Trigonometry

4

Airport runways are named on the basis of the angles they form with due north, measured in a clockwise direction. These angles are called bearings and can be determined using trigonometry.

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SELECTED APPLICATIONS

Trigonometric functions have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Speed of a Bicycle, Exercise 108, page 293
- Machine Shop Calculations, Exercise 69, page 310
- Sales, Exercise 88, page 320
- Respiratory Cycle, Exercise 73, page 330
- Data Analysis: Meteorology, Exercise 75, page 330
- Predator-Prey Model, Exercise 77, page 341
- Security Patrol, Exercise 97, page 351
- Navigation, Exercise 29, page 360
- Wave Motion, Exercise 60, page 362

4.1 Radian and Degree Measure

What you should learn

- Describe angles.
- Use radian measure.
- Use degree measure.
- Use angles to model and solve real-life problems.

Why you should learn it

You can use angles to model and solve real-life problems. For instance, in Exercise 108 on page 293, you are asked to use angles to find the speed of a bicycle.

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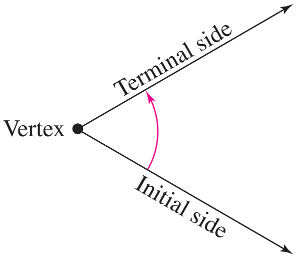
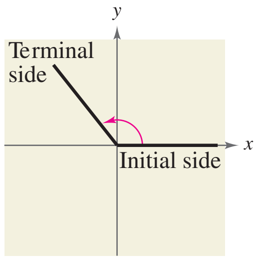
Angles

As derived from the Greek language, the word **trigonometry** means "measurement of triangles." Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying. With the development of calculus and the physical sciences in the 17th century, a different perspective arose—one that viewed the classic trigonometric relationships as *functions* with the set of real numbers as their domains. Consequently, the applications of trigonometry expanded to include a vast number of physical phenomena involving rotations and vibrations. These phenomena include sound waves, light rays, planetary orbits, vibrating strings, pendulums, and orbits of atomic particles.

The approach in this text incorporates *both* perspectives, starting with angles and their measure.



Angle
FIGURE 4.1



Angle in Standard Position
FIGURE 4.2

An **angle** is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side,** as shown in Figure 4.1. The endpoint of the ray is the **vertex** of the angle. This perception of an angle fits a coordinate system in which the origin is the vertex and the initial side coincides with the positive x-axis. Such an angle is in **standard position,** as shown in Figure 4.2. **Positive angles** are generated by counterclockwise rotation, and **negative angles** by clockwise rotation, as shown in Figure 4.3. Angles are labeled with Greek letters α (alpha), β (beta), and θ (theta), as well as uppercase letters A, B, and C. In Figure 4.4, note that angles α and β have the same initial and terminal sides. Such angles are **coterminal.**

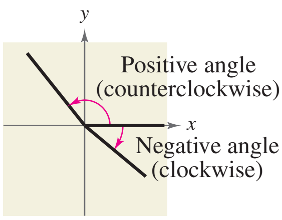


FIGURE 4.3

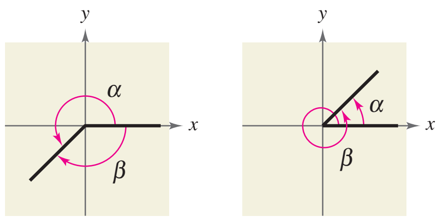
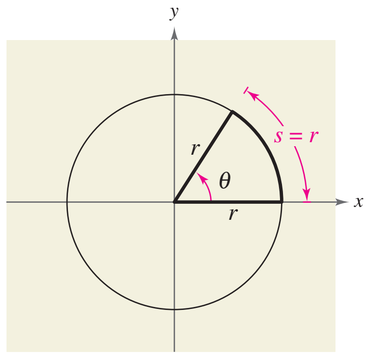


FIGURE 4.4 *Coterminal Angles*

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

Radian Measure

The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side. One way to measure angles is in *radians*. This type of measure is especially useful in calculus. To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle, as shown in Figure 4.5.



Arc length = radius when θ = 1 radian
FIGURE 4.5

Definition of Radian

One **radian** is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. See Figure 4.5. Algebraically, this means that

$$\theta = \frac{s}{r}$$

where θ is measured in radians.

Because the circumference of a circle is $2\pi r$ units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of

$$s = 2\pi r.$$

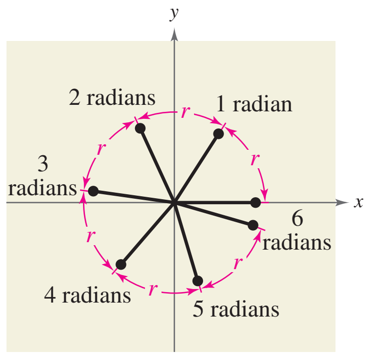


FIGURE 4.6

Moreover, because $2\pi \approx 6.28$, there are just over six radius lengths in a full circle, as shown in Figure 4.6. Because the units of measure for s and r are the same, the ratio s/r has no units—it is simply a real number.

Because the radian measure of an angle of one full revolution is 2π, you can obtain the following.

$$\frac{1}{2}\text{ revolution} = \frac{2\pi}{2} = \pi\text{ radians}$$

$$\frac{1}{4}\text{ revolution} = \frac{2\pi}{4} = \frac{\pi}{2}\text{ radians}$$

$$\frac{1}{6}\text{ revolution} = \frac{2\pi}{6} = \frac{\pi}{3}\text{ radians}$$

These and other common angles are shown in Figure 4.7.

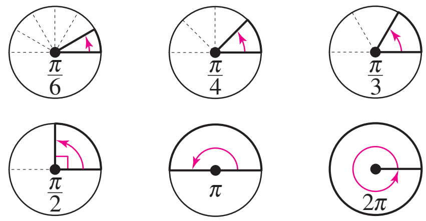


FIGURE 4.7

STUDY TIP

One revolution around a circle of radius r corresponds to an angle of 2π radians because

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi\text{ radians.}$$

Recall that the four quadrants in a coordinate system are numbered I, II, III, and IV. Figure 4.8 on page 284 shows which angles between 0 and 2π lie in each of the four quadrants. Note that angles between 0 and $\pi/2$ are **acute** angles and angles between $\pi/2$ and π are **obtuse** angles.

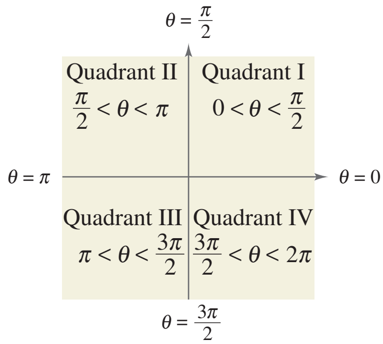


FIGURE 4.8

STUDY TIP

The phrase "the terminal side of θ lies in a quadrant" is often abbreviated by simply saying that "θ lies in a quadrant." The terminal sides of the "quadrant angles" 0, $\pi/2$, π, and $3\pi/2$ do not lie within quadrants.

Two angles are coterminal if they have the same initial and terminal sides. For instance, the angles 0 and 2π are coterminal, as are the angles $\pi/6$ and $13\pi/6$. You can find an angle that is coterminal to a given angle θ by adding or subtracting 2π (one revolution), as demonstrated in Example 1. A given angle θ has infinitely many coterminal angles. For instance, $\theta = \pi/6$ is coterminal with

$$\frac{\pi}{6} + 2n\pi$$

where n is an integer.

Example 1 Sketching and Finding Coterminal Angles

a. For the positive angle $13\pi/6$, subtract 2π to obtain a coterminal angle

$$\frac{13\pi}{6} - 2\pi = \frac{\pi}{6}.$$ See Figure 4.9.

b. For the positive angle $3\pi/4$, subtract 2π to obtain a coterminal angle

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4}.$$ See Figure 4.10.

c. For the negative angle $-2\pi/3$, add 2π to obtain a coterminal angle

$$-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}.$$ See Figure 4.11.

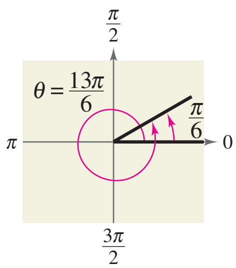


FIGURE 4.9

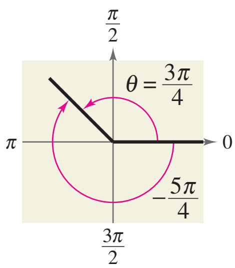


FIGURE 4.10

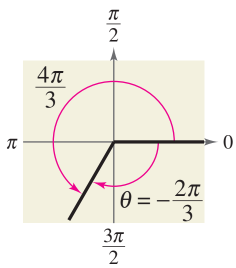


FIGURE 4.11

CHECKPOINT Now try Exercise 17.

Two positive angles α and β are **complementary** (complements of each other) if their sum is $\pi/2$. Two positive angles are **supplementary** (supplements of each other) if their sum is π. See Figure 4.12.

Complementary Angles *Supplementary Angles*

FIGURE 4.12

Example 2 Complementary and Supplementary Angles

If possible, find the complement and the supplement of (a) $2\pi/5$ and (b) $4\pi/5$.

Solution

a. The complement of $2\pi/5$ is

$$\frac{\pi}{2} - \frac{2\pi}{5} = \frac{5\pi}{10} - \frac{4\pi}{10} = \frac{\pi}{10}.$$

The supplement of $2\pi/5$ is

$$\pi - \frac{2\pi}{5} = \frac{5\pi}{5} - \frac{2\pi}{5} = \frac{3\pi}{5}.$$

b. Because $4\pi/5$ is greater than $\pi/2$, it has no complement. (Remember that complements are *positive* angles.) The supplement is

$$\pi - \frac{4\pi}{5} = \frac{5\pi}{5} - \frac{4\pi}{5} = \frac{\pi}{5}.$$

CHECKPOINT Now try Exercise 21.

Degree Measure

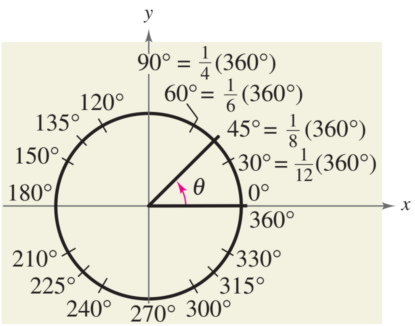


FIGURE 4.13

A second way to measure angles is in terms of **degrees,** denoted by the symbol °. A measure of one degree (1°) is equivalent to a rotation of $\frac{1}{360}$ of a complete revolution about the vertex. To measure angles, it is convenient to mark degrees on the circumference of a circle, as shown in Figure 4.13. So, a full revolution (counterclockwise) corresponds to 360°, a half revolution to 180°, a quarter revolution to 90°, and so on.

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations

$$360° = 2\pi \text{ rad} \qquad \text{and} \qquad 180° = \pi \text{ rad}.$$

From the latter equation, you obtain

$$1° = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \left(\frac{180°}{\pi}\right)$$

which lead to the conversion rules at the top of the next page.

Conversions Between Degrees and Radians

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ rad}}{180^\circ}$.

2. To convert radians to degrees, multiply radians by $\dfrac{180^\circ}{\pi \text{ rad}}$.

To apply these two conversion rules, use the basic relationship $\pi \text{ rad} = 180^\circ$. (See Figure 4.14.)

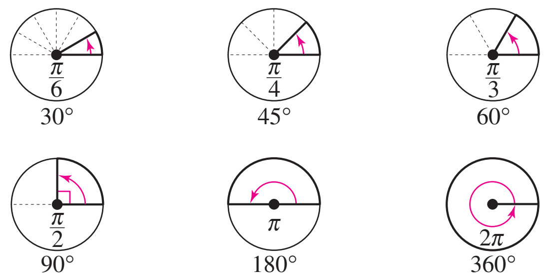


FIGURE 4.14

When no units of angle measure are specified, *radian measure is implied*. For instance, if you write $\theta = 2$, you imply that $\theta = 2$ radians.

Example 3 Converting from Degrees to Radians

a. $135^\circ = (135 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = \dfrac{3\pi}{4}$ radians Multiply by $\pi/180$.

b. $540^\circ = (540 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = 3\pi$ radians Multiply by $\pi/180$.

c. $-270^\circ = (-270 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = -\dfrac{3\pi}{2}$ radians Multiply by $\pi/180$.

CHECKPOINT Now try Exercise 47.

Example 4 Converting from Radians to Degrees

a. $-\dfrac{\pi}{2} \text{ rad} = \left(-\dfrac{\pi}{2} \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right) = -90^\circ$ Multiply by $180/\pi$.

b. $\dfrac{9\pi}{2} \text{ rad} = \left(\dfrac{9\pi}{2} \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right) = 810^\circ$ Multiply by $180/\pi$.

c. $2 \text{ rad} = (2 \cancel{\text{rad}})\left(\dfrac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right) = \dfrac{360^\circ}{\pi} \approx 114.59^\circ$ Multiply by $180/\pi$.

CHECKPOINT Now try Exercise 51.

If you have a calculator with a "radian-to-degree" conversion key, try using it to verify the result shown in part (c) of Example 4.

Technology

With calculators it is convenient to use *decimal* degrees to denote fractional parts of degrees. Historically, however, fractional parts of degrees were expressed in *minutes* and *seconds*, using the prime (′) and double prime (″) notations, respectively. That is,

$$1' = \text{one minute} = \tfrac{1}{60}(1^\circ)$$
$$1'' = \text{one second} = \tfrac{1}{3600}(1^\circ)$$

Consequently, an angle of 64 degrees, 32 minutes, and 47 seconds is represented by $\theta = 64^\circ\, 32'\, 47''$. Many calculators have special keys for converting an angle in degrees, minutes, and seconds (D° M′ S″) to decimal degree form, and vice versa.

Applications

The *radian measure* formula, $\theta = s/r$, can be used to measure arc length along a circle.

Arc Length

For a circle of radius r, a central angle θ intercepts an arc of length s given by

$$s = r\theta \qquad \text{Length of circular arc}$$

where θ is measured in radians. Note that if $r = 1$, then $s = \theta$, and the radian measure of θ equals the arc length.

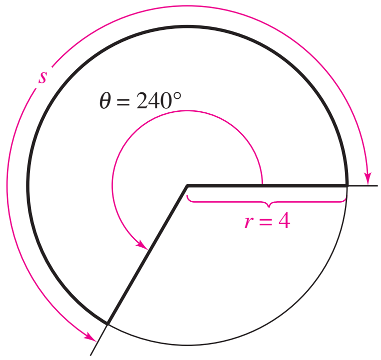


FIGURE 4.15

Example 5 Finding Arc Length

A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 240°, as shown in Figure 4.15.

Solution

To use the formula $s = r\theta$, first convert 240° to radian measure.

$$240° = (240 \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = \frac{4\pi}{3} \text{ radians}$$

Then, using a radius of $r = 4$ inches, you can find the arc length to be

$$s = r\theta = 4\left(\frac{4\pi}{3}\right) = \frac{16\pi}{3} \approx 16.76 \text{ inches.}$$

Note that the units for $r\theta$ are determined by the units for r because θ is given in radian measure, which has no units.

CHECKPOINT Now try Exercise 87.

The formula for the length of a circular arc can be used to analyze the motion of a particle moving at a *constant speed* along a circular path.

Linear and Angular Speeds

Consider a particle moving at a constant speed along a circular arc of radius r. If s is the length of the arc traveled in time t, then the **linear speed** v of the particle is

$$\text{Linear speed } v = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}.$$

Moreover, if θ is the angle (in radian measure) corresponding to the arc length s, then the **angular speed** ω (the lowercase Greek letter omega) of the particle is

$$\text{Angular speed } \omega = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}.$$

STUDY TIP

Linear speed measures how fast the particle moves, and angular speed measures how fast the angle changes. By dividing the formula for arc length by t, you can establish a relationship between linear speed v and angular speed ω, as shown.

$$s = r\theta$$

$$\frac{s}{t} = \frac{r\theta}{t}$$

$$v = r\omega$$

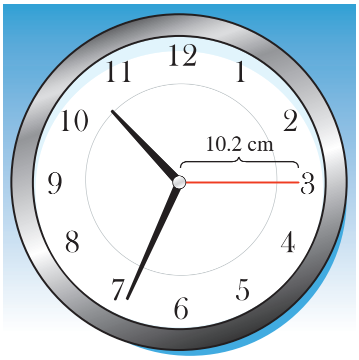


FIGURE 4.16

Example 6 Finding Linear Speed

The second hand of a clock is 10.2 centimeters long, as shown in Figure 4.16. Find the linear speed of the tip of this second hand as it passes around the clock face.

Solution

In one revolution, the arc length traveled is

$$\begin{aligned} s &= 2\pi r \\ &= 2\pi(10.2) && \text{Substitute for } r. \\ &= 20.4\pi \text{ centimeters.} \end{aligned}$$

The time required for the second hand to travel this distance is

$$t = 1 \text{ minute} = 60 \text{ seconds.}$$

So, the linear speed of the tip of the second hand is

$$\begin{aligned} \text{Linear speed} &= \frac{s}{t} \\ &= \frac{20.4\pi \text{ centimeters}}{60 \text{ seconds}} \\ &\approx 1.068 \text{ centimeters per second.} \end{aligned}$$

CHECKPOINT Now try Exercise 103.

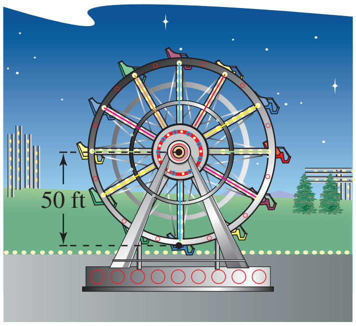


FIGURE 4.17

Example 7 Finding Angular and Linear Speeds

A Ferris wheel with a 50-foot radius (see Figure 4.17) makes 1.5 revolutions per minute.

a. Find the angular speed of the Ferris wheel in radians per minute.

b. Find the linear speed of the Ferris wheel.

Solution

a. Because each revolution generates 2π radians, it follows that the wheel turns $(1.5)(2\pi) = 3\pi$ radians per minute. In other words, the angular speed is

$$\begin{aligned} \text{Angular speed} &= \frac{\theta}{t} \\ &= \frac{3\pi \text{ radians}}{1 \text{ minute}} = 3\pi \text{ radians per minute.} \end{aligned}$$

b. The linear speed is

$$\begin{aligned} \text{Linear speed} &= \frac{s}{t} \\ &= \frac{r\theta}{t} \\ &= \frac{50(3\pi) \text{ feet}}{1 \text{ minute}} \approx 471.2 \text{ feet per minute.} \end{aligned}$$

CHECKPOINT Now try Exercise 105.

A **sector** of a circle is the region bounded by two radii of the circle and their intercepted arc (see Figure 4.18).

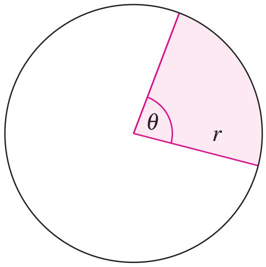


FIGURE 4.18

Area of a Sector of a Circle

For a circle of radius r, the area A of a sector of the circle with central angle θ is given by

$$A = \frac{1}{2}r^2\theta$$

where θ is measured in radians.

Example 8 Area of a Sector of a Circle

A sprinkler on a golf course fairway is set to spray water over a distance of 70 feet and rotates through an angle of 120° (see Figure 4.19). Find the area of the fairway watered by the sprinkler.

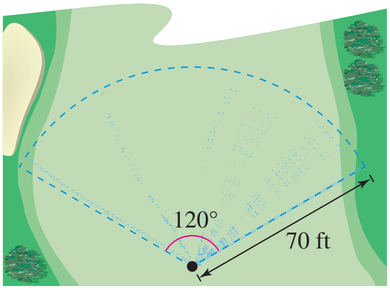


FIGURE 4.19

Solution

First convert 120° to radian measure as follows.

$$\begin{aligned}\theta &= 120^\circ \\ &= (120\ \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) && \text{Multiply by } \pi/180. \\ &= \frac{2\pi}{3} \text{ radians}\end{aligned}$$

Then, using $\theta = 2\pi/3$ and $r = 70$, the area is

$$\begin{aligned}A &= \frac{1}{2}r^2\theta && \text{Formula for the area of a sector of a circle} \\ &= \frac{1}{2}(70)^2\left(\frac{2\pi}{3}\right) && \text{Substitute for } r \text{ and } \theta. \\ &= \frac{4900\pi}{3} && \text{Simplify.} \\ &\approx 5131 \text{ square feet.} && \text{Simplify.}\end{aligned}$$

CHECKPOINT Now try Exercise 107.

4.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

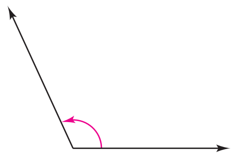
VOCABULARY CHECK: Fill in the blanks.

1. ________ means "measurement of triangles."
2. An ________ is determined by rotating a ray about its endpoint.
3. Two angles that have the same initial and terminal sides are ________.
4. One ________ is the measure of a central angle that intercepts an arc equal to the radius of the circle.
5. Angles that measure between 0 and $\pi/2$ are ________ angles, and angles that measure between $\pi/2$ and π are ________ angles.
6. Two positive angles that have a sum of $\pi/2$ are ________ angles, whereas two positive angles that have a sum of π are ________ angles.
7. The angle measure that is equivalent to $\frac{1}{360}$ of a complete revolution about an angle's vertex is one ________.
8. The ________ speed of a particle is the ratio of the arc length traveled to the time traveled.
9. The ________ speed of a particle is the ratio of the change in the central angle to time.
10. The area of a sector of a circle with radius r and central angle θ, where θ is measured in radians, is given by the formula ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, estimate the angle to the nearest one-half radian.

1.



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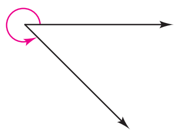

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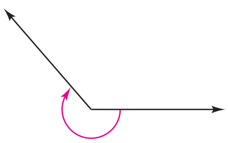

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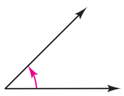

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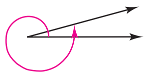


In Exercises 7–12, determine the quadrant in which each angle lies. (The angle measure is given in radians.)

7. (a) $\dfrac{\pi}{5}$ (b) $\dfrac{7\pi}{5}$

8. (a) $\dfrac{11\pi}{8}$ (b) $\dfrac{9\pi}{8}$

9. (a) $-\dfrac{\pi}{12}$ (b) -2

10. (a) -1 (b) $-\dfrac{11\pi}{9}$

11. (a) 3.5 (b) 2.25

12. (a) 6.02 (b) -4.25

In Exercises 13–16, sketch each angle in standard position.

13. (a) $\dfrac{5\pi}{4}$ (b) $-\dfrac{2\pi}{3}$

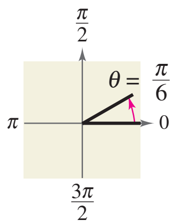
14. (a) $-\dfrac{7\pi}{4}$ (b) $\dfrac{5\pi}{2}$

15. (a) $\dfrac{11\pi}{6}$ (b) -3

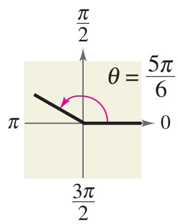
16. (a) 4 (b) 7π

In Exercises 17–20, determine two coterminal angles (one positive and one negative) for each angle. Give your answers in radians.

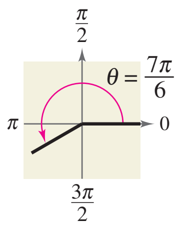
17. (a)



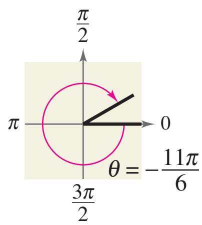
(b)



18. (a)



(b)



19. (a) $\theta = \dfrac{2\pi}{3}$ (b) $\theta = \dfrac{\pi}{12}$

20. (a) $\theta = -\dfrac{9\pi}{4}$ (b) $\theta = -\dfrac{2\pi}{15}$

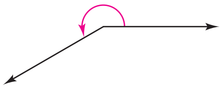
In Exercises 21–24, find (if possible) the complement and supplement of each angle.

21. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ **22.** (a) $\frac{\pi}{12}$ (b) $\frac{11\pi}{12}$

23. (a) 1 (b) 2 **24.** (a) 3 (b) 1.5

In Exercises 25–30, estimate the number of degrees in the angle.

25.



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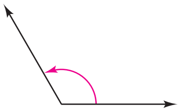

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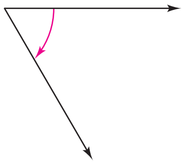
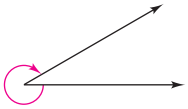
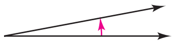


28.



29.

30.



In Exercises 31–34, determine the quadrant in which each angle lies.

31. (a) $130°$ (b) $285°$

32. (a) $8.3°$ (b) $257° 30'$

33. (a) $-132° 50'$ (b) $-336°$

34. (a) $-260°$ (b) $-3.4°$

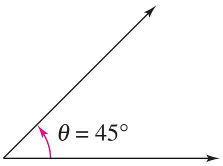
In Exercises 35–38, sketch each angle in standard position.

35. (a) $30°$ (b) $150°$ **36.** (a) $-270°$ (b) $-120°$

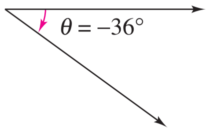
37. (a) $405°$ (b) $480°$ **38.** (a) $-750°$ (b) $-600°$

In Exercises 39–42, determine two coterminal angles (one positive and one negative) for each angle. Give your answers in degrees.

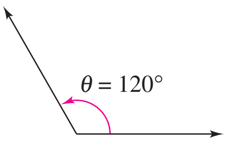
39. (a)



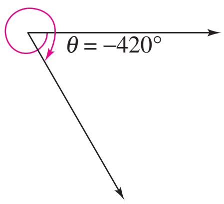
(b)



40. (a)



(b)



41. (a) $\theta = 240°$ (b) $\theta = -180°$

42. (a) $\theta = -420°$ (b) $\theta = 230°$

In Exercises 43–46, find (if possible) the complement and supplement of each angle.

43. (a) $18°$ (b) $115°$ **44.** (a) $3°$ (b) $64°$

45. (a) $79°$ (b) $150°$ **46.** (a) $130°$ (b) $170°$

In Exercises 47–50, rewrite each angle in radian measure as a multiple of π. (Do not use a calculator.)

47. (a) $30°$ (b) $150°$ **48.** (a) $315°$ (b) $120°$

49. (a) $-20°$ (b) $-240°$ **50.** (a) $-270°$ (b) $144°$

In Exercises 51–54, rewrite each angle in degree measure. (Do not use a calculator.)

51. (a) $\frac{3\pi}{2}$ (b) $\frac{7\pi}{6}$ **52.** (a) $-\frac{7\pi}{12}$ (b) $\frac{\pi}{9}$

53. (a) $\frac{7\pi}{3}$ (b) $-\frac{11\pi}{30}$ **54.** (a) $\frac{11\pi}{6}$ (b) $\frac{34\pi}{15}$

In Exercises 55–62, convert the angle measure from degrees to radians. Round to three decimal places.

55. $115°$ **56.** $87.4°$

57. $-216.35°$ **58.** $-48.27°$

59. $532°$ **60.** $345°$

61. $-0.83°$ **62.** $0.54°$

In Exercises 63–70, convert the angle measure from radians to degrees. Round to three decimal places.

63. $\frac{\pi}{7}$ **64.** $\frac{5\pi}{11}$

65. $\frac{15\pi}{8}$ **66.** $\frac{13\pi}{2}$

67. -4.2π **68.** 4.8π

69. -2 **70.** -0.57

In Exercises 71–74, convert each angle measure to decimal degree form.

71. (a) $54° 45'$ (b) $-128° 30'$

72. (a) $245° 10'$ (b) $2° 12'$

73. (a) $85° 18' 30''$ (b) $330° 25''$

74. (a) $-135° 36''$ (b) $-408° 16' 20''$

In Exercises 75–78, convert each angle measure to D° M′ S″ form.

75. (a) $240.6°$ (b) $-145.8°$

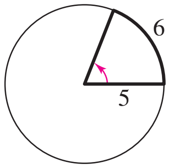
76. (a) $-345.12°$ (b) $0.45°$

77. (a) $2.5°$ (b) $-3.58°$

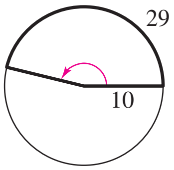
78. (a) $-0.355°$ (b) $0.7865°$

In Exercises 79–82, find the angle in radians.

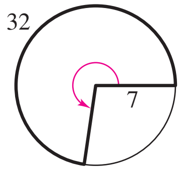
79.



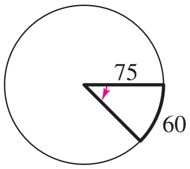
80.



81.



82.



In Exercises 83–86, find the radian measure of the central angle of a circle of radius *r* that intercepts an arc of length *s*.

	Radius r	*Arc Length s*
83.	27 inches	6 inches
84.	14 feet	8 feet
85.	14.5 centimeters	25 centimeters
86.	80 kilometers	160 kilometers

In Exercises 87–90, find the length of the arc on a circle of radius *r* intercepted by a central angle θ.

	Radius r	*Central Angle θ*
87.	15 inches	180°
88.	9 feet	60°
89.	3 meters	1 radian
90.	20 centimeters	$\pi/4$ radian

In Exercises 91–94, find the area of the sector of the circle with radius *r* and central angle θ.

	Radius r	*Central Angle θ*
91.	4 inches	$\dfrac{\pi}{3}$
92.	12 millimeters	$\dfrac{\pi}{4}$
93.	2.5 feet	225°
94.	1.4 miles	330°

Distance Between Cities **In Exercises 95 and 96, find the distance between the cities. Assume that Earth is a sphere of radius 4000 miles and that the cities are on the same longitude (one city is due north of the other).**

	City	*Latitude*
95.	Dallas, Texas	$32^\circ\, 47'\, 39''$ N
	Omaha, Nebraska	$41^\circ\, 15'\, 50''$ N
96.	San Francisco, California	$37^\circ\, 47'\, 36''$ N
	Seattle, Washington	$47^\circ\, 37'\, 18''$ N

97. ***Difference in Latitudes*** Assuming that Earth is a sphere of radius 6378 kilometers, what is the difference in the latitudes of Syracuse, New York and Annapolis, Maryland, where Syracuse is 450 kilometers due north of Annapolis?

98. ***Difference in Latitudes*** Assuming that Earth is a sphere of radius 6378 kilometers, what is the difference in the latitudes of Lynchburg, Virginia and Myrtle Beach, South Carolina, where Lynchburg is 400 kilometers due north of Myrtle Beach?

99. ***Instrumentation*** The pointer on a voltmeter is 6 centimeters in length (see figure). Find the angle through which the pointer rotates when it moves 2.5 centimeters on the scale.

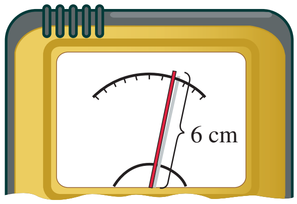


FIGURE FOR 99

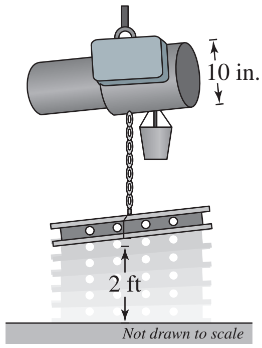


FIGURE FOR 100

100. ***Electric Hoist*** An electric hoist is being used to lift a beam (see figure). The diameter of the drum on the hoist is 10 inches, and the beam must be raised 2 feet. Find the number of degrees through which the drum must rotate.

101. ***Angular Speed*** A car is moving at a rate of 65 miles per hour, and the diameter of its wheels is 2.5 feet.

(a) Find the number of revolutions per minute the wheels are rotating.

(b) Find the angular speed of the wheels in radians per minute.

102. ***Angular Speed*** A two-inch-diameter pulley on an electric motor that runs at 1700 revolutions per minute is connected by a belt to a four-inch-diameter pulley on a saw arbor.

(a) Find the angular speed (in radians per minute) of each pulley.

(b) Find the revolutions per minute of the saw.

103. ***Linear and Angular Speeds*** A $7\frac{1}{4}$-inch circular power saw rotates at 5200 revolutions per minute.

(a) Find the angular speed of the saw blade in radians per minute.

(b) Find the linear speed (in feet per minute) of one of the 24 cutting teeth as they contact the wood being cut.

104. ***Linear and Angular Speeds*** A carousel with a 50-foot diameter makes 4 revolutions per minute.

(a) Find the angular speed of the carousel in radians per minute.

(b) Find the linear speed of the platform rim of the carousel.

105. ***Linear and Angular Speeds*** The diameter of a DVD is approximately 12 centimeters. The drive motor of the DVD player is controlled to rotate precisely between 200 and 500 revolutions per minute, depending on what track is being read.

(a) Find an interval for the angular speed of a DVD as it rotates.

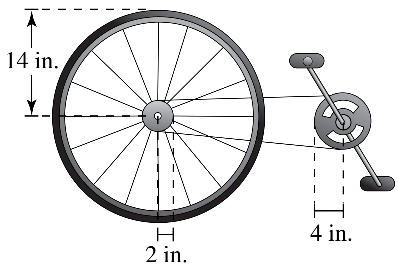
(b) Find an interval for the linear speed of a point on the outermost track as the DVD rotates.

106. ***Area*** A car's rear windshield wiper rotates 125°. The total length of the wiper mechanism is 25 inches and wipes the windshield over a distance of 14 inches. Find the area covered by the wiper.

107. ***Area*** A sprinkler system on a farm is set to spray water over a distance of 35 meters and to rotate through an angle of 140°. Draw a diagram that shows the region that can be irrigated with the sprinkler. Find the area of the region.

Model It

108. ***Speed of a Bicycle*** The radii of the pedal sprocket, the wheel sprocket, and the wheel of the bicycle in the figure are 4 inches, 2 inches, and 14 inches, respectively. A cyclist is pedaling at a rate of 1 revolution per second.



(a) Find the speed of the bicycle in feet per second and miles per hour.

(b) Use your result from part (a) to write a function for the distance d (in miles) a cyclist travels in terms of the number n of revolutions of the pedal sprocket.

Model It (continued)

(c) Write a function for the distance d (in miles) a cyclist travels in terms of the time t (in seconds). Compare this function with the function from part (b).

(d) Classify the types of functions you found in parts (b) and (c). Explain your reasoning.

Synthesis

True or False? **In Exercises 109–111, determine whether the statement is true or false. Justify your answer.**

109. A measurement of 4 radians corresponds to two complete revolutions from the initial side to the terminal side of an angle.

110. The difference between the measures of two coterminal angles is always a multiple of 360° if expressed in degrees and is always a multiple of 2π radians if expressed in radians.

