Appendix A Review of Fundamental Concepts of Algebra

A.1 Real Numbers and Their Properties

What you should learn

- Represent and classify real numbers.
- Order real numbers and use inequalities.
- Find the absolute values of real numbers and find the distance between two real numbers.
- Evaluate algebraic expressions.
- Use the basic rules and properties of algebra.

Why you should learn it

Real numbers are used to represent many real-life quantities. For example, in Exercise 65 on page A9, you will use real numbers to represent the federal deficit.

The *HM mathSpace®* CD-ROM and *Eduspace®* for this text contain additional resources related to the concepts discussed in this chapter.

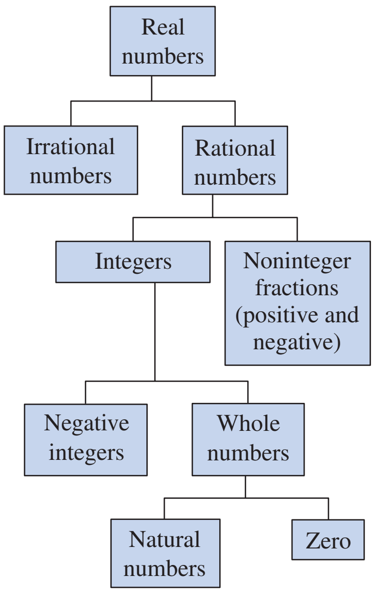


FIGURE A.1 *Subsets of real numbers*

Real Numbers

Real numbers are used in everyday life to describe quantities such as age, miles per gallon, and population. Real numbers are represented by symbols such as

$$-5, 9, 0, \frac{4}{3}, 0.666 \ldots, 28.21, \sqrt{2}, \pi, \text{ and } \sqrt[3]{-32}.$$

Here are some important **subsets** (each member of subset B is also a member of set A) of the real numbers. The three dots, called *ellipsis points*, indicate that the pattern continues indefinitely.

$$\{1, 2, 3, 4, \ldots\} \qquad \text{Set of natural numbers}$$

$$\{0, 1, 2, 3, 4, \ldots\} \qquad \text{Set of whole numbers}$$

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \qquad \text{Set of integers}$$

A real number is **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\frac{1}{3} = 0.3333 \ldots = 0.\overline{3}, \frac{1}{8} = 0.125, \text{ and } \frac{125}{111} = 1.126126 \ldots = 1.\overline{126}$$

are rational. The decimal representation of a rational number either repeats $\left(\text{as in } \frac{173}{55} = 3.1\overline{45}\right)$ or terminates $\left(\text{as in } \frac{1}{2} = 0.5\right)$. A real number that cannot be written as the ratio of two integers is called **irrational.** Irrational numbers have infinite nonrepeating decimal representations. For instance, the numbers

$$\sqrt{2} = 1.4142135 \ldots \approx 1.41 \quad \text{and} \quad \pi = 3.1415926 \ldots \approx 3.14$$

are irrational. (The symbol $\approx$ means "is approximately equal to.") Figure A.1 shows subsets of real numbers and their relationships to each other.

Real numbers are represented graphically by a **real number line.** The point 0 on the real number line is the **origin.** Numbers to the right of 0 are positive, and numbers to the left of 0 are negative, as shown in Figure A.2. The term **nonnegative** describes a number that is either positive or zero.

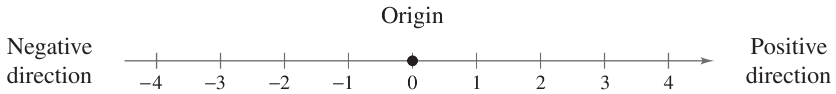


FIGURE A.2 *The real number line*

As illustrated in Figure A.3, there is a *one-to-one correspondence* between real numbers and points on the real number line.

Every real number corresponds to exactly one point on the real number line.

Every point on the real number line corresponds to exactly one real number.

FIGURE A.3 *One-to-one*

Ordering Real Numbers

One important property of real numbers is that they are *ordered*.

Definition of Order on the Real Number Line

If a and b are real numbers, a is less than b if $b - a$ is positive. The **order** of a and b is denoted by the **inequality** $a < b$. This relationship can also be described by saying that b is *greater than* a and writing $b > a$. The inequality $a \le b$ means that a is *less than or equal to* b, and the inequality $b \ge a$ means that b is *greater than or equal to* a. The symbols $<$, $>$, $\le$, and $\ge$ are *inequality symbols*.

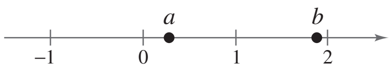


FIGURE A.4 *$a < b$ if and only if a lies to the left of b.*

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line, as shown in Figure A.4.

Example 1 Interpreting Inequalities

Describe the subset of real numbers represented by each inequality.

a. $x \le 2$ **b.** $-2 \le x < 3$

Solution

a. The inequality $x \le 2$ denotes all real numbers less than or equal to 2, as shown in Figure A.5.

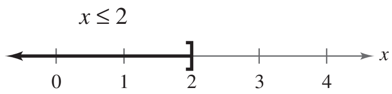


FIGURE A.5

b. The inequality $-2 \le x < 3$ means that $x \ge -2$ *and* $x < 3$. This “double inequality” denotes all real numbers between -2 and 3, including -2 but not including 3, as shown in Figure A.6.

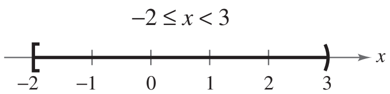
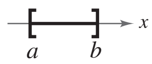
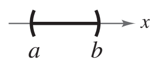
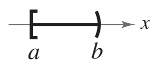
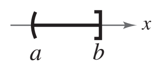


FIGURE A.6

CHECKPOINT Now try Exercise 19.

Inequalities can be used to describe subsets of real numbers called **intervals.** In the bounded intervals below, the real numbers a and b are the **endpoints** of each interval. The endpoints of a closed interval are included in the interval, whereas the endpoints of an open interval are not included in the interval.

Bounded Intervals on the Real Number Line

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \le x \le b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \le x < b$	
$(a, b]$		$a < x \le b$	

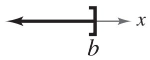
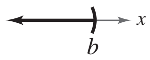
STUDY TIP

The reason that the four types of intervals at the right are called *bounded* is that each has a finite length. An interval that does not have a finite length is *unbounded* (see page A3).

> **STUDY TIP**
>
> Note that whenever you write intervals containing ∞ or $-\infty$, you always use a parenthesis and never a bracket. This is because these symbols are never an endpoint of an interval and therefore not included in the interval.

The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

Unbounded Intervals on the Real Number Line

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

Example 2 Using Inequalities to Represent Intervals

Use inequality notation to describe each of the following.

a. c is at most 2. **b.** m is at least -3.

c. All x in the interval $(-3, 5]$

Solution

a. The statement "c is at most 2" can be represented by $c \leq 2$.

b. The statement "m is at least -3" can be represented by $m \geq -3$.

c. "All x in the interval $(-3, 5]$" can be represented by $-3 < x \leq 5$.

CHECKPOINT Now try Exercise 31.

Example 3 Interpreting Intervals

Give a verbal description of each interval.

a. $(-1, 0)$ **b.** $[2, \infty)$ **c.** $(-\infty, 0)$

Solution

a. This interval consists of all real numbers that are greater than -1 and less than 0.

b. This interval consists of all real numbers that are greater than or equal to 2.

c. This interval consists of all negative real numbers.

CHECKPOINT Now try Exercise 29.

The **Law of Trichotomy** states that for any two real numbers a and b, *precisely* one of three relationships is possible:

$$a = b, \quad a < b, \quad \text{or} \quad a > b. \qquad \text{Law of Trichotomy}$$

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude*, or the distance between the origin and the point representing the real number on the real number line.

Definition of Absolute Value

If a is a real number, then the absolute value of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

Notice in this definition that the absolute value of a real number is never negative. For instance, if $a = -5$, then $|-5| = -(-5) = 5$. The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. So, $|0| = 0$.

Example 4 Evaluating the Absolute Value of a Number

Evaluate $\dfrac{|x|}{x}$ for (a) $x > 0$ and (b) $x < 0$.

Solution

a. If $x > 0$, then $|x| = x$ and $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$.

b. If $x < 0$, then $|x| = -x$ and $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$.

CHECKPOINT Now try Exercise 47.

Properties of Absolute Values

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a||b|$ **4.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

Absolute value can be used to define the distance between two points on the real number line. For instance, the distance between -3 and 4 is

$$\begin{aligned} |-3 - 4| &= |-7| \\ &= 7 \end{aligned}$$

as shown in Figure A.7.

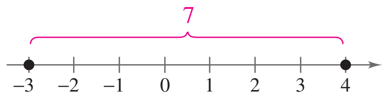


FIGURE A.7 *The distance between* -3 *and 4 is 7.*

Distance Between Two Points on the Real Number Line

Let a and b be real numbers. The **distance between a and b** is

$$d(a, b) = |b - a| = |a - b|.$$

Algebraic Expressions

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables,** and combinations of letters and numbers are **algebraic expressions.** Here are a few examples of algebraic expressions.

$$5x, \qquad 2x - 3, \qquad \frac{4}{x^2 + 2}, \qquad 7x + y$$

Definition of an Algebraic Expression

An **algebraic expression** is a collection of letters **(variables)** and real numbers **(constants)** combined using the operations of addition, subtraction, multiplication, division, and exponentiation.

The **terms** of an algebraic expression are those parts that are separated by *addition*. For example,

$$x^2 - 5x + 8 = x^2 + (-5x) + 8$$

has three terms: x^2 and $-5x$ are the **variable terms** and 8 is the **constant term.** The numerical factor of a variable term is the **coefficient** of the variable term. For instance, the coefficient of $-5x$ is -5, and the coefficient of x^2 is 1.

To **evaluate** an algebraic expression, substitute numerical values for each of the variables in the expression. Here are two examples.

Expression	*Value of Variable*	*Substitute*	*Value of Expression*
$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$

When an algebraic expression is evaluated, the **Substitution Principle** is used. It states that "If $a = b$, then a can be replaced by b in any expression involving a." In the first evaluation shown above, for instance, 3 is *substituted* for x in the expression $-3x + 5$.

Basic Rules of Algebra

There are four arithmetic operations with real numbers: *addition*, *multiplication*, *subtraction*, and *division*, denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$ or /. Of these, addition and multiplication are the two primary operations. Subtraction and division are the inverse operations of addition and multiplication, respectively.

Definitions of Subtraction and Division

Subtraction: Add the opposite.

$$a - b = a + (-b)$$

Division: Multiply by the reciprocal.

If $b \neq 0$, then $a/b = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

In these definitions, $-b$ is the **additive inverse** (or opposite) of b, and $1/b$ is the **multiplicative inverse** (or reciprocal) of b. In the fractional form a/b, a is the **numerator** of the fraction and b is the **denominator.**

Because the properties of real numbers below are true for variables and algebraic expressions as well as for real numbers, they are often called the **Basic Rules of Algebra.** Try to formulate a verbal description of each property. For instance, the first property states that *the order in which two real numbers are added does not affect their sum.*

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

	Property	*Example*
Commutative Property of Addition:	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication:	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x + (5 + x^2)$
Associative Property of Multiplication:	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Properties:	$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
	$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property:	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property:	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property:	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Properties are also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is $a(b - c) = ab - ac$.

STUDY TIP

Notice the difference between the *opposite of a number* and a *negative number*. If a is already negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then

$$-a = -(-5) = 5.$$

Properties of Negation and Equality

Let a and b be real numbers, variables, or algebraic expressions.

	Property	*Example*
1.	$(-1)a = -a$	$(-1)7 = -7$
2.	$-(-a) = a$	$-(-6) = 6$
3.	$(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4.	$(-a)(-b) = ab$	$(-2)(-x) = 2x$
5.	$-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8)$
		$= -x - 8$
6.	If $a = b$, then $a \pm c = b \pm c$.	$\frac{1}{2} + 3 = 0.5 + 3$
7.	If $a = b$, then $ac = bc$.	$4^2 \cdot 2 = 16 \cdot 2$
8.	If $a \pm c = b \pm c$, then $a = b$.	$1.4 - 1 = \frac{7}{5} - 1 \implies 1.4 = \frac{7}{5}$
9.	If $ac = bc$ and $c \neq 0$, then $a = b$.	$3x = 3 \cdot 4 \implies x = 4$

STUDY TIP

The "or" in the Zero-Factor Property includes the possibility that either or both factors may be zero. This is an **inclusive or,** and it is the way the word "or" is generally used in mathematics.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$

2. $a \cdot 0 = 0$

3. $\dfrac{0}{a} = 0, \qquad a \neq 0$

4. $\dfrac{a}{0}$ is undefined.

5. Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.

Properties and Operations of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. Equivalent Fractions: $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$.

2. Rules of Signs: $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$

3. Generate Equivalent Fractions: $\dfrac{a}{b} = \dfrac{ac}{bc}, \qquad c \neq 0$

4. Add or Subtract with Like Denominators: $\dfrac{a}{b} \pm \dfrac{c}{b} = \dfrac{a \pm c}{b}$

5. Add or Subtract with Unlike Denominators: $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$

6. Multiply Fractions: $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$

7. Divide Fractions: $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}, \qquad c \neq 0$

STUDY TIP

In Property 1 of fractions, the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

Example 5 **Properties and Operations of Fractions**

a. Equivalent fractions: $\dfrac{x}{5} = \dfrac{3 \cdot x}{3 \cdot 5} = \dfrac{3x}{15}$

b. Divide fractions: $\dfrac{7}{x} \div \dfrac{3}{2} = \dfrac{7}{x} \cdot \dfrac{2}{3} = \dfrac{14}{3x}$

c. Add fractions with unlike denominators: $\dfrac{x}{3} + \dfrac{2x}{5} = \dfrac{5 \cdot x + 3 \cdot 2x}{3 \cdot 5} = \dfrac{11x}{15}$

CHECKPOINT Now try Exercise 103.

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. A **prime number** is an integer that has exactly two positive factors—itself and 1—such as 2, 3, 5, 7, and 11. The numbers 4, 6, 8, 9, and 10 are **composite** because each can be written as the product of two or more prime numbers. The number 1 is neither prime nor composite. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

A.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

VOCABULARY CHECK: Fill in the blanks.

1. A real number is ________ if it can be written as the ratio $\frac{p}{q}$ of two integers, where $q \neq 0$.
2. ________ numbers have infinite nonrepeating decimal representations.
3. The distance between a point on the real number line and the origin is the ________ ________ of the real number.
4. A number that can be written as the product of two or more prime numbers is called a ________ number.
5. An integer that has exactly two positive factors, the integer itself and 1, is called a ________ number.
6. An algebraic expression is a collection of letters called ________ and real numbers called ________.
7. The ________ of an algebraic expression are those parts separated by addition.
8. The numerical factor of a variable term is the ________ of the variable term.
9. The ________ ________ states that if $ab = 0$, then $a = 0$ or $b = 0$.

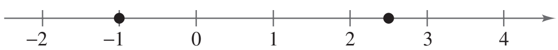
In Exercises 1–6, determine which numbers in the set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

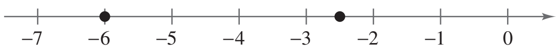
1. $-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1, -4, 2, -11$
2. $\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}, -3, 12, 5$
3. $2.01, 0.666\ldots, -13, 0.010110111\ldots, 1, -6$
4. $2.3030030003\ldots, 0.7575, -4.63, \sqrt{10}, -75, 4$
5. $-\pi, -\frac{1}{3}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5, -1, 8, -22$
6. $25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi, 7, -11.1, 13$

In Exercises 7–10, use a calculator to find the decimal form of the rational number. If it is a nonterminating decimal, write the repeating pattern.

7. $\frac{5}{8}$
8. $\frac{1}{3}$
9. $\frac{41}{333}$
10. $\frac{6}{11}$

In Exercises 11 and 12, approximate the numbers and place the correct symbol (< or >) between them.

11. 

12. 

In Exercises 13–18, plot the two real numbers on the real number line. Then place the appropriate inequality symbol (< or >) between them.

13. $-4, -8$
14. $-3.5, 1$
15. $\frac{3}{2}, 7$
16. $1, \frac{16}{3}$
17. $\frac{5}{6}, \frac{2}{3}$
18. $-\frac{8}{7}, -\frac{3}{7}$

In Exercises 19–30, (a) give a verbal description of the subset of real numbers represented by the inequality or the interval, (b) sketch the subset on the real number line, and (c) state whether the interval is bounded or unbounded.

19. $x \leq 5$
20. $x \geq -2$
21. $x < 0$
22. $x > 3$
23. $[4, \infty)$
24. $(-\infty, 2)$
25. $-2 < x < 2$
26. $0 \leq x \leq 5$
27. $-1 \leq x < 0$
28. $0 < x \leq 6$
29. $[-2, 5)$
30. $(-1, 2]$

In Exercises 31–38, use inequality notation to describe the set.

31. All x in the interval $(-2, 4]$
32. All y in the interval $[-6, 0)$
33. y is nonnegative.
34. y is no more than 25.
35. t is at least 10 and at most 22.
36. k is less than 5 but no less than -3.
37. The dog's weight W is more than 65 pounds.
38. The annual rate of inflation r is expected to be at least 2.5% but no more than 5%.

In Exercises 39–48, evaluate the expression.

39. $|-10|$
40. $|0|$
41. $|3 - 8|$
42. $|4 - 1|$
43. $|-1| - |-2|$
44. $-3 - |-3|$
45. $\dfrac{-5}{|-5|}$
46. $-3|-3|$
47. $\dfrac{|x + 2|}{x + 2}, \quad x < -2$
48. $\dfrac{|x - 1|}{x - 1}, \quad x > 1$

In Exercises 49–54, place the correct symbol (<, >, or =) between the pair of real numbers.

49. $|-3| \quad \square \quad -|-3|$

50. $|-4| \quad \square \quad |4|$

51. $-5 \quad \square \quad -|5|$

52. $-|-6| \quad \square \quad |-6|$

53. $-|-2| \quad \square \quad -|2|$

54. $-(-2) \quad \square \quad -2$

In Exercises 55–60, find the distance between *a* and *b*.

55. $a = 126,\ b = 75$

56. $a = -126,\ b = -75$

57. $a = -\frac{5}{2},\ b = 0$

58. $a = \frac{1}{4},\ b = \frac{11}{4}$

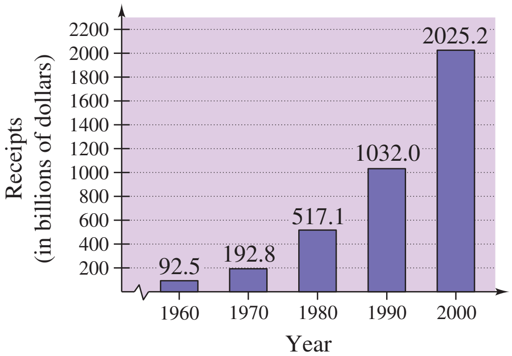
59. $a = \frac{16}{5},\ b = \frac{112}{75}$

60. $a = 9.34,\ b = -5.65$

Budget Variance **In Exercises 61–64, the accounting department of a sports drink bottling company is checking to see whether the actual expenses of a department differ from the budgeted expenses by more than \$500 or by more than 5%. Fill in the missing parts of the table, and determine whether each actual expense passes the "budget variance test."**

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	$0.05b$
61. Wages	\$112,700	\$113,356		
62. Utilities	\$9,400	\$9,772		
63. Taxes	\$37,640	\$37,335		
64. Insurance	\$2,575	\$2,613		

65. ***Federal Deficit*** The bar graph shows the federal government receipts (in billions of dollars) for selected years from 1960 through 2000. (Source: U.S. Office of Management and Budget)



(a) Complete the table. (*Hint:* Find $\lvert$Receipts − Expenditures$\rvert$.)

Year	Expenditures (in billions)	Surplus or deficit (in billions)
1960	\$92.2	
1970	\$195.6	
1980	\$590.9	
1990	\$1253.2	
2000	\$1788.8	

(b) Use the table in part (a) to construct a bar graph showing the magnitude of the surplus or deficit for each year.

66. ***Veterans*** The table shows the number of living veterans (in thousands) in the United States in 2002 by age group. Construct a circle graph showing the percent of living veterans by age group as a fraction of the total number of living veterans. (Source: Department of Veteran Affairs)

Age group	Number of veterans
Under 35	2213
35–44	3290
45–54	4666
55–64	5665
65 and older	9784

In Exercises 67–72, use absolute value notation to describe the situation.

67. The distance between x and 5 is no more than 3.

68. The distance between x and -10 is at least 6.

69. y is at least six units from 0.

70. y is at most two units from a.

71. While traveling on the Pennsylvania Turnpike, you pass milepost 326 near Valley Forge, then milepost 351 near Philadelphia. How many miles do you travel during that time period?

72. The temperature in Chicago, Illinois was 48° last night at midnight, then 82° at noon today. What was the change in temperature over the 12-hour period?

In Exercises 73–78, identify the terms. Then identify the coefficients of the variable terms of the expression.

73. $7x + 4$

74. $6x^3 - 5x$

75. $\sqrt{3}x^2 - 8x - 11$

76. $3\sqrt{3}x^2 + 1$

77. $4x^3 + \frac{x}{2} - 5$

78. $3x^4 - \frac{x^2}{4}$

In Exercises 79–84, evaluate the expression for each value of *x*. (If not possible, state the reason.)

	Expression	*Values*	
79.	$4x - 6$	(a) $x = -1$	(b) $x = 0$
80.	$9 - 7x$	(a) $x = -3$	(b) $x = 3$
81.	$x^2 - 3x + 4$	(a) $x = -2$	(b) $x = 2$
82.	$-x^2 + 5x - 4$	(a) $x = -1$	(b) $x = 1$
83.	$\frac{x + 1}{x - 1}$	(a) $x = 1$	(b) $x = -1$
84.	$\frac{x}{x + 2}$	(a) $x = 2$	(b) $x = -2$

In Exercises 85–96, identify the rule(s) of algebra illustrated by the statement.

85. $x + 9 = 9 + x$

86. $2\left(\frac{1}{2}\right) = 1$

87. $\frac{1}{h + 6}(h + 6) = 1, \quad h \neq -6$

88. $(x + 3) - (x + 3) = 0$

89. $2(x + 3) = 2 \cdot x + 2 \cdot 3$

90. $(z - 2) + 0 = z - 2$

91. $1 \cdot (1 + x) = 1 + x$

92. $(z + 5)x = z \cdot x + 5 \cdot x$

93. $x + (y + 10) = (x + y) + 10$

94. $x(3y) = (x \cdot 3)y = (3x)y$

95. $3(t - 4) = 3 \cdot t - 3 \cdot 4$

96. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

In Exercises 97–104, perform the operation(s). (Write fractional answers in simplest form.)

97. $\frac{3}{16} + \frac{5}{16}$

98. $\frac{6}{7} - \frac{4}{7}$

99. $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

100. $\frac{10}{11} + \frac{6}{33} - \frac{13}{66}$

101. $12 \div \frac{1}{4}$

102. $-\left(6 \cdot \frac{4}{8}\right)$

103. $\frac{2x}{3} - \frac{x}{4}$

104. $\frac{5x}{6} \cdot \frac{2}{9}$

105. (a) Use a calculator to complete the table.

n	1	0.5	0.01	0.0001	0.000001
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $5/n$ as n approaches 0.

106. (a) Use a calculator to complete the table.

n	1	10	100	10,000	100,000
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $5/n$ as n increases without bound.

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. If $a < b$, then $\frac{1}{a} < \frac{1}{b}$, where $a \neq b \neq 0$.

108. Because $\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$, then $\frac{c}{a + b} = \frac{c}{a} + \frac{c}{b}$.

109. ***Exploration*** Consider $|u + v|$ and $|u| + |v|$, where $u \neq v \neq 0$.

(a) Are the values of the expressions always equal? If not, under what conditions are they unequal?

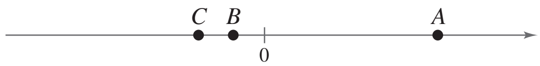
(b) If the two expressions are not equal for certain values of u and v, is one of the expressions always greater than the other? Explain.

110. ***Think About It*** Is there a difference between saying that a real number is positive and saying that a real number is nonnegative? Explain.

111. ***Think About It*** Because every even number is divisible by 2, is it possible that there exist any even prime numbers? Explain.

