3.) $$A = \begin{pmatrix} 1 & -5 & -20 \\ -4 & 11 & -1 \\ 8 & -4 & 2 \end{pmatrix}$$

Elimination der ersten Spalte $a = (1, -4, 8)^T$

$\|a\|_2 = \sqrt{1^2 + (-4)^2 + 8^2} = \sqrt{81} = 9$; $a_{11} = 1 > 0$

$$\Rightarrow v = a + 9 e_1 = \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 8 \end{pmatrix}$$

$\Rightarrow \|v\|_2^2 = 10^2 + (-4)^2 + 8^2 = 180$

$$Q_1 = I - \frac{2}{\|v\|_2^2} v v^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{180} \begin{pmatrix} 10 \\ -4 \\ 8 \end{pmatrix} \cdot (10, -4, 8)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{90} \begin{pmatrix} 100 & -40 & 80 \\ -40 & 16 & -32 \\ 80 & -32 & 64 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{10}{9} & -4/9 & 8/9 \\ -4/9 & 8/45 & -16/45 \\ 8/9 & -16/45 & 32/45 \end{pmatrix} = \begin{pmatrix} -1/9 & 4/9 & -8/9 \\ 4/9 & 37/45 & 16/45 \\ -8/9 & 16/45 & 13/45 \end{pmatrix}$$

$$Q_1 A = \begin{pmatrix} -9 & 9 & 0 \\ 0 & 27/5 & -9 \\ 0 & 36/5 & 18 \end{pmatrix} \Rightarrow \begin{array}{c|ccc} 1 & 9 & 0 & -9 \\ -4 & 27/5 & -9 & * \\ 8 & 36/5 & 18 & * \end{array}$$

Elimination der zweiten Spalte $a = (27/5, 36/5)^T$

$\|a\|_2 = \sqrt{(27/5)^2 + (36/5)^2} = \sqrt{81} = 9$, $a_{22} = 27/5 > 0$

$$\Rightarrow v = a + 9 e^1 = \begin{pmatrix} 27/5 \\ 36/5 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \end{pmatrix} = \begin{pmatrix} 72/5 \\ 36/5 \end{pmatrix}$$

$\|v\|_2^2 = (72/5)^2 + (36/5)^2 = 1296/5$

$$\tilde{Q}_2 = I - \frac{2}{\|v\|_2^2} v v^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{5}{648} \begin{pmatrix} 72/5 \\ 36/5 \end{pmatrix} (72/5, 36/5)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 8/5 & 4/5 \\ 4/5 & 2/5 \end{pmatrix} = \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

3.) $Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}$

$$\Rightarrow Q_2 Q_1 A = \begin{pmatrix} -9 & 9 & 0 \\ 0 & -9 & -9 \\ 0 & 0 & 18 \end{pmatrix} = R$$

$$Q = (Q_2 Q_1)^T = \begin{pmatrix} -1/9 & 4/9 & -8/9 \\ 4/9 & -7/9 & -4/9 \\ -8/9 & -4/9 & -1/9 \end{pmatrix}$$

4.) $d = -\operatorname{sign}(a_{11}) \|a_1\| \qquad v_1 = a_{11} - d$

$\|v\|_2^2 = -2 v_1 d$

$v = (\underbrace{(a_{11} \pm \|a_1\|)}_{v_1}, \underbrace{a_{21}}_{v_2}, \dots, \underbrace{a_{n1}}_{v_n})^T$

$$\|v\|_2^2 = (a_{11} \pm \|a_1\|)^2 + a_{21}^2 + \dots + a_{n1}^2$$

$$= a_{11}^2 \pm 2\|a_1\| a_{11} + \|a_1\|^2 + a_{21}^2 + \dots a_{n1}^2$$

$$= \|a_1\|^2 + 2\|a_1\| |a_{11}| + \|a_1\|^2 = 2(\|a_1\|^2 + \underbrace{\|a_1\| |a_{11}|}_{|a_{11}|})$$

$$= 2(\|a_1\|^2 + \|a_1\| \operatorname{sign}(a_{11}) a_{11})$$

$$= 2(d^2 - a_{11} d)$$

$$= 2d^2 - 2a_{11} d$$

$$= -2(a_{11} - d) d$$

$$= -2 v_1 d$$

$\square$