23

Triangular Plate Displacement Elements

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§23.1. Introduction

This Chapter presents an overview of finite element models for thin plates using the Kirchhoff Plate Bending (KPB) model and the Total Potential Energy (TPE) vaariational principle.The derivation of shape functions for the triangle geometry is covered in the next Chapter.

The purpose of this Chapter is to explain the construction of displacement-based triangular Kirchhoff plate bending elements through the development of the necessary interpolation formulas. The resulting element is complete but does not satisfy full normal-slope conformity.

§23.2. Triangular Element Properties

We recall the following properties of a straight-sided triangular element, taken from Chapter 15 of IFEM.

§23.2.1. Triangle Geometry

The geometry of the 3-node triangle shown in Figure 23.1(a) is specified by the location of its three corner nodes on the $\{x, y\}$ plane. The nodes are labeled 1, 2, 3 while traversing the sides in *counterclockwise* fashion. The location of the corners is defined by their coordinates:

$$x_i,\ y_i, \qquad i = 1, 2, 3 \tag{23.1}$$

The area of the triangle is denoted by A and is given by

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1). \tag{23.2}$$

It is important to realize that the area given by formula (23.2) is a *signed* quantity. It is positive if the corners are numbered in counterclockwise order as shown in Figure 23.1(b). This convention is followed in the sequel.

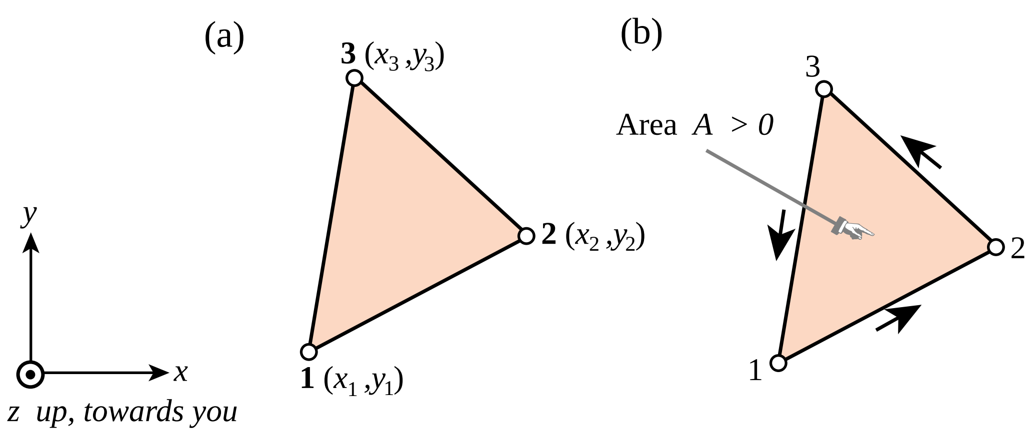

Figure 23.1. Geometry of stright-sided triangular element.

§23.2.2. Triangular Coordinates

Points of the triangle may also be located in terms of a *parametric* coordinate system:

$$\zeta_1,\ \zeta_2,\ \zeta_3. \tag{23.3}$$

In the literature these three parameters receive many names. In the sequel the name *triangular coordinates* will be used to stress its close association with this particular geometry.

§23.2.3. Triangular CoordinatesProperties

Equations

$$\zeta_i = constant \tag{23.4}$$

represent a set of straight lines parallel to the side opposite to the i^{th} corner. See Figure 23.2. The equation of sides 12, 23 and 31 are $\zeta_1 = 0$, $\zeta_2 = 0$ and $\zeta_3 = 0$, respectively. The three corners have coordinates (1,0,0), (0,1,0) and (0,0,1). The three midpoints of the sides have coordinates $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$, the centroid $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and so on. The coordinates are not independent because their sum is unity:

$$\zeta_1 + \zeta_2 + \zeta_3 = 1. \tag{23.5}$$

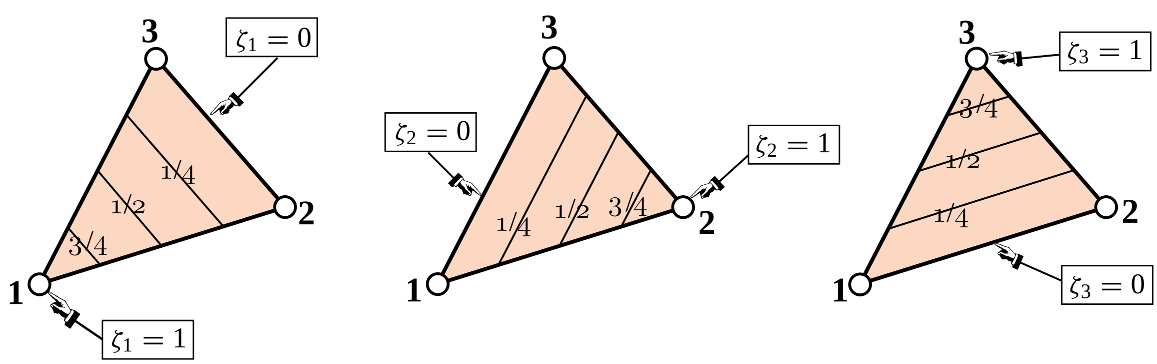

Figure 23.2. Triangular coordinates.

§23.2.4. Linear Interpolation

Consider a function $w(x, y)$ that varies *linearly* over the triangle domain. In terms of Cartesian coordinates it may be expressed as

$$w(x, y) = a_0 + a_1 x + a_2 y, \tag{23.6}$$

where a_0, a_1 and a_2 are coefficients to be determined from three conditions. In finite element work such conditions are often the *nodal values* taken by w at the corners:

$$w_1,\ \ w_2,\ \ w_3 \tag{23.7}$$

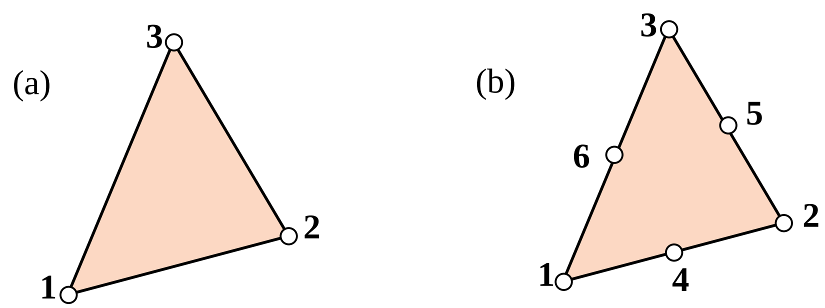

Figure 23.3. Nodal configurations for: (a) linear interpolation of w by three values w_i, at corners $i = 1, 2, 3$; (b) quadratic interpolation of w by six values w_i at corners $i = 1, 2, 3$ and midpoints $i = 4, 5, 6$.

The expression in triangular coordinates makes direct use of these three values:

$$w(\zeta_1, \zeta_2, \zeta_3) = w_1\zeta_1 + w_2\zeta_2 + w_3\zeta_3 = [\, w_1 \quad w_2 \quad w_3 \,] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = [\, \zeta_1 \quad \zeta_2 \quad \zeta_3 \,] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}. \tag{23.8}$$

Expression (23.8) is called a *linear interpolant* for w. See Figure 23.3(a).

§23.2.5. Coordinate Transformations

Quantities which are closely linked with the element geometry are naturally expressed in triangular coordinates. On the other hand, quantities such as displacements, strains and stresses are often expressed in the Cartesian system x, y. We therefore need transformation equations through which we can pass from one coordinate system to the other.

Cartesian and triangular coordinates are linked by the relation

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}. \tag{23.9}$$

The first equation says that the sum of the three coordinates is one. The second and third express x and y linearly as homogeneous forms in the triangular coordinates. These simply apply the linear interpolant formula (23.8) to the Cartesian coordinates: $x = x_1\zeta_1 + x_2\zeta_2 + x_3\zeta_3$ and $y = y_1\zeta_1 + y_2\zeta_2 + y_3\zeta_3$.

Inversion of (23.9) yields

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}. \tag{23.10}$$

Here $x_{jk} = x_j - x_k$, $y_{jk} = y_j - y_k$, A is the triangle area given by (23.2) and A_{jk} denotes the area subtended by corners j, k and the origin of the x–y system. If this origin is taken at the centroid of the triangle, $A_{23} = A_{31} = A_{12} = A/3$.