112. ***Writing*** Describe the differences among the sets of natural numbers, whole numbers, integers, rational numbers, and irrational numbers.

In Exercises 113 and 114, use the real numbers *A*, *B*, and *C* shown on the number line. Determine the sign of each expression.



113. (a) $-A$

(b) $B - A$

114. (a) $-C$

(b) $A - C$

115. ***Writing*** Can it ever be true that $|a| = -a$ for a real number a? Explain.

A.2 Exponents and Radicals

What you should learn

- Use properties of exponents.
- Use scientific notation to represent real numbers.
- Use properties of radicals.
- Simplify and combine radicals.
- Rationalize denominators and numerators.
- Use properties of rational exponents.

Why you should learn it

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 105 on page A22, you will use an expression involving rational exponents to find the time required for a funnel to empty for different water heights.

Integer Exponents

Repeated *multiplication* can be written in **exponential form.**

Repeated Multiplication	*Exponential Form*
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

Exponential Notation

If a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdot \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read "a to the nth **power.**"

An exponent can also be negative. In Property 3 below, be sure you see how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	*Example*
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert (-2)^2 \rvert = \lvert -2 \rvert^2 = (2)^2 = 4$

Technology

You can use a calculator to evaluate exponential expressions. When doing so, it is important to know when to use parentheses because the calculator follows the order of operations. For instance, evaluate $(-2)^4$ as follows

Scientific:

(2 +/−) y^x 4 =

Graphing:

((−) 2) ^ 4 ENTER

The display will be 16. If you omit the parentheses, the display will be -16.

It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$ and $-2^4 = -16$.

The properties of exponents listed on the preceding page apply to *all* integers m and n, not just to positive integers as shown in the examples in this section.

Example 1 Using Properties of Exponents

Use the properties of exponents to simplify each expression.

a. $(-3ab^4)(4ab^{-3})$ **b.** $(2xy^2)^3$ **c.** $3a(-4a^2)^0$ **d.** $\left(\dfrac{5x^3}{y}\right)^2$

Solution

a. $(-3ab^4)(4ab^{-3}) = (-3)(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

d. $\left(\dfrac{5x^3}{x}\right)^2 = \dfrac{5^2(x^3)^2}{y^2} = \dfrac{25x^6}{y^2}$

CHECKPOINT Now try Exercise 25.

Example 2 Rewriting with Positive Exponents

Rewrite each expression with positive exponents.

a. x^{-1} **b.** $\dfrac{1}{3x^{-2}}$ **c.** $\dfrac{12a^3b^{-4}}{4a^{-2}b}$ **d.** $\left(\dfrac{3x^2}{y}\right)^{-2}$

Solution

a. $x^{-1} = \dfrac{1}{x}$ Property 3

b. $\dfrac{1}{3x^{-2}} = \dfrac{1(x^2)}{3} = \dfrac{x^2}{3}$ The exponent -2 does not apply to 3.

c. $\dfrac{12a^3b^{-4}}{4a^{-2}b} = \dfrac{12a^3 \cdot a^2}{4b \cdot b^4} = \dfrac{3a^5}{b^5}$ Properties 3 and 1

d. $\left(\dfrac{3x^2}{y}\right)^{-2} = \dfrac{3^{-2}(x^2)^{-2}}{y^{-2}}$ Properties 5 and 7

$= \dfrac{3^{-2}x^{-4}}{y^{-2}}$ Property 6

$= \dfrac{y^2}{3^2x^4}$ Property 3

$= \dfrac{y^2}{9x^4}$ Simplify.

CHECKPOINT Now try Exercise 33.

STUDY TIP

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented in this text. The important thing is to use steps that you understand and, of course, steps that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 2(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Note how Property 3 is used in the first step of this solution. The fractional form of this property is

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

$$359{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \le c < 10$ and n is an integer. So, the number of gallons of water on Earth can be written in scientific notation as

$$3.59 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.59 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.\underbrace{0000000000000000000000000009}_{\text{28 decimal places}}.$$

Example 3 Scientific Notation

Write each number in scientific notation.

a. 0.0000782 **b.** $836{,}100{,}000$

Solution

a. $0.0000782 = 7.82 \times 10^{-5}$ **b.** $836{,}100{,}000 = 8.361 \times 10^{8}$

CHECKPOINT Now try Exercise 37.

Example 4 Decimal Notation

Write each number in decimal notation.

a. 9.36×10^{-6} **b.** 1.345×10^{2}

Solution

a. $9.36 \times 10^{-6} = 0.00000936$ **b.** $1.345 \times 10^{2} = 134.5$

CHECKPOINT Now try Exercise 41.

Technology

Most calculators automatically switch to scientific notation when they are showing large (or small) numbers that exceed the display range.

To *enter* numbers in scientific notation, your calculator should have an exponential entry key labeled

$\boxed{\text{EE}}$ or $\boxed{\text{EXP}}$.

Consult the user's guide for your calculator for instructions on keystrokes and how numbers in scientific notation are displayed.

Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Definition of *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If $n = 2$, the root is a **square root.** If $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square root* of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer n is the **index** of the radical, and the number a is the **radicand.** If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices*.)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: $\cancel{\sqrt{4} = \pm 2}$ *Correct:* $-\sqrt{4} = -2$ *and* $\sqrt{4} = 2$

Example 5 **Evaluating Expressions Involving Radicals**

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-(\sqrt{36}) = -(\sqrt{6^2}) = -(6) = -6$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

CHECKPOINT Now try Exercise 51.

Here are some generalizations about the nth roots of real numbers.

Generalizations About nth Roots of Real Numbers

Real Number a	Integer n	Root(s) of a	Example
$a > 0$	$n > 0$, is even.	$\sqrt[n]{a},\ -\sqrt[n]{a}$	$\sqrt[4]{81} = 3,\ -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd.	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even.	No real roots	$\sqrt{-4}$ is not a real number.
$a = 0$	n is even or odd.	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common special case of Property 6 is $\sqrt{a^2} = \lvert a \rvert$.

Example 6 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$ **c.** $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$

b. $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$

d. $\sqrt[6]{y^6} = \lvert y \rvert$

CHECKPOINT Now try Exercise 61.

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

STUDY TIP

When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 7(b), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 7(c), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

Example 7 Simplifying Even Roots

a. $\sqrt[4]{48} = \sqrt[4]{\overset{\text{Perfect 4th power}}{16} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

b. $\sqrt{75x^3} = \sqrt{\overset{\text{Perfect square}}{25x^2} \cdot \overset{\text{Leftover factor}}{3x}}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$

$= 5x\sqrt{3x}$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

CHECKPOINT Now try Exercise 63(a).

Example 8 Simplifying Odd Roots

a. $\sqrt[3]{24} = \sqrt[3]{\overset{\text{Perfect cube}}{8} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

b. $\sqrt[3]{24a^4} = \sqrt[3]{\overset{\text{Perfect cube}}{8a^3} \cdot \overset{\text{Leftover factor}}{3a}}$ Find largest cube factor.

$= \sqrt[3]{(2a)^3 \cdot 3a}$

$= 2a\sqrt[3]{3a}$ Find root of perfect cube.

c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5}$ Find largest cube factor.

$= \sqrt[3]{(-2x^2)^3 \cdot 5}$

$= -2x^2\sqrt[3]{5}$ Find root of perfect cube.

CHECKPOINT Now try Exercise 63(b).

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

Example 9 Combining Radicals

a. $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3}$ Find square factors.

$= 8\sqrt{3} - 9\sqrt{3}$ Find square roots and multiply by coefficients.

$= (8 - 9)\sqrt{3}$ Combine like terms.

$= -\sqrt{3}$ Simplify.

b. $\sqrt[3]{16x} - \sqrt[3]{54x^4} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x}$ Find cube factors.

$= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x}$ Find cube roots.

$= (2 - 3x)\sqrt[3]{2x}$ Combine like terms.

CHECKPOINT Now try Exercise 71.

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$. For cube roots, choose a rationalizing factor that generates a perfect cube.

Example 10 Rationalizing Single-Term Denominators

Rationalize the denominator of each expression.

a. $\dfrac{5}{2\sqrt{3}}$ **b.** $\dfrac{2}{\sqrt[3]{5}}$

Solution

a. $\dfrac{5}{2\sqrt{3}} = \dfrac{5}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ $\sqrt{3}$ is rationalizing factor.

$= \dfrac{5\sqrt{3}}{2(3)}$ Multiply.

$= \dfrac{5\sqrt{3}}{6}$ Simplify.

b. $\dfrac{2}{\sqrt[3]{5}} = \dfrac{2}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$ $\sqrt[3]{5^2}$ is rationalizing factor.

$= \dfrac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}}$ Multiply.

$= \dfrac{2\sqrt[3]{25}}{5}$ Simplify.

CHECKPOINT Now try Exercise 79.

Example 11 Rationalizing a Denominator with Two Terms

$$\frac{2}{3+\sqrt{7}} = \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}}$$ Multiply numerator and denominator by conjugate of denominator.

$$= \frac{2(3-\sqrt{7})}{3(3)+3(-\sqrt{7})+\sqrt{7}(3)-(\sqrt{7})(\sqrt{7})}$$ Use Distributive Property.

$$= \frac{2(3-\sqrt{7})}{(3)^2-(\sqrt{7})^2}$$ Simplify.

$$= \frac{2(3-\sqrt{7})}{9-7}$$ Square terms of denominator.

$$= \frac{2(3-\sqrt{7})}{2} = 3-\sqrt{7}$$ Simplify.

CHECKPOINT Now try Exercise 81.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Appendix A.4 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

STUDY TIP

Do not confuse the expression $\sqrt{5}+\sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x}+\sqrt{y}$. Similarly, $\sqrt{x^2+y^2}$ does not equal $x+y$.

Example 12 Rationalizing a Numerator

$$\frac{\sqrt{5}-\sqrt{7}}{2} = \frac{\sqrt{5}-\sqrt{7}}{2} \cdot \frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$ Multiply numerator and denominator by conjugate of numerator.

$$= \frac{(\sqrt{5})^2-(\sqrt{7})^2}{2(\sqrt{5}+\sqrt{7})}$$ Simplify.

$$= \frac{5-7}{2(\sqrt{5}+\sqrt{7})}$$ Square terms of numerator.

$$= \frac{-2}{2(\sqrt{5}+\sqrt{7})} = \frac{-1}{\sqrt{5}+\sqrt{7}}$$ Simplify.

CHECKPOINT Now try Exercise 85.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$a^{1/n} = \sqrt[n]{a}$, where $1/n$ is the **rational exponent** of a.

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

> **STUDY TIP**
>
> Rational exponents can be tricky, and you must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual-looking results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

> **Technology**
>
> There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\boxed{\sqrt{\ }}$. For cube roots, you can use the *cube root key* $\boxed{\sqrt[3]{\ }}$. For other roots, you can first convert the radical to exponential form and then use the *exponential key* $\boxed{\wedge}$, or you can use the *xth root key* $\boxed{\sqrt[x]{\ }}$. Consult the user's guide for your calculator for specific keystrokes.

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

Power

Index

When you are working with rational exponents, the properties of integer exponents still apply. For instance,

$$2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}.$$

Example 13 Changing from Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$

b. $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{(5/2)}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

CHECKPOINT Now try Exercise 87.

Example 14 Changing from Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$ **d.** $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

CHECKPOINT Now try Exercise 89.

Rational exponents are useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions in calculus.

Example 15 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \quad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$ Reduce index.

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x-1)^{4/3}(2x-1)^{-1/3} = (2x-1)^{(4/3)-(1/3)}$

$$= 2x - 1, \quad x \neq \frac{1}{2}$$

f.
$$\begin{aligned}\frac{x-1}{(x-1)^{-1/2}} &= \frac{x-1}{(x-1)^{-1/2}} \cdot \frac{(x-1)^{1/2}}{(x-1)^{1/2}}\\ &= \frac{(x-1)^{3/2}}{(x-1)^0}\\ &= (x-1)^{3/2}, \quad x \neq 1\end{aligned}$$

CHECKPOINT Now try Exercise 99.

A.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. In the exponential form a^n, n is the ________ and a is the ________.
2. A convenient way of writing very large or very small numbers is called ________ ________.
3. One of the two equal factors of a number is called a __________ __________ of the number.
4. The ________ ________ ________ of a number is the nth root that has the same sign as a, and is denoted by $\sqrt[n]{a}$.
5. In the radical form, $\sqrt[n]{a}$ the positive integer n is called the ________ of the radical and the number a is called the ________.
6. When an expression involving radicals has all possible factors removed, radical-free denominators, and a reduced index, it is in ________ ________.
7. The expressions $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are ________ of each other.
8. The process used to create a radical-free denominator is know as ________ the denominator.
9. In the expression $b^{m/n}$, m denotes the ________ to which the base is raised and n denotes the ________ or root to be taken.

In Exercises 1 and 2, write the expression as a repeated multiplication problem.

1. 8^5
2. $(-2)^7$

In Exercises 3 and 4, write the expression using exponential notation.

3. $(4.9)(4.9)(4.9)(4.9)(4.9)(4.9)$
4. $(-10)(-10)(-10)(-10)(-10)$

In Exercises 5–12, evaluate each expression.

5. (a) $3^2 \cdot 3$ (b) $3 \cdot 3^3$
6. (a) $\dfrac{5^5}{5^2}$ (b) $\dfrac{3^2}{3^4}$
7. (a) $(3^3)^0$ (b) -3^2
8. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$
9. (a) $\dfrac{3 \cdot 4^{-4}}{3^{-4} \cdot 4^{-1}}$ (b) $32(-2)^{-5}$
10. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$
11. (a) $2^{-1} + 3^{-1}$ (b) $(2^{-1})^{-2}$
12. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

In Exercises 13–16, use a calculator to evaluate the expression. (If necessary, round your answer to three decimal places.)

13. $(-4)^3(5^2)$
14. $(8^{-4})(10^3)$
15. $\dfrac{3^6}{7^3}$
16. $\dfrac{4^3}{3^{-4}}$

In Exercises 17–24, evaluate the expression for the given value of x.

	Expression	*Value*
17.	$-3x^3$	$x = 2$
18.	$7x^{-2}$	$x = 4$
19.	$6x^0$	$x = 10$
20.	$5(-x)^3$	$x = 3$
21.	$2x^3$	$x = -3$
22.	$-3x^4$	$x = -2$
23.	$4x^2$	$x = -\frac{1}{2}$
24.	$5(-x)^3$	$x = -\frac{1}{3}$

In Exercises 25–30, simplify each expression.

25. (a) $(-5z)^3$ (b) $5x^4(x^2)$
26. (a) $(3x)^2$ (b) $(4x^3)^0$
27. (a) $6y^2(2y^0)^2$ (b) $\dfrac{3x^5}{x^3}$
28. (a) $(-z)^3(3z^4)$ (b) $\dfrac{25y^8}{10y^4}$
29. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x+y)^3}{9(x+y)}$
30. (a) $\dfrac{r^4}{r^6}$ (b) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$

In Exercises 31–36, rewrite each expression with positive exponents and simplify.

31. (a) $(x + 5)^0, \quad x \neq -5$ (b) $(2x^2)^{-2}$
32. (a) $(2x^5)^0, \quad x \neq 0$ (b) $(z + 2)^{-3}(z + 2)^{-1}$

33. (a) $(-2x^2)^3(4x^3)^{-1}$ (b) $\left(\dfrac{x}{10}\right)^{-1}$

34. (a) $(4y^{-2})(8y^4)$ (b) $\left(\dfrac{x^{-3}y^4}{5}\right)^{-3}$

35. (a) $3^n \cdot 3^{2n}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$

36. (a) $\dfrac{x^2 \cdot x^n}{x^3 \cdot x^n}$ (b) $\left(\dfrac{a^{-3}}{b^{-3}}\right)\left(\dfrac{a}{b}\right)^3$

In Exercises 37–40, write the number in scientific notation.

37. Land area of Earth: 57,300,000 square miles

38. Light year: 9,460,000,000,000 kilometers

39. Relative density of hydrogen: 0.0000899 gram per cubic centimeter

40. One micron (millionth of a meter): 0.00003937 inch

In Exercises 41–44, write the number in decimal notation.

41. Worldwide daily consumption of Coca-Cola: 4.568×10^9 ounces (Source: The Coca-Cola Company)

42. Interior temperature of the sun: 1.5×10^7 degrees Celsius

43. Charge of an electron: 1.6022×10^{-19} coulomb

44. Width of a human hair: 9.0×10^{-5} meter

In Exercises 45 and 46, evaluate each expression without using a calculator.

45. (a) $\sqrt{25 \times 10^8}$ (b) $\sqrt[3]{8 \times 10^{15}}$

46. (a) $(1.2 \times 10^7)(5 \times 10^{-3})$ (b) $\dfrac{(6.0 \times 10^8)}{(3.0 \times 10^{-3})}$

In Exercises 47–50, use a calculator to evaluate each expression. (Round your answer to three decimal places.)

47. (a) $750\left(1 + \dfrac{0.11}{365}\right)^{800}$

(b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

48. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$

(b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

49. (a) $\sqrt{4.5 \times 10^9}$ (b) $\sqrt[3]{6.3 \times 10^4}$

50. (a) $(2.65 \times 10^{-4})^{1/3}$ (b) $\sqrt{9 \times 10^{-4}}$

In Exercises 51–56, evaluate each expression without using a calculator.

51. (a) $\sqrt{9}$ (b) $\sqrt[3]{\frac{27}{8}}$

52. (a) $27^{1/3}$ (b) $36^{3/2}$

53. (a) $32^{-3/5}$ (b) $\left(\frac{16}{81}\right)^{-3/4}$

54. (a) $100^{-3/2}$ (b) $\left(\frac{9}{4}\right)^{-1/2}$

55. (a) $\left(-\dfrac{1}{64}\right)^{-1/3}$ (b) $\left(\dfrac{1}{\sqrt{32}}\right)^{-2/5}$

56. (a) $\left(-\dfrac{125}{27}\right)^{-1/3}$ (b) $-\left(\dfrac{1}{125}\right)^{-4/3}$

In Exercises 57–60, use a calculator to approximate the number. (Round your answer to three decimal places.)

57. (a) $\sqrt{57}$ (b) $\sqrt[5]{-27^3}$

58. (a) $\sqrt[3]{45^2}$ (b) $\sqrt[6]{125}$

59. (a) $(-12.4)^{-1.8}$ (b) $(5\sqrt{3})^{-2.5}$

60. (a) $\dfrac{7 - (4.1)^{-3.2}}{2}$ (b) $\left(\dfrac{13}{3}\right)^{-3/2} - \left(-\dfrac{3}{2}\right)^{13/3}$

In Exercises 61 and 62, use the properties of radicals to simplify each expression.

61. (a) $(\sqrt[3]{4})^3$ (b) $\sqrt[5]{96x^5}$

62. (a) $\sqrt{12} \cdot \sqrt{3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 63–74, simplify each radical expression.

63. (a) $\sqrt{8}$ (b) $\sqrt[3]{54}$

64. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$

65. (a) $\sqrt{72x^3}$ (b) $\sqrt{\dfrac{18^2}{z^3}}$

66. (a) $\sqrt{54xy^4}$ (b) $\sqrt{\dfrac{32a^4}{b^2}}$

67. (a) $\sqrt[3]{16x^5}$ (b) $\sqrt{75x^2y^{-4}}$

68. (a) $\sqrt[4]{3x^4y^2}$ (b) $\sqrt[5]{160x^8z^4}$

69. (a) $2\sqrt{50} + 12\sqrt{8}$ (b) $10\sqrt{32} - 6\sqrt{18}$

70. (a) $4\sqrt{27} - \sqrt{75}$ (b) $\sqrt[3]{16} + 3\sqrt[3]{54}$

71. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$

72. (a) $8\sqrt{49x} - 14\sqrt{100x}$ (b) $-3\sqrt{48x^2} + 7\sqrt{75x^2}$

73. (a) $3\sqrt{x+1} + 10\sqrt{x+1}$ (b) $7\sqrt{80x} - 2\sqrt{125x}$

74. (a) $-\sqrt{x^3 - 7} + 5\sqrt{x^3 - 7}$ (b) $11\sqrt{245x^3} - 9\sqrt{45x^3}$

In Exercises 75–78, complete the statement with <, =, or >.

75. $\sqrt{5} + \sqrt{3} \;\square\; \sqrt{5 + 3}$ **76.** $\sqrt{\dfrac{3}{11}} \;\square\; \dfrac{\sqrt{3}}{\sqrt{11}}$

77. $5 \;\square\; \sqrt{3^2 + 2^2}$ **78.** $5 \;\square\; \sqrt{3^2 + 4^2}$

In Exercises 79–82, rationalize the denominator of the expression. Then simplify your answer.

79. $\dfrac{1}{\sqrt{3}}$ **80.** $\dfrac{5}{\sqrt{10}}$

81. $\dfrac{2}{5 - \sqrt{3}}$ **82.** $\dfrac{3}{\sqrt{5} + \sqrt{6}}$

In Exercises 83–86, rationalize the numerator of the expression. Then simplify your answer.

83. $\dfrac{\sqrt{8}}{2}$ **84.** $\dfrac{\sqrt{2}}{3}$

85. $\dfrac{\sqrt{5} + \sqrt{3}}{3}$ **86.** $\dfrac{\sqrt{7} - 3}{4}$

In Exercises 87–94, fill in the missing form of the expression.

	Radical Form	*Rational Exponent Form*
87.	$\sqrt{9}$	
88.	$\sqrt[3]{64}$	
89.		$32^{1/5}$
90.		$-(144^{1/2})$
91.	$\sqrt[3]{-216}$	
92.		$(-243)^{1/5}$
93.	$\left(\sqrt[4]{81}\right)^3$	
94.		$16^{5/4}$

In Exercises 95–98, perform the operations and simplify.

95. $\dfrac{(2x^2)^{3/2}}{2^{1/2}x^4}$ **96.** $\dfrac{x^{4/3}y^{2/3}}{(xy)^{1/3}}$

97. $\dfrac{x^{-3} \cdot x^{1/2}}{x^{3/2} \cdot x^{-1}}$ **98.** $\dfrac{5^{-1/2} \cdot 5x^{5/2}}{(5x)^{3/2}}$

In Exercises 99 and 100, reduce the index of each radical.

99. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x + 1)^4}$

100. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 101 and 102, write each expression as a single radical. Then simplify your answer.

101. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

102. (a) $\sqrt{\sqrt{243(x + 1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

103. ***Period of a Pendulum*** The period T (in seconds) of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L is the length of the pendulum (in feet). Find the period of a pendulum whose length is 2 feet.

104. ***Erosion*** A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the largest particle that can be carried by a stream flowing at the rate of $\frac{3}{4}$ foot per second.

105. ***Mathematical Modeling*** A funnel is filled with water to a height of h centimeters. The formula

$$t = 0.03[12^{5/2} - (12 - h)^{5/2}], \quad 0 \le h \le 12$$

represents the amount of time t (in seconds) that it will take for the funnel to empty.

(a) Use the table feature of a graphing utility to find the times required for the funnel to empty for water heights of $h = 0$, $h = 1$, $h = 2$, . . . $h = 12$ centimeters.

(b) What value does t appear to be approaching as the height of the water becomes closer and closer to 12 centimeters?

106. ***Speed of Light*** The speed of light is approximately 11,180,000 miles per minute. The distance from the sun to Earth is approximately 93,000,000 miles. Find the time for light to travel from the sun to Earth.

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. $\dfrac{x^{k+1}}{x} = x^k$ **108.** $(a^n)^k = a^{n^k}$

109. Verify that $a^0 = 1$, $a \ne 0$. (*Hint:* Use the property of exponents $a^m/a^n = a^{m-n}$.)

110. Explain why each of the following pairs is not equal.

(a) $(3x)^{-1} \ne \dfrac{3}{x}$ (b) $y^3 \cdot y^2 \ne y^6$

(c) $(a^2b^3)^4 \ne a^6b^7$ (d) $(a + b)^2 \ne a^2 + b^2$

(e) $\sqrt{4x^2} \ne 2x$ (f) $\sqrt{2} + \sqrt{3} \ne \sqrt{5}$

111. ***Exploration*** List all possible digits that occur in the units place of the square of a positive integer. Use that list to determine whether $\sqrt{5233}$ is an integer.

