APPENDIXES

A NUMBERS, INEQUALITIES, AND ABSOLUTE VALUES

Calculus is based on the real number system. We start with the **integers**:

$$\ldots, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

Then we construct the **rational numbers**, which are ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n} \qquad \text{where } m \text{ and } n \text{ are integers and } n \neq 0$$

Examples are

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) Some real numbers, such as $\sqrt{2}$, can't be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that the following are also irrational numbers:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \sin 1° \qquad \log_{10} 2$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we mean "real number."

Every number has a decimal representation. If the number is rational, then the corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.317171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) On the other hand, if the number is irrational, the decimal is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better the approximation we get.

The real numbers can be represented by points on a line as in Figure 1. The positive direction (to the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. Thus every real number is represented by a point on the line, and every point P on the line corresponds to exactly one real number. The number associated with the point P is called the **coordinate** of P and the line is then called a **coor-**

dinate line, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

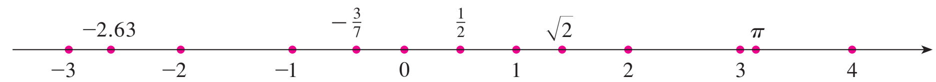


FIGURE 1

The real numbers are ordered. We say *a is less than b* and write $a < b$ if $b - a$ is a positive number. Geometrically this means that a lies to the left of b on the number line. (Equivalently, we say *b is greater than a* and write $b > a$.) The symbol $a \leqslant b$ (or $b \geqslant a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities:

$$7 < 7.4 < 7.5 \qquad -3 > -\pi \qquad \sqrt{2} < 2 \qquad \sqrt{2} \leqslant 2 \qquad 2 \leqslant 2$$

In what follows we need to use *set notation*. A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$. If S and T are sets, then their **union** $S \cup T$ is the set consisting of all elements that are in S or T (or in both S and T). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. In other words, $S \cap T$ is the common part of S and T. The empty set, denoted by $\varnothing$, is the set that contains no element.

Some sets can be described by listing their elements between braces. For instance, the set A consisting of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in *set-builder notation* as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of x such that x is an integer and $0 < x < 7$."

INTERVALS

Certain sets of real numbers, called **intervals,** occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

FIGURE 2
Open interval (a, b)

Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the round brackets ( ) and by the open dots in Figure 2. The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$

FIGURE 3
Closed interval $[a, b]$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 3. It is also possible to include only one endpoint in an interval, as shown in Table 1.

1 TABLE OF INTERVALS

■ Table 1 lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that $a < b$.

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$

This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

INEQUALITIES

When working with inequalities, note the following rules.

2 RULES FOR INEQUALITIES

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 1 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set.*

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\tfrac{4}{6} = -\tfrac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 2 Solve the inequalities $4 \leqslant 3x - 2 < 13$.

SOLUTION Here the solution set consists of all values of x that satisfy both inequalities. Using the rules given in (2), we see that the following inequalities are equivalent:

$$4 \leqslant 3x - 2 < 13$$

$$6 \leqslant 3x < 15 \qquad \text{(add 2)}$$

$$2 \leqslant x < 5 \qquad \text{(divide by 3)}$$

Therefore the solution set is $[2, 5)$. ■

EXAMPLE 3 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

■ A visual method for solving Example 3 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 4) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

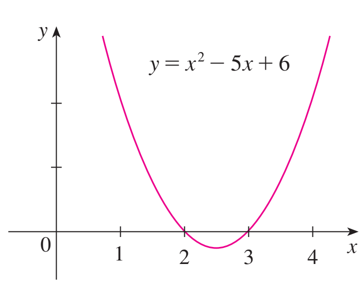


FIGURE 4

Another method for obtaining the information in the chart is to use *test values*. For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

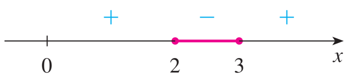


FIGURE 5

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 5. ■

EXAMPLE 4 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x - 1)(x + 4) > 0$$

As in Example 3 we solve the corresponding equation $x(x - 1)(x + 4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign as shown in the following chart:

Interval	x	$x - 1$	$x + 4$	$x(x - 1)(x + 4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

FIGURE 6

The solution is illustrated in Figure 6. ■

ABSOLUTE VALUE

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

■ Remember that if a is negative, then $-a$ is positive.

3
$$\boxed{\begin{aligned} |a| &= a \qquad \text{if } a \geqslant 0 \\ |a| &= -a \qquad \text{if } a < 0 \end{aligned}}$$

EXAMPLE 5 Express $|3x - 2|$ without using the absolute-value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore, the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (3), we then have the equation

4 $$\boxed{\sqrt{a^2} = |a|}$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

5 PROPERTIES OF ABSOLUTE VALUES Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

6 Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

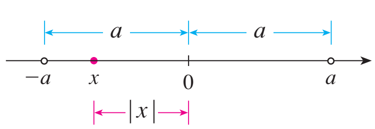


FIGURE 7

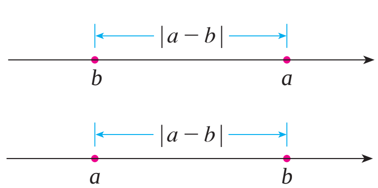


FIGURE 8
Length of a line segment $= |a - b|$

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 7 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 8.)

EXAMPLE 6 Solve $|2x - 5| = 3$.

SOLUTION By Property 4 of (6), $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus $x = 4$ or $x = 1$.

EXAMPLE 7 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of (6), $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval $(3, 7)$.

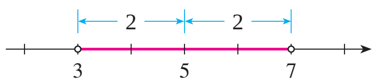


FIGURE 9

SOLUTION 2 Geometrically the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$. ■

EXAMPLE 8 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of (6), $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \qquad \text{or} \qquad 3x + 2 \leqslant -4$$

In the first case $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

Another important property of absolute value, called the Triangle Inequality, is used frequently not only in calculus but throughout mathematics in general.

7 THE TRIANGLE INEQUALITY If a and b are any real numbers, then

$$|a + b| \leqslant |a| + |b|$$

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leqslant a \leqslant |a|$$

is always true because a equals either $|a|$ or $-|a|$. The corresponding statement for b is

$$-|b| \leqslant b \leqslant |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$

If we now apply Properties 4 and 5 (with x replaced by $a + b$ and a by $|a| + |b|$), we obtain

$$|a + b| \leqslant |a| + |b|$$

which is what we wanted to show.

EXAMPLE 9 If $|x - 4| < 0.1$ and $|y - 7| < 0.2$, use the Triangle Inequality to estimate $|(x + y) - 11|$.

SOLUTION In order to use the given information, we use the Triangle Inequality with $a = x - 4$ and $b = y - 7$:

$$\begin{aligned} |(x + y) - 11| &= |(x - 4) + (y - 7)| \\ &\leqslant |x - 4| + |y - 7| \\ &< 0.1 + 0.2 = 0.3 \end{aligned}$$

Thus $$|(x + y) - 11| < 0.3$$ ■

A EXERCISES

1–12 Rewrite the expression without using the absolute value symbol.

1. $|5 - 23|$

2. $|5| - |-23|$

3. $|-\pi|$

4. $|\pi - 2|$

5. $|\sqrt{5} - 5|$

6. $||-2| - |-3||$

7. $|x - 2|$ if $x < 2$

8. $|x - 2|$ if $x > 2$

9. $|x + 1|$

10. $|2x - 1|$

11. $|x^2 + 1|$

12. $|1 - 2x^2|$

13–38 Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

13. $2x + 7 > 3$

14. $3x - 11 < 4$

15. $1 - x \leqslant 2$

16. $4 - 3x \geqslant 6$

17. $2x + 1 < 5x - 8$

18. $1 + 5x > 5 - 3x$

19. $-1 < 2x - 5 < 7$

20. $1 < 3x + 4 \leqslant 16$

21. $0 \leqslant 1 - x < 1$

22. $-5 \leqslant 3 - 2x \leqslant 9$

23. $4x < 2x + 1 \leqslant 3x + 2$

24. $2x - 3 < x + 4 < 3x - 2$

25. $(x - 1)(x - 2) > 0$

26. $(2x + 3)(x - 1) \leqslant 0$

27. $2x^2 + x \leqslant 1$

28. $(2x + 3)(x - 1) \geqslant 0$

29. $x^2 + x + 1 > 0$

30. $x^2 + x > 1$

31. $x^2 < 3$

32. $x^2 \geqslant 5$

33. $x^3 - x^2 \leqslant 0$

34. $(x + 1)(x - 2)(x + 3) \geqslant 0$

35. $x^3 > x$

36. $x^3 + 3x < 4x^2$

37. $\dfrac{1}{x} < 4$

38. $-3 < \dfrac{1}{x} \leqslant 1$

39. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

40. Use the relationship between C and F given in Exercise 39 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

41. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.
(a) If the ground temperature is 20°C, write a formula for the temperature at height h.
(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

42. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

43–46 Solve the equation for x.

43. $|2x| = 3$

44. $|3x + 5| = 1$

45. $|x + 3| = |2x + 1|$

46. $\left|\dfrac{2x - 1}{x + 1}\right| = 3$

47–56 Solve the inequality.

47. $|x| < 3$

48. $|x| \geqslant 3$

49. $|x - 4| < 1$

50. $|x - 6| < 0.1$

51. $|x + 5| \geqslant 2$

52. $|x + 1| \geqslant 3$

53. $|2x - 3| \leqslant 0.4$

54. $|5x - 2| < 6$

55. $1 \leqslant |x| \leqslant 4$

56. $0 < |x - 5| < \frac{1}{2}$

57–58 Solve for x, assuming a, b, and c are positive constants.

57. $a(bx - c) \geqslant bc$

58. $a \leqslant bx + c < 2a$

59–60 Solve for x, assuming a, b, and c are negative constants.

59. $ax + b < c$

60. $\dfrac{ax + b}{c} \leqslant b$

61. Suppose that $|x - 2| < 0.01$ and $|y - 3| < 0.04$. Use the Triangle Inequality to show that $|(x + y) - 5| < 0.05$.

62. Show that if $|x + 3| < \frac{1}{2}$, then $|4x + 13| < 3$.

63. Show that if $a < b$, then $a < \dfrac{a + b}{2} < b$.

64. Use Rule 3 to prove Rule 5 of (2).

65. Prove that $|ab| = |a||b|$. [*Hint:* Use Equation 4.]

66. Prove that $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. Show that if $0 < a < b$, then $a^2 < b^2$.

68. Prove that $|x - y| \geqslant |x| - |y|$. [*Hint:* Use the Triangle Inequality with $a = x - y$ and $b = y$.]

69. Show that the sum, difference, and product of rational numbers are rational numbers.

70. (a) Is the sum of two irrational numbers always an irrational number?