111. An angle that measures $-1260°$ lies in Quadrant III.

112. ***Writing*** In your own words, explain the meanings of (a) an angle in standard position, (b) a negative angle, (c) coterminal angles, and (d) an obtuse angle.

113. ***Think About It*** A fan motor turns at a given angular speed. How does the speed of the tips of the blades change if a fan of greater diameter is installed on the motor? Explain.

114. ***Think About It*** Is a degree or a radian the larger unit of measure? Explain.

115. ***Writing*** If the radius of a circle is increasing and the magnitude of a central angle is held constant, how is the length of the intercepted arc changing? Explain your reasoning.

116. ***Proof*** Prove that the area of a circular sector of radius r with central angle θ is $A = \frac{1}{2}\theta r^2$, where θ is measured in radians.

Skills Review

In Exercises 117–120, simplify the radical expression.

117. $\dfrac{4}{4\sqrt{2}}$

118. $\dfrac{5\sqrt{5}}{2\sqrt{10}}$

119. $\sqrt{2^2 + 6^2}$

120. $\sqrt{17^2 - 9^2}$

In Exercises 121–124, sketch the graphs of $y = x^5$ and the specified transformation.

121. $f(x) = (x - 2)^5$

122. $f(x) = x^5 - 4$

123. $f(x) = 2 - x^5$

124. $f(x) = -(x + 3)^5$

4.2 Trigonometric Functions: The Unit Circle

What you should learn

- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use the domain and period to evaluate sine and cosine functions.
- Use a calculator to evaluate trigonometric functions.

Why you should learn it

Trigonometric functions are used to model the movement of an oscillating weight. For instance, in Exercise 57 on page 300, the displacement from equilibrium of an oscillating weight suspended by a spring is modeled as a function of time.

Richard Megna/Fundamental Photographs

The Unit Circle

The two historical perspectives of trigonometry incorporate different methods for introducing the trigonometric functions. Our first introduction to these functions is based on the unit circle.

Consider the **unit circle** given by

$$x^2 + y^2 = 1 \qquad \text{Unit circle}$$

as shown in Figure 4.20.

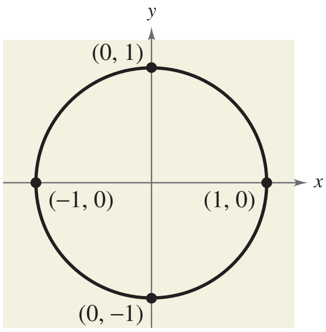


FIGURE 4.20

Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping, as shown in Figure 4.21.

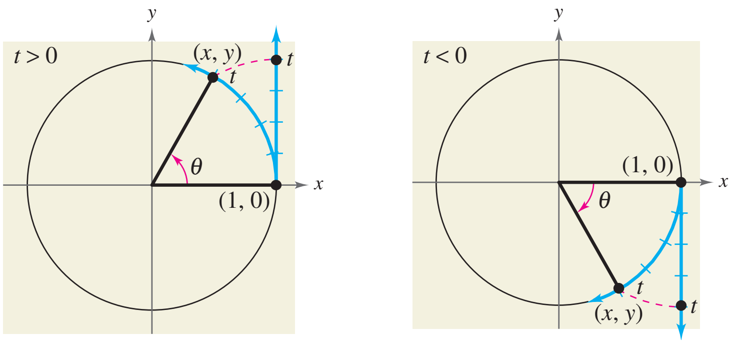


FIGURE 4.21

As the real number line is wrapped around the unit circle, each real number t corresponds to a point (x, y) on the circle. For example, the real number 0 corresponds to the point $(1, 0)$. Moreover, because the unit circle has a circumference of 2π, the real number 2π also corresponds to the point $(1, 0)$.

In general, each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t. With this interpretation of t, the arc length formula $s = r\theta$ (with $r = 1$) indicates that the real number t is the length of the arc intercepted by the angle θ, given in radians.

The Trigonometric Functions

From the preceding discussion, it follows that the coordinates x and y are two functions of the real variable t. You can use these coordinates to define the six trigonometric functions of t.

sine **cosecant** **cosine** **secant** **tangent** **cotangent**

These six functions are normally abbreviated sin, csc, cos, sec, tan, and cot, respectively.

STUDY TIP

Note in the definition at the right that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$. For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*. Similarly, the cotangent and cosecant are not defined when $y = 0$. For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, cot 0 and csc 0 are *undefined*.

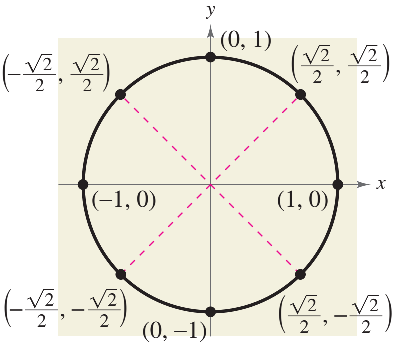


FIGURE 4.22

In Figure 4.22, the unit circle has been divided into eight equal arcs, corresponding to t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

Similarly, in Figure 4.23, the unit circle has been divided into 12 equal arcs, corresponding to t-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

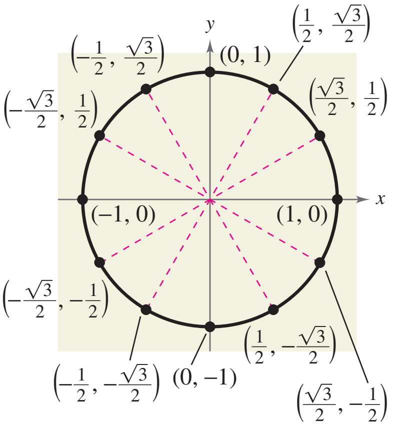


FIGURE 4.23

To verify the points on the unit circle in Figure 4.22, note that $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ also lies on the line $y = x$. So, substituting x for y in the equation of the unit circle produces the following.

$$x^2 + x^2 = 1 \quad \Rightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{\sqrt{2}}{2}$$

Because the point is in the first quadrant, $x = \frac{\sqrt{2}}{2}$ and because $y = x$, you also have $y = \frac{\sqrt{2}}{2}$. You can use similar reasoning to verify the rest of the points in Figure 4.22 and the points in Figure 4.23.

Using the (x, y) coordinates in Figures 4.22 and 4.23, you can easily evaluate the trigonometric functions for common t-values. This procedure is demonstrated in Examples 1 and 2. You should study and learn these exact function values for common t-values because they will help you in later sections to perform calculations quickly and easily.

Example 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at each real number.

a. $t = \dfrac{\pi}{6}$ **b.** $t = \dfrac{5\pi}{4}$ **c.** $t = 0$ **d.** $t = \pi$

Solution

For each t-value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 295.

a. $t = \dfrac{\pi}{6}$ corresponds to the point $(x, y) = \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

$$\sin\frac{\pi}{6} = y = \frac{1}{2} \qquad \csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b. $t = \dfrac{5\pi}{4}$ corresponds to the point $(x, y) = \left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec\frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1 \qquad \cot\frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

c. $t = 0$ corresponds to the point $(x, y) = (1, 0)$.

$$\sin 0 = y = 0 \qquad \csc 0 = \frac{1}{y} \text{ is undefined.}$$

$$\cos 0 = x = 1 \qquad \sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad \cot 0 = \frac{x}{y} \text{ is undefined.}$$

d. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0 \qquad \csc \pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos \pi = x = -1 \qquad \sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad \cot \pi = \frac{x}{y} \text{ is undefined.}$$

CHECKPOINT Now try Exercise 23.

Exploration

With your graphing utility in *radian* and *parametric* modes, enter the equations

X1T = cos T and Y1T = sin T

and use the following settings.

Tmin = 0, Tmax = 6.3,
Tstep = 0.1
Xmin = -1.5, Xmax = 1.5,
Xscl = 1
Ymin = -1, Ymax = 1,
Yscl = 1

1. Graph the entered equations and describe the graph.
2. Use the *trace* feature to move the cursor around the graph. What do the t-values represent? What do the x- and y-values represent?
3. What are the least and greatest values of x and y?

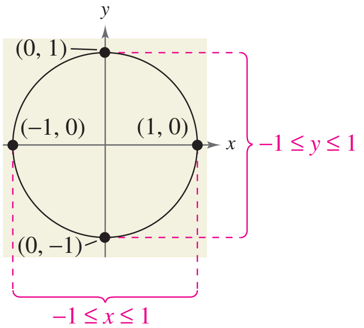


FIGURE 4.24

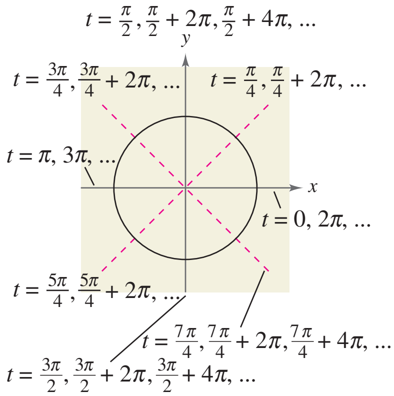


FIGURE 4.25

Example 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at $t = -\dfrac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sec\left(-\frac{\pi}{3}\right) = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} \qquad \cot\left(-\frac{\pi}{3}\right) = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

CHECKPOINT Now try Exercise 25.

Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 4.24. Because $r = 1$, it follows that $\sin t = y$ and $\cos t = x$. Moreover, because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

$$\begin{array}{ccc} -1 \le y \le 1 & & -1 \le x \le 1 \\ & \text{and} & \\ -1 \le \sin t \le 1 & & -1 \le \cos t \le 1 \end{array}$$

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.25. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

Recall from Section 1.5 that a function f is *even* if $f(-t) = f(t)$, and is *odd* if $f(-t) = -f(t)$.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

STUDY TIP

From the definition of periodic function, it follows that the sine and cosine functions are periodic and have a period of 2π. The other four trigonometric functions are also periodic, and will be discussed further in Section 4.6.

Example 3 Using the Period to Evaluate the Sine and Cosine

a. Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have $\sin\frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$.

b. Because $-\frac{7\pi}{2} = -4\pi + \frac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0.$$

c. For $\sin t = \frac{4}{5}$, $\sin(-t) = -\frac{4}{5}$ because the sine function is odd.

CHECKPOINT Now try Exercise 31.

Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (*degree* or *radian*).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the x^{-1} key with their respective reciprocal functions sine, cosine, and tangent. For example, to evaluate $\csc(\pi/8)$, use the fact that

$$\csc\frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

(SIN (π ÷ 8)) x^{-1} ENTER Display 2.6131259

Technology

When evaluating trigonometric functions with a calculator, remember to enclose all fractional angle measures in parentheses. For instance, if you want to evaluate $\sin\theta$ for $\theta = \pi/6$, you should enter

SIN (π ÷ 6) ENTER.

These keystrokes yield the correct value of 0.5. Note that some calculators automatically place a left parenthesis after trigonometric functions. Check the user's guide for your calculator for specific keystrokes on how to evaluate trigonometric functions.

Example 4 Using a Calculator

	Function	*Mode*	*Calculator Keystrokes*	*Display*
a.	$\sin\frac{2\pi}{3}$	Radian	SIN (2 π ÷ 3) ENTER	0.8660254
b.	$\cot 1.5$	Radian	(TAN (1.5)) x^{-1} ENTER	0.0709148

CHECKPOINT Now try Exercise 45.

4.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. Each real number t corresponds to a point (x, y) on the ________ ________.

2. A function f is ________ if there exists a positive real number c such that $f(t + c) = f(t)$ for all t in the domain of f.

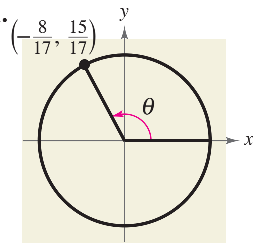
3. The smallest number c for which a function f is periodic is called the ________ of f.

4. A function f is ________ if $f(-t) = -f(t)$ and ________ if $f(-t) = f(t)$.

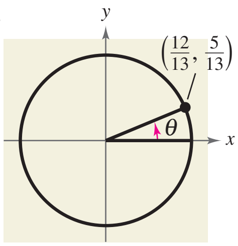
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ.

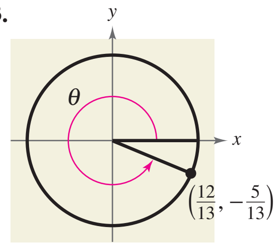
1.



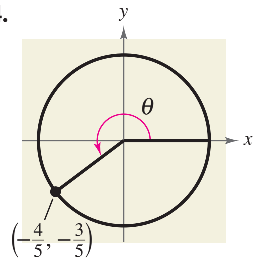
2.



3.



4.



In Exercises 5–12, find the point (x, y) on the unit circle that corresponds to the real number t.

5. $t = \frac{\pi}{4}$ **6.** $t = \frac{\pi}{3}$

7. $t = \frac{7\pi}{6}$ **8.** $t = \frac{5\pi}{4}$

9. $t = \frac{4\pi}{3}$ **10.** $t = \frac{5\pi}{3}$

11. $t = \frac{3\pi}{2}$ **12.** $t = \pi$

In Exercises 13–22, evaluate (if possible) the sine, cosine, and tangent of the real number.

13. $t = \frac{\pi}{4}$ **14.** $t = \frac{\pi}{3}$

15. $t = -\frac{\pi}{6}$ **16.** $t = -\frac{\pi}{4}$

17. $t = -\frac{7\pi}{4}$ **18.** $t = -\frac{4\pi}{3}$

19. $t = \frac{11\pi}{6}$ **20.** $t = \frac{5\pi}{3}$

21. $t = -\frac{3\pi}{2}$ **22.** $t = -2\pi$

In Exercises 23–28, evaluate (if possible) the six trigonometric functions of the real number.

23. $t = \frac{3\pi}{4}$ **24.** $t = \frac{5\pi}{6}$

25. $t = -\frac{\pi}{2}$ **26.** $t = \frac{3\pi}{2}$

27. $t = \frac{4\pi}{3}$ **28.** $t = \frac{7\pi}{4}$

In Exercises 29–36, evaluate the trigonometric function using its period as an aid.

29. $\sin 5\pi$ **30.** $\cos 5\pi$

31. $\cos \frac{8\pi}{3}$ **32.** $\sin \frac{9\pi}{4}$

33. $\cos\left(-\frac{15\pi}{2}\right)$ **34.** $\sin \frac{19\pi}{6}$

35. $\sin\left(-\frac{9\pi}{4}\right)$ **36.** $\cos\left(-\frac{8\pi}{3}\right)$

In Exercises 37–42, use the value of the trigonometric function to evaluate the indicated functions.

37. $\sin t = \frac{1}{3}$
(a) $\sin(-t)$
(b) $\csc(-t)$

38. $\sin(-t) = \frac{3}{8}$
(a) $\sin t$
(b) $\csc t$

39. $\cos(-t) = -\frac{1}{5}$
(a) $\cos t$
(b) $\sec(-t)$

40. $\cos t = -\frac{3}{4}$
(a) $\cos(-t)$
(b) $\sec(-t)$

41. $\sin t = \frac{4}{5}$
(a) $\sin(\pi - t)$
(b) $\sin(t + \pi)$

42. $\cos t = \frac{4}{5}$
(a) $\cos(\pi - t)$
(b) $\cos(t + \pi)$

In Exercises 43–52, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places. (Be sure the calculator is set in the correct angle mode.)

43. $\sin \dfrac{\pi}{4}$

44. $\tan \dfrac{\pi}{3}$

45. $\csc 1.3$

46. $\cot 1$

47. $\cos(-1.7)$

48. $\cos(-2.5)$

49. $\csc 0.8$

50. $\sec 1.8$

51. $\sec 22.8$

52. $\sin(-0.9)$

Estimation **In Exercises 53 and 54, use the figure and a straightedge to approximate the value of each trigonometric function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

53. (a) $\sin 5$ (b) $\cos 2$

54. (a) $\sin 0.75$ (b) $\cos 2.5$

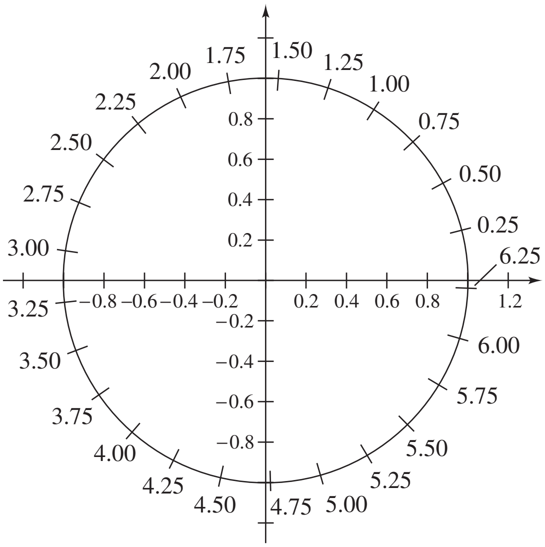


FIGURE FOR 53–56

Estimation **In Exercises 55 and 56, use the figure and a straightedge to approximate the solution of each equation, where $0 \le t < 2\pi$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

55. (a) $\sin t = 0.25$ (b) $\cos t = -0.25$

56. (a) $\sin t = -0.75$ (b) $\cos t = 0.75$

Model It

57. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is given by $y(t) = \frac{1}{4}e^{-t} \cos 6t$ where y is the displacement (in feet) and t is the time (in seconds).

Model It (continued)

(a) Complete the table.

t	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y					

(b) Use the *table* feature of a graphing utility to approximate the time when the weight reaches equilibrium.

(c) What appears to happen to the displacement as t increases?

58. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring is given by $y(t) = \frac{1}{4} \cos 6t$, where y is the displacement (in feet) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

Synthesis

True or False? **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

59. Because $\sin(-t) = -\sin t$, it can be said that the sine of a negative angle is a negative number.

60. $\tan a = \tan(a - 6\pi)$

61. ***Exploration*** Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = \pi - t_1$, respectively.

(a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2).

(b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(\pi - t_1)$.

(c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(\pi - t_1)$.

62. Use the unit circle to verify that the cosine and secant functions are even and that the sine, cosecant, tangent, and cotangent functions are odd.

Skills Review

In Exercises 63–66, find the inverse function f^{-1} of the one-to-one function f.

63. $f(x) = \frac{1}{2}(3x - 2)$

64. $f(x) = \frac{1}{4}x^3 + 1$

65. $f(x) = \sqrt{x^2 - 4}, \quad x \ge 2$

66. $f(x) = \dfrac{x + 2}{x - 4}$

In Exercises 67–70, sketch the graph of the rational function by hand. Show all asymptotes.

67. $f(x) = \dfrac{2x}{x - 3}$

68. $f(x) = \dfrac{5x}{x^2 + x - 6}$

69. $f(x) = \dfrac{x^2 + 3x - 10}{2x^2 - 8}$

70. $f(x) = \dfrac{x^3 - 6x^2 + x - 1}{2x^2 - 5x - 8}$

4.3 Right Triangle Trigonometry

What you should learn

- Evaluate trigonometric functions of acute angles.
- Use the fundamental trigonometric identities.
- Use a calculator to evaluate trigonometric functions.
- Use trigonometric functions to model and solve real-life problems.

Why you should learn it

Trigonometric functions are often used to analyze real-life situations. For instance, in Exercise 71 on page 311, you can use trigonometric functions to find the height of a helium-filled balloon.

Joseph Sohm; Chromosohm

The Six Trigonometric Functions

Our second look at the trigonometric functions is from a *right triangle* perspective. Consider a right triangle, with one acute angle labeled θ, as shown in Figure 4.26. Relative to the angle θ, the three sides of the triangle are the **hypotenuse,** the **opposite side** (the side opposite the angle θ), and the **adjacent side** (the side adjacent to the angle θ).

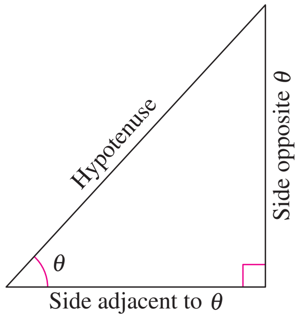


FIGURE 4.26

Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle θ.

sine cosecant cosine secant tangent cotangent

In the following definitions, it is important to see that $0° < \theta < 90°$ (θ lies in the first quadrant) and that for such angles the value of each trigonometric function is *positive*.

Right Triangle Definitions of Trigonometric Functions

Let θ be an *acute* angle of a right triangle. The six trigonometric functions of the angle θ are defined as follows. (Note that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle.

opp = the length of the side *opposite* θ

adj = the length of the side *adjacent to* θ

hyp = the length of the *hypotenuse*

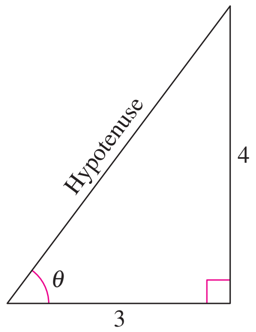


FIGURE 4.27

Example 1 Evaluating Trigonometric Functions

Use the triangle in Figure 4.27 to find the values of the six trigonometric functions of θ.

Solution

By the Pythagorean Theorem, $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$, it follows that

$$\begin{aligned}\text{hyp} &= \sqrt{4^2 + 3^2}\\ &= \sqrt{25}\\ &= 5.\end{aligned}$$

So, the six trigonometric functions of θ are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}.$$

CHECKPOINT Now try Exercise 3.

Historical Note
Georg Joachim Rhaeticus (1514–1576) was the leading Teutonic mathematical astronomer of the 16th century. He was the first to define the trigonometric functions as ratios of the sides of a right triangle.

In Example 1, you were given the lengths of two sides of the right triangle, but not the angle θ. Often, you will be asked to find the trigonometric functions of a *given* acute angle θ. To do this, construct a right triangle having θ as one of its angles.

Example 2 Evaluating Trigonometric Functions of 45°

Find the values of sin 45°, cos 45°, and tan 45°.

Solution

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 4.28. Choose the length of the adjacent side to be 1. From geometry, you know that the other acute angle is also 45°. So, the triangle is isosceles and the length of the opposite side is also 1. Using the Pythagorean Theorem, you find the length of the hypotenuse to be $\sqrt{2}$.

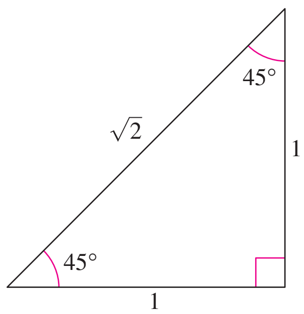


FIGURE 4.28

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

CHECKPOINT Now try Exercise 17.

STUDY TIP

Because the angles 30°, 45°, and 60° ($\pi/6$, $\pi/4$, and $\pi/3$) occur frequently in trigonometry, you should learn to construct the triangles shown in Figures 4.28 and 4.29.

Technology

You can use a calculator to convert the answers in Example 3 to decimals. However, the radical form is the exact value and in most cases, the exact value is preferred.

Example 3 Evaluating Trigonometric Functions of 30° and 60°

Use the equilateral triangle shown in Figure 4.29 to find the values of sin 60°, cos 60°, sin 30°, and cos 30°.

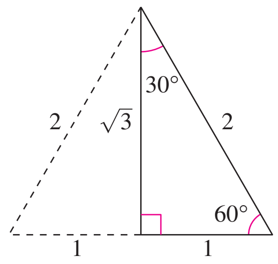


FIGURE 4.29

Solution

Use the Pythagorean Theorem and the equilateral triangle in Figure 4.29 to verify the lengths of the sides shown in the figure. For $\theta = 60°$, you have adj $= 1$, opp $= \sqrt{3}$, and hyp $= 2$. So,

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

For $\theta = 30°$, adj $= \sqrt{3}$, opp $= 1$, and hyp $= 2$. So,

$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \quad \text{and} \quad \cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

CHECKPOINT Now try Exercise 19.

Sines, Cosines, and Tangents of Special Angles

$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan 45° = \tan\frac{\pi}{4} = 1$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

In the box, note that $\sin 30° = \frac{1}{2} = \cos 60°$. This occurs because 30° and 60° are complementary angles. In general, it can be shown from the right triangle definitions that *cofunctions of complementary angles are equal*. That is, if θ is an acute angle, the following relationships are true.

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sec(90° - \theta) = \csc\theta \qquad \csc(90° - \theta) = \sec\theta$$

Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric functions (identities).

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Note that $\sin^2\theta$ represents $(\sin\theta)^2$, $\cos^2\theta$ represents $(\cos\theta)^2$, and so on.

Example 4 Applying Trigonometric Identities

Let θ be an acute angle such that $\sin\theta = 0.6$. Find the values of (a) $\cos\theta$ and (b) $\tan\theta$ using trigonometric identities.

Solution

a. To find the value of $\cos\theta$, use the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

So, you have

$$(0.6)^2 + \cos^2\theta = 1 \qquad \text{Substitute 0.6 for } \sin\theta.$$

$$\cos^2\theta = 1 - (0.6)^2 = 0.64 \qquad \text{Subtract } (0.6)^2 \text{ from each side.}$$

$$\cos\theta = \sqrt{0.64} = 0.8. \qquad \text{Extract the positive square root.}$$

b. Now, knowing the sine and cosine of θ, you can find the tangent of θ to be

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75.$$

Use the definitions of $\cos\theta$ and $\tan\theta$, and the triangle shown in Figure 4.30, to check these results.

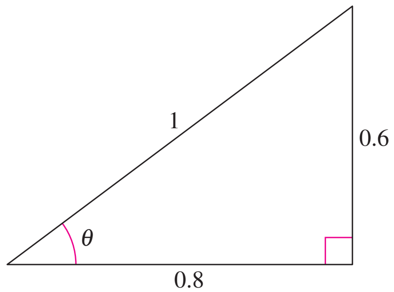


FIGURE 4.30

CHECKPOINT Now try Exercise 29.

Example 5 Applying Trigonometric Identities

Let θ be an acute angle such that $\tan\theta = 3$. Find the values of (a) $\cot\theta$ and (b) $\sec\theta$ using trigonometric identities.

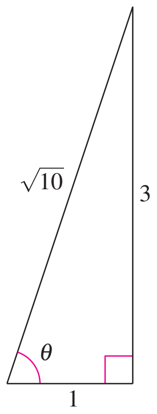


FIGURE 4.31

Solution

a. $\cot\theta = \dfrac{1}{\tan\theta}$ Reciprocal identity

$\cot\theta = \dfrac{1}{3}$

b. $\sec^2\theta = 1 + \tan^2\theta$ Pythagorean identity

$\sec^2\theta = 1 + 3^2$

$\sec^2\theta = 10$

$\sec\theta = \sqrt{10}$

Use the definitions of $\cot\theta$ and $\sec\theta$, and the triangle shown in Figure 4.31, to check these results.

CHECKPOINT Now try Exercise 31.

Evaluating Trigonometric Functions with a Calculator

To use a calculator to evaluate trigonometric functions of angles measured in degrees, first set the calculator to *degree* mode and then proceed as demonstrated in Section 4.2. For instance, you can find values of cos 28° and sec 28° as follows.

Function	*Mode*	*Calculator Keystrokes*	*Display*
a. cos 28°	Degree	$\boxed{\text{COS}}$ 28 $\boxed{\text{ENTER}}$	0.8829476
b. sec 28°	Degree	$\boxed{(}$ $\boxed{\text{COS}}$ $\boxed{(}$ 28 $\boxed{)}$ $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$	1.1325701

Throughout this text, angles are assumed to be measured in radians unless noted otherwise. For example, sin 1 means the sine of 1 radian and sin 1° means the sine of 1 degree.

STUDY TIP

You can also use the reciprocal identities for sine, cosine, and tangent to evaluate the cosecant, secant, and cotangent functions with a calculator. For instance, you could use the following keystroke sequence to evaluate sec 28°.

1 $\boxed{\div}$ $\boxed{\text{COS}}$ 28 $\boxed{\text{ENTER}}$

The calculator should display 1.1325701.

Example 6 Using a Calculator

Use a calculator to evaluate $\sec(5°\,40'\,12'')$.

Solution

Begin by converting to decimal degree form. [Recall that $1' = \frac{1}{60}(1°)$ and $1'' = \frac{1}{3600}(1°)$].

$$5°\,40'\,12'' = 5° + \left(\frac{40}{60}\right)^{\circ} + \left(\frac{12}{3600}\right)^{\circ} = 5.67°$$

Then, use a calculator to evaluate sec 5.67°.

Function	*Calculator Keystrokes*	*Display*
$\sec(5°\,40'\,12'') = \sec 5.67°$	$\boxed{(}$ $\boxed{\text{COS}}$ $\boxed{(}$ 5.67 $\boxed{)}$ $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$	1.0049166

CHECKPOINT Now try Exercise 47.

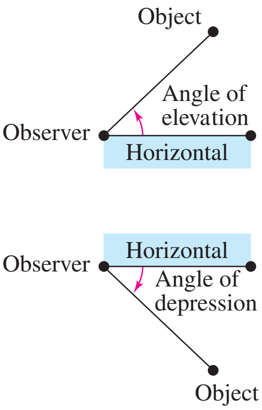


FIGURE 4.32

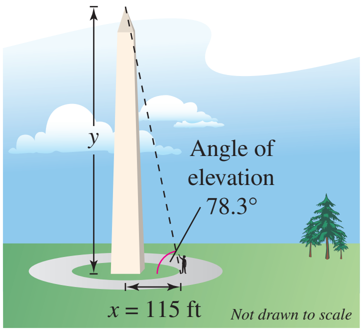


FIGURE 4.33

Applications Involving Right Triangles

Many applications of trigonometry involve a process called **solving right triangles.** In this type of application, you are usually given one side of a right triangle and one of the acute angles and are asked to find one of the other sides, *or* you are given two sides and are asked to find one of the acute angles.

In Example 7, the angle you are given is the **angle of elevation,** which represents the angle from the horizontal upward to an object. For objects that lie below the horizontal, it is common to use the term **angle of depression,** as shown in Figure 4.32.

Example 7 Using Trigonometry to Solve a Right Triangle

A surveyor is standing 115 feet from the base of the Washington Monument, as shown in Figure 4.33. The surveyor measures the angle of elevation to the top of the monument as 78.3°. How tall is the Washington Monument?

Solution

From Figure 4.33, you can see that

$$\tan 78.3° = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

where $x = 115$ and y is the height of the monument. So, the height of the Washington Monument is

$$y = x \tan 78.3° \approx 115(4.82882) \approx 555 \text{ feet.}$$

CHECKPOINT Now try Exercise 63.

Example 8 Using Trigonometry to Solve a Right Triangle

An historic lighthouse is 200 yards from a bike path along the edge of a lake. A walkway to the lighthouse is 400 yards long. Find the acute angle θ between the bike path and the walkway, as illustrated in Figure 4.34.

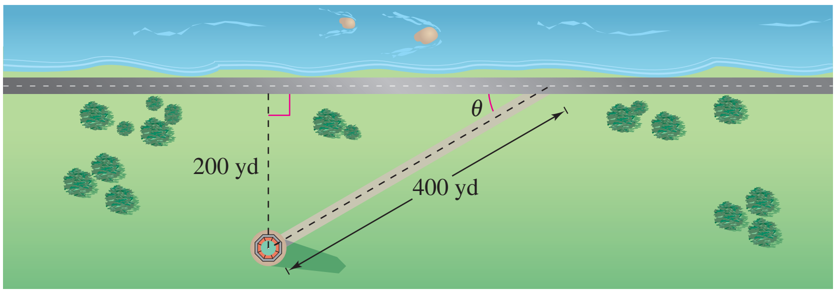


FIGURE 4.34

Solution

From Figure 4.34, you can see that the sine of the angle θ is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{200}{400} = \frac{1}{2}.$$

Now you should recognize that $\theta = 30°$.

CHECKPOINT Now try Exercise 65.

By now you are able to recognize that $\theta = 30°$ is the acute angle that satisfies the equation $\sin \theta = \frac{1}{2}$. Suppose, however, that you were given the equation $\sin \theta = 0.6$ and were asked to find the acute angle θ. Because

$$\sin 30° = \frac{1}{2}$$
$$= 0.5000$$

and

$$\sin 45° = \frac{1}{\sqrt{2}}$$
$$\approx 0.7071$$

you might guess that θ lies somewhere between 30° and 45°. In a later section, you will study a method by which a more precise value of θ can be determined.

Example 9 Solving a Right Triangle

Find the length c of the skateboard ramp shown in Figure 4.35.

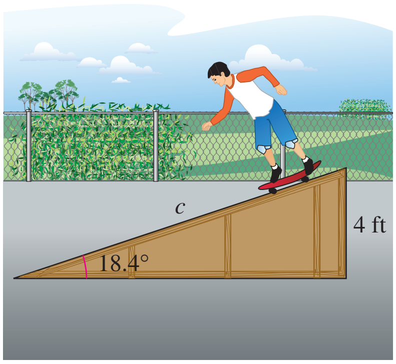


FIGURE 4.35

Solution

From Figure 4.35, you can see that

$$\sin 18.4° = \frac{\text{opp}}{\text{hyp}}$$
$$= \frac{4}{c}.$$

So, the length of the skateboard ramp is

$$c = \frac{4}{\sin 18.4°}$$
$$\approx \frac{4}{0.3156}$$
$$\approx 12.7 \text{ feet.}$$

CHECKPOINT Now try Exercise 67.

4.3 Exercises

VOCABULARY CHECK:

1. Match the trigonometric function with its right triangle definition.

(a) Sine (b) Cosine (c) Tangent (d) Cosecant (e) Secant (f) Cotangent

(i) $\dfrac{\text{hypotenuse}}{\text{adjacent}}$ (ii) $\dfrac{\text{adjacent}}{\text{opposite}}$ (iii) $\dfrac{\text{hypotenuse}}{\text{opposite}}$ (iv) $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ (v) $\dfrac{\text{opposite}}{\text{hypotenuse}}$ (vi) $\dfrac{\text{opposite}}{\text{adjacent}}$

In Exercises 2 and 3, fill in the blanks.

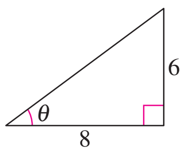
2. Relative to the angle θ, the three sides of a right triangle are the ________ side, the ________ side, and the ________.

3. An angle that measures from the horizontal upward to an object is called the angle of ________, whereas an angle that measures from the horizontal downward to an object is called the angle of ________.

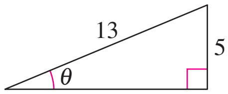
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, find the exact values of the six trigonometric functions of the angle θ shown in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)

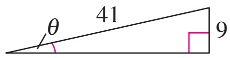
1.



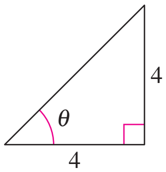
2.



3.

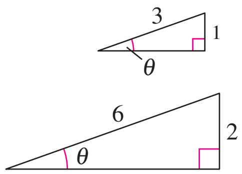


4.

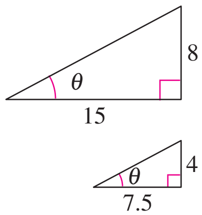


In Exercises 5–8, find the exact values of the six trigonometric functions of the angle θ for each of the two triangles. Explain why the function values are the same.

