22

Thin Plate Bending Elements: Overview

TABLE OF CONTENTS

		Page
§22.1.	**Introduction**	22–3
§22.2.	**Overview of KTP FE Models**	22–3
	§22.2.1. Triangles	22–3
	§22.2.2. A Potpourri of Freedom Configurations	22–3
	§22.2.3. Connecting Plate Bending Elements	22–5
§22.3.	**Convergence Conditions**	22–5
	§22.3.1. Completeness	22–6
	§22.3.2. Continuity Games	22–6
§22.4.	**Kinematic Limitation Principles**	22–8
	§22.4.1. Limitation Theorem I	22–9
	§22.4.2. Limitation Theorem II	22–9
	§22.4.3. Limitation Theorem III	22–10
§22.5.	**A Brief History Of Plate Bending FEM**	22–11
	§22.5.1. Early Work	22–11
	§22.5.2. Rectangular Elements	22–11
	§22.5.3. Triangular Elements	22–12
	§22.5.4. Quadrilateral Elements	22–13
	§22.5.5. More Recent Work	22–13

§22.1. Introduction

This Chapter presents an overview of finite element models for thin plates usinf the Kirchhoff Plate Bending (KPB) model. The derivation of shape functions for the triangle geometry is covered in the next Chapter.

§22.2. Overview of KTP FE Models

The plate domain Ω is subdivided into finite elements in the usual way, as illustrated in Figure 22.1. The most widely used geometries are triangles and quadrilaterals with straight sides. Curved side KPB elements are rare. They are more widely seen in shear-endowed C^0 models derived by the degenerate solid approach.[1]

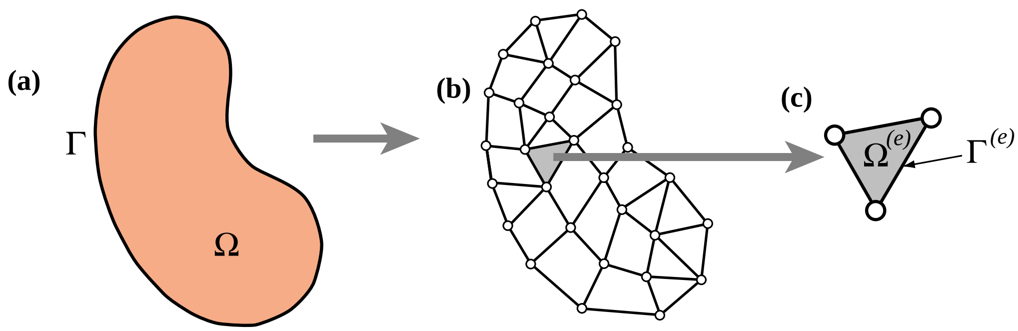

FIGURE 22.1. A thin plate subdivided into finite elements.

§22.2.1. Triangles

This and following Chapters will focus on KPB *triangular* elements only. These triangles will invariably have straight sides. Their geometry is defined by the position of the three corners as pictured in Figure 22.2(a). The positive sense of traversal of the boundary is shown in Figure 22.2. This sense defines three side directions: s_1, s_2 and s_3, which are aligned with the sides opposite corners 1, 2 and 3, respectively. The *external* normal directions n_1, n_2 and n_3 shown there are oriented at -90° from s_1, s_2 and s_3.[2]

The area of the triangle, denoted by A, is a signed quantity given by

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1). \tag{22.1}$$

We shall require that $A > 0$.

[1] Those are 3D solid elements one of which natural directions is designated as "thickness direction" and shrinked.

[2] This means that $\{n_i, s_i, z\}$ for $i = 1, 2, 3$ form three right-handed RCC systems, one for each side.

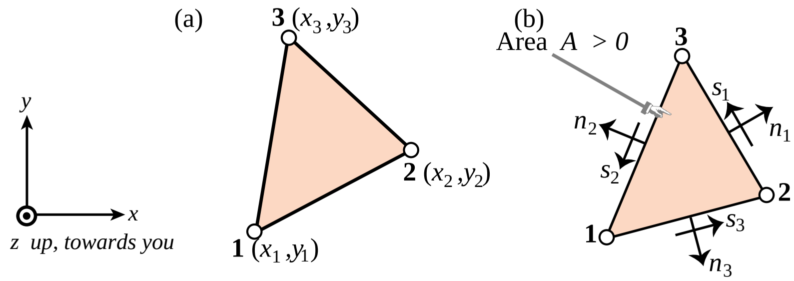

FIGURE 22.2. Triangular geometry and intrinsic directions.

§22.2.2. A Potpourri of Freedom Configurations

In KPB elements treated by assumed transverse displacements, the minimum polynomial expansion of w to achieve at least partly the compatibility requirements, is cubic. A complete cubic has 10 terms and consequently can accomodate 10 element degrees of freedom (DOFs). Figure 22.3 shows several 10-DOF configurations from which the cubic interpolation over the complete triangle can be written as an interpolation formula, with shape functions expressed in terms of the geometry data and the triangular coordinates. Those interpolation formulas are studied in the next Chapter.

