2, \mathcal{L}^{xx}_3, \mathcal{L}^{xy}_3, \mathcal{L}^{yy}_3,\\ &\mathcal{L}^n_1, \mathcal{L}^n_2, \mathcal{L}^n_3\end{aligned}$$

The functionals $\widehat{\mathcal{L}}^\circ_\alpha$ and $\widehat{\mathcal{L}}^n_\alpha$ are the corresponding ones on the reference triangle.

The basis functions for the Argyris element are the unique elements N_j of $\mathbb{P}_5(K)$, the space of bivariate polynomials of degree up to 5, such that

$$\mathcal{L}_i(N_j) = \delta_{ij}, \qquad i, j \in \{1, \ldots, 21\}.$$

Likewise, we consider the unique $\widehat{N}_j \in \mathbb{P}_5(\widehat{K})$ such that

$$\widehat{\mathcal{L}}_i(\widehat{N}_j) = \delta_{ij}, \qquad i, j \in \{1, \ldots, 21\}.$$

We define

$$\widetilde{\mathcal{L}}_i(\phi) := \widehat{\mathcal{L}}_i(\phi \circ F),$$

namely, $\widetilde{\mathcal{L}}_i(\phi \circ F^{-1}) = \widehat{\mathcal{L}}_i(\phi)$. Since both sets $\{\mathcal{L}_i\}$ and $\{\widetilde{\mathcal{L}}_i\}$ are bases of $\mathbb{P}_5^*(K)$, the dual space to $\mathbb{P}_5(K)$, there exists a nonsingular matrix $C = (c_{ij})$ such that

$$\widetilde{\mathcal{L}}_i = \sum_{j=1}^{21} c_{ij}\mathcal{L}_j, \qquad \text{in } \mathbb{P}_5^*(K), \qquad i = 1, \ldots, 21. \tag{3}$$

By an elementary transposition argument, it follows that

$$N_i \circ F = \sum_{k=1}^{21} c_{ki}\widehat{N}_k, \qquad i = 1, \ldots, 21.$$

The aim of the following section is to give a simple expression for the matrix C.

3. COMPUTATION OF THE CHANGE OF BASES

We consider the matrix

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which rotates vectors about an angle $\pi/2$ in the positive sense. Next, we introduce a new set of functionals

$$\mathcal{L}_i^*, \qquad i = 1, \ldots, 24$$

doing as follows. We keep the vertex-associated functionals $\mathcal{L}_i^* := \mathcal{L}_i$, $i = 1, \ldots, 18$ in their original order. We then introduce the functionals

$$\mathcal{L}_\alpha^\circ, \qquad \alpha \in \{1, 2, 3\}, \qquad \circ \in \{\perp, \shortparallel\}$$

by the expressions

$$\mathcal{L}_\alpha^\perp(\phi) := \nabla_{\mathbf{x}}\phi(\mathbf{m}_\alpha) \cdot R\mathbf{v}_\alpha, \qquad \mathcal{L}_\alpha^\shortparallel(\phi) := \nabla_{\mathbf{x}}\phi(\mathbf{m}_\alpha) \cdot \mathbf{v}_\alpha$$

ordered in the following way: $\mathcal{L}_1^\perp, \mathcal{L}_2^\perp, \mathcal{L}_3^\perp, \mathcal{L}_1^\shortparallel, \mathcal{L}_2^\shortparallel, \mathcal{L}_3^\shortparallel$.

If we write gradients columnwise $\nabla\phi := (\partial_x\phi, \partial_y\phi)^\top$, then

$$\nabla_{\widehat{\mathbf{x}}}(\phi \circ F) = B^\top \nabla_{\mathbf{x}}\phi \circ F.$$