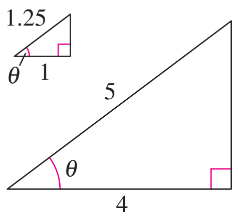
5.



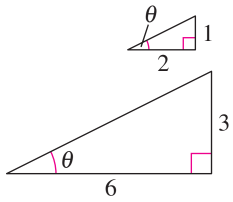
6.



7.



8.



In Exercises 9–16, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side and then find the other five trigonometric functions of θ.

9. $\sin \theta = \frac{3}{4}$

10. $\cos \theta = \frac{5}{7}$

11. $\sec \theta = 2$

12. $\cot \theta = 5$

13. $\tan \theta = 3$

14. $\sec \theta = 6$

15. $\cot \theta = \frac{3}{2}$

16. $\csc \theta = \frac{17}{4}$

In Exercises 17–26, construct an appropriate triangle to complete the table. ($0 \le \theta \le 90°$, $0 \le \theta \le \pi/2$)

	Function	*θ (deg)*	*θ (rad)*	*Function Value*
17.	sin	30°		
18.	cos	45°		
19.	tan		$\dfrac{\pi}{3}$	
20.	sec		$\dfrac{\pi}{4}$	
21.	cot			$\dfrac{\sqrt{3}}{3}$
22.	csc			$\sqrt{2}$
23.	cos		$\dfrac{\pi}{6}$	
24.	sin		$\dfrac{\pi}{4}$	
25.	cot			1
26.	tan			$\dfrac{\sqrt{3}}{3}$

In Exercises 27–32, use the given function value(s), and trigonometric identities (including the cofunction identities), to find the indicated trigonometric functions.

27. $\sin 60° = \dfrac{\sqrt{3}}{2}$, $\cos 60° = \dfrac{1}{2}$

(a) $\tan 60°$ (b) $\sin 30°$

(c) $\cos 30°$ (d) $\cot 60°$

28. $\sin 30° = \dfrac{1}{2}$, $\tan 30° = \dfrac{\sqrt{3}}{3}$

(a) $\csc 30°$ (b) $\cot 60°$

(c) $\cos 30°$ (d) $\cot 30°$

29. $\csc \theta = \dfrac{\sqrt{13}}{2}$, $\sec \theta = \dfrac{\sqrt{13}}{3}$

(a) $\sin \theta$ (b) $\cos \theta$

(c) $\tan \theta$ (d) $\sec(90° - \theta)$

30. $\sec \theta = 5$, $\tan \theta = 2\sqrt{6}$

(a) $\cos \theta$ (b) $\cot \theta$

(c) $\cot(90° - \theta)$ (d) $\sin \theta$

31. $\cos \alpha = \frac{1}{3}$

(a) $\sec \alpha$ (b) $\sin \alpha$

(c) $\cot \alpha$ (d) $\sin(90° - \alpha)$

32. $\tan \beta = 5$

(a) $\cot \beta$ (b) $\cos \beta$

(c) $\tan(90° - \beta)$ (d) $\csc \beta$

In Exercises 33–42, use trigonometric identities to transform the left side of the equation into the right side $(0 < \theta < \pi/2)$.

33. $\tan \theta \cot \theta = 1$

34. $\cos \theta \sec \theta = 1$

35. $\tan \alpha \cos \alpha = \sin \alpha$

36. $\cot \alpha \sin \alpha = \cos \alpha$

37. $(1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta$

38. $(1 + \sin \theta)(1 - \sin \theta) = \cos^2 \theta$

39. $(\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = 1$

40. $\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta - 1$

41. $\dfrac{\sin \theta}{\cos \theta} + \dfrac{\cos \theta}{\sin \theta} = \csc \theta \sec \theta$

42. $\dfrac{\tan \beta + \cot \beta}{\tan \beta} = \csc^2 \beta$

In Exercises 43–52, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct angle mode.)

43. (a) $\sin 10°$ (b) $\cos 80°$

44. (a) $\tan 23.5°$ (b) $\cot 66.5°$

45. (a) $\sin 16.35°$ (b) $\csc 16.35°$

46. (a) $\cos 16° \, 18'$ (b) $\sin 73° \, 56'$

47. (a) $\sec 42° \, 12'$ (b) $\csc 48° \, 7'$

48. (a) $\cos 4° \, 50' \, 15''$ (b) $\sec 4° \, 50' \, 15''$

49. (a) $\cot 11° \, 15'$ (b) $\tan 11° \, 15'$

50. (a) $\sec 56° \, 8 \, 10''$ (b) $\cos 56° \, 8 \, 10''$

51. (a) $\csc 32° \, 40' \, 3''$ (b) $\tan 44° \, 28 \, 16''$

52. (a) $\sec\left(\frac{9}{5} \cdot 20 + 32\right)°$ (b) $\cot\left(\frac{9}{5} \cdot 30 + 32\right)°$

In Exercises 53–58, find the values of θ in degrees $(0° < \theta < 90°)$ and radians $(0 < \theta < \pi/2)$ without the aid of a calculator.

53. (a) $\sin \theta = \dfrac{1}{2}$ (b) $\csc \theta = 2$

54. (a) $\cos \theta = \dfrac{\sqrt{2}}{2}$ (b) $\tan \theta = 1$

55. (a) $\sec \theta = 2$ (b) $\cot \theta = 1$

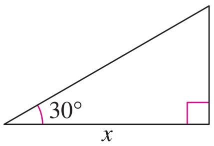
56. (a) $\tan \theta = \sqrt{3}$ (b) $\cos \theta = \dfrac{1}{2}$

57. (a) $\csc \theta = \dfrac{2\sqrt{3}}{3}$ (b) $\sin \theta = \dfrac{\sqrt{2}}{2}$

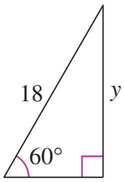
58. (a) $\cot \theta = \dfrac{\sqrt{3}}{3}$ (b) $\sec \theta = \sqrt{2}$

In Exercises 59–62, solve for *x*, *y*, or *r* as indicated.

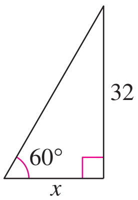
59. Solve for x.



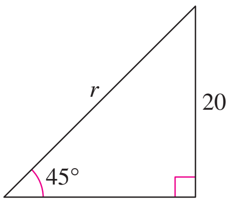
60. Solve for y.



61. Solve for x.



62. Solve for r.



63. ***Empire State Building*** You are standing 45 meters from the base of the Empire State Building. You estimate that the angle of elevation to the top of the 86th floor (the observatory) is 82°. If the total height of the building is another 123 meters above the 86th floor, what is the approximate height of the building? One of your friends is on the 86th floor. What is the distance between you and your friend?

64. ***Height*** A six-foot person walks from the base of a broadcasting tower directly toward the tip of the shadow cast by the tower. When the person is 132 feet from the tower and 3 feet from the tip of the shadow, the person's shadow starts to appear beyond the tower's shadow.

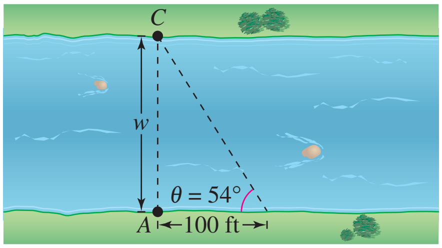
(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the tower.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

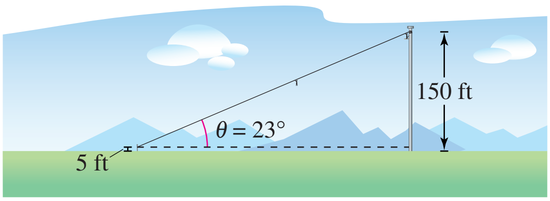
(c) What is the height of the tower?

65. ***Angle of Elevation*** You are skiing down a mountain with a vertical height of 1500 feet. The distance from the top of the mountain to the base is 3000 feet. What is the angle of elevation from the base to the top of the mountain?

66. ***Width of a River*** A biologist wants to know the width w of a river so in order to properly set instruments for studying the pollutants in the water. From point A, the biologist walks downstream 100 feet and sights to point C (see figure). From this sighting, it is determined that $\theta = 54°$. How wide is the river?



67. ***Length*** A steel cable zip-line is being constructed for a competition on a reality television show. One end of the zip-line is attached to a platform on top of a 150-foot pole. The other end of the zip-line is attached to the top of a 5-foot stake. The angle of elevation to the platform is 23° (see figure).

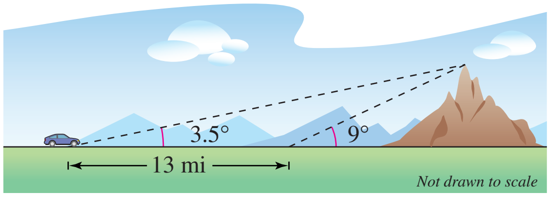


(a) How long is the zip-line?

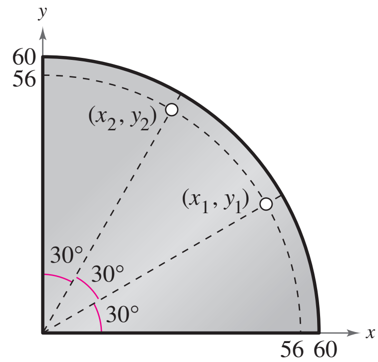
(b) How far is the stake from the pole?

(c) Contestants take an average of 6 seconds to reach the ground from the top of the zip-line. At what rate are contestants moving down the line? At what rate are they dropping vertically?

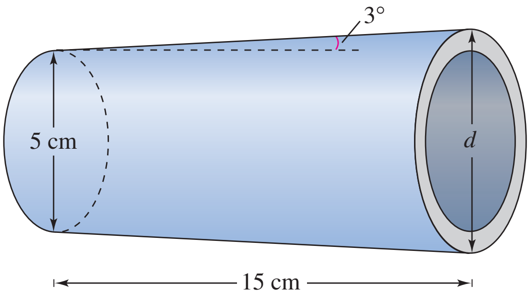
68. ***Height of a Mountain*** In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9°. Approximate the height of the mountain.



69. ***Machine Shop Calculations*** A steel plate has the form of one-fourth of a circle with a radius of 60 centimeters. Two two-centimeter holes are to be drilled in the plate positioned as shown in the figure. Find the coordinates of the center of each hole.



70. ***Machine Shop Calculations*** A tapered shaft has a diameter of 5 centimeters at the small end and is 15 centimeters long (see figure). The taper is 3°. Find the diameter d of the large end of the shaft.



Model It

71. ***Height*** A 20-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 85° with the ground.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the balloon.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the balloon?

(d) The breeze becomes stronger and the angle the balloon makes with the ground decreases. How does this affect the triangle you drew in part (a)?

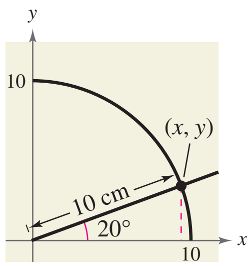
(e) Complete the table, which shows the heights (in meters) of the balloon for decreasing angle measures θ.

Angle, θ	80°	70°	60°	50°
Height				

Angle, θ	40°	30°	20°	10°
Height				

(f) As the angle the balloon makes with the ground approaches 0°, how does this affect the height of the balloon? Draw a right triangle to explain your reasoning.

72. ***Geometry*** Use a compass to sketch a quarter of a circle of radius 10 centimeters. Using a protractor, construct an angle of 20° in standard position (see figure). Drop a perpendicular line from the point of intersection of the terminal side of the angle and the arc of the circle. By actual measurement, calculate the coordinates (x, y) of the point of intersection and use these measurements to approximate the six trigonometric functions of a 20° angle.



Synthesis

True or False? **In Exercises 73–78, determine whether the statement is true or false. Justify your answer.**

73. $\sin 60° \csc 60° = 1$

74. $\sec 30° = \csc 60°$

75. $\sin 45° + \cos 45° = 1$

76. $\cot^2 10° - \csc^2 10° = -1$

77. $\dfrac{\sin 60°}{\sin 30°} = \sin 2°$

78. $\tan[(5°)^2] = \tan^2(5°)$

79. ***Writing*** In right triangle trigonometry, explain why $\sin 30° = \frac{1}{2}$ regardless of the size of the triangle.

80. ***Think About It*** You are given only the value $\tan \theta$. Is it possible to find the value of $\sec \theta$ without finding the measure of θ? Explain.

81. ***Exploration***

(a) Complete the table.

θ	0.1	0.2	0.3	0.4	0.5
$\sin \theta$					

(b) Is θ or $\sin \theta$ greater for θ in the interval $(0, 0.5]$?

(c) As θ approaches 0, how do θ and $\sin \theta$ compare? Explain.

82. ***Exploration***

(a) Complete the table.

θ	0°	18°	36°	54°	72°	90°
$\sin \theta$						
$\cos \theta$						

(b) Discuss the behavior of the sine function for θ in the range from 0° to 90°.

(c) Discuss the behavior of the cosine function for θ in the range from 0° to 90°.

(d) Use the definitions of the sine and cosine functions to explain the results of parts (b) and (c).

Skills Review

In Exercises 83–86, perform the operations and simplify.

83. $\dfrac{x^2 - 6x}{x^2 + 4x - 12} \cdot \dfrac{x^2 + 12x + 36}{x^2 - 36}$

84. $\dfrac{2t^2 + 5t - 12}{9 - 4t^2} \div \dfrac{t^2 - 16}{4t^2 + 12t + 9}$

85. $\dfrac{3}{x + 2} - \dfrac{2}{x - 2} + \dfrac{x}{x^2 + 4x + 4}$

86. $\dfrac{\left(\dfrac{3}{x} - \dfrac{1}{4}\right)}{\left(\dfrac{12}{x} - 1\right)}$

4.4 Trigonometric Functions of Any Angle

What you should learn

- Evaluate trigonometric functions of any angle.
- Use reference angles to evaluate trigonometric functions.
- Evaluate trigonometric functions of real numbers.

Why you should learn it

You can use trigonometric functions to model and solve real-life problems. For instance, in Exercise 87 on page 319, you can use trigonometric functions to model the monthly normal temperatures in New York City and Fairbanks, Alaska.

James Urbach/SuperStock

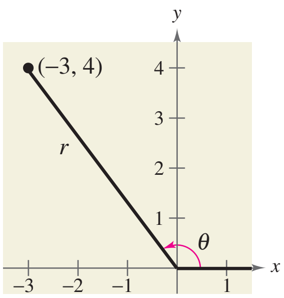


FIGURE 4.36

Introduction

In Section 4.3, the definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. If θ is an *acute* angle, these definitions coincide with those given in the preceding section.

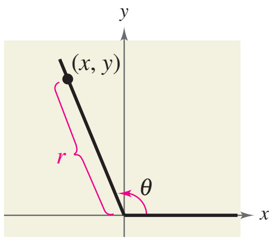
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

$$\sec\theta = \frac{r}{x}, \quad x \neq 0 \qquad \csc\theta = \frac{r}{y}, \quad y \neq 0$$



Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, if $x = 0$, the tangent and secant of θ are undefined. For example, the tangent of $90°$ is undefined. Similarly, if $y = 0$, the cotangent and cosecant of θ are undefined.

Example 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

Solution

Referring to Figure 4.36, you can see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

So, you have the following.

$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{r} = -\frac{3}{5}$$

$$\tan\theta = \frac{y}{x} = -\frac{4}{3}$$

CHECKPOINT Now try Exercise 1.

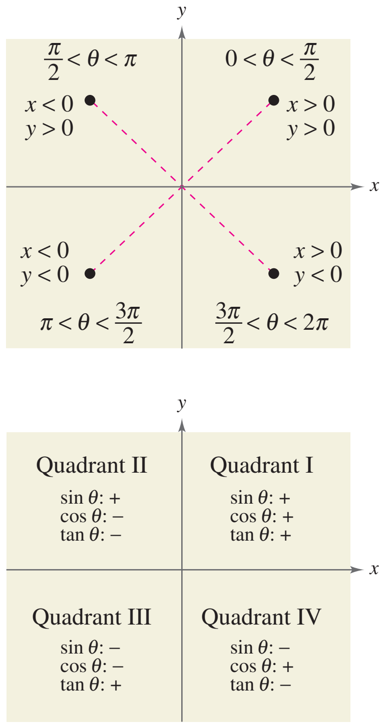


FIGURE 4.37

The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because $\cos\theta = x/r$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.) In a similar manner, you can verify the results shown in Figure 4.37.

Example 2 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{5}{4}$ and $\cos\theta > 0$, find $\sin\theta$ and $\sec\theta$.

Solution

Note that θ lies in Quadrant IV because that is the only quadrant in which the tangent is negative and the cosine is positive. Moreover, using

$$\tan\theta = \frac{y}{x} = -\frac{5}{4}$$

and the fact that y is negative in Quadrant IV, you can let $y = -5$ and $x = 4$. So, $r = \sqrt{16 + 25} = \sqrt{41}$ and you have

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{41}}$$

$$\approx -0.7809$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{4}$$

$$\approx 1.6008.$$

CHECKPOINT Now try Exercise 17.

Example 3 Trigonometric Functions of Quadrant Angles

Evaluate the cosine and tangent functions at the four quadrant angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Solution

To begin, choose a point on the terminal side of each angle, as shown in Figure 4.38. For each of the four points, $r = 1$, and you have the following.

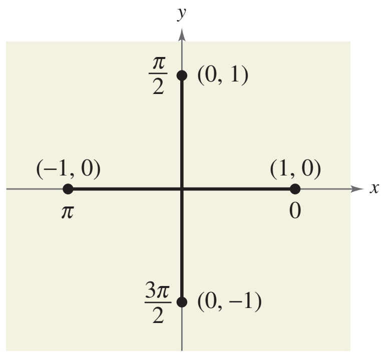


FIGURE 4.38

$$\cos 0 = \frac{x}{r} = \frac{1}{1} = 1 \qquad \tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad (x, y) = (1, 0)$$

$$\cos\frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} \implies \text{undefined} \qquad (x, y) = (0, 1)$$

$$\cos\pi = \frac{x}{r} = \frac{-1}{1} = -1 \qquad \tan\pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad (x, y) = (-1, 0)$$

$$\cos\frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0} \implies \text{undefined} \qquad (x, y) = (0, -1)$$

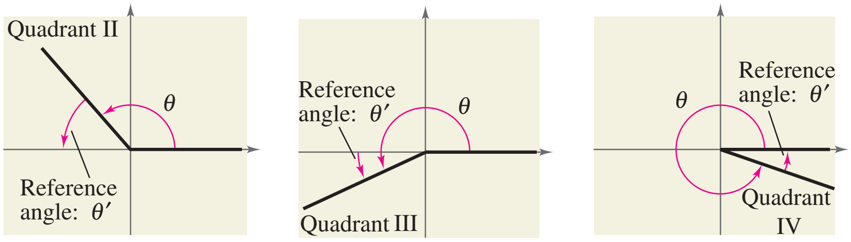
CHECKPOINT Now try Exercise 29.

Reference Angles

The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

> ### Definition of Reference Angle
> Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 4.39 shows the reference angles for θ in Quadrants II, III, and IV.



$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

FIGURE 4.39

Example 4 Finding Reference Angles

Find the reference angle θ'.

a. $\theta = 300°$ **b.** $\theta = 2.3$ **c.** $\theta = -135°$

Solution

a. Because 300° lies in Quadrant IV, the angle it makes with the x-axis is

$$\theta' = 360° - 300°$$
$$= 60°. \qquad \text{Degrees}$$

Figure 4.40 shows the angle $\theta = 300°$ and its reference angle $\theta' = 60°$.

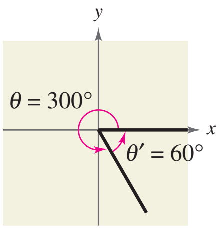


FIGURE 4.40

b. Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$
$$\approx 0.8416. \qquad \text{Radians}$$

Figure 4.41 shows the angle $\theta = 2.3$ and its reference angle $\theta' = \pi - 2.3$.

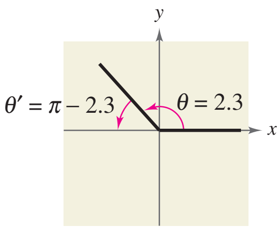


FIGURE 4.41

c. First, determine that $-135°$ is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180°$$
$$= 45°. \qquad \text{Degrees}$$

Figure 4.42 shows the angle $\theta = -135°$ and its reference angle $\theta' = 45°$.

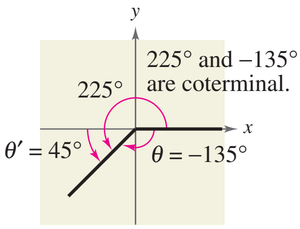
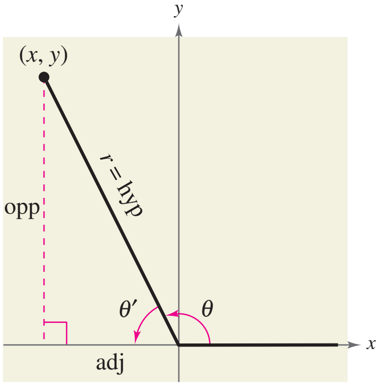


FIGURE 4.42

CHECKPOINT Now try Exercise 37.



opp $= |y|$, adj $= |x|$

FIGURE 4.43

Trigonometric Functions of Real Numbers

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ, as shown in Figure 4.43. By definition, you know that

$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

So, it follows that $\sin\theta$ and $\sin\theta'$ are equal, *except possibly in sign*. The same is true for $\tan\theta$ and $\tan\theta'$ *and* for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.

2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

STUDY TIP

Learning the table of values at the right is worth the effort because doing so will increase both your efficiency and your confidence. Here is a pattern for the sine function that may help you remember the values.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Reverse the order to get cosine values of the same angles.

By using reference angles and the special angles discussed in the preceding section, you can greatly extend the scope of *exact* trigonometric values. For instance, knowing the function values of $30°$ means that you know the function values of all angles for which $30°$ is a reference angle. For convenience, the table below shows the exact values of the trigonometric functions of special angles and quadrant angles.

Trigonometric Values of Common Angles

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

Example 5 Using Reference Angles

Evaluate each trigonometric function.

a. $\cos \dfrac{4\pi}{3}$ **b.** $\tan(-210^\circ)$ **c.** $\csc \dfrac{11\pi}{4}$

Solution

a. Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is $\theta' = (4\pi/3) - \pi = \pi/3$, as shown in Figure 4.44. Moreover, the cosine is negative in Quadrant III, so

$$\cos \frac{4\pi}{3} = (-) \cos \frac{\pi}{3}$$
$$= -\frac{1}{2}.$$

b. Because $-210^\circ + 360^\circ = 150^\circ$, it follows that -210° is coterminal with the second-quadrant angle 150°. So, the reference angle is $\theta' = 180^\circ - 150^\circ = 30^\circ$, as shown in Figure 4.45. Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210^\circ) = (-) \tan 30^\circ$$
$$= -\frac{\sqrt{3}}{3}.$$

c. Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. So, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 4.46. Because the cosecant is positive in Quadrant II, you have

$$\csc \frac{11\pi}{4} = (+) \csc \frac{\pi}{4}$$
$$= \frac{1}{\sin(\pi/4)}$$
$$= \sqrt{2}.$$

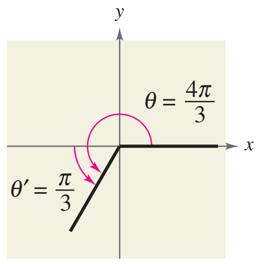


FIGURE 4.44

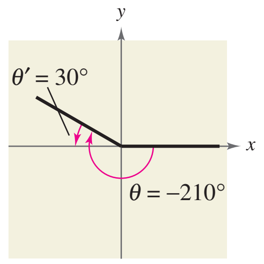


FIGURE 4.45

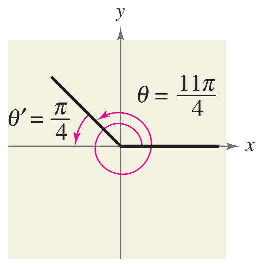


FIGURE 4.46

CHECKPOINT Now try Exercise 51.

Example 6 Using Trigonometric Identities

Let θ be an angle in Quadrant II such that $\sin\theta = \frac{1}{3}$. Find (a) $\cos\theta$ and (b) $\tan\theta$ by using trigonometric identities.

Solution

a. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, you obtain

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1 \qquad \text{Substitute } \tfrac{1}{3} \text{ for } \sin\theta.$$

$$\cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}.$$

Because $\cos\theta < 0$ in Quadrant II, you can use the negative root to obtain

$$\cos\theta = -\frac{\sqrt{8}}{\sqrt{9}}$$

$$= -\frac{2\sqrt{2}}{3}.$$

b. Using the trigonometric identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, you obtain

$$\tan\theta = \frac{1/3}{-2\sqrt{2}/3} \qquad \text{Substitute for } \sin\theta \text{ and } \cos\theta.$$

$$= -\frac{1}{2\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{4}.$$

CHECKPOINT Now try Exercise 59.

You can use a calculator to evaluate trigonometric functions, as shown in the next example.

Example 7 Using a Calculator

Use a calculator to evaluate each trigonometric function.

a. $\cot 410°$ **b.** $\sin(-7)$ **c.** $\sec\dfrac{\pi}{9}$

Solution

	Function	*Mode*	*Calculator Keystrokes*	*Display*
a.	$\cot 410°$	Degree	(TAN (410)) x^{-1} ENTER	0.8390996
b.	$\sin(-7)$	Radian	SIN ((−) 7) ENTER	−0.6569866
c.	$\sec\dfrac{\pi}{9}$	Radian	(COS (π ÷ 9)) x^{-1} ENTER	1.0641778

CHECKPOINT Now try Exercise 69.

4.4 Exercises

VOCABULARY CHECK:

In Exercises 1–6, let θ be an angle in standard position, with (x, y) a point on the terminal side of θ and $r\sqrt{x^2 + y^2} \neq 0$.

1. $\sin\theta =$ ________ **2.** $\dfrac{r}{y} =$ ________

3. $\tan\theta =$ ________ **4.** $\sec\theta =$ ________

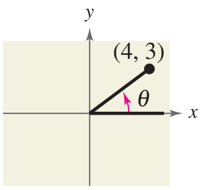
5. $\dfrac{x}{r} =$ ________ **6.** $\dfrac{x}{y} =$ ________

7. The acute positive angle that is formed by the terminal side of the angle θ and the horizontal axis is called the ________ angle of θ and is denoted by θ'.

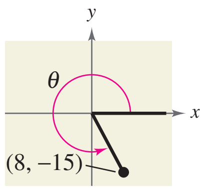
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ.

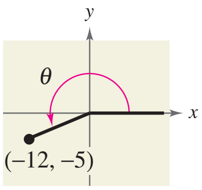
1. (a)



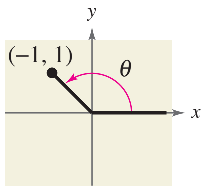
(b)



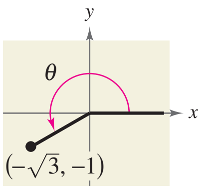
2. (a)



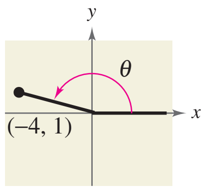
(b)



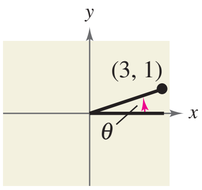
3. (a)



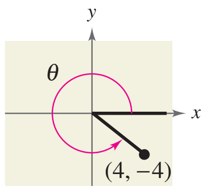
(b)



4. (a)



(b)



In Exercises 5–10, the point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

5. $(7, 24)$ **6.** $(8, 15)$

7. $(-4, 10)$ **8.** $(-5, -2)$

9. $(-3.5, 6.8)$ **10.** $\left(3\frac{1}{2}, -7\frac{3}{4}\right)$

In Exercises 11–14, state the quadrant in which θ lies.

11. $\sin\theta < 0$ and $\cos\theta < 0$

12. $\sin\theta > 0$ and $\cos\theta > 0$

13. $\sin\theta > 0$ and $\tan\theta < 0$

14. $\sec\theta > 0$ and $\cot\theta < 0$

In Exercises 15–24, find the values of the six trigonometric functions of θ with the given constraint.

	Function Value	*Constraint*
15.	$\sin\theta = \frac{3}{5}$	θ lies in Quadrant II.
16.	$\cos\theta = -\frac{4}{5}$	θ lies in Quadrant III.
17.	$\tan\theta = -\frac{15}{8}$	$\sin\theta < 0$
18.	$\cos\theta = \frac{8}{17}$	$\tan\theta < 0$
19.	$\cot\theta = -3$	$\cos\theta > 0$
20.	$\csc\theta = 4$	$\cot\theta < 0$
21.	$\sec\theta = -2$	$\sin\theta > 0$
22.	$\sin\theta = 0$	$\sec\theta = -1$
23.	$\cot\theta$ is undefined.	$\pi/2 \leq \theta \leq 3\pi/2$
24.	$\tan\theta$ is undefined.	$\pi \leq \theta \leq 2\pi$

In Exercises 25–28, the terminal side of θ lies on the given line in the specified quadrant. Find the values of the six trigonometric functions of θ by finding a point on the line.

	Line	*Quadrant*
25.	$y = -x$	II
26.	$y = \frac{1}{3}x$	III
27.	$2x - y = 0$	III
28.	$4x + 3y = 0$	IV

In Exercises 29–36, evaluate the trigonometric function of the quadrant angle.

29. $\sin \pi$ **30.** $\csc \dfrac{3\pi}{2}$

31. $\sec \dfrac{3\pi}{2}$ **32.** $\sec \pi$

33. $\sin \dfrac{\pi}{2}$ **34.** $\cot \pi$

35. $\csc \pi$ **36.** $\cot \dfrac{\pi}{2}$

In Exercises 37–44, find the reference angle θ', and sketch θ and θ' in standard position.

37. $\theta = 203°$ **38.** $\theta = 309°$

39. $\theta = -245°$ **40.** $\theta = -145°$

41. $\theta = \dfrac{2\pi}{3}$ **42.** $\theta = \dfrac{7\pi}{4}$

43. $\theta = 3.5$ **44.** $\theta = \dfrac{11\pi}{3}$

In Exercises 45–58, evaluate the sine, cosine, and tangent of the angle without using a calculator.

45. $225°$ **46.** $300°$

47. $750°$ **48.** $-405°$

49. $-150°$ **50.** $-840°$

51. $\dfrac{4\pi}{3}$ **52.** $\dfrac{\pi}{4}$

53. $-\dfrac{\pi}{6}$ **54.** $-\dfrac{\pi}{2}$

55. $\dfrac{11\pi}{4}$ **56.** $\dfrac{10\pi}{3}$

57. $-\dfrac{3\pi}{2}$

58. $-\dfrac{25\pi}{4}$

In Exercises 59–64, find the indicated trigonometric value in the specified quadrant.

	Function	*Quadrant*	*Trigonometric Value*
59.	$\sin \theta = -\frac{3}{5}$	IV	$\cos \theta$
60.	$\cot \theta = -3$	II	$\sin \theta$
61.	$\tan \theta = \frac{3}{2}$	III	$\sec \theta$
62.	$\csc \theta = -2$	IV	$\cot \theta$
63.	$\cos \theta = \frac{5}{8}$	I	$\sec \theta$
64.	$\sec \theta = -\frac{9}{4}$	III	$\tan \theta$

In Exercises 65–80, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places. (Be sure the calculator is set in the correct angle mode.)

65. $\sin 10°$ **66.** $\sec 225°$

67. $\cos(-110°)$ **68.** $\csc(-330°)$

69. $\tan 304°$ **70.** $\cot 178°$

71. $\sec 72°$ **72.** $\tan(-188°)$

73. $\tan 4.5$ **74.** $\cot 1.35$

75. $\tan \dfrac{\pi}{9}$ **76.** $\tan\left(-\dfrac{\pi}{9}\right)$

77. $\sin(-0.65)$ **78.** $\sec 0.29$

79. $\cot\left(-\dfrac{11\pi}{8}\right)$ **80.** $\csc\left(-\dfrac{15\pi}{14}\right)$

In Exercises 81–86, find two solutions of the equation. Give your answers in degrees ($0° \le \theta < 360°$) and in radians ($0 \le \theta < 2\pi$). Do not use a calculator.

81. (a) $\sin \theta = \frac{1}{2}$ (b) $\sin \theta = -\frac{1}{2}$

82. (a) $\cos \theta = \dfrac{\sqrt{2}}{2}$ (b) $\cos \theta = -\dfrac{\sqrt{2}}{2}$

83. (a) $\csc \theta = \dfrac{2\sqrt{3}}{3}$ (b) $\cot \theta = -1$

84. (a) $\sec \theta = 2$ (b) $\sec \theta = -2$

85. (a) $\tan \theta = 1$ (b) $\cot \theta = -\sqrt{3}$

86. (a) $\sin \theta = \dfrac{\sqrt{3}}{2}$ (b) $\sin \theta = -\dfrac{\sqrt{3}}{2}$

Model It

87. ***Data Analysis: Meteorology*** The table shows the monthly normal temperatures (in degrees Fahrenheit) for selected months for New York City (N) and Fairbanks, Alaska (F). (Source: National Climatic Data Center)

Month	New York City, N	Fairbanks, F
January	33	−10
April	52	32
July	77	62
October	58	24
December	38	−6

(a) Use the *regression* feature of a graphing utility to find a model of the form

$y = a \sin(bt + c) + d$

for each city. Let t represent the month, with $t = 1$ corresponding to January.

Model It (continued)

(b) Use the models from part (a) to find the monthly normal temperatures for the two cities in February, March, May, June, August, September, and November.

(c) Compare the models for the two cities.

88. ***Sales*** A company that produces snowboards, which are seasonal products, forecasts monthly sales over the next 2 years to be

$$S = 23.1 + 0.442t + 4.3 \cos \frac{\pi t}{6}$$

where S is measured in thousands of units and t is the time in months, with $t = 1$ representing January 2006. Predict sales for each of the following months.

(a) February 2006

(b) February 2007

(c) June 2006

(d) June 2007

89. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring is given by

$$y(t) = 2 \cos 6t$$

where y is the displacement (in centimeters) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

90. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is given by

$$y(t) = 2e^{-t} \cos 6t$$

where y is the displacement (in centimeters) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

91. ***Electric Circuits*** The current I (in amperes) when 100 volts is applied to a circuit is given by

$$I = 5e^{-2t} \sin t$$

where t is the time (in seconds) after the voltage is applied. Approximate the current at $t = 0.7$ second after the voltage is applied.

92. ***Distance*** An airplane, flying at an altitude of 6 miles, is on a flight path that passes directly over an observer (see figure). If θ is the angle of elevation from the observer to the plane, find the distance d from the observer to the plane when (a) $\theta = 30°$, (b) $\theta = 90°$, and (c) $\theta = 120°$.