Because a complete polynomial is invariant with respect to a change in basis, all of the configurations depicted in Figure 22.3 are *equivalent* in providing the *same* interpolation over the triangle. They differ, however, when connecting to adjacent elements. Only configuration (f) is practical for connecting elements over arbitrary meshes using the Direct Stiffness Method (DSM). The other configurations are valuable in intermediate derivations, or for various theoretical studies.

The 10-node configuration (a) specifies the cubic by the 10 values w_i, $i = 1, \ldots 10$, of the deflection at corners, thirdpoints of sides, and centroid. This is a useful starting point because the associated shape functions can be constructed directly using the technique explained in Chapter 17 of IFEM. The resulting plate element is useless, however, because it does not enforce interelement C^1 continuity at any boundary point.

From (a) one can pass to any of (b) through (d), the choice being primarily a matter of taste or objectives. Configurations (b) and (d) use the six corner partial derivatives of w along the side directions or the normal to the sides, respectively. The notation is $w_{sij} = (\partial w/\partial s_i)_j$ and $w_{nij} = (\partial w/\partial n_i)_j$, where i is the side index and j the corner index. (Sides are identified by the number of the opposite corner.) For example, $w_{s21} = (\partial w/\partial s_2)_1$. These partials are briefly called *side slopes* and *normal slopes*, respectively, on account of their physical meaning.

According to the fundamental kinematic assumption of the KPB model, a w slope along a midsurface direction is equivalent for small deflections to a *midsurface rotation* about a line perpendicular to and forming a -90° angle with that direction. Rotations about the s_i and n_i directions are called *side rotations* and *normal rotations*, respectively, for brevity.[3]

For example at corner 1, normal rotation θ_{n21} equals side slope w_{s21}. Replacing the six side-slope DOF w_{sij} of Figure 22.3(b) by the normal rotations θ_{nij} produces configuration (c). Similarly, replacing the six normal-slope DOF w_{nij} of Figure 22.3(d) by the side rotations θ_{sij} produces

[3] Note that in passing from slopes to equivalent rotations, the qualifiers "side" and "normal" exchange.

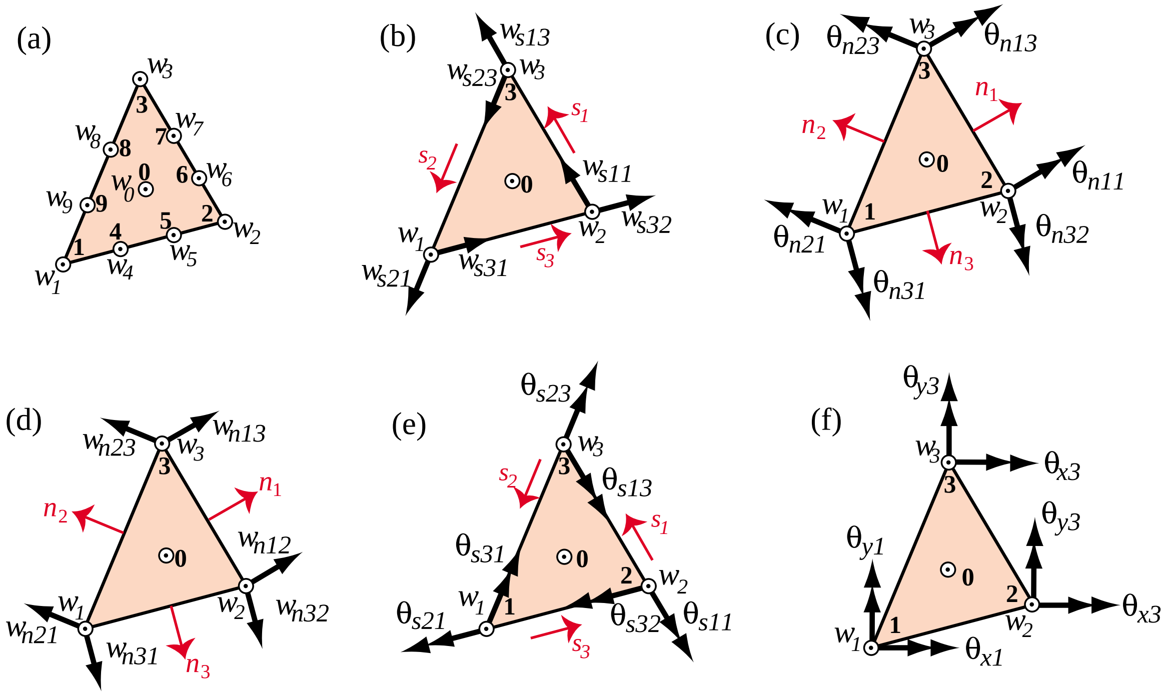

FIGURE 22.3. 10-DOF configurations for expressing the complete cubic interpolation of the lateral deflection w over a KPB triangle. Normal to sides and tangential directions identified by red arrows to avoid confusion with DOFs shown in black.

configuration (e). Note that the positive sense of the θ_{sij}, viewed as vectors, is opposite that of s_i; this is a consequence of the sign conventions and positive-rotation rule.