(b) Is the product of two irrational numbers always an irrational number?

B COORDINATE GEOMETRY AND LINES

Just as the points on a line can be identified with real numbers by assigning them coordinates, as described in Appendix A, so the points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axis.

Any point P in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates a and b as shown in Figure 1. Then the point P is assigned the ordered pair (a, b). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in Figure 2.

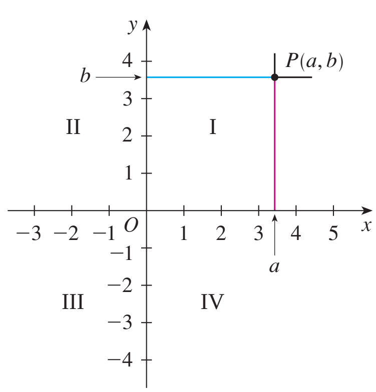


FIGURE 1

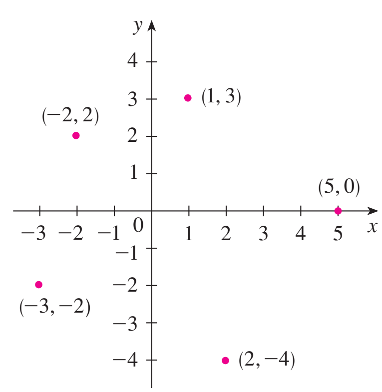


FIGURE 2

By reversing the preceding process we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." [Although the notation used for an open interval (a, b) is the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre Fermat (1601–1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane** and is denoted by $\mathbb{R}^2$.

The x- and y-axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

EXAMPLE 1 Describe and sketch the regions given by the following sets.

(a) $\{(x, y) \mid x \geqslant 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

SOLUTION

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 3(a).

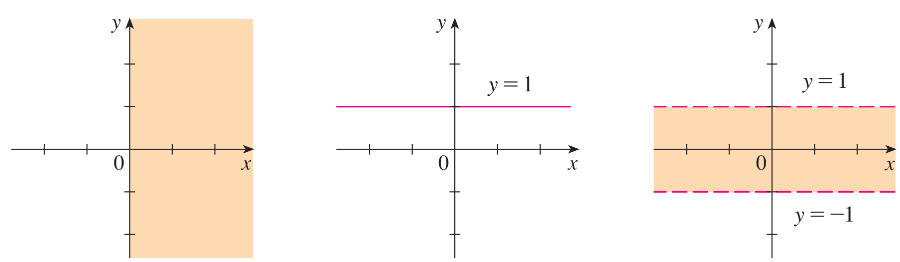


FIGURE 3 (a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 3(b)].

(c) Recall from Appendix A that

$$|y| < 1 \qquad \text{if and only if} \qquad -1 < y < 1$$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.] ■

Recall from Appendix A that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus the distance between points $P_1(x_1, y_1)$ and $P_3(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $P_2(x_2, y_2)$ and $P_3(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. (See Figure 4.)

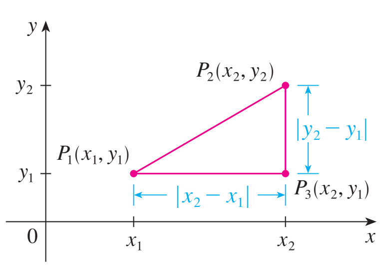


FIGURE 4

To find the distance $|P_1P_2|$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 4 is a right triangle, and so by the Pythagorean Theorem we have

$$|P_1P_2| = \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1 DISTANCE FORMULA The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 The distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

LINES

We want to find an equation of a given line L; such an equation is satisfied by the coordinates of the points on L and by no other point. To find the equation of L we use its *slope*, which is a measure of the steepness of the line.

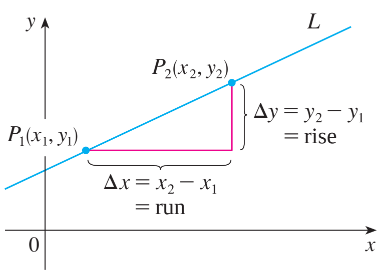


FIGURE 5

2 DEFINITION The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Thus the slope of a line is the ratio of the change in y, Δy, to the change in x, Δx. (See Figure 5.) The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

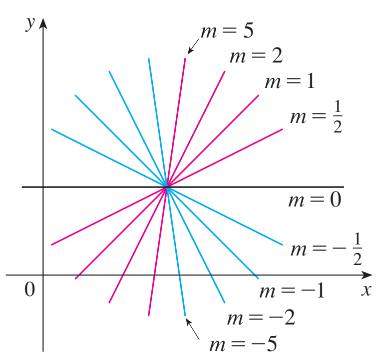


FIGURE 6

Figure 6 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

3 POINT-SLOPE FORM OF THE EQUATION OF A LINE An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 3 Find an equation of the line through $(1, -7)$ with slope $-\frac{1}{2}$.

SOLUTION Using (3) with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -7$, we obtain an equation of the line as

$$y + 7 = -\tfrac{1}{2}(x - 1)$$

which we can rewrite as

$$2y + 14 = -x + 1 \qquad \text{or} \qquad x + 2y + 13 = 0$$

EXAMPLE 4 Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION By Definition 2 the slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1)$$

which simplifies to

$$3x + 2y = 1$$

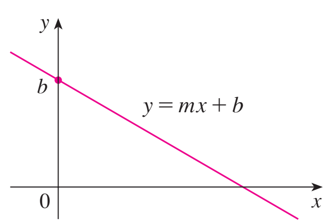


FIGURE 7

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 7.) This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = b$, becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

4 SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

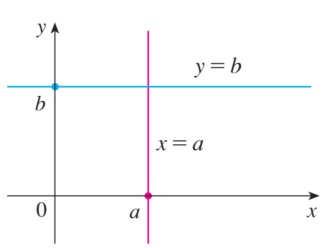


FIGURE 8

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Observe that the equation of every line can be written in the form

5 $$\boxed{Ax + By + C = 0}$$

because a vertical line has the equation $x = a$ or $x - a = 0$ ($A = 1$, $B = 0$, $C = -a$) and a nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$ ($A = -m$, $B = 1$, $C = -b$). Conversely, if we start with a general first-degree equation, that is, an equation of the form (5), where A, B, and C are constants and A and B are not both 0, then we can show that it is the equation of a line. If $B = 0$, the equation becomes $Ax + C = 0$ or $x = -C/A$, which represents a vertical line with x-intercept $-C/A$. If $B \neq 0$, the equation can be rewritten by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and we recognize this as being the slope-intercept form of the equation of a line ($m = -A/B$, $b = -C/B$). Therefore an equation of the form (5) is called a **linear equation** or the **general equation of a line**. For brevity, we often refer to "the line $Ax + By + C = 0$" instead of "the line whose equation is $Ax + By + C = 0$."

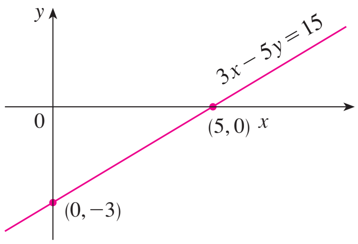


FIGURE 9

EXAMPLE 5 Sketch the graph of the equation $3x - 5y = 15$.

SOLUTION Since the equation is linear, its graph is a line. To draw the graph, we can simply find two points on the line. It's easiest to find the intercepts. Substituting $y = 0$ (the equation of the x-axis) in the given equation, we get $3x = 15$, so $x = 5$ is the x-intercept. Substituting $x = 0$ in the equation, we see that the y-intercept is -3. This allows us to sketch the graph as in Figure 9. ■

EXAMPLE 6 Graph the inequality $x + 2y > 5$.

SOLUTION We are asked to sketch the graph of the set $\{(x, y) \mid x + 2y > 5\}$ and we do so by solving the inequality for y:

$$x + 2y > 5$$

$$2y > -x + 5$$

$$y > -\tfrac{1}{2}x + \tfrac{5}{2}$$

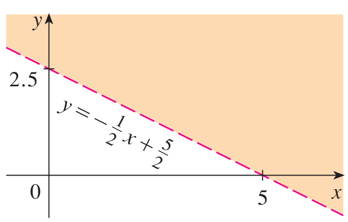


FIGURE 10

Compare this inequality with the equation $y = -\frac{1}{2}x + \frac{5}{2}$, which represents a line with slope $-\frac{1}{2}$ and y-intercept $\frac{5}{2}$. We see that the given graph consists of points whose y-coordinates are *larger* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$. Thus the graph is the region that lies *above* the line, as illustrated in Figure 10. ■

PARALLEL AND PERPENDICULAR LINES

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus, Fifth Edition* by Stewart, Redlin, and Watson (Thomson Brooks/Cole, Belmont, CA, 2006).

6 PARALLEL AND PERPENDICULAR LINES

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

EXAMPLE 7 Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION The given line can be written in the form

$$y = -\tfrac{2}{3}x - \tfrac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\tfrac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$. ■

EXAMPLE 8 Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

SOLUTION The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular. ■

B EXERCISES

1–6 Find the distance between the points.

1. $(1, 1)$, $(4, 5)$
2. $(1, -3)$, $(5, 7)$
3. $(6, -2)$, $(-1, 3)$
4. $(1, -6)$, $(-1, -3)$
5. $(2, 5)$, $(4, -7)$
6. (a, b), (b, a)

7–10 Find the slope of the line through P and Q.

7. $P(1, 5)$, $Q(4, 11)$
8. $P(-1, 6)$, $Q(4, -3)$
9. $P(-3, 3)$, $Q(-1, -6)$
10. $P(-1, -4)$, $Q(6, 0)$

11. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.
12. (a) Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle using the converse of the Pythagorean Theorem.
    (b) Use slopes to show that ABC is a right triangle.
    (c) Find the area of the triangle.
13. Show that the points $(-2, 9)$, $(4, 6)$, $(1, 0)$, and $(-5, 3)$ are the vertices of a square.
14. (a) Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear (lie on the same line) by showing that $|AB| + |BC| = |AC|$.
    (b) Use slopes to show that A, B, and C are collinear.
15. Show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.
16. Show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

17–20 Sketch the graph of the equation.