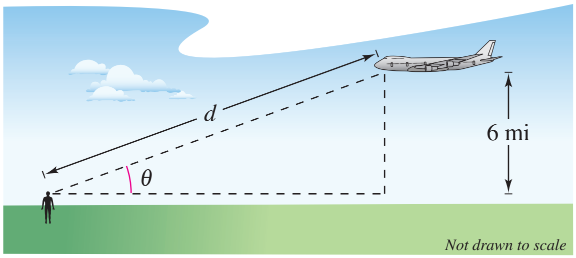


FIGURE FOR 92

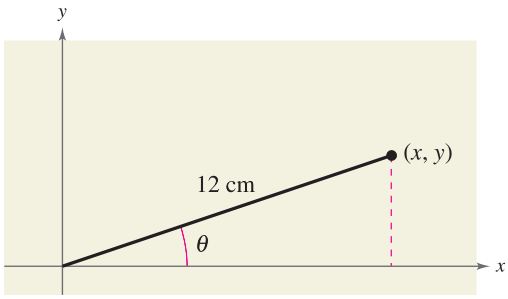
Synthesis

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. In each of the four quadrants, the signs of the secant function and sine function will be the same.

94. To find the reference angle for an angle θ (given in degrees), find the integer n such that $0 \le 360°n - \theta \le 360°$. The difference $360°n - \theta$ is the reference angle.

95. ***Writing*** Consider an angle in standard position with $r = 12$ centimeters, as shown in the figure. Write a short paragraph describing the changes in the values of x, y, $\sin \theta$, $\cos \theta$, and $\tan \theta$ as θ increases continuously from 0° to 90°.



96. ***Writing*** Explain how reference angles are used to find the trigonometric functions of obtuse angles.

Skills Review

In Exercises 97–106, graph the function. Identify the domain and any intercepts and asymptotes of the function.

97. $y = x^2 + 3x - 4$

98. $y = 2x^2 - 5x$

99. $f(x) = x^3 + 8$

100. $g(x) = x^4 + 2x^2 - 3$

101. $f(x) = \dfrac{x - 7}{x^2 + 4x + 4}$

102. $h(x) = \dfrac{x^2 - 1}{x + 5}$

103. $y = 2^{x-1}$

104. $y = 3^{x+1} + 2$

105. $y = \ln x^4$

106. $y = \log_{10}(x + 2)$

4.5 Graphs of Sine and Cosine Functions

What you should learn

- Sketch the graphs of basic sine and cosine functions.
- Use amplitude and period to help sketch the graphs of sine and cosine functions.
- Sketch translations of the graphs of sine and cosine functions.
- Use sine and cosine functions to model real-life data.

Why you should learn it

Sine and cosine functions are often used in scientific calculations. For instance, in Exercise 73 on page 330, you can use a trigonometric function to model the airflow of your respiratory cycle.

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Basic Sine and Cosine Curves

In this section, you will study techniques for sketching the graphs of the sine and cosine functions. The graph of the sine function is a **sine curve.** In Figure 4.47, the black portion of the graph represents one period of the function and is called **one cycle** of the sine curve. The gray portion of the graph indicates that the basic sine curve repeats indefinitely in the positive and negative directions. The graph of the cosine function is shown in Figure 4.48.

Recall from Section 4.2 that the domain of the sine and cosine functions is the set of all real numbers. Moreover, the range of each function is the interval $[-1, 1]$, and each function has a period of 2π. Do you see how this information is consistent with the basic graphs shown in Figures 4.47 and 4.48?

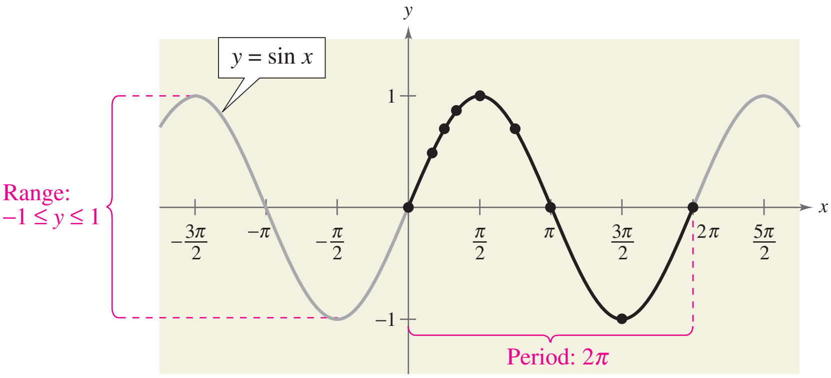


FIGURE 4.47

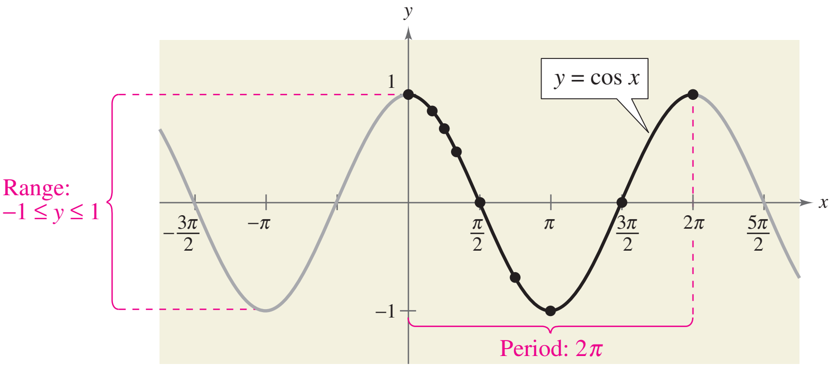


FIGURE 4.48

Note in Figures 4.47 and 4.48 that the sine curve is symmetric with respect to the *origin*, whereas the cosine curve is symmetric with respect to the *y-axis*. These properties of symmetry follow from the fact that the sine function is odd and the cosine function is even.

To sketch the graphs of the basic sine and cosine functions by hand, it helps to note five **key points** in one period of each graph: the *intercepts*, *maximum points*, and *minimum points* (see Figure 4.49).

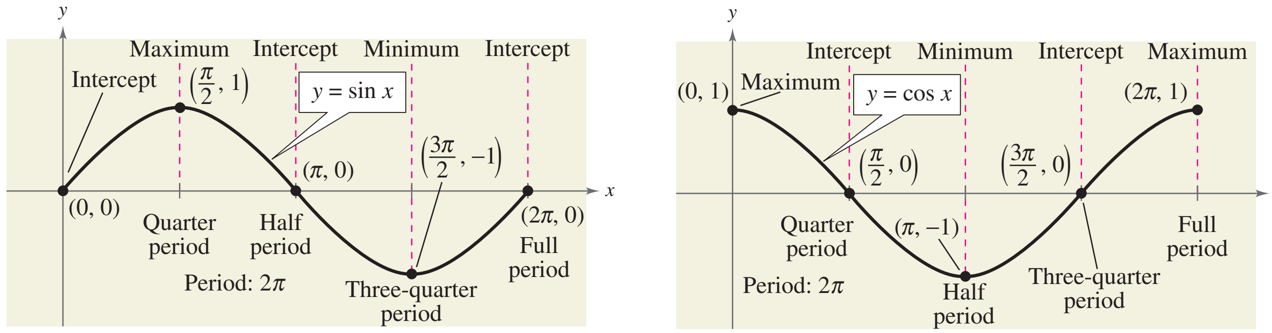


FIGURE 4.49

Example 1 Using Key Points to Sketch a Sine Curve

Sketch the graph of $y = 2 \sin x$ on the interval $[-\pi, 4\pi]$.

Solution

Note that

$$y = 2 \sin x = 2(\sin x)$$

indicates that the y-values for the key points will have twice the magnitude of those on the graph of $y = \sin x$. Divide the period 2π into four equal parts to get the key points for $y = 2 \sin x$.

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$\left(\frac{\pi}{2}, 2\right)$,	$(\pi, 0)$,	$\left(\frac{3\pi}{2}, -2\right)$,	and	$(2\pi, 0)$

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in Figure 4.50.

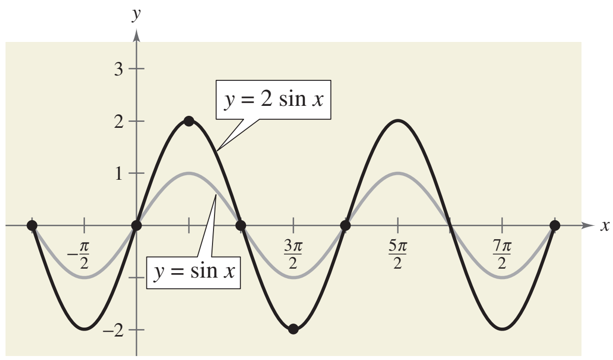


FIGURE 4.50

CHECKPOINT Now try Exercise 35.

Technology

When using a graphing utility to graph trigonometric functions, pay special attention to the viewing window you use. For instance, try graphing $y = [\sin(10x)]/10$ in the standard viewing window in *radian* mode. What do you observe? Use the *zoom* feature to find a viewing window that displays a good view of the graph.

Amplitude and Period

In the remainder of this section you will study the graphic effect of each of the constants a, b, c, and d in equations of the forms

$$y = d + a\sin(bx - c)$$

and

$$y = d + a\cos(bx - c).$$

A quick review of the transformations you studied in Section 1.7 should help in this investigation.

The constant factor a in $y = a\sin x$ acts as a *scaling factor*—a *vertical stretch* or *vertical shrink* of the basic sine curve. If $|a| > 1$, the basic sine curve is stretched, and if $|a| < 1$, the basic sine curve is shrunk. The result is that the graph of $y = a\sin x$ ranges between $-a$ and a instead of between -1 and 1. The absolute value of a is the **amplitude** of the function $y = a\sin x$. The range of the function $y = a\sin x$ for $a > 0$ is $-a \le y \le a$.

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of $y = a\sin x$ and $y = a\cos x$ represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a|.$$

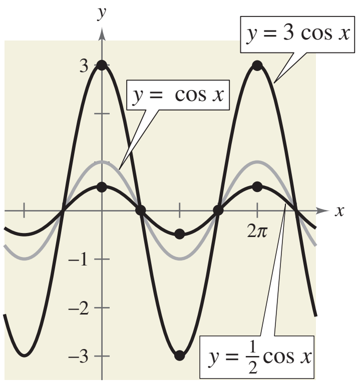


FIGURE 4.51

Example 2 Scaling: Vertical Shrinking and Stretching

On the same coordinate axes, sketch the graph of each function.

a. $y = \dfrac{1}{2}\cos x$ **b.** $y = 3\cos x$

Solution

a. Because the amplitude of $y = \frac{1}{2}\cos x$ is $\frac{1}{2}$, the maximum value is $\frac{1}{2}$ and the minimum value is $-\frac{1}{2}$. Divide one cycle, $0 \le x \le 2\pi$, into four equal parts to get the key points

Maximum	Intercept	Minimum	Intercept		Maximum
$\left(0, \frac{1}{2}\right)$,	$\left(\frac{\pi}{2}, 0\right)$,	$\left(\pi, -\frac{1}{2}\right)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$\left(2\pi, \frac{1}{2}\right)$.

b. A similar analysis shows that the amplitude of $y = 3\cos x$ is 3, and the key points are

Maximum	Intercept	Minimum	Intercept		Maximum
$(0, 3)$,	$\left(\frac{\pi}{2}, 0\right)$,	$(\pi, -3)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$(2\pi, 3)$.

The graphs of these two functions are shown in Figure 4.51. Notice that the graph of $y = \frac{1}{2}\cos x$ is a vertical *shrink* of the graph of $y = \cos x$ and the graph of $y = 3\cos x$ is a vertical *stretch* of the graph of $y = \cos x$.

CHECKPOINT Now try Exercise 37.

Exploration

Sketch the graph of $y = \cos bx$ for $b = \frac{1}{2}$, 2, and 3. How does the value of b affect the graph? How many complete cycles occur between 0 and 2π for each value of b?

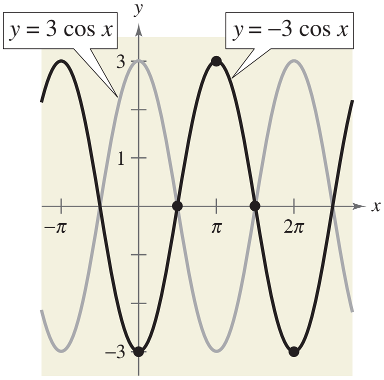


FIGURE 4.52

You know from Section 1.7 that the graph of $y = -f(x)$ is a **reflection** in the x-axis of the graph of $y = f(x)$. For instance, the graph of $y = -3\cos x$ is a reflection of the graph of $y = 3\cos x$, as shown in Figure 4.52.

Because $y = a\sin x$ completes one cycle from $x = 0$ to $x = 2\pi$, it follows that $y = a\sin bx$ completes one cycle from $x = 0$ to $x = 2\pi/b$.

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of $y = a\sin bx$ and $y = a\cos bx$ is given by

$$\text{Period} = \frac{2\pi}{b}.$$

Note that if $0 < b < 1$, the period of $y = a\sin bx$ is greater than 2π and represents a *horizontal stretching* of the graph of $y = a\sin x$. Similarly, if $b > 1$, the period of $y = a\sin bx$ is less than 2π and represents a *horizontal shrinking* of the graph of $y = a\sin x$. If b is negative, the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ are used to rewrite the function.

Exploration

Sketch the graph of

$$y = \sin(x - c)$$

where $c = -\pi/4$, 0, and $\pi/4$. How does the value of c affect the graph?

Example 3 Scaling: Horizontal Stretching

Sketch the graph of $y = \sin\dfrac{x}{2}$.

Solution

The amplitude is 1. Moreover, because $b = \frac{1}{2}$, the period is

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi. \qquad \text{Substitute for } b.$$

Now, divide the period-interval $[0, 4\pi]$ into four equal parts with the values π, 2π, and 3π to obtain the key points on the graph.

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$(\pi, 1)$,	$(2\pi, 0)$,	$(3\pi, -1)$,	and	$(4\pi, 0)$

The graph is shown in Figure 4.53.

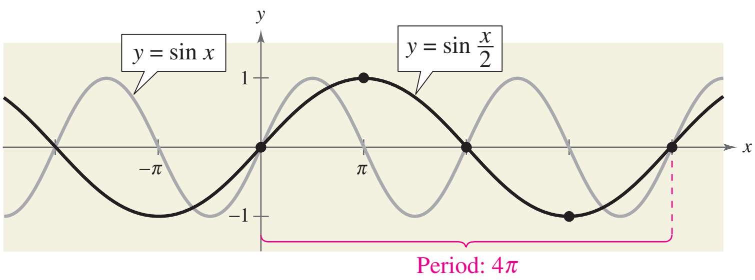


FIGURE 4.53

CHECKPOINT Now try Exercise 39.

STUDY TIP

In general, to divide a period-interval into four equal parts, successively add "period/4," starting with the left endpoint of the interval. For instance, for the period-interval $[-\pi/6, \pi/2]$ of length $2\pi/3$, you would successively add

$$\frac{2\pi/3}{4} = \frac{\pi}{6}$$

to get $-\pi/6$, 0, $\pi/6$, $\pi/3$, and $\pi/2$ as the x-values for the key points on the graph.

Translations of Sine and Cosine Curves

The constant c in the general equations

$$y = a\sin(bx - c) \qquad \text{and} \qquad y = a\cos(bx - c)$$

creates a *horizontal translation* (shift) of the basic sine and cosine curves. Comparing $y = a\sin bx$ with $y = a\sin(bx - c)$, you find that the graph of $y = a\sin(bx - c)$ completes one cycle from $bx - c = 0$ to $bx - c = 2\pi$. By solving for x, you can find the interval for one cycle to be

$$\overbrace{\frac{c}{b}}^{\text{Left endpoint}} \le x \le \overbrace{\frac{c}{b} + \underbrace{\frac{2\pi}{b}}_{\text{Period}}}^{\text{Right endpoint}}.$$

This implies that the period of $y = a\sin(bx - c)$ is $2\pi/b$, and the graph of $y = a\sin bx$ is shifted by an amount c/b. The number c/b is the **phase shift.**

Graphs of Sine and Cosine Functions

The graphs of $y = a\sin(bx - c)$ and $y = a\cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)

$$\text{Amplitude} = |a| \qquad \text{Period} = \frac{2\pi}{b}$$

The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

Example 4 **Horizontal Translation**

Sketch the graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{3}\right)$.

Solution

The amplitude is $\frac{1}{2}$ and the period is 2π. By solving the equations

$$x - \frac{\pi}{3} = 0 \quad \Rightarrow \quad x = \frac{\pi}{3}$$

and

$$x - \frac{\pi}{3} = 2\pi \quad \Rightarrow \quad x = \frac{7\pi}{3}$$

you see that the interval $[\pi/3, 7\pi/3]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the key points

Intercept	Maximum	Intercept	Minimum		Intercept
$\left(\frac{\pi}{3}, 0\right)$,	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$,	$\left(\frac{4\pi}{3}, 0\right)$,	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$,	and	$\left(\frac{7\pi}{3}, 0\right)$.

The graph is shown in Figure 4.54.

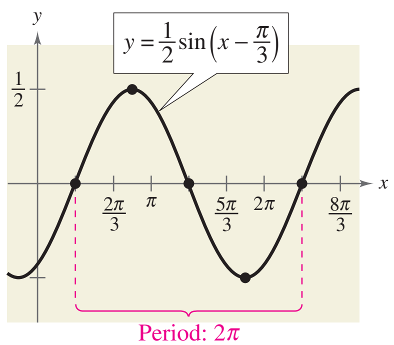


FIGURE 4.54

CHECKPOINT Now try Exercise 45.

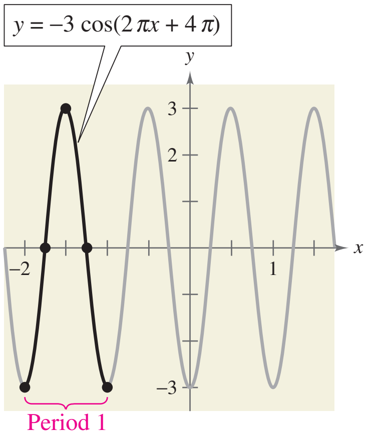


FIGURE 4.55

Example 5 Horizontal Translation

Sketch the graph of

$$y = -3\cos(2\pi x + 4\pi).$$

Solution

The amplitude is 3 and the period is $2\pi/2\pi = 1$. By solving the equations

$$2\pi x + 4\pi = 0$$
$$2\pi x = -4\pi$$
$$x = -2$$

and

$$2\pi x + 4\pi = 2\pi$$
$$2\pi x = -2\pi$$
$$x = -1$$

you see that the interval $[-2, -1]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the key points

Minimum	*Intercept*	*Maximum*	*Intercept*		*Minimum*
$(-2, -3)$,	$\left(-\frac{7}{4}, 0\right)$,	$\left(-\frac{3}{2}, 3\right)$,	$\left(-\frac{5}{4}, 0\right)$,	and	$(-1, -3)$.

The graph is shown in Figure 4.55.

CHECKPOINT Now try Exercise 47.

The final type of transformation is the *vertical translation* caused by the constant d in the equations

$$y = d + a\sin(bx - c)$$

and

$$y = d + a\cos(bx - c).$$

The shift is d units upward for $d > 0$ and d units downward for $d < 0$. In other words, the graph oscillates about the horizontal line $y = d$ instead of about the x-axis.

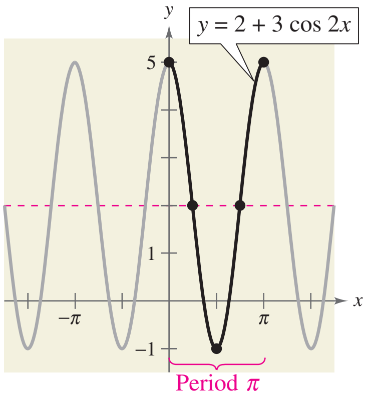


FIGURE 4.56

Example 6 Vertical Translation

Sketch the graph of

$$y = 2 + 3\cos 2x.$$

Solution

The amplitude is 3 and the period is π. The key points over the interval $[0, \pi]$ are

$$(0, 5), \quad \left(\frac{\pi}{4}, 2\right), \quad \left(\frac{\pi}{2}, -1\right), \quad \left(\frac{3\pi}{4}, 2\right), \quad \text{and} \quad (\pi, 5).$$

The graph is shown in Figure 4.56. Compared with the graph of $f(x) = 3\cos 2x$, the graph of $y - 2 + 3\cos 2x$ is shifted upward two units.

CHECKPOINT Now try Exercise 53.

Mathematical Modeling

Sine and cosine functions can be used to model many real-life situations, including electric currents, musical tones, radio waves, tides, and weather patterns.

Example 7 Finding a Trigonometric Model

Time, t	Depth, y
Midnight	3.4
2 A.M.	8.7
4 A.M.	11.3
6 A.M.	9.1
8 A.M.	3.8
10 A.M.	0.1
Noon	1.2

Throughout the day, the depth of water at the end of a dock in Bar Harbor, Maine varies with the tides. The table shows the depths (in feet) at various times during the morning. (Source: Nautical Software, Inc.)

a. Use a trigonometric function to model the data.

b. Find the depths at 9 A.M. and 3 P.M.

c. A boat needs at least 10 feet of water to moor at the dock. During what times in the afternoon can it safely dock?

Solution

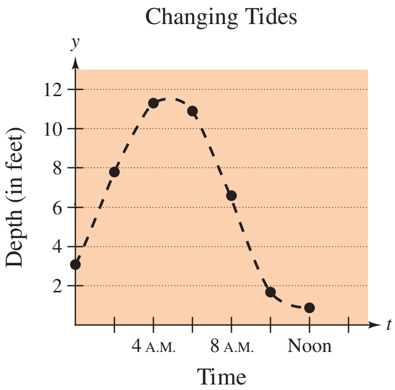


FIGURE 4.57

a. Begin by graphing the data, as shown in Figure 4.57. You can use either a sine or cosine model. Suppose you use a cosine model of the form

$$y = a\cos(bt - c) + d.$$

The difference between the maximum height and the minimum height of the graph is twice the amplitude of the function. So, the amplitude is

$$a = \frac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})] = \frac{1}{2}(11.3 - 0.1) = 5.6.$$

The cosine function completes one half of a cycle between the times at which the maximum and minimum depths occur. So, the period is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})] = 2(10 - 4) = 12$$

which implies that $b = 2\pi/p \approx 0.524$. Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 2.094$. Moreover, because the average depth is $\frac{1}{2}(11.3 + 0.1) = 5.7$, it follows that $d = 5.7$. So, you can model the depth with the function given by

$$y = 5.6\cos(0.524t - 2.094) + 5.7.$$

b. The depths at 9 A.M. and 3 P.M. are as follows.

$$y = 5.6\cos(0.524 \cdot 9 - 2.094) + 5.7$$

$$\approx 0.84 \text{ foot} \qquad \text{9 A.M.}$$

$$y = 5.6\cos(0.524 \cdot 15 - 2.094) + 5.7$$

$$\approx 10.57 \text{ feet} \qquad \text{3 P.M.}$$

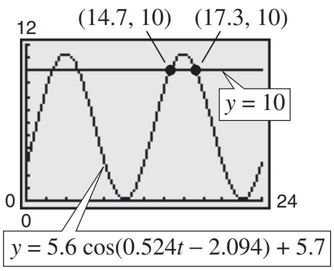


FIGURE 4.58

c. To find out when the depth y is at least 10 feet, you can graph the model with the line $y = 10$ using a graphing utility, as shown in Figure 4.58. Using the *intersect* feature, you can determine that the depth is at least 10 feet between 2:42 P.M. ($t \approx 14.7$) and 5:18 P.M. ($t \approx 17.3$).

CHECKPOINT Now try Exercise 77.

4.5 Exercises

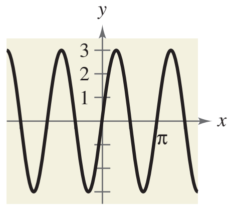
VOCABULARY CHECK: Fill in the blanks.

1. One period of a sine or cosine function function is called one ________ of the sine curve or cosine curve.
2. The ________ of a sine or cosine curve represents half the distance between the maximum and minimum values of the function.
3. The period of a sine or cosine function is given by ________.
4. For the function given by $y = a\sin(bx - c)$, $\frac{c}{b}$ represents the ________ ________ of the graph of the function.
5. For the function given by $y = d + a\cos(bx - c)$, d represents a ________ ________ of the graph of the function.

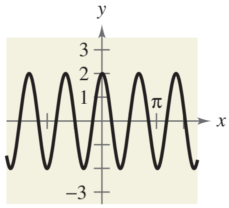
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–14, find the period and amplitude.

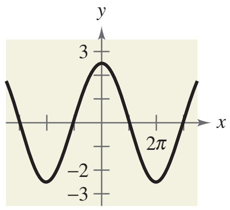
1. $y = 3\sin 2x$



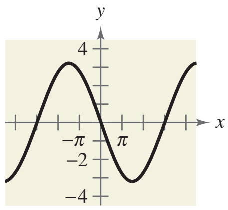
2. $y = 2\cos 3x$



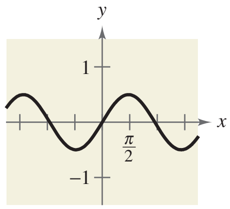
3. $y = \frac{5}{2}\cos\frac{x}{2}$



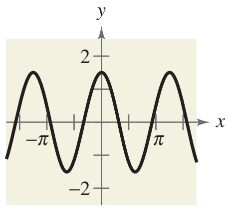
4. $y = -3\sin\frac{x}{3}$



5. $y = \frac{1}{2}\sin\frac{\pi x}{3}$



6. $y = \frac{3}{2}\cos\frac{\pi x}{2}$



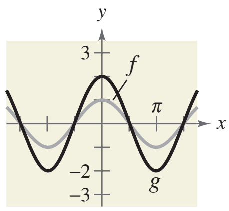
7. $y = -2\sin x$
8. $y = -\cos\frac{2x}{3}$
9. $y = 3\sin 10x$
10. $y = \frac{1}{3}\sin 8x$
11. $y = \frac{1}{2}\cos\frac{2x}{3}$
12. $y = \frac{5}{2}\cos\frac{x}{4}$
13. $y = \frac{1}{4}\sin 2\pi x$
14. $y = \frac{2}{3}\cos\frac{\pi x}{10}$

In Exercises 15–22, describe the relationship between the graphs of *f* and *g*. Consider amplitude, period, and shifts.

15. $f(x) = \sin x$
    $g(x) = \sin(x - \pi)$
16. $f(x) = \cos x$
    $g(x) = \cos(x + \pi)$
17. $f(x) = \cos 2x$
    $g(x) = -\cos 2x$
18. $f(x) = \sin 3x$
    $g(x) = \sin(-3x)$
19. $f(x) = \cos x$
    $g(x) = \cos 2x$
20. $f(x) = \sin x$
    $g(x) = \sin 3x$
21. $f(x) = \sin 2x$
    $g(x) = 3 + \sin 2x$
22. $f(x) = \cos 4x$
    $g(x) = -2 + \cos 4x$

In Exercises 23–26, describe the relationship between the graphs of *f* and *g*. Consider amplitude, period, and shifts.

23.



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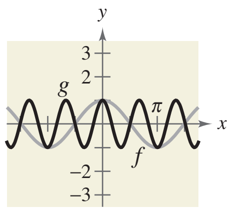

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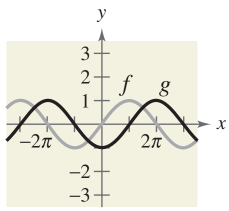

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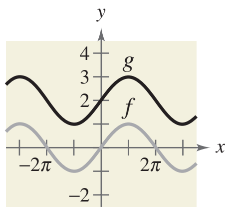


In Exercises 27–34, graph f and g on the same set of coordinate axes. (Include two full periods.)

27. $f(x) = -2 \sin x$
$g(x) = 4 \sin x$

28. $f(x) = \sin x$
$g(x) = \sin \dfrac{x}{3}$

29. $f(x) = \cos x$
$g(x) = 1 + \cos x$

30. $f(x) = 2 \cos 2x$
$g(x) = -\cos 4x$

31. $f(x) = -\dfrac{1}{2} \sin \dfrac{x}{2}$
$g(x) = 3 - \dfrac{1}{2} \sin \dfrac{x}{2}$

32. $f(x) = 4 \sin \pi x$
$g(x) = 4 \sin \pi x - 3$

33. $f(x) = 2 \cos x$
$g(x) = 2 \cos(x + \pi)$

34. $f(x) = -\cos x$
$g(x) = -\cos(x - \pi)$

In Exercises 35–56, sketch the graph of the function. (Include two full periods.)

35. $y = 3 \sin x$

36. $y = \frac{1}{4} \sin x$

37. $y = \frac{1}{3} \cos x$

38. $y = 4 \cos x$

39. $y = \cos \dfrac{x}{2}$

40. $y = \sin 4x$

41. $y = \cos 2\pi x$

42. $y = \sin \dfrac{\pi x}{4}$

43. $y = -\sin \dfrac{2\pi x}{3}$

44. $y = -10 \cos \dfrac{\pi x}{6}$

45. $y = \sin\left(x - \dfrac{\pi}{4}\right)$

46. $y = \sin(x - \pi)$

47. $y = 3 \cos(x + \pi)$

48. $y = 4 \cos\left(x + \dfrac{\pi}{4}\right)$

49. $y = 2 - \sin \dfrac{2\pi x}{3}$

50. $y = -3 + 5 \cos \dfrac{\pi t}{12}$

51. $y = 2 + \frac{1}{10} \cos 60\pi x$

52. $y = 2 \cos x - 3$

53. $y = 3 \cos(x + \pi) - 3$

54. $y = 4 \cos\left(x + \dfrac{\pi}{4}\right) + 4$

55. $y = \dfrac{2}{3} \cos\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right)$

56. $y = -3 \cos(6x + \pi)$

In Exercises 57–62, use a graphing utility to graph the function. Include two full periods. Be sure to choose an appropriate viewing window.

57. $y = -2 \sin(4x + \pi)$

58. $y = -4 \sin\left(\dfrac{2}{3}x - \dfrac{\pi}{3}\right)$

59. $y = \cos\left(2\pi x - \dfrac{\pi}{2}\right) + 1$

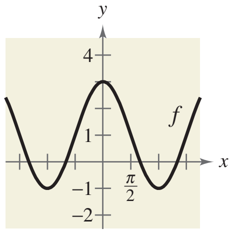
60. $y = 3 \cos\left(\dfrac{\pi x}{2} + \dfrac{\pi}{2}\right) - 2$

61. $y = -0.1 \sin\left(\dfrac{\pi x}{10} + \pi\right)$

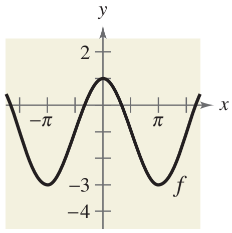
62. $y = \frac{1}{100} \sin 120\pi t$

Graphical Reasoning **In Exercises 63–66, find a and d for the function $f(x) = a \cos x + d$ such that the graph of f matches the figure.**

63.



64.



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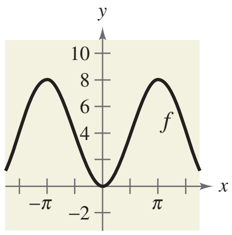
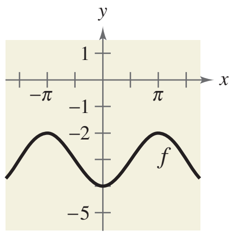
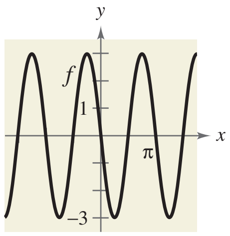


66.

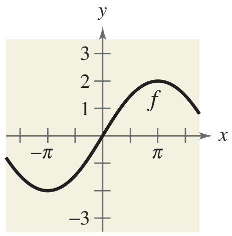


Graphical Reasoning **In Exercises 67–70, find a, b, and c for the function $f(x) = a \sin(bx - c)$ such that the graph of f matches the figure.**

67.



68.



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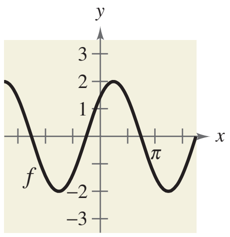
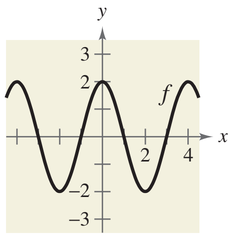


70.



In Exercises 71 and 72, use a graphing utility to graph y_1 and y_2 in the interval $[-2\pi, 2\pi]$. Use the graphs to find real numbers x such that $y_1 = y_2$.

71. $y_1 = \sin x$
$y_2 = -\frac{1}{2}$

72. $y_1 = \cos x$
$y_2 = -1$

73. ***Respiratory Cycle*** For a person at rest, the velocity v (in liters per second) of air flow during a respiratory cycle (the time from the beginning of one breath to the beginning of the next) is given by $v = 0.85 \sin \frac{\pi t}{3}$, where t is the time (in seconds). (Inhalation occurs when $v > 0$, and exhalation occurs when $v < 0$.)

(a) Find the time for one full respiratory cycle.

(b) Find the number of cycles per minute.

(c) Sketch the graph of the velocity function.

74. ***Respiratory Cycle*** After exercising for a few minutes, a person has a respiratory cycle for which the velocity of air flow is approximated by $v = 1.75 \sin \frac{\pi t}{2}$, where t is the time (in seconds). (Inhalation occurs when $v > 0$, and exhalation occurs when $v < 0$.)

(a) Find the time for one full respiratory cycle.

(b) Find the number of cycles per minute.

(c) Sketch the graph of the velocity function.

75. ***Data Analysis: Meteorology*** The table shows the maximum daily high temperatures for Tallahassee T and Chicago C (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. (Source: National Climatic Data Center)

Month, t	Tallahassee, T	Chicago, C
1	63.8	29.6
2	67.4	34.7
3	74.0	46.1
4	80.0	58.0
5	86.5	69.9
6	90.9	79.2
7	92.0	83.5
8	91.5	81.2
9	88.5	73.9
10	81.2	62.1
11	72.9	47.1
12	65.8	34.4

(a) A model for the temperature in Tallahassee is given by

$$T(t) = 77.90 + 14.10 \cos\left(\frac{\pi t}{6} - 3.67\right).$$

Find a trigonometric model for Chicago.

(b) Use a graphing utility to graph the data points and the model for the temperatures in Tallahassee. How well does the model fit the data?

(c) Use a graphing utility to graph the data points and the model for the temperatures in Chicago. How well does the model fit the data?

(d) Use the models to estimate the average maximum temperature in each city. Which term of the models did you use? Explain.