§22.2.3. Connecting Plate Bending Elements

If corner slopes along two noncoincident directions are given, the slope along any other corner direction is known. The same is true for corner rotations. It follows that for any of the configurations of Figures 22.3(b) through (e), *the deflection and tangent plane at the 3 corners are known*. However that information *cannot* be readily communicated to adjacent elements.

The difficulties are illustrated with Figure 22.4(a), which shows two adjacent triangles: red element $(e1)$ and blue element $(e2)$, possessing the DOF configuration of Figure 22.3(b). The deflections w_1 and w_2 match without problems because direction z is shared. But the color-coded side slopes do not match.[4] For more elements meeting at a corner the result is chaotic.

To make the element suitable for implementation in a DSM-based program, it is necessary to transform slope or rotational DOFs to *global directions*. The obvious choices are the midsurface axes $\{x, y\}$. Most FEM codes use rotations instead of slopes since that simplifies connection of different element types (e.g., shells to beams) in three dimensions. Choosing corner rotations θ_{xi} and θ_{yi} as DOF we are led to the configuration of Figure 22.3(f). As illustrated in Figure 22.4(b), the connection problem is solved and the elements are now suitable for the DSM.

[4] Positive slopes along the common side point in opposite directions.

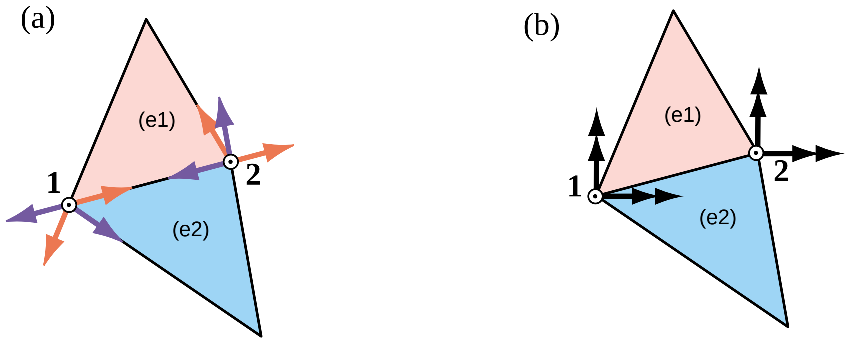

FIGURE 22.4. Connecting KPB elements.

§22.3. Convergence Conditions

The foregoing exposition has centered on displacement assumed elements where w is the master field. Element stiffness equations are obtained through the Total Potential Energy (TPE) variational principle presented in the previous Chapter. The completeness and continuity requirements are summarized in §21.3 on the basis of a variational index $m_w = 2$. These are now studied in more detail for cubic triangles.

§22.3.1. Completeness

The TPE variational index $m_w = 2$ requires that all w-polynomials of order 0, 1 and 2 in $\{x, y\}$ be exactly represented over each element. Constant and linear polynomials represent rigid body motions, whereas quadratic polynomials represent constant curvature states.

Now if w is interpolated by a complete cubic, the ten terms $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$ are automatically present for any freedom configuration. This appears to be more than enough. Nothing to worry about, right? Wrong. Preservation of such terms over each triangle is guaranteed only if full C^1 continuity is verified. But, as discussed below, attaining C^1 continuity continuity is difficult. To get it one while sticking to cubics one must make substantial changes in the construction of w. Since those changes do not necessarily preserve completeness, that requirement appears as an *a posteriori* constraint. Alternatively, to make life easier C^1 continuity may be abandoned except at corners. If so completeness may again be lost, for example by a seemingly harmless static condensation of w_0. Again this has to be kept as a constraint.

The conclusion is that *completeness cannot be taken for granted* in displacement-assumed KPB elements. Gone is the "IFEM easy ride" of isoparametric elements for variational index 1.

§22.3.2. Continuity Games

To explain what C^1 interelement continuity entails, it is convenient to break this condition into two levels:[5]

C^0 *Continuity*. The element is C^0 compatible if w over any side is completely specified by DOFs on that side.

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[5] It is tacitly assumed that the condition is satisfied inside the element.

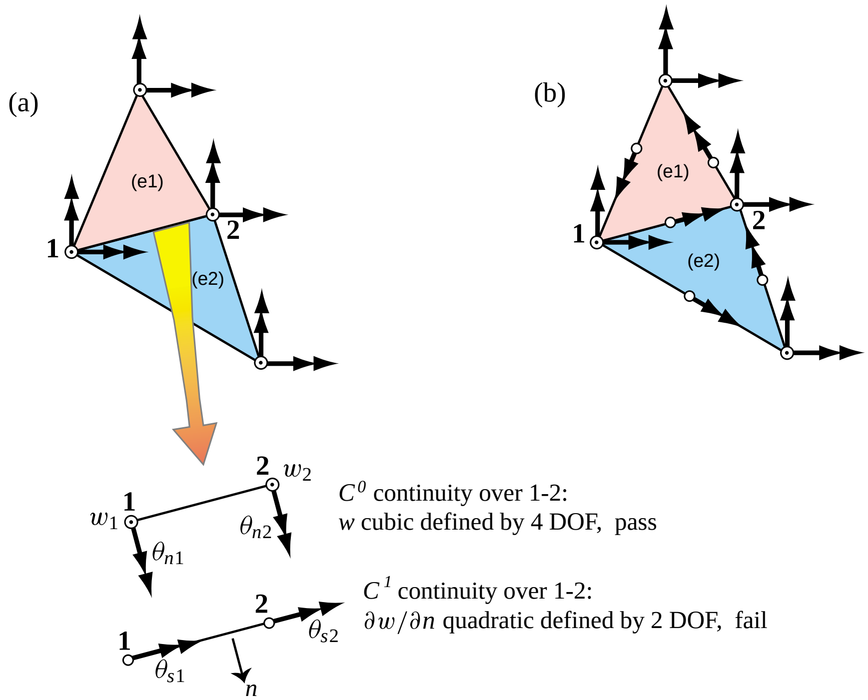

FIGURE 22.5. Checking interelement continuity of KPB triangles along common side 1-2.