Also, by using the following column form of the Hessian matrix $\mathcal{H}(\phi) := (\partial_{xx}\phi, \partial_{xy}\phi, \partial_{yy}\phi)^\top$, it follows that

$$\mathcal{H}_{\widehat{\mathbf{x}}}(\phi \circ F) = \Theta\, \mathcal{H}_{\mathbf{x}}(\phi) \circ F,$$

where

$$\Theta := \begin{bmatrix} b_{11}^2 & 2b_{11}b_{21} & b_{21}^2 \\ b_{12}b_{11} & b_{12}b_{21}+b_{11}b_{22} & b_{21}b_{22} \\ b_{12}^2 & 2b_{22}b_{12} & b_{22}^2 \end{bmatrix}.$$

We employ these matrices to construct a 21×24 matrix D in block-diagonal form. We have

$$D := \mathrm{diag}(I_3, B^\top, B^\top, B^\top, \Theta, \Theta, \Theta, Q),$$

where I_3 is a 3×3 identity matrix and the block Q is a 3×6 matrix

$$Q := \left[\begin{array}{ccc|ccc} f_1 & & & g_1 & & \\ & f_2 & & & g_2 & \\ & & f_3 & & & g_3 \end{array}\right],$$

constructed as follows. If $\ell_\alpha := |\mathbf{v}_\alpha|$ is the length of the αth side and

$$\begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \mathbf{a}_3^\top \end{bmatrix} := \begin{bmatrix} \ell_1^{-2} & & \\ & \ell_2^{-2} & \\ & & \ell_3^{-2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} B^\top, \tag{4}$$

then

$$f_\alpha := \mathbf{a}_\alpha \cdot R\mathbf{v}_\alpha, \qquad g_\alpha := \mathbf{a}_\alpha \cdot \mathbf{v}_\alpha, \qquad \alpha \in \{1,2,3\}. \tag{5}$$

The diag instruction corresponds to the Matlab `blckdiag` command and creates a matrix by adding each block to the right lowermost corner of the matrix created by the preceding blocks.

PROPOSITION 1. *If $\mathbb{P}(K)$ is the space of bivariate polynomials of arbitrary order on K and $\mathbb{P}^*(K)$ is its dual space, then*

$$\widetilde{\mathcal{L}}_i = \sum_{j=1}^{24} d_{ij}\mathcal{L}_j^*, \qquad \text{in } \mathbb{P}^*(K), \qquad i = 1, \dots, 21. \tag{6}$$

PROOF. We first remark that

$$\widetilde{\mathcal{L}}_\alpha^0 = \mathcal{L}_\alpha^0, \qquad \begin{bmatrix} \widetilde{\mathcal{L}}_\alpha^x \\ \widetilde{\mathcal{L}}_\alpha^y \end{bmatrix} = B^\top \begin{bmatrix} \mathcal{L}_\alpha^x \\ \mathcal{L}_\alpha^y \end{bmatrix}, \qquad \begin{bmatrix} \widetilde{\mathcal{L}}_\alpha^{xx} \\ \widetilde{\mathcal{L}}_\alpha^{xy} \\ \widetilde{\mathcal{L}}_\alpha^{yy} \end{bmatrix} = \Theta \begin{bmatrix} \mathcal{L}_\alpha^{xx} \\ \mathcal{L}_\alpha^{xy} \\ \mathcal{L}_\alpha^{yy} \end{bmatrix}.$$

Notice that $\mathbf{v}_\alpha = B\widehat{\mathbf{v}}_\alpha$ for $\alpha \in \{1,2,3\}$, where $\widehat{\mathbf{v}}_\alpha$'s are defined in the reference triangle as in (2). Then

$$\widetilde{\mathcal{L}}_\alpha^n = \frac{1}{|\widehat{\mathbf{v}}_\alpha|} R\widehat{\mathbf{v}}_\alpha \cdot \ell_\alpha^{-2} B^\top \begin{bmatrix} -v_\alpha^y & v_\alpha^x \\ v_\alpha^x & v_\alpha^y \end{bmatrix} \begin{bmatrix} \mathcal{L}_{\alpha\beta}^\perp \\ \mathcal{L}_{\alpha\beta}^{\shortparallel} \end{bmatrix}. \tag{7}$$