17. $x = 3$
18. $y = -2$
19. $xy = 0$
20. $|y| = 1$

21–36 Find an equation of the line that satisfies the given conditions.

21. Through $(2, -3)$, slope 6
22. Through $(-1, 4)$, slope -3
23. Through $(1, 7)$, slope $\frac{2}{3}$
24. Through $(-3, -5)$, slope $-\frac{7}{2}$
25. Through $(2, 1)$ and $(1, 6)$
26. Through $(-1, -2)$ and $(4, 3)$
27. Slope 3, y-intercept -2
28. Slope $\frac{2}{5}$, y-intercept 4
29. x-intercept 1, y-intercept -3
30. x-intercept -8, y-intercept 6
31. Through $(4, 5)$, parallel to the x-axis
32. Through $(4, 5)$, parallel to the y-axis
33. Through $(1, -6)$, parallel to the line $x + 2y = 6$
34. y-intercept 6, parallel to the line $2x + 3y + 4 = 0$
35. Through $(-1, -2)$, perpendicular to the line $2x + 5y + 8 = 0$
36. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$, perpendicular to the line $4x - 8y = 1$

37–42 Find the slope and y-intercept of the line and draw its graph.

37. $x + 3y = 0$
38. $2x - 5y = 0$

39. $y = -2$

40. $2x - 3y + 6 = 0$

41. $3x - 4y = 12$

42. $4x + 5y = 10$

43–52 Sketch the region in the xy-plane.

43. $\{(x, y) \mid x < 0\}$

44. $\{(x, y) \mid y > 0\}$

45. $\{(x, y) \mid xy < 0\}$

46. $\{(x, y) \mid x \geqslant 1 \text{ and } y < 3\}$

47. $\{(x, y) \mid |x| \leqslant 2\}$

48. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

49. $\{(x, y) \mid 0 \leqslant y \leqslant 4 \text{ and } x \leqslant 2\}$

50. $\{(x, y) \mid y > 2x - 1\}$

51. $\{(x, y) \mid 1 + x \leqslant y \leqslant 1 - 2x\}$

52. $\{(x, y) \mid -x \leqslant y < \frac{1}{2}(x + 3)\}$

53. Find a point on the y-axis that is equidistant from $(5, -5)$ and $(1, 1)$.

54. Show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

55. Find the midpoint of the line segment joining the given points.
(a) $(1, 3)$ and $(7, 15)$ (b) $(-1, 6)$ and $(8, -12)$

56. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A median is a line segment from a vertex to the midpoint of the opposite side.)

57. Show that the lines $2x - y = 4$ and $6x - 2y = 10$ are not parallel and find their point of intersection.

58. Show that the lines $3x - 5y + 19 = 0$ and $10x + 6y - 50 = 0$ are perpendicular and find their point of intersection.

59. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

60. (a) Find equations for the sides of the triangle with vertices $P(1, 0)$, $Q(3, 4)$, and $R(-1, 6)$.
(b) Find equations for the medians of this triangle. Where do they intersect?

61. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.
(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

62. A car leaves Detroit at 2:00 PM, traveling at a constant speed west along I-96. It passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?

C GRAPHS OF SECOND-DEGREE EQUATIONS

In Appendix B we saw that a first-degree, or linear, equation $Ax + By + C = 0$ represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \qquad y = x^2 + 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

The graph of such an equation in x and y is the set of all points (x, y) that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the xy-plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the geometric problem.

CIRCLES

As an example of this type of problem, let's find an equation of the circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from

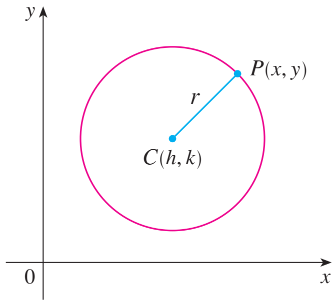


FIGURE 1

the center $C(h, k)$ is r. (See Figure 1.) Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the desired equation.

1 EQUATION OF A CIRCLE An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Find an equation of the circle with radius 3 and center $(2, -5)$.

SOLUTION From Equation 1 with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

EXAMPLE 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or

$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle (1), we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. It is sketched in Figure 2.

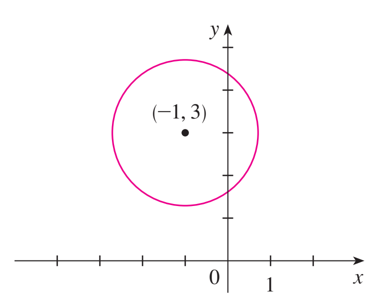


FIGURE 2
$x^2 + y^2 + 2x - 6y + 7 = 0$

PARABOLAS

The geometric properties of parabolas are reviewed in Section 10.5. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

EXAMPLE 3 Draw the graph of the parabola $y = x^2$.

SOLUTION We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 3.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
±1	1
±2	4
±3	9

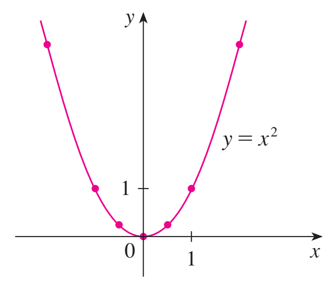


FIGURE 3

Figure 4 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. In each case the *vertex*, the point where the parabola changes direction, is the origin. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure 5).

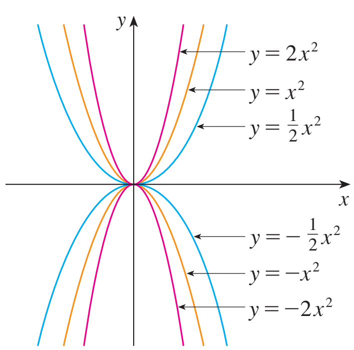
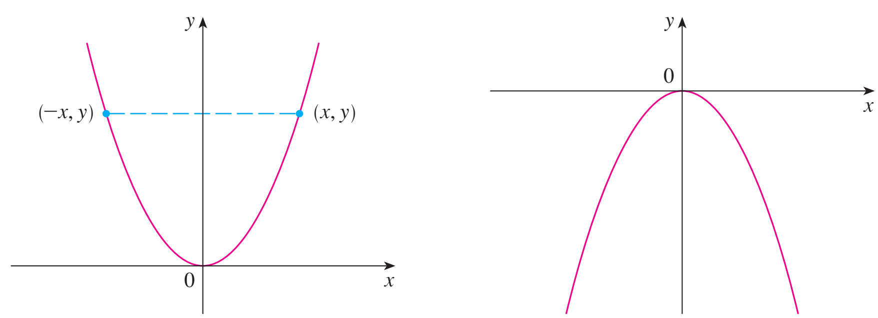


FIGURE 4



(a) $y = ax^2,\ a > 0$

(b) $y = ax^2,\ a < 0$

FIGURE 5

Notice that if (x, y) satisfies $y = ax^2$, then so does $(-x, y)$. This corresponds to the geometric fact that if the right half of the graph is reflected about the y-axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the *y*-axis**.

> The graph of an equation is symmetric with respect to the y-axis if the equation is unchanged when x is replaced by $-x$.

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See

Figure 6.) This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.

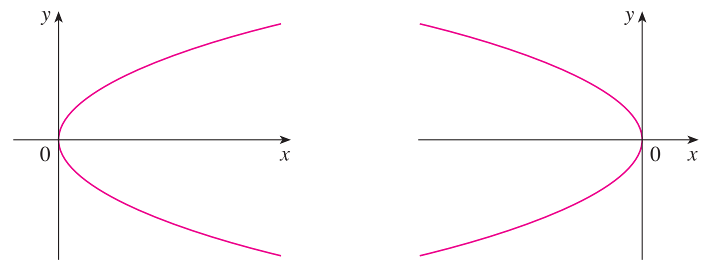


FIGURE 6

(a) $x = ay^2,\ a > 0$ (b) $x = ay^2,\ a < 0$

> The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

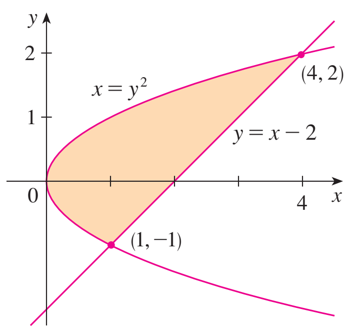


FIGURE 7

EXAMPLE 4 Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$.

SOLUTION First we find the points of intersection by solving the two equations. Substituting $x = y + 2$ into the equation $x = y^2$, we get $y + 2 = y^2$, which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so $y = 2$ or -1. Thus the points of intersection are $(4, 2)$ and $(1, -1)$, and we draw the line $y = x - 2$ passing through these points. We then sketch the parabola $x = y^2$ by referring to Figure 6(a) and having the parabola pass through $(4, 2)$ and $(1, -1)$. The region bounded by $x = y^2$ and $y = x - 2$ means the finite region whose boundaries are these curves. It is sketched in Figure 7. ■

ELLIPSES

The curve with equation

2 $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

where a and b are positive numbers, is called an **ellipse** in standard position. (Geometric properties of ellipses are discussed in Section 10.5.) Observe that Equation 2 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.

> The **x-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.
>
> The **y-intercepts** are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in its equation.

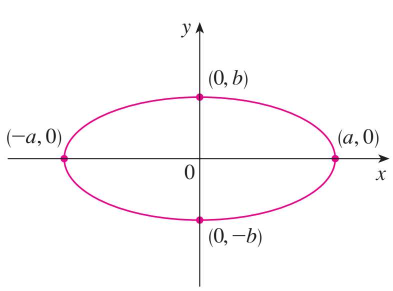


FIGURE 8

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

If we set $y = 0$ in Equation 2, we get $x^2 = a^2$ and so the x-intercepts are $\pm a$. Setting $x = 0$, we get $y^2 = b^2$, so the y-intercepts are $\pm b$. Using this information, together with symmetry, we sketch the ellipse in Figure 8. If $a = b$, the ellipse is a circle with radius a.

EXAMPLE 5 Sketch the graph of $9x^2 + 16y^2 = 144$.

SOLUTION We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse (2), so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4; the y-intercepts are ± 3. The graph is sketched in Figure 9.