C^1 *Continuity*. The element is C^1 compatible if it is C^0 compatible, *and* the normal slope $\partial w/\partial n$ over any side is completely specified by DOFs on that side.

The first level: C^0 continuity, is straightforward. In the IFEM course, which for 2D problems deals with variational index $m = 1$ only, the condition is easily achieved with the isoparametric formulation. The second level is far more difficult. Attaining it is the exception rather than the rule, and elements that make it are not necessarily the best performing ones. Nevertheless it is worth studying since so many theory advances in FEM: hybrid principles, the patch test, etc., came as a result of research in C^1 plate elements.

To appreciate the difficulties in attaining C^1 continuity consider two cubic triangles with the freedom configuration of Figure 22.3(f) connected as shown in Figure 22.5(a). At corners 1 and 2 the rotational freedoms are rotated to align with common side 1-2 and its normal as shown underneath Figure 22.5(a). Over side 1-2 the deflection w varies cubically. This variation is defined by four DOF on that wide: w_1, w_2, θ_{n1} and θ_{n2}. Consequently C^0 continuity holds. Over side 1-2 the normal slope $\partial w/\partial n$ varies quadratically since it comes from differentiating w once. A quadratic is defined by three values; but there are only two DOF that can control the normal slope: θ_{s1} and θ_{s2}. Consequently C^1 continuity is violated between corner points.

To control a quadratic variation of $\partial w/\partial n = \theta_s$ an additional DOF on the side is needed. The simplest implementation of this idea is illustrated in Figure 22.5(b): add a side rotation DOF at the midpoint. But this increases the total number of DOF of the triangle to at least 12: 9 at the corners and 3 at the midpoints.[6] Because a cubic has only 10 independent terms, terms from a quartic polynomial are needed if we want to keep just a polynomial expansion over the full triangle. But

[6] The number climbs to 13 if the centroid deflection w_0 is kept as a DOF.

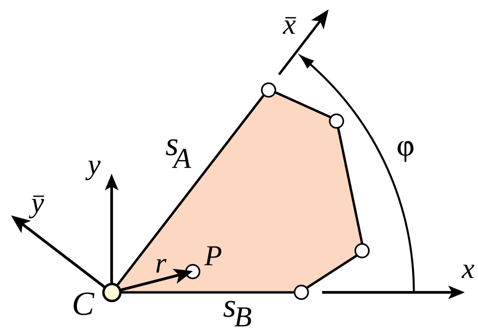

FIGURE 22.6. A corner C of a polygonal KPB element.

that raises the side variation orders of w and $\partial w/\partial n$ to 4 and 3, respectively, and again we are short. Limitation Theorem III given below state that it is impossible to "catch up" under these conditions.

§22.4. Kinematic Limitation Principles

This section examines *kinematic limitation principles* that place constraints on the construction of KPB displacement-assumed elements. The principles are useful in ruling out once and for all the easy road to constructing such elements, and in explaining why researchers turned their attention elsewhere. The presentation below largely follows the author's 1966 thesis [203], which in turn was inspired by a 1965 Technical Note [393].

Limitation principles 1 and 2 are valid for an arbitrary element polygonal shape as illustrated in Figure 22.6, which has only corner DOF on its boundary.[7]

Select a corner C bounded by sides s_A and s_B, which subtend angle φ. We use the abbreviations $s_\varphi = \sin\varphi$ and $c_\varphi = \cos\varphi$. Select a rectangular Cartesian coordinate (RCC) system: $\{x, y\}$ with origin at C and x along side s_A. Another RCC system $\{\bar{x}, \bar{y}\}$ is placed with $\bar{x}$ along side s_B. The systems are related by $\{\bar{x} = xc_\varphi + ys_\varphi,\ \bar{y} = -xs_\varphi + yc_\varphi\}$ and $\{x = \bar{x}c_\varphi - \bar{y}s_\varphi,\ y = \bar{x}s_\varphi + \bar{y}c_\varphi\}$

We focus on limitations related to assuming that w has *continuous second derivatives* at C. That is, the following Taylor expansion holds at a point $P(x, y)$ at a distance r from C:

$$w = a_0 + a_1\,x + a_2\,y + a_3\,x^2 + a_4\,xy + a_5\,y^2 + O(r^3) \tag{22.2}$$