(e) What is the period of each model? Are the periods what you expected? Explain.

(f) Which city has the greater variability in temperature throughout the year? Which factor of the models determines this variability? Explain.

76. ***Health*** The function given by $P = 100 - 20 \cos \frac{5\pi t}{3}$ approximates the blood pressure P (in millimeters) of mercury at time t (in seconds) for a person at rest.

(a) Find the period of the function.

(b) Find the number of heartbeats per minute.

77. ***Piano Tuning*** When tuning a piano, a technician strikes a tuning fork for the A above middle C and sets up a wave motion that can be approximated by $y = 0.001 \sin 880\pi t$, where t is the time (in seconds).

(a) What is the period of the function?

(b) The frequency f is given by $f = 1/p$. What is the frequency of the note?

Model It

78. ***Data Analysis: Astronomy*** The percent y of the moon's face that is illuminated on day x of the year 2007, where $x = 1$ represents January 1, is shown in the table. (Source: U.S. Naval Observatory)

x	y
3	1.0
11	0.5
19	0.0
26	0.5
32	1.0
40	0.5

(a) Create a scatter plot of the data.

(b) Find a trigonometric model that fits the data.

(c) Add the graph of your model in part (b) to the scatter plot. How well does the model fit the data?

(d) What is the period of the model?

(e) Estimate the moon's percent illumination for March 12, 2007.

79. ***Fuel Consumption*** The daily consumption C (in gallons) of diesel fuel on a farm is modeled by

$$C = 30.3 + 21.6 \sin\left(\frac{2\pi t}{365} + 10.9\right)$$

where t is the time (in days), with $t = 1$ corresponding to January 1.

(a) What is the period of the model? Is it what you expected? Explain.

(b) What is the average daily fuel consumption? Which term of the model did you use? Explain.

(c) Use a graphing utility to graph the model. Use the graph to approximate the time of the year when consumption exceeds 40 gallons per day.

80. ***Ferris Wheel*** A Ferris wheel is built such that the height h (in feet) above ground of a seat on the wheel at time t (in seconds) can be modeled by

$$h(t) = 53 + 50 \sin\left(\frac{\pi}{10}t - \frac{\pi}{2}\right).$$

(a) Find the period of the model. What does the period tell you about the ride?

(b) Find the amplitude of the model. What does the amplitude tell you about the ride?

(c) Use a graphing utility to graph one cycle of the model.

Synthesis

True or False? **In Exercises 81–83, determine whether the statement is true or false. Justify your answer.**

81. The graph of the function given by $f(x) = \sin(x + 2\pi)$ translates the graph of $f(x) = \sin x$ exactly one period to the right so that the two graphs look identical.

82. The function given by $y = \frac{1}{2}\cos 2x$ has an amplitude that is twice that of the function given by $y = \cos x$.

83. The graph of $y = -\cos x$ is a reflection of the graph of $y = \sin(x + \pi/2)$ in the x-axis.

84. ***Writing*** Use a graphing utility to graph the function given by $y = d + a\sin(bx - c)$, for several different values of a, b, c, and d. Write a paragraph describing the changes in the graph corresponding to changes in each constant.

Conjecture **In Exercises 85 and 86, graph f and g on the same set of coordinate axes. Include two full periods. Make a conjecture about the functions.**

85. $f(x) = \sin x, \quad g(x) = \cos\left(x - \dfrac{\pi}{2}\right)$

86. $f(x) = \sin x, \quad g(x) = -\cos\left(x + \dfrac{\pi}{2}\right)$

87. ***Exploration*** Using calculus, it can be shown that the sine and cosine functions can be approximated by the polynomials

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad \text{and} \quad \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

where x is in radians.

(a) Use a graphing utility to graph the sine function and its polynomial approximation in the same viewing window. How do the graphs compare?

(b) Use a graphing utility to graph the cosine function and its polynomial approximation in the same viewing window. How do the graphs compare?

(c) Study the patterns in the polynomial approximations of the sine and cosine functions and predict the next term in each. Then repeat parts (a) and (b). How did the accuracy of the approximations change when an additional term was added?

88. ***Exploration*** Use the polynomial approximations for the sine and cosine functions in Exercise 87 to approximate the following function values. Compare the results with those given by a calculator. Is the error in the approximation the same in each case? Explain.

(a) $\sin \dfrac{1}{2}$ (b) $\sin 1$ (c) $\sin \dfrac{\pi}{6}$

(d) $\cos(-0.5)$ (e) $\cos 1$ (f) $\cos \dfrac{\pi}{4}$

Skills Review

In Exercises 89–92, use the properties of logarithms to write the expression as a sum, difference, and/or constant multiple of a logarithm.

89. $\log_{10} \sqrt{x - 2}$

90. $\log_2[x^2(x - 3)]$

91. $\ln \dfrac{t^3}{t - 1}$

92. $\ln \sqrt{\dfrac{z}{z^2 + 1}}$

In Exercises 93–96, write the expression as the logarithm of a single quantity.

93. $\frac{1}{2}(\log_{10} x + \log_{10} y)$

94. $2\log_2 x + \log_2(xy)$

95. $\ln 3x - 4\ln y$

96. $\frac{1}{2}(\ln 2x - 2\ln x) + 3\ln x$

97. **Make a Decision** To work an extended application analyzing the normal daily maximum temperature and normal precipitation in Honolulu, Hawaii, visit this text's website at *college.hmco.com*. *(Data Source: NOAA)*

4.6 Graphs of Other Trigonometric Functions

What you should learn

- Sketch the graphs of tangent functions.
- Sketch the graphs of cotangent functions.
- Sketch the graphs of secant and cosecant functions.
- Sketch the graphs of damped trigonometric functions.

Why you should learn it

Trigonometric functions can be used to model real-life situations such as the distance from a television camera to a unit in a parade as in Exercise 76 on page 341.

Photodisc/Getty Images

Graph of the Tangent Function

Recall that the tangent function is odd. That is, $\tan(-x) = -\tan x$. Consequently, the graph of $y = \tan x$ is symmetric with respect to the origin. You also know from the identity $\tan x = \sin x/\cos x$ that the tangent is undefined for values at which $\cos x = 0$. Two such values are $x = \pm\pi/2 \approx \pm 1.5708$.

x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	Undef.	-1255.8	-14.1	-1	0	1	14.1	1255.8	Undef.

As indicated in the table, $\tan x$ increases without bound as x approaches $\pi/2$ from the left, and decreases without bound as x approaches $-\pi/2$ from the right. So, the graph of $y = \tan x$ has *vertical asymptotes* at $x = \pi/2$ and $x = -\pi/2$, as shown in Figure 4.59. Moreover, because the period of the tangent function is π, vertical asymptotes also occur when $x = \pi/2 + n\pi$, where n is an integer. The domain of the tangent function is the set of all real numbers other than $x = \pi/2 + n\pi$, and the range is the set of all real numbers.

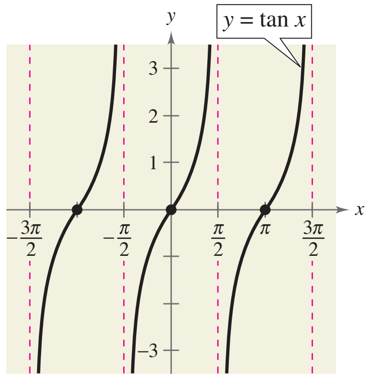


FIGURE 4.59

PERIOD: π
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, \infty)$
VERTICAL ASYMPTOTES: $x = \frac{\pi}{2} + n\pi$

Sketching the graph of $y = a\tan(bx - c)$ is similar to sketching the graph of $y = a\sin(bx - c)$ in that you locate key points that identify the intercepts and asymptotes. Two consecutive vertical asymptotes can be found by solving the equations

$$bx - c = -\frac{\pi}{2} \qquad \text{and} \qquad bx - c = \frac{\pi}{2}.$$

The midpoint between two consecutive vertical asymptotes is an x-intercept of the graph. The period of the function $y = a\tan(bx - c)$ is the distance between two consecutive vertical asymptotes. The amplitude of a tangent function is not defined. After plotting the asymptotes and the x-intercept, plot a few additional points between the two asymptotes and sketch one cycle. Finally, sketch one or two additional cycles to the left and right.

Example 1 Sketching the Graph of a Tangent Function

Sketch the graph of $y = \tan \frac{x}{2}$.

Solution

By solving the equations

$$\frac{x}{2} = -\frac{\pi}{2} \quad \text{and} \quad \frac{x}{2} = \frac{\pi}{2}$$

$$x = -\pi \qquad\qquad x = \pi$$

you can see that two consecutive vertical asymptotes occur at $x = -\pi$ and $x = \pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 4.60.

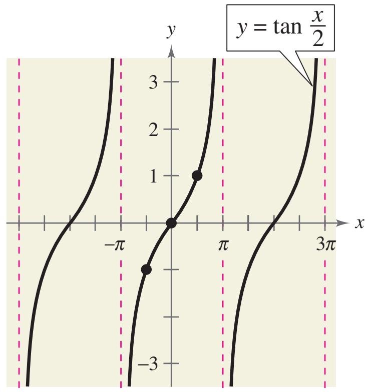


FIGURE 4.60

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\tan \frac{x}{2}$	Undef.	-1	0	1	Undef.

CHECKPOINT Now try Exercise 7.

Example 2 Sketching the Graph of a Tangent Function

Sketch the graph of $y = -3 \tan 2x$.

Solution

By solving the equations

$$2x = -\frac{\pi}{2} \quad \text{and} \quad 2x = \frac{\pi}{2}$$

$$x = -\frac{\pi}{4} \qquad\qquad x = \frac{\pi}{4}$$

you can see that two consecutive vertical asymptotes occur at $x = -\pi/4$ and $x = \pi/4$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 4.61.

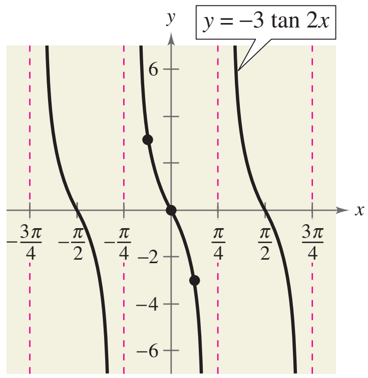


FIGURE 4.61

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$-3 \tan 2x$	Undef.	3	0	-3	Undef.

CHECKPOINT Now try Exercise 9.

By comparing the graphs in Examples 1 and 2, you can see that the graph of $y = a \tan(bx - c)$ increases between consecutive vertical asymptotes when $a > 0$, and decreases between consecutive vertical asymptotes when $a < 0$. In other words, the graph for $a < 0$ is a reflection in the x-axis of the graph for $a > 0$.

Graph of the Cotangent Function

The graph of the cotangent function is similar to the graph of the tangent function. It also has a period of π. However, from the identity

$$y = \cot x = \frac{\cos x}{\sin x}$$

you can see that the cotangent function has vertical asymptotes when $\sin x$ is zero, which occurs at $x = n\pi$, where n is an integer. The graph of the cotangent function is shown in Figure 4.62. Note that two consecutive vertical asymptotes of the graph of $y = a\cot(bx - c)$ can be found by solving the equations $bx - c = 0$ and $bx - c = \pi$.

Technology

Some graphing utilities have difficulty graphing trigonometric functions that have vertical asymptotes. Your graphing utility may connect parts of the graphs of tangent, cotangent, secant, and cosecant functions that are not supposed to be connected. To eliminate this problem, change the mode of the graphing utility to *dot* mode.

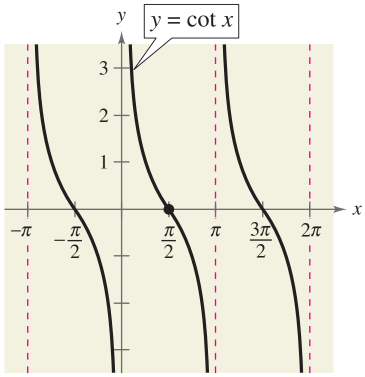


FIGURE 4.62

PERIOD: π
DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, \infty)$
VERTICAL ASYMPTOTES: $x = n\pi$

Example 3 Sketching the Graph of a Cotangent Function

Sketch the graph of $y = 2\cot\frac{x}{3}$.

Solution

By solving the equations

$$\frac{x}{3} = 0 \quad \text{and} \quad \frac{x}{3} = \pi$$

$$x = 0 \qquad\qquad x = 3\pi$$

you can see that two consecutive vertical asymptotes occur at $x = 0$ and $x = 3\pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 4.63. Note that the period is 3π, the distance between consecutive asymptotes.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2\cot\frac{x}{3}$	Undef.	2	0	-2	Undef.

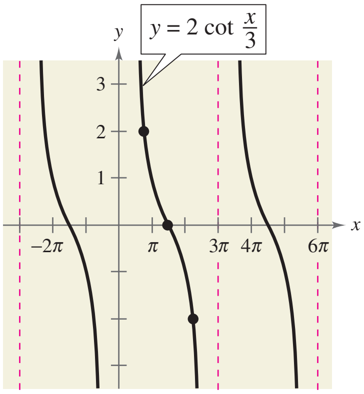


FIGURE 4.63

CHECKPOINT Now try Exercise 19.

Graphs of the Reciprocal Functions

The graphs of the two remaining trigonometric functions can be obtained from the graphs of the sine and cosine functions using the reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}.$$

For instance, at a given value of x, the y-coordinate of $\sec x$ is the reciprocal of the y-coordinate of $\cos x$. Of course, when $\cos x = 0$, the reciprocal does not exist. Near such values of x, the behavior of the secant function is similar to that of the tangent function. In other words, the graphs of

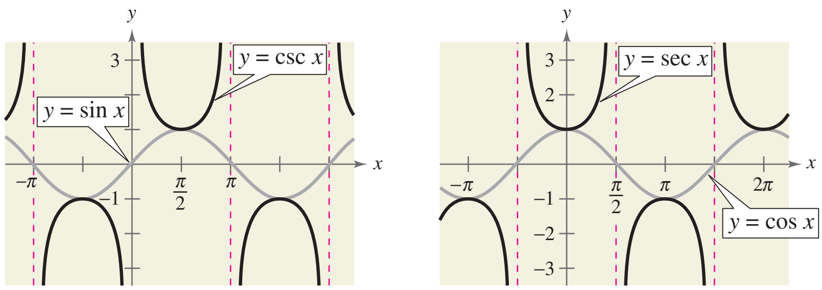
$$\tan x = \frac{\sin x}{\cos x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}$$

have vertical asymptotes at $x = \pi/2 + n\pi$, where n is an integer, and the cosine is zero at these x-values. Similarly,

$$\cot x = \frac{\cos x}{\sin x} \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

have vertical asymptotes where $\sin x = 0$—that is, at $x = n\pi$.

To sketch the graph of a secant or cosecant function, you should first make a sketch of its reciprocal function. For instance, to sketch the graph of $y = \csc x$, first sketch the graph of $y = \sin x$. Then take reciprocals of the y-coordinates to obtain points on the graph of $y = \csc x$. This procedure is used to obtain the graphs shown in Figure 4.64.



Period: 2π
Domain: all $x \neq n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Vertical asymptotes: $x = n\pi$
Symmetry: origin

Period: 2π
Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$
Symmetry: y-axis

Figure 4.64

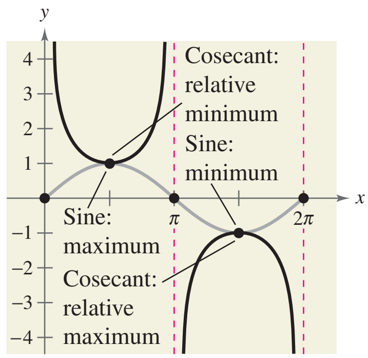


Figure 4.65

In comparing the graphs of the cosecant and secant functions with those of the sine and cosine functions, note that the "hills" and "valleys" are interchanged. For example, a hill (or maximum point) on the sine curve corresponds to a valley (a relative minimum) on the cosecant curve, and a valley (or minimum point) on the sine curve corresponds to a hill (a relative maximum) on the cosecant curve, as shown in Figure 4.65. Additionally, x-intercepts of the sine and cosine functions become vertical asymptotes of the cosecant and secant functions, respectively (see Figure 4.65).

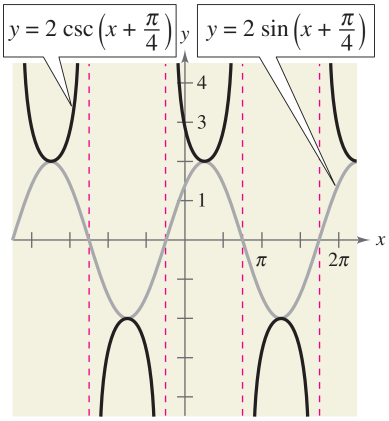


FIGURE 4.66

Example 4 Sketching the Graph of a Cosecant Function

Sketch the graph of $y = 2\csc\left(x + \frac{\pi}{4}\right)$.

Solution

Begin by sketching the graph of

$$y = 2\sin\left(x + \frac{\pi}{4}\right).$$

For this function, the amplitude is 2 and the period is 2π. By solving the equations

$$x + \frac{\pi}{4} = 0 \qquad \text{and} \qquad x + \frac{\pi}{4} = 2\pi$$

$$x = -\frac{\pi}{4} \qquad\qquad x = \frac{7\pi}{4}$$

you can see that one cycle of the sine function corresponds to the interval from $x = -\pi/4$ to $x = 7\pi/4$. The graph of this sine function is represented by the gray curve in Figure 4.66. Because the sine function is zero at the midpoint and endpoints of this interval, the corresponding cosecant function

$$y = 2\csc\left(x + \frac{\pi}{4}\right)$$

$$= 2\left(\frac{1}{\sin[x + (\pi/4)]}\right)$$

has vertical asymptotes at $x = -\pi/4$, $x = 3\pi/4$, $x = 7\pi/4$, etc. The graph of the cosecant function is represented by the black curve in Figure 4.66.

CHECKPOINT Now try Exercise 25.

Example 5 Sketching the Graph of a Secant Function

Sketch the graph of $y = \sec 2x$.

Solution

Begin by sketching the graph of $y = \cos 2x$, as indicated by the gray curve in Figure 4.67. Then, form the graph of $y = \sec 2x$ as the black curve in the figure. Note that the x-intercepts of $y = \cos 2x$

$$\left(-\frac{\pi}{4}, 0\right), \quad \left(\frac{\pi}{4}, 0\right), \quad \left(\frac{3\pi}{4}, 0\right), \ldots$$

correspond to the vertical asymptotes

$$x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}, \ldots$$

of the graph of $y = \sec 2x$. Moreover, notice that the period of $y = \cos 2x$ and $y = \sec 2x$ is π.

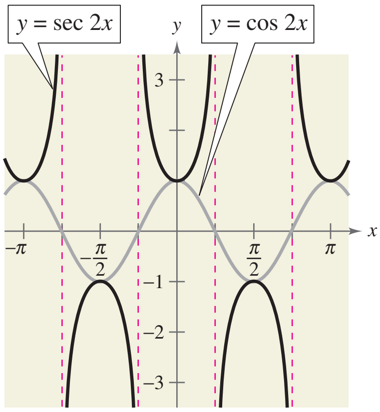


FIGURE 4.67

CHECKPOINT Now try Exercise 27.

Damped Trigonometric Graphs

A *product* of two functions can be graphed using properties of the individual functions. For instance, consider the function

$$f(x) = x \sin x$$

as the product of the functions $y = x$ and $y = \sin x$. Using properties of absolute value and the fact that $|\sin x| \leq 1$, you have $0 \leq |x||\sin x| \leq |x|$. Consequently,

$$-|x| \leq x \sin x \leq |x|$$

which means that the graph of $f(x) = x \sin x$ lies between the lines $y = -x$ and $y = x$. Furthermore, because

$$f(x) = x \sin x = \pm x \qquad \text{at} \qquad x = \frac{\pi}{2} + n\pi$$

and

$$f(x) = x \sin x = 0 \qquad \text{at} \qquad x = n\pi$$

the graph of f touches the line $y = -x$ or the line $y = x$ at $x = \pi/2 + n\pi$ and has x-intercepts at $x = n\pi$. A sketch of f is shown in Figure 4.68. In the function $f(x) = x \sin x$, the factor x is called the **damping factor.**

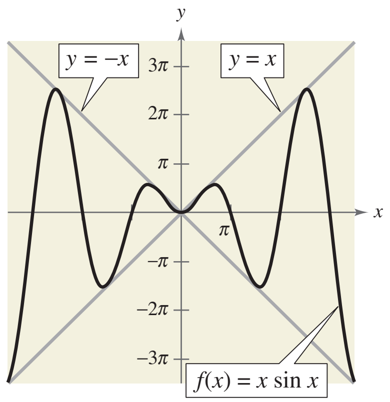


FIGURE 4.68

STUDY TIP

Do you see why the graph of $f(x) = x \sin x$ touches the lines $y = \pm x$ at $x = \pi/2 + n\pi$ and why the graph has x-intercepts at $x = n\pi$? Recall that the sine function is equal to 1 at $\pi/2$, $3\pi/2, 5\pi/2, \ldots$ (odd multiples of $\pi/2$) and is equal to 0 at π, $2\pi, 3\pi, \ldots$ (multiples of π).

Example 6 Damped Sine Wave

Sketch the graph of

$$f(x) = e^{-x} \sin 3x.$$

Solution

Consider $f(x)$ as the product of the two functions

$$y = e^{-x} \qquad \text{and} \qquad y = \sin 3x$$

each of which has the set of real numbers as its domain. For any real number x, you know that $e^{-x} \geq 0$ and $|\sin 3x| \leq 1$. So, $e^{-x}|\sin 3x| \leq e^{-x}$, which means that

$$-e^{-x} \leq e^{-x} \sin 3x \leq e^{-x}.$$

Furthermore, because

$$f(x) = e^{-x} \sin 3x = \pm e^{-x} \qquad \text{at} \qquad x = \frac{\pi}{6} + \frac{n\pi}{3}$$

and

$$f(x) = e^{-x} \sin 3x = 0 \qquad \text{at} \qquad x = \frac{n\pi}{3}$$

the graph of f touches the curves $y = -e^{-x}$ and $y = e^{-x}$ at $x = \pi/6 + n\pi/3$ and has intercepts at $x = n\pi/3$. A sketch is shown in Figure 4.69.

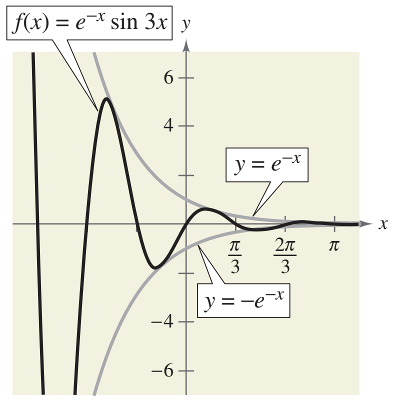
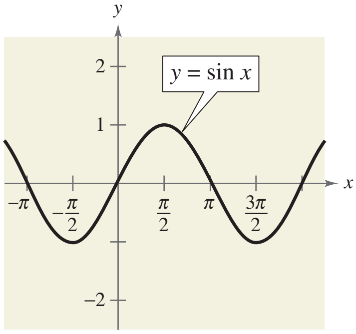


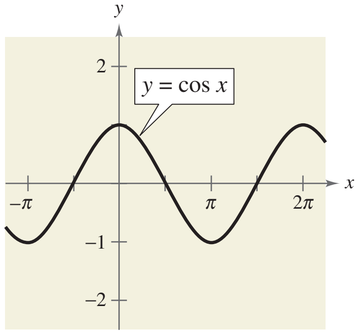
FIGURE 4.69

CHECKPOINT Now try Exercise 65.

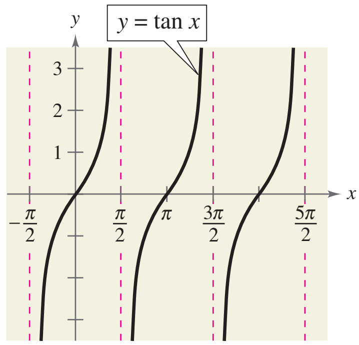
Figure 4.70 summarizes the characteristics of the six basic trigonometric functions.



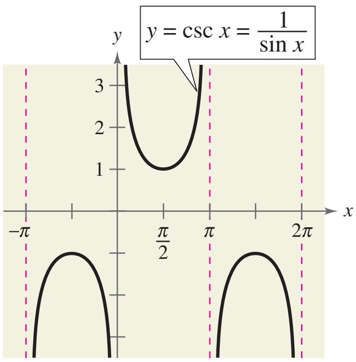
DOMAIN: ALL REALS
RANGE: $[-1, 1]$
PERIOD: 2π



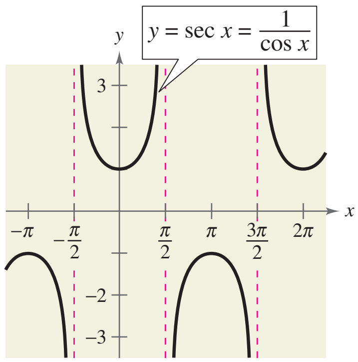
DOMAIN: ALL REALS
RANGE: $[-1, 1]$
PERIOD: 2π



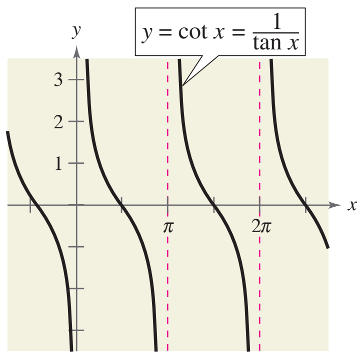
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π

FIGURE 4.70

WRITING ABOUT MATHEMATICS

Combining Trigonometric Functions Recall from Section 1.8 that functions can be combined arithmetically. This also applies to trigonometric functions. For each of the functions

$$h(x) = x + \sin x \quad \text{and} \quad h(x) = \cos x - \sin 3x$$

(a) identify two simpler functions f and g that comprise the combination, (b) use a table to show how to obtain the numerical values of $h(x)$ from the numerical values of $f(x)$ and $g(x)$, and (c) use graphs of f and g to show how h may be formed.

Can you find functions

$$f(x) = d + a\sin(bx + c) \quad \text{and} \quad g(x) = d + a\cos(bx + c)$$

such that $f(x) + g(x) = 0$ for all x?

4.6 Exercises

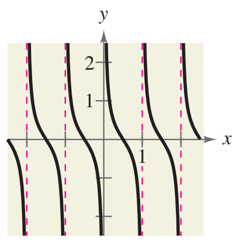
VOCABULARY CHECK: Fill in the blanks.

1. The graphs of the tangent, cotangent, secant, and cosecant functions all have ________ asymptotes.
2. To sketch the graph of a secant or cosecant function, first make a sketch of its corresponding ________ function.
3. For the functions given by $f(x) = g(x) \cdot \sin x$, $g(x)$ is called the ________ factor of the function $f(x)$.
4. The period of $y = \tan x$ is ________.
5. The domain of $y = \cot x$ is all real numbers such that ________.
6. The range of $y = \sec x$ is ________.
7. The period of $y = \csc x$ is ________.

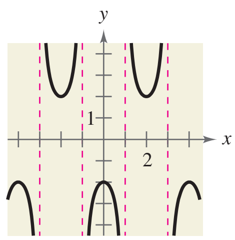
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, match the function with its graph. State the period of the function. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

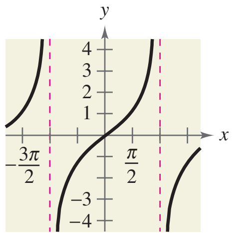
(a)



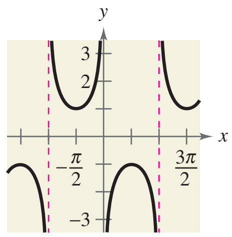
(b)



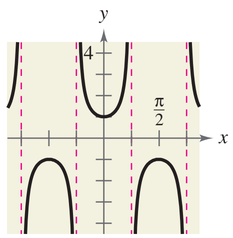
(c)



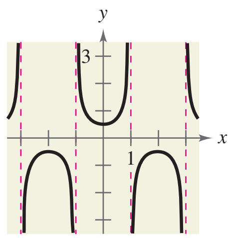
(d)



(e)



(f)



1. $y = \sec 2x$
2. $y = \tan \dfrac{x}{2}$
3. $y = \dfrac{1}{2} \cot \pi x$
4. $y = -\csc x$
5. $y = \dfrac{1}{2} \sec \dfrac{\pi x}{2}$
6. $y = -2 \sec \dfrac{\pi x}{2}$

In Exercises 7–30, sketch the graph of the function. Include two full periods.

7. $y = \frac{1}{3} \tan x$
8. $y = \frac{1}{4} \tan x$
9. $y = \tan 3x$
10. $y = -3 \tan \pi x$
11. $y = -\frac{1}{2} \sec x$
12. $y = \frac{1}{4} \sec x$
13. $y = \csc \pi x$
14. $y = 3 \csc 4x$
15. $y = \sec \pi x - 1$
16. $y = -2 \sec 4x + 2$
17. $y = \csc \dfrac{x}{2}$
18. $y = \csc \dfrac{x}{3}$
19. $y = \cot \dfrac{x}{2}$
20. $y = 3 \cot \dfrac{\pi x}{2}$
21. $y = \frac{1}{2} \sec 2x$
22. $y = -\frac{1}{2} \tan x$
23. $y = \tan \dfrac{\pi x}{4}$
24. $y = \tan(x + \pi)$
25. $y = \csc(\pi - x)$
26. $y = \csc(2x - \pi)$
27. $y = 2 \sec(x + \pi)$
28. $y = -\sec \pi x + 1$
29. $y = \dfrac{1}{4} \csc\left(x + \dfrac{\pi}{4}\right)$
30. $y = 2 \cot\left(x + \dfrac{\pi}{2}\right)$

In Exercises 31–40, use a graphing utility to graph the function. Include two full periods.

31. $y = \tan \dfrac{x}{3}$
32. $y = -\tan 2x$
33. $y = -2 \sec 4x$
34. $y = \sec \pi x$
35. $y = \tan\left(x - \dfrac{\pi}{4}\right)$
36. $y = \dfrac{1}{4} \cot\left(x - \dfrac{\pi}{2}\right)$
37. $y = -\csc(4x - \pi)$
38. $y = 2 \sec(2x - \pi)$
39. $y = 0.1 \tan\left(\dfrac{\pi x}{4} + \dfrac{\pi}{4}\right)$
40. $y = \dfrac{1}{3} \sec\left(\dfrac{\pi x}{2} + \dfrac{\pi}{2}\right)$

In Exercises 41–48, use a graph to solve the equation on the interval $[-2\pi, 2\pi]$.

41. $\tan x = 1$

42. $\tan x = \sqrt{3}$

43. $\cot x = -\dfrac{\sqrt{3}}{3}$

44. $\cot x = 1$

45. $\sec x = -2$

46. $\sec x = 2$

47. $\csc x = \sqrt{2}$

48. $\csc x = -\dfrac{2\sqrt{3}}{3}$

In Exercises 49 and 50, use the graph of the function to determine whether the function is even, odd, or neither.

49. $f(x) = \sec x$

50. $f(x) = \tan x$

51. ***Graphical Reasoning*** Consider the functions given by

$$f(x) = 2\sin x \quad \text{and} \quad g(x) = \frac{1}{2}\csc x$$

on the interval $(0, \pi)$.

(a) Graph f and g in the same coordinate plane.

(b) Approximate the interval in which $f > g$.

(c) Describe the behavior of each of the functions as x approaches π. How is the behavior of g related to the behavior of f as x approaches π?

52. ***Graphical Reasoning*** Consider the functions given by

$$f(x) = \tan\frac{\pi x}{2} \quad \text{and} \quad g(x) = \frac{1}{2}\sec\frac{\pi x}{2}$$

on the interval $(-1, 1)$.

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Approximate the interval in which $f < g$.

(c) Approximate the interval in which $2f < 2g$. How does the result compare with that of part (b)? Explain.

In Exercises 53–56, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to determine whether the expressions are equivalent. Verify the results algebraically.

53. $y_1 = \sin x \csc x, \quad y_2 = 1$

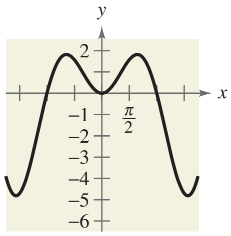
54. $y_1 = \sin x \sec x, \quad y_2 = \tan x$

55. $y_1 = \dfrac{\cos x}{\sin x}, \quad y_2 = \cot x$

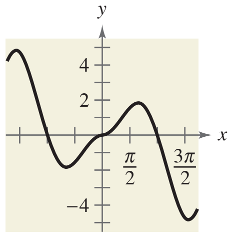
56. $y_1 = \sec^2 x - 1, \quad y_2 = \tan^2 x$

In Exercises 57–60, match the function with its graph. Describe the behavior of the function as x approaches zero. [The graphs are labeled (a), (b), (c), and (d).]

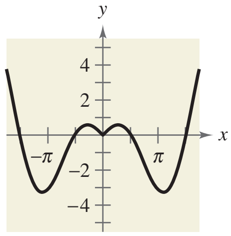
(a)



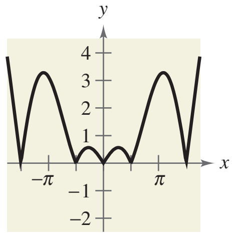
(b)



(c)



(d)



57. $f(x) = |x \cos x|$

58. $f(x) = x \sin x$

59. $g(x) = |x| \sin x$

60. $g(x) = |x| \cos x$

Conjecture **In Exercises 61–64, graph the functions f and g. Use the graphs to make a conjecture about the relationship between the functions.**

61. $f(x) = \sin x + \cos\left(x + \dfrac{\pi}{2}\right), \quad g(x) = 0$

62. $f(x) = \sin x - \cos\left(x + \dfrac{\pi}{2}\right), \quad g(x) = 2\sin x$

63. $f(x) = \sin^2 x, \quad g(x) = \frac{1}{2}(1 - \cos 2x)$

64. $f(x) = \cos^2 \dfrac{\pi x}{2}, \quad g(x) = \dfrac{1}{2}(1 + \cos \pi x)$

In Exercises 65–68, use a graphing utility to graph the function and the damping factor of the function in the same viewing window. Describe the behavior of the function as x increases without bound.

