2) $x_0, \dots, x_n \in [-1, 1]$ paarweise verschieden

$\beta_0, \dots, \beta_n \in \mathbb{R}$ für beliebige Polynome mit Grad $\leq n$ soll gelten:

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=0}^{n} \beta_i\, p(x_i)$$

$$g(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$$

und $\int_a^b g(x)\,dx = \sum_{k=0}^{n} f(x_k) \int_a^b L_k(x)\,dx = \sum_{k=0}^{n} \beta_k f(x_k)$

mit $\beta_k = \int_a^b L_k(x)\,dx$

hier: $p(x) = g(x)$ und $f(x_k) = y_i$ und $a=-1,\ b=1$

$$\Rightarrow \int_{-1}^{1} p(x)\,dx = \int_a^b p(x)\,dx = \sum_{i=0}^{n} \beta_i f(x_i) = \sum_{i=0}^{n} \beta_i y_i$$

$$= \sum_{i=0}^{n} \beta_i\, p(x_i)$$

da $p_n(x_i) = y_i$

4.) $\int_0^h f(t)\,dt \quad , \; l_0 = h \quad , \; l_1 = \frac{h}{2}$

$m_0 = 1 \quad , \quad m_1 = 2$

$$T_{m_0} = \frac{h}{2}\left(f(0) + f(h)\right)$$

$$T_{m_1} = \frac{h}{4}\left(f(0) + 2f\left(\frac{h}{2}\right) + f(h)\right)$$

$$T = \alpha_0 T_{m_0} + \alpha_1 T_{m_1}$$

$$\alpha_0 = \frac{l_1^2}{l_1^2 - l_0^2} = -\frac{1}{3} \qquad \alpha_1 = -\frac{l_0^2}{l_1^2 - l_0^2} = \frac{4}{3}$$

$$\begin{aligned} T &= -\frac{1}{3}\cdot\frac{h}{2}\left(f(0) + f(h)\right) + \frac{4}{3}\cdot\frac{h}{4}\left(f(0) + 2f\left(\frac{h}{2}\right) + f(h)\right) \\ &= \frac{h}{6}\left(f(0) + f(h)\right) + \frac{2h}{3} f\left(\frac{h}{2}\right) \\ &= h\left(\frac{1}{6}f(0) + \frac{4}{6}f\left(\frac{h}{2}\right) + \frac{1}{6}f(h)\right) \end{aligned}$$

$\square$

6 a)

Nach Satz 242 gilt:

$$E = \int_{a_i}^{b_i} f(x)\,dx - T = -\frac{1}{12} h^3 f''(\xi)$$

f'' ist idR unbekannt.

Wir teilen das Intervall in der Mitte

$$[a_l, b_l] = [a_i, m_i], \ [a_r, b_r] = [m_i, b_i]$$

$$m_i = \frac{a_i + b_i}{2}$$

Wir wenden auf beide Hälften die Trapezregel an, d.h. $h_l = h_r = \frac{h}{2}$

$$E_l = \int_{a_i}^{m_i} f(x)\,dx - T_l = -\frac{1}{12}\left(\frac{h}{2}\right)^3 f''(\xi_l)$$

$$E_r = \int_{m_i}^{b_i} f(x)\,dx - T_r = -\frac{1}{12}\left(\frac{h}{2}\right)^3 f''(\xi_r)$$

und erhalten die Näherung $\tilde{T} = T_l + T_r$

mit Fehlerabschätzung

$$\tilde{E} = \int_{a_i}^{b_i} f(x)\,dx - \tilde{T} = E_l + E_r = -\frac{1}{12} \cdot \frac{1}{4} \cdot \frac{f''(\xi_l) + f''(\xi_r)}{2}$$