We need the following results derivable from (22.2). The lateral deflections over s_A and s_B are

$$\begin{aligned} w_A &= a_0 + a_1 x + a_3 x^2 + O(r^3),\\ w_B &= a_0 + (a_1 c_\varphi + a_2 s_\varphi)\bar{x} + (a_3 c_\varphi^2 + a_4 s_\varphi c_\varphi + a_5 s_\varphi^2)\bar{x}^2 + O(r^3). \end{aligned} \tag{22.3}$$

The along-the-side slopes over s_A and s_B are obtained by evaluating $\partial w/\partial x$ at $y = 0$ and $\partial w/\bar{x}$ at $\bar{y} = 0$:

$$\begin{aligned} w_{sA} &= a_1 + 2a_3\,x + O(r^2),\\ w_{sB} &= a_1 c_\varphi + a_2 s_\varphi + 2(a_3 c_\varphi^2 + a_4 s_\varphi c_\varphi + a_5 c_\varphi^2)\bar{x} + O(r^2). \end{aligned} \tag{22.4}$$

[7] The presence of internal DOFs is not excluded.

The normal slopes over s_A and s_B are obtained by evaluating $-w_y = -\partial w/\partial y = -a_2 - a_4\,x - 2a_5\,y + O(r^2)$ at $y = 0$, and $w_{\bar{y}} = \partial w/\partial\bar{y} = -(a_1 + 2a_3\,x + a_4\,y)s_\varphi + (a_2 + a_4\,x + 2a_5\,y)\,c_\varphi + O(r^2)$ at $\bar{y} = 0$. This gives

$$w_{nA} = -a_2 - a_4\,x + O(r^2),$$
$$w_{nB} = -a_1\,s_\varphi + a_2\,c_\varphi + \big(a_4(c_\varphi^2 - s_\varphi^2) - 2(a_3 - a_5)s_\varphi c_\varphi\big)\bar{x} + O(r^2) \tag{22.5}$$

Assume that the element satisfies the following four assumptions.

(I) The Taylor series (22.2) at C is valid; thus the deflection w has second derivatives at C.

(II) Three nodal values are chosen at C: $w_C = a_0$, $\theta_{xC} = (\partial w/\partial y)_C = a_1$ and $\theta_{yC} = -(\partial w/\partial x)_C = -a_2$. This is the standard choice for plate elements.

(III) Completeness is satisfied in that the six states $w = \{1, x, y, x^2, xy, y^2\}$ are exactly representable over the element.

(IV) The variation of the normal slope $\partial w/\partial n$ along the element sides is linear.

§22.4.1. Limitation Theorem I

A KPB element cannot satisfy (I), (II), (III) and (IV) simultaneously.

Proof. Choose three set of corner DOF at C to satisfy:

$$\begin{aligned}
&\text{Set 1:} \quad w_C = 1, \quad \left(\frac{\partial w}{\partial n_A}\right)_C = 0, \quad \left(\frac{\partial w}{\partial n_B}\right)_C = 0,\\
&\text{Set 2:} \quad w_C = 0, \quad \left(\frac{\partial w}{\partial n_A}\right)_C = 1, \quad \left(\frac{\partial w}{\partial n_B}\right)_C = 0,\\
&\text{Set 3:} \quad w_C = 0, \quad \left(\frac{\partial w}{\partial n_A}\right)_C = 0, \quad \left(\frac{\partial w}{\partial n_B}\right)_C = 1.
\end{aligned} \tag{22.6}$$

while all other DOF are set to zero.

Set 1 imposes $a_0 = 1$ and $a_1 = a_2 = 0$. Both normal slopes at C are zero, and so are at other corners. Because of the linear variation assumption (IV), $w_{nA} = \partial w/\partial n_A \equiv 0$ and $w_{nB} = \partial w/\partial n_B \equiv= 0$. Expressions (22.5) require $a_4 = 0$ and $a_3 = a_5$.

Set 2 imposes $a_0 = 0$, $a_1 = 1$ and $a_2 = c_\varphi/s_\varphi$. Now $w_{nB} \equiv 0$. This requires $a_4 = 0$ and $a_3 = a_5$, as above. Replacing gives $w_{nA} = 1$, which contradicts (IV).

Set 3 imposes $a_0 = a_1 = 0$, and $a_2 = 1$. Now $w_{nA} = 0$ identically, which forces $a_4 = 0$.

An arbitrary set of values for the DOFs at C can be written as a linear combination of (22.6). But any such combination requires $a_4 = 0$, making the twist term vanish identically in (22.1). Thus the assumption (IV) of completeness cannot be satisfied.

Oddly enough the proof needs no assumption about how w varies along the sides; that is, C^0 compatibility. Just the assumption that the normal slope varies linearly is enough to kill completeness.

This theorem says that to get a C^1 compatible element while retaining assumptions (I), (II) and (III) the normal slope variation cannot be linear. Such conforming elements can be constructed, for example, using product of cubic Hermitian functions along side directions with sutable damping factors along the other directions. But this approach runs into serious trouble as shown by the next limitation principle.

§22.4.2. Limitation Theorem II

Any C^1-compatible, non rectangular KPB element that satisfies conditions (I) and (II) cannot represent exactly all constant curvature states.