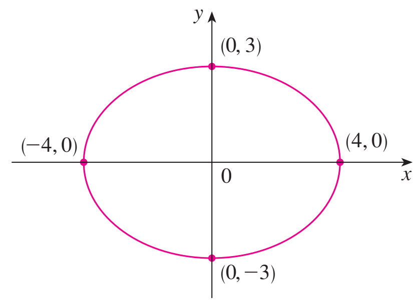


FIGURE 9
$9x^2 + 16y^2 = 144$

HYPERBOLAS

The curve with equation

3
$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

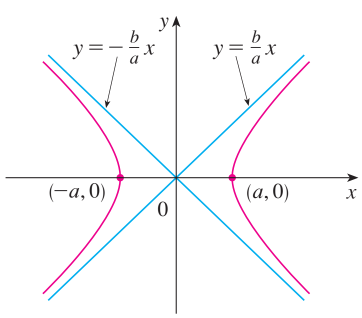


FIGURE 10
The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is called a **hyperbola** in standard position. Again, Equation 3 is unchanged when x is replaced by $-x$ or y is replaced by $-y$, so the hyperbola is symmetric with respect to both axes. To find the x-intercepts we set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. However, if we put $x = 0$ in Equation 3, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. In fact, from Equation 3 we obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

which shows that $x^2 \geqslant a^2$ and so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*. It is sketched in Figure 10.

In drawing a hyperbola it is useful to draw first its *asymptotes*, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 10. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in Chapter 2. (See also Exercise 55 in Section 4.5.)

By interchanging the roles of x and y we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

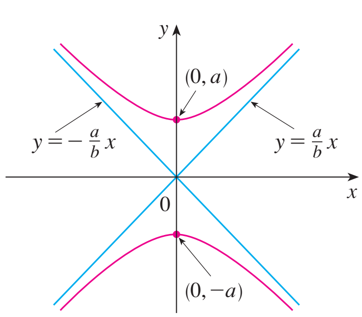


FIGURE 11
The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

which also represents a hyperbola and is sketched in Figure 11.

EXAMPLE 6 Sketch the curve $9x^2 - 4y^2 = 36$.

SOLUTION Dividing both sides by 36, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola (Equation 3). Since $a^2 = 4$, the x-intercepts are ± 2. Since $b^2 = 9$, we have $b = 3$ and the asymptotes are $y = \pm\left(\frac{3}{2}\right)x$. The hyperbola is sketched in Figure 12.

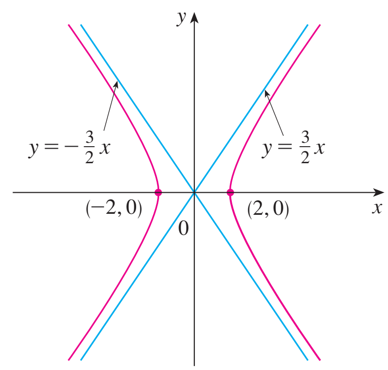
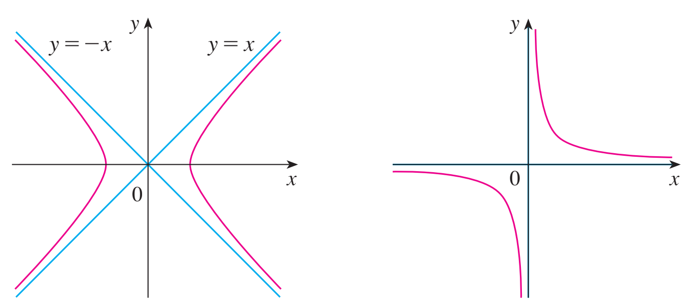


FIGURE 12
The hyperbola $9x^2 - 4y^2 = 36$

If $b = a$, a hyperbola has the equation $x^2 - y^2 = a^2$ (or $y^2 - x^2 = a^2$) and is called an *equilateral hyperbola* [see Figure 13(a)]. Its asymptotes are $y = \pm x$, which are perpendicular. If an equilateral hyperbola is rotated by 45°, the asymptotes become the x- and y-axes, and it can be shown that the new equation of the hyperbola is $xy = k$, where k is a constant [see Figure 13(b)].



(a) $x^2 - y^2 = a^2$

(b) $xy = k \quad (k > 0)$

FIGURE 13
Equilateral hyperbolas

SHIFTED CONICS

Recall that an equation of the circle with center the origin and radius r is $x^2 + y^2 = r^2$, but if the center is the point (h, k), then the equation of the circle becomes

$$(x - h)^2 + (y - k)^2 = r^2$$

Similarly, if we take the ellipse with equation

4 $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

and translate it (shift it) so that its center is the point (h, k), then its equation becomes

5
$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

(See Figure 14.)

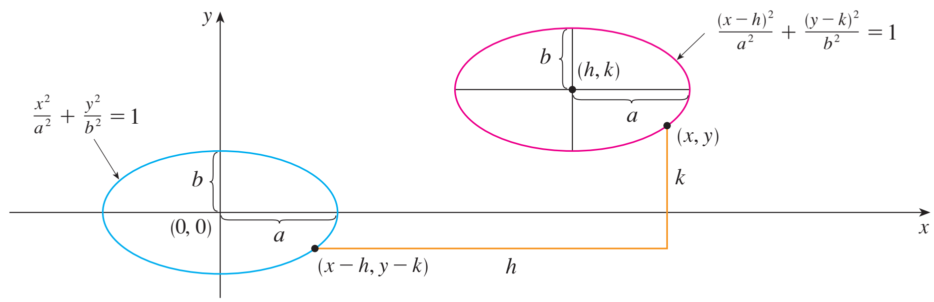


FIGURE 14

Notice that in shifting the ellipse, we replaced x by $x - h$ and y by $y - k$ in Equation 4 to obtain Equation 5. We use the same procedure to shift the parabola $y = ax^2$ so that its vertex (the origin) becomes the point (h, k) as in Figure 15. Replacing x by $x - h$ and y by $y - k$, we see that the new equation is

$$y - k = a(x - h)^2 \qquad \text{or} \qquad y = a(x - h)^2 + k$$

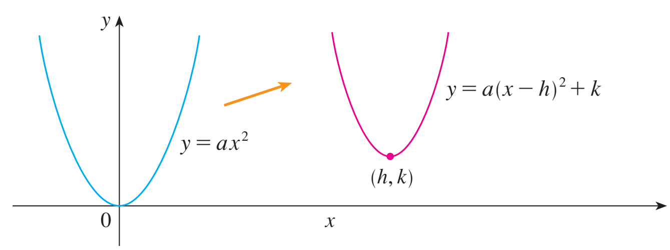


FIGURE 15

EXAMPLE 7 Sketch the graph of the equation $y = 2x^2 - 4x + 1$.

SOLUTION First we complete the square:

$$y = 2(x^2 - 2x) + 1 = 2(x - 1)^2 - 1$$

In this form we see that the equation represents the parabola obtained by shifting $y = 2x^2$ so that its vertex is at the point $(1, -1)$. The graph is sketched in Figure 16.

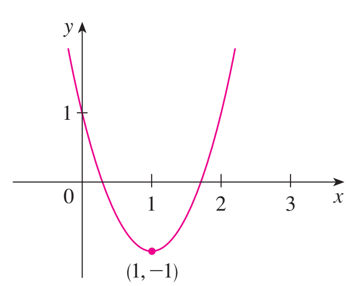


FIGURE 16
$y = 2x^2 - 4x + 1$

■

EXAMPLE 8 Sketch the curve $x = 1 - y^2$.

SOLUTION This time we start with the parabola $x = -y^2$ (as in Figure 6 with $a = -1$) and shift one unit to the right to get the graph of $x = 1 - y^2$. (See Figure 17.)

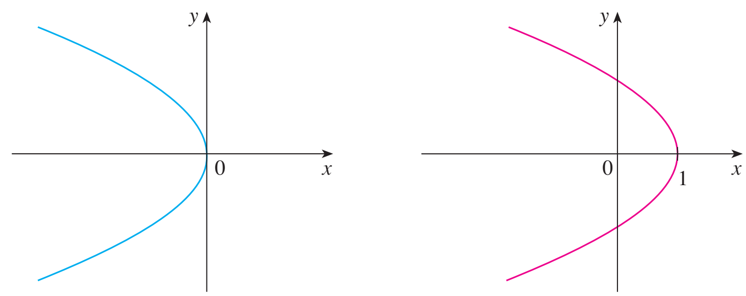


FIGURE 17 (a) $x = -y^2$ (b) $x = 1 - y^2$

C EXERCISES

1–4 Find an equation of a circle that satisfies the given conditions.

1. Center $(3, -1)$, radius 5

2. Center $(-2, -8)$, radius 10

3. Center at the origin, passes through $(4, 7)$

4. Center $(-1, 5)$, passes through $(-4, -6)$

5–9 Show that the equation represents a circle and find the center and radius.

5. $x^2 + y^2 - 4x + 10y + 13 = 0$

6. $x^2 + y^2 + 6y + 2 = 0$

7. $x^2 + y^2 + x = 0$

8. $16x^2 + 16y^2 + 8x + 32y + 1 = 0$

9. $2x^2 + 2y^2 - x + y = 1$

10. Under what condition on the coefficients a, b, and c does the equation $x^2 + y^2 + ax + by + c = 0$ represent a circle? When that condition is satisfied, find the center and radius of the circle.

11–32 Identify the type of curve and sketch the graph. Do not plot points. Just use the standard graphs given in Figures 5, 6, 8, 10, and 11 and shift if necessary.

11. $y = -x^2$

12. $y^2 - x^2 = 1$

13. $x^2 + 4y^2 = 16$

14. $x = -2y^2$

15. $16x^2 - 25y^2 = 400$

16. $25x^2 + 4y^2 = 100$

17. $4x^2 + y^2 = 1$

18. $y = x^2 + 2$

19. $x = y^2 - 1$

20. $9x^2 - 25y^2 = 225$

21. $9y^2 - x^2 = 9$

22. $2x^2 + 5y^2 = 10$

23. $xy = 4$

24. $y = x^2 + 2x$

25. $9(x - 1)^2 + 4(y - 2)^2 = 36$

26. $16x^2 + 9y^2 - 36y = 108$

27. $y = x^2 - 6x + 13$

28. $x^2 - y^2 - 4x + 3 = 0$

29. $x = 4 - y^2$

30. $y^2 - 2x + 6y + 5 = 0$

31. $x^2 + 4y^2 - 6x + 5 = 0$

32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

33–34 Sketch the region bounded by the curves.