112. ***Think About It*** Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

The symbol indicates an exercise or a part of an exercise in which you are instructed to use a graphing utility.

A.3 Polynomials and Factoring

What you should learn

- Write polynomials in standard form.
- Add, subtract, and multiply polynomials.
- Use special products to multiply polynomials.
- Remove common factors from polynomials.
- Factor special polynomial forms.
- Factor trinomials as the product of two binomials.
- Factor polynomials by grouping.

Why you should learn it

Polynomials can be used to model and solve real-life problems. For instance, in Exercise 210 on page A34, a polynomial is used to model the stopping distance of an automobile.

Polynomials

The most common type of algebraic expression is the **polynomial.** Some examples are $2x + 5$, $3x^4 - 7x^2 + 2x + 4$, and $5x^2y^2 - xy + 3$. The first two are *polynomials in x* and the third is a *polynomial in x and y*. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. For instance, the polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, -5, 0, and 1.

Definition of a Polynomial in x

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and let n be a nonnegative integer. A polynomial in x is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, a_n is the **leading coefficient,** and a_0 is the **constant term.**

Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively. In **standard form,** a polynomial is written with descending powers of x.

Example 1 Writing Polynomials in Standard Form

	Polynomial	*Standard Form*	*Degree*
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7
b.	$4 - 9x^2$	$-9x^2 + 4$	2
c.	8	$8 \quad (8 = 8x^0)$	0

CHECKPOINT Now try Exercise 11.

A polynomial that has all zero coefficients is called the zero polynomial, denoted by 0. No degree is assigned to this particular polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms. For instance, the degree of the polynomial $-2x^3y^6 + 4xy - x^7y^4$ is 11 because the sum of the exponents in the last term is the greatest. The leading coefficient of the polynomial is the coefficient of the highest-degree term. Expressions are not polynomials if a variable is underneath a radical or if a polynomial expression (with degree greater than 0) is in the denominator of a term. The following expressions are not polynomials.

$$x^3 - \sqrt{3x} = x^3 - (3x)^{1/2}$$ The exponent "1/2" is not an integer.

$$x^2 + \frac{5}{x} = x^2 + 5x^{-1}$$ The exponent "-1" is not a nonnegative integer.

Operations with Polynomials

You can add and subtract polynomials in much the same way you add and subtract real numbers. Simply add or subtract the *like terms* (terms having the same variables to the same powers) by adding their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms and their sum is

$$\begin{aligned} -3xy^2 + 5xy^2 &= (-3 + 5)xy^2 \\ &= 2xy^2. \end{aligned}$$

STUDY TIP

When an expression inside parentheses is preceded by a negative sign, remember to distribute the negative sign to each term inside the parentheses, as shown.

$$\begin{aligned} -(x^2 - x + 3) \\ = -x^2 + x - 3 \end{aligned}$$

Example 2 Sums and Differences of Polynomials

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (-7x^2 + 2x^2) - x + (-3 + 8)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distributive Property

$= (7x^4 - 3x^4) + (-x^2 + 4x^2) + (-4x - 3x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

CHECKPOINT Now try Exercise 33.

To find the *product* of two polynomials, use the left and right Distributive Properties. For example, if you treat $5x + 7$ as a single quantity, you can multiply $3x - 2$ by $5x + 7$ as follows.

$$\begin{aligned} (3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7) \\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7) \\ &= \underbrace{15x^2}_{\substack{\text{Product of} \\ \text{First terms}}} + \underbrace{21x}_{\substack{\text{Product of} \\ \text{Outer terms}}} - \underbrace{10x}_{\substack{\text{Product of} \\ \text{Inner terms}}} - \underbrace{14}_{\substack{\text{Product of} \\ \text{Last terms}}} \\ &= 15x^2 + 11x - 14 \end{aligned}$$

Note in this **FOIL Method** (which can only be used to multiply two binomials) that the outer (O) and inner (I) terms are like terms and can be combined.

Example 3 Finding a Product by the FOIL Method

Use the FOIL Method to find the product of $2x - 4$ and $x + 5$.

Solution

$$\begin{aligned} (2x - 4)(x + 5) &= \overset{\text{F}}{2x^2} + \overset{\text{O}}{10x} - \overset{\text{I}}{4x} - \overset{\text{L}}{20} \\ &= 2x^2 + 6x - 20 \end{aligned}$$

CHECKPOINT Now try Exercise 47.

Special Products

Some binomial products have special forms that occur frequently in algebra. You do not need to memorize these formulas because you can use the Distributive Property to multiply. However, becoming familiar with these formulas will enable you to manipulate the algebra more quickly.

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	*Example*
Sum and Difference of Same Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 4^2$ $= x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 2(x)(3) + 3^2$ $= x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2$ $= 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3$ $= x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3$ $= x^3 - 3x^2 + 3x - 1$

Example 4 Special Products

Find each product.

a. $5x + 9$ and $5x - 9$ **b.** $x + y - 2$ and $x + y + 2$

Solution

a. The product of a sum and a difference of the *same* two terms has no middle term and takes the form $(u + v)(u - v) = u^2 - v^2$.

$$(5x + 9)(5x - 9) = (5x)^2 - 9^2 = 25x^2 - 81$$

b. By grouping $x + y$ in parentheses, you can write the product of the trinomials as a special product.

$$(x + y - 2)(x + y + 2) = [\underbrace{(x + y) - 2}_{\text{Difference}}][\underbrace{(x + y) + 2}_{\text{Sum}}]$$

$$= (x + y)^2 - 2^2 \quad \text{Sum and difference of same terms}$$

$$= x^2 + 2xy + y^2 - 4$$

CHECKPOINT Now try Exercise 67.

Polynomials with Common Factors

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and for simplifying rational expressions.

Unless noted otherwise, when you are asked to factor a polynomial, you can assume that you are looking for factors with integer coefficients. If a polynomial cannot be factored using integer coefficients, then it is **prime** or **irreducible over the integers.** For instance, the polynomial $x^2 - 3$ is irreducible over the integers. Over the *real numbers*, this polynomial can be factored as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

A polynomial is **completely factored** when each of its factors is prime. For instance

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4) \qquad \text{Completely factored}$$

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4) \qquad \text{Not completely factored}$$

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction.

$$ab + ac = a(b + c) \qquad a \text{ is a common factor.}$$

Removing (factoring out) any common factors is the first step in completely factoring a polynomial.

Example 5 Removing Common Factors

Factor each expression.

a. $6x^3 - 4x$

b. $-4x^2 + 12x - 16$

c. $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $6x^3 - 4x = 2x(3x^2) - 2x(2)$ — $2x$ is a common factor.

$= 2x(3x^2 - 2)$

b. $-4x^2 + 12x - 16 = -4(x^2) + (-4)(-3x) + (-4)4$ — -4 is a common factor.

$= -4(x^2 - 3x + 4)$

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$ — $(x - 2)$ is a common factor.

CHECKPOINT Now try Exercise 91.

Factoring Special Polynomial Forms

Some polynomials have special forms that arise from the special product forms on page A25. You should learn to recognize these forms so that you can factor such polynomials easily.

Factoring Special Polynomial Forms

Factored Form	*Example*
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

One of the easiest special polynomial forms to factor is the difference of two squares. The factored form is always a set of *conjugate pairs*.

$$\underbrace{u^2 - v^2}_{\text{Difference}} = \underbrace{(u + v)(u - v)}_{\text{Opposite signs}}$$

Conjugate pairs

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers*.

STUDY TIP

In Example 6, note that the first step in factoring a polynomial is to check for any common factors. Once the common factors are removed, it is often possible to recognize patterns that were not immediately obvious.

Example 6 **Removing a Common Factor First**

$$\begin{aligned} 3 - 12x^2 &= 3(1 - 4x^2) && \text{3 is a common factor.} \\ &= 3[1^2 - (2x)^2] \\ &= 3(1 + 2x)(1 - 2x) && \text{Difference of two squares} \end{aligned}$$

CHECKPOINT Now try Exercise 105.

Example 7 **Factoring the Difference of Two Squares**

a. $(x + 2)^2 - y^2 = [(x + 2) + y][(x + 2) - y]$

$= (x + 2 + y)(x + 2 - y)$

b.
$$\begin{aligned} 16x^4 - 81 &= (4x^2)^2 - 9^2 \\ &= (4x^2 + 9)(4x^2 - 9) && \text{Difference of two squares} \\ &= (4x^2 + 9)[(2x)^2 - 3^2] \\ &= (4x^2 + 9)(2x + 3)(2x - 3) && \text{Difference of two squares} \end{aligned}$$

CHECKPOINT Now try Exercise 109.

A **perfect square trinomial** is the square of a binomial, and it has the following form.

$$u^2 + 2uv + v^2 = (u + v)^2 \qquad \text{or} \qquad u^2 - 2uv + v^2 = (u - v)^2$$

Like signs   Like signs

Note that the first and last terms are squares and the middle term is twice the product of u and v.

Example 8 Factoring Perfect Square Trinomials

Factor each trinomial.

a. $x^2 - 10x + 25$

b. $16x^2 + 24x + 9$

Solution

a. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2 = (x - 5)^2$

b. $16x^2 + 24x + 9 = (4x)^2 + 2(4x)(3) + 3^2 = (4x + 3)^2$

CHECKPOINT Now try Exercise 115.

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

Like signs   Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs   Unlike signs

Example 9 Factoring the Difference of Cubes

Factor $x^3 - 27$.

Solution

$$x^3 - 27 = x^3 - 3^3 \qquad \text{Rewrite 27 as } 3^3.$$

$$= (x - 3)(x^2 + 3x + 9) \qquad \text{Factor.}$$

CHECKPOINT Now try Exercise 123.

Example 10 Factoring the Sum of Cubes

a. $y^3 + 8 = y^3 + 2^3$ Rewrite 8 as 2^3.

$= (y + 2)(y^2 - 2y + 4)$ Factor.

b. $3(x^3 + 64) = 3(x^3 + 4^3)$ Rewrite 64 as 4^3.

$= 3(x + 4)(x^2 - 4x + 16)$ Factor.

CHECKPOINT Now try Exercise 125.

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

Factors of a

$$ax^2 + bx + c = (\boxed{}x + \boxed{})(\boxed{}x + \boxed{})$$

Factors of c

The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx. For instance, in the trinomial $6x^2 + 17x + 5$, you can write all possible factorizations and determine which one has outer and inner products that add up to $17x$.

$$(6x + 5)(x + 1),\ (6x + 1)(x + 5),\ (2x + 1)(3x + 5),\ (2x + 5)(3x + 1)$$

You can see that $(2x + 5)(3x + 1)$ is the correct factorization because the outer (O) and inner (I) products add up to $17x$.

$$(2x + 5)(3x + 1) = \overset{\text{F}}{6x^2} + \overset{\text{O}}{2x} + \overset{\text{I}}{15x} + \overset{\text{L}}{5} = 6x^2 + \overset{\text{O + I}}{17x} + 5.$$

Example 11 Factoring a Trinomial: Leading Coefficient Is 1

Factor $x^2 - 7x + 12$.

Solution

The possible factorizations are

$$(x - 2)(x - 6), \quad (x - 1)(x - 12), \quad \text{and} \quad (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$

CHECKPOINT Now try Exercise 131.

Example 12 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor $2x^2 + x - 15$.

Solution

The eight possible factorizations are as follows.

$$(2x - 1)(x + 15) \qquad (2x + 1)(x - 15)$$
$$(2x - 3)(x + 5) \qquad (2x + 3)(x - 5)$$
$$(2x - 5)(x + 3) \qquad (2x + 5)(x - 3)$$
$$(2x - 15)(x + 1) \qquad (2x + 15)(x - 1)$$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3). \qquad \text{O + I} = 6x - 5x = x$$

CHECKPOINT Now try Exercise 139.

STUDY TIP

Factoring a trinomial can involve trial and error. However, once you have produced the factored form, it is an easy matter to check your answer. For instance, you can verify the factorization in Example 11 by multiplying out the expression $(x - 3)(x - 4)$ to see that you obtain the original trinomial, $x^2 - 7x + 12$.

Factoring by Grouping

Sometimes polynomials with more than three terms can be factored by a method called **factoring by grouping.** It is not always obvious which terms to group, and sometimes several different groupings will work.

Example 13 Factoring by Grouping

Use factoring by grouping to factor $x^3 - 2x^2 - 3x + 6$.

Solution

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor each group.} \\ &= (x - 2)(x^2 - 3) && \text{Distributive Property} \end{aligned}$$

CHECKPOINT Now try Exercise 147.

STUDY TIP

Another way to factor the polynomial in Example 13 is to group the terms as follows.

$$\begin{aligned} &x^3 - 2x^2 - 3x + 6 \\ &= (x^3 - 3x) - (2x^2 - 6) \\ &= x(x^2 - 3) - 2(x^2 - 3) \\ &= (x^2 - 3)(x - 2) \end{aligned}$$

As you can see, you obtain the same result as in Example 13.

Factoring a trinomial can involve quite a bit of trial and error. Some of this trial and error can be lessened by using factoring by grouping. The key to this method of factoring is knowing how to rewrite the middle term. In general, to factor a trinomial $ax^2 + bx + c$ by grouping, choose factors of the product ac that add up to b and use these factors to rewrite the middle term. This technique is illustrated in Example 14.

Example 14 Factoring a Trinomial by Grouping

Use factoring by grouping to factor $2x^2 + 5x - 3$.

Solution

In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, which implies that the product ac is -6. Now, -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$. So, you can rewrite the middle term as $5x = 6x - x$. This produces the following.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && \text{Distributive Property} \end{aligned}$$

So, the trinomial factors as $2x^2 + 5x - 3 = (x + 3)(2x - 1)$.

CHECKPOINT Now try Exercise 153.

Guidelines for Factoring Polynomials

1. Factor out any common factors using the Distributive Property.
2. Factor according to one of the special polynomial forms.
3. Factor as $ax^2 + bx + c = (mx + r)(nx + s)$.
4. Factor by grouping.

A.3 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. For the polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$, the degree is ________, the leading coefficient is ________, and the constant term is ________.

2. A polynomial in x in standard form is written with ________ powers of x.

3. A polynomial with one term is called a ________, while a polynomial with two terms is called a ________, and a polynomial with three terms is called a ________.

4. To add or subtract polynomials, add or subtract the ________ ________ by adding their coefficients.

5. The letters in "FOIL" stand for the following.

F ________ O ________ I ________ L ________

6. The process of writing a polynomial as a product is called ________.

7. A polynomial is ________ ________ when each of its factors is prime.

In Exercises 1–6, match the polynomial with its description. [The polynomials are labeled (a), (b), (c), (d), (e), and (f).]

(a) $3x^2$ (b) $1 - 2x^3$

(c) $x^3 + 3x^2 + 3x + 1$ (d) 12

(e) $-3x^5 + 2x^3 + x$ (f) $\frac{2}{3}x^4 + x^2 + 10$

1. A polynomial of degree 0

2. A trinomial of degree 5

3. A binomial with leading coefficient -2

4. A monomial of positive degree

5. A trinomial with leading coefficient $\frac{2}{3}$

6. A third-degree polynomial with leading coefficient 1

In Exercises 7–10, write a polynomial that fits the description. (There are many correct answers.)

7. A third-degree polynomial with leading coefficient -2

8. A fifth-degree polynomial with leading coefficient 6

9. A fourth-degree binomial with a negative leading coefficient

10. A third-degree binomial with an even leading coefficient

In Exercises 11–22, (a) write the polynomial in standard form, (b) identify the degree and leading coefficient of the polynomial, and (c) state whether the polynomial is a monomial, a binomial, or a trinomial.

11. $14x - \frac{1}{2}x^5$ **12.** $2x^2 - x + 1$

13. $-3x^4 + 2x^2 - 5$ **14.** $7x$

15. $x^5 - 1$ **16.** $-y + 25y^2 + 1$

17. 3 **18.** $t^2 + 9$

19. $1 + 6x^4 - 4x^5$ **20.** $3 + 2x$

21. $4x^3y$ **22.** $-x^5y + 2x^2y^2 + xy^4$

In Exercises 23–28, determine whether the expression is a polynomial. If so, write the polynomial in standard form.

23. $2x - 3x^3 + 8$ **24.** $2x^3 + x - 3x^{-1}$

25. $\dfrac{3x + 4}{x}$ **26.** $\dfrac{x^2 + 2x - 3}{2}$

27. $y^2 - y^4 + y^3$

28. $\sqrt{y^2 - y^4}$

In Exercises 29–46, perform the operation and write the result in standard form.

29. $(6x + 5) - (8x + 15)$

30. $(2x^2 + 1) - (x^2 - 2x + 1)$

31. $-(x^3 - 2) + (4x^3 - 2x)$

32. $-(5x^2 - 1) - (-3x^2 + 5)$

33. $(15x^2 - 6) - (-8.3x^3 - 14.7x^2 - 17)$

34. $(15.2x^4 - 18x - 19.1) - (13.9x^4 - 9.6x + 15)$

35. $5z - [3z - (10z + 8)]$

36. $(y^3 + 1) - [(y^2 + 1) + (3y - 7)]$

37. $3x(x^2 - 2x + 1)$

38. $y^2(4y^2 + 2y - 3)$

39. $-5z(3z - 1)$

40. $(-3x)(5x + 2)$

41. $(1 - x^3)(4x)$

42. $-4x(3 - x^3)$

43. $(2.5x^2 + 3)(3x)$

44. $(2 - 3.5y)(2y^3)$

45. $-4x\left(\frac{1}{8}x + 3\right)$

46. $2y\left(4 - \frac{7}{8}y\right)$

In Exercises 47–84, multiply or find the special product.

47. $(x + 3)(x + 4)$
48. $(x - 5)(x + 10)$
49. $(3x - 5)(2x + 1)$
50. $(7x - 2)(4x - 3)$
51. $(x^2 - x + 1)(x^2 + x + 1)$
52. $(x^2 + 3x - 2)(x^2 - 3x - 2)$
53. $(x + 10)(x - 10)$ **54.** $(2x + 3)(2x - 3)$
55. $(x + 2y)(x - 2y)$ **56.** $(2x + 3y)(2x - 3y)$
57. $(2x + 3)^2$ **58.** $(4x + 5)^2$
59. $(2x - 5y)^2$ **60.** $(5 - 8x)^2$
61. $(x + 1)^3$ **62.** $(x - 2)^3$
63. $(2x - y)^3$ **64.** $(3x + 2y)^3$
65. $(4x^3 - 3)^2$ **66.** $(8x + 3)^2$
67. $[(m - 3) + n][(m - 3) - n]$
68. $[(x + y) + 1][(x + y) - 1]$
69. $[(x - 3) + y]^2$ **70.** $[(x + 1) - y]^2$
71. $(2r^2 - 5)(2r^2 + 5)$ **72.** $(3a^3 - 4b^2)(3a^3 + 4b^2)$
73. $\left(\frac{1}{2}x - 3\right)^2$ **74.** $\left(\frac{2}{3}t + 5\right)^2$
75. $\left(\frac{1}{3}x - 2\right)\left(\frac{1}{3}x + 2\right)$ **76.** $\left(2x + \frac{1}{5}\right)\left(2x - \frac{1}{5}\right)$
77. $(1.2x + 3)^2$ **78.** $(1.5y - 3)^2$
79. $(1.5x - 4)(1.5x + 4)$ **80.** $(2.5y + 3)(2.5y - 3)$
81. $5x(x + 1) - 3x(x + 1)$
82. $(2x - 1)(x + 3) + 3(x + 3)$
83. $(u + 2)(u - 2)(u^2 + 4)$
84. $(x + y)(x - y)(x^2 + y^2)$

In Exercises 85–88, find the product. (The expressions are not polynomials, but the formulas can still be used.)

85. $\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right)$ **86.** $\left(5 + \sqrt{x}\right)\left(5 - \sqrt{x}\right)$
87. $\left(x - \sqrt{5}\right)^2$ **88.** $\left(x + \sqrt{3}\right)^2$

In Exercises 89–96, factor out the common factor.

89. $3x + 6$ **90.** $5y - 30$
91. $2x^3 - 6x$ **92.** $4x^3 - 6x^2 + 12x$
93. $x(x - 1) + 6(x - 1)$ **94.** $3x(x + 2) - 4(x + 2)$
95. $(x + 3)^2 - 4(x + 3)$ **96.** $(3x - 1)^2 + (3x - 1)$

In Exercises 97–102, find the greatest common factor such that the remaining factors have only integer coefficients.

97. $\frac{1}{2}x + 4$ **98.** $\frac{1}{3}y + 5$
99. $\frac{1}{2}x^3 + 2x^2 - 5x$ **100.** $\frac{1}{3}y^4 - 5y^2 + 2y$
101. $\frac{2}{3}x(x - 3) - 4(x - 3)$ **102.** $\frac{4}{5}y(y + 1) - 2(y + 1)$

In Exercises 103–112, completely factor the difference of two squares.

103. $x^2 - 81$
104. $x^2 - 49$
105. $32y^2 - 18$
106. $4 - 36y^2$
107. $16x^2 - \frac{1}{9}$
108. $\frac{4}{25}y^2 - 64$
109. $(x - 1)^2 - 4$ **110.** $25 - (z + 5)^2$
111. $9u^2 - 4v^2$ **112.** $25x^2 - 16y^2$

In Exercises 113–122, factor the perfect square trinomial.

113. $x^2 - 4x + 4$ **114.** $x^2 + 10x + 25$
115. $4t^2 + 4t + 1$ **116.** $9x^2 - 12x + 4$
117. $25y^2 - 10y + 1$ **118.** $36y^2 - 108y + 81$
119. $9u^2 + 24uv + 16v^2$ **120.** $4x^2 - 4xy + y^2$
121. $x^2 - \frac{4}{3}x + \frac{4}{9}$ **122.** $z^2 + z + \frac{1}{4}$

In Exercises 123–130, factor the sum or difference of cubes.

123. $x^3 - 8$ **124.** $x^3 - 27$
125. $y^3 + 64$ **126.** $z^3 + 125$
127. $8t^3 - 1$ **128.** $27x^3 + 8$
129. $u^3 + 27v^3$ **130.** $64x^3 - y^3$

In Exercises 131–144, factor the trinomial.

131. $x^2 + x - 2$ **132.** $x^2 + 5x + 6$
133. $s^2 - 5s + 6$ **134.** $t^2 - t - 6$
135. $20 - y - y^2$ **136.** $24 + 5z - z^2$
137. $x^2 - 30x + 200$ **138.** $x^2 - 13x + 42$
139. $3x^2 - 5x + 2$ **140.** $2x^2 - x - 1$
141. $5x^2 + 26x + 5$ **142.** $12x^2 + 7x + 1$
143. $-9z^2 + 3z + 2$ **144.** $-5u^2 - 13u + 6$

In Exercises 145–152, factor by grouping.

145. $x^3 - x^2 + 2x - 2$ **146.** $x^3 + 5x^2 - 5x - 25$
147. $2x^3 - x^2 - 6x + 3$ **148.** $5x^3 - 10x^2 + 3x - 6$
149. $6 + 2x - 3x^3 - x^4$ **150.** $x^5 + 2x^3 + x^2 + 2$
151. $6x^3 - 2x + 3x^2 - 1$ **152.** $8x^5 - 6x^2 + 12x^3 - 9$

In Exercises 153–158, factor the trinomial by grouping.

153. $3x^2 + 10x + 8$ **154.** $2x^2 + 9x + 9$
155. $6x^2 + x - 2$ **156.** $6x^2 - x - 15$
157. $15x^2 - 11x + 2$ **158.** $12x^2 - 13x + 1$

In Exercises 159–192, completely factor the expression.