Proof. If the element is exactly in a constant curvature state, the deflection w must be quadratic in $\{x, y\}$. Hence the normal slope variation must be linear. But according to Limitation Theorem I the element cannot represent the constant twist state.

This theorem shows that (I), (II) and (III) are incompatible. A more detailed study shows that for a C^1 compatible rectangular element with sides aligned with $\{x, y\}$ only the twist state is lost but that x^2 and y^2 can be exactly represented. For non-rectangular geometries all constant curvature states are lost.

If one insists in C^1 continuity there are two ways out:

Abandon (I): Keep a single polynomial over the element but admit higher order derivatives as corner degrees of freedom.

Abandon (II): Permit discontinuous second derivatives at corners through the use of non-polynomial assumptions, or macroelements.

Both techniques have been tried with success. The use of second derivatives as DOFs, if (I) is abandoned, is forced by the next limitation principle.

§22.4.3. Limitation Theorem III

Suppose that a simple complete polynomial expansion of order $n \geq 3$ is assumed for w over a triangle. At each corner i the deflection w_i, the slopes w_{xi}, w_{yi} and all midsurface derivatives up to order $m \geq 1$ are taken as degrees of freedom. Then C^1 continuity requires $m \geq 2$ and $n \geq 5$.

Proof. Proven in the writer's thesis.[8]

Here is an informal summary of the proof. The total number of DOFs for a complete polynomial is $P_n = (n+1)(n+2)/2 = F_n + B_n$. Of these $F_n = \min\big((n+1)(n+2)/2, 6n-9\big)$ are called *fundamental freedoms* in the sense that they affect interelement compatibility. The $B_n = \max\big(0, (n-5)(n-4)/2\big)$ are called *bubble freedoms*, which have zero value and normal slopes along the three sides. Bubbles occur only if $n \geq 6$.

Over each side the variation of w is of order n and that of $w_n = \partial w/\partial n$ of order $n-1$. This requires $n+1$ and n control DOF on the side, respectively. The number of corner freedoms is $N_c = (m+1)(m+2)/2$, which provides $2(m+1)$ and $2m$ controls on w and w_n, respectively. Within the side (e.g. at a midpoint) one need to add $N_{ws} = n+1-2(m+1) \geq 0$ control DOFs for C^0 continuity in w and $N_{wns} = n-2m \geq 0$ control DOFs for C^1 continuity in w_n. The grand total of boundary DOFs is $N_b = 3(N_c + N_{ws} + N_{wns})$. This has to be equal to the number of fundamental freedoms F_n. Here is a tabulation for various values of n and m. N/A means that the

[8] C. A. Felippa, Refined finite element analysos of linear and nonlinear two-dimensional structures, *Ph.D. Dissertation*, Department of Civil Engineering, University of California at Berkeley, 1966.

interpolation order is not applicable as being too low as it gives $N_{ws} < 0$ and/or $N_{wns} < 0$.

	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
	$F_n = 10$	$F_n = 15$	$F_n = 21$	$F_n = 27$	$F_n = 33$
$m = 1$	$N_b = 12$	$N_b = 18$	$N_b = 24$	$N_b = 30$	$N_b = 36$
$m = 2$	N/A	N/A	$\boxed{N_b = 21}$	$N_b = 27$	$N_b = 33$
$m = 3$	N/A	N/A	N/A	N/A	$N_b = 33$

The first interesting solution is boxed. It corresponds to a complete quintic polynomial $n = 5$ with 21 DOFs, all of which are fundamental. Six degrees of freedom are required at each corner: w_i, w_{xi}, w_{yi}, w_{xxi}, w_{xyi}, w_{yyi}, $i = 1, 2, 3$, plus one normal slope (or side rotation) at each midpoint. The resulting element, called CCT-21, was presented in the author's thesis [203]. The element, however, is too complex for use in standard FEM codes because it does not mix easily with other elements such as beams. So it has only been used in special purpose codes for plates only.

§22.5. A Brief History Of Plate Bending FEM

§22.5.1. Early Work

By the late 1950s the success of the Finite Element Method with membrane-type structural problems in Aerospace (for example, for wing covers and fuselage panels) led to high hopes for its application to plate bending and shell problems. The first results were published by the late 1950s.. But until 1965 only rectangular models gave satisfactory results. The construction of successful triangular elements to model plates and shells of arbitrary geometry proved far more difficult than expected. Early failures, however, led to a more complete understanding of the theoretical basis of FEM, and motivated theoretical advances taken for granted today.

The major source of difficulties in plate beding — as opposed to membranes in plane stress — is due to stricter continuity requirements. The objective of attaining normal slope interelement compatibility posed serious problems, documented in the form of Limitation Theorems in §22.4. By 1963 researchers were looking around "escape ways" to bypass those problems. It was recognized that completeness, in the form of exact representation of rigid body and constant curvature modes, was fundamental for convergence to the analytical solution, a criterion first enunciated by Melosh [485]. The effect of compatibility violations was more difficult to understand until the patch test came along, still unnamed, by 1966 [64].