33. $y = 3x, \quad y = x^2$

34. $y = 4 - x^2, \quad x - 2y = 2$

35. Find an equation of the parabola with vertex $(1, -1)$ that passes through the points $(-1, 3)$ and $(3, 3)$.

36. Find an equation of the ellipse with center at the origin that passes through the points $\left(1, -10\sqrt{2}/3\right)$ and $\left(-2, 5\sqrt{5}/3\right)$.

37–40 Sketch the graph of the set.

37. $\{(x, y) \mid x^2 + y^2 \leqslant 1\}$

38. $\{(x, y) \mid x^2 + y^2 > 4\}$

39. $\{(x, y) \mid y \geqslant x^2 - 1\}$

40. $\{(x, y) \mid x^2 + 4y^2 \leqslant 4\}$

D TRIGONOMETRY

ANGLES

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore

1
$$\boxed{\pi \text{ rad} = 180^\circ}$$

and

2
$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ \qquad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$$

EXAMPLE 1
(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.

SOLUTION
(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60^\circ = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225^\circ$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

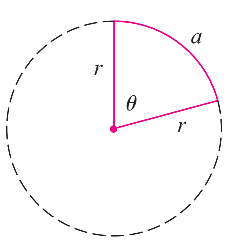


FIGURE 1

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

3
$$\boxed{\theta = \frac{a}{r}} \qquad \boxed{a = r\theta}$$

Remember that Equations 3 are valid only when θ is measured in radians.

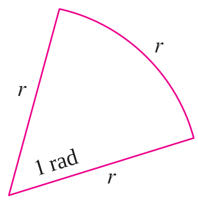


FIGURE 2

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

EXAMPLE 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

SOLUTION

(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \tfrac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

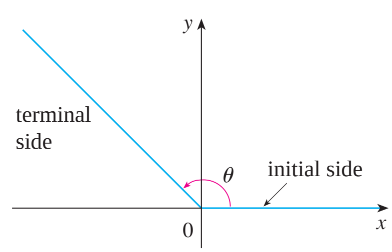


FIGURE 3 $\theta \geqslant 0$

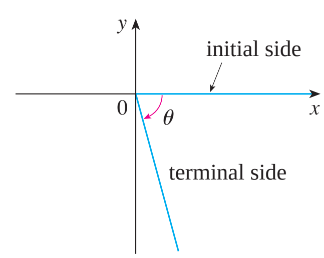


FIGURE 4 $\theta < 0$

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

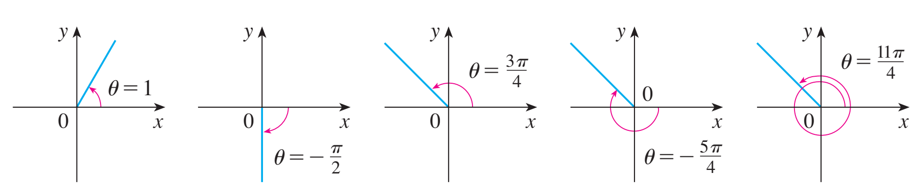


FIGURE 5
Angles in standard position

THE TRIGONOMETRIC FUNCTIONS

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

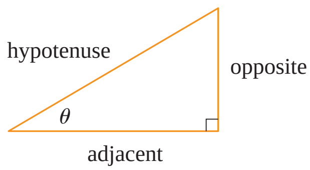


FIGURE 6

4
$$\boxed{\begin{aligned} \sin\theta &= \frac{\text{opp}}{\text{hyp}} &\qquad \csc\theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos\theta &= \frac{\text{adj}}{\text{hyp}} & \sec\theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan\theta &= \frac{\text{opp}}{\text{adj}} & \cot\theta &= \frac{\text{adj}}{\text{opp}} \end{aligned}}$$

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

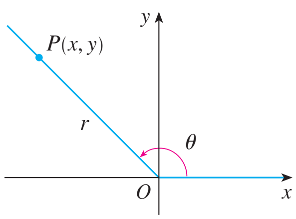


FIGURE 7

5
$$\boxed{\begin{aligned} \sin\theta &= \frac{y}{r} &\qquad \csc\theta &= \frac{r}{y} \\ \cos\theta &= \frac{x}{r} & \sec\theta &= \frac{r}{x} \\ \tan\theta &= \frac{y}{x} & \cot\theta &= \frac{x}{y} \end{aligned}}$$

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression sin 3 implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number, we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write sin 3° and, with our calculator in degree mode, we find that

$$\sin 3^\circ \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 8. For instance,

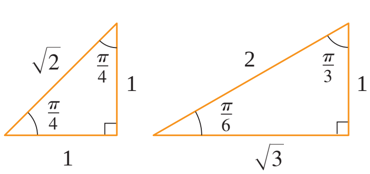


FIGURE 8

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

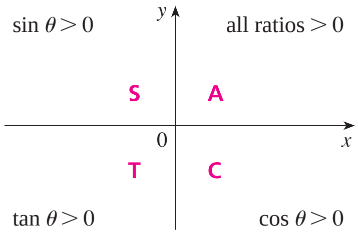


FIGURE 9

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" shown in Figure 9.

EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

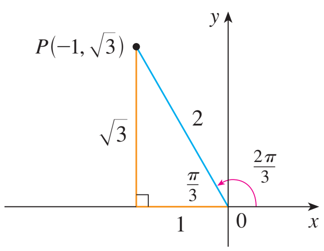


FIGURE 10

SOLUTION From Figure 10 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$ ■

The following table gives some values of $\sin\theta$ and $\cos\theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

EXAMPLE 4 If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

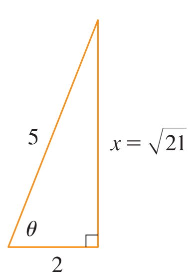


FIGURE 11

SOLUTION Since $\cos\theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 11. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$ ■

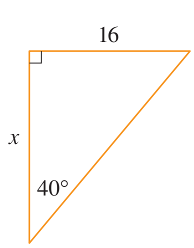


FIGURE 12

EXAMPLE 5 Use a calculator to approximate the value of x in Figure 12.

SOLUTION From the diagram we see that

$$\tan 40° = \frac{16}{x}$$

Therefore

$$x = \frac{16}{\tan 40°} \approx 19.07$$ ■

TRIGONOMETRIC IDENTITIES

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

6
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

7
$$\sin^2\theta + \cos^2\theta = 1$$

If we now divide both sides of Equation 7 by $\cos^2\theta$ and use Equations 6, we get

8
$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if we divide both sides of Equation 7 by $\sin^2\theta$, we get

9
$$1 + \cot^2\theta = \csc^2\theta$$

The identities

10a
$$\sin(-\theta) = -\sin\theta$$
10b
$$\cos(-\theta) = \cos\theta$$

■ Odd functions and even functions are discussed in Section 1.1.

show that sin is an odd function and cos is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 39).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

11
$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

These identities show that the sine and cosine functions are periodic with period 2π.

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

$$\text{12a} \qquad \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\text{12b} \qquad \cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are outlined in Exercises 85, 86, and 87.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

$$\text{13a} \qquad \sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\text{13b} \qquad \cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for $\tan(x \pm y)$:

$$\text{14a} \qquad \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\text{14b} \qquad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas (12), we get the **double-angle formulas**:

$$\text{15a} \qquad \sin 2x = 2 \sin x \cos x$$

$$\text{15b} \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

$$\text{16a} \qquad \cos 2x = 2 \cos^2 x - 1$$

$$\text{16b} \qquad \cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

$$\text{17a} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\text{17b} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from Equations 12 and 13:

$$\boxed{\begin{aligned} &\textbf{18a} \quad & \sin x \cos y &= \tfrac{1}{2}[\sin(x+y) + \sin(x-y)] \\ &\textbf{18b} \quad & \cos x \cos y &= \tfrac{1}{2}[\cos(x+y) + \cos(x-y)] \\ &\textbf{18c} \quad & \sin x \sin y &= \tfrac{1}{2}[\cos(x-y) - \cos(x+y)] \end{aligned}}$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of them, remember that they can all be deduced from Equations 12a and 12b.

EXAMPLE 6 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

SOLUTION Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x\,(1 - 2\cos x) = 0$$

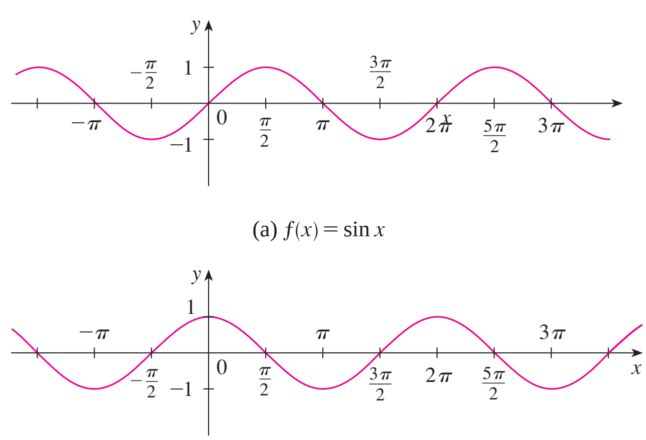
Therefore, there are two possibilities:

$$\begin{aligned} \sin x &= 0 & \qquad \text{or} \qquad 1 - 2\cos x &= 0 \\ x &= 0, \pi, 2\pi & \cos x &= \tfrac{1}{2} \\ & & x &= \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

The given equation has five solutions: 0, $\pi/3$, π, $5\pi/3$, and 2π. ■

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

The graph of the function $f(x) = \sin x$, shown in Figure 13(a), is obtained by plotting points for $0 \leqslant x \leqslant 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

FIGURE 13

integer multiples of π, that is,

$$\sin x = 0 \qquad \text{whenever } x = n\pi, \quad n \text{ an integer}$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 13(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

The graphs of the remaining four trigonometric functions are shown in Figure 14 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π.