159. $6x^2 - 54$

160. $12x^2 - 48$

161. $x^3 - 4x^2$

162. $x^3 - 9x$

163. $x^2 - 2x + 1$

164. $16 + 6x - x^2$

165. $1 - 4x + 4x^2$

166. $-9x^2 + 6x - 1$

167. $2x^2 + 4x - 2x^3$

168. $2y^3 - 7y^2 - 15y$

169. $9x^2 + 10x + 1$

170. $13x + 6 + 5x^2$

171. $\frac{1}{81}x^2 + \frac{2}{9}x - 8$

172. $\frac{1}{8}x^2 - \frac{1}{96}x - \frac{1}{16}$

173. $3x^3 + x^2 + 15x + 5$

174. $5 - x + 5x^2 - x^3$

175. $x^4 - 4x^3 + x^2 - 4x$

176. $3u - 2u^2 + 6 - u^3$

177. $\frac{1}{4}x^3 + 3x^2 + \frac{3}{4}x + 9$

178. $\frac{1}{5}x^3 + x^2 - x - 5$

179. $(t - 1)^2 - 49$

180. $(x^2 + 1)^2 - 4x^2$

181. $(x^2 + 8)^2 - 36x^2$

182. $2t^3 - 16$

183. $5x^3 + 40$

184. $4x(2x - 1) + (2x - 1)^2$

185. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$

186. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$

187. $7(3x + 2)^2(1 - x)^2 + (3x + 2)(1 - x)^3$

188. $7x(2)(x^2 + 1)(2x) - (x^2 + 1)^2(7)$

189. $3(x - 2)^2(x + 1)^4 + (x - 2)^3(4)(x + 1)^3$

190. $2x(x - 5)^4 - x^2(4)(x - 5)^3$

191. $5(x^6 + 1)^4(6x^5)(3x + 2)^3 + 3(3x + 2)^2(3)(x^6 + 1)^5$

192. $\frac{x^2}{2}(x^2 + 1)^4 - (x^2 + 1)^5$

In Exercises 193–196, find all values of *b* for which the trinomial can be factored.

193. $x^2 + bx - 15$

194. $x^2 + bx + 50$

195. $x^2 + bx - 12$

196. $x^2 + bx + 24$

In Exercises 197–200, find two integer values of *c* such that the trinomial can be factored. (There are many correct answers.)

197. $2x^2 + 5x + c$

198. $3x^2 - 10x + c$

199. $3x^2 - x + c$

200. $2x^2 + 9x + c$

201. ***Cost, Revenue, and Profit*** An electronics manufacturer can produce and sell x radios per week. The total cost C (in dollars) of producing x radios is

$C = 73x + 25{,}000$

and the total revenue R (in dollars) is

$R = 95x.$

(a) Find the profit P in terms of x.

(b) Find the profit obtained by selling 5000 radios per week.

202. ***Cost, Revenue, and Profit*** An artisan can produce and sell x hats per month. The total cost C (in dollars) of producing x hats is

$C = 460 + 12x$

and the total revenue R (in dollars) is

$R = 36x.$

(a) Find the profit P in terms of x.

(b) Find the profit obtained by selling 42 hats per month.

203. ***Compound Interest*** After 2 years, an investment of \$500 compounded annually at an interest rate r will yield an amount of $500(1 + r)^2$.

(a) Write this polynomial in standard form.

(b) Use a calculator to evaluate the polynomial for the values of r shown in the table.

r	$2\frac{1}{2}\%$	3%	4%	$4\frac{1}{2}\%$	5%
$500(1 + r)^2$					

(c) What conclusion can you make from the table?

204. ***Compound Interest*** After 3 years, an investment of \$1200 compounded annually at an interest rate r will yield an amount of $1200(1 + r)^3$.

(a) Write this polynomial in standard form.

(b) Use a calculator to evaluate the polynomial for the values of r shown in the table.

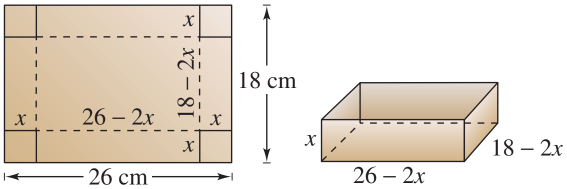
r	2%	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$
$1200(1 + r)^3$					

(c) What conclusion can you make from the table?

205. ***Volume of a Box*** A take-out fast-food restaurant is constructing an open box by cutting squares from the corners of a piece of cardboard that is 18 centimeters by 26 centimeters (see figure). The edge of each cut-out square is x centimeters.

(a) Find the volume of the box in terms of x.

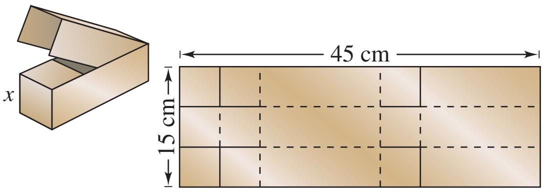
(b) Find the volume when $x = 1$, $x = 2$, and $x = 3$.



206. ***Volume of a Box*** An overnight shipping company is designing a closed box by cutting along the solid lines and folding along the broken lines on the rectangular piece of corrugated cardboard shown in the figure. The length and width of the rectangle are 45 centimeters and 15 centimeters, respectively.

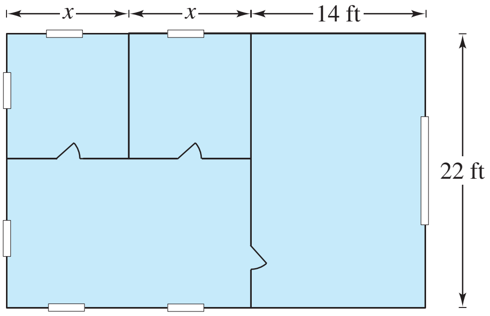
(a) Find the volume of the shipping box in terms of x.

(b) Find the volume when $x = 3$, $x = 5$, and $x = 7$.

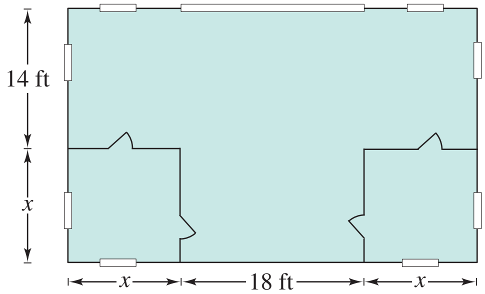


Geometry **In Exercises 207 and 208, find a polynomial that represents the total number of square feet for the floor plan shown in the figure.**

207.

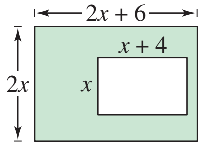


208.

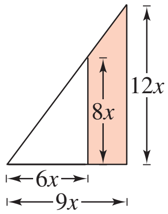


209. ***Geometry*** Find the area of the shaded region in each figure. Write your result as a polynomial in standard form.

(a)



(b)



210. ***Stopping Distance*** The stopping distance of an automobile is the distance traveled during the driver's reaction time plus the distance traveled after the brakes are applied. In an experiment, these distances were measured (in feet) when the automobile was traveling at a speed of x miles per hour on dry, level pavement, as shown in the bar graph. The distance traveled during the reaction time R was

$$R = 1.1x$$

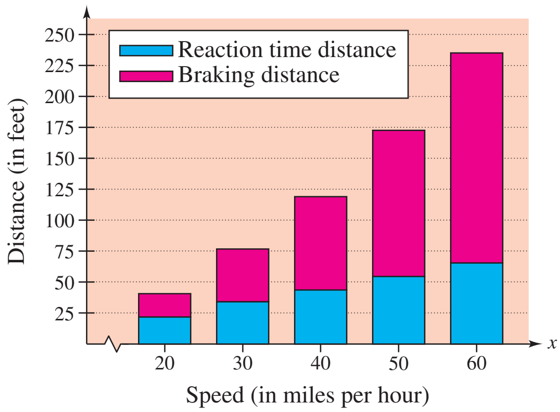
and the braking distance B was

$$B = 0.0475x^2 - 0.001x + 0.23.$$

(a) Determine the polynomial that represents the total stopping distance T.

(b) Use the result of part (a) to estimate the total stopping distance when $x = 30$, $x = 40$, and $x = 55$ miles per hour.

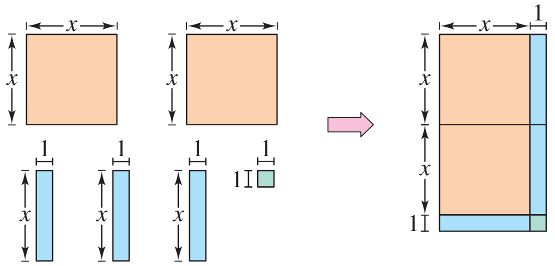
(c) Use the bar graph to make a statement about the total stopping distance required for increasing speeds.



Geometric Modeling **In Exercises 211–214, draw a "geometric factoring model" to represent the factorization. For instance, a factoring model for**

$$2x^2 + 3x + 1 = (2x + 1)(x + 1)$$

is shown in the figure.



211. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

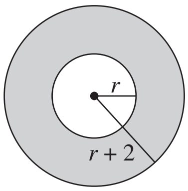
212. $x^2 + 4x + 3 = (x + 3)(x + 1)$

213. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

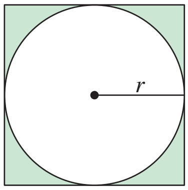
214. $x^2 + 3x + 2 = (x + 2)(x + 1)$

Geometry **In Exercises 215–218, write an expression in factored form for the area of the shaded portion of the figure.**

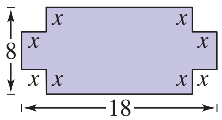
215.



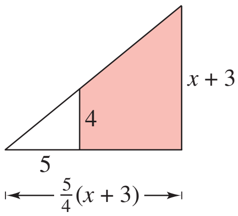
216.



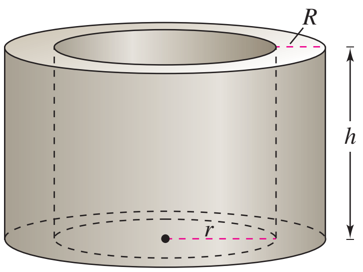
217.



218.



219. ***Geometry*** The volume V of concrete used to make the cylindrical concrete storage tank shown in the figure is $V = \pi R^2h - \pi r^2h$ where R is the outside radius, r is the inside radius, and h is the height of the storage tank.



(a) Factor the expression for the volume.

(b) From the result of part (a), show that the volume of concrete is

2π(average radius)(thickness of the tank)h.

220. ***Chemistry*** The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.

Synthesis

True or False? **In Exercises 221–224, determine whether the statement is true or false. Justify your answer.**

221. The product of two binomials is always a second-degree polynomial.

222. The sum of two binomials is always a binomial.

223. The difference of two perfect squares can be factored as the product of conjugate pairs.

224. The sum of two perfect squares can be factored as the binomial sum squared.

225. Find the degree of the product of two polynomials of degrees m and n.

226. Find the degree of the sum of two polynomials of degrees m and n if $m < n$.

227. ***Think About It*** When the polynomial

$$-x^3 + 3x^2 + 2x - 1$$

is subtracted from an unknown polynomial, the difference is

$$5x^2 + 8.$$

If it is possible, find the unknown polynomial.

228. ***Logical Reasoning*** Verify that $(x + y)^2$ is not equal to $x^2 + y^2$ by letting $x = 3$ and $y = 4$ and evaluating both expressions. Are there any values of x and y for which $(x + y)^2 = x^2 + y^2$? Explain.

229. Factor $x^{2n} - y^{2n}$ completely.

230. Factor $x^{3n} + y^{3n}$ completely.

231. Factor $x^{3n} - y^{2n}$ completely.

232. ***Writing*** Explain what is meant when it is said that a polynomial is in factored form.

233. Give an example of a polynomial that is prime with respect to the integers.

A.4 Rational Expressions

What you should learn

- Find domains of algebraic expressions.
- Simplify rational expressions.
- Add, subtract, multiply, and divide rational expressions.
- Simplify complex fractions and rewrite difference quotients.

Why you should learn it

Rational expressions can be used to solve real-life problems. For instance, in Exercise 84 on page A45, a rational expression is used to model the projected number of households banking and paying bills online from 2002 through 2007.

Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, $(x + 1) + (x + 2)$ and $2x + 3$ are equivalent because

$$\begin{aligned}(x + 1) + (x + 2) &= x + 1 + x + 2\\ &= x + x + 1 + 2\\ &= 2x + 3.\end{aligned}$$

Example 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, unless the domain is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or equal to 2, because the square root of a negative number is not a real number.

c. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would result in division by zero, which is undefined.

CHECKPOINT Now try Exercise 1.

The quotient of two algebraic expressions is a *fractional expression*. Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \qquad \frac{2x - 1}{x + 1}, \qquad \text{or} \qquad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.** Recall that a fraction is in simplest form if its numerator and denominator have no factors in common aside from ± 1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \qquad c \neq 0$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials.

Simplifying Rational Expressions

When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common.

In this text, when a rational expression is written, the domain is usually not listed with the expression. It is *implied* that the real numbers that make the denominator zero are excluded from the expression. Also, when performing operations with rational expressions, this text follows the convention of listing *by the simplified expression* all values of x that must be specifically excluded from the domain in order to make the domains of the simplified and original expressions agree.

Example 2 Simplifying a Rational Expression

Write $\dfrac{x^2 + 4x - 12}{3x - 6}$ in simplest form.

Solution

$$\frac{x^2 + 4x - 12}{3x - 6} = \frac{(x + 6)\cancel{(x - 2)}}{3\cancel{(x - 2)}} \qquad \text{Factor completely.}$$

$$= \frac{x + 6}{3}, \quad x \neq 2 \qquad \text{Divide out common factors.}$$

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make sure that the simplified expression is *equivalent* to the original expression, you must restrict the domain of the simplified expression by excluding the value $x = 2$.

CHECKPOINT Now try Exercise 19.

STUDY TIP

In Example 2, do not make the mistake of trying to simplify further by dividing out terms.

$$\cancel{\frac{x + 6}{3} = \frac{x + \cancel{6}}{\cancel{3}} = x + 2}$$

Remember that to simplify fractions, divide out common *factors*, not terms.

Sometimes it may be necessary to change the sign of a factor to simplify a rational expression, as shown in Example 3.

Example 3 Simplifying Rational Expressions

Write $\dfrac{12 + x - x^2}{2x^2 - 9x + 4}$ in simplest form.

Solution

$$\frac{12 + x - x^2}{2x^2 - 9x + 4} = \frac{(4 - x)(3 + x)}{(2x - 1)(x - 4)} \qquad \text{Factor completely.}$$

$$= \frac{-\cancel{(x - 4)}(3 + x)}{(2x - 1)\cancel{(x - 4)}} \qquad (4 - x) = -(x - 4)$$

$$= -\frac{3 + x}{2x - 1}, \quad x \neq 4 \qquad \text{Divide out common factors.}$$

CHECKPOINT Now try Exercise 25.

Operations with Rational Expressions

To multiply or divide rational expressions, use the properties of fractions discussed in Appendix A.1. Recall that to divide fractions, you invert the divisor and multiply.

Example 4 Multiplying Rational Expressions

$$\frac{2x^2+x-6}{x^2+4x-5}\cdot\frac{x^3-3x^2+2x}{4x^2-6x}=\frac{\cancel{(2x-3)}(x+2)}{(x+5)\cancel{(x-1)}}\cdot\frac{\cancel{x}(x-2)\cancel{(x-1)}}{2\cancel{x}\cancel{(2x-3)}}$$

$$=\frac{(x+2)(x-2)}{2(x+5)},\qquad x\neq 0,\ x\neq 1,\ x\neq \tfrac{3}{2}$$

CHECKPOINT Now try Exercise 39.

In Example 4 the restrictions $x \neq 0$, $x \neq 1$, and $x \neq \frac{3}{2}$ are listed with the simplified expression in order to make the two domains agree. Note that the value $x = -5$ is excluded from both domains, so it is not necessary to list this value.

Example 5 Dividing Rational Expressions

$$\frac{x^3-8}{x^2-4}\div\frac{x^2+2x+4}{x^3+8}=\frac{x^3-8}{x^2-4}\cdot\frac{x^3+8}{x^2+2x+4}\qquad\text{Invert and multiply.}$$

$$=\frac{\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+2)}\cancel{(x-2)}}\cdot\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x^2+2x+4)}}$$

$$=x^2-2x+4,\qquad x\neq\pm 2\qquad\text{Divide out common factors.}$$

CHECKPOINT Now try Exercise 41.

To add or subtract rational expressions, you can use the LCD (least common denominator) method or the *basic definition*

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd},\qquad b\neq 0,\ d\neq 0.\qquad\text{Basic definition}$$

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

Example 6 Subtracting Rational Expressions

$$\frac{x}{x-3}-\frac{2}{3x+4}=\frac{x(3x+4)-2(x-3)}{(x-3)(3x+4)}\qquad\text{Basic definition}$$

$$=\frac{3x^2+4x-2x+6}{(x-3)(3x+4)}\qquad\text{Distributive Property}$$

$$=\frac{3x^2+2x+6}{(x-3)(3x+4)}\qquad\text{Combine like terms.}$$

CHECKPOINT Now try Exercise 49.

STUDY TIP

When subtracting rational expressions, remember to distribute the negative sign to all the terms in the quantity that is being subtracted.

For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\frac{1}{6}+\frac{3}{4}-\frac{2}{3}=\frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4}\qquad \text{The LCD is 12.}$$

$$=\frac{2}{12}+\frac{9}{12}-\frac{8}{12}$$

$$=\frac{3}{12}$$

$$=\frac{1}{4}$$

Sometimes the numerator of the answer has a factor in common with the denominator. In such cases the answer should be simplified. For instance, in the example above, $\frac{3}{12}$ was simplified to $\frac{1}{4}$.

Example 7 Combining Rational Expressions: The LCD Method

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution

Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the LCD is $x(x+1)(x-1)$.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)}\qquad \text{Distributive Property}$$

$$=\frac{3x^2-2x^2+x^2+3x+3x+2}{x(x+1)(x-1)}\qquad \text{Group like terms.}$$

$$=\frac{2x^2+6x+2}{x(x+1)(x-1)}\qquad \text{Combine like terms.}$$

$$=\frac{2(x^2+3x+1)}{x(x+1)(x-1)}\qquad \text{Factor.}$$

CHECKPOINT Now try Exercise 51.

Complex Fractions and the Difference Quotient

Fractional expressions with separate fractions in the numerator, denominator, or both are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

A complex fraction can be simplified by combining the fractions in its numerator into a single fraction and then combining the fractions in its denominator into a single fraction. Then invert the denominator and multiply.

Example 8 Simplifying a Complex Fraction

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]} \qquad \text{Combine fractions.}$$

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)} \qquad \text{Simplify.}$$

$$= \frac{2-3x}{x}\cdot\frac{x-1}{x-2} \qquad \text{Invert and multiply.}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \qquad x \neq 1$$

CHECKPOINT Now try Exercise 57.

Another way to simplify a complex fraction is to multiply its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)}\cdot\frac{x(x-1)}{x(x-1)} \qquad \text{LCD is } x(x-1).$$

$$= \frac{\left(\frac{2-3x}{\cancel{x}}\right)\cdot \cancel{x}(x-1)}{\left(\frac{x-2}{\cancel{x-1}}\right)\cdot x\cancel{(x-1)}}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \qquad x \neq 1$$

The next three examples illustrate some methods for simplifying rational expressions involving negative exponents and radicals. These types of expressions occur frequently in calculus.

To simplify an expression with negative exponents, one method is to begin by factoring out the common factor with the *smaller* exponent. Remember that when factoring, you *subtract* exponents. For instance, in $3x^{-5/2} + 2x^{-3/2}$ the smaller exponent is $-\frac{5}{2}$ and the common factor is $x^{-5/2}$.

$$\begin{aligned}3x^{-5/2} + 2x^{-3/2} &= x^{-5/2}[3(1) + 2x^{-3/2-(-5/2)}]\\ &= x^{-5/2}(3 + 2x^1)\\ &= \frac{3 + 2x}{x^{5/2}}\end{aligned}$$

Example 9 **Simplifying an Expression**

Simplify the following expression containing negative exponents.

$$x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2}$$

Solution

Begin by factoring out the common factor with the *smaller exponent*.

$$\begin{aligned}x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} &= (1 - 2x)^{-3/2}[x + (1 - 2x)^{(-1/2)-(-3/2)}]\\ &= (1 - 2x)^{-3/2}[x + (1 - 2x)^1]\\ &= \frac{1 - x}{(1 - 2x)^{3/2}}\end{aligned}$$

CHECKPOINT Now try Exercise 65.

A second method for simplifying an expression with negative exponents is shown in the next example.

Example 10 **Simplifying an Expression with Negative Exponents**

$$\begin{aligned}&\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2}\\ &\qquad= \frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \cdot \frac{(4 - x^2)^{1/2}}{(4 - x^2)^{1/2}}\\ &\qquad= \frac{(4 - x^2)^1 + x^2(4 - x^2)^0}{(4 - x^2)^{3/2}}\\ &\qquad= \frac{4 - x^2 + x^2}{(4 - x^2)^{3/2}}\\ &\qquad= \frac{4}{(4 - x^2)^{3/2}}\end{aligned}$$

CHECKPOINT Now try Exercise 67.

Example 11 Rewriting a Difference Quotient

The following expression from calculus is an example of a *difference quotient*.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rewrite this expression by rationalizing its numerator.

Solution

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \qquad h \neq 0$$

Notice that the original expression is undefined when $h = 0$. So, you must exclude $h = 0$ from the domain of the simplified expression so that the expressions are equivalent.

CHECKPOINT Now try Exercise 73.

Difference quotients, such as that in Example 11, occur frequently in calculus. Often, they need to be rewritten in an equivalent form that can be evaluated when $h = 0$. Note that the equivalent form is not simpler than the original form, but it has the advantage that it is defined when $h = 0$.

A.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The set of real numbers for which an algebraic expression is defined is the ________ of the expression.

2. The quotient of two algebraic expressions is a fractional expression and the quotient of two polynomials is a ________ ________.

3. Fractional expressions with separate fractions in the numerator, denominator, or both are called ________ fractions.

4. To simplify an expression with negative exponents, it is possible to begin by factoring out the common factor with the ________ exponent.

5. Two algebraic expressions that have the same domain and yield the same values for all numbers in their domains are called ________.

6. An important rational expression, such as $\frac{(x+h)^2 - x^2}{h}$, that occurs in calculus is called a ________ ________.

In Exercises 1–8, find the domain of the expression.

1. $3x^2 - 4x + 7$

2. $2x^2 + 5x - 2$

3. $4x^3 + 3, \quad x \geq 0$

4. $6x^2 - 9, \quad x > 0$

5. $\dfrac{1}{x - 2}$

6. $\dfrac{x + 1}{2x + 1}$

7. $\sqrt{x + 1}$

8. $\sqrt{6 - x}$

In Exercises 9 and 10, find the missing factor in the numerator such that the two fractions are equivalent.

9. $\dfrac{5}{2x} = \dfrac{5(\square)}{6x^2}$

10. $\dfrac{3}{4} = \dfrac{3(\square)}{4(x + 1)}$

In Exercises 11–28, write the rational expression in simplest form.

11. $\dfrac{15x^2}{10x}$

12. $\dfrac{18y^2}{60y^5}$

13. $\dfrac{3xy}{xy + x}$

14. $\dfrac{2x^2y}{xy - y}$

15. $\dfrac{4y - 8y^2}{10y - 5}$

16. $\dfrac{9x^2 + 9x}{2x + 2}$

17. $\dfrac{x - 5}{10 - 2x}$

18. $\dfrac{12 - 4x}{x - 3}$

19. $\dfrac{y^2 - 16}{y + 4}$

20. $\dfrac{x^2 - 25}{5 - x}$

21. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - 4}$

22. $\dfrac{x^2 + 8x - 20}{x^2 + 11x + 10}$

23. $\dfrac{y^2 - 7y + 12}{y^2 + 3y - 18}$

24. $\dfrac{x^2 - 7x + 6}{x^2 + 11x + 10}$

25. $\dfrac{2 - x + 2x^2 - x^3}{x^2 - 4}$

26. $\dfrac{x^2 - 9}{x^3 + x^2 - 9x - 9}$

27. $\dfrac{z^3 - 8}{z^2 + 2z + 4}$

28. $\dfrac{y^3 - 2y^2 - 3y}{y^3 + 1}$

In Exercises 29 and 30, complete the table. What can you conclude?

29.

x	0	1	2	3	4	5	6
$\dfrac{x^2 - 2x - 3}{x - 3}$							
$x + 1$							

30.

x	0	1	2	3	4	5	6
$\dfrac{x - 3}{x^2 - x - 6}$							
$\dfrac{1}{x + 2}$							

31. ***Error Analysis*** Describe the error.

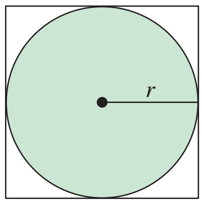
$$\frac{5x^3}{2x^3 + 4} = \frac{5x^3}{2x^3 + 4} = \frac{5}{2 + 4} = \frac{5}{6}$$

32. ***Error Analysis*** Describe the error.