§23.2.6. Partial Derivatives

From equations (23.9) and (23.10) we immediately obtain the following relations between partial derivatives:

$$\frac{\partial x}{\partial \zeta_i} = x_i, \qquad \frac{\partial y}{\partial \zeta_i} = y_i, \tag{23.11}$$

$$2A\frac{\partial \zeta_i}{\partial x} = y_{jk}, \qquad 2A\frac{\partial \zeta_i}{\partial y} = x_{kj}. \tag{23.12}$$

In (23.12) j and k denote the *cyclic permutations* of i. For example, if $i = 2$, then $j = 3$ and $k = 1$. The derivatives of a function $w(\zeta_1, \zeta_2, \zeta_3)$ with respect to x or y follow immediately from (23.12) and application of the chain rule:

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{2A}\left(\frac{\partial w}{\partial \zeta_1} y_{23} + \frac{\partial w}{\partial \zeta_2} y_{31} + \frac{\partial w}{\partial \zeta_3} y_{12}\right) \\ \frac{\partial w}{\partial y} &= \frac{1}{2A}\left(\frac{\partial w}{\partial \zeta_1} x_{32} + \frac{\partial w}{\partial \zeta_2} x_{13} + \frac{\partial w}{\partial \zeta_3} x_{21}\right) \end{aligned} \tag{23.13}$$

which in matrix form is

$$\begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \dfrac{\partial w}{\partial \zeta_1} \\ \dfrac{\partial w}{\partial \zeta_2} \\ \dfrac{\partial w}{\partial \zeta_3} \end{bmatrix}. \tag{23.14}$$

§23.2.7. Six Node Quadratic Interpolation

Consider next the six-node triangle shown in Figure 23.3(b). This element still has straight sides, and nodes 4, 5 and 6 are located at the midpoint of the sides. In Chapter 16 of the IFEM course it was shown a function $w(x, y)$ that varies quadratically over the element and takes on the node values w_i, $i = 1, 2, 3, 4, 5, 6$, can be interpolated in terms of the triangular coordinates by the formula

$$w = [\, w_1 \;\; w_2 \;\; w_3 \;\; w_4 \;\; w_5 \;\; w_6 \,] \begin{bmatrix} N_1^q \\ N_2^q \\ N_3^q \\ N_4^q \\ N_5^q \\ N_6^q \end{bmatrix}. \tag{23.15}$$

where the N_i^q are the quadratic shape functions

$$\begin{aligned} &N_1^q = \zeta_1(2\zeta_1 - 1) \quad && N_2^q = \zeta_2(2\zeta_2 - 1) \quad && N_3^q = \zeta_3(2\zeta_3 - 1) \\ &N_4^q = 4\zeta_1\zeta_2 && N_5^q = 4\zeta_2\zeta_3 && N_6^q = 4\zeta_3\zeta_1 \end{aligned} \tag{23.16}$$

The geometry is still defined by (23.9) because we are not permitting nodes 4,5,6 to be away from the midpoint positions. Similarly, the partial derivative expressions of §23.1.6 remain valid.

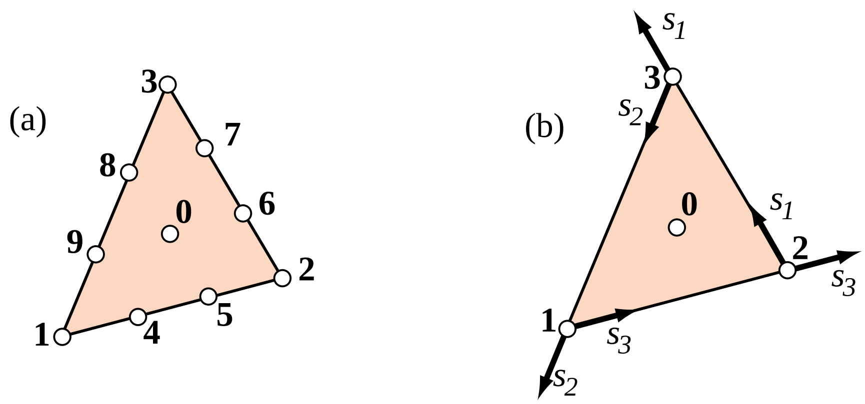

Figure 23.4. Nodal configurations for two forms of cubic interpolation of $w(x, y)$ over triangle: (a) interpolation by ten values w_i at corners $i = 1, 2, 3$, thirdpoints $i = 4, 5, 6, 7, 8, 9$ and centroid $i = 0$; (b) interpolation by four w_i values at corners $i = 1, 2, 3$ and centroid $i = 0$, plus six side slopes $(\partial w/\partial s_2)_1$, $(\partial w/\partial s_3)_1$, $(\partial w/\partial s_3)_2$, $(\partial w/\partial s_1)_2$, $(\partial w/\partial s_1)_3$ and $(\partial w/\partial s_2)_3$.