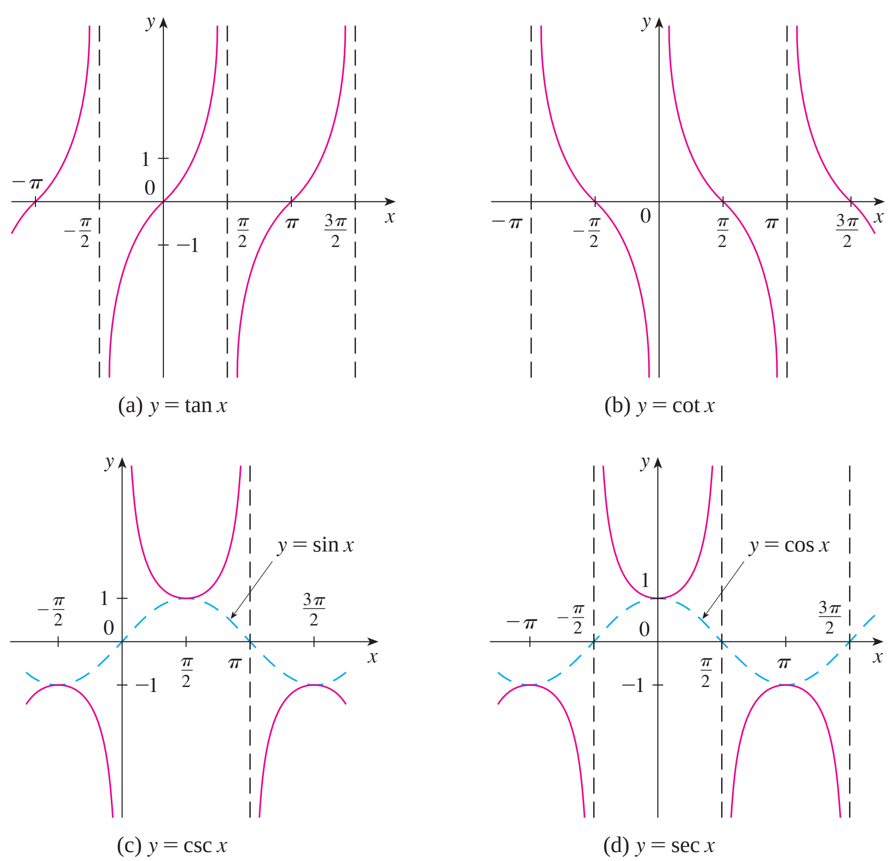


FIGURE 14

D EXERCISES

1–6 Convert from degrees to radians.

1. $210°$ **2.** $300°$ **3.** $9°$

4. $-315°$ **5.** $900°$ **6.** $36°$

7–12 Convert from radians to degrees.

7. 4π **8.** $-\dfrac{7\pi}{2}$ **9.** $\dfrac{5\pi}{12}$

10. $\dfrac{8\pi}{3}$ **11.** $-\dfrac{3\pi}{8}$ **12.** 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of $72°$.

15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

17–22 Draw, in standard position, the angle whose measure is given.

17. $315°$ **18.** $-150°$ **19.** $-\dfrac{3\pi}{4}$ rad

20. $\dfrac{7\pi}{3}$ rad **21.** 2 rad **22.** -3 rad

23–28 Find the exact trigonometric ratios for the angle whose radian measure is given.

23. $\dfrac{3\pi}{4}$ **24.** $\dfrac{4\pi}{3}$ **25.** $\dfrac{9\pi}{2}$

26. -5π **27.** $\dfrac{5\pi}{6}$ **28.** $\dfrac{11\pi}{4}$

29–34 Find the remaining trigonometric ratios.

29. $\sin\theta = \dfrac{3}{5}, \quad 0 < \theta < \dfrac{\pi}{2}$

30. $\tan\alpha = 2, \quad 0 < \alpha < \dfrac{\pi}{2}$

31. $\sec\phi = -1.5, \quad \dfrac{\pi}{2} < \phi < \pi$

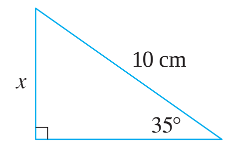
32. $\cos x = -\dfrac{1}{3}, \quad \pi < x < \dfrac{3\pi}{2}$

33. $\cot\beta = 3, \quad \pi < \beta < 2\pi$

34. $\csc\theta = -\dfrac{4}{3}, \quad \dfrac{3\pi}{2} < \theta < 2\pi$

35–38 Find, correct to five decimal places, the length of the side labeled x.

35.



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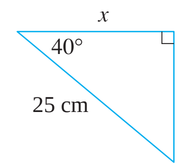

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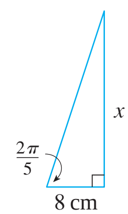

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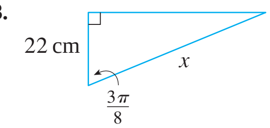


39–41 Prove each equation.

39. (a) Equation 10a (b) Equation 10b

40. (a) Equation 14a (b) Equation 14b

41. (a) Equation 18a (b) Equation 18b
(c) Equation 18c

42–58 Prove the identity.

42. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

43. $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$

44. $\sin(\pi - x) = \sin x$

45. $\sin\theta\cot\theta = \cos\theta$

46. $(\sin x + \cos x)^2 = 1 + \sin 2x$

47. $\sec y - \cos y = \tan y \sin y$

48. $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha \sin^2\alpha$

49. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$

50. $2\csc 2t = \sec t \csc t$

51. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

52. $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

53. $\sin x \sin 2x + \cos x \cos 2x = \cos x$

54. $\sin^2 x - \sin^2 y = \sin(x + y)\sin(x - y)$

55. $\dfrac{\sin\phi}{1 - \cos\phi} = \csc\phi + \cot\phi$

56. $\tan x + \tan y = \dfrac{\sin(x + y)}{\cos x \cos y}$

57. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$

58. $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$

59–64 If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

59. $\sin(x + y)$

60. $\cos(x + y)$

61. $\cos(x - y)$

62. $\sin(x - y)$

63. $\sin 2y$

64. $\cos 2y$

65–72 Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

65. $2 \cos x - 1 = 0$

66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$

68. $|\tan x| = 1$

69. $\sin 2x = \cos x$

70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$

72. $2 + \cos 2x = 3 \cos x$

73–76 Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

73. $\sin x \leqslant \frac{1}{2}$

74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$

76. $\sin x > \cos x$

77–82 Graph the function by starting with the graphs in Figures 13 and 14 and applying the transformations of Section 1.3 where appropriate.

77. $y = \cos\left(x - \dfrac{\pi}{3}\right)$

78. $y = \tan 2x$

79. $y = \dfrac{1}{3} \tan\left(x - \dfrac{\pi}{2}\right)$

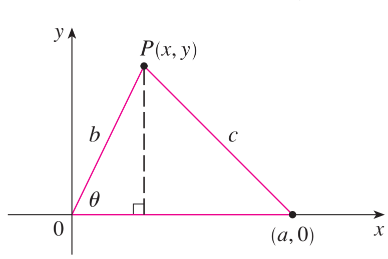
80. $y = 1 + \sec x$

81. $y = |\sin x|$

82. $y = 2 + \sin\left(x + \dfrac{\pi}{4}\right)$

83. Prove the **Law of Cosines**: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

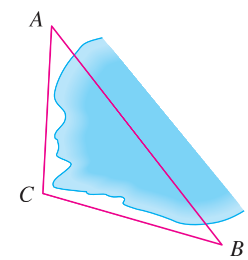


[*Hint:* Introduce a coordinate system so that θ is in standard position as in the figure. Express x and y in terms of θ and then use the distance formula to compute c.]

84. In order to find the distance $|AB|$ across a small inlet, a point C is located as in the figure and the following measurements were recorded:

$$\angle C = 103° \qquad |AC| = 820 \text{ m} \qquad |BC| = 910 \text{ m}$$

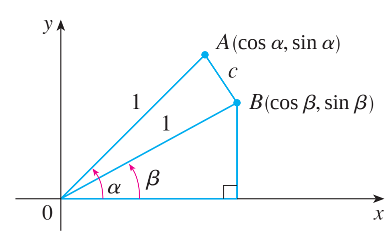
Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

[*Hint:* Compute c^2 in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

to prove the subtraction formula for the sine function.

88. Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \tfrac{1}{2}ab \sin \theta$$

89. Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \qquad |BC| = 3 \text{ cm} \qquad \angle ABC = 107°$$

E SIGMA NOTATION

A convenient way of writing sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} a_i$$

This tells us to start with $i = m$.

1 DEFINITION If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on consecutive integer values beginning with m and ending with n, that is, $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

EXAMPLE 1

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$ ■

EXAMPLE 2 Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$ ■

The following theorem gives three simple rules for working with sigma notation.

2 THEOREM If c is any constant (that is, it does not depend on i), then

(a) $\displaystyle\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$

(b) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\displaystyle\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

PROOF To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$\begin{aligned}(a_m + b_m) &+ (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)\\ &= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)\end{aligned}$$

Rule (c) is proved similarly. ■

EXAMPLE 3 Find $\displaystyle\sum_{i=1}^{n} 1$.

SOLUTION

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$ ■

EXAMPLE 4 Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION This formula can be proved by mathematical induction (see page 77) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$\begin{aligned}S &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + (n-1) + n\\ S &= n + (n-1) + (n-2) + \cdots + \quad 2 \quad + 1\end{aligned}$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$ ■

EXAMPLE 5 Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + [(n+1)^3 - \cancel{n^3}]$$
$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 3S + 3\,\frac{n(n+1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or
$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

■ **PRINCIPLE OF MATHEMATICAL INDUCTION**

Let S_n be a statement involving the positive integer n. Suppose that

1. S_1 is true.
2. If S_k is true, then S_{k+1} is true.

Then S_n is true for all positive integers n.

■ See pages 55 and 58 for a more thorough discussion of mathematical induction.

SOLUTION 2 Let S_n be the given formula.

1. S_1 is true because $\quad 1^2 = \dfrac{1(1+1)(2\cdot 1+1)}{6}$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= (k+1)\,\frac{k(2k+1) + 6(k+1)}{6}$$
$$= (k+1)\,\frac{2k^2 + 7k + 6}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n. ■

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 5.

3 THEOREM Let c be a constant and n a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$

(b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 6 Evaluate $\displaystyle\sum_{i=1}^{n} i(4i^2 - 3)$.

SOLUTION Using Theorems 2 and 3, we have

$$\begin{aligned}
\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\
&= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\
&= \frac{n(n+1)[2n(n+1) - 3]}{2} \\
&= \frac{n(n+1)(2n^2 + 2n - 3)}{2}
\end{aligned}$$

■ The type of calculation in Example 7 arises in Chapter 5 when we compute areas.

EXAMPLE 7 Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

SOLUTION

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\
&= \tfrac{1}{2}\cdot 1 \cdot 1 \cdot 2 + 3 = 4
\end{aligned}$$

E EXERCISES

1–10 Write the sum in expanded form.