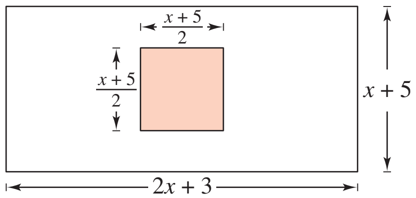
$$\frac{x^3 + 25x}{x^2 - 2x - 15} = \frac{x(x^2 + 25)}{(x - 5)(x + 3)}$$
$$= \frac{x(x + 5)(x - 5)}{(x - 5)(x + 3)} = \frac{x(x + 5)}{x + 3}$$

Geometry **In Exercises 33 and 34, find the ratio of the area of the shaded portion of the figure to the total area of the figure.**

33.



34.



In Exercises 35–42, perform the multiplication or division and simplify.

35. $\dfrac{5}{x - 1} \cdot \dfrac{x - 1}{25(x - 2)}$

36. $\dfrac{x + 13}{x^3(3 - x)} \cdot \dfrac{x(x - 3)}{5}$

37. $\dfrac{r}{r - 1} \cdot \dfrac{r^2 - 1}{r^2}$

38. $\dfrac{4y - 16}{5y + 15} \cdot \dfrac{2y + 6}{4 - y}$

39. $\dfrac{t^2 - t - 6}{t^2 + 6t + 9} \cdot \dfrac{t + 3}{t^2 - 4}$

40. $\dfrac{x^2 + xy - 2y^2}{x^3 + x^2y} \cdot \dfrac{x}{x^2 + 3xy + 2y^2}$

41. $\dfrac{x^2 - 36}{x} \div \dfrac{x^3 - 6x^2}{x^2 + x}$

42. $\dfrac{x^2 - 14x + 49}{x^2 - 49} \div \dfrac{3x - 21}{x + 7}$

In Exercises 43–52, perform the addition or subtraction and simplify.

43. $\dfrac{5}{x-1}+\dfrac{x}{x-1}$

44. $\dfrac{2x-1}{x+3}+\dfrac{1-x}{x+3}$

45. $6-\dfrac{5}{x+3}$

46. $\dfrac{3}{x-1}-5$

47. $\dfrac{3}{x-2}+\dfrac{5}{2-x}$

48. $\dfrac{2x}{x-5}-\dfrac{5}{5-x}$

49. $\dfrac{1}{x^2-x-2}-\dfrac{x}{x^2-5x+6}$

50. $\dfrac{2}{x^2-x-2}+\dfrac{10}{x^2+2x-8}$

51. $-\dfrac{1}{x}+\dfrac{2}{x^2+1}+\dfrac{1}{x^3+x}$

52. $\dfrac{2}{x+1}+\dfrac{2}{x-1}+\dfrac{1}{x^2-1}$

Error Analysis **In Exercises 53 and 54, describe the error.**

53.
$$\frac{x+4}{x+2}-\frac{3x-8}{x+2}=\frac{x+4-3x-8}{x+2}=\frac{-2x-4}{x+2}=\frac{-2(x+2)}{x+2}=-2$$

54.
$$\frac{6-x}{x(x+2)}+\frac{x+2}{x^2}+\frac{8}{x^2(x+2)}=\frac{x(6-x)+(x+2)^2+8}{x^2(x+2)}=\frac{6x-x^2+x^2+4+8}{x^2(x+2)}=\frac{6(x+2)}{x^2(x+2)}=\frac{6}{x^2}$$

In Exercises 55–60, simplify the complex fraction.

55. $\dfrac{\left(\dfrac{x}{2}-1\right)}{(x-2)}$

56. $\dfrac{(x-4)}{\left(\dfrac{x}{4}-\dfrac{4}{x}\right)}$

57. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$

58. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

59. $\dfrac{\left(\sqrt{x}-\dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

60. $\dfrac{\left(\dfrac{t^2}{\sqrt{t^2+1}}-\sqrt{t^2+1}\right)}{t^2}$

In Exercises 61–66, factor the expression by removing the common factor with the smaller exponent.

61. x^5-2x^{-2}

62. x^5-5x^{-3}

63. $x^2(x^2+1)^{-5}-(x^2+1)^{-4}$

64. $2x(x-5)^{-3}-4x^2(x-5)^{-4}$

65. $2x^2(x-1)^{1/2}-5(x-1)^{-1/2}$

66. $4x^3(2x-1)^{3/2}-2x(2x-1)^{-1/2}$

In Exercises 67 and 68, simplify the expression.

67. $\dfrac{3x^{1/3}-x^{-2/3}}{3x^{-2/3}}$

68. $\dfrac{-x^3(1-x^2)^{-1/2}-2x(1-x^2)^{1/2}}{x^4}$

In Exercises 69–72, simplify the difference quotient.

69. $\dfrac{\left(\dfrac{1}{x+h}-\dfrac{1}{x}\right)}{h}$

70. $\dfrac{\left[\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}\right]}{h}$

71. $\dfrac{\left(\dfrac{1}{x+h-4}-\dfrac{1}{x-4}\right)}{h}$

72. $\dfrac{\left(\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}\right)}{h}$

In Exercises 73–76, simplify the difference quotient by rationalizing the numerator.

73. $\dfrac{\sqrt{x+2}-\sqrt{x}}{2}$

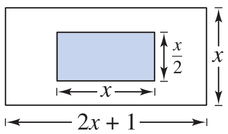
74. $\dfrac{\sqrt{z-3}-\sqrt{z}}{3}$

75. $\dfrac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$

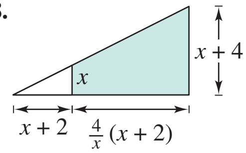
76. $\dfrac{\sqrt{x+h-2}-\sqrt{x-2}}{h}$

Probability **In Exercises 77 and 78, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.**

77.



78.



79. ***Rate*** A photocopier copies at a rate of 16 pages per minute.

(a) Find the time required to copy one page.

(b) Find the time required to copy x pages.

(c) Find the time required to copy 60 pages.

80. ***Rate*** After working together for t hours on a common task, two workers have done fractional parts of the job equal to $t/3$ and $t/5$, respectively. What fractional part of the task has been completed?

Finance **In Exercises 81 and 82, the formula that approximates the annual interest rate r of a monthly installment loan is given by**

$$r = \frac{\left[\dfrac{24(NM - P)}{N}\right]}{\left(P + \dfrac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

81. (a) Approximate the annual interest rate for a four-year car loan of \$16,000 that has monthly payments of \$400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

82. (a) Approximate the annual interest rate for a five-year car loan of \$28,000 that has monthly payments of \$525.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

83. ***Refrigeration*** When food (at room temperature) is placed in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. The model that gives the temperature of food that has an original temperature of 75°F and is placed in a 40°F refrigerator is

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours).

(a) Complete the table.

t	0	2	4	6	8	10
T						

t	12	14	16	18	20	22
T						

(b) What value of T does the mathematical model appear to be approaching?

84. ***Interactive Money Management*** The table shows the projected numbers of U.S. households (in millions) banking online and paying bills online for the years 2002 through 2007. (Source: eMarketer; Forrester Research)

Year	Banking	Paying Bills
2002	21.9	13.7
2003	26.8	17.4
2004	31.5	20.9
2005	35.0	23.9
2006	40.0	26.7
2007	45.0	29.1

Mathematical models for these data are

$$\text{Number banking online} = \frac{-0.728t^2 + 23.81t - 0.3}{-0.049t^2 + 0.61t + 1.0}$$

and

$$\text{Number paying bills online} = \frac{4.39t + 5.5}{0.002t^2 + 0.01t + 1.0}$$

where t represents the year, with $t = 2$ corresponding to 2002.

(a) Using the models, create a table to estimate the projected number of households banking online and the projected number of households paying bills online for the given years.

(b) Compare the values given by the models with the actual data.

(c) Determine a model for the ratio of the projected number of households paying bills online to the projected number of households banking online.

(d) Use the model from part (c) to find the ratio over the given years. Interpret your results.

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. $\dfrac{x^{2n} - 1^{2n}}{x^n - 1^n} = x^n + 1^n$

86. $\dfrac{x^2 - 3x + 2}{x - 1} = x - 2$, for all values of x.

87. ***Think About It*** How do you determine whether a rational expression is in simplest form?

A.5 Solving Equations

What you should learn

- Identify different types of equations.
- Solve linear equations in one variable and equations that lead to linear equations.
- Solve quadratic equations by factoring, extracting square roots, completing the square, and using the Quadratic Formula.
- Solve polynomial equations of degree three or greater.
- Solve equations involving radicals.
- Solve equations with absolute values.

Why you should learn it

Linear equations are used in many real-life applications. For example, in Exercise 185 on page A58, linear equations can be used to model the relationship between the length of a thigh-bone and the height of a person, helping researchers learn about ancient cultures.

Equations and Solutions of Equations

An **equation** in x is a statement that two algebraic expressions are equal. For example

$$3x - 5 = 7, \; x^2 - x - 6 = 0, \text{ and } \sqrt{2x} = 4$$

are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation

$$3x - 5 = 7$$

because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers, the equation has the two solutions $x = \sqrt{10}$ and $x = -\sqrt{10}$.

An equation that is true for *every* real number in the *domain* of the variable is called an **identity.** The domain is the set of all real numbers for which the equation is defined. For example

$$x^2 - 9 = (x + 3)(x - 3) \qquad \text{Identity}$$

is an identity because it is a true statement for any real value of x. The equation

$$\frac{x}{3x^2} = \frac{1}{3x} \qquad \text{Identity}$$

where $x \neq 0$, is an identity because it is true for any nonzero real value of x.

An equation that is true for just *some* (or even none) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation

$$x^2 - 9 = 0 \qquad \text{Conditional equation}$$

is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation. The equation $2x - 4 = 2x + 1$ is conditional because there are no real values of x for which the equation is true. Learning to solve conditional equations is the primary focus of this section.

Linear Equations in One Variable

Definition of a Linear Equation

A **linear equation in one variable** x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$.

A linear equation has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$$ax + b = 0 \qquad \text{Write original equation.}$$

$$ax = -b \qquad \text{Subtract } b \text{ from each side.}$$

$$x = -\frac{b}{a} \qquad \text{Divide each side by } a.$$

To solve a conditional equation in x, isolate x on one side of the equation by a sequence of **equivalent** (and usually simpler) **equations,** each having the same solution(s) as the original equation. The operations that yield equivalent equations come from the Substitution Principle (see Appendix A.1) and simplification techniques.

Generating Equivalent Equations

An equation can be transformed into an *equivalent equation* by one or more of the following steps.

	Given Equation	*Equivalent Equation*
1. Remove symbols of grouping, combine like terms, or simplify fractions on one or both sides of the equation.	$2x - x = 4$	$x = 4$
2. Add (or subtract) the same quantity to (from) *each* side of the equation.	$x + 1 = 6$	$x = 5$
3. Multiply (or divide) *each* side of the equation by the same *nonzero* quantity.	$2x = 6$	$x = 3$
4. Interchange the two sides of the equation.	$2 = x$	$x = 2$

STUDY TIP

After solving an equation, you should check each solution in the original equation. For instance, you can check the solution to Example 1(a) as follows.

$$3x - 6 = 0 \qquad \text{Write original equation.}$$

$$3(2) - 6 \stackrel{?}{=} 0 \qquad \text{Substitute 2 for } x.$$

$$0 = 0 \qquad \text{Solution checks. } \checkmark$$

Try checking the solution to Example 1(b).

Example 1 Solving a Linear Equation

a. $3x - 6 = 0$ Original equation

$3x = 6$ Add 6 to each side.

$x = 2$ Divide each side by 3.

b. $5x + 4 = 3x - 8$ Original equation

$2x + 4 = -8$ Subtract $3x$ from each side.

$2x = -12$ Subtract 4 from each side.

$x = -6$ Divide each side by 2.

CHECKPOINT Now try Exercise 13.

STUDY TIP

An equation with a *single fraction* on each side can be cleared of denominators by **cross multiplying,** which is equivalent to multiplying by the LCD and then dividing out. To do this, multiply the left numerator by the right denominator and the right numerator by the left denominator as follows.

$$\frac{a}{b} = \frac{c}{d} \qquad \text{LCD is } bd.$$

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd \qquad \text{Multiply by LCD.}$$

$$ad = cb \qquad \text{Divide out common factors.}$$

To solve an equation involving fractional expressions, find the least common denominator (LCD) of all terms and multiply every term by the LCD. This process will clear the original equation of fractions and produce a simpler equation to work with.

Example 2 An Equation Involving Fractional Expressions

Solve $\frac{x}{3} + \frac{3x}{4} = 2$.

Solution

$$\frac{x}{3} + \frac{3x}{4} = 2 \qquad \text{Write original equation.}$$

$$(12)\frac{x}{3} + (12)\frac{3x}{4} = (12)2 \qquad \text{Multiply each term by the LCD of 12.}$$

$$4x + 9x = 24 \qquad \text{Divide out and multiply.}$$

$$13x = 24 \qquad \text{Combine like terms.}$$

$$x = \frac{24}{13} \qquad \text{Divide each side by 13.}$$

The solution is $x = \frac{24}{13}$. Check this in the original equation.

CHECKPOINT Now try Exercise 21.

When multiplying or dividing an equation by a *variable* quantity, it is possible to introduce an extraneous solution. An **extraneous solution** is one that does not satisfy the original equation. Therefore, it is essential that you check your solutions.

Example 3 An Equation with an Extraneous Solution

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

Solution

The LCD is $x^2 - 4$, or $(x + 2)(x - 2)$. Multiply each term by this LCD.

$$\frac{1}{x-2}(x+2)(x-2) = \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x + 2 = 3(x - 2) - 6x, \qquad x \neq \pm 2$$

$$x + 2 = 3x - 6 - 6x$$

$$x + 2 = -3x - 6$$

$$4x = -8 \quad \Rightarrow \quad x = -2 \qquad \text{Extraneous solution}$$

In the original equation, $x = -2$ yields a denominator of zero. So, $x = -2$ is an extraneous solution, and the original equation has *no solution*.

CHECKPOINT Now try Exercise 37.

STUDY TIP

Recall that the least common denominator of two or more fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. For instance, in Example 3, by factoring each denominator you can determine that the LCD is $(x + 2)(x - 2)$.

Quadratic Equations

A **quadratic equation** in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers, with $a \neq 0$. A quadratic equation in x is also known as a **second-degree polynomial equation** in x.

You should be familiar with the following four methods of solving quadratic equations.

Solving a Quadratic Equation

Factoring: If $ab = 0$, then $a = 0$ or $b = 0$.

Example:

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x - 3 = 0 \quad \Rightarrow \quad x = 3$$

$$x + 2 = 0 \quad \Rightarrow \quad x = -2$$

Square Root Principle: If $u^2 = c$, where $c > 0$, then $u = \pm\sqrt{c}$.

Example:

$$(x + 3)^2 = 16$$

$$x + 3 = \pm 4$$

$$x = -3 \pm 4$$

$$x = 1 \quad \text{or} \quad x = -7$$

Completing the Square: If $x^2 + bx = c$, then

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2 \qquad \text{Add } \left(\frac{b}{2}\right)^2 \text{ to each side.}$$

$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}.$$

Example:

$$x^2 + 6x = 5$$

$$x^2 + 6x + 3^2 = 5 + 3^2 \qquad \text{Add } \left(\frac{6}{2}\right)^2 \text{ to each side.}$$

$$(x + 3)^2 = 14$$

$$x + 3 = \pm\sqrt{14}$$

$$x = -3 \pm \sqrt{14}$$

Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example:

$$2x^2 + 3x - 1 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{17}}{4}$$

STUDY TIP

The Square Root Principle is also referred to as *extracting square roots*.

STUDY TIP

You can solve every quadratic equation by completing the square or using the Quadratic Formula.

Example 4 Solving a Quadratic Equation by Factoring

a. $2x^2 + 9x + 7 = 3$ — Original equation

$2x^2 + 9x + 4 = 0$ — Write in general form.

$(2x + 1)(x + 4) = 0$ — Factor.

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ — Set 1st factor equal to 0.

$x + 4 = 0 \Rightarrow x = -4$ — Set 2nd factor equal to 0.

The solutions are $x = -\frac{1}{2}$ and $x = -4$. Check these in the original equation.

b. $6x^2 - 3x = 0$ — Original equation

$3x(2x - 1) = 0$ — Factor.

$3x = 0 \Rightarrow x = 0$ — Set 1st factor equal to 0.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ — Set 2nd factor equal to 0.

The solutions are $x = 0$ and $x = \frac{1}{2}$. Check these in the original equation.

CHECKPOINT Now try Exercise 57.

Note that the method of solution in Example 4 is based on the Zero-Factor Property from Appendix A.1. Be sure you see that this property works *only* for equations written in general form (in which the right side of the equation is zero). So, all terms must be collected on one side *before* factoring. For instance, in the equation $(x - 5)(x + 2) = 8$, it is *incorrect* to set each factor equal to 8. Try to solve this equation correctly.

Example 5 Extracting Square Roots

Solve each equation by extracting square roots.

a. $4x^2 = 12$ **b.** $(x - 3)^2 = 7$

Solution

a. $4x^2 = 12$ — Write original equation.

$x^2 = 3$ — Divide each side by 4.

$x = \pm\sqrt{3}$ — Extract square roots.

When you take the square root of a variable expression, you must account for both positive and negative solutions. So, the solutions are $x = \sqrt{3}$ and $x = -\sqrt{3}$. Check these in the original equation.

b. $(x - 3)^2 = 7$ — Write original equation.

$x - 3 = \pm\sqrt{7}$ — Extract square roots.

$x = 3 \pm \sqrt{7}$ — Add 3 to each side.

The solutions are $x = 3 \pm \sqrt{7}$. Check these in the original equation.

CHECKPOINT Now try Exercise 77.

When solving quadratic equations by completing the square, you must add $(b/2)^2$ to *each side* in order to maintain equality. If the leading coefficient is *not* 1, you must divide each side of the equation by the leading coefficient *before* completing the square, as shown in Example 7.

Example 6 Completing the Square: Leading Coefficient Is 1

Solve $x^2 + 2x - 6 = 0$ by completing the square.

Solution

$$x^2 + 2x - 6 = 0 \qquad \text{Write original equation.}$$

$$x^2 + 2x = 6 \qquad \text{Add 6 to each side.}$$

$$x^2 + \underbrace{2x + 1^2}_{(\text{half of } 2)^2} = 6 + 1^2 \qquad \text{Add } 1^2 \text{ to each side.}$$

$$(x + 1)^2 = 7 \qquad \text{Simplify.}$$

$$x + 1 = \pm\sqrt{7} \qquad \text{Take square root of each side.}$$

$$x = -1 \pm \sqrt{7} \qquad \text{Subtract 1 from each side.}$$

The solutions are $x = -1 \pm \sqrt{7}$. Check these in the original equation.

CHECKPOINT Now try Exercise 85.

Example 7 Completing the Square: Leading Coefficient Is Not 1

$$3x^2 - 4x - 5 = 0 \qquad \text{Original equation}$$

$$3x^2 - 4x = 5 \qquad \text{Add 5 to each side.}$$

$$x^2 - \frac{4}{3}x = \frac{5}{3} \qquad \text{Divide each side by 3.}$$

$$x^2 - \underbrace{\frac{4}{3}x + \left(-\frac{2}{3}\right)^2}_{\left(\text{half of } -\frac{4}{3}\right)^2} = \frac{5}{3} + \left(-\frac{2}{3}\right)^2 \qquad \text{Add } \left(-\tfrac{2}{3}\right)^2 \text{ to each side.}$$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = \frac{19}{9} \qquad \text{Simplify.}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{19}{9} \qquad \text{Perfect square trinomial.}$$

$$x - \frac{2}{3} = \pm\frac{\sqrt{19}}{3} \qquad \text{Extract square roots.}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{19}}{3} \qquad \text{Solutions}$$

CHECKPOINT Now try Exercise 91.

STUDY TIP

When using the Quadratic Formula, remember that *before* the formula can be applied, you must first write the quadratic equation in general form.

Example 8 The Quadratic Formula: Two Distinct Solutions

Use the Quadratic Formula to solve $x^2 + 3x = 9$.

Solution

$$x^2 + 3x = 9$$ Write original equation.

$$x^2 + 3x - 9 = 0$$ Write in general form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)}$$ Substitute $a = 1$, $b = 3$, and $c = -9$.

$$x = \frac{-3 \pm \sqrt{45}}{2}$$ Simplify.

$$x = \frac{-3 \pm 3\sqrt{5}}{2}$$ Simplify.

The equation has two solutions:

$$x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

CHECKPOINT Now try Exercise 101.

Example 9 The Quadratic Formula: One Solution

Use the Quadratic Formula to solve $8x^2 - 24x + 18 = 0$.

Solution

$$8x^2 - 24x + 18 = 0$$ Write original equation.

$$4x^2 - 12x + 9 = 0$$ Divide out common factor of 2.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$ Substitute $a = 4$, $b = -12$, and $c = 9$.

$$x = \frac{12 \pm \sqrt{0}}{8} = \frac{3}{2}$$ Simplify.

This quadratic equation has only one solution: $x = \frac{3}{2}$. Check this in the original equation.

CHECKPOINT Now try Exercise 105.

Note that Example 9 could have been solved without first dividing out a common factor of 2. Substituting $a = 8$, $b = -24$, and $c = 18$ into the Quadratic Formula produces the same result.

Polynomial Equations of Higher Degree

The methods used to solve quadratic equations can sometimes be extended to solve polynomial equations of higher degree.

Example 10 Solving a Polynomial Equation by Factoring

Solve $3x^4 = 48x^2$.

Solution

First write the polynomial equation in general form with zero on one side, factor the other side, and then set each factor equal to zero and solve.

$$3x^4 = 48x^2 \qquad \text{Write original equation.}$$

$$3x^4 - 48x^2 = 0 \qquad \text{Write in general form.}$$

$$3x^2(x^2 - 16) = 0 \qquad \text{Factor out common factor.}$$

$$3x^2(x + 4)(x - 4) = 0 \qquad \text{Write in factored form.}$$

$$3x^2 = 0 \Rightarrow x = 0 \qquad \text{Set 1st factor equal to 0.}$$

$$x + 4 = 0 \Rightarrow x = -4 \qquad \text{Set 2nd factor equal to 0.}$$

$$x - 4 = 0 \Rightarrow x = 4 \qquad \text{Set 3rd factor equal to 0.}$$

You can check these solutions by substituting in the original equation, as follows.

Check

$$3(0)^4 = 48(0)^2 \qquad 0 \text{ checks. } \checkmark$$

$$3(-4)^4 = 48(-4)^2 \qquad -4 \text{ checks. } \checkmark$$

$$3(4)^4 = 48(4)^2 \qquad 4 \text{ checks. } \checkmark$$

So, you can conclude that the solutions are $x = 0$, $x = -4$, and $x = 4$.

✓CHECKPOINT Now try Exercise 135.

Example 11 Solving a Polynomial Equation by Factoring

Solve $x^3 - 3x^2 - 3x + 9 = 0$.

Solution

$$x^3 - 3x^2 - 3x + 9 = 0 \qquad \text{Write original equation.}$$

$$x^2(x - 3) - 3(x - 3) = 0 \qquad \text{Factor by grouping.}$$

$$(x - 3)(x^2 - 3) = 0 \qquad \text{Distributive Property}$$

$$x - 3 = 0 \Rightarrow x = 3 \qquad \text{Set 1st factor equal to 0.}$$

$$x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3} \qquad \text{Set 2nd factor equal to 0.}$$

The solutions are $x = 3$, $x = \sqrt{3}$, and $x = -\sqrt{3}$. Check these in the original equation.

✓CHECKPOINT Now try Exercise 143.

STUDY TIP

A common mistake that is made in solving an equation such as that in Example 10 is to divide each side of the equation by the variable factor x^2. This loses the solution $x = 0$. When solving an equation, always write the equation in general form, then factor the equation and set each factor equal to zero. Do not divide each side of an equation by a variable factor in an attempt to simplify the equation.

Equations Involving Radicals

Operations such as squaring each side of an equation, raising each side of an equation to a rational power, and multiplying each side of an equation by a variable quantity all can introduce extraneous solutions. So, when you use any of these operations, checking your solutions is crucial.