§23.3. Cubic Interpolation

Cubic interpolants for w are fundamental in the construction of the simplest Kirchhoff plate bending elements using the TPE functional. The complete two-dimensional cubic polynomial has 10 terms. There are several choices for the selection of nodal values that determine that interpolation. Several important ones are examined next.

§23.3.1. Cubic Interpolation Choice 1: 10 Nodes

Next consider the ten-node triangle shown in Figure 23.4(a). This element still has straight sides. Nodes 4 through 9 are placed at the thirdpoints of the sides as indicated. The tenth node is placed at the centroid and is labeled 0.

In a Homework of the IFEM course it was shown that a function $w(x, y)$ that varies cubically over the element and takes on the node values w_i, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 0$, can be interpolated in terms of the triangular coordinates by the formula

$$w = [\, w_1 \;\; w_2 \;\; w_3 \;\; w_4 \;\; w_5 \;\; w_6 \;\; w_7 \;\; w_8 \;\; w_9 \;\; w_0 \,] \begin{bmatrix} N_1^{c1} \\ N_2^{c1} \\ N_3^{c1} \\ N_4^{c1} \\ N_5^{c1} \\ N_6^{c1} \\ \vdots \\ N_0^{c1} \end{bmatrix}. \tag{23.17}$$

where the N_i^{c1} are the 10-node cubic shape functions

$$
\begin{array}{lll}
N_1^{c1} = \frac{1}{2}\zeta_1(3\zeta_1-1)(3\zeta_1-2) & N_2^{c1} = \frac{1}{2}\zeta_2(3\zeta_2-1)(3\zeta_2-2) & N_3^{c1} = \frac{1}{2}\zeta_3(3\zeta_3-1)(3\zeta_3-2) \\
N_4^{c1} = \frac{9}{2}\zeta_1\zeta_2(3\zeta_1-1) & N_5^{c1} = \frac{9}{2}\zeta_1\zeta_2(3\zeta_2-1) & N_6^{c1} = \frac{9}{2}\zeta_2\zeta_3(3\zeta_2-1) \\
N_7^{c1} = \frac{9}{2}\zeta_2\zeta_3(3\zeta_3-1) & N_8^{c1} = \frac{9}{2}\zeta_3\zeta_1(3\zeta_3-1) & N_9^{c1} = \frac{9}{2}\zeta_3\zeta_1(3\zeta_1-1) \\
N_0^{c1} = 27\zeta_1\zeta_2\zeta_3 & &
\end{array}
\tag{23.18}
$$

The geometry is still defined by (23.9) because we are not permitting nodes 4 through 9 to be away from the thirdpoint positions as well as forcing 0 to be exactly at the centroid. Similarly, the partial derivative expressions of §23.1.6 remain valid.

§23.3.2. Cubic Interpolation Choice 2: 4 Nodes plus 6 Side Slopes

Consider now a variant of the cubic interpolation over the triangle as shown in Figure 23.4(b). This element still has straight sides. In addition to values at the corners 1,2,3 and the centroid 0, we we specify the side-slope corner derivatives $w_{jk} = (\partial w/\partial s_j)_k$. There are six combinations: $w_{21} = (\partial w/\partial s_2)_1$, $w_{31} = (\partial w/\partial s_3)_1$, $w_{32} = (\partial w/\partial s_3)_2$, $w_{12} = (\partial w/\partial s_1)_2$, $w_{13} = (\partial w/\partial s_1)_3$ and $w_{23} = (\partial w/\partial s_2)_3$. The resulting interpolation is

$$
w = [\, w_1 \;\; w_{21} \;\; w_{31} \;\; w_2 \;\; w_{32} \;\; w_{12} \;\; w_3 \;\; w_{13} \;\; w_{23} \;\; w_0 \,] \begin{bmatrix} N_1^{c2} \\ N_2^{c2} \\ N_3^{c2} \\ N_4^{c2} \\ N_5^{c2} \\ N_6^{c2} \\ \vdots \\ N_0^{c2} \end{bmatrix}. \tag{23.19}
$$