1. $\sum_{i=1}^{5} \sqrt{i}$

2. $\sum_{i=1}^{6} \dfrac{1}{i+1}$

3. $\sum_{i=4}^{6} 3^i$

4. $\sum_{i=4}^{6} i^3$

5. $\sum_{k=0}^{4} \dfrac{2k-1}{2k+1}$

6. $\sum_{k=5}^{8} x^k$

7. $\sum_{i=1}^{n} i^{10}$

8. $\sum_{j=n}^{n+3} j^2$

9. $\sum_{j=0}^{n-1} (-1)^j$

10. $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

11–20 Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

21–35 Find the value of the sum.

21. $\sum_{i=4}^{8} (3i - 2)$

22. $\sum_{i=3}^{6} i(i + 2)$

23. $\sum_{j=1}^{6} 3^{j+1}$

24. $\sum_{k=0}^{8} \cos k\pi$

25. $\sum_{n=1}^{20} (-1)^n$

26. $\sum_{i=1}^{100} 4$

27. $\sum_{i=0}^{4} (2^i + i^2)$

28. $\sum_{i=-2}^{4} 2^{3-i}$

29. $\sum_{i=1}^{n} 2i$

30. $\sum_{i=1}^{n} (2 - 5i)$

31. $\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\sum_{i=1}^{n} (3 + 2i)^2$

33. $\sum_{i=1}^{n} (i + 1)(i + 2)$

34. $\sum_{i=1}^{n} i(i + 1)(i + 2)$

35. $\sum_{i=1}^{n} (i^3 - i - 2)$

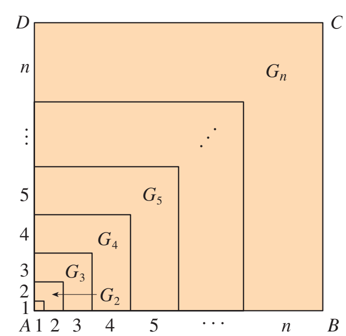
36. Find the number n such that $\sum_{i=1}^{n} i = 78$.

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with $(1 + i)^4 - i^4$].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about AD 1010. The figure shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths 1, 2, 3, . . . , n. Thus the side of the square has length $n(n + 1)/2$ so the area is $[n(n + 1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\sum_{i=1}^{n} [i^4 - (i - 1)^4]$

(b) $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\sum_{i=3}^{99} \left(\dfrac{1}{i} - \dfrac{1}{i+1}\right)$

(d) $\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality:

$$\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n} |a_i|$$

43–46 Find the limit.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{1}{n}\left(\dfrac{i}{n}\right)^2$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{1}{n}\left[\left(\dfrac{i}{n}\right)^3 + 1\right]$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{2}{n}\left[\left(\dfrac{2i}{n}\right)^3 + 5\left(\dfrac{2i}{n}\right)\right]$

46. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(1+\frac{3i}{n}\right)^3-2\left(1+\frac{3i}{n}\right)\right]$

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

48. Evaluate $\displaystyle\sum_{i=1}^{n}\frac{3}{2^{i-1}}$.

49. Evaluate $\displaystyle\sum_{i=1}^{n}(2i + 2^i)$.

50. Evaluate $\displaystyle\sum_{i=1}^{m}\left[\sum_{j=1}^{n}(i + j)\right]$.

F PROOFS OF THEOREMS

In this appendix we present proofs of several theorems that are stated in the main body of the text. The sections in which they occur are indicated in the margin.

SECTION 2.3

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\displaystyle\lim_{x\to a}[f(x) + g(x)] = L + M$

2. $\displaystyle\lim_{x\to a}[f(x) - g(x)] = L - M$

3. $\displaystyle\lim_{x\to a}[cf(x)] = cL$

4. $\displaystyle\lim_{x\to a}[f(x)g(x)] = LM$

5. $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

PROOF OF LAW 4 Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |[f(x) - L]g(x) + L[g(x) - M]| \\
&\leqslant |[f(x) - L]g(x)| + |L[g(x) - M]| \qquad \text{(Triangle Inequality)} \\
&= |f(x) - L||g(x)| + |L||g(x) - M|
\end{aligned}$$

We want to make each of these terms less than $\varepsilon/2$.

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta_1 \qquad \text{then} \qquad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leqslant |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta_3 \qquad \text{then} \qquad |f(x) - L| < \frac{\varepsilon}{2\bigl(1 + |M|\bigr)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then we have $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$, so we can combine the inequalities to obtain

$$\begin{aligned} |f(x)g(x) - LM| &\leqslant |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\varepsilon}{2\bigl(1 + |M|\bigr)}\bigl(1 + |M|\bigr) + |L|\,\frac{\varepsilon}{2\bigl(1 + |L|\bigr)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that $\lim_{x\to a} f(x)g(x) = LM$. ■

PROOF OF LAW 3 If we take $g(x) = c$ in Law 4, we get

$$\begin{aligned} \lim_{x\to a} [cf(x)] &= \lim_{x\to a} [g(x)f(x)] = \lim_{x\to a} g(x) \cdot \lim_{x\to a} f(x) \\ &= \lim_{x\to a} c \cdot \lim_{x\to a} f(x) \\ &= c \lim_{x\to a} f(x) \qquad \text{(by Law 7)} \end{aligned}$$ ■

PROOF OF LAW 2 Using Law 1 and Law 3 with $c = -1$, we have

$$\begin{aligned} \lim_{x\to a} [f(x) - g(x)] &= \lim_{x\to a} [f(x) + (-1)g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} (-1)g(x) \\ &= \lim_{x\to a} f(x) + (-1) \lim_{x\to a} g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \end{aligned}$$ ■

PROOF OF LAW 5 First let us show that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$$

To do this we must show that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$$

Observe that
$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|}$$

We know that we can make the numerator small. But we also need to know that the denominator is not small when x is near a. Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that, whenever $0 < |x - a| < \delta_1$, we have

$$|g(x) - M| < \frac{|M|}{2}$$

and therefore
$$\begin{aligned} |M| &= |M - g(x) + g(x)| \leqslant |M - g(x)| + |g(x)| \\ &< \frac{|M|}{2} + |g(x)| \end{aligned}$$

This shows that

$$\text{if} \qquad 0 < |x - a| < \delta_1 \qquad \text{then} \qquad |g(x)| > \frac{|M|}{2}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M||g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there exists $\delta_2 > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta_2 \qquad \text{then} \qquad |g(x) - M| < \frac{M^2}{2} \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - a| < \delta$, we have

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

It follows that $\lim_{x \to a} 1/g(x) = 1/M$. Finally, using Law 4, we obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \left(\frac{1}{g(x)} \right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$ ■

2 THEOREM If $f(x) \leqslant g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M$$

then $L \leqslant M$.

PROOF We use the method of proof by contradiction. Suppose, if possible, that $L > M$. Law 2 of limits says that

$$\lim_{x \to a} [g(x) - f(x)] = M - L$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |[g(x) - f(x)] - (M - L)| < \varepsilon$$

In particular, taking $\varepsilon = L - M$ (noting that $L - M > 0$ by hypothesis), we have a number $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |[g(x) - f(x)] - (M - L)| < L - M$$

Since $a \leqslant |a|$ for any number a, we have

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad [g(x) - f(x)] - (M - L) < L - M$$

which simplifies to

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad g(x) < f(x)$$

But this contradicts $f(x) \leqslant g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leqslant M$. ■

$\boxed{3}$ **THE SQUEEZE THEOREM** If $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

PROOF Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |h(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so

$$L - \varepsilon < f(x) \leqslant g(x) \leqslant h(x) < L + \varepsilon$$

In particular, $$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \to a} g(x) = L$. ■

SECTION 2.5

THEOREM If f is a one-to-one continuous function defined on an interval (a, b), then its inverse function f^{-1} is also continuous.

PROOF First we show that if f is both one-to-one and continuous on (a, b), then it must be either increasing or decreasing on (a, b). If it were neither increasing nor decreasing, then there would exist numbers x_1, x_2, and x_3 in (a, b) with $x_1 < x_2 < x_3$ such that $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$. There are two possibilities: either (1) $f(x_3)$ lies between $f(x_1)$ and $f(x_2)$ or (2) $f(x_1)$ lies between $f(x_2)$ and $f(x_3)$. (Draw a picture.) In case (1) we apply the Intermediate Value Theorem to the continuous function f to get a number c between x_1 and x_2 such that $f(c) = f(x_3)$. In case (2) the Intermediate Value Theorem gives a number c between x_2 and x_3 such that $f(c) = f(x_1)$. In either case we have contradicted the fact that f is one-to-one.

Let us assume, for the sake of definiteness, that f is increasing on (a, b). We take any number y_0 in the domain of f^{-1} and we let $f^{-1}(y_0) = x_0$; that is, x_0 is the number in (a, b) such that $f(x_0) = y_0$. To show that f^{-1} is continuous at y_0 we take any $\varepsilon > 0$ such that the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is contained in the interval (a, b). Since f is increasing, it maps the numbers in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ onto the numbers in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and f^{-1} reverses the correspondence. If we let δ denote the smaller of the numbers $\delta_1 = y_0 - f(x_0 - \varepsilon)$ and $\delta_2 = f(x_0 + \varepsilon) - y_0$, then the interval $(y_0 - \delta, y_0 + \delta)$ is contained in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and so is mapped into the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ by f^{-1}. (See the arrow diagram in Figure 1.) We have therefore found a number $\delta > 0$ such that

$$\text{if} \qquad |y - y_0| < \delta \qquad \text{then} \qquad |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$$

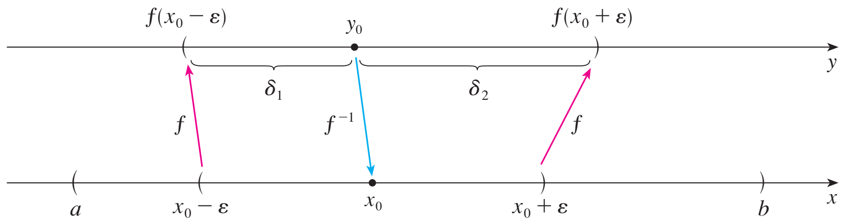


FIGURE 1

This shows that $\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0)$ and so f^{-1} is continuous at any number y_0 in its domain. ■

> **8 THEOREM** If f is continuous at b and $\lim_{x \to a} g(x) = b$, then
>
> $$\lim_{x \to a} f(g(x)) = f(b)$$

PROOF Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$\text{if} \qquad 0 < |y - b| < \delta_1 \qquad \text{then} \qquad |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |g(x) - b| < \delta_1$$

Combining these two statements, we see that whenever $0 < |x - a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x \to a} f(g(x)) = f(b)$. ■

SECTION 3.3

The proof of the following result was promised when we proved that $\displaystyle\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

> **THEOREM** If $0 < \theta < \pi/2$, then $\theta \leqslant \tan \theta$.