Example 12 Solving Equations Involving Radicals

a.

$\sqrt{2x+7} - x = 2$	Original equation
$\sqrt{2x+7} = x + 2$	Isolate radical.
$2x + 7 = x^2 + 4x + 4$	Square each side.
$0 = x^2 + 2x - 3$	Write in general form.
$0 = (x+3)(x-1)$	Factor.
$x + 3 = 0 \Rightarrow x = -3$	Set 1st factor equal to 0.
$x - 1 = 0 \Rightarrow x = 1$	Set 2nd factor equal to 0.

By checking these values, you can determine that the only solution is $x = 1$.

b.

$\sqrt{2x-5} - \sqrt{x-3} = 1$	Original equation
$\sqrt{2x-5} = \sqrt{x-3} + 1$	Isolate $\sqrt{2x-5}$.
$2x - 5 = x - 3 + 2\sqrt{x-3} + 1$	Square each side.
$2x - 5 = x - 2 + 2\sqrt{x-3}$	Combine like terms.
$x - 3 = 2\sqrt{x-3}$	Isolate $2\sqrt{x-3}$.
$x^2 - 6x + 9 = 4(x-3)$	Square each side.
$x^2 - 10x + 21 = 0$	Write in general form.
$(x-3)(x-7) = 0$	Factor.
$x - 3 = 0 \Rightarrow x = 3$	Set 1st factor equal to 0.
$x - 7 = 0 \Rightarrow x = 7$	Set 2nd factor equal to 0.

The solutions are $x = 3$ and $x = 7$. Check these in the original equation.

CHECKPOINT Now try Exercise 155.

STUDY TIP

When an equation contains two radicals, it may not be possible to isolate both. In such cases, you may have to raise each side of the equation to a power at *two* different stages in the solution, as shown in Example 12(b).

Example 13 Solving an Equation Involving a Rational Exponent

$(x-4)^{2/3} = 25$	Original equation
$\sqrt[3]{(x-4)^2} = 25$	Rewrite in radical form.
$(x-4)^2 = 15{,}625$	Cube each side.
$x - 4 = \pm 125$	Take square root of each side.
$x = 129, \quad x = -121$	Add 4 to each side.

CHECKPOINT Now try Exercise 163.

Equations with Absolute Values

To solve an equation involving an absolute value, remember that the expression inside the absolute value signs can be positive or negative. This results in *two* separate equations, each of which must be solved. For instance, the equation

$$|x - 2| = 3$$

results in the two equations $x - 2 = 3$ and $-(x - 2) = 3$, which implies that the equation has two solutions: $x = 5$ and $x = -1$.

Example 14 Solving an Equation Involving Absolute Value

Solve $|x^2 - 3x| = -4x + 6$.

Solution

Because the variable expression inside the absolute value signs can be positive or negative, you must solve the following two equations.

First Equation

$$x^2 - 3x = -4x + 6 \qquad \text{Use positive expression.}$$
$$x^2 + x - 6 = 0 \qquad \text{Write in general form.}$$
$$(x + 3)(x - 2) = 0 \qquad \text{Factor.}$$
$$x + 3 = 0 \Rightarrow x = -3 \qquad \text{Set 1st factor equal to 0.}$$
$$x - 2 = 0 \Rightarrow x = 2 \qquad \text{Set 2nd factor equal to 0.}$$

Second Equation

$$-(x^2 - 3x) = -4x + 6 \qquad \text{Use negative expression.}$$
$$x^2 - 7x + 6 = 0 \qquad \text{Write in general form.}$$
$$(x - 1)(x - 6) = 0 \qquad \text{Factor.}$$
$$x - 1 = 0 \Rightarrow x = 1 \qquad \text{Set 1st factor equal to 0.}$$
$$x - 6 = 0 \Rightarrow x = 6 \qquad \text{Set 2nd factor equal to 0.}$$

Check

$$|(-3)^2 - 3(-3)| \stackrel{?}{=} -4(-3) + 6 \qquad \text{Substitute } -3 \text{ for } x.$$
$$18 = 18 \qquad -3 \text{ checks.} \checkmark$$
$$|(2)^2 - 3(2)| \stackrel{?}{=} -4(2) + 6 \qquad \text{Substitute 2 for } x.$$
$$2 \neq -2 \qquad \text{2 does not check.}$$
$$|(1)^2 - 3(1)| \stackrel{?}{=} -4(1) + 6 \qquad \text{Substitute 1 for } x.$$
$$2 = 2 \qquad \text{1 checks.} \checkmark$$
$$|(6)^2 - 3(6)| \stackrel{?}{=} -4(6) + 6 \qquad \text{Substitute 6 for } x.$$
$$18 \neq -18 \qquad \text{6 does not check.}$$

The solutions are $x = -3$ and $x = 1$.

CHECKPOINT Now try Exercise 181.

A.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. An ________ is a statement that equates two algebraic expressions.
2. To find all values that satisfy an equation is to ________ the equation.
3. There are two types of equations, ________ and ________ equations.
4. A linear equation in one variable is an equation that can be written in the standard from ________.
5. When solving an equation, it is possible to introduce an ________ solution, which is a value that does not satisfy the original equation.
6. An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ is a ________ ________, or a second-degree polynomial equation in x.
7. The four methods that can be used to solve a quadratic equation are ________, ________, ________, and the ________.

In Exercises 1–10, determine whether the equation is an identity or a conditional equation.

1. $2(x - 1) = 2x - 2$
2. $3(x + 2) = 5x + 4$
3. $-6(x - 3) + 5 = -2x + 10$
4. $3(x + 2) - 5 = 3x + 1$
5. $4(x + 1) - 2x = 2(x + 2)$
6. $-7(x - 3) + 4x = 3(7 - x)$
7. $x^2 - 8x + 5 = (x - 4)^2 - 11$
8. $x^2 + 2(3x - 2) = x^2 + 6x - 4$
9. $3 + \dfrac{1}{x + 1} = \dfrac{4x}{x + 1}$
10. $\dfrac{5}{x} + \dfrac{3}{x} = 24$

In Exercises 11–26, solve the equation and check your solution.

11. $x + 11 = 15$
12. $7 - x = 19$
13. $7 - 2x = 25$
14. $7x + 2 = 23$
15. $8x - 5 = 3x + 20$
16. $7x + 3 = 3x - 17$
17. $2(x + 5) - 7 = 3(x - 2)$
18. $3(x + 3) = 5(1 - x) - 1$
19. $x - 3(2x + 3) = 8 - 5x$
20. $9x - 10 = 5x + 2(2x - 5)$
21. $\dfrac{5x}{4} + \dfrac{1}{2} = x - \dfrac{1}{2}$
22. $\dfrac{x}{5} - \dfrac{x}{2} = 3 + \dfrac{3x}{10}$

23 $\frac{3}{2}(z + 5) - \frac{1}{4}(z + 24) = 0$

24. $\dfrac{3x}{2} + \dfrac{1}{4}(x - 2) = 10$
25. $0.25x + 0.75(10 - x) = 3$
26. $0.60x + 0.40(100 - x) = 50$

In Exercises 27–48, solve the equation and check your solution. (If not possible, explain why.)

27. $x + 8 = 2(x - 2) - x$
28. $8(x + 2) - 3(2x + 1) = 2(x + 5)$
29. $\dfrac{100 - 4x}{3} = \dfrac{5x + 6}{4} + 6$
30. $\dfrac{17 + y}{y} + \dfrac{32 + y}{y} = 100$
31. $\dfrac{5x - 4}{5x + 4} = \dfrac{2}{3}$
32. $\dfrac{10x + 3}{5x + 6} = \dfrac{1}{2}$
33. $10 - \dfrac{13}{x} = 4 + \dfrac{5}{x}$
34. $\dfrac{15}{x} - 4 = \dfrac{6}{x} + 3$
35. $3 = 2 + \dfrac{2}{z + 2}$
36. $\dfrac{1}{x} + \dfrac{2}{x - 5} = 0$
37. $\dfrac{x}{x + 4} + \dfrac{4}{x + 4} + 2 = 0$
38. $\dfrac{7}{2x + 1} - \dfrac{8x}{2x - 1} = -4$
39. $\dfrac{2}{(x - 4)(x - 2)} = \dfrac{1}{x - 4} + \dfrac{2}{x - 2}$
40. $\dfrac{4}{x - 1} + \dfrac{6}{3x + 1} = \dfrac{15}{3x + 1}$
41. $\dfrac{1}{x - 3} + \dfrac{1}{x + 3} = \dfrac{10}{x^2 - 9}$
42. $\dfrac{1}{x - 2} + \dfrac{3}{x + 3} = \dfrac{4}{x^2 + x - 6}$
43. $\dfrac{3}{x^2 - 3x} + \dfrac{4}{x} = \dfrac{1}{x - 3}$
44. $\dfrac{6}{x} - \dfrac{2}{x + 3} = \dfrac{3(x + 5)}{x^2 + 3x}$

45. $(x + 2)^2 + 5 = (x + 3)^2$

46. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

47. $(x + 2)^2 - x^2 = 4(x + 1)$

48. $(2x + 1)^2 = 4(x^2 + x + 1)$

In Exercises 49–54, write the quadratic equation in general form.

49. $2x^2 = 3 - 8x$

50. $x^2 = 16x$

51. $(x - 3)^2 = 3$

52. $13 - 3(x + 7)^2 = 0$

53. $\frac{1}{5}(3x^2 - 10) = 18x$

54. $x(x + 2) = 5x^2 + 1$

In Exercises 55–68, solve the quadratic equation by factoring.

55. $6x^2 + 3x = 0$

56. $9x^2 - 1 = 0$

57. $x^2 - 2x - 8 = 0$

58. $x^2 - 10x + 9 = 0$

59. $x^2 + 10x + 25 = 0$

60. $4x^2 + 12x + 9 = 0$

61. $3 + 5x - 2x^2 = 0$

62. $2x^2 = 19x + 33$

63. $x^2 + 4x = 12$

64. $-x^2 + 8x = 12$

65. $\frac{3}{4}x^2 + 8x + 20 = 0$

66. $\frac{1}{8}x^2 - x - 16 = 0$

67. $x^2 + 2ax + a^2 = 0$, a is a real number

68. $(x + a)^2 - b^2 = 0$, a and b are real numbers

In Exercises 69–82, solve the equation by extracting square roots.

69. $x^2 = 49$

70. $x^2 = 169$

71. $x^2 = 11$

72. $x^2 = 32$

73. $3x^2 = 81$

74. $9x^2 = 36$

75. $(x - 12)^2 = 16$

76. $(x + 13)^2 = 25$

77. $(x + 2)^2 = 14$

78. $(x - 5)^2 = 30$

79. $(2x - 1)^2 = 18$

80. $(4x + 7)^2 = 44$

81. $(x - 7)^2 = (x + 3)^2$

82. $(x + 5)^2 = (x + 4)^2$

In Exercises 83–92, solve the quadratic equation by completing the square.

83. $x^2 + 4x - 32 = 0$

84. $x^2 - 2x - 3 = 0$

85. $x^2 + 12x + 25 = 0$

86. $x^2 + 8x + 14 = 0$

87. $9x^2 - 18x = -3$

88. $9x^2 - 12x = 14$

89. $8 + 4x - x^2 = 0$

90. $-x^2 + x - 1 = 0$

91. $2x^2 + 5x - 8 = 0$

92. $4x^2 - 4x - 99 = 0$

In Exercises 93–116, use the Quadratic Formula to solve the equation.

93. $2x^2 + x - 1 = 0$

94. $2x^2 - x - 1 = 0$

95. $16x^2 + 8x - 3 = 0$

96. $25x^2 - 20x + 3 = 0$

97. $2 + 2x - x^2 = 0$

98. $x^2 - 10x + 22 = 0$

99. $x^2 + 14x + 44 = 0$

100. $6x = 4 - x^2$

101. $x^2 + 8x - 4 = 0$

102. $4x^2 - 4x - 4 = 0$

103. $12x - 9x^2 = -3$

104. $16x^2 + 22 = 40x$

105. $9x^2 + 24x + 16 = 0$

106. $36x^2 + 24x - 7 = 0$

107. $4x^2 + 4x = 7$

108. $16x^2 - 40x + 5 = 0$

109. $28x - 49x^2 = 4$

110. $3x + x^2 - 1 = 0$

111. $8t = 5 + 2t^2$

112. $25h^2 + 80h + 61 = 0$

113. $(y - 5)^2 = 2y$

114. $(z + 6)^2 = -2z$

115. $\frac{1}{2}x^2 + \frac{3}{8}x = 2$

116. $\left(\frac{5}{7}x - 14\right)^2 = 8x$

In Exercises 117–124, use the Quadratic Formula to solve the equation. (Round your answer to three decimal places.)

117. $5.1x^2 - 1.7x - 3.2 = 0$

118. $2x^2 - 2.50x - 0.42 = 0$

119. $-0.067x^2 - 0.852x + 1.277 = 0$

120. $-0.005x^2 + 0.101x - 0.193 = 0$

121. $422x^2 - 506x - 347 = 0$

122. $1100x^2 + 326x - 715 = 0$

123. $12.67x^2 + 31.55x + 8.09 = 0$

124. $-3.22x^2 - 0.08x + 28.651 = 0$

In Exercises 125–134, solve the equation using any convenient method.

125. $x^2 - 2x - 1 = 0$

126. $11x^2 + 33x = 0$

127. $(x + 3)^2 = 81$

128. $x^2 - 14x + 49 = 0$

129. $x^2 - x - \frac{11}{4} = 0$

130. $x^2 + 3x - \frac{3}{4} = 0$

131. $(x + 1)^2 = x^2$

132. $a^2x^2 - b^2 = 0$, a and b are real numbers

133. $3x + 4 = 2x^2 - 7$

134. $4x^2 + 2x + 4 = 2x + 8$

In Exercises 135–152, find all solutions of the equation. Check your solutions in the original equation.

135. $4x^4 - 18x^2 = 0$

136. $20x^3 - 125x = 0$

137. $x^4 - 81 = 0$

138. $x^6 - 64 = 0$

139. $x^3 + 216 = 0$

140. $27x^3 - 512 = 0$

141. $5x^3 + 30x^2 + 45x = 0$

142. $9x^4 - 24x^3 + 16x^2 = 0$

143. $x^3 - 3x^2 - x + 3 = 0$

144. $x^3 + 2x^2 + 3x + 6 = 0$

145. $x^4 - x^3 + x - 1 = 0$

146. $x^4 + 2x^3 - 8x - 16 = 0$

147. $x^4 - 4x^2 + 3 = 0$

148. $x^4 + 5x^2 - 36 = 0$

149. $4x^4 - 65x^2 + 16 = 0$

150. $36t^4 + 29t^2 - 7 = 0$

151. $x^6 + 7x^3 - 8 = 0$

152. $x^6 + 3x^3 + 2 = 0$

In Exercises 153–184, find all solutions of the equation. Check your solutions in the original equation.

153. $\sqrt{2x} - 10 = 0$

154. $4\sqrt{x} - 3 = 0$

155. $\sqrt{x - 10} - 4 = 0$

156. $\sqrt{5 - x} - 3 = 0$

157. $\sqrt[3]{2x + 5} + 3 = 0$

158. $\sqrt[3]{3x + 1} - 5 = 0$

159. $-\sqrt{26 - 11x} + 4 = x$

160. $x + \sqrt{31 - 9x} = 5$

161. $\sqrt{x + 1} = \sqrt{3x + 1}$

162. $\sqrt{x + 5} = \sqrt{x - 5}$

163. $(x - 5)^{3/2} = 8$

164. $(x + 3)^{3/2} = 8$

165. $(x + 3)^{2/3} = 8$

166. $(x + 2)^{2/3} = 9$

167. $(x^2 - 5)^{3/2} = 27$

168. $(x^2 - x - 22)^{3/2} = 27$

169. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$

170. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

171. $x = \dfrac{3}{x} + \dfrac{1}{2}$

172. $\dfrac{4}{x} - \dfrac{5}{3} = \dfrac{x}{6}$

173. $\dfrac{1}{x} - \dfrac{1}{x + 1} = 3$

174. $\dfrac{4}{x + 1} - \dfrac{3}{x + 2} = 1$

175. $\dfrac{20 - x}{x} = x$

176. $4x + 1 = \dfrac{3}{x}$

177. $\dfrac{x}{x^2 - 4} + \dfrac{1}{x + 2} = 3$

178. $\dfrac{x + 1}{3} - \dfrac{x + 1}{x + 2} = 0$

179. $|2x - 1| = 5$

180. $|3x + 2| = 7$

181. $|x| = x^2 + x - 3$

182. $|x^2 + 6x| = 3x + 18$

183. $|x + 1| = x^2 - 5$

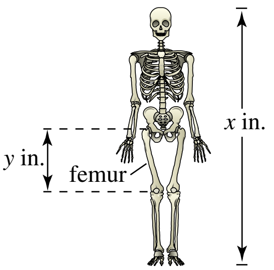
184. $|x - 10| = x^2 - 10x$

185. ***Anthropology*** The relationship between the length of an adult's femur (thigh bone) and the height of the adult can be approximated by the linear equations

$y = 0.432x - 10.44$ Female

$y = 0.449x - 12.15$ Male

where y is the length of the femur in inches and x is the height of the adult in inches (see figure).



(a) An anthropologist discovers a femur belonging to an adult human female. The bone is 16 inches long. Estimate the height of the female.

(b) From the foot bones of an adult human male, an anthropologist estimates that the person's height was 69 inches. A few feet away from the site where the foot bones were discovered, the anthropologist discovers a male adult femur that is 19 inches long. Is it likely that both the foot bones and the thigh bone came from the same person?

(c) Complete the table to determine if there is a height of an adult for which an anthropologist would not be able to determine whether the femur belonged to a male or a female.

Height, x	Female femur length, y	Male femur length, y
60		
70		
80		
90		
100		
110		

186. ***Operating Cost*** A delivery company has a fleet of vans. The annual operating cost C per van is

$C = 0.32m + 2500$

where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost of $10,000?

187. ***Flood Control*** A river has risen 8 feet above its flood stage. The water begins to recede at a rate of 3 inches per hour. Write a mathematical model that shows the number of feet above flood stage after t hours. If the water continually recedes at this rate, when will the river be 1 foot above its flood stage?

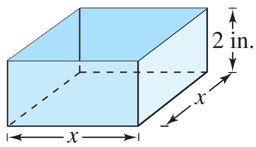
188. ***Floor Space*** The floor of a one-story building is 14 feet longer than it is wide. The building has 1632 square feet of floor space.

(a) Draw a diagram that gives a visual representation of the floor space. Represent the width as w and show the length in terms of w.

(b) Write a quadratic equation in terms of w.

(c) Find the length and width of the floor of the building.

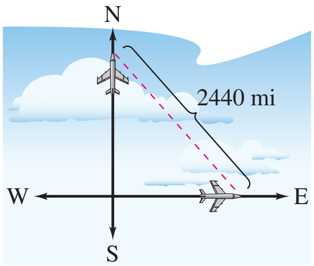
189. ***Packaging*** An open box with a square base (see figure) is to be constructed from 84 square inches of material. The height of the box is 2 inches. What are the dimensions of the box? (*Hint:* The surface area is $S = x^2 + 4xh$.)



190. ***Geometry*** The hypotenuse of an isosceles right triangle is 5 centimeters long. How long are its sides?

191. ***Geometry*** An equilateral triangle has a height of 10 inches. How long is one of its sides? (*Hint:* Use the height of the triangle to partition the triangle into two congruent right triangles.)

192. ***Flying Speed*** Two planes leave simultaneously from Chicago's O'Hare Airport, one flying due north and the other due east (see figure). The northbound plane is flying 50 miles per hour faster than the eastbound plane. After 3 hours, the planes are 2440 miles apart. Find the speed of each plane.



193. ***Voting Population*** The total voting-age population P (in millions) in the United States from 1990 to 2002 can be modeled by

$$P = \frac{182.45 - 3.189t}{1.00 - 0.026t}, \quad 0 \le t \le 12$$

where t represents the year, with $t = 0$ corresponding to 1990. (Source: U.S. Census Bureau)

(a) In which year did the total voting-age population reach 200 million?

(b) Use the model to predict when the total voting-age population will reach 230 million. Is this prediction reasonable? Explain.

194. ***Airline Passengers*** An airline offers daily flights between Chicago and Denver. The total monthly cost C (in millions of dollars) of these flights is $C = \sqrt{0.2x + 1}$ where x is the number of passengers (in thousands). The total cost of the flights for June is 2.5 million dollars. How many passengers flew in June?

195. ***Demand*** The demand equation for a video game is modeled by $p = 40 - \sqrt{0.01x + 1}$ where x is the number of units demanded per day and p is the price per unit. Approximate the demand when the price is \$37.55.

196. ***Demand*** The demand equation for a high definition television set is modeled by

$$p = 800 - \sqrt{0.01x + 1}$$

where x is the number of units demanded per month and p is the price per unit. Approximate the demand when the price is \$750.

Synthesis

True or False? **In Exercises 197–200, determine whether the statement is true or false. Justify your answer.**

197. The equation $x(3 - x) = 10$ is a linear equation.

198. If $(2x - 3)(x + 5) = 8$, then either $2x - 3 = 8$ or $x + 5 = 8$.

199. An equation can never have more than one extraneous solution.

200. When solving an absolute value equation, you will always have to check more than one solution.

201. ***Think About It*** What is meant by *equivalent equations*? Give an example of two equivalent equations.

202. ***Writing*** Describe the steps used to transform an equation into an equivalent equation.

203. To solve the equation $2x^2 + 3x = 15x$, a student divides each side by x and solves the equation $2x + 3 = 15$. The resulting solution $(x = 6)$ satisfies the original equation. Is there an error? Explain.

204. Solve $3(x + 4)^2 + (x + 4) - 2 = 0$ in two ways.

(a) Let $u = x + 4$, and solve the resulting equation for u. Then solve the u-solution for x.

(b) Expand and collect like terms in the equation, and solve the resulting equation for x.

(c) Which method is easier? Explain.

Think About It **In Exercises 205–210, write a quadratic equation that has the given solutions. (There are many correct answers.)**

205. -3 and 6

206. -4 and -11

207. 8 and 14

208. $\frac{1}{6}$ and $-\frac{2}{5}$

209. $1 + \sqrt{2}$ and $1 - \sqrt{2}$

210. $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$

In Exercises 211 and 212, consider an equation of the form $x + |x - a| = b$, where a and b are constants.

211. Find a and b when the solution of the equation is $x = 9$. (There are many correct answers.)

212. ***Writing*** Write a short paragraph listing the steps required to solve this equation involving absolute values and explain why it is important to check your solutions.

213. Solve each equation, given that a and b are not zero.

(a) $ax^2 + bx = 0$

(b) $ax^2 - ax = 0$

A.6 Linear Inequalities in One Variable

What you should learn

- Represent solutions of linear inequalities in one variable.
- Solve linear inequalities in one variable.
- Solve inequalities involving absolute values.
- Use inequalities to model and solve real-life problems.

Why you should learn it

Inequalities can be used to model and solve real-life problems. For instance, in Exercise 101 on page A68, you will use a linear inequality to analyze the average salary for elementary school teachers.

Introduction

Simple inequalities were discussed in Appendix A.1. There, you used the inequality symbols $<$, $\leq$, $>$, and $\geq$ to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

Now, you will expand your work with inequalities to include more involved statements such as

$$5x - 7 < 3x + 9$$

and

$$-3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of

$$x + 1 < 4$$

is all real numbers that are less than 3.

The set of all points on the real number line that represent the solution set is the **graph of the inequality.** Graphs of many types of inequalities consist of intervals on the real number line. See Appendix A.1 to review the nine basic types of intervals on the real number line. Note that each type of interval can be classified as *bounded* or *unbounded*.

Example 1 Intervals and Inequalities

Write an inequality to represent each interval, and state whether the interval is bounded or unbounded.

a. $(-3, 5]$

b. $(-3, \infty)$

c. $[0, 2]$

d. $(-\infty, \infty)$

Solution

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-3, \infty)$ corresponds to $-3 < x$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

d. $(-\infty, \infty)$ corresponds to $-\infty < x < \infty$. Unbounded

CHECKPOINT Now try Exercise 1.

Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **Properties of Inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$$-2 < 5 \qquad \text{Original inequality}$$

$$(-3)(-2) > (-3)(5) \qquad \text{Multiply each side by } -3 \text{ and reverse inequality.}$$

$$6 > -15 \qquad \text{Simplify.}$$

Notice that if the inequality was not reversed you would obtain the false statement $6 < -15$.

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$$x + 2 < 5$$

and

$$x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes the operations that can be used to create equivalent inequalities.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. Transitive Property

$$a < b \text{ and } b < c \implies a < c$$

2. Addition of Inequalities

$$a < b \text{ and } c < d \implies a + c < b + d$$

3. Addition of a Constant

$$a < b \implies a + c < b + c$$

4. Multiplication by a Constant

$$\text{For } c > 0,\ a < b \implies ac < bc$$

$$\text{For } c < 0,\ a < b \implies ac > bc \qquad \text{Reverse the inequality.}$$

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and the symbol $>$ is replaced by $\geq$. For instance, another form of the multiplication property would be as follows.

$$\text{For } c > 0,\ a \leq b \implies ac \leq bc$$

$$\text{For } c < 0,\ a \leq b \implies ac \geq bc$$

Solving a Linear Inequality in One Variable

The simplest type of inequality is a **linear inequality** in one variable. For instance, $2x + 3 > 4$ is a linear inequality in x.

In the following examples, pay special attention to the steps in which the inequality symbol is reversed. Remember that when you multiply or divide by a negative number, you must reverse the inequality symbol.

Example 2 Solving Linear Inequalities

Solve each inequality.

a. $5x - 7 > 3x + 9$

b. $1 - \dfrac{3x}{2} \geq x - 4$

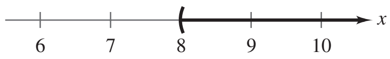
> **STUDY TIP**
>
> Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x.

Solution

a.

$5x - 7 > 3x + 9$	Write original inequality.
$2x - 7 > 9$	Subtract $3x$ from each side.
$2x > 16$	Add 7 to each side.
$x > 8$	Divide each side by 2.

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph of this solution set is shown in Figure A.8. Note that a parenthesis at 8 on the real number line indicates that 8 *is not* part of the solution set.



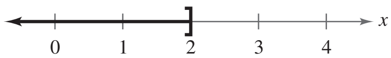
Solution interval: $(8, \infty)$

FIGURE A.8

b.

$1 - \dfrac{3x}{2} \geq x - 4$	Write original inequality.
$2 - 3x \geq 2x - 8$	Multiply each side by 2.
$2 - 5x \geq -8$	Subtract $2x$ from each side.
$-5x \geq -10$	Subtract 2 from each side.
$x \leq 2$	Divide each side by -5 and reverse the inequality.

The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph of this solution set is shown in Figure A.9. Note that a bracket at 2 on the real number line indicates that 2 *is* part of the solution set.



Solution interval: $(-\infty, 2]$

FIGURE A.9

CHECKPOINT Now try Exercise 25.

Sometimes it is possible to write two inequalities as a **double inequality.** For instance, you can write the two inequalities $-4 \le 5x - 2$ and $5x - 2 < 7$ more simply as

$$-4 \le 5x - 2 < 7. \qquad \text{Double inequality}$$

This form allows you to solve the two inequalities together, as demonstrated in Example 3.

Example 3 Solving a Double Inequality

To solve a double inequality, you can isolate x as the middle term.

$$-3 \le 6x - 1 < 3 \qquad \text{Original inequality}$$

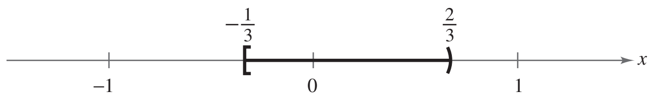
$$-3 + 1 \le 6x - 1 + 1 < 3 + 1 \qquad \text{Add 1 to each part.}$$

$$-2 \le 6x < 4 \qquad \text{Simplify.}$$

$$\frac{-2}{6} \le \frac{6x}{6} < \frac{4}{6} \qquad \text{Divide each part by 6.}$$

$$-\frac{1}{3} \le x < \frac{2}{3} \qquad \text{Simplify.}$$

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$, which is denoted by $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The graph of this solution set is shown in Figure A.10.



Solution interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$

FIGURE A.10

CHECKPOINT Now try Exercise 37.

The double inequality in Example 3 could have been solved in two parts as follows.

$$-3 \le 6x - 1 \quad \text{and} \quad 6x - 1 < 3$$

$$-2 \le 6x \qquad\qquad 6x < 4$$

$$-\frac{1}{3} \le x \qquad\qquad x < \frac{2}{3}$$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which

$$-\frac{1}{3} \le x < \frac{2}{3}.$$

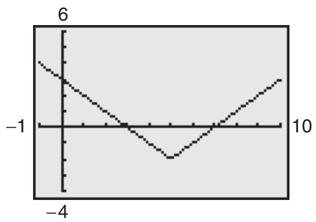
When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \le -1$ as $3 < x \le -1$. This "inequality" is wrong because 3 is not less than -1.

Technology

A graphing utility can be used to identify the solution set of the graph of an inequality. For instance, to find the solution set of $|x - 5| < 2$ (see Example 4), rewrite the inequality as $|x - 5| - 2 < 0$, enter

Y1 = abs (X − 5) − 2,

and press the *graph* key. The graph should look like the one shown below.



Notice that the graph is below the x-axis on the interval $(3, 7)$.

Inequalities Involving Absolute Values

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.

   $|x| < a$ if and only if $-a < x < a$. Double inequality

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.

   $|x| > a$ if and only if $x < -a$ or $x > a$. Compound inequality

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 4 Solving an Absolute Value Inequality

Solve each inequality.

a. $|x - 5| < 2$ **b.** $|x + 3| \geq 7$

Solution

a. $|x - 5| < 2$ Write original inequality.

$-2 < x - 5 < 2$ Write equivalent inequalities.

$-2 + 5 < x - 5 + 5 < 2 + 5$ Add 5 to each part.

$3 < x < 7$ Simplify.

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph of this solution set is shown in Figure A.11.

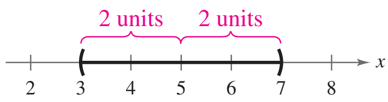
b. $|x + 3| \geq 7$ Write original inequality.

$x + 3 \leq -7$ or $x + 3 \geq 7$ Write equivalent inequalities.

$x + 3 - 3 \leq -7 - 3$ $x + 3 - 3 \geq 7 - 3$ Subtract 3 from each side.

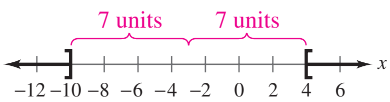
$x \leq -10$ $x \geq 4$ Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4. The interval notation for this solution set is $(-\infty, -10] \cup [4, \infty)$. The symbol $\cup$ is called a *union* symbol and is used to denote the combining of two sets. The graph of this solution set is shown in Figure A.12.



$|x - 5| < 2$: *Solutions lie inside* $(3, 7)$

FIGURE A.11



$|x + 3| \geq 7$: *Solutions lie outside* $(-10, 4)$

FIGURE A.12

CHECKPOINT Now try Exercise 49.

STUDY TIP

Note that the graph of the inequality $|x - 5| < 2$ can be described as all real numbers *within* two units of 5, as shown in Figure A.11.

Applications

A problem-solving plan can be used to model and solve real-life problems that involve inequalities, as illustrated in Example 5.

Example 5 Comparative Shopping

You are choosing between two different cell phone plans. Plan A costs \$49.99 per month for 500 minutes plus \$0.40 for each additional minute. Plan B costs \$45.99 per month for 500 minutes plus \$0.45 for each additional minute. How many *additional* minutes must you use in one month for plan B to cost more than plan A?

Solution

Verbal Model: Monthly cost for plan B $>$ Monthly cost for plan A

Labels: Minutes used (over 500) in one month $= m$ (minutes)
Monthly cost for plan A $= 0.40m + 49.99$ (dollars)
Monthly cost for plan B $= 0.45m + 45.99$ (dollars)

Inequality:
$$0.45m + 45.99 > 0.40m + 49.99$$
$$0.05m > 4$$
$$m > 80 \text{ minutes}$$

Plan B costs more if you use more than 80 additional minutes in one month.

CHECKPOINT Now try Exercise 91.

Example 6 Accuracy of a Measurement

You go to a candy store to buy chocolates that cost \$9.89 per pound. The scale that is used in the store has a state seal of approval that indicates the scale is accurate to within half an ounce (or $\frac{1}{32}$ of a pound). According to the scale, your purchase weighs one-half pound and costs \$4.95. How much might you have been undercharged or overcharged as a result of inaccuracy in the scale?

Solution

Let x represent the *true* weight of the candy. Because the scale is accurate to within half an ounce (or $\frac{1}{32}$ of a pound), the difference between the exact weight (x) and the scale weight $\left(\frac{1}{2}\right)$ is less than or equal to $\frac{1}{32}$ of a pound. That is, $\left|x - \frac{1}{2}\right| \le \frac{1}{32}$. You can solve this inequality as follows.

$$-\frac{1}{32} \le x - \frac{1}{2} \le \frac{1}{32}$$
$$\frac{15}{32} \le x \le \frac{17}{32}$$
$$0.46875 \le x \le 0.53125$$

In other words, your "one-half pound" of candy could have weighed as little as 0.46875 pound (which would have cost \$4.64) or as much as 0.53125 pound (which would have cost \$5.25). So, you could have been overcharged by as much as \$0.31 or undercharged by as much as \$0.30.

CHECKPOINT Now try Exercise 105.

A.6 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The set of all real numbers that are solutions to an inequality is the ________ ________ of the inequality.

2. The set of all points on the real number line that represent the solution set of an inequality is the ________ of the inequality.

3. To solve a linear inequality in one variable, you can use the properties of inequalities, which are identical to those used to solve equations, with the exception of multiplying or dividing each side by a ________ number.

4. Two inequalities that have the same solution set are ________ ________.

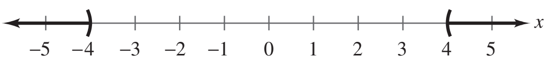
5. It is sometimes possible to write two inequalities as one inequality, called a ________ inequality.

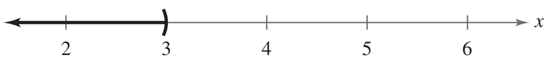
6. The symbol $\cup$ is called a ________ symbol and is used to denote the combining of two sets.

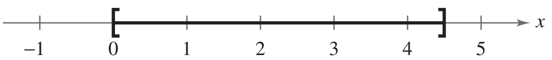
In Exercises 1–6, (a) write an inequality that represents the interval and (b) state whether the interval is bounded or unbounded.

1. $[-1, 5]$
2. $(2, 10]$
3. $(11, \infty)$
4. $[-5, \infty)$
5. $(-\infty, -2)$
6. $(-\infty, 7]$

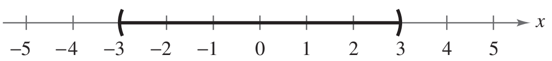
In Exercises 7–12, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

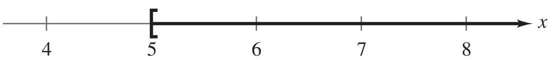
(a)


(b)


(c)


(d)


(e)


(f)


7. $x < 3$
8. $x \geq 5$
9. $-3 < x \leq 4$
10. $0 \leq x \leq \frac{9}{2}$
11. $|x| < 3$
12. $|x| > 4$

In Exercises 13–18, determine whether each value of *x* is a solution of the inequality.

Inequality	*Values*	
13. $5x - 12 > 0$	(a) $x = 3$	(b) $x = -3$
	(c) $x = \frac{5}{2}$	(d) $x = \frac{3}{2}$
14. $2x + 1 < -3$	(a) $x = 0$	(b) $x = -\frac{1}{4}$
	(c) $x = -4$	(d) $x = -\frac{3}{2}$
15. $0 < \dfrac{x-2}{4} < 2$	(a) $x = 4$	(b) $x = 10$
	(c) $x = 0$	(d) $x = \frac{7}{2}$
16. $-1 < \dfrac{3-x}{2} \leq 1$	(a) $x = 0$	(b) $x = -5$
	(c) $x = 1$	(d) $x = 5$
17. $\lvert x - 10 \rvert \geq 3$	(a) $x = 13$	(b) $x = -1$
	(c) $x = 14$	(d) $x = 9$
18. $\lvert 2x - 3 \rvert < 15$	(a) $x = -6$	(b) $x = 0$
	(c) $x = 12$	(d) $x = 7$

In Exercises 19–44, solve the inequality and sketch the solution on the real number line. (Some inequalities have no solutions.)

19. $4x < 12$
20. $10x < -40$
21. $-2x > -3$
22. $-6x > 15$
23. $x - 5 \geq 7$
24. $x + 7 \leq 12$
25. $2x + 7 < 3 + 4x$
26. $3x + 1 \geq 2 + x$
27. $2x - 1 \geq 1 - 5x$
28. $6x - 4 \leq 2 + 8x$
29. $4 - 2x < 3(3 - x)$
30. $4(x + 1) < 2x + 3$
31. $\frac{3}{4}x - 6 \leq x - 7$
32. $3 + \frac{2}{7}x > x - 2$
33. $\frac{1}{2}(8x + 1) \geq 3x + \frac{5}{2}$
34. $9x - 1 < \frac{3}{4}(16x - 2)$
35. $3.6x + 11 \geq -3.4$
36. $15.6 - 1.3x < -5.2$
37. $1 < 2x + 3 < 9$
38. $-8 \leq -(3x + 5) < 13$

39. $-4 < \dfrac{2x - 3}{3} < 4$

40. $0 \leq \dfrac{x + 3}{2} < 5$

41. $\frac{3}{4} > x + 1 > \frac{1}{4}$

42. $-1 < 2 - \dfrac{x}{3} < 1$

43. $3.2 \leq 0.4x - 1 \leq 4.4$

44. $4.5 > \dfrac{1.5x + 6}{2} > 10.5$

In Exercises 45–60, solve the inequality and sketch the solution on the real number line. (Some inequalities have no solution.)

45. $|x| < 6$

46. $|x| > 4$

47. $\left|\dfrac{x}{2}\right| > 1$

48. $\left|\dfrac{x}{5}\right| > 3$

49. $|x - 5| < -1$

50. $|x - 7| < -5$

51. $|x - 20| \leq 6$

52. $|x - 8| \geq 0$

53. $|3 - 4x| \geq 9$

54. $|1 - 2x| < 5$

55. $\left|\dfrac{x - 3}{2}\right| \geq 4$

56. $\left|1 - \dfrac{2x}{3}\right| < 1$

57. $|9 - 2x| - 2 < -1$

58. $|x + 14| + 3 > 17$

59. $2|x + 10| \geq 9$

60. $3|4 - 5x| \leq 9$

Graphical Analysis **In Exercises 61–68, use a graphing utility to graph the inequality and identify the solution set.**

61. $6x > 12$

62. $3x - 1 \leq 5$

63. $5 - 2x \geq 1$

64. $3(x + 1) < x + 7$

65. $|x - 8| \leq 14$

66. $|2x + 9| > 13$

67. $2|x + 7| \geq 13$

68. $\frac{1}{2}|x + 1| \leq 3$

Graphical Analysis **In Exercises 69–74, use a graphing utility to graph the equation. Use the graph to approximate the values of *x* that satisfy each inequality.**

	Equation	*Inequalities*	
69.	$y = 2x - 3$	(a) $y \geq 1$	(b) $y \leq 0$
70.	$y = \frac{2}{3}x + 1$	(a) $y \leq 5$	(b) $y \geq 0$
71.	$y = -\frac{1}{2}x + 2$	(a) $0 \leq y \leq 3$	(b) $y \geq 0$
72.	$y = -3x + 8$	(a) $-1 \leq y \leq 3$	(b) $y \leq 0$
73.	$y = \|x - 3\|$	(a) $y \leq 2$	(b) $y \geq 4$
74.	$y = \left\|\frac{1}{2}x + 1\right\|$	(a) $y \leq 4$	(b) $y \geq 1$

In Exercises 75–80, find the interval(s) on the real number line for which the radicand is nonnegative.

75. $\sqrt{x - 5}$

76. $\sqrt{x - 10}$

77. $\sqrt{x + 3}$

78. $\sqrt{3 - x}$

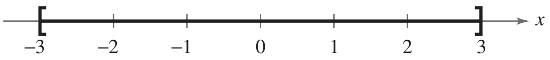
79. $\sqrt[4]{7 - 2x}$

80. $\sqrt[4]{6x + 15}$

81. ***Think About It*** The graph of $|x - 5| < 3$ can be described as all real numbers within three units of 5. Give a similar description of $|x - 10| < 8$.

82. ***Think About It*** The graph of $|x - 2| > 5$ can be described as all real numbers more than five units from 2. Give a similar description of $|x - 8| > 4$.

In Exercises 83–90, use absolute value notation to define the interval (or pair of intervals) on the real number line.

83.


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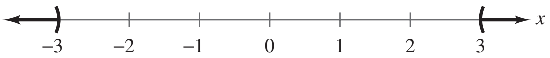

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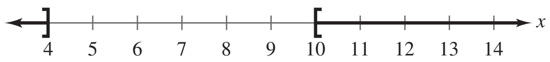

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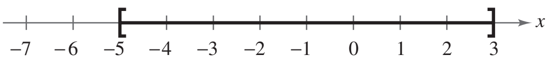


87. All real numbers within 10 units of 12

88. All real numbers at least five units from 8

89. All real numbers more than four units from -3

90. All real numbers no more than seven units from -6

91. ***Checking Account*** You can choose between two types of checking accounts at your local bank. Type A charges a monthly service fee of \$6 plus \$0.25 for each check written. Type B charges a monthly service fee of \$4.50 plus \$0.50 for each check written. How many checks must you write in a month in order for the monthly charges for type A to be less than that for type B?

92. ***Copying Costs*** Your department sends its copying to the photocopy center of your company. The center bills your department \$0.10 per page. You have investigated the possibility of buying a departmental copier for \$3000. With your own copier, the cost per page would be \$0.03. The expected life of the copier is 4 years. How many copies must you make in the four-year period to justify buying the copier?

93. ***Investment*** In order for an investment of \$1000 to grow to more than \$1062.50 in 2 years, what must the annual interest rate be? $[A = P(1 + rt)]$

94. ***Investment*** In order for an investment of \$750 to grow to more than \$825 in 2 years, what must the annual interest rate be? $[A = P(1 + rt)]$

95. ***Cost, Revenue, and Profit*** The revenue for selling x units of a product is $R = 115.95x$. The cost of producing x units is

$$C = 95x + 750.$$

To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

96. ***Cost, Revenue, and Profit*** The revenue for selling x units of a product is $R = 24.55x$. The cost of producing x units is

$$C = 15.4x + 150{,}000.$$

To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

97. ***Daily Sales*** A doughnut shop sells a dozen doughnuts for \$2.95. Beyond the fixed costs (rent, utilities, and insurance) of \$150 per day, it costs \$1.45 for enough materials (flour, sugar, and so on) and labor to produce a dozen doughnuts. The daily profit from doughnut sales varies between \$50 and \$200. Between what levels (in dozens) do the daily sales vary?

98. ***Weight Loss Program*** A person enrolls in a diet and exercise program that guarantees a loss of at least $1\frac{1}{2}$ pounds per week. The person's weight at the beginning of the program is 164 pounds. Find the maximum number of weeks before the person attains a goal weight of 128 pounds.

99. ***Data Analysis: IQ Scores and GPA*** The admissions office of a college wants to determine whether there is a relationship between IQ scores x and grade-point averages y after the first year of school. An equation that models the data the admissions office obtained is

$$y = 0.067x - 5.638.$$

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of x that predict a grade-point average of at least 3.0.

100. ***Data Analysis: Weightlifting*** You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). The table shows a sample of data from 12 athletes.

(a) Use a graphing utility to plot the data.

Athlete's weight, x	Bench-press weight, y
165	170
184	185
150	200
210	255
196	205
240	295
202	190
170	175
185	195
190	185
230	250
160	155

(b) A model for the data is $y = 1.3x - 36$. Use a graphing utility to graph the model in the same viewing window used in part (a).

(c) Use the graph to estimate the values of x that predict a maximum bench-press weight of at least 200 pounds.

(d) Verify your estimate from part (c) algebraically.

(e) Use the graph to write a statement about the accuracy of the model. If you think the graph indicates that an athlete's weight is not a particularly good indicator of the athlete's maximum bench-press weight, list other factors that might influence an individual's maximum bench-press weight.

101. ***Teachers' Salaries*** The average salary S (in thousands of dollars) for elementary school teachers in the United States from 1990 to 2002 is approximated by the model

$$S = 1.05t + 31.0, \qquad 0 \le t \le 12$$

where t represents the year, with $t = 0$ corresponding to 1990. (Source: National Education Association)

(a) According to this model, when was the average salary at least \$32,000, but not more than \$42,000?

(b) According to this model, when will the average salary exceed \$48,000?

102. ***Egg Production*** The number of eggs E (in billions) produced in the United States from 1990 to 2002 can be modeled by

$$E = 1.64t + 67.2, \quad 0 \leq t \leq 12$$

where t represents the year, with $t = 0$ corresponding to 1990. (Source: U.S. Department of Agriculture)

(a) According to this model, when was the annual egg production 70 billion, but no more than 80 billion?

(b) According to this model, when will the annual egg production exceed 95 billion?

103. ***Geometry*** The side of a square is measured as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

104. ***Geometry*** The side of a square is measured as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

105. ***Accuracy of Measurement*** You stop at a self-service gas station to buy 15 gallons of 87-octane gasoline at \$1.89 a gallon. The gas pump is accurate to within $\frac{1}{10}$ of a gallon. How much might you be undercharged or overcharged?

106. ***Accuracy of Measurement*** You buy six T-bone steaks that cost \$14.99 per pound. The weight that is listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{2}$ ounce. How much might you be undercharged or overcharged?

107. ***Time Study*** A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$\left|\frac{t - 15.6}{1.9}\right| < 1$$

where t is time in minutes. Determine the interval on the real number line in which these times lie.

108. ***Height*** The heights h of two-thirds of the members of a population satisfy the inequality

$$\left|\frac{h - 68.5}{2.7}\right| \leq 1$$

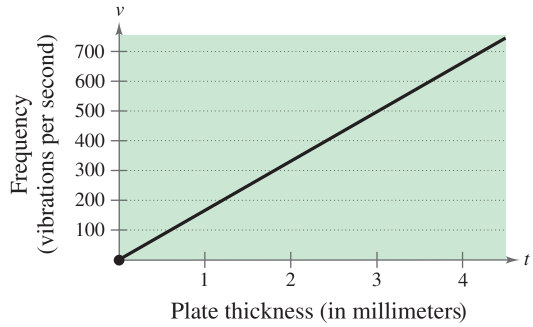
where h is measured in inches. Determine the interval on the real number line in which these heights lie.

109. ***Meteorology*** An electronic device is to be operated in an environment with relative humidity h in the interval defined by $|h - 50| \leq 30$. What are the minimum and maximum relative humidities for the operation of this device?

110. ***Music*** Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. He used the model for the frequency of the vibrations on a circular plate

$$v = \frac{2.6t}{d^2}\sqrt{\frac{E}{\rho}}$$

where v is the frequency (in vibrations per second), t is the plate thickness (in millimeters), d is the diameter of the plate, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d, E, and ρ, the graph of the equation is a line (see figure).



(a) Estimate the frequency when the plate thickness is 2 millimeters.

(b) Estimate the plate thickness when the frequency is 600 vibrations per second.

(c) Approximate the interval for the plate thickness when the frequency is between 200 and 400 vibrations per second.

(d) Approximate the interval for the frequency when the plate thickness is less than 3 millimeters.

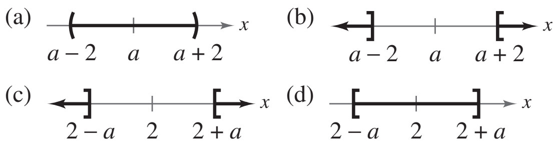
Synthesis

True or False? **In Exercises 111 and 112, determine whether the statement is true or false. Justify your answer.**

111. If a, b, and c are real numbers, and $a \leq b$, then $ac \leq bc$.

112. If $-10 \leq x \leq 8$, then $-10 \geq -x$ and $-x \geq -8$.

113. Identify the graph of the inequality $|x - a| \geq 2$.



114. Find sets of values of a, b, and c such that $0 \leq x \leq 10$ is a solution of the inequality $|ax - b| \leq c$.

A.7 Errors and the Algebra of Calculus

What you should learn

- Avoid common algebraic errors.
- Recognize and use algebraic techniques that are common in calculus.