in which the N_i^{c2} are the shape functions

$$
\begin{array}{ll}
N_1^{c2} = \zeta_1^2(\zeta_1+3\zeta_2+3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_2^{c2} = -L_{31}(\zeta_3\zeta_1^2 - \zeta_1\zeta_2\zeta_3) \\
N_3^{c2} = L_{12}(\zeta_1\zeta_2^2 - \zeta_1\zeta_2\zeta_3) & N_4^{c2} = \zeta_2^2(3\zeta_1+\zeta_2+3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\
N_5^{c2} = -L_{12}(\zeta_1\zeta_2^2 - \zeta_1\zeta_2\zeta_3) & N_6^{c2} = L_{23}(\zeta_2\zeta_3^2 - \zeta_1\zeta_2\zeta_3) \\
N_7^{c2} = \zeta_3^2(3\zeta_1+3\zeta_2+\zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_8^{c2} = -L_{23}(\zeta_2\zeta_3^2 - \zeta_1\zeta_2\zeta_3) \\
N_9^{c2} = L_{31}(\zeta_3\zeta_1^2 - \zeta_1\zeta_2\zeta_3) & N_0^{c2} = 27\zeta_1\zeta_2\zeta_3
\end{array}
\tag{23.20}
$$

and L_{jk} denotes the length of the side joining corners j and k. The function of the corrective terms $\zeta_1\zeta_2\zeta_3$ is to make the first nine functions (23.20) vanish at 0.

The $c1$ and $c2$ formulas are related by the transformation matrix

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_0 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 \\ 20 & 0 & 4L_{12} & 7 & -2L_{12} & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 2L_{12} & 20 & -4L_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 4L_{23} & 7 & -2L_{23} & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 2L_{23} & 20 & -4L_{23} & 0 & 0 \\ 7 & -2L_{31} & 0 & 0 & 0 & 0 & 20 & 0 & 4L_{31} & 0 \\ 20 & -4L_{31} & 0 & 0 & 0 & 0 & 7 & 0 & 2L_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 \end{bmatrix} \begin{bmatrix} w_1 \\ w_{21} \\ w_{31} \\ w_2 \\ w_{32} \\ w_{12} \\ w_3 \\ w_{13} \\ w_{23} \\ w_0 \end{bmatrix} \tag{23.21}$$

§23.3.3. Cubic Interpolation Choice 3: 4 Nodes plus 6 Cartesian Slopes

This choice is similar to the previous one but the corner derivatives are taken with respect to the directions x, y of the Cartesian reference system. The notation used for the slopes is $w_{x1} = (\partial w/\partial x)_1$, $w_{y1} = (\partial w/\partial y)_1$, $w_{x2} = (\partial w/\partial x)_2$, $w_{y2} = (\partial w/\partial y)_2$, $w_{x3} = (\partial w/\partial x)_3$ and $w_{y3} = (\partial w/\partial y)_3$. The resulting interpolation is

$$w = [\, w_1 \;\; w_{x1} \;\; w_{y1} \;\; w_2 \;\; w_{x2} \;\; w_{y2} \;\; w_3 \;\; w_{x3} \;\; w_{y3} \;\; w_0 \,] \begin{bmatrix} N_1^{c3} \\ N_2^{c3} \\ N_3^{c3} \\ N_4^{c3} \\ N_5^{c3} \\ N_6^{c3} \\ \vdots \\ N_0^{c3} \end{bmatrix}. \tag{23.22}$$