65. $g(x) = e^{-x^2/2} \sin x$

66. $f(x) = e^{-x} \cos x$

67. $f(x) = 2^{-x/4} \cos \pi x$

68. $h(x) = 2^{-x^2/4} \sin x$

Exploration **In Exercises 69–74, use a graphing utility to graph the function. Describe the behavior of the function as x approaches zero.**

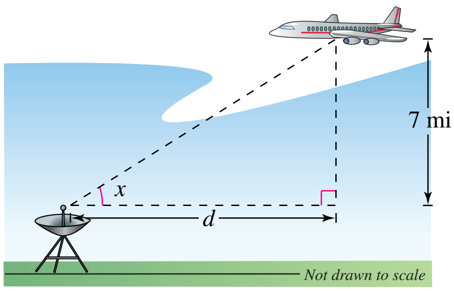
69. $y = \dfrac{6}{x} + \cos x, \quad x > 0$

70. $y = \dfrac{4}{x} + \sin 2x, \quad x > 0$

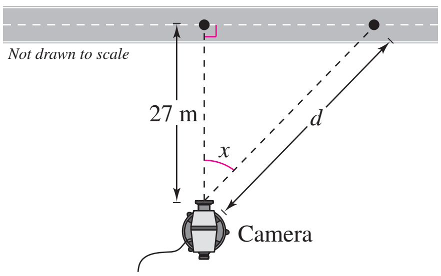
71. $g(x) = \dfrac{\sin x}{x}$ **72.** $f(x) = \dfrac{1 - \cos x}{x}$

73. $f(x) = \sin \dfrac{1}{x}$ **74.** $h(x) = x \sin \dfrac{1}{x}$

75. ***Distance*** A plane flying at an altitude of 7 miles above a radar antenna will pass directly over the radar antenna (see figure). Let d be the ground distance from the antenna to the point directly under the plane and let x be the angle of elevation to the plane from the antenna. (d is positive as the plane approaches the antenna.) Write d as a function of x and graph the function over the interval $0 < x < \pi$.



76. ***Television Coverage*** A television camera is on a reviewing platform 27 meters from the street on which a parade will be passing from left to right (see figure). Write the distance d from the camera to a particular unit in the parade as a function of the angle x, and graph the function over the interval $-\pi/2 < x < \pi/2$. (Consider x as negative when a unit in the parade approaches from the left.)



Model It

77. ***Predator-Prey Model*** The population C of coyotes (a predator) at time t (in months) in a region is estimated to be

$$C = 5000 + 2000 \sin \frac{\pi t}{12}$$

and the population R of rabbits (its prey) is estimated to be

Model It (continued)

$$R = 25{,}000 + 15{,}000 \cos \frac{\pi t}{12}.$$

(a) Use a graphing utility to graph both models in the same viewing window. Use the window setting $0 \le t \le 100$.

(b) Use the graphs of the models in part (a) to explain the oscillations in the size of each population.

(c) The cycles of each population follow a periodic pattern. Find the period of each model and describe several factors that could be contributing to the cyclical patterns.

78. ***Sales*** The projected monthly sales S (in thousands of units) of lawn mowers (a seasonal product) are modeled by $S = 74 + 3t - 40 \cos(\pi t/6)$, where t is the time (in months), with $t = 1$ corresponding to January. Graph the sales function over 1 year.

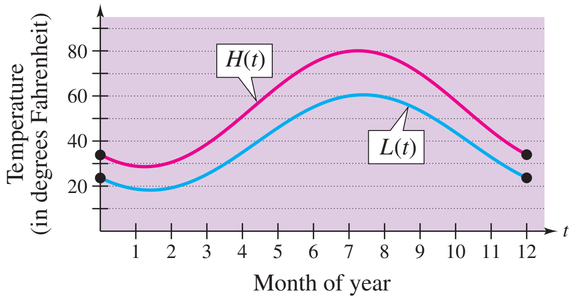
79. ***Meterology*** The normal monthly high temperatures H (in degrees Fahrenheit) for Erie, Pennsylvania are approximated by

$$H(t) = 54.33 - 20.38 \cos \frac{\pi t}{6} - 15.69 \sin \frac{\pi t}{6}$$

and the normal monthly low temperatures L are approximated by

$$L(t) = 39.36 - 15.70 \cos \frac{\pi t}{6} - 14.16 \sin \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January (see figure). (Source: National Oceanic and Atmospheric Administration)



(a) What is the period of each function?

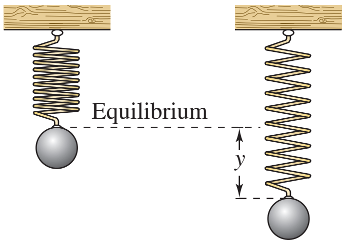
(b) During what part of the year is the difference between the normal high and normal low temperatures greatest? When is it smallest?

(c) The sun is northernmost in the sky around June 21, but the graph shows the warmest temperatures at a later date. Approximate the lag time of the temperatures relative to the position of the sun.

80. ***Harmonic Motion*** An object weighing W pounds is suspended from the ceiling by a steel spring (see figure). The weight is pulled downward (positive direction) from its equilibrium position and released. The resulting motion of the weight is described by the function

$$y = \frac{1}{2}e^{-t/4}\cos 4t, \quad t > 0$$

where y is the distance (in feet) and t is the time (in seconds).



(a) Use a graphing utility to graph the function.

(b) Describe the behavior of the displacement function for increasing values of time t.

Synthesis

True or False? **In Exercises 81 and 82, determine whether the statement is true or false. Justify your answer.**

81. The graph of $y = \csc x$ can be obtained on a calculator by graphing the reciprocal of $y = \sin x$.

82. The graph of $y = \sec x$ can be obtained on a calculator by graphing a translation of the reciprocal of $y = \sin x$.

83. ***Writing*** Describe the behavior of $f(x) = \tan x$ as x approaches $\pi/2$ from the left and from the right.

84. ***Writing*** Describe the behavior of $f(x) = \csc x$ as x approaches π from the left and from the right.

85. ***Exploration*** Consider the function given by

$$f(x) = x - \cos x.$$

(a) Use a graphing utility to graph the function and verify that there exists a zero between 0 and 1. Use the graph to approximate the zero.

(b) Starting with $x_0 = 1$, generate a sequence $x_1, x_2, x_3, \ldots,$ where $x_n = \cos(x_{n-1})$. For example,

$$x_0 = 1$$
$$x_1 = \cos(x_0)$$
$$x_2 = \cos(x_1)$$
$$x_3 = \cos(x_2)$$
$$\vdots$$

What value does the sequence approach?

86. ***Approximation*** Using calculus, it can be shown that the tangent function can be approximated by the polynomial

$$\tan x \approx x + \frac{2x^3}{3!} + \frac{16x^5}{5!}$$

where x is in radians. Use a graphing utility to graph the tangent function and its polynomial approximation in the same viewing window. How do the graphs compare?

87. ***Approximation*** Using calculus, it can be shown that the secant function can be approximated by the polynomial

$$\sec x \approx 1 + \frac{x^2}{2!} + \frac{5x^4}{4!}$$

where x is in radians. Use a graphing utility to graph the secant function and its polynomial approximation in the same viewing window. How do the graphs compare?

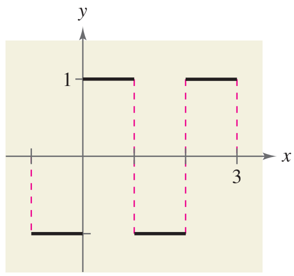
88. ***Pattern Recognition***

(a) Use a graphing utility to graph each function.

$$y_1 = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x\right)$$
$$y_2 = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x\right)$$

(b) Identify the pattern started in part (a) and find a function y_3 that continues the pattern one more term. Use a graphing utility to graph y_3.

(c) The graphs in parts (a) and (b) approximate the periodic function in the figure. Find a function y_4 that is a better approximation.



Skills Review

In Exercises 89–92, solve the exponential equation. Round your answer to three decimal places.

89. $e^{2x} = 54$

90. $8^{3x} = 98$

91. $\dfrac{300}{1 + e^{-x}} = 100$

92. $\left(1 + \dfrac{0.15}{365}\right)^{365t} = 5$

In Exercises 93–98, solve the logarithmic equation. Round your answer to three decimal places.

93. $\ln(3x - 2) = 73$

94. $\ln(14 - 2x) = 68$

95. $\ln(x^2 + 1) = 3.2$

96. $\ln \sqrt{x + 4} = 5$

97. $\log_8 x + \log_8(x - 1) = \frac{1}{3}$

98. $\log_6 x + \log_6(x^2 - 1) = \log_6 64x$

4.7 Inverse Trigonometric Functions

What you should learn

- Evaluate and graph the inverse sine function.
- Evaluate and graph the other inverse trigonometric functions.
- Evaluate and graph the compositions of trigonometric functions.

Why you should learn it

You can use inverse trigonometric functions to model and solve real-life problems. For instance, in Exercise 92 on page 351, an inverse trigonometric function can be used to model the angle of elevation from a television camera to a space shuttle launch.

NASA

Inverse Sine Function

Recall from Section 1.9 that, for a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test. From Figure 4.71, you can see that $y = \sin x$ does not pass the test because different values of x yield the same y-value.

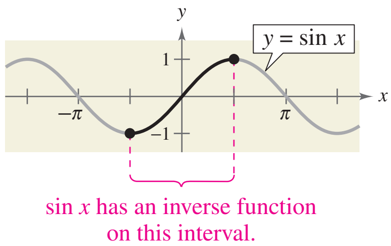


FIGURE 4.71

However, if you restrict the domain to the interval $-\pi/2 \le x \le \pi/2$ (corresponding to the black portion of the graph in Figure 4.71), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.
2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

So, on the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.** It is denoted by

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The $\arcsin x$ notation (read as "the arcsine of x") comes from the association of a central angle with its intercepted *arc length* on a unit circle. So, $\arcsin x$ means the angle (or arc) whose sine is x. Both notations, $\arcsin x$ and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than $1/\sin x$. The values of $\arcsin x$ lie in the interval $-\pi/2 \le \arcsin x \le \pi/2$. The graph of $y = \arcsin x$ is shown in Example 2.

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.

STUDY TIP

When evaluating the inverse sine function, it helps to remember the phrase "the arcsine of x is the angle (or number) whose sine is x."

STUDY TIP

As with the trigonometric functions, much of the work with the inverse trigonometric functions can be done by *exact* calculations rather than by calculator approximations. Exact calculations help to increase your understanding of the inverse functions by relating them to the right triangle definitions of the trigonometric functions.

Example 1 Evaluating the Inverse Sine Function

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

Solution

a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, it follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$ Angle whose sine is $-\frac{1}{2}$

b. Because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, it follows that

$$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$ Angle whose sine is $\sqrt{3}/2$

c. It is not possible to evaluate $y = \sin^{-1} x$ when $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is $[-1, 1]$.

CHECKPOINT Now try Exercise 1.

Example 2 Graphing the Arcsine Function

Sketch a graph of

$$y = \arcsin x.$$

Solution

By definition, the equations $y = \arcsin x$ and $\sin y = x$ are equivalent for $-\pi/2 \le y \le \pi/2$. So, their graphs are the same. From the interval $[-\pi/2, \pi/2]$, you can assign values to y in the second equation to make a table of values. Then plot the points and draw a smooth curve through the points.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

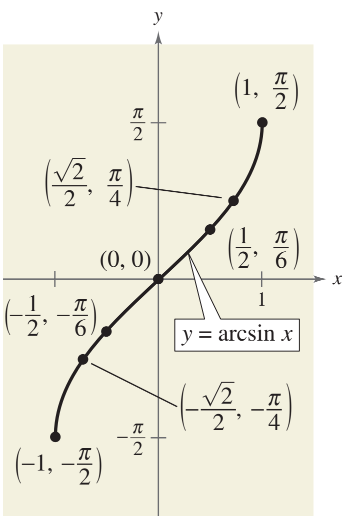


FIGURE 4.72

The resulting graph for $y = \arcsin x$ is shown in Figure 4.72. Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 4.71. Be sure you see that Figure 4.72 shows the *entire* graph of the inverse sine function. Remember that the domain of $y = \arcsin x$ is the closed interval $[-1, 1]$ and the range is the closed interval $[-\pi/2, \pi/2]$.

CHECKPOINT Now try Exercise 17.

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \le x \le \pi$, as shown in Figure 4.73.

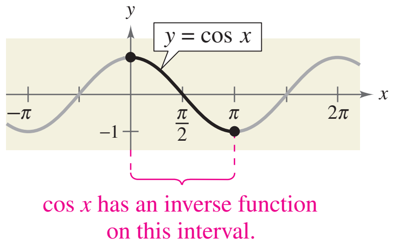


FIGURE 4.73

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

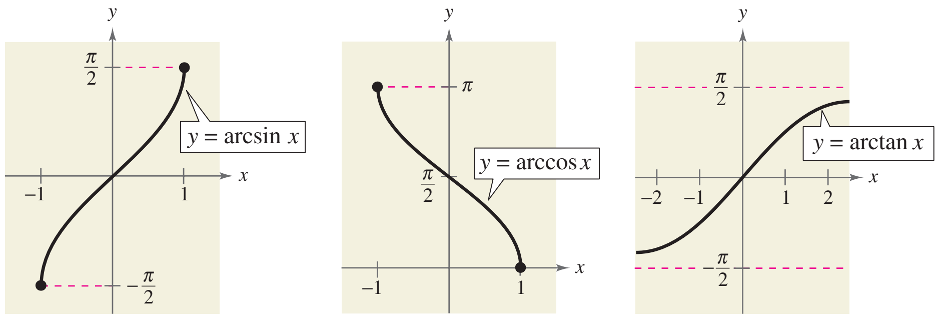
$$y = \arccos x \qquad \text{or} \qquad y = \cos^{-1} x.$$

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$. The following list summarizes the definitions of the three most common inverse trigonometric functions. The remaining three are defined in Exercises 101–103.

Definitions of the Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

The graphs of these three inverse trigonometric functions are shown in Figure 4.74.



DOMAIN: $[-1, 1]$
RANGE: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

DOMAIN: $[-1, 1]$
RANGE: $[0, \pi]$

DOMAIN: $(-\infty, \infty)$
RANGE: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

FIGURE 4.74

Example 3 Evaluating Inverse Trigonometric Functions

Find the exact value.

a. $\arccos \frac{\sqrt{2}}{2}$ **b.** $\cos^{-1}(-1)$

c. $\arctan 0$ **d.** $\tan^{-1}(-1)$

Solution

a. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}. \qquad \text{Angle whose cosine is } \sqrt{2}/2$$

b. Because $\cos \pi = -1$, and π lies in $[0, \pi]$, it follows that

$$\cos^{-1}(-1) = \pi. \qquad \text{Angle whose cosine is } -1$$

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan 0 = 0. \qquad \text{Angle whose tangent is } 0$$

d. Because $\tan(-\pi/4) = -1$, and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\tan^{-1}(-1) = -\frac{\pi}{4}. \qquad \text{Angle whose tangent is } -1$$

CHECKPOINT Now try Exercise 11.

Example 4 Calculators and Inverse Trigonometric Functions

Use a calculator to approximate the value (if possible).

a. $\arctan(-8.45)$

b. $\sin^{-1} 0.2447$

c. $\arccos 2$

Solution

	Function	*Mode*	*Calculator Keystrokes*
a.	$\arctan(-8.45)$	Radian	TAN⁻¹ ((–) 8.45) ENTER

From the display, it follows that $\arctan(-8.45) \approx -1.453001$.

b.	$\sin^{-1} 0.2447$	Radian	SIN⁻¹ (0.2447) ENTER

From the display, it follows that $\sin^{-1} 0.2447 \approx 0.2472103$.

c.	$\arccos 2$	Radian	COS⁻¹ (2) ENTER

In *real number* mode, the calculator should display an *error message* because the domain of the inverse cosine function is $[-1, 1]$.

CHECKPOINT Now try Exercise 25.

STUDY TIP

It is important to remember that the domain of the inverse sine function and the inverse cosine function is $[-1, 1]$, as indicated in Example 4(c).

In Example 4, if you had set the calculator to *degree* mode, the displays would have been in degrees rather than radians. This convention is peculiar to calculators. By definition, the values of inverse trigonometric functions are *always in radians*.

Compositions of Functions

Recall from Section 1.9 that for all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Inverse Properties of Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

Keep in mind that these inverse properties do not apply for arbitrary values of x and y. For instance,

$$\arcsin\left(\sin \frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

In other words, the property

$$\arcsin(\sin y) = y$$

is not valid for values of y outside the interval $[-\pi/2, \pi/2]$.

Example 5 Using Inverse Properties

If possible, find the exact value.

a. $\tan[\arctan(-5)]$ **b.** $\arcsin\left(\sin \frac{5\pi}{3}\right)$ **c.** $\cos(\cos^{-1} \pi)$

Solution

a. Because -5 lies in the domain of the arctan function, the inverse property applies, and you have

$$\tan[\arctan(-5)] = -5.$$

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \le y \le \pi/2$. However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

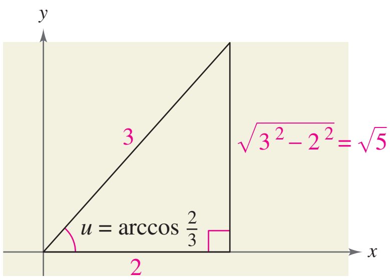
which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin \frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

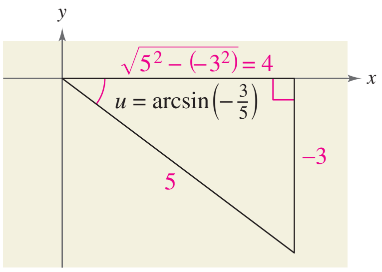
c. The expression $\cos(\cos^{-1} \pi)$ is not defined because $\cos^{-1} \pi$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.

CHECKPOINT Now try Exercise 43.

Example 6 shows how to use right triangles to find exact values of compositions of inverse functions. Then, Example 7 shows how to use right triangles to convert a trigonometric expression into an algebraic expression. This conversion technique is used frequently in calculus.



Angle whose cosine is $\frac{2}{3}$
FIGURE 4.75



Angle whose sine is $-\frac{3}{5}$
FIGURE 4.76

Example 6 Evaluating Compositions of Functions

Find the exact value.

a. $\tan\left(\arccos \dfrac{2}{3}\right)$ **b.** $\cos\left[\arcsin\left(-\dfrac{3}{5}\right)\right]$

Solution

a. If you let $u = \arccos \frac{2}{3}$, then $\cos u = \frac{2}{3}$. Because $\cos u$ is positive, u is a *first*-quadrant angle. You can sketch and label angle u as shown in Figure 4.75. Consequently,

$$\tan\left(\arccos \frac{2}{3}\right) = \tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

b. If you let $u = \arcsin\left(-\frac{3}{5}\right)$, then $\sin u = -\frac{3}{5}$. Because $\sin u$ is negative, u is a *fourth*-quadrant angle. You can sketch and label angle u as shown in Figure 4.76. Consequently,

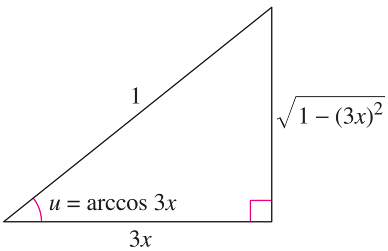
$$\cos\left[\arcsin\left(-\frac{3}{5}\right)\right] = \cos u = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$

CHECKPOINT Now try Exercise 51.

Example 7 Some Problems from Calculus

Write each of the following as an algebraic expression in x.

a. $\sin(\arccos 3x), \quad 0 \le x \le \dfrac{1}{3}$ **b.** $\cot(\arccos 3x), \quad 0 \le x < \dfrac{1}{3}$



Angle whose cosine is $3x$
FIGURE 4.77

Solution

If you let $u = \arccos 3x$, then $\cos u = 3x$, where $-1 \le 3x \le 1$. Because

$$\cos u = \frac{\text{adj}}{\text{hyp}} = \frac{3x}{1}$$

you can sketch a right triangle with acute angle u, as shown in Figure 4.77. From this triangle, you can easily convert each expression to algebraic form.

a. $\sin(\arccos 3x) = \sin u = \dfrac{\text{opp}}{\text{hyp}} = \sqrt{1 - 9x^2}, \quad 0 \le x \le \dfrac{1}{3}$

b. $\cot(\arccos 3x) = \cot u = \dfrac{\text{adj}}{\text{opp}} = \dfrac{3x}{\sqrt{1 - 9x^2}}, \quad 0 \le x < \dfrac{1}{3}$

CHECKPOINT Now try Exercise 59.

In Example 7, similar arguments can be made for x-values lying in the interval $\left[-\frac{1}{3}, 0\right]$.

4.7 Exercises

VOCABULARY CHECK: Fill in the blanks.

	Function	*Alternative Notation*	*Domain*	*Range*
1.	$y = \arcsin x$	________	________	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2.	________	$y = \cos^{-1} x$	$-1 \le x \le 1$	________
3.	$y = \arctan x$	________	________	________

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–16, evaluate the expression without using a calculator.

1. $\arcsin \frac{1}{2}$ **2.** $\arcsin 0$

3. $\arccos \frac{1}{2}$ **4.** $\arccos 0$

5. $\arctan \frac{\sqrt{3}}{3}$ **6.** $\arctan(-1)$

7. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **8.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

9. $\arctan(-\sqrt{3})$ **10.** $\arctan \sqrt{3}$

11. $\arccos\left(-\frac{1}{2}\right)$ **12.** $\arcsin \frac{\sqrt{2}}{2}$

13. $\sin^{-1} \frac{\sqrt{3}}{2}$ **14.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

15. $\tan^{-1} 0$ **16.** $\cos^{-1} 1$

In Exercises 17 and 18, use a graphing utility to graph *f*, *g*, and $y = x$ in the same viewing window to verify geometrically that *g* is the inverse function of *f*. (Be sure to restrict the domain of *f* properly.)

17. $f(x) = \sin x, \quad g(x) = \arcsin x$

18. $f(x) = \tan x, \quad g(x) = \arctan x$

In Exercises 19–34, use a calculator to evaluate the expression. Round your result to two decimal places.

19. $\arccos 0.28$ **20.** $\arcsin 0.45$

21. $\arcsin(-0.75)$ **22.** $\arccos(-0.7)$

23. $\arctan(-3)$ **24.** $\arctan 15$

25. $\sin^{-1} 0.31$ **26.** $\cos^{-1} 0.26$

27. $\arccos(-0.41)$ **28.** $\arcsin(-0.125)$

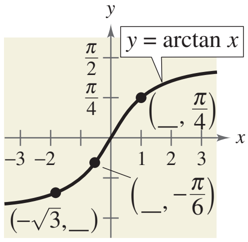
29. $\arctan 0.92$ **30.** $\arctan 2.8$

31. $\arcsin \frac{3}{4}$ **32.** $\arccos\left(-\frac{1}{3}\right)$

33. $\tan^{-1} \frac{7}{2}$ **34.** $\tan^{-1}\left(-\frac{95}{7}\right)$

In Exercises 35 and 36, determine the missing coordinates of the points on the graph of the function.

35.



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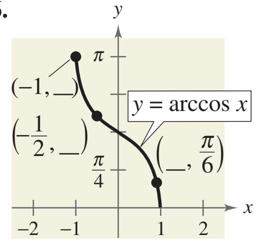
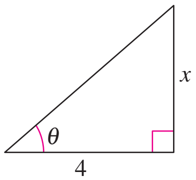


In Exercises 37–42, use an inverse trigonometric function to write θ as a function of x.

37.



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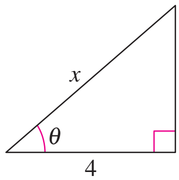

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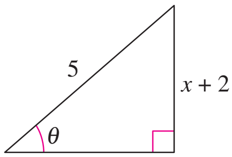

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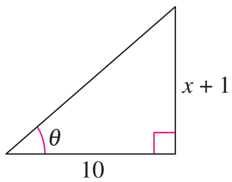

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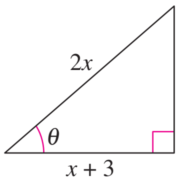

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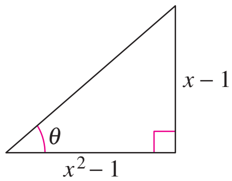


In Exercises 43–48, use the properties of inverse trigonometric functions to evaluate the expression.

43. $\sin(\arcsin 0.3)$ **44.** $\tan(\arctan 25)$

45. $\cos[\arccos(-0.1)]$ **46.** $\sin[\arcsin(-0.2)]$

47. $\arcsin(\sin 3\pi)$ **48.** $\arccos\left(\cos \frac{7\pi}{2}\right)$

In Exercises 49–58, find the exact value of the expression. (*Hint:* Sketch a right triangle.)

49. $\sin\left(\arctan \frac{3}{4}\right)$

50. $\sec\left(\arcsin \frac{4}{5}\right)$

51. $\cos(\tan^{-1} 2)$

52. $\sin\left(\cos^{-1} \frac{\sqrt{5}}{5}\right)$

53. $\cos\left(\arcsin \frac{5}{13}\right)$

54. $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

55. $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$

56. $\tan\left[\arcsin\left(-\frac{3}{4}\right)\right]$

57. $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$

58. $\cot\left(\arctan \frac{5}{8}\right)$

In Exercises 59–68, write an algebraic expression that is equivalent to the expression. (*Hint:* Sketch a right triangle, as demonstrated in Example 7.)

59. $\cot(\arctan x)$

60. $\sin(\arctan x)$

61. $\cos(\arcsin 2x)$

62. $\sec(\arctan 3x)$

63. $\sin(\arccos x)$

64. $\sec[\arcsin(x - 1)]$

65. $\tan\left(\arccos \frac{x}{3}\right)$

66. $\cot\left(\arctan \frac{1}{x}\right)$

67. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

68. $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 69 and 70, use a graphing utility to graph *f* and *g* in the same viewing window to verify that the two functions are equal. Explain why they are equal. Identify any asymptotes of the graphs.

69. $f(x) = \sin(\arctan 2x), \quad g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

70. $f(x) = \tan\left(\arccos \dfrac{x}{2}\right), \quad g(x) = \dfrac{\sqrt{4 - x^2}}{x}$

In Exercises 71–74, fill in the blank.

71. $\arctan \dfrac{9}{x} = \arcsin(\quad), \quad x \neq 0$

72. $\arcsin \dfrac{\sqrt{36 - x^2}}{6} = \arccos(\quad), \quad 0 \leq x \leq 6$

73. $\arccos \dfrac{3}{\sqrt{x^2 - 2x + 10}} = \arcsin(\quad)$

74. $\arccos \dfrac{x - 2}{2} = \arctan(\quad), \quad |x - 2| \leq 2$

In Exercises 75 and 76, sketch a graph of the function and compare the graph of *g* with the graph of $f(x) = \arcsin x$.

75. $g(x) = \arcsin(x - 1)$

76. $g(x) = \arcsin \dfrac{x}{2}$

In Exercises 77–82, sketch a graph of the function.

77. $y = 2 \arccos x$

78. $g(t) = \arccos(t + 2)$

79. $f(x) = \arctan 2x$

80. $f(x) = \dfrac{\pi}{2} + \arctan x$

81. $h(v) = \tan(\arccos v)$

82. $f(x) = \arccos \dfrac{x}{4}$

In Exercises 83–88, use a graphing utility to graph the function.

83. $f(x) = 2 \arccos(2x)$

84. $f(x) = \pi \arcsin(4x)$

85. $f(x) = \arctan(2x - 3)$

86. $f(x) = -3 + \arctan(\pi x)$

87. $f(x) = \pi - \sin^{-1}\left(\dfrac{2}{3}\right)$

88. $f(x) = \dfrac{\pi}{2} + \cos^{-1}\left(\dfrac{1}{\pi}\right)$

In Exercises 89 and 90, write the function in terms of the sine function by using the identity

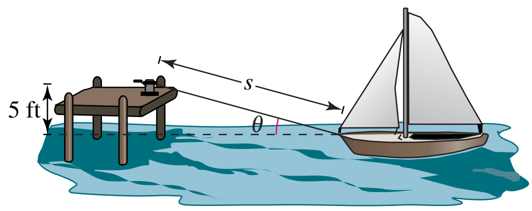
$$A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \sin\left(\omega t + \arctan \frac{A}{B}\right).$$

Use a graphing utility to graph both forms of the function. What does the graph imply?

89. $f(t) = 3 \cos 2t + 3 \sin 2t$

90. $f(t) = 4 \cos \pi t + 3 \sin \pi t$

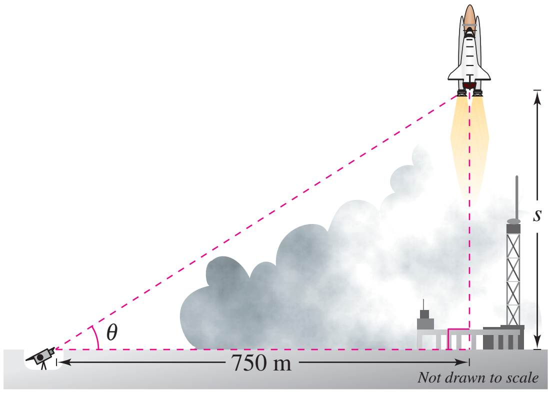
91. ***Docking a Boat*** A boat is pulled in by means of a winch located on a dock 5 feet above the deck of the boat (see figure). Let θ be the angle of elevation from the boat to the winch and let s be the length of the rope from the winch to the boat.



(a) Write θ as a function of s.

(b) Find θ when $s = 40$ feet and $s = 20$ feet.

92. ***Photography*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad (see figure). Let θ be the angle of elevation to the shuttle and let s be the height of the shuttle.



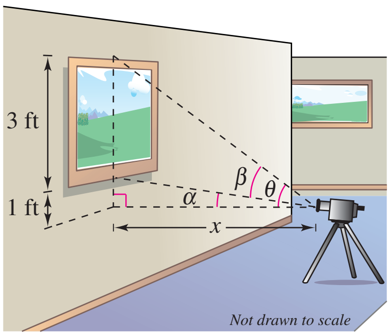
(a) Write θ as a function of s.

(b) Find θ when $s = 300$ meters and $s = 1200$ meters.

Model It

93. ***Photography*** A photographer is taking a picture of a three-foot-tall painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting (see figure). The angle β subtended by the camera lens x feet from the painting is

$$\beta = \arctan \frac{3x}{x^2 + 4}, \quad x > 0.$$

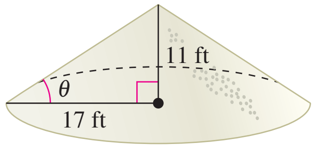


(a) Use a graphing utility to graph β as a function of x.

(b) Move the cursor along the graph to approximate the distance from the picture when β is maximum.

(c) Identify the asymptote of the graph and discuss its meaning in the context of the problem.

94. ***Granular Angle of Repose*** Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose* (see figure). When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. (Source: Bulk-Store Structures, Inc.)



(a) Find the angle of repose for rock salt.

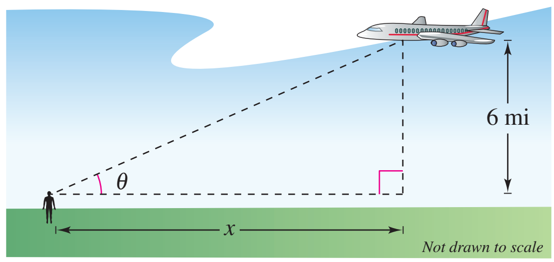
(b) How tall is a pile of rock salt that has a base diameter of 40 feet?

95. ***Granular Angle of Repose*** When whole corn is stored in a cone-shaped pile 20 feet high, the diameter of the pile's base is about 82 feet.

(a) Find the angle of repose for whole corn.

(b) How tall is a pile of corn that has a base diameter of 100 feet?

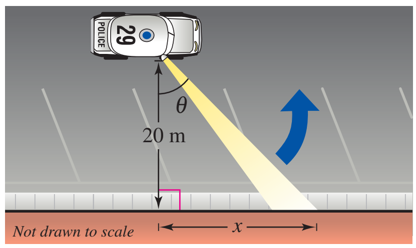
96. ***Angle of Elevation*** An airplane flies at an altitude of 6 miles toward a point directly over an observer. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) Find θ when $x = 7$ miles and $x = 1$ mile.

97. ***Security Patrol*** A security car with its spotlight on is parked 20 meters from a warehouse. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) Find θ when $x = 5$ meters and $x = 12$ meters.

Synthesis

True or False? **In Exercises 98–100, determine whether the statement is true or false. Justify your answer.**

98. $\sin \frac{5\pi}{6} = \frac{1}{2} \Rightarrow \arcsin \frac{1}{2} = \frac{5\pi}{6}$

99. $\tan \frac{5\pi}{4} = 1 \Rightarrow \arctan 1 = \frac{5\pi}{4}$

100. $\arctan x = \dfrac{\arcsin x}{\arccos x}$

101. Define the inverse cotangent function by restricting the domain of the cotangent function to the interval $(0, \pi)$, and sketch its graph.

102. Define the inverse secant function by restricting the domain of the secant function to the intervals $[0, \pi/2)$ and $(\pi/2, \pi]$, and sketch its graph.

103. Define the inverse cosecant function by restricting the domain of the cosecant function to the intervals $[-\pi/2, 0)$ and $(0, \pi/2]$, and sketch its graph.

104. Use the results of Exercises 101–103 to evaluate each expression without using a calculator.

(a) $\operatorname{arcsec} \sqrt{2}$ (b) $\operatorname{arcsec} 1$

(c) $\operatorname{arccot}\left(-\sqrt{3}\right)$ (d) $\operatorname{arccsc} 2$

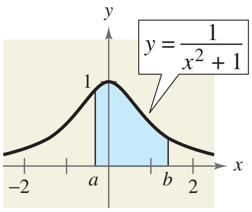
105. ***Area*** In calculus, it is shown that the area of the region bounded by the graphs of $y = 0$, $y = 1/(x^2 + 1)$, $x = a$, and $x = b$ is given by

$$\text{Area} = \arctan b - \arctan a$$

(see figure). Find the area for the following values of a and b.

(a) $a = 0, b = 1$ (b) $a = -1, b = 1$

(c) $a = 0, b = 3$ (d) $a = -1, b = 3$



106. ***Think About It*** Use a graphing utility to graph the functions

$$f(x) = \sqrt{x} \text{ and } g(x) = 6 \arctan x.$$

For $x > 0$, it appears that $g > f$. Explain why you know that there exists a positive real number a such that $g < f$ for $x > a$. Approximate the number a.

107. ***Think About It*** Consider the functions given by

$$f(x) = \sin x \quad \text{and} \quad f^{-1}(x) = \arcsin x.$$

(a) Use a graphing utility to graph the composite functions $f \circ f^{-1}$ and $f^{-1} \circ f$.

(b) Explain why the graphs in part (a) are not the graph of the line $y = x$. Why do the graphs of $f \circ f^{-1}$ and $f^{-1} \circ f$ differ?

108. ***Proof*** Prove each identity.

(a) $\arcsin(-x) = -\arcsin x$

(b) $\arctan(-x) = -\arctan x$

(c) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$

(d) $\arcsin x + \arccos x = \frac{\pi}{2}$

(e) $\arcsin x = \arctan \dfrac{x}{\sqrt{1 - x^2}}$

Skills Review

In Exercises 109–112, evaluate the expression. Round your result to three decimal places.