Why you should learn it

An efficient command of algebra is critical in mastering this course and in the study of calculus.

Algebraic Errors to Avoid

This section contains five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving dividing out. Many of these errors are made because they seem to be the *easiest* things to do. For instance, the operations of subtraction and division are often believed to be commutative and associative. The following examples illustrate the fact that subtraction and division are neither commutative nor associative.

Not commutative	*Not associative*
$4 - 3 \neq 3 - 4$	$8 - (6 - 2) \neq (8 - 6) - 2$
$15 \div 5 \neq 5 \div 15$	$20 \div (4 \div 2) \neq (20 \div 4) \div 2$

Errors Involving Parentheses

Potential Error	*Correct Form*	*Comment*
$\cancel{a - (x - b) = a - x - b}$	$a - (x - b) = a - x + b$	Change all signs when distributing minus sign.
$\cancel{(a + b)^2 = a^2 + b^2}$	$(a + b)^2 = a^2 + 2ab + b^2$	Remember the middle term when squaring binomials.
$\xcancel{\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{2}(ab)}$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{4}(ab) = \frac{ab}{4}$	$\frac{1}{2}$ occurs twice as a factor.
$\xcancel{(3x + 6)^2 = 3(x + 2)^2}$	$(3x + 6)^2 = [3(x + 2)]^2 = 3^2(x + 2)^2$	When factoring, apply exponents to all factors.

Errors Involving Fractions

Potential Error	*Correct Form*	*Comment*
$\xcancel{\frac{a}{x + b} = \frac{a}{x} + \frac{a}{b}}$	Leave as $\frac{a}{x + b}$.	Do not add denominators when adding fractions.
$\xcancel{\frac{\left(\frac{x}{a}\right)}{b} = \frac{bx}{a}}$	$\frac{\left(\frac{x}{a}\right)}{b} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$	Multiply by the reciprocal when dividing fractions.
$\xcancel{\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b}}$	$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$	Use the property for adding fractions.
$\xcancel{\frac{1}{3x} = \frac{1}{3}x}$	$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	Use the property for multiplying fractions.
$\xcancel{(1/3)x = \frac{1}{3x}}$	$(1/3)x = \frac{1}{3} \cdot x = \frac{x}{3}$	Be careful when using a slash to denote division.
$\xcancel{(1/x) + 2 = \frac{1}{x + 2}}$	$(1/x) + 2 = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$	Be careful when using a slash to denote division and be sure to find a common denominator before you add fractions.

Errors Involving Exponents

Potential Error	*Correct Form*	*Comment*
$\cancel{(x^2)^3 = x^5}$	$(x^2)^3 = x^{2 \cdot 3} = x^6$	Multiply exponents when raising a power to a power.
$\cancel{x^2 \cdot x^3 = x^6}$	$x^2 \cdot x^3 = x^{2+3} = x^5$	Add exponents when multiplying powers with like bases.
$\cancel{2x^3 = (2x)^3}$	$2x^3 = 2(x^3)$	Exponents have priority over coefficients.
$\xcancel{\dfrac{1}{x^2 - x^3} = x^{-2} - x^{-3}}$	Leave as $\dfrac{1}{x^2 - x^3}$.	Do not move term-by-term from denominator to numerator.

Errors Involving Radicals

Potential Error	*Correct Form*	*Comment*
$\cancel{\sqrt{5x} = 5\sqrt{x}}$	$\sqrt{5x} = \sqrt{5}\sqrt{x}$	Radicals apply to every factor inside the radical.
$\cancel{\sqrt{x^2 + a^2} = x + a}$	Leave as $\sqrt{x^2 + a^2}$.	Do not apply radicals term-by-term.
$\cancel{\sqrt{-x + a} = -\sqrt{x - a}}$	Leave as $\sqrt{-x + a}$.	Do not factor minus signs out of square roots.

Errors Involving Dividing Out

Potential Error	*Correct Form*	*Comment*
$\xcancel{\dfrac{a + bx}{a} = 1 + bx}$	$\dfrac{a + bx}{a} = \dfrac{a}{a} + \dfrac{bx}{a} = 1 + \dfrac{b}{a}x$	Divide out common factors, not common terms.
$\xcancel{\dfrac{a + ax}{a} = a + x}$	$\dfrac{a + ax}{a} = \dfrac{a(1 + x)}{a} = 1 + x$	Factor before dividing out.
$\xcancel{1 + \dfrac{x}{2x} = 1 + \dfrac{1}{x}}$	$1 + \dfrac{x}{2x} = 1 + \dfrac{1}{2} = \dfrac{3}{2}$	Divide out common factors.

A good way to avoid errors is to *work slowly*, *write neatly*, and *talk to yourself*. Each time you write a step, ask yourself why the step is algebraically legitimate. You can justify the step below because *dividing the numerator and denominator by the same nonzero number produces an equivalent fraction*.

$$\frac{2x}{6} = \frac{\cancel{2} \cdot x}{\cancel{2} \cdot 3} = \frac{x}{3}$$

Example 1 Using the Property for Adding Fractions

Describe and correct the error. $\xcancel{\dfrac{1}{2x} + \dfrac{1}{3x} = \dfrac{1}{5x}}$

Solution

When adding fractions, use the property for adding fractions: $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{b + a}{ab}$.

$$\frac{1}{2x} + \frac{1}{3x} = \frac{3x + 2x}{6x^2} = \frac{5x}{6x^2} = \frac{5}{6x}$$

CHECKPOINT Now try Exercise 17.

Some Algebra of Calculus

In calculus it is often necessary to take a simplified algebraic expression and "unsimplify" it. See the following lists, taken from a standard calculus text.

Unusual Factoring

Expression	*Useful Calculus Form*	*Comment*
$\dfrac{5x^4}{8}$	$\dfrac{5}{8}x^4$	Write with fractional coefficient.
$\dfrac{x^2 + 3x}{-6}$	$-\dfrac{1}{6}(x^2 + 3x)$	Write with fractional coefficient.
$2x^2 - x - 3$	$2\left(x^2 - \dfrac{x}{2} - \dfrac{3}{2}\right)$	Factor out the leading coefficient.
$\dfrac{x}{2}(x + 1)^{-1/2} + (x + 1)^{1/2}$	$\dfrac{(x + 1)^{-1/2}}{2}[x + 2(x + 1)]$	Factor out factor with lowest power.

Writing with Negative Exponents

Expression	*Useful Calculus Form*	*Comment*
$\dfrac{9}{5x^3}$	$\dfrac{9}{5}x^{-3}$	Move the factor to the numerator and change the sign of the exponent.
$\dfrac{7}{\sqrt{2x - 3}}$	$7(2x - 3)^{-1/2}$	Move the factor to the numerator and change the sign of the exponent.

Writing a Fraction as a Sum

Expression	*Useful Calculus Form*	*Comment*
$\dfrac{x + 2x^2 + 1}{\sqrt{x}}$	$x^{1/2} + 2x^{3/2} + x^{-1/2}$	Divide each term by $x^{1/2}$.
$\dfrac{1 + x}{x^2 + 1}$	$\dfrac{1}{x^2 + 1} + \dfrac{x}{x^2 + 1}$	Rewrite the fraction as the sum of fractions.
$\dfrac{2x}{x^2 + 2x + 1}$	$\dfrac{2x + 2 - 2}{x^2 + 2x + 1}$	Add and subtract the same term.
	$= \dfrac{2x + 2}{x^2 + 2x + 1} - \dfrac{2}{(x + 1)^2}$	Rewrite the fraction as the difference of fractions.
$\dfrac{x^2 - 2}{x + 1}$	$x - 1 - \dfrac{1}{x + 1}$	Use long division. (See Section 2.3.)
$\dfrac{x + 7}{x^2 - x - 6}$	$\dfrac{2}{x - 3} - \dfrac{1}{x + 2}$	Use the method of partial fractions. (See Section 7.4.)

Inserting Factors and Terms

Expression	*Useful Calculus Form*	*Comment*
$(2x-1)^3$	$\frac{1}{2}(2x-1)^3(2)$	Multiply and divide by 2.
$7x^2(4x^3-5)^{1/2}$	$\frac{7}{12}(4x^3-5)^{1/2}(12x^2)$	Multiply and divide by 12.
$\frac{4x^2}{9}-4y^2=1$	$\frac{x^2}{9/4}-\frac{y^2}{1/4}=1$	Write with fractional denominators.
$\frac{x}{x+1}$	$\frac{x+1-1}{x+1}=1-\frac{1}{x+1}$	Add and subtract the same term.

The next five examples demonstrate many of the steps in the preceding lists.

Example 2 Factors Involving Negative Exponents

Factor $x(x+1)^{-1/2}+(x+1)^{1/2}$.

Solution

When multiplying factors with like bases, you add exponents. When factoring, you are undoing multiplication, and so you *subtract* exponents.

$$\begin{aligned} x(x+1)^{-1/2}+(x+1)^{1/2} &= (x+1)^{-1/2}[x(x+1)^0+(x+1)^1] \\ &= (x+1)^{-1/2}[x+(x+1)] \\ &= (x+1)^{-1/2}(2x+1) \end{aligned}$$

CHECKPOINT Now try Exercise 23.

Another way to simplify the expression in Example 2 is to multiply the expression by a fractional form of 1 and then use the Distributive Property.

$$\begin{aligned} x(x+1)^{-1/2}+(x+1)^{1/2} &= [x(x+1)^{-1/2}+(x+1)^{1/2}]\cdot\frac{(x+1)^{1/2}}{(x+1)^{1/2}} \\ &= \frac{x(x+1)^0+(x+1)^1}{(x+1)^{1/2}} = \frac{2x+1}{\sqrt{x+1}} \end{aligned}$$

Example 3 Inserting Factors in an Expression

Insert the required factor: $\frac{x+2}{(x^2+4x-3)^2}=(\quad)\frac{1}{(x^2+4x-3)^2}(2x+4)$.

Solution

The expression on the right side of the equation is twice the expression on the left side. To make both sides equal, insert a factor of $\frac{1}{2}$.

$$\frac{x+2}{(x^2+4x-3)^2}=\left(\frac{1}{2}\right)\frac{1}{(x^2+4x-3)^2}(2x+4)$$

Right side is multiplied and divided by 2.

CHECKPOINT Now try Exercise 25.

Example 4 **Rewriting Fractions**

Explain why the two expressions are equivalent.

$$\frac{4x^2}{9} - 4y^2 = \frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{1}{4}}$$

Solution

To write the expression on the left side of the equation in the form given on the right side, multiply the numerators and denominators of both terms by $\frac{1}{4}$.

$$\frac{4x^2}{9} - 4y^2 = \frac{4x^2}{9}\left(\frac{\frac{1}{4}}{\frac{1}{4}}\right) - 4y^2\left(\frac{\frac{1}{4}}{\frac{1}{4}}\right) = \frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{1}{4}}$$

CHECKPOINT Now try Exercise 29.

Example 5 **Rewriting with Negative Exponents**

Rewrite each expression using negative exponents.

a. $\dfrac{-4x}{(1-2x^2)^2}$ **b.** $\dfrac{2}{5x^3} - \dfrac{1}{\sqrt{x}} + \dfrac{3}{5(4x)^2}$

Solution

a. $\dfrac{-4x}{(1-2x^2)^2} = -4x(1-2x^2)^{-2}$

b. Begin by writing the second term in exponential form.

$$\begin{aligned}\frac{2}{5x^3} - \frac{1}{\sqrt{x}} + \frac{3}{5(4x)^2} &= \frac{2}{5x^3} - \frac{1}{x^{1/2}} + \frac{3}{5(4x)^2}\\ &= \frac{2}{5}x^{-3} - x^{-1/2} + \frac{3}{5}(4x)^{-2}\end{aligned}$$

CHECKPOINT Now try Exercise 39.

Example 6 **Writing a Fraction as a Sum of Terms**

Rewrite each fraction as the sum of three terms.

a. $\dfrac{x^2 - 4x + 8}{2x}$ **b.** $\dfrac{x + 2x^2 + 1}{\sqrt{x}}$

Solution

a. $$\begin{aligned}\frac{x^2 - 4x + 8}{2x} &= \frac{x^2}{2x} - \frac{4x}{2x} + \frac{8}{2x}\\ &= \frac{x}{2} - 2 + \frac{4}{x}\end{aligned}$$

b. $$\begin{aligned}\frac{x + 2x^2 + 1}{\sqrt{x}} &= \frac{x}{x^{1/2}} + \frac{2x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\\ &= x^{1/2} + 2x^{3/2} + x^{-1/2}\end{aligned}$$

CHECKPOINT Now try Exercise 43.

A.7 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. To write the expression $\frac{2}{\sqrt{x}}$ with negative exponents, move $\sqrt{x}$ to the ________ and change the sign of the exponent.

2. When dividing fractions, multiply by the ________.

In Exercises 1–18, describe and correct the error.

1. $2x - (3y + 4) = 2x - 3y + 4$

2. $5z + 3(x - 2) = 5z + 3x - 2$

3. $\dfrac{4}{16x - (2x + 1)} = \dfrac{4}{14x + 1}$

4. $\dfrac{1 - x}{(5 - x)(-x)} = \dfrac{x - 1}{x(x - 5)}$

5. $(5z)(6z) = 30z$

6. $x(yz) = (xy)(xz)$

7. $a\left(\dfrac{x}{y}\right) = \dfrac{ax}{ay}$

8. $(4x)^2 = 4x^2$

9. $\sqrt{x + 9} = \sqrt{x} + 3$

10. $\sqrt{25 - x^2} = 5 - x$

11. $\dfrac{2x^2 + 1}{5x} = \dfrac{2x + 1}{5}$

12. $\dfrac{6x + y}{6x - y} = \dfrac{x + y}{x - y}$

13. $\dfrac{1}{a^{-1} + b^{-1}} = \left(\dfrac{1}{a + b}\right)^{-1}$

14. $\dfrac{1}{x + y^{-1}} = \dfrac{y}{x + 1}$

15. $(x^2 + 5x)^{1/2} = x(x + 5)^{1/2}$

16. $x(2x - 1)^2 = (2x^2 - x)^2$

17. $\dfrac{3}{x} + \dfrac{4}{y} = \dfrac{7}{x + y}$

18. $\dfrac{1}{2y} = (1/2)y$

In Exercises 19–38, insert the required factor in the parentheses.

19. $\dfrac{3x + 2}{5} = \dfrac{1}{5}(\boxed{\quad})$

20. $\dfrac{7x^2}{10} = \dfrac{7}{10}(\boxed{\quad})$

21. $\frac{2}{3}x^2 + \frac{1}{3}x + 5 = \frac{1}{3}(\boxed{\quad})$

22. $\frac{3}{4}x + \frac{1}{2} = \frac{1}{4}(\boxed{\quad})$

23. $x^2(x^3 - 1)^4 = (\boxed{\quad})(x^3 - 1)^4(3x^2)$

24. $x(1 - 2x^2)^3 = (\boxed{\quad})(1 - 2x^2)^3(-4x)$

25. $\dfrac{4x + 6}{(x^2 + 3x + 7)^3} = (\boxed{\quad})\dfrac{1}{(x^2 + 3x + 7)^3}(2x + 3)$

26. $\dfrac{x + 1}{(x^2 + 2x - 3)^2} = (\boxed{\quad})\dfrac{1}{(x^2 + 2x - 3)^2}(2x + 2)$

27. $\dfrac{3}{x} + \dfrac{5}{2x^2} - \dfrac{3}{2}x = (\boxed{\quad})(6x + 5 - 3x^3)$

28. $\dfrac{(x - 1)^2}{169} + (y + 5)^2 = \dfrac{(x - 1)^3}{169(\boxed{\quad})} + (y + 5)^2$

29. $\dfrac{9x^2}{25} + \dfrac{16y^2}{49} = \dfrac{x^2}{(\boxed{\quad})} + \dfrac{y^2}{(\boxed{\quad})}$

30. $\dfrac{3x^2}{4} - \dfrac{9y^2}{16} = \dfrac{x^2}{(\boxed{\quad})} - \dfrac{y^2}{(\boxed{\quad})}$

31. $\dfrac{x^2}{1/12} - \dfrac{y^2}{2/3} = \dfrac{12x^2}{(\boxed{\quad})} - \dfrac{3y^2}{(\boxed{\quad})}$

32. $\dfrac{x^2}{4/9} + \dfrac{y^2}{7/8} = \dfrac{9x^2}{(\boxed{\quad})} + \dfrac{8y^2}{(\boxed{\quad})}$

33. $x^{1/3} - 5x^{4/3} = x^{1/3}(\boxed{\quad})$

34. $3(2x + 1)x^{1/2} + 4x^{3/2} = x^{1/2}(\boxed{\quad})$

35. $(1 - 3x)^{4/3} - 4x(1 - 3x)^{1/3} = (1 - 3x)^{1/3}(\boxed{\quad})$

36. $\dfrac{1}{2\sqrt{x}} + 5x^{3/2} - 10x^{5/2} = \dfrac{1}{2\sqrt{x}}(\boxed{\quad})$

37. $\dfrac{1}{10}(2x + 1)^{5/2} - \dfrac{1}{6}(2x + 1)^{3/2} = \dfrac{(2x + 1)^{3/2}}{15}(\boxed{\quad})$

38. $\dfrac{3}{7}(t + 1)^{7/3} - \dfrac{3}{4}(t + 1)^{4/3} = \dfrac{3(t + 1)^{4/3}}{28}(\boxed{\quad})$

In Exercises 39–42, write the expression using negative exponents.

39. $\dfrac{3x^2}{(2x - 1)^3}$

40. $\dfrac{x + 1}{x(6 - x)^{1/2}}$

41. $\dfrac{4}{3x} + \dfrac{4}{x^4} - \dfrac{7x}{\sqrt[3]{2x}}$

42. $\dfrac{x}{x - 2} + \dfrac{1}{x^2} + \dfrac{8}{3(9x)^3}$

In Exercises 43–48, write the fraction as a sum of two or more terms.

43. $\dfrac{16 - 5x - x^2}{x}$

44. $\dfrac{x^3 - 5x^2 + 4}{x^2}$

45. $\dfrac{4x^3 - 7x^2 + 1}{x^{1/3}}$

46. $\dfrac{2x^5 - 3x^3 + 5x - 1}{x^{3/2}}$

47. $\dfrac{3 - 5x^2 - x^4}{\sqrt{x}}$

48. $\dfrac{x^3 - 5x^4}{3x^2}$

In Exercises 49–60, simplify the expression.

49. $\dfrac{-2(x^2 - 3)^{-3}(2x)(x + 1)^3 - 3(x + 1)^2(x^2 - 3)^{-2}}{[(x + 1)^3]^2}$

50. $\dfrac{x^5(-3)(x^2 + 1)^{-4}(2x) - (x^2 + 1)^{-3}(5)x^4}{(x^5)^2}$

51. $\dfrac{(6x + 1)^3(27x^2 + 2) - (9x^3 + 2x)(3)(6x + 1)^2(6)}{[(6x + 1)^3]^2}$

52. $\dfrac{(4x^2+9)^{1/2}(2)-(2x+3)\left(\frac{1}{2}\right)(4x^2+9)^{-1/2}(8x)}{[(4x^2+9)^{1/2}]^2}$

53. $\dfrac{(x+2)^{3/4}(x+3)^{-2/3}-(x+3)^{1/3}(x+2)^{-1/4}}{[(x+2)^{3/4}]^2}$

54. $(2x-1)^{1/2}-(x+2)(2x-1)^{-1/2}$

55. $\dfrac{2(3x-1)^{1/3}-(2x+1)\left(\frac{1}{3}\right)(3x-1)^{-2/3}(3)}{(3x-1)^{2/3}}$

56. $\dfrac{(x+1)\left(\frac{1}{2}\right)(2x-3x^2)^{-1/2}(2-6x)-(2x-3x^2)^{1/2}}{(x+1)^2}$

57. $\dfrac{1}{(x^2+4)^{1/2}}\cdot\dfrac{1}{2}(x^2+4)^{-1/2}(2x)$

58. $\dfrac{1}{x^2-6}(2x)+\dfrac{1}{2x+5}(2)$

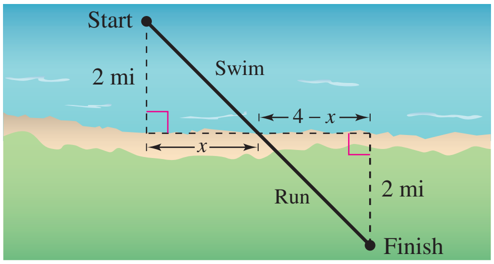
59. $(x^2+5)^{1/2}\left(\frac{3}{2}\right)(3x-2)^{1/2}(3)$
$\quad + (3x-2)^{3/2}\left(\frac{1}{2}\right)(x^2+5)^{-1/2}(2x)$

60. $(3x+2)^{-1/2}(3)(x-6)^{1/2}(1)$
$\quad + (x-6)^3\left(-\frac{1}{2}\right)(3x+2)^{-3/2}(3)$

61. ***Athletics*** An athlete has set up a course for training as part of her regimen in preparation for an upcoming triathlon. She is dropped off by a boat 2 miles from the nearest point on shore. The finish line is 4 miles down the coast and 2 miles inland (see figure). She can swim 2 miles per hour and run 6 miles per hour. The time t (in hours) required for her to reach the finish line can be approximated by the model

$$t=\frac{\sqrt{x^2+4}}{2}+\frac{\sqrt{(4-x)^2+4}}{6}$$

where x is the distance down the coast (in miles) to which she swims and then leaves the water to start her run.



(a) Find the times required for the triathlete to finish when she swims to the points $x = 0.5$, $x = 1.0, \ldots,$ $x = 3.5$, and $x = 4.0$ miles down the coast.

(b) Use your results from part (a) to determine the distance down the coast that will yield the minimum amount of time required for the triathlete to reach the finish line.

(c) The expression below was obtained using calculus. It can be used to find the minimum amount of time required for the triathlete to reach the finish line. Simplify the expression.

$$\tfrac{1}{2}x(x^2+4)^{-1/2}+\tfrac{1}{6}(x-4)(x^2-8x+20)^{-1/2}$$

62. (a) Verify that $y_1 = y_2$ analytically.

$$y_1 = x^2\left(\frac{1}{3}\right)(x^2+1)^{-2/3}(2x)+(x^2+1)^{1/3}(2x)$$

$$y_2 = \frac{2x(4x^2+3)}{3(x^2+1)^{2/3}}$$

(b) Complete the table and demonstrate the equality in part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{5}{2}$
y_1							
y_2							

Synthesis

True or False? **In Exercises 63–66, determine whether the statement is true or false. Justify your answer.**

63. $x^{-1}+y^{-2}=\dfrac{y^2+x}{xy^2}$

64. $\dfrac{1}{x^{-2}+y^{-1}}=x^2+y$

65. $\dfrac{1}{\sqrt{x}+4}=\dfrac{\sqrt{x}-4}{x-16}$

66. $\dfrac{x^2-9}{\sqrt{x}-3}=\sqrt{x}+3$

In Exercises 67–70, find and correct any errors. If the problem is correct, state that it is correct.

67. $x^n\cdot x^{3n}=x^{3n^2}$

68. $(x^n)^{2n}+(x^{2n})^n=2x^{2n^2}$

69. $x^{2n}+y^{2n}=(x^n+y^n)^2$

70. $\dfrac{x^{2n}\cdot x^{3n}}{x^{3n}+x^2}=\dfrac{x^{5n}}{x^{3n}+x^2}$

71. ***Think About It*** You are taking a course in calculus, and for one of the homework problems you obtain the following answer.

$$\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}$$

The answer in the back of the book is $\frac{1}{15}(2x-1)^{3/2}(3x+1)$. Show how the second answer can be obtained from the first. Then use the same technique to simplify each of the following expressions.

(a) $\dfrac{2}{3}x(2x-3)^{3/2}-\dfrac{2}{15}(2x-3)^{5/2}$

(b) $\dfrac{2}{3}x(4+x)^{3/2}-\dfrac{2}{15}(4+x)^{5/2}$