Unter der Annahme $f''(\xi_l) \approx f''(\xi_r) \approx f''(\xi)$ folgt

$$\tilde{E} = \frac{1}{4} E$$

und damit

$$\tilde{T} - T = E - \tilde{E} \approx \frac{3}{4} E \approx 3\tilde{E}$$

Daraus erhalten wir $F, \tilde{F}$ für $T, \tilde{T}$

$$F = \frac{4}{3} |\tilde{T} - T| \approx E$$

$$\tilde{F} = \frac{|\tilde{T} - T|}{3} \approx \tilde{E}$$

$\square$

6.) b) $T = \frac{b_i - a_i}{2}\left(f(a_i) + f(b_i)\right)$

$$\tilde{T} = \frac{b_i - a_i}{4}\left(f(a_i) + 2f\left(\frac{a_i + b_i}{2}\right) + f(b_i)\right)$$

$$\tilde{F} = \frac{1}{3}\left|\tilde{T} - T\right|$$

$$\int_1^5 \frac{1}{x}\,dx$$

[1;5]:

$$T_0 = \frac{5-1}{2}\left(1 + \frac{1}{5}\right) = \frac{12}{5}$$

$$\tilde{T}_0 = \frac{5-1}{4}\left(1 + 2\cdot\frac{1}{3} + \frac{1}{5}\right) = \frac{28}{15}$$

$$\Rightarrow \tilde{F} = \frac{1}{3}\left|\frac{28}{15} - \frac{12}{5}\right| = \frac{1}{3}\cdot\frac{8}{15} = \frac{8}{45} > 0{,}02$$

[1;3]:

$$T_{11} = 1\cdot\left(1 + \frac{1}{3}\right) = \frac{4}{3}$$

$$\tilde{T}_{11} = \frac{1}{2}\left(1 + 1 + \frac{1}{3}\right) = \frac{7}{6}$$

$$\Rightarrow \tilde{F}_{11} = \frac{1}{3}\left|\frac{7}{6} - \frac{4}{3}\right| = \frac{1}{3}\cdot\frac{1}{6} = \frac{1}{18}$$

$$\Rightarrow \tilde{F}_{11} > 0{,}02$$

[3;5]:

$$T_{12} = 1\cdot\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{8}{15}$$

$$\tilde{T}_{12} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{2}\cdot\frac{31}{30} = \frac{31}{60}$$

$$\Rightarrow \tilde{F}_{12} = \frac{1}{3}\left|\frac{8}{15} - \frac{31}{60}\right| = \frac{1}{3}\cdot\frac{1}{60} = \frac{1}{180}$$

$$\Rightarrow \tilde{F}_{12} < 0{,}02 \checkmark$$

[1;2]:

$$T_{21} = \frac{1}{2}\cdot\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

$$\tilde{T}_{21} = \frac{1}{4}\cdot\left(1 + 2\cdot\frac{2}{3} + \frac{1}{2}\right) = \frac{1}{4}\cdot\left(\frac{6}{6} + \frac{8}{6} + \frac{3}{6}\right) = \frac{17}{24}$$

$$\Rightarrow \tilde{F}_{21} = \frac{1}{3}\left|\frac{17}{24} - \frac{3}{4}\right| = \frac{1}{3}\cdot\frac{1}{24} = \frac{1}{72}$$

$$\Rightarrow \tilde{F}_{21} < 0{,}02 \checkmark$$

[2;3]:

$$T_{22} = \frac{1}{2}\cdot\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$$

$$\tilde{T}_{22} = \frac{1}{4}\left(\frac{1}{2} + 2\left(\frac{2}{5}\right) + \frac{1}{3}\right) = \frac{1}{4}\left(\frac{15}{30} + \frac{24}{30} + \frac{10}{30}\right) = \frac{49}{120}$$

$$\Rightarrow \tilde{F}_{22} = \frac{1}{3}\left|\frac{49}{120} - \frac{5}{12}\right| = \frac{1}{3}\cdot\frac{1}{120} = \frac{1}{360}$$

$$\Rightarrow \tilde{F}_{22} < 0{,}02 \checkmark$$

$$\Rightarrow \tilde{T}_{12} + \tilde{T}_{21} + \tilde{T}_{22} = \frac{31}{60} + \frac{17}{24} + \frac{49}{120} = \frac{49}{30} = 1{,}6\overline{3}$$

exakt: $\int_1^5 \frac{1}{x}\,dx = \ln(5) - \ln(1) \approx 1{,}609$