in which the N_i^{c3} are the shape functions

$$\begin{aligned} N_1^{c3} &= \zeta_1^2(\zeta_1 + 3\zeta_2 + 3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_2^{c3} &= \zeta_1^2(x_{21}\zeta_2 - x_{13}\zeta_3) + (x_{13} - x_{21})\zeta_1\zeta_2\zeta_3 \\ N_3^{c3} &= \zeta_1^2(y_{21}\zeta_2 - y_{13}\zeta_3) + (y_{13} - y_{21})\zeta_1\zeta_2\zeta_3 & N_4^{c3} &= \zeta_2^2(3\zeta_1 + \zeta_2 + 3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\ N_5^{c3} &= \zeta_2^2(x_{32}\zeta_3 - x_{21}\zeta_1) + (x_{21} - x_{32})\zeta_1\zeta_2\zeta_3 & N_6^{c3} &= \zeta_2^2(y_{32}\zeta_3 - y_{21}\zeta_1) + (y_{21} - y_{32})\zeta_1\zeta_2\zeta_3 \\ N_7^{c3} &= \zeta_3^2(3\zeta_1 + 3\zeta_2 + \zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_8^{c3} &= \zeta_3^2(x_{13}\zeta_1 - x_{32}\zeta_2) + (x_{32} - x_{13})\zeta_1\zeta_2\zeta_3 \\ N_9^{c3} &= \zeta_3^2(y_{13}\zeta_1 - y_{32}\zeta_2) + (y_{32} - y_{13})\zeta_1\zeta_2\zeta_3 & N_0^{c3} &= 27\zeta_1\zeta_2\zeta_3 \end{aligned} \tag{23.23}$$

in which $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$.

§23.3.4. Cubic Interpolation Choice 4: 4 Nodes plus 6 Cartesian Rotations

This interpolation is the same as the previous one, with some rearrangements. The rotational degrees of freedom at the corners are introduced using the relations $\theta_x = \partial w/\partial y$ and $\theta_y = -\partial w/\partial x$.. The resulting interpolation is

$$w = [\, w_1 \ \ \theta_{x1} \ \ \theta_{y1} \ \ w_2 \ \ \theta_{x2} \ \ \theta_{y2} \ \ w_3 \ \ \theta_{x3} \ \ \theta_{y3} \ \ w_0 \,] \begin{bmatrix} N_1^{c4} \\ N_2^{c4} \\ N_3^{c4} \\ N_4^{c4} \\ N_5^{c4} \\ N_6^{c4} \\ \vdots \\ N_0^{c4} \end{bmatrix}. \tag{23.24}$$

in which the N_i^{c4} are the same shape functions of Choice 3 with some sign and index changes:

$$\begin{array}{lll} N_1^{c4} = N_1^{c3}, & N_2^{c4} = N_3^{c3}, & N_3^{c4} = -N_2^{c3}, \\ N_4^{c4} = N_4^{c3}, & N_5^{c4} = N_6^{c3}, & N_6^{c4} = -N_5^{c3}, \\ N_7^{c4} = N_7^{c3}, & N_8^{c4} = N_9^{c3}, & N_9^{c4} = -N_8^{c3}, \\ N_0^{c4} = N_0^{c3}. & & \end{array} \tag{23.25}$$

§23.4. The BCIZ Plate Bending Element

One of the simplest Kirchhoff plate bending elements[1] was presented in 1966 by Bazeley, Cheung, Irons and Zienkiewicz [66]. This is called the “BCIZ element” after the authors’ initials.

This element can be derived from the cubic interpolation choice 4 (23.24). The technique appears a bit mysterious at first. Basically one can construct a 10×10 plate stiffness matrix. Freedom w_0 at the centroid can be statically condensed out and the resulting 9×9 stiffness used in finite element analysis. Unfortunately the static condensation destroys curvature completeness and the solutions will not generally converge, as emperically shown in Tocher’s thesis [770]. The idea behing the BCIZ element is that elimination of the centroidal DOF is done in such a way that completeness is maintained, using a kinemtic constraint.

§23.4.1. The Kinematic Constraint

As explained above, we seek a kinematic constraint to eliminate w_0 in terms of the nine connector DOFs:

$$w_0 = [\, a_{w1} \ \ a_{\theta x1} \ \ a_{\theta y1} \ \ a_{w2} \ \ a_{\theta x2} \ \ a_{\theta y2} \ \ a_{w3} \ \ a_{\theta x3} \ \ a_{\theta y3} \,] \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \end{bmatrix} \tag{23.26}$$

[1] Historically the first triangular plate bending element that satisfied completeness and invariance. This element is also important as the motivation for development of the patch test.