To prove (7) we just have to notice that

$$\widetilde{\mathcal{L}}^n_\alpha(\phi) = \widehat{\mathcal{L}}^n_\alpha(\phi \circ F) = \frac{1}{|\widehat{\mathbf{v}}_\alpha|} R\,\widehat{\mathbf{v}}_\alpha \cdot \nabla_{\widehat{\mathbf{x}}}(\phi \circ F)(\widehat{\mathbf{m}}_\alpha) = \frac{1}{|\widehat{\mathbf{v}}_\alpha|} R\mathbf{v}_\alpha \cdot B^\top \nabla_{\mathbf{x}}\phi(\mathbf{m}_\alpha)$$

and that, since

$$\begin{bmatrix} \mathcal{L}^\perp_\alpha \\ \mathcal{L}^{\shortparallel}_\alpha \end{bmatrix} = \begin{bmatrix} -v^y_\alpha & v^x_\alpha \\ v^x_\alpha & v^y_\alpha \end{bmatrix} \begin{bmatrix} \mathcal{L}^x_\alpha \\ \mathcal{L}^y_\alpha \end{bmatrix},$$

then

$$\begin{bmatrix} \mathcal{L}^x_\alpha \\ \mathcal{L}^y_\alpha \end{bmatrix} = \frac{1}{|\mathbf{v}_\alpha|^2} \begin{bmatrix} -v^y_\alpha & v^x_\alpha \\ v^x_\alpha & v^y_\alpha \end{bmatrix} \begin{bmatrix} \mathcal{L}^\perp_\alpha \\ \mathcal{L}^{\shortparallel}_\alpha \end{bmatrix}.$$

Finally, it follows readily that (7) can be also read as

$$\widetilde{\mathcal{L}}^n_\alpha = f_\alpha \mathcal{L}^\perp_\alpha + g_\alpha \mathcal{L}^{\shortparallel}_\alpha,$$

with

$$f_\alpha = \frac{1}{\ell^2_\alpha\,|\widehat{\mathbf{v}}_\alpha|} R\,\widehat{\mathbf{v}}_\alpha \cdot B^\top R\mathbf{v}_\alpha, \qquad g_\alpha = \frac{1}{\ell^2_\alpha\,|\widehat{\mathbf{v}}_\alpha|} R\,\widehat{\mathbf{v}}_\alpha \cdot B^\top \mathbf{v}_\alpha$$

and that these expressions of f_α and g_α correspond to the ones given in the construction of D. □

Consider now a new matrix E, 24×21, with the structure

$$E = \begin{bmatrix} I_{18} & 0 \\ 0 & L \\ T & 0 \end{bmatrix}, \qquad L = \mathrm{diag}(\ell_1, \ell_2, \ell_3),$$

where the last block (3×18) is composed by concatenation of three subblocks (3×3, 3×6, and 3×9, respectively). Hence

$$\frac{15}{18}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad -\frac{7}{16}\begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_1^\top & \mathbf{0} \\ \mathbf{v}_2^\top & \mathbf{0} & \mathbf{v}_2^\top \\ \mathbf{0} & \mathbf{v}_3^\top & \mathbf{v}_3^\top \end{bmatrix}, \qquad \frac{1}{32}\begin{bmatrix} -\mathbf{w}_1^\top & \mathbf{w}_1^\top & \mathbf{0} \\ -\mathbf{w}_2^\top & \mathbf{0} & \mathbf{w}_2^\top \\ \mathbf{0} & -\mathbf{w}_3^\top & \mathbf{w}_3^\top \end{bmatrix},$$

with $\mathbf{w}_\alpha^\top = (v^x_\alpha\, v^x_\alpha,\, 2v^x_\alpha\, v^y_\alpha,\, v^y_\alpha\, v^y_\alpha)$.