PROOF Figure 2 shows a sector of a circle with center O, central angle θ, and radius 1. Then

$$|AD| = |OA| \tan \theta = \tan \theta$$

We approximate the arc AB by an inscribed polygon consisting of n equal line segments

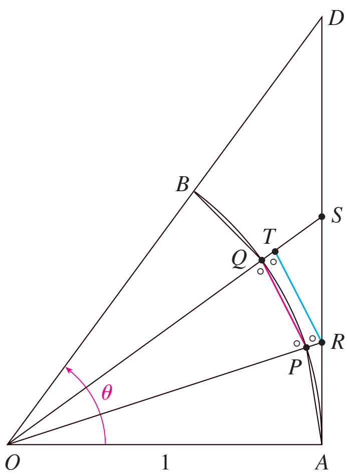


FIGURE 2

and we look at a typical segment PQ. We extend the lines OP and OQ to meet AD in the points R and S. Then we draw $RT \parallel PQ$ as in Figure 2. Observe that

$$\angle RTO = \angle PQO < 90^\circ$$

and so $\angle RTS > 90^\circ$. Therefore we have

$$|PQ| < |RT| < |RS|$$

If we add n such inequalities, we get

$$L_n < |AD| = \tan\theta$$

where L_n is the length of the inscribed polygon. Thus, by Theorem 2.3.2, we have

$$\lim_{n\to\infty} L_n \leqslant \tan\theta$$

But the arc length is defined in Equation 8.1.1 as the limit of the lengths of inscribed polygons, so

$$\theta = \lim_{n\to\infty} L_n \leqslant \tan\theta$$

■

SECTION 4.3

CONCAVITY TEST

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

PROOF OF (a) Let a be any number in I. We must show that the curve $y = f(x)$ lies above the tangent line at the point $(a, f(a))$. The equation of this tangent is

$$y = f(a) + f'(a)(x - a)$$

So we must show that

$$f(x) > f(a) + f'(a)(x - a)$$

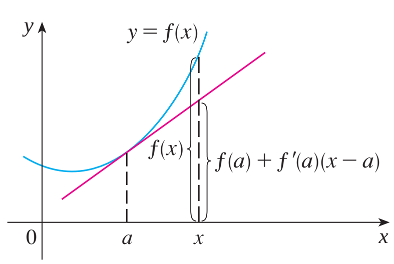


FIGURE 3

whenever $x \in I$ $(x \neq a)$. (See Figure 3.)

First let us take the case where $x > a$. Applying the Mean Value Theorem to f on the interval $[a, x]$, we get a number c, with $a < c < x$, such that

$$f(x) - f(a) = f'(c)(x - a) \tag{1}$$

Since $f'' > 0$ on I, we know from the Increasing/Decreasing Test that f' is increasing on I. Thus, since $a < c$, we have

$$f'(a) < f'(c)$$

and so, multiplying this inequality by the positive number $x - a$, we get

$$f'(a)(x - a) < f'(c)(x - a) \tag{2}$$

Now we add $f(a)$ to both sides of this inequality:

$$f(a) + f'(a)(x - a) < f(a) + f'(c)(x - a)$$

But from Equation 1 we have $f(x) = f(a) + f'(c)(x - a)$. So this inequality becomes

$$\boxed{3} \qquad f(x) > f(a) + f'(a)(x - a)$$

which is what we wanted to prove.

For the case where $x < a$ we have $f'(c) < f'(a)$, but multiplication by the negative number $x - a$ reverses the inequality, so we get (2) and (3) as before. ■

SECTION 4.4

In order to give the promised proof of l'Hospital's Rule, we first need a generalization of the Mean Value Theorem. The following theorem is named after another French mathematician, Augustin-Louis Cauchy (1789–1857).

■ See the biographical sketch of Cauchy on page 113.

1 CAUCHY'S MEAN VALUE THEOREM Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Notice that if we take the special case in which $g(x) = x$, then $g'(c) = 1$ and Theorem 1 is just the ordinary Mean Value Theorem. Furthermore, Theorem 1 can be proved in a similar manner. You can verify that all we have to do is change the function h given by Equation 4.2.4 to the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

and apply Rolle's Theorem as before.

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

PROOF OF L'HOSPITAL'S RULE We are assuming that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \to a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{y \to a^+} \frac{F'(y)}{G'(y)} = \lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\begin{aligned} \lim_{x \to \infty} \frac{f(x)}{g(x)} &= \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)} \\ &= \lim_{t \to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} && \text{(by l'Hospital's Rule for finite } a\text{)} \\ &= \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \end{aligned}$$

■

SECTION 11.8

In order to prove Theorem 11.8.3, we first need the following results.

THEOREM

1. If a power series $\Sigma\, c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\Sigma\, c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it diverges whenever $|x| > |d|$.

PROOF OF 1 Suppose that $\sum c_n b^n$ converges. Then, by Theorem 11.2.6, we have $\lim_{n\to\infty} c_n b^n = 0$. According to Definition 11.1.2 with $\varepsilon = 1$, there is a positive integer N such that $|c_n b^n| < 1$ whenever $n \geqslant N$. Thus, for $n \geqslant N$, we have

$$|c_n x^n| = \left|\frac{c_n b^n x^n}{b^n}\right| = |c_n b^n| \left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n$$

If $|x| < |b|$, then $|x/b| < 1$, so $\sum |x/b|^n$ is a convergent geometric series. Therefore, by the Comparison Test, the series $\sum_{n=N}^{\infty} |c_n x^n|$ is convergent. Thus the series $\sum c_n x^n$ is absolutely convergent and therefore convergent. ■

PROOF OF 2 Suppose that $\sum c_n d^n$ diverges. If x is any number such that $|x| > |d|$, then $\sum c_n x^n$ cannot converge because, by part 1, the convergence of $\sum c_n x^n$ would imply the convergence of $\sum c_n d^n$. Therefore $\sum c_n x^n$ diverges whenever $|x| > |d|$. ■

THEOREM For a power series $\sum c_n x^n$ there are only three possibilities:

1. The series converges only when $x = 0$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

PROOF Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers b and d such that $\sum c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore the set $S = \{x \mid \sum c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \leqslant |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set S. Thus, by the Completeness Axiom (see Section 11.1), S has a least upper bound R. If $|x| > R$, then $x \notin S$, so $\sum c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\sum c_n b^n$ converges, so by the preceding theorem $\sum c_n x^n$ converges. ■

3 **THEOREM** For a power series $\sum c_n (x - a)^n$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

PROOF If we make the change of variable $u = x - a$, then the power series becomes $\sum c_n u^n$ and we can apply the preceding theorem to this series. In case 3 we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$. ■

SECTION 14.3

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

PROOF For small values of h, $h \neq 0$, consider the difference

$$\Delta(h) = [f(a + h, b + h) - f(a + h, b)] - [f(a, b + h) - f(a, b)]$$

Notice that if we let $g(x) = f(x, b + h) - f(x, b)$, then

$$\Delta(h) = g(a + h) - g(a)$$

By the Mean Value Theorem, there is a number c between a and $a + h$ such that

$$g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b + h$ such that

$$f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d) \to (a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b), we obtain

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$. ■

SECTION 14.4

$\boxed{8}$ **THEOREM** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

PROOF Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to (14.4.7), to prove that f is differentiable at (a, b) we have to show that we can write Δz in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Referring to Figure 4, we write

$$\boxed{1} \quad \Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

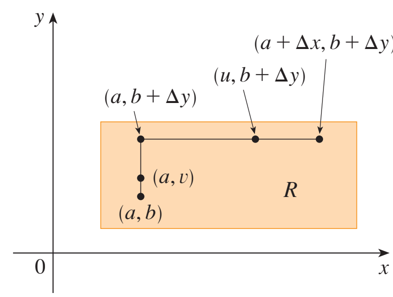


FIGURE 4

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval $[a, a + \Delta x]$ and $g'(x) = f_x(x, b + \Delta y)$. If we apply the Mean Value Theorem to g, we get

$$g(a + \Delta x) - g(a) = g'(u)\,\Delta x$$

where u is some number between a and $a + \Delta x$. In terms of f, this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y)\,\Delta x$$

This gives us an expression for the first part of the right side of Equation 1. For the second part we let $h(y) = f(a, y)$. Then h is a function of a single variable defined on the interval $[b, b + \Delta y]$ and $h'(y) = f_y(a, y)$. A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v)\,\Delta y$$

where v is some number between b and $b + \Delta y$. In terms of f, this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v)\,\Delta y$$

We now substitute these expressions into Equation 1 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y)\,\Delta x + f_y(a, v)\,\Delta y \\ &= f_x(a, b)\,\Delta x + [f_x(u, b + \Delta y) - f_x(a, b)]\,\Delta x + f_y(a, b)\,\Delta y \\ &\qquad + [f_y(a, v) - f_y(a, b)]\,\Delta y \\ &= f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y\end{aligned}$$

where
$$\varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b)$$

$$\varepsilon_2 = f_y(a, v) - f_y(a, b)$$

Since $(u, b + \Delta y) \to (a, b)$ and $(a, v) \to (a, b)$ as $(\Delta x, \Delta y) \to (0, 0)$ and since f_x and f_y are continuous at (a, b), we see that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Therefore f is differentiable at (a, b). ■

G THE LOGARITHM DEFINED AS AN INTEGRAL

Our treatment of exponential and logarithmic functions until now has relied on our intuition, which is based on numerical and visual evidence. (See Sections 1.5, 1.6, and 3.1.) Here we use the Fundamental Theorem of Calculus to give an alternative treatment that provides a surer footing for these functions.

Instead of starting with a^x and defining $\log_a x$ as its inverse, this time we start by defining $\ln x$ as an integral and then define the exponential function as its inverse. You should bear in mind that we do not use any of our previous definitions and results concerning exponential and logarithmic functions.

THE