Substituting this into any of the 4 choices of the previous section, w_0 is eliminated. For choice 4:

$$w = [\, w_1 \;\; \theta_{x1} \;\; \theta_{y1} \;\; w_2 \;\; \theta_{x2} \;\; \theta_{y2} \;\; w_3 \;\; \theta_{x3} \;\; \theta_{y3} \,] \begin{bmatrix} N_1^{c4} + a_{w1} N_0 \\ N_2^{c4} + a_{\theta x1} N_0 \\ N_3^{c4} + a_{\theta y1} N_0 \\ \vdots \\ N_9^{c4} + a_{\theta y3} N_0 \end{bmatrix}. \tag{23.27}$$

in which $N_0 = 27\zeta_1\zeta_2\zeta_3$ is the same for all choices. To determine the a's in (23.27) we impose the condition that *all constant curvature states must be exactly represented*. This is a completeness condition. Constant curvature states are associated with quadratic variations of w, which are defined by the 6-node interpolation (23.16). The procedure is as follows:

(i) Use (23.16) to express w_0 in terms of w_1 through w_6 by setting $\zeta_1 = \zeta_2 = \zeta_3 = 1/3$.

(ii) Take the cubic interpolation choice 4 (23.24) and express the value of w_4, w_5 and w_6 in terms of the nine connection DOF w_1 through θ_{y3} by setting ζ_1, ζ_2 and ζ_3 to appropriate midpoint coordinates.

(iii) Eliminate w_4, w_5 and w_6 from (i) and (ii) to get (23.26) and hence (23.27). The resulting formula is called the BCIZ interpolation. The associated $N's$ are the BCIZ shape functions.

It can be shown that the constraint is

$$\begin{aligned} \mathbf{q} = [&1/3, (-2x_1 + x_2 + x_3)/18, (-2y_1 + y_2 + y_3)/18, \\ &1/3, (x_1 - 2x_2 + x_3)/18, (y_1 - 2y_2 + y_3)/18, \\ &1/3, (x_1 + x_2 - 2x_3)/18, (y_1 + y_2 - 2y_3)/18] \end{aligned} \tag{23.28}$$

The derivation of the entries of (23.28) is the subject of a homework Exercise.

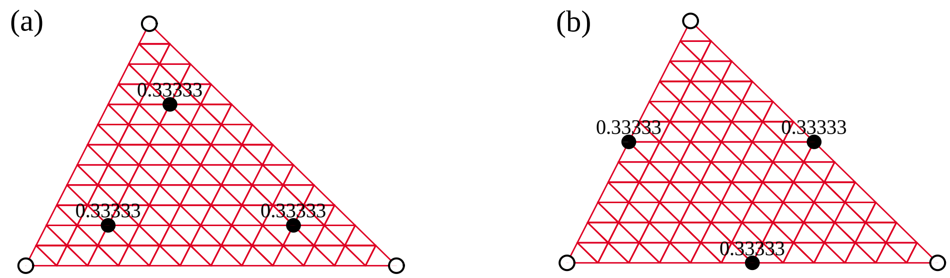

Figure 23.5. The two 3-point Gauss integration rules for triangles, which are useful in the computation of the plate element stiffness matrix **K**. Numbers annotated near sample points are the weights.

§23.4.2. The Curvature Displacement Matrix

Once we have an interpolation formula for w, as in (23.27), the curvatures over the plate element can be evaluated by double differentiation of the shape functions with respect to x and y. The resulting relation can be expressed in matrix form as

$$\begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix} = \begin{bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial y \partial y \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & B_{19} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} & B_{27} & B_{28} & B_{29} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & B_{38} & B_{39} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \end{bmatrix} \tag{23.29}$$

or

$$\boldsymbol{\kappa} = \mathbf{B}\mathbf{u} \tag{23.30}$$

The entries B_{ij} are *linear* in the triangular coordinates ζ_1, ζ_2 and ζ_3.

The derivation of the entries of the **B** matrix is the subject of a homework Exercise.

§23.4.3. The Element Stiffness Matrix

For this displacement element, the stiffness matrix is given by the usual formula

$$\mathbf{K} = \int_{\Omega^{(e)}} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \tag{23.31}$$

where $\Omega^{(e)}$ is the element area, **D** is the plate rigidity matrix that relates moment to curvatures (see §9.3.2) and **B** is the curvature-displacement matrix defined in (23.30).