PROPOSITION 2. *It holds that*

$$\mathcal{L}^*_i = \sum_{j=1}^{21} e_{ij}\mathcal{L}_j, \qquad in\ \mathbb{P}^*_5(K), \qquad i = 1, \ldots, 24. \tag{8}$$

PROOF. It is clear that $\mathcal{L}^*_j = \mathcal{L}_j$ for $j = 1, \ldots, 18$ and that since $\ell_\alpha\,\mathbf{n}_\alpha = R\mathbf{v}_\alpha$, then $\mathcal{L}^\perp_\alpha = \ell_\alpha \mathcal{L}^n_\alpha$. Take now an arbitrary $\phi \in \mathbb{P}_5(K)$ and define

$$\psi(t) := \phi(t\mathbf{x}_\beta + (1-t)\mathbf{x}_\alpha) \in \mathbb{P}_5(t), \quad \alpha < \beta.$$

Notice that for all $\psi \in \mathbb{P}_5(t)$,

$$\psi'(1/2) = \frac{15}{8}(\psi(1) - \psi(0)) - \frac{7}{16}(\psi'(1) + \psi'(0)) + \frac{1}{32}(\psi''(1) - \psi''(0)). \tag{9}$$

Take now the index γ so that $\mathbf{v}_\gamma = \mathbf{x}_\beta - \mathbf{x}_\alpha$ (see (2) for the correspondence). Hence

$$\begin{aligned}
\psi'(1/2) &= \mathcal{L}^{\shortparallel}_\gamma(\phi)\\
\psi(0) &= \mathcal{L}^0_\alpha(\phi)\\
\psi(1) &= \mathcal{L}^0_\beta(\phi)\\
\psi'(0) &= v^x_\gamma \mathcal{L}^x_\alpha(\phi) + v^y_\gamma \mathcal{L}^y_\alpha(\phi)\\
\psi'(1) &= v^x_\gamma \mathcal{L}^x_\beta(\phi) + v^y_\gamma \mathcal{L}^y_\beta(\phi)\\
\psi''(0) &= (v^x_\gamma)^2 \mathcal{L}^{xx}_\alpha(\phi) + 2 v^x_\gamma v^x_\gamma \mathcal{L}^{xy}_\alpha(\phi) + (v^y_\gamma)^2 \mathcal{L}^{yy}_\alpha(\phi)\\
\psi''(1) &= (v^x_\gamma)^2 \mathcal{L}^{xx}_\beta(\phi) + 2 v^x_\gamma v^x_\gamma \mathcal{L}^{xy}_\beta(\phi) + (v^y_\gamma)^2 \mathcal{L}^{yy}_\beta(\phi).
\end{aligned}$$

Applying (9) we obtain the expression

$$\begin{aligned}
\mathcal{L}^{\shortparallel}_\gamma = {} & \frac{15}{8}(-\mathcal{L}^0_\alpha + \mathcal{L}^0_\beta) - \frac{7}{16}\left(v^x_\gamma \mathcal{L}^x_\alpha + v^y_\gamma \mathcal{L}^y_\alpha + v^x_\gamma \mathcal{L}^x_\beta + v^y_\gamma \mathcal{L}^y_\beta\right)\\
& + \frac{1}{32}\Big(- (v^x_\gamma)^2 \mathcal{L}^{xx}_\alpha - 2 v^x_\gamma v^y_\gamma \mathcal{L}^{xy}_\alpha - (v^y_\gamma)^2 \mathcal{L}^{yy}_\alpha + (v^x_\gamma)^2 \mathcal{L}^{xx}_\beta + 2 v^x_\gamma v^y_\gamma \mathcal{L}^{xy}_\beta + (v^y_\gamma)^2 \mathcal{L}^{yy}_\beta \Big).
\end{aligned}$$

This identity finishes the proof. □

Notice that by combining (6) and (8) we obtain that the matrix in (3) is simply

$$C = D\,E. \tag{10}$$

This gives an expression of the change of bases from the reference element to any triangle factored as the product of two rectangular very sparse matrices constructed with simple geometric elements of the triangle.

4. NUMERICAL TESTS

The aim of this section is to check the code by developing two applications that use it.