§22.5.2. Rectangular Elements

The first successful rectangular plate bending element was developed by Adini and Clough [5]. It has has 12 degrees of freedom (DOF) and uses a complete third order polynomial expansion in x and y, aligned with the rectangle sides, plus two additional x^3y and xy^3 terms. It satisfies completeness as well as transverse deflection continuity, but normal slope continuity is only maintained at the four corner points. The same element results from another expansion proposed by Melosh [485], who erroneously stated that the element satisfies C^1 continuity. The error was noted in a subsquent discussion [743]. In 1961 he had proposed [483] a rectangular plate element constructed with beam-like edge functions damped linearly toward the opposite side, plus a uniform twisting mode. Again C^0 continuity was achieved but not C^1 except at corners.

Both of the foregoing elements displayed good convergence characteristics when used for rectangular plates. However the search for a compatible displacement field was underway to try to achieve monotonic convergence. A fully compatible 12-DOF rectangular element was apparently first developed by Papenfuss in an obscure (MS Thesis) reference [541]. This element was rediscovered several times. The simplest derivation can be carried out with products of Hermite cubic polynomials, as noted below. Unfortunately the uniform twist state is not included in the expansion and consequently the element fails the completeness requirement, converging monotonically to a zero twist-curvature solution. This is the right answer to the wrong problem.

In a brief but important paper, Irons and Draper [393] stressed the importance of *completeness* for constant strain modes (constant curvature modes in the case of plate bending). They proved that it is impossible to construct any polygonal-shape plate element with only 3 DOFs per corner and continuous corner curvatures that can simultaneously maintain normal slope conformity and inclusion of the uniform twist mode. This negative result, presented in §22.4 as Limitation Theorem II, effectively closed the door to the construction of the analog of isoparametric elements in plate bending.

The construction of fully compatible polynomials expansions of various orders for rectangular shapes was solved by Bogner *et al.* [96] through Hermitian interpolation functions. In the 1965 conference paper they rederived Papenfuss' non-convergent element, but in an Addendum published as an Appendix to the 1966 Proceedings volume [96] they recognised the lack of the twist mode and an additional degree of freedom: the twist curvature, was added at each corner. The 16-DOF element is complete and compatible, and produced excellent results; unfortunetly a similar construction for quadrilateral shapes is not possible. More refined rectangular elements with 36 DOFs have been also developed using fifth order Hermite polynomials.

§22.5.3. Triangular Elements

Flat triangular plate elements have a wider range of application than rectangular elements since they naturally conform to the analysis of plates and shells of arbitrary geometry for both small and large deflections. As previously noted, however, the development of adequate kinematic expansions was not an easy problem and kept researchers busy for decades.

The success of incompatible rectangular elements such as Adini's elements can be explained by noting that the assumed polynonial expansions for w can be viewed as "natural" deformation modes, following a trivial reduction to nondimensional form. They are intrisically related to element geometry because the local system is chosen along two preferred directions. Lack of C^1 continuity between corners disppears in the limit of a mesh refinement, and causes no harm.

Early attempts to construct triangular elements tried to mimic that scheme, using a Rectangular Cartesian Coordinate (RCC) frame arbitrarily oriented with respect to the element. This leads to an unpleasant lack of invariance whenever an *incomplete* polynomial was selected, since kinematic constraints were artificially imposed. Furthermore the role of completeness was not understood. Thus the first suggested expansion for a triangular element with 9 DOFs in Tocher's 1962 thesis [742]

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 x y^2 + \alpha_9 y^3 \tag{22.7}$$

in which the xy term is missing, violates compatibility, completeness and invariance requirements. The element converges to the wrong solution with zero twist curvature. The thesis [742] then tries two variants of the cubic expansion:

1 Keep xy but combine the two cubic terms: $x^2y + xy^2$.

2 Use a complete 10-term cubic polynomial. The first choice satisfies completeness but violates compatibility *and* invariance. The second assumption satisfies completeness and invariance but grossly violates compatibility while posing the problem: what to do with the extra DOF? Tocher decided to eliminate it by a generalized inversion process akin to static condensation, which unfortunately discards a fundamental DOF. This led to an extremely flexible and non-convergent element. The kinematic constraint elimination technique of Bazeley et. al. in [64], which is discussed in the next Chapter, was more successful and produced an element which, although finicky, is still in use today.

The first fully compatible 9-DOF cubic triangle was finally constructed by the macroelement technique as described in [139]. The triangle was divided into three subtriangles, over each of which a 9-term cubic expansion with linear variation along the exterior side was assumed; applicaation of internal kinematic constraints reduced the exterior DOF to 9. A similar element with quadratic slope variation and 12 DOF was constructed by the writer[203]; a 4 triangle assembly then produced a quadrilateral element [140] that has been extensively used as bending component of a shell element as noted below. The original derivation of [139], carried out in x, y coordinates were considerably simplified in [203,140] through the use of triangular natural coordinates.

Th already cited paper by Bazeley *et al.* [64], was a crucial milestone. Three plate bending triangles were presented therein. Two compatible elements were developed using rational functions. Numerical tests showed them to be way overstiff and have no interest today but the third one was a different story. An incompatible element called (since) the BCIZ triangle was obtained by eliminating the 10th DOF from a complete cubic in such as way that completeness was maintained. The element is incompatible. Computational experiments showed that it converged for some mesh patterns but not for others. This puzzling behavior led to the invention of the patch test, which first appeared (without name) in an Addendum to the Proceedings paper [64]. As noted in a previous Chapter, the patch test was further developed by Irons and coworkers in the 1970s [396,397]. A mathematical version is presented in the Strang-Fix monograph [698].

