1) $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} \qquad Ax = b$

a)

$$B_J = I - D^{-1} A$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} \lambda & 0 & \frac{1}{2} \\ 0 & \lambda & 0 \\ \frac{1}{2} & 0 & \lambda \end{vmatrix} = 0 \iff \lambda^3 - \frac{1}{4}\lambda = 0 \iff \lambda(\lambda^2 - \frac{1}{4}) = 0$$

$$\iff \lambda = 0 \lor \lambda = -\frac{1}{2} \lor \lambda = \frac{1}{2}$$

$$\Rightarrow \rho(B_J) = \frac{1}{2}$$

$$B_{GS} = (D - E)^{-1} F$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & & \\ 0 & 1 & 0 & & 1 & \\ 1 & 0 & 2 & & & 1 \\ \hline 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -\frac{1}{2} & 0 & 1 \\ \hline 1 & & & \frac{1}{2} & & \\ & 1 & & & 1 & \\ & & 1 & -\frac{1}{4} & & \frac{1}{2} \end{array}$$

$$\begin{vmatrix} \lambda & 0 & \frac{1}{2} \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda - \frac{1}{4} \end{vmatrix} = 0 \iff \lambda^3 - \frac{1}{4}\lambda^2 = 0 \iff \lambda^2(\lambda - \frac{1}{4}) = 0$$

$$\iff \lambda = 0 \lor \lambda = \frac{1}{4}$$

$$\Rightarrow \rho(B_{GS}) = \frac{1}{4}$$

1 b)

Jakobi $\quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right)$

$$x_1^{(1)} = \frac{1}{2} \cdot (4 - 0 \cdot 0 - 1 \cdot 0) = 2$$
$$x_2^{(1)} = 1 \cdot (0 - 0 \cdot 0 - 0 \cdot 0) = 0$$
$$x_3^{(1)} = \frac{1}{2} \cdot (5 - 1 \cdot 0 - 0 \cdot 0) = \frac{5}{2}$$

$$x_1^{(2)} = \frac{1}{2} \cdot \left(4 - 0 \cdot 0 - 1 \cdot \frac{5}{2}\right) = \frac{3}{4}$$
$$x_2^{(2)} = 1 \cdot \left(0 - 0 \cdot 2 - 0 \cdot \frac{5}{2}\right) = 0$$
$$x_3^{(2)} = \frac{1}{2} \cdot (5 - 1 \cdot 2 - 0 \cdot 0) = \frac{3}{2}$$

$$x_1^{(3)} = \frac{1}{2} \cdot \left(4 - 0 \cdot 0 - 1 \cdot \frac{3}{2}\right) = \frac{5}{4}$$
$$x_2^{(3)} = \qquad 0$$
$$x_3^{(3)} = \frac{1}{2} \cdot \left(5 - 1 \cdot \frac{3}{4} - 0 \cdot 0\right) = \frac{17}{8}$$

$$x_1^{(4)} = \frac{1}{2} \cdot \left(4 - 0 \cdot 0 - 1 \cdot \frac{17}{8}\right) = \frac{15}{16}$$
$$x_2^{(3)} = \qquad 0$$
$$x_3^{(3)} = \frac{1}{2} \cdot \left(5 - 1 \cdot \frac{5}{4} - 0 \cdot 0\right) = \frac{15}{8}$$

Gauß-Seidel $x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right)$

$$x_1^{(1)} = \frac{1}{2} \cdot (4 - 0 \cdot 0 - 1 \cdot 0) = 2$$
$$x_2^{(1)} = 1 \cdot (0 - 0 \cdot 2 - 0 \cdot 0) = 0$$
$$x_3^{(1)} = \frac{1}{2} \cdot (5 - 1 \cdot 2 - 0 \cdot 0) = \frac{3}{2}$$

$$x_1^{(2)} = \frac{1}{2} \cdot \left(4 - 0 \cdot 0 - 1 \cdot \frac{3}{2}\right) = \frac{5}{4}$$
$$x_2^{(2)} = \quad 0$$
$$x_3^{(2)} = \frac{1}{2} \cdot \left(5 - 1 \cdot \frac{5}{4} - 0 \cdot 0\right) = \frac{15}{8}$$

$$x_1^{(3)} = \frac{1}{2} \cdot \left(4 - 0 \cdot 0 - 1 \cdot \frac{15}{8}\right) = \frac{17}{16}$$
$$x_2^{(2)} = \quad 0$$
$$x_3^{(2)} = \frac{1}{2} \cdot \left(5 - 1 \cdot \frac{17}{16} - 0 \cdot 0\right) = \frac{63}{32}$$

$$x_1^{(4)} = \frac{1}{2} \cdot \left(4 - 0 \cdot 0 - 1 \cdot \frac{63}{32}\right) = \frac{65}{64}$$
$$x_2^{(4)} = \quad 0$$
$$x_3^{(4)} = \frac{1}{2} \cdot \left(5 - 1 \cdot \frac{65}{64} - 0 \cdot 0\right) = \frac{255}{128}$$