The first experiment concerns the interpolation of a function on a particular triangle. Specifically, given a triangle K, we define for any function f that is, smooth enough the interpolation operator

$$\mathbb{P}_5 \ni p \quad \text{such that} \quad \mathcal{L}_j(p - f) = 0, \qquad j = 1, \ldots, 21.$$

In our test, we have taken $f = \cos(xy)\exp(x - y)$ and the triangle K with vertices $\{(1, 0), (0, 1), (-1, 1)\}$. As an L^∞ error estimate we introduce

$$\begin{aligned}
E &:= \max_{(x,y)\in\mathcal{E}} |f(x, y) - p(x, y)|,\\
E_\circ &:= \max_{(x,y)\in\mathcal{E}} |\partial_\circ f(x, y) - \partial_\circ p(x, y)|, \qquad \circ \in \{x, y, xx, xy, yy\},
\end{aligned}$$

Table I. L^∞ Error Estimate of the Evaluation of the Argyris Basis

E	e.c.r	E_x	e.c.r	E_y	e.c.r
2.10E−02		8.26E−02		3.98E−02	
1.38E−03	3.94	5.87E−03	3.82	5.77E−03	2.78
3.79E−05	5.18	2.84E−04	4.37	3.35E−04	4.10
6.97E−07	5.76	1.02E−05	4.79	1.31E−05	4.68
1.13E−08	5.94	3.35E−07	4.93	4.39E−07	4.90
1.84E−10	5.94	1.08E−08	4.97	1.40E−08	4.97
2.91E−12	5.98	3.38E−10	4.98	4.44E−10	4.98

E_{xx}	e.c.r	E_{xy}	e.c.r	E_{yy}	e.c.r
7.72E−01		3.17E−01		2.91E−01	
1.05E−01	2.88	5.41E−02	2.55	7.63E−02	1.93
1.07E−02	3.29	5.64E−03	3.26	1.04E−02	2.87
7.77E−04	3.79	4.34E−04	3.70	8.64E−04	3.59
5.00E−05	3.96	3.04E−05	3.84	6.01E−05	3.85
3.23E−06	3.95	1.98E−06	3.94	3.93E−06	3.93
2.05E−07	3.98	1.26E−07	3.97	2.51E−07	3.97

$\mathcal{E}$ being the set of points with barycentric coordinates

$$\left(\frac{i}{32}, \frac{j}{32}, 1 - \frac{i}{32} - \frac{j}{32}\right), \qquad i, j \geq 0,\ i + j \leq 32. \tag{11}$$

We then proceed to split the triangle K into four new triangles by uniform red refinement, that is, by connecting the midpoints of each edge, and compute the interpolate on each triangle. Hence, we have a $\mathcal{C}^1$ piecewise polynomial on K. The L^∞ error for the function and its derivatives up to order 2 is estimated as in (11) on each of the new triangles. This process of refinement is repeated up to 6 times. Notice that the finest grid consists of 4096 triangles. The results obtained are collected in Table I where we show also the estimated convergence rates (e.c.r.), defined by

$$\log_2(E(h)/E(h/2)),$$

where $E(h)$ and $E(h/2)$ are the errors on two consecutive grids.

We point out that the theoretical orders of convergence are 6 for the function, 5 for its first derivatives, and 4 for the derivatives of second order (see Ciarlet [1975]).

In the next experiment we have implemented the finite-element solution of the Dirichlet problem for the biharmonic equation. Let Ω be the regular hexagon inscribed in the unit circle and $\mathcal{T}_h$, the set of 34 triangles of the grid depicted in Figure 3. This initial grid was computed using the function `initmesh` of the partial differential equation toolbox of Matlab. On $\mathcal{T}_h$ and on its successively uniform refinements, in the same way as in the first experiment, we construct the $\mathcal{C}^1$ finite element spaces

$$V_h := \left\{u_h \in C^1(\Omega) \,\middle|\, u_h|_T \in \mathbb{P}_5,\ \ \forall T \in T_h\right\}, \qquad V_h^0 := \left\{u_h \in V_h \,\middle|\, \partial_\nu u_h = \gamma_\Gamma u_h = 0\right\}.$$