§22.5.4. Quadrilateral Elements

Arbitrary quadrilaterals can be constructed as macroelements formed by 2 or 4 triangles, and eleimnating internal DOF (if any) by static condensation. Those assemblies are suitable for inclusion into DSM programs. This turns out to be a natural scheme for modeling thin shell structures, in which bending and membrane components re combined, since for arbitrray shell geometries the 4 corners of the quadrilateral need not be coplanar. The above cited article by Clough and Felippa [140] presents the first quadrilateral macroelement constructed that way. That element was included in the open-source SAP family of FEM codes [804], and used for production shell analysis since 1968 — many shell structures around the world were designed through that element.

A direct construction of an arbitrary quadrilateral with 16 DOFs was presented by Fraaeijs de Veubeke [276]. A macroassembly of four triangles is produced by cutting the quadrilateral by the two diagonals without a center node. Several coordinate systems are used to develop the finite element fields. A brief description of this element is provided in [397, §16.4.1].

§22.5.5. More Recent Work

The fully conforming elements developed in the mid 1960s proved “safe” for FEM program users in that convergence could be guaranteed. Performance was another matter. Triangular elements proved to be excessively stiff, particularly for high aspect ratios. A significant improvement in performance was achieved by Razzaque [620] who replaced the shape function curvatures with least-square-fitted smooth functions. This technique was later shown to be equivalent to the stress-hybrid formulation.

The first application of mixed functionals to finite elements was actually to the plate bending problem. Herrman [354] developed a mixed triangular model in which transverse displacements *and* bending moments are selected as master fields. A linear variation was assumed for both variables. This work was based on the HR variational principle and included the transversal shear energy. The element did not perform well, however, in practice.

Successful plate bending elements have also been been constructed by Pian’s assumed-stress hybrid method [573,575] The 9-DOF triangles in this class are normally derived by assuming cubic deflection and linear slope variations along the element sides, and a linear variation of the internal moment field. Efficient formulations of such elements have been published; see e.g., [12,827]. Hybrid elements generally give better moment accuracy than conforming displacement elements. The derivation of these elements, however, is more involved in that it depends on finding equilibrium moments fields within the element, which is not a straightforward matter if the moments vary within the element, or if or large deflections are considered.

Much of the recent research on displacement-assumed models has focused on relaxing or abandoning the assumptions of Kirchhoff thin-plate theory. Relaxing these assumptions has produced elements based on the so-called discrete Kirchhoff theory.[702,187]. In this method the primary expansion is made for the plate rotations. The rotations are linked to the nodal freedoms by introduction of thin-plate normality conditions at selected boundary points, and then interpolating displacements and rotations along the boundary. The initial applications of this method appear unduly complicated. A clear and relatively simple account is given by Batoz, Bathe and Ho [59]. The most successful of these elements to date is the DKT (Discrete Kirchhoff Triangle), an explicit formulation of which has been presented by Batoz [60].

A more drastic step consists of abandoning the Kirchhoff theory in favor of the Reissner-Mindlin theory of moderately thick plates. The continuity requirements for the displacement assumption are lowered to C^0 (hence the name "C^0 bending elements"), but the transverse shear becomes an integral part of the formulation.

Historically the first fully conforming triangular plate elements were not Clough-Tocher's but C^0 elements called "facet" elements that were derived by Melosh in the late 1950s, although an account of their formulation was not published until 1966 [487]. Facet elements, however, suffer from severe numerical problems for thin-plate and obtuse-angle conditions. The approach was revived later by Argyris *et. al.* [30] within the context of degenerated "brick" elements.

The construction of robust C^0 triangular bending elements is delicate, as they are susceptible to 'shear locking' effects in the thin-plate regime if fully integrated, and to kinematic deficiencies (spurious modes) if they are not. Successful quadrilateral C^0 elements have been developed by Hughes, Taylor and Kanolkulchai [377], Pugh, Hinton and Zienkiewicz [597], MacNeal [456,455] Crisfield [165], Tessler and Hughes [728], and Dvorkin and Bathe, [193]. A continuum mechanics based four-node shell element for general nonlinear analysis was developed by Park and Stanley [690,555,691]. Triangular elements in this class have been presented by Belytschko, Stolarski and Carpenter [75] as well as Tessler and Hughes [728].

A different path has been taken by Bergan and coworkers, who retained the classical Kirchhoff formulation but in conjunction with the use of highly nonconforming (C^{-1}) shape functions. They have shown that interelement continuity is not an obstacle to convergence provided the shape functions satisfy certain energy and force orthogonality conditions [82]. The stiffness matrix is constructed using the *free formulation* [86] rather than the standard potential energy formulation. A characteristic feature of these formulations is the careful separation between basic and higher order assumed displacement functions or "modes". Results for triangular bending elements derived through this approach have reported satisfactory performance [80,337,89,216]. One of these elements, which is based on force-orthogonal higher order functions, was rated in 1983 as the best performer in its class [398].