109. $(8.2)^{3.4}$ **110.** $10(14)^{-2}$

111. $(1.1)^{50}$ **112.** $16^{-2\pi}$

In Exercises 113–116, sketch a right triangle corresponding to the trigonometric function of the acute angle *θ*. Use the Pythagorean Theorem to determine the third side. Then find the other five trigonometric functions of *θ*.

113. $\sin \theta = \frac{3}{4}$ **114.** $\tan \theta = 2$

115. $\cos \theta = \frac{5}{6}$ **116.** $\sec \theta = 3$

117. ***Partnership Costs*** A group of people agree to share equally in the cost of a \$250,000 endowment to a college. If they could find two more people to join the group, each person's share of the cost would decrease by \$6250. How many people are presently in the group?

118. ***Speed*** A boat travels at a speed of 18 miles per hour in still water. It travels 35 miles upstream and then returns to the starting point in a total of 4 hours. Find the speed of the current.

119. ***Compound Interest*** A total of \$15,000 is invested in an account that pays an annual interest rate of 3.5%. Find the balance in the account after 10 years, if interest is compounded (a) quarterly, (b) monthly, (c) daily, and (d) continuously.

120. ***Profit*** Because of a slump in the economy, a department store finds that its annual profits have dropped from \$742,000 in 2002 to \$632,000 in 2004. The profit follows an exponential pattern of decline. What is the expected profit for 2008? (Let $t = 2$ represent 2002.)

4.8 Applications and Models

What you should learn

- Solve real-life problems involving right triangles.
- Solve real-life problems involving directional bearings.
- Solve real-life problems involving harmonic motion.

Why you should learn it

Right triangles often occur in real-life situations. For instance, in Exercise 62 on page 362, right triangles are used to determine the shortest grain elevator for a grain storage bin on a farm.

Applications Involving Right Triangles

In this section, the three angles of a right triangle are denoted by the letters A, B, and C (where C is the right angle), and the lengths of the sides opposite these angles by the letters a, b, and c (where c is the hypotenuse).

Example 1 **Solving a Right Triangle**

Solve the right triangle shown in Figure 4.78 for all unknown sides and angles.

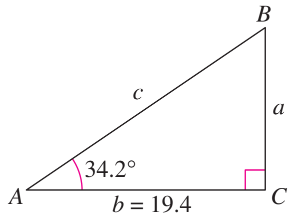


FIGURE 4.78

Solution

Because $C = 90°$, it follows that $A + B = 90°$ and $B = 90° - 34.2° = 55.8°$. To solve for a, use the fact that

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{a}{b} \quad \Rightarrow \quad a = b \tan A.$$

So, $a = 19.4 \tan 34.2° \approx 13.18$. Similarly, to solve for c, use the fact that

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c} \quad \Rightarrow \quad c = \frac{b}{\cos A}.$$

So, $c = \dfrac{19.4}{\cos 34.2°} \approx 23.46.$

CHECKPOINT Now try Exercise 1.

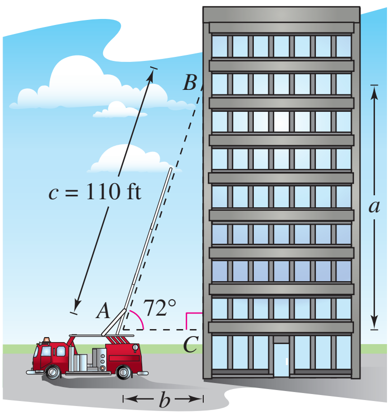


FIGURE 4.79

Example 2 **Finding a Side of a Right Triangle**

A safety regulation states that the maximum angle of elevation for a rescue ladder is 72°. A fire department's longest ladder is 110 feet. What is the maximum safe rescue height?

Solution

A sketch is shown in Figure 4.79. From the equation $\sin A = a/c$, it follows that

$$a = c \sin A = 110 \sin 72° \approx 104.6.$$

So, the maximum safe rescue height is about 104.6 feet above the height of the fire truck.

CHECKPOINT Now try Exercise 15.

Example 3 Finding a Side of a Right Triangle

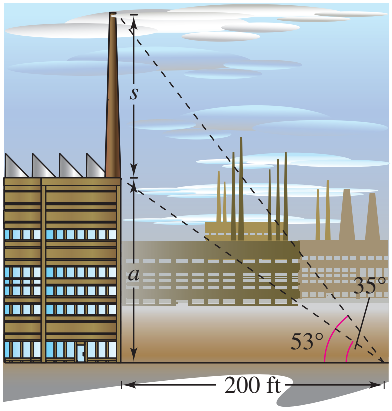


FIGURE 4.80

At a point 200 feet from the base of a building, the angle of elevation to the *bottom* of a smokestack is 35°, whereas the angle of elevation to the *top* is 53°, as shown in Figure 4.80. Find the height s of the smokestack alone.

Solution

Note from Figure 4.80 that this problem involves two right triangles. For the smaller right triangle, use the fact that

$$\tan 35° = \frac{a}{200}$$

to conclude that the height of the building is

$$a = 200 \tan 35°.$$

For the larger right triangle, use the equation

$$\tan 53° = \frac{a + s}{200}$$

to conclude that $a + s = 200 \tan 53°$. So, the height of the smokestack is

$$\begin{aligned} s &= 200 \tan 53° - a \\ &= 200 \tan 53° - 200 \tan 35° \\ &\approx 125.4 \text{ feet.} \end{aligned}$$

CHECKPOINT Now try Exercise 19.

Example 4 Finding an Acute Angle of a Right Triangle

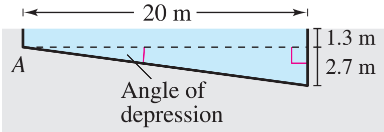


FIGURE 4.81

A swimming pool is 20 meters long and 12 meters wide. The bottom of the pool is slanted so that the water depth is 1.3 meters at the shallow end and 4 meters at the deep end, as shown in Figure 4.81. Find the angle of depression of the bottom of the pool.

Solution

Using the tangent function, you can see that

$$\begin{aligned} \tan A &= \frac{\text{opp}}{\text{adj}} \\ &= \frac{2.7}{20} \\ &= 0.135. \end{aligned}$$

So, the angle of depression is

$$\begin{aligned} A &= \arctan 0.135 \\ &\approx 0.13419 \text{ radian} \\ &\approx 7.69°. \end{aligned}$$

CHECKPOINT Now try Exercise 25.

Trigonometry and Bearings

In surveying and navigation, directions are generally given in terms of **bearings.** A bearing measures the acute angle that a path or line of sight makes with a fixed north-south line, as shown in Figure 4.82. For instance, the bearing S 35° E in Figure 4.82 means 35 degrees east of south.

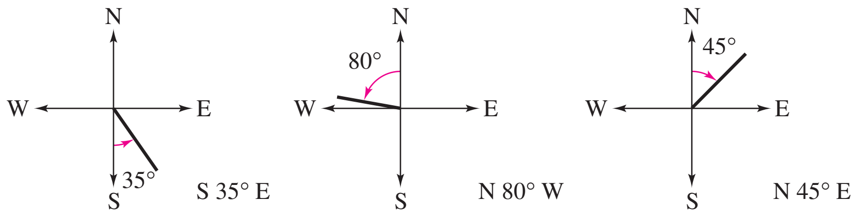


FIGURE 4.82

Example 5 Finding Directions in Terms of Bearings

A ship leaves port at noon and heads due west at 20 knots, or 20 nautical miles (nm) per hour. At 2 P.M. the ship changes course to N 54° W, as shown in Figure 4.83. Find the ship's bearing and distance from the port of departure at 3 P.M.

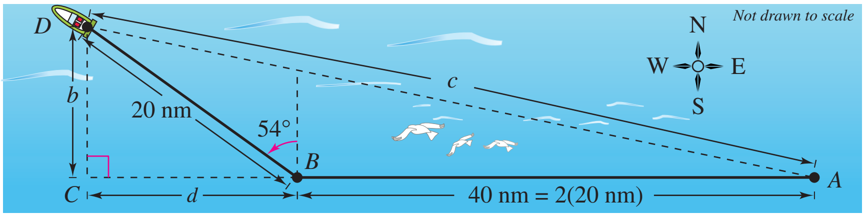


FIGURE 4.83

Solution

For triangle BCD, you have $B = 90° - 54° = 36°$. The two sides of this triangle can be determined to be

$$b = 20 \sin 36° \qquad \text{and} \qquad d = 20 \cos 36°.$$

For triangle ACD, you can find angle A as follows.

$$\tan A = \frac{b}{d + 40} = \frac{20 \sin 36°}{20 \cos 36° + 40} \approx 0.2092494$$

$$A \approx \arctan 0.2092494 \approx 0.2062732 \text{ radian} \approx 11.82°$$

The angle with the north-south line is $90° - 11.82° = 78.18°$. So, the bearing of the ship is N 78.18° W. Finally, from triangle ACD, you have $\sin A = b/c$, which yields

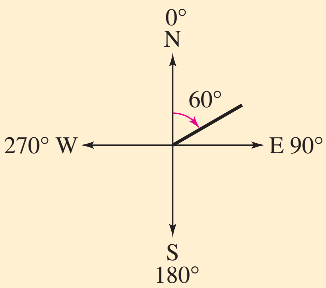
$$c = \frac{b}{\sin A} = \frac{20 \sin 36°}{\sin 11.82°}$$

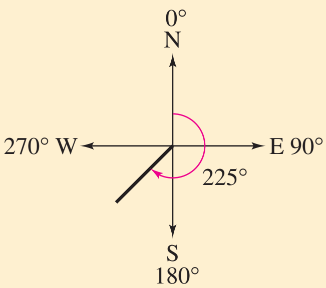
$$\approx 57.4 \text{ nautical miles.} \qquad \text{Distance from port}$$

CHECKPOINT Now try Exercise 31.

STUDY TIP

In *air navigation*, bearings are measured in degrees *clockwise* from north. Examples of air navigation bearings are shown below.





Harmonic Motion

The periodic nature of the trigonometric functions is useful for describing the motion of a point on an object that vibrates, oscillates, rotates, or is moved by wave motion.

For example, consider a ball that is bobbing up and down on the end of a spring, as shown in Figure 4.84. Suppose that 10 centimeters is the maximum distance the ball moves vertically upward or downward from its equilibrium (at rest) position. Suppose further that the time it takes for the ball to move from its maximum displacement above zero to its maximum displacement below zero and back again is $t = 4$ seconds. Assuming the ideal conditions of perfect elasticity and no friction or air resistance, the ball would continue to move up and down in a uniform and regular manner.

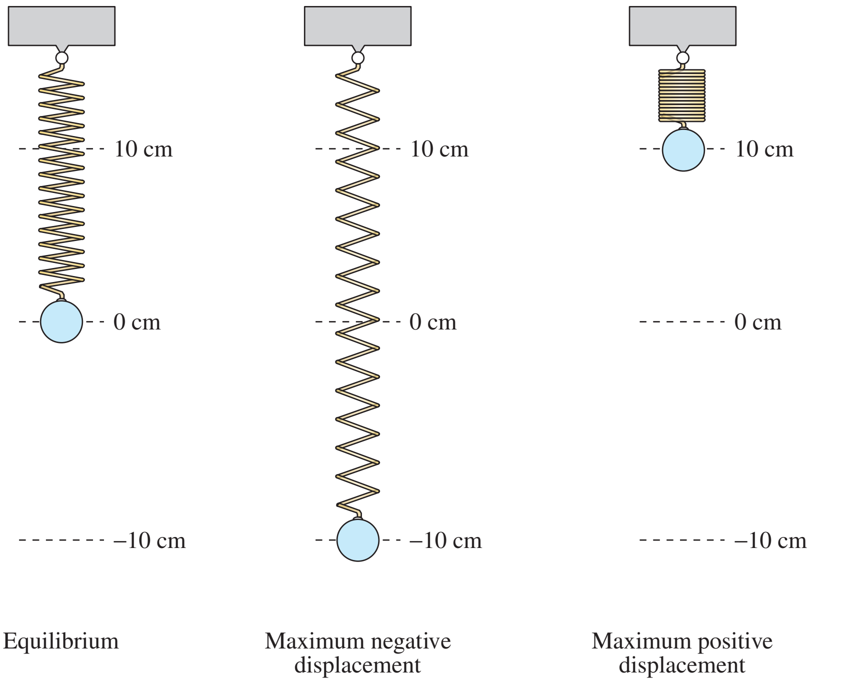


FIGURE 4.84

From this spring you can conclude that the period (time for one complete cycle) of the motion is

$$\text{Period} = 4 \text{ seconds}$$

its amplitude (maximum displacement from equilibrium) is

$$\text{Amplitude} = 10 \text{ centimeters}$$

and its **frequency** (number of cycles per second) is

$$\text{Frequency} = \frac{1}{4} \text{ cycle per second.}$$

Motion of this nature can be described by a sine or cosine function, and is called **simple harmonic motion.**

Definition of Simple Harmonic Motion

A point that moves on a coordinate line is said to be in **simple harmonic motion** if its distance d from the origin at time t is given by either

$$d = a \sin \omega t \qquad \text{or} \qquad d = a \cos \omega t$$

where a and ω are real numbers such that $\omega > 0$. The motion has amplitude $|a|$, period $2\pi/\omega$, and frequency $\omega/(2\pi)$.

Example 6 Simple Harmonic Motion

Write the equation for the simple harmonic motion of the ball described in Figure 4.84, where the period is 4 seconds. What is the frequency of this harmonic motion?

Solution

Because the spring is at equilibrium ($d = 0$) when $t = 0$, you use the equation

$$d = a \sin \omega t.$$

Moreover, because the maximum displacement from zero is 10 and the period is 4, you have

$$\text{Amplitude} = |a| = 10$$

$$\text{Period} = \frac{2\pi}{\omega} = 4 \quad \Rightarrow \quad \omega = \frac{\pi}{2}.$$

Consequently, the equation of motion is

$$d = 10 \sin \frac{\pi}{2} t.$$

Note that the choice of $a = 10$ or $a = -10$ depends on whether the ball initially moves up or down. The frequency is

$$\begin{aligned} \text{Frequency} &= \frac{\omega}{2\pi} \\ &= \frac{\pi/2}{2\pi} \\ &= \frac{1}{4} \text{ cycle per second.} \end{aligned}$$

CHECKPOINT Now try Exercise 51.

One illustration of the relationship between sine waves and harmonic motion can be seen in the wave motion resulting when a stone is dropped into a calm pool of water. The waves move outward in roughly the shape of sine (or cosine) waves, as shown in Figure 4.85. As an example, suppose you are fishing and your fishing bob is attached so that it does not move horizontally. As the waves move outward from the dropped stone, your fishing bob will move up and down in simple harmonic motion, as shown in Figure 4.86.

FIGURE 4.85

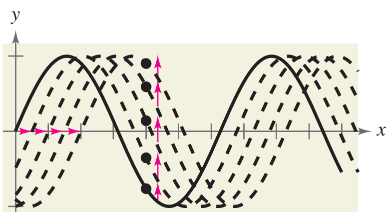


FIGURE 4.86

Example 7 Simple Harmonic Motion

Given the equation for simple harmonic motion

$$d = 6 \cos \frac{3\pi}{4} t$$

find (a) the maximum displacement, (b) the frequency, (c) the value of d when $t = 4$, and (d) the least positive value of t for which $d = 0$.

Algebraic Solution

The given equation has the form $d = a \cos \omega t$, with $a = 6$ and $\omega = 3\pi/4$.

a. The maximum displacement (from the point of equilibrium) is given by the amplitude. So, the maximum displacement is 6.

b. $$\text{Frequency} = \frac{\omega}{2\pi}$$

$$= \frac{3\pi/4}{2\pi} = \frac{3}{8} \text{ cycle per unit of time}$$

c. $$d = 6 \cos\left[\frac{3\pi}{4}(4)\right]$$

$$= 6 \cos 3\pi$$

$$= 6(-1)$$

$$= -6$$

d. To find the least positive value of t for which $d = 0$, solve the equation

$$d = 6 \cos \frac{3\pi}{4} t = 0.$$

First divide each side by 6 to obtain

$$\cos \frac{3\pi}{4} t = 0.$$

This equation is satisfied when

$$\frac{3\pi}{4} t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots.$$

Multiply these values by $4/(3\pi)$ to obtain

$$t = \frac{2}{3}, 2, \frac{10}{3}, \ldots.$$

So, the least positive value of t is $t = \frac{2}{3}$.

CHECKPOINT Now try Exercise 55.

Graphical Solution

Use a graphing utility set in *radian* mode to graph

$$y = 6 \cos \frac{3\pi}{4} x.$$

a. Use the *maximum* feature of the graphing utility to estimate that the maximum displacement from the point of equilibrium $y = 0$ is 6, as shown in Figure 4.87.

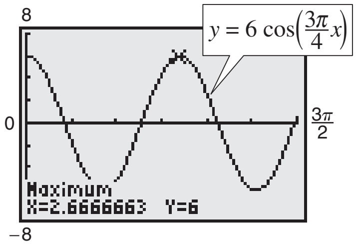


FIGURE 4.87

b. The period is the time for the graph to complete one cycle, which is $x \approx 2.667$. You can estimate the frequency as follows.

$$\text{Frequency} \approx \frac{1}{2.667} \approx 0.375 \text{ cycle per unit of time}$$

c. Use the *trace* feature to estimate that the value of y when $x = 4$ is $y = -6$, as shown in Figure 4.88.

d. Use the *zero* or *root* feature to estimate that the least positive value of x for which $y = 0$ is $x \approx 0.6667$, as shown in Figure 4.89.

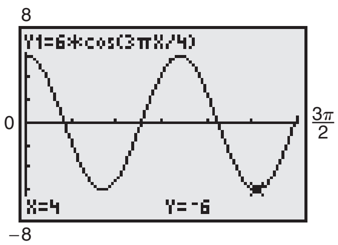


FIGURE 4.88

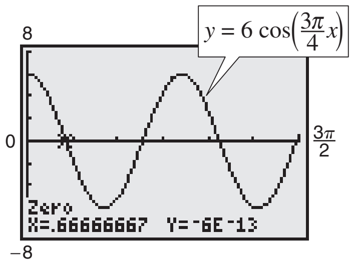


FIGURE 4.89

4.8 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. An angle that measures from the horizontal upward to an object is called the angle of ________, whereas an angle that measures from the horizontal downward to an object is called the angle of ________.
2. A ________ measures the acute angle a path or line of sight makes with a fixed north-south line.
3. A point that moves on a coordinate line is said to be in simple ________ ________ if its distance d from the origin at time t is given by either $d = a \sin \omega t$ or $d = a \cos \omega t$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, solve the right triangle shown in the figure. Round your answers to two decimal places.

1. $A = 20°, \quad b = 10$
2. $B = 54°, \quad c = 15$
3. $B = 71°, \quad b = 24$
4. $A = 8.4°, \quad a = 40.5$
5. $a = 6, \quad b = 10$
6. $a = 25, \quad c = 35$
7. $b = 16, \quad c = 52$
8. $b = 1.32, \quad c = 9.45$
9. $A = 12°15', \quad c = 430.5$
10. $B = 65°12', \quad a = 14.2$

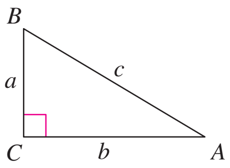


FIGURE FOR 1–10

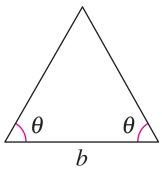
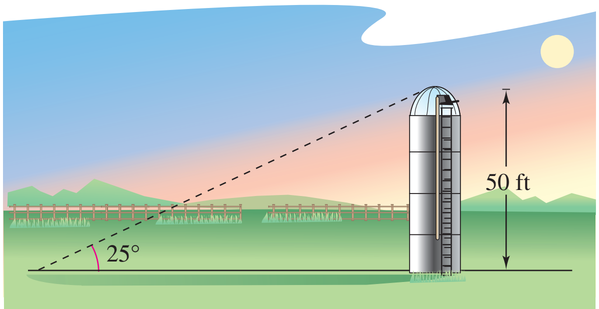


FIGURE FOR 11–14

In Exercises 11–14, find the altitude of the isosceles triangle shown in the figure. Round your answers to two decimal places.

11. $\theta = 52°, \quad b = 4$ inches
12. $\theta = 18°, \quad b = 10$ meters
13. $\theta = 41°, \quad b = 46$ inches
14. $\theta = 27°, \quad b = 11$ feet

15. ***Length*** The sun is 25° above the horizon. Find the length of a shadow cast by a silo that is 50 feet tall (see figure).



16. ***Length*** The sun is 20° above the horizon. Find the length of a shadow cast by a building that is 600 feet tall.

17. ***Height*** A ladder 20 feet long leans against the side of a house. Find the height from the top of the ladder to the ground if the angle of elevation of the ladder is 80°.

18. ***Height*** The length of a shadow of a tree is 125 feet when the angle of elevation of the sun is 33°. Approximate the height of the tree.

19. ***Height*** From a point 50 feet in front of a church, the angles of elevation to the base of the steeple and the top of the steeple are 35° and 47° 40′, respectively.

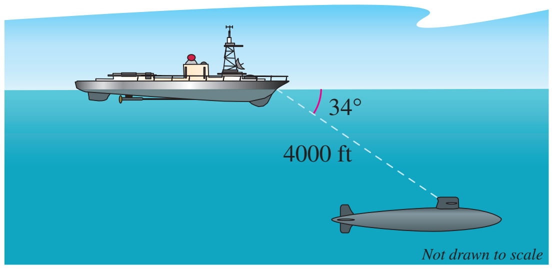
    (a) Draw right triangles that give a visual representation of the problem. Label the known and unknown quantities.

    (b) Use a trigonometric function to write an equation involving the unknown quantity.

    (c) Find the height of the steeple.

20. ***Height*** You are standing 100 feet from the base of a platform from which people are bungee jumping. The angle of elevation from your position to the top of the platform from which they jump is 51°. From what height are the people jumping?

21. ***Depth*** The sonar of a navy cruiser detects a submarine that is 4000 feet from the cruiser. The angle between the water line and the submarine is 34° (see figure). How deep is the submarine?



22. ***Angle of Elevation*** An engineer erects a 75-foot cellular telephone tower. Find the angle of elevation to the top of the tower at a point on level ground 50 feet from its base.

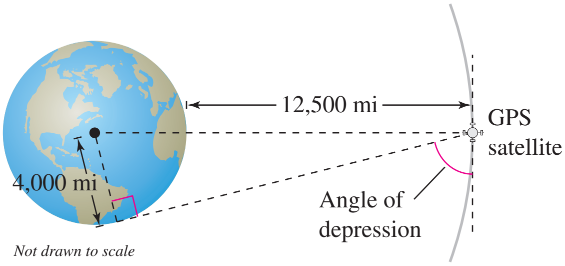
23. ***Angle of Elevation*** The height of an outdoor basketball backboard is $12\frac{1}{2}$ feet, and the backboard casts a shadow $17\frac{1}{3}$ feet long.

(a) Draw a right triangle that gives a visual representation of the problem. Label the known and unknown quantities.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Find the angle of elevation of the sun.

24. ***Angle of Depression*** A Global Positioning System satellite orbits 12,500 miles above Earth's surface (see figure). Find the angle of depression from the satellite to the horizon. Assume the radius of Earth is 4000 miles.



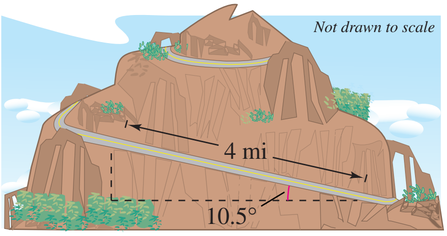
25. ***Angle of Depression*** A cellular telephone tower that is 150 feet tall is placed on top of a mountain that is 1200 feet above sea level. What is the angle of depression from the top of the tower to a cell phone user who is 5 horizontal miles away and 400 feet above sea level?

26. ***Airplane Ascent*** During takeoff, an airplane's angle of ascent is 18° and its speed is 275 feet per second.

(a) Find the plane's altitude after 1 minute.

(b) How long will it take the plane to climb to an altitude of 10,000 feet?

27. ***Mountain Descent*** A sign on a roadway at the top of a mountain indicates that for the next 4 miles the grade is 10.5° (see figure). Find the change in elevation over that distance for a car descending the mountain.



28. ***Mountain Descent*** A roadway sign at the top of a mountain indicates that for the next 4 miles the grade is 12%. Find the angle of the grade and the change in elevation over the 4 miles for a car descending the mountain.

29. ***Navigation*** An airplane flying at 600 miles per hour has a bearing of 52°. After flying for 1.5 hours, how far north and how far east will the plane have traveled from its point of departure?

30. ***Navigation*** A jet leaves Reno, Nevada and is headed toward Miami, Florida at a bearing of 100°. The distance between the two cities is approximately 2472 miles.

(a) How far north and how far west is Reno relative to Miami?

(b) If the jet is to return directly to Reno from Miami, at what bearing should it travel?

31. ***Navigation*** A ship leaves port at noon and has a bearing of S 29° W. The ship sails at 20 knots.

(a) How many nautical miles south and how many nautical miles west will the ship have traveled by 6:00 P.M.?

(b) At 6:00 P.M., the ship changes course to due west. Find the ship's bearing and distance from the port of departure at 7:00 P.M.

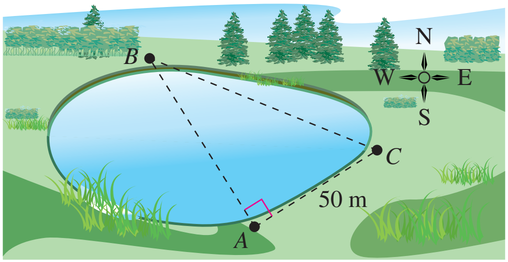
32. ***Navigation*** A privately owned yacht leaves a dock in Myrtle Beach, South Carolina and heads toward Freeport in the Bahamas at a bearing of S 1.4° E. The yacht averages a speed of 20 knots over the 428 nautical-mile trip.

(a) How long will it take the yacht to make the trip?

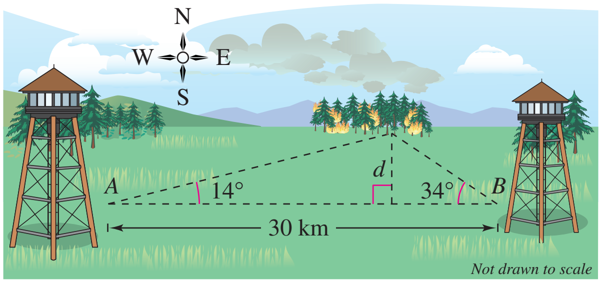
(b) How far east and south is the yacht after 12 hours?

(c) If a plane leaves Myrtle Beach to fly to Freeport, what bearing should be taken?

33. ***Surveying*** A surveyor wants to find the distance across a swamp (see figure). The bearing from A to B is N 32° W. The surveyor walks 50 meters from A, and at the point C the bearing to B is N 68° W. Find (a) the bearing from A to C and (b) the distance from A to B.



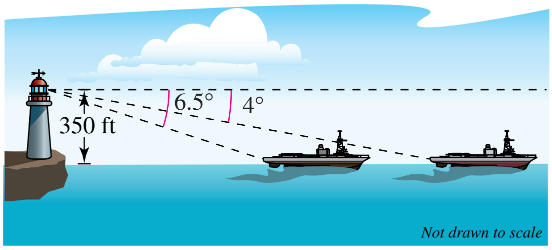
34. ***Location of a Fire*** Two fire towers are 30 kilometers apart, where tower A is due west of tower B. A fire is spotted from the towers, and the bearings from A and B are E 14° N and W 34° N, respectively (see figure). Find the distance d of the fire from the line segment AB.



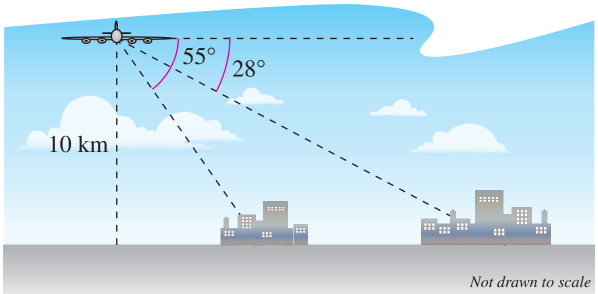
35. ***Navigation*** A ship is 45 miles east and 30 miles south of port. The captain wants to sail directly to port. What bearing should be taken?

36. ***Navigation*** An airplane is 160 miles north and 85 miles east of an airport. The pilot wants to fly directly to the airport. What bearing should be taken?

37. ***Distance*** An observer in a lighthouse 350 feet above sea level observes two ships directly offshore. The angles of depression to the ships are 4° and 6.5° (see figure). How far apart are the ships?



38. ***Distance*** A passenger in an airplane at an altitude of 10 kilometers sees two towns directly to the east of the plane. The angles of depression to the towns are 28° and 55° (see figure). How far apart are the towns?



39. ***Altitude*** A plane is observed approaching your home and you assume that its speed is 550 miles per hour. The angle of elevation of the plane is 16° at one time and 57° one minute later. Approximate the altitude of the plane.

40. ***Height*** While traveling across flat land, you notice a mountain directly in front of you. The angle of elevation to the peak is 2.5°. After you drive 17 miles closer to the mountain, the angle of elevation is 9°. Approximate the height of the mountain.

Geometry **In Exercises 41 and 42, find the angle α between two nonvertical lines L_1 and L_2. The angle α satisfies the equation**

$$\tan \alpha = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectively. (Assume that $m_1 m_2 \neq -1$.)

41. L_1: $3x - 2y = 5$
L_2: $x + y = 1$

42. L_1: $2x - y = 8$
L_2: $x - 5y = -4$

43. ***Geometry*** Determine the angle between the diagonal of a cube and the diagonal of its base, as shown in the figure.

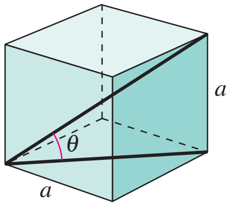


FIGURE FOR 43

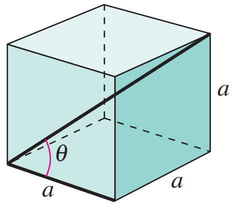


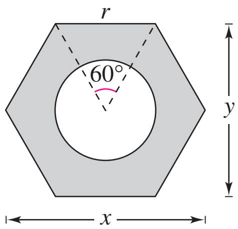
FIGURE FOR 44

44. ***Geometry*** Determine the angle between the diagonal of a cube and its edge, as shown in the figure.

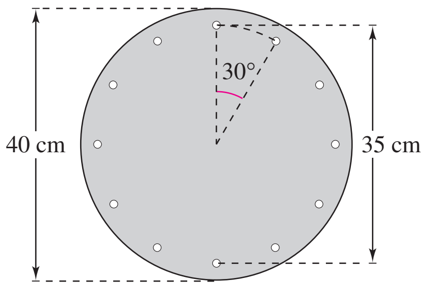
45. ***Geometry*** Find the length of the sides of a regular pentagon inscribed in a circle of radius 25 inches.

46. ***Geometry*** Find the length of the sides of a regular hexagon inscribed in a circle of radius 25 inches.

47. ***Hardware*** Write the distance y across the flat sides of a hexagonal nut as a function of r, as shown in the figure.

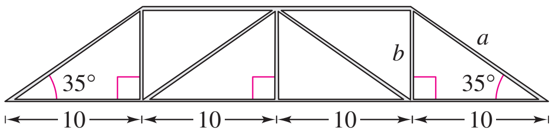


48. ***Bolt Holes*** The figure shows a circular piece of sheet metal that has a diameter of 40 centimeters and contains 12 equally spaced bolt holes. Determine the straight-line distance between the centers of consecutive bolt holes.

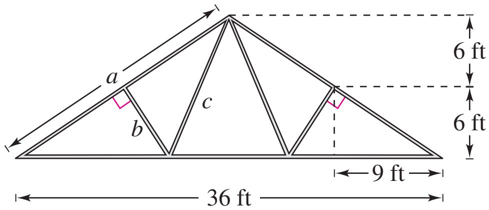


Trusses **In Exercises 49 and 50, find the lengths of all the unknown members of the truss.**

49.



50.



Harmonic Motion **In Exercises 51–54, find a model for simple harmonic motion satisfying the specified conditions.**

	Displacement $(t = 0)$	*Amplitude*	*Period*
51.	0	4 centimeters	2 seconds
52.	0	3 meters	6 seconds
53.	3 inches	3 inches	1.5 seconds
54.	2 feet	2 feet	10 seconds

Harmonic Motion **In Exercises 55–58, for the simple harmonic motion described by the trigonometric function, find (a) the maximum displacement, (b) the frequency, (c) the value of d when $t = 5$, and (d) the least positive value of t for which $d = 0$. Use a graphing utility to verify your results.**

55. $d = 4 \cos 8\pi t$

56. $d = \frac{1}{2} \cos 20\pi t$

57. $d = \frac{1}{16} \sin 120\pi t$

58. $d = \frac{1}{64} \sin 792\pi t$

59. ***Tuning Fork*** A point on the end of a tuning fork moves in simple harmonic motion described by $d = a \sin \omega t$. Find ω given that the tuning fork for middle C has a frequency of 264 vibrations per second.

60. ***Wave Motion*** A buoy oscillates in simple harmonic motion as waves go past. It is noted that the buoy moves a total of 3.5 feet from its low point to its high point (see figure), and that it returns to its high point every 10 seconds. Write an equation that describes the motion of the buoy if its high point is at $t = 0$.

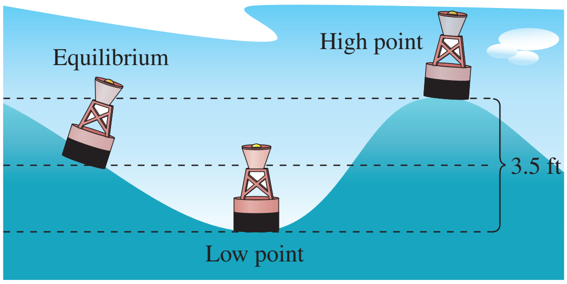


FIGURE FOR 60

61. ***Oscillation of a Spring*** A ball that is bobbing up and down on the end of a spring has a maximum displacement of 3 inches. Its motion (in ideal conditions) is modeled by $y = \frac{1}{4} \cos 16t$ $(t > 0)$, where y is measured in feet and t is the time in seconds.

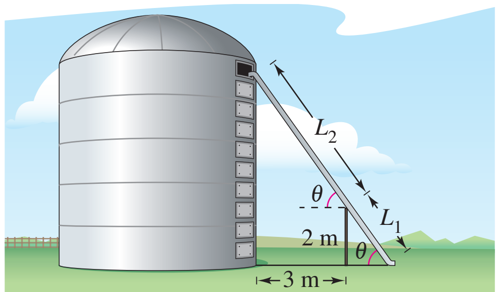
(a) Graph the function.