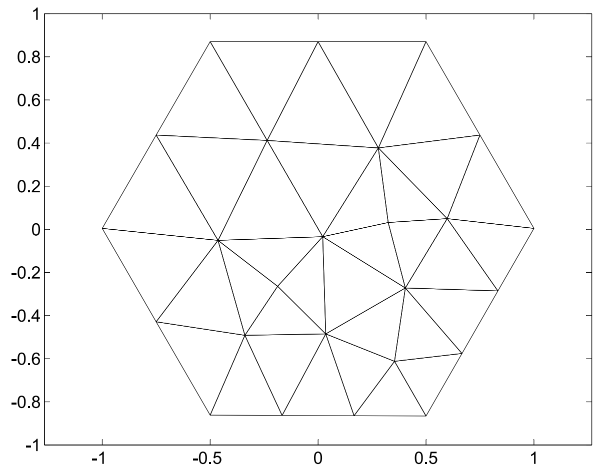

Fig. 3. Domain of the differential problem and initial grid.

Here γ_Γ and ∂_ν denote trace and outward normal derivative on Γ, the boundary of Ω. We consider the variational problem

$$u_h \in V_h^0, \qquad \text{such that} \qquad a(u_h, v_h) = \int_\Omega f\, v_h, \qquad \forall v_h \in V_h^0, \tag{12}$$

with

$$a(u, v) := \int_\Omega \left(\partial^2_{xx} u\, \partial^2_{xx} v + 2\, \partial^2_{xy} u\, \partial^2_{xy} v + \partial^2_{yy} u\, \partial^2_{yy} v\right).$$

The solution u_h of the numerical scheme is the finite-element solution of the boundary value problem

$$\Delta^2 u = f, \quad \text{in } \Omega, \qquad \gamma_\Gamma u = 0, \quad \partial_\nu u = 0. \tag{13}$$

In our test, the righthand side f is taken so that

$$u(x, y) = \prod_{i=1}^{6} (\alpha_i + \beta_i x - y)^2$$

is the exact solution of (13). In this expression, $y = \alpha_i + \beta_i x$ is the equation of the line containing the ith side of the hexagon.

The assembly of the matrix is done element-by-element in the usual finite element way. This is possible, as we remark in the Introduction, because for any triangle the elements of the global basis of V_h, whose support has nontrivial intersection with K, give the local Argyris basis of this triangle.

The computation of the matrix and of the righthand side requires the evaluation of some integrals involving the elements of the local basis of the Argyris element on each triangle and their second derivatives. These integrals are computed using the Dunavant rule of degree 7 (see Dunavant [1985]) which uses 13 evaluations of the function. Hence, we have a practical example where we can test the performance of our algorithms when they are put in the context of a practical problem. In our experiments the profile tool of Matlab has been

Table II. $L^2(\Omega)$ and $H^2(\Omega)$ Error of the Finite-Element Solution

	L^2−error	e.c.r.	H^2−error	e.c.r.
$\mathcal{T}_h^0$	3.59E-02		7.06E-01	
$\mathcal{T}_h^1$	2.27E-04	7.31	2.64E-02	4.74
$\mathcal{T}_h^2$	2.55E-06	6.48	1.35E-03	4.29
$\mathcal{T}_h^3$	1.90E-08	7.07	6.35E-05	4.41
$\mathcal{T}_h^4$	2.71E-10	6.13	2.89E-06	4.46

used to track the performance of the different functions involved. We have observed that the evaluation of the Argyris functions consumes a minor fraction of the CPU time spent in the assembly of the matrix. Moreover, the finer the grid, the less the percentage of CPU time used in the evaluation. In any case, the bulk of the computational time is spent when the local contributions of each triangle are transferred to the corresponding entries of the matrix.

Starting from an initial grid $\mathcal{T}_h^0$ of 34 triangles, Table II shows the error in $L^2(\Omega)$ and $H^2(\Omega)$ norms for different uniform refinements ($\mathcal{T}_h^j$ is the grid obtained after j steps of uniform refinement). Notice that the numerical solution seems to converge to the exact one somewhat faster than what the theory predicts: order 4 in H^2-norm and order 6 in L^2-norm. One possible explanation for this behavior is that finer grids are needed to make the higher-order components in the error negligible with respect to the leading terms.

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