1.) $p(x) = a + bx^2$

$$\|e\|_2 = \|A\cancel{p} - y\|_2$$

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad p = \begin{pmatrix} a \\ b \end{pmatrix}, \quad y = \begin{pmatrix} 4 \\ 0 \\ 1 \\ -4 \end{pmatrix}$$

$\min \|e_2\|$

$\Rightarrow$

$$A^T A \hat{p} = A^T y$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} \hat{p} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 9 \\ 9 & 33 \end{pmatrix} \hat{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{array}{cc|c} 4 & 9 & 1 \\ 9 & 33 & 1 \\ \hline 1 & \frac{9}{4} & \frac{1}{4} \\ 0 & \frac{51}{4} & -\frac{5}{4} \\ \hline 1 & \frac{9}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{5}{51} \\ \hline 1 & 0 & \frac{8}{17} \\ 0 & 1 & -\frac{5}{51} \end{array}$$

$a = \frac{8}{17} \qquad b = -\frac{5}{51}$

$$p(x) = \frac{8}{17} - \frac{5}{51} x^2$$

3)

a) $A^T A x = A^T b$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} b$$

$$\Rightarrow \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{pmatrix} b$$

$$\begin{array}{ccc|cc} 2 & 2 & 0 & 1 & 1 \\ 2 & 4 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 & -1 \end{array} \Rightarrow \begin{array}{ccc|cc} 2 & 2 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 & -1 \\ 0 & 2 & 2 & 0 & -1 \end{array}$$

$$\Rightarrow \begin{array}{ccc|cc} 2 & 2 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & -1 & -2 \; (*) \end{array} \Rightarrow \begin{array}{ccc|cc} 1 & 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1 & 1/2 & -1/2 \\ 0 & 0 & 1 & & \lambda \end{array}$$

$$\Rightarrow \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & \lambda \\ 0 & 1 & 0 & 1/2 & -1/2 & -\lambda \\ 0 & 0 & 1 & & & \lambda \end{array} \qquad (*) \Rightarrow b_1 = -2 b_2$$

$$\Rightarrow x = \underbrace{\begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \\ 0 & 0 \end{pmatrix} b}_{:= -B} + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{:= L} \lambda$$

b) $\min \| L\lambda - B \|_2 \Rightarrow L^T L \lambda = L^T B$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \lambda = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1/2 & 1/2 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow 3\lambda = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \end{pmatrix} b$$

$$\Rightarrow \lambda = \begin{pmatrix} \frac{1}{6} & -\frac{1}{2} \end{pmatrix} b$$

$$x = \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \\ 0 & 0 \end{pmatrix} b + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{6} & -\frac{1}{2} \end{pmatrix} b$$

$$= \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \\ 0 & 0 \end{pmatrix} b + \begin{pmatrix} 1/6 & -1/2 \\ -1/6 & 1/2 \\ 1/6 & -1/2 \end{pmatrix} b$$

$$= \begin{pmatrix} 1/6 & 1/2 \\ 2/6 & 0 \\ 1/6 & -1/2 \end{pmatrix} b$$

$$A^+ = \frac{1}{6} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 1 & -3 \end{pmatrix}$$

4.)

$$A^T A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\begin{vmatrix} \lambda - 2 & -2 & 0 \\ -2 & \lambda - 4 & -2 \\ 0 & -2 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 (\lambda - 4) - 8(\lambda - 2) \overset{!}{=} 0$$

$$\Rightarrow (\lambda - 2) \left[(\lambda - 2)(\lambda - 4) - 8 \right] = 0$$

$$\Rightarrow (\lambda - 2) \quad \lambda (\lambda - 6) = 0$$

$$\Rightarrow \lambda = 2 \quad \vee \lambda = 0 \quad \vee \quad \lambda = 6$$

$$A A^T = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} \lambda - 6 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 6)(\lambda - 2) \overset{!}{=} 0$$

$$\Rightarrow \quad \lambda = 6 \quad \vee \quad \lambda = 2$$

Singulärwerte:

$$\sigma_1 = \sqrt{6} \qquad \sigma_2 = \sqrt{2}$$