(b) What is the period of the oscillations?

(c) Determine the first time the weight passes the point of equilibrium $(y = 0)$.

Model It

62. ***Numerical and Graphical Analysis*** A two-meter-high fence is 3 meters from the side of a grain storage bin. A grain elevator must reach from ground level outside the fence to the storage bin (see figure). The objective is to determine the shortest elevator that meets the constraints.



(a) Complete four rows of the table.

θ	L_1	L_2	$L_1 + L_2$
0.1	$\frac{2}{\sin 0.1}$	$\frac{3}{\cos 0.1}$	23.0
0.2	$\frac{2}{\sin 0.2}$	$\frac{3}{\cos 0.2}$	13.1

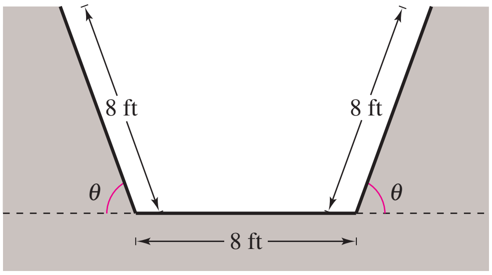
Model It (continued)

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum length of the elevator.

(c) Write the length $L_1 + L_2$ as a function of θ.

(d) Use a graphing utility to graph the function. Use the graph to estimate the minimum length. How does your estimate compare with that of part (b)?

63. ***Numerical and Graphical Analysis*** The cross section of an irrigation canal is an isosceles trapezoid of which three of the sides are 8 feet long (see figure). The objective is to find the angle θ that maximizes the area of the cross section. [*Hint:* The area of a trapezoid is $(h/2)(b_1 + b_2)$.]



(a) Complete seven additional rows of the table.

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10°$	$8 \sin 10°$	22.1
8	$8 + 16 \cos 20°$	$8 \sin 20°$	42.5

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the maximum cross-sectional area.

(c) Write the area A as a function of θ.

(d) Use a graphing utility to graph the function. Use the graph to estimate the maximum cross-sectional area. How does your estimate compare with that of part (b)?

64. ***Data Analysis*** The table shows the average sales S (in millions of dollars) of an outerwear manufacturer for each month t, where $t = 1$ represents January.

Time, t	1	2	3	4	5	6
Sales, s	13.46	11.15	8.00	4.85	2.54	1.70

Time, t	7	8	9	10	11	12
Sales, s	2.54	4.85	8.00	11.15	13.46	14.3

(a) Create a scatter plot of the data.

(b) Find a trigonometric model that fits the data. Graph the model with your scatter plot. How well does the model fit the data?

(c) What is the period of the model? Do you think it is reasonable given the context? Explain your reasoning.

(d) Interpret the meaning of the model's amplitude in the context of the problem.

Synthesis

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

65. The Leaning Tower of Pisa is not vertical, but if you know the exact angle of elevation θ to the 191-foot tower when you stand near it, then you can determine the exact distance to the tower d by using the formula

$$\tan \theta = \frac{191}{d}.$$

66. For the harmonic motion of a ball bobbing up and down on the end of a spring, one period can be described as the length of one coil of the spring.

67. ***Writing*** Is it true that N 24° E means 24 degrees north of east? Explain.

68. ***Writing*** Explain the difference between bearings used in nautical navigation and bearings used in air navigation.

Skills Review

In Exercises 69–72, write the slope-intercept form of the equation of the line with the specified characteristics. Then sketch the line.

69. $m = 4$, passes through $(-1, 2)$

70. $m = -\frac{1}{2}$, passes through $\left(\frac{1}{3}, 0\right)$

71. Passes through $(-2, 6)$ and $(3, 2)$

72. Passes through $\left(\frac{1}{4}, -\frac{2}{3}\right)$ and $\left(-\frac{1}{2}, \frac{1}{3}\right)$

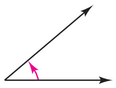
4 Chapter Summary

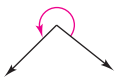
What did you learn?

Section 4.1	Review Exercises
☐ Describe angles *(p. 282)*.	1, 2
☐ Use radian measure *(p. 283)*.	3–6, 11–18
☐ Use degree measure *(p. 285)*.	7–18
☐ Use angles to model and solve real-life problems *(p. 287)*.	19–24
Section 4.2	
☐ Identify a unit circle and describe its relationship to real numbers *(p. 294)*.	25–28
☐ Evaluate trigonometric functions using the unit circle *(p. 295)*.	29–32
☐ Use domain and period to evaluate sine and cosine functions *(p. 297)*.	33–36
☐ Use a calculator to evaluate trigonometric functions *(p. 298)*.	37–40
Section 4.3	
☐ Evaluate trigonometric functions of acute angles *(p. 301)*.	41–44
☐ Use the fundamental trigonometric identities *(p. 304)*.	45–48
☐ Use a calculator to evaluate trigonometric functions *(p. 305)*.	49–54
☐ Use trigonometric functions to model and solve real-life problems *(p. 306)*.	55, 56
Section 4.4	
☐ Evaluate trigonometric functions of any angle *(p. 312)*.	57–70
☐ Use reference angles to evaluate trigonometric functions *(p. 314)*.	71–82
☐ Evaluate trigonometric functions of real numbers *(p. 315)*.	83–88
Section 4.5	
☐ Use amplitude and period to help sketch the graphs of sine and cosine functions *(p. 323)*.	89–92
☐ Sketch translations of the graphs of sine and cosine functions *(p. 325)*.	93–96
☐ Use sine and cosine functions to model real-life data *(p. 327)*.	97, 98
Section 4.6	
☐ Sketch the graphs of tangent *(p. 332)* and cotangent *(p. 334)* functions.	99–102
☐ Sketch the graphs of secant and cosecant functions *(p. 335)*.	103–106
☐ Sketch the graphs of damped trigonometric functions *(p. 337)*.	107, 108
Section 4.7	
☐ Evaluate and graph the inverse sine function *(p. 343)*.	109–114, 123, 126
☐ Evaluate and graph the other inverse trigonometric functions *(p. 345)*.	115–122, 124, 125
☐ Evaluate compositions of trigonometric functions *(p. 347)*.	127–132
Section 4.8	
☐ Solve real-life problems involving right triangles *(p. 353)*.	133, 134
☐ Solve real-life problems involving directional bearings *(p. 355)*.	135
☐ Solve real-life problems involving harmonic motion *(p. 356)*.	136

4 Review Exercises

4.1 In Exercises 1 and 2, estimate the angle to the nearest one-half radian.

1. 

2. 

In Exercises 3–10, (a) sketch the angle in standard position, (b) determine the quadrant in which the angle lies, and (c) determine one positive and one negative coterminal angle.

3. $\frac{11\pi}{4}$ **4.** $\frac{2\pi}{9}$

5. $-\frac{4\pi}{3}$ **6.** $-\frac{23\pi}{3}$

7. $70°$ **8.** $280°$

9. $-110°$ **10.** $-405°$

In Exercises 11–14, convert the angle measure from degrees to radians. Round your answer to three decimal places.

11. $480°$ **12.** $-127.5°$

13. $-33° 45'$ **14.** $196° 77'$

In Exercises 15–18, convert the angle measure from radians to degrees. Round your answer to three decimal places.

15. $\frac{5\pi}{7}$ **16.** $-\frac{11\pi}{6}$

17. -3.5 **18.** 5.7

19. ***Arc Length*** Find the length of the arc on a circle with a radius of 20 inches intercepted by a central angle of $138°$.

20. ***Arc Length*** Find the length of the arc on a circle with a radius of 11 meters intercepted by a central angle of $60°$.

21. ***Phonograph*** Compact discs have all but replaced phonograph records. Phonograph records are vinyl discs that rotate on a turntable. A typical record album is 12 inches in diameter and plays at $33\frac{1}{3}$ revolutions per minute.

(a) What is the angular speed of a record album?

(b) What is the linear speed of the outer edge of a record album?

22. ***Bicycle*** At what speed is a bicyclist traveling when his 27-inch-diameter tires are rotating at an angular speed of 5π radians per second?

23. ***Circular Sector*** Find the area of the sector of a circle with a radius of 18 inches and central angle $\theta = 120°$.

24. ***Circular Sector*** Find the area of the sector of a circle with a radius of 6.5 millimeters and central angle $\theta = 5\pi/6$.

4.2 In Exercises 25–28, find the point (x, y) on the unit circle that corresponds to the real number t.

25. $t = \frac{2\pi}{3}$ **26.** $t = \frac{3\pi}{4}$

27. $t = \frac{5\pi}{6}$ **28.** $t = -\frac{4\pi}{3}$

In Exercises 29–32, evaluate (if possible) the six trigonometric functions of the real number.

29. $t = \frac{7\pi}{6}$ **30.** $t = \frac{\pi}{4}$

31. $t = -\frac{2\pi}{3}$ **32.** $t = 2\pi$

In Exercises 33–36, evaluate the trigonometric function using its period as an aid.

33. $\sin \frac{11\pi}{4}$ **34.** $\cos 4\pi$

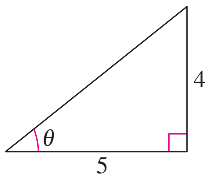
35. $\sin\left(-\frac{17\pi}{6}\right)$ **36.** $\cos\left(-\frac{13\pi}{3}\right)$

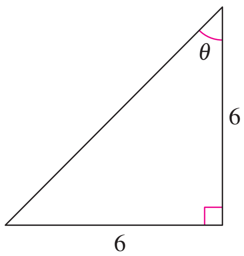
In Exercises 37–40, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places.

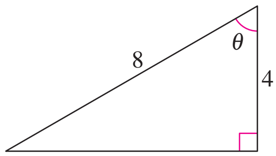
37. $\tan 33$ **38.** $\csc 10.5$

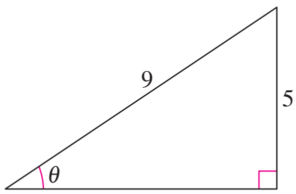
39. $\sec \frac{12\pi}{5}$ **40.** $\sin\left(-\frac{\pi}{9}\right)$

4.3 In Exercises 41–44, find the exact values of the six trigonometric functions of the angle θ shown in the figure.

41. 

42. 

43. 

44. 

In Exercises 45–48, use the given function value and trigonometric identities (including the cofunction identities) to find the indicated trigonometric functions.

45. $\sin\theta = \frac{1}{3}$ (a) $\csc\theta$ (b) $\cos\theta$ (c) $\sec\theta$ (d) $\tan\theta$

46. $\tan\theta = 4$ (a) $\cot\theta$ (b) $\sec\theta$ (c) $\cos\theta$ (d) $\csc\theta$

47. $\csc\theta = 4$ (a) $\sin\theta$ (b) $\cos\theta$ (c) $\sec\theta$ (d) $\tan\theta$

48. $\csc\theta = 5$ (a) $\sin\theta$ (b) $\cot\theta$ (c) $\tan\theta$ (d) $\sec(90° - \theta)$

In Exercises 49–54, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places.

49. $\tan 33°$

50. $\csc 11°$

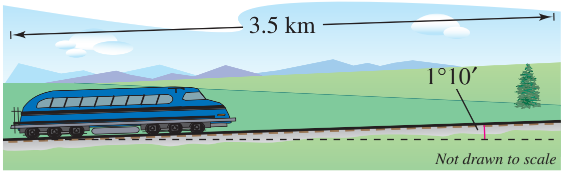
51. $\sin 34.2°$

52. $\sec 79.3°$

53. $\cot 15°\,14'$

54. $\cos 78°\,11'58''$

55. ***Railroad Grade*** A train travels 3.5 kilometers on a straight track with a grade of $1°\,10'$ (see figure). What is the vertical rise of the train in that distance?



56. ***Guy Wire*** A guy wire runs from the ground to the top of a 25-foot telephone pole. The angle formed between the wire and the ground is 52°. How far from the base of the pole is the wire attached to the ground?

4.4 In Exercises 57–64, the point is on the terminal side of an angle θ in standard position. Determine the exact values of the six trigonometric functions of the angle θ.

57. $(12, 16)$

58. $(3, -4)$

59. $\left(\frac{2}{3}, \frac{5}{2}\right)$

60. $\left(-\frac{10}{3}, -\frac{2}{3}\right)$

61. $(-0.5, 4.5)$

62. $(0.3, 0.4)$

63. $(x, 4x),\ x > 0$

64. $(-2x, -3x),\ x > 0$

In Exercises 65–70, find the values of the six trigonometric functions of θ.

	Function Value	*Constraint*
65.	$\sec\theta = \frac{6}{5}$	$\tan\theta < 0$
66.	$\csc\theta = \frac{3}{2}$	$\cos\theta < 0$
67.	$\sin\theta = \frac{3}{8}$	$\cos\theta < 0$
68.	$\tan\theta = \frac{5}{4}$	$\cos\theta < 0$
69.	$\cos\theta = -\frac{2}{5}$	$\sin\theta > 0$
70.	$\sin\theta = -\frac{2}{4}$	$\cos\theta > 0$

In Exercises 71–74, find the reference angle θ', and sketch θ and θ' in standard position.

71. $\theta = 264°$

72. $\theta = 635°$

73. $\theta = -\dfrac{6\pi}{5}$

74. $\theta = \dfrac{17\pi}{3}$

In Exercises 75–82, evaluate the sine, cosine, and tangent of the angle without using a calculator.

75. $\dfrac{\pi}{3}$

76. $\dfrac{\pi}{4}$

77. $-\dfrac{7\pi}{3}$

78. $-\dfrac{5\pi}{4}$

79. $495°$

80. $-150°$

81. $-240°$

82. $315°$

In Exercises 83–88, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places.

83. $\sin 4$

84. $\tan 3$

85. $\sin(-3.2)$

86. $\cot(-4.8)$

85. $\sin\dfrac{12\pi}{5}$

88. $\tan\left(-\dfrac{25\pi}{7}\right)$

4.5 In Exercises 89–96, sketch the graph of the function. Include two full periods.

89. $y = \sin x$

90. $y = \cos x$

91. $f(x) = 5\sin\dfrac{2x}{5}$

92. $f(x) = 8\cos\left(-\dfrac{x}{4}\right)$

93. $y = 2 + \sin x$

94. $y = -4 - \cos\pi x$

95. $g(t) = \frac{5}{2}\sin(t - \pi)$

96. $g(t) = 3\cos(t + \pi)$

97. ***Sound Waves*** Sound waves can be modeled by sine functions of the form $y = a\sin bx$, where x is measured in seconds.

(a) Write an equation of a sound wave whose amplitude is 2 and whose period is $\frac{1}{264}$ second.

(b) What is the frequency of the sound wave described in part (a)?

98. ***Data Analysis: Meteorology*** The times S of sunset (Greenwich Mean Time) at 40° north latitude on the 15th of each month are: 1(16:59), 2(17:35), 3(18:06), 4(18:38), 5(19:08), 6(19:30), 7(19:28), 8(18:57), 9(18:09), 10(17:21), 11(16:44), 12(16:36). The month is represented by t, with $t = 1$ corresponding to January. A model (in which minutes have been converted to the decimal parts of an hour) for the data is

$$S(t) = 18.09 + 1.41 \sin\left(\frac{\pi t}{6} + 4.60\right).$$

(a) Use a graphing utility to graph the data points and the model in the same viewing window.

(b) What is the period of the model? Is it what you expected? Explain.

(c) What is the amplitude of the model? What does it represent in the model? Explain.

4.6 **In Exercises 99–106, sketch a graph of the function. Include two full periods.**

99. $f(x) = \tan x$

100. $f(t) = \tan\left(t - \frac{\pi}{4}\right)$

101. $f(x) = \cot x$

102. $g(t) = 2 \cot 2t$

103. $f(x) = \sec x$

104. $h(t) = \sec\left(t - \frac{\pi}{4}\right)$

105. $f(x) = \csc x$

106. $f(t) = 3 \csc\left(2t + \frac{\pi}{4}\right)$

In Exercises 107 and 108, use a graphing utility to graph the function and the damping factor of the function in the same viewing window. Describe the behavior of the function as x increases without bound.

107. $f(x) = x \cos x$

108. $g(x) = x^4 \cos x$

4.7 **In Exercises 109–114, evaluate the expression. If necessary, round your answer to two decimal places.**

109. $\arcsin\left(-\frac{1}{2}\right)$

110. $\arcsin(-1)$

111. $\arcsin 0.4$

112. $\arcsin 0.213$

113. $\sin^{-1}(-0.44)$

114. $\sin^{-1} 0.89$

In Exercises 115–118, evaluate the expression without the aid of a calculator.

115. $\arccos \frac{\sqrt{3}}{2}$

116. $\arccos \frac{\sqrt{2}}{2}$

117. $\cos^{-1}(-1)$

118. $\cos^{-1} \frac{\sqrt{3}}{2}$

In Exercises 119–122, use a calculator to evaluate the expression. Round your answer to two decimal places.

119. $\arccos 0.324$

120. $\arccos(-0.888)$

121. $\tan^{-1}(-1.5)$

122. $\tan^{-1} 8.2$

In Exercises 123–126, use a graphing utility to graph the function.

123. $f(x) = 2 \arcsin x$

124. $f(x) = 3 \arccos x$

125. $f(x) = \arctan \frac{x}{2}$

126. $f(x) = -\arcsin 2x$

In Exercises 127–130, find the exact value of the expression.

127. $\cos\left(\arctan \frac{3}{4}\right)$

128. $\tan\left(\arccos \frac{3}{5}\right)$

129. $\sec\left(\arctan \frac{12}{5}\right)$

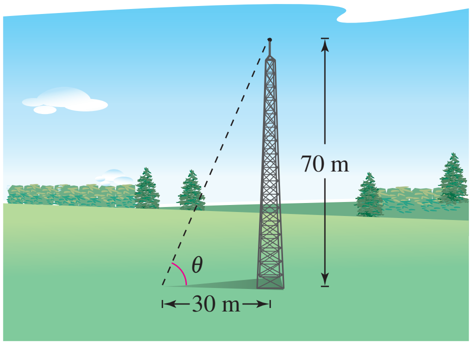
130. $\cot\left[\arcsin\left(-\frac{12}{13}\right)\right]$

In Exercises 131 and 132, write an algebraic expression that is equivalent to the expression.

131. $\tan\left(\arccos \frac{x}{2}\right)$

132. $\sec[\arcsin(x - 1)]$

4.8 **133.** ***Angle of Elevation*** The height of a radio transmission tower is 70 meters, and it casts a shadow of length 30 meters (see figure). Find the angle of elevation of the sun.



134. ***Height*** Your football has landed at the edge of the roof of your school building. When you are 25 feet from the base of the building, the angle of elevation to your football is 21°. How high off the ground is your football?

135. ***Distance*** From city A to city B, a plane flies 650 miles at a bearing of 48°. From city B to city C, the plane flies 810 miles at a bearing of 115°. Find the distance from city A to city C and the bearing from city A to city C.

136. ***Wave Motion*** Your fishing bobber oscillates in simple harmonic motion from the waves in the lake where you fish. Your bobber moves a total of 1.5 inches from its high point to its low point and returns to its high point every 3 seconds. Write an equation modeling the motion of your bobber if it is at its high point at time $t = 0$.

Synthesis

True or False? **In Exercises 137–140, determine whether the statement is true or false. Justify your answer.**

137. The tangent function is often useful for modeling simple harmonic motion.

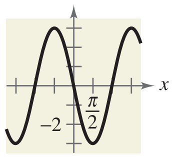
138. The inverse sine function $y = \arcsin x$ cannot be defined as a function over any interval that is greater than the interval defined as $-\pi/2 \le y \le \pi/2$.

139. $y = \sin \theta$ is not a function because $\sin 30° = \sin 150°$.

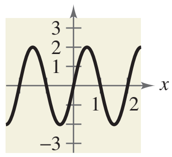
140. Because $\tan 3\pi/4 = -1$, $\arctan(-1) = 3\pi/4$.

In Exercises 141–144, match the function $y = a \sin bx$ with its graph. Base your selection solely on your interpretation of the constants a and b. Explain your reasoning. [The graphs are labeled (a), (b), (c), and (d).]

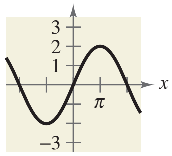
(a)



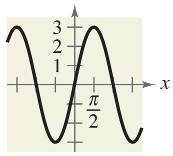
(b)



(c)



(d)



141. $y = 3 \sin x$

142. $y = -3 \sin x$

143. $y = 2 \sin \pi x$

144. $y = 2 \sin \dfrac{x}{2}$

145. ***Writing*** Describe the behavior of $f(\theta) = \sec \theta$ at the zeros of $g(\theta) = \cos \theta$. Explain your reasoning.

146. ***Conjecture***

(a) Use a graphing utility to complete the table.

θ	0.1	0.4	0.7	1.0	1.3
$\tan\left(\theta - \dfrac{\pi}{2}\right)$					
$-\cot \theta$					

(b) Make a conjecture about the relationship between $\tan\left(\theta - \dfrac{\pi}{2}\right)$ and $-\cot \theta$.

147. ***Writing*** When graphing the sine and cosine functions, determining the amplitude is part of the analysis. Explain why this is not true for the other four trigonometric functions.

148. ***Oscillation of a Spring*** A weight is suspended from a ceiling by a steel spring. The weight is lifted (positive direction) from the equilibrium position and released. The resulting motion of the weight is modeled by

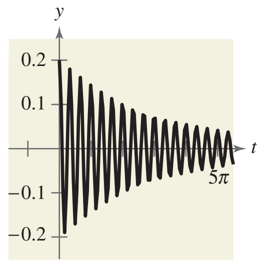
$$y = Ae^{-kt} \cos bt = \tfrac{1}{5}e^{-t/10} \cos 6t$$

where y is the distance in feet from equilibrium and t is the time in seconds. The graph of the function is shown in the figure. For each of the following, describe the change in the system without graphing the resulting function.

(a) A is changed from $\frac{1}{5}$ to $\frac{1}{3}$.

(b) k is changed from $\frac{1}{10}$ to $\frac{1}{3}$.

(c) b is changed from 6 to 9.



149. ***Graphical Reasoning*** The formulas for the area of a circular sector and arc length are $A = \frac{1}{2}r^2\theta$ and $s = r\theta$, respectively. (r is the radius and θ is the angle measured in radians.)

(a) For $\theta = 0.8$, write the area and arc length as functions of r. What is the domain of each function? Use a graphing utility to graph the functions. Use the graphs to determine which function changes more rapidly as r increases. Explain.

(b) For $r = 10$ centimeters, write the area and arc length as functions of θ. What is the domain of each function? Use a graphing utility to graph and identify the functions.

150. ***Writing*** Describe a real-life application that can be represented by a simple harmonic motion model and is different from any that you've seen in this chapter. Explain which function you would use to model your application and why. Explain how you would determine the amplitude, period, and frequency of the model for your application.

4 Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

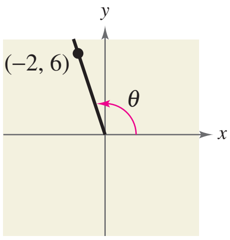


FIGURE FOR 4

1. Consider an angle that measures $\frac{5\pi}{4}$ radians.

(a) Sketch the angle in standard position.

(b) Determine two coterminal angles (one positive and one negative).

(c) Convert the angle to degree measure.

2. A truck is moving at a rate of 90 kilometers per hour, and the diameter of its wheels is 1 meter. Find the angular speed of the wheels in radians per minute.

3. A water sprinkler sprays water on a lawn over a distance of 25 feet and rotates through an angle of 130°. Find the area of the lawn watered by the sprinkler.

4. Find the exact values of the six trigonometric functions of the angle θ shown in the figure.

5. Given that $\tan\theta = \frac{3}{2}$, find the other five trigonometric functions of θ.

6. Determine the reference angle θ' of the angle $\theta = 290°$ and sketch θ and θ' in standard position.

7. Determine the quadrant in which θ lies if $\sec\theta < 0$ and $\tan\theta > 0$.

8. Find two exact values of θ in degrees $(0 \leq \theta < 360°)$ if $\cos\theta = -\sqrt{3}/2$. (Do not use a calculator.)

9. Use a calculator to approximate two values of θ in radians $(0 \leq \theta < 2\pi)$ if $\csc\theta = 1.030$. Round the results to two decimal places.

In Exercises 10 and 11, find the remaining five trigonometric functions of θ satisfying the conditions.

10. $\cos\theta = \frac{3}{5}, \quad \tan\theta < 0$

11. $\sec\theta = -\frac{17}{8}, \quad \sin\theta > 0$

In Exercises 12 and 13, sketch the graph of the function. (Include two full periods.)

12. $g(x) = -2\sin\left(x - \frac{\pi}{4}\right)$

13. $f(\alpha) = \frac{1}{2}\tan 2\alpha$

In Exercises 14 and 15, use a graphing utility to graph the function. If the function is periodic, find its period.

14. $y = \sin 2\pi x + 2\cos\pi x$

15. $y = 6e^{-0.12t}\cos(0.25t), \quad 0 \leq t \leq 32$

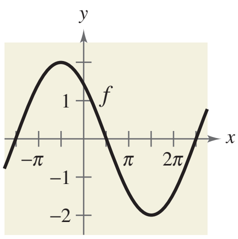


FIGURE FOR 16

16. Find a, b, and c for the function $f(x) = a\sin(bx + c)$ such that the graph of f matches the figure.

17. Find the exact value of $\tan\left(\arccos\frac{2}{3}\right)$ without the aid of a calculator.

18. Graph the function $f(x) = 2\arcsin\left(\frac{1}{2}x\right)$.

19. A plane is 80 miles south and 95 miles east of Cleveland Hopkins International Airport. What bearing should be taken to fly directly to the airport?

20. Write the equation for the simple harmonic motion of a ball on a spring that starts at its lowest point of 6 inches below equilibrium, bounces to its maximum height of 6 inches above equilibrium, and returns to its lowest point in a total of 2 seconds.

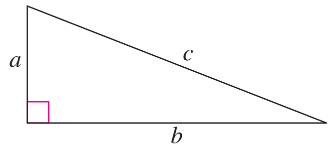
The Pythagorean Theorem

The Pythagorean Theorem is one of the most famous theorems in mathematics. More than 100 different proofs now exist. James A. Garfield, the twentieth president of the United States, developed a proof of the Pythagorean Theorem in 1876. His proof, shown below, involved the fact that a trapezoid can be formed from two congruent right triangles and an isosceles right triangle.

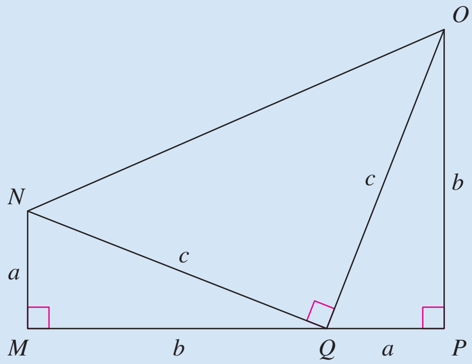
The Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse, where a and b are the legs and c is the hypotenuse.

$$a^2 + b^2 = c^2$$



Proof



$$\begin{matrix}\text{Area of} \\ \text{trapezoid } MNOP\end{matrix} = \begin{matrix}\text{Area of} \\ \triangle MNQ\end{matrix} + \begin{matrix}\text{Area of} \\ \triangle PQO\end{matrix} + \begin{matrix}\text{Area of} \\ \triangle NOQ\end{matrix}$$

$$\frac{1}{2}(a + b)(a + b) = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$$

$$\frac{1}{2}(a + b)(a + b) = ab + \frac{1}{2}c^2$$

$$(a + b)(a + b) = 2ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

P.S. Problem Solving

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

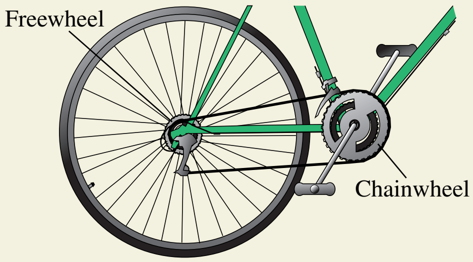
1. The restaurant at the top of the Space Needle in Seattle, Washington is circular and has a radius of 47.25 feet. The dining part of the restaurant revolves, making about one complete revolution every 48 minutes. A dinner party was seated at the edge of the revolving restaurant at 6:45 P.M. and was finished at 8:57 P.M.

(a) Find the angle through which the dinner party rotated.

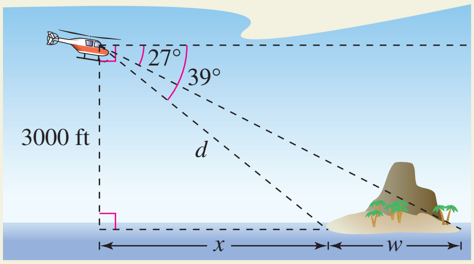
(b) Find the distance the party traveled during dinner.

2. A bicycle's gear ratio is the number of times the freewheel turns for every one turn of the chainwheel (see figure). The table shows the numbers of teeth in the freewheel and chainwheel for the first five gears of an 18-speed touring bicycle. The chainwheel completes one rotation for each gear. Find the angle through which the freewheel turns for each gear. Give your answers in both degrees and radians.

Gear number	Number of teeth in freewheel	Number of teeth in chainwheel
1	32	24
2	26	24
3	22	24
4	32	40
5	19	24



3. A surveyor in a helicopter is trying to determine the width of an island, as shown in the figure.



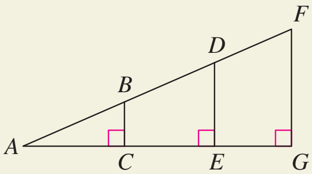
Not drawn to scale

(a) What is the shortest distance d the helicopter would have to travel to land on the island?

(b) What is the horizontal distance x that the helicopter would have to travel before it would be directly over the nearer end of the island?

(c) Find the width w of the island. Explain how you obtained your answer.

4. Use the figure below.



(a) Explain why $\triangle ABC$, $\triangle ADE$, and $\triangle AFG$ are similar triangles.

(b) What does similarity imply about the ratios

$$\frac{BC}{AB}, \frac{DE}{AD}, \text{ and } \frac{FG}{AF}?$$

(c) Does the value of sin A depend on which triangle from part (a) is used to calculate it? Would the value of sin A change if it were found using a different right triangle that was similar to the three given triangles?

(d) Do your conclusions from part (c) apply to the other five trigonometric functions? Explain.

5. Use a graphing utility to graph h, and use the graph to decide whether h is even, odd, or neither.

(a) $h(x) = \cos^2 x$

(b) $h(x) = \sin^2 x$

6. If f is an even function and g is an odd function, use the results of Exercise 5 to make a conjecture about h, where

(a) $h(x) = [f(x)]^2$

(b) $h(x) = [g(x)]^2$.

7. The model for the height h (in feet) of a Ferris wheel car is

$$h = 50 + 50 \sin 8\pi t$$

where t is the time (in minutes). (The Ferris wheel has a radius of 50 feet.) This model yields a height of 50 feet when $t = 0$. Alter the model so that the height of the car is 1 foot when $t = 0$.

8. The pressure P (in millimeters of mercury) against the walls of the blood vessels of a patient is modeled by

$$P = 100 - 20\cos\left(\frac{8\pi}{3}t\right)$$

where t is time (in seconds).

(a) Use a graphing utility to graph the model.

(b) What is the period of the model? What does the period tell you about this situation?

(c) What is the amplitude of the model? What does it tell you about this situation?

(d) If one cycle of this model is equivalent to one heartbeat, what is the pulse of this patient?

(e) If a physician wants this patient's pulse rate to be 64 beats per minute or less, what should the period be? What should the coefficient of t be?

9. A popular theory that attempts to explain the ups and downs of everyday life states that each of us has three cycles, called biorhythms, which begin at birth. These three cycles can be modeled by sine waves.

Physical (23 days): $P = \sin\dfrac{2\pi t}{23}, \quad t \geq 0$

Emotional (28 days): $E = \sin\dfrac{2\pi t}{28}, \quad t \geq 0$

Intellectual (33 days): $I = \sin\dfrac{2\pi t}{33}, \quad t \geq 0$

where t is the number of days since birth. Consider a person who was born on July 20, 1986.

(a) Use a graphing utility to graph the three models in the same viewing window for $7300 \leq t \leq 7380$.

(b) Describe the person's biorhythms during the month of September 2006.

(c) Calculate the person's three energy levels on September 22, 2006.

10. (a) Use a graphing utility to graph the functions given by

$$f(x) = 2\cos 2x + 3\sin 3x$$

and

$$g(x) = 2\cos 2x + 3\sin 4x.$$

(b) Use the graphs from part (a) to find the period of each function.

(c) If α and β are positive integers, is the function given by

$$h(x) = A\cos\alpha x + B\sin\beta x$$

periodic? Explain your reasoning.

11. Two trigonometric functions f and g have periods of 2, and their graphs intersect at $x = 5.35$.

(a) Give one smaller and one larger positive value of x at which the functions have the same value.

(b) Determine one negative value of x at which the graphs intersect.

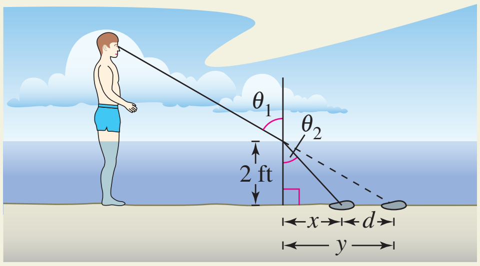
(c) Is it true that $f(13.35) = g(-4.65)$? Explain your reasoning.

12. The function f is periodic, with period c. So, $f(t + c) = f(t)$. Are the following equal? Explain.

(a) $f(t - 2c) = f(t)$ (b) $f\left(t + \frac{1}{2}c\right) = f\left(\frac{1}{2}t\right)$

(c) $f\left(\frac{1}{2}(t + c)\right) = f\left(\frac{1}{2}t\right)$

13. If you stand in shallow water and look at an object below the surface of the water, the object will look farther away from you than it really is. This is because when light rays pass between air and water, the water refracts, or bends, the light rays. The index of refraction for water is 1.333. This is the ratio of the sine of θ_1 and the sine of θ_2 (see figure).



(a) You are standing in water that is 2 feet deep and are looking at a rock at angle $\theta_1 = 60°$ (measured from a line perpendicular to the surface of the water). Find θ_2.

(b) Find the distances x and y.

(c) Find the distance d between where the rock is and where it appears to be.

(d) What happens to d as you move closer to the rock? Explain your reasoning.

14. In calculus, it can be shown that the arctangent function can be approximated by the polynomial

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

where x is in radians.

(a) Use a graphing utility to graph the arctangent function and its polynomial approximation in the same viewing window. How do the graphs compare?

(b) Study the pattern in the polynomial approximation of the arctangent function and guess the next term. Then repeat part (a). How does the accuracy of the approximation change when additional terms are added?