If **D** is constant over the element, as frequently assumed, the integrand of **K** is quadratic in the triangular coordinates. If so the integral can be done exactly using one of the two 3-point Gauss rule presented in Chapter 23 of IFEM. The two rules are reproduced below for convenience:

$$\frac{1}{A}\int_{\Omega^{(e)}} F(\zeta_1, \zeta_2, \zeta_3)\, d\Omega = \tfrac{1}{3}F(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}) + \tfrac{1}{3}F(\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{6}) + \tfrac{1}{3}F(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{2}{3}). \tag{23.32}$$

$$\frac{1}{A}\int_{\Omega^{(e)}} F(\zeta_1,\zeta_2,\zeta_3)\,d\Omega = \tfrac{1}{3}F(\tfrac{1}{2},\tfrac{1}{2},0)+\tfrac{1}{3}F(0,\tfrac{1}{2},\tfrac{1}{2})+\tfrac{1}{3}F(\tfrac{1}{2},0,\tfrac{1}{2}). \tag{23.33}$$

The second one is called *the midpoint rule* because the three sample points are at the triangle midpoints. These rules are depicted in Figures 23.5(a) and (b), respectively. Both are exact up to quadratic polynomials in the triangular coordinates, which is what is needed for the plate stiffness matrix.

Notes and Bibliography

The foregoing shape functions were first developed and presented in terms of triangular coordinates in the 1966 thesis [211].

Homework Exercises for Chapter 23
Triangular Plate Displacement Elements

EXERCISE 23.1 [A:15] Check that the shape functions given for cubic interpolation choice 4 satisfy the conditions of being 1 for the associated DOF, and zero for all others. For example N_2, which is associated with θ_{x1}, must satisfy

$$N_2(1,0,0) = 0, \qquad \frac{\partial N_2}{\partial y}(1,0,0) = 1 = \theta_{x1}, \qquad -\frac{\partial N_2}{\partial x}(1,0,0) = 0 = \theta_{y1}, \qquad \text{etc.} \tag{E23.1}$$

Note: It is sufficient to verify N_1, N_2, and N_3 because the next six shape functions are obtained by cyclic permutation of subscripts. N_0 is easy to check since its corner value and corner slopes are zero.

EXERCISE 23.2 [A:15] Using the chain rule and the results of §23.1.6, show that the following formula can be used to compute the plate curvatures from an interpolation formula in w:

$$\begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23}^2 & x_{32}^2 & 2x_{32}y_{23} \\ y_{31}^2 & x_{13}^2 & 2x_{13}y_{31} \\ y_{12}^2 & x_{21}^2 & 2x_{21}y_{12} \\ 2y_{23}y_{31} & 2x_{32}x_{13} & 2(x_{32}y_{31} + x_{13}y_{23}) \\ 2y_{31}y_{12} & 2x_{13}x_{21} & 2(x_{13}y_{12} + x_{21}y_{31}) \\ 2y_{12}y_{23} & 2x_{21}x_{32} & 2(x_{21}y_{23} + x_{32}y_{12}) \end{bmatrix}^T \begin{bmatrix} \dfrac{\partial^2 w}{\partial \zeta_1^2} \\ \dfrac{\partial^2 w}{\partial \zeta_2^2} \\ \dfrac{\partial^2 w}{\partial \zeta_3^2} \\ \dfrac{\partial^2 w}{\partial \zeta_1 \partial \zeta_2} \\ \dfrac{\partial^2 w}{\partial \zeta_2 \partial \zeta_3} \\ \dfrac{\partial^2 w}{\partial \zeta_3 \partial \zeta_1} \end{bmatrix}, \tag{E23.2}$$

EXERCISE 23.3 [A:15] Derive the expression for the $a's$ in (23.27) following the procedure outline in that section, and derive the modified shape functions appearing in (23.28).

EXERCISE 23.4 [A:25] Program the BCIZ element starting from the interpolation formula provided by the previous exercise, the expression (E23.2) to form the curvature-displacement matrix $\mathbf{B}$ and one of the 3-point Gauss rules given in §233.3 to evaluate $\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} \mathbf{B}^T\, \mathbf{D}\, \mathbf{B}\, d\Omega$. Assume that $\mathbf{D} = (h^3/12)\mathbf{E}$ in which both the thickness h and the 3×3 stress-strain matrix $\mathbf{E}$ are constant over the element, and supplied through arguments.

EXERCISE 23.5 [A:15] Evaluate $\mathbf{K}^{(e)}$ of the BCIZ element for a triangle with corners at $(0,0)$, $(2,1)$ and $(1,3)$, fabricated with isotropic material of $E = 120$ and $\nu = 0$, and constant thickness $h = 1$. As a check, compute the 9 eigenvalues; three of them should be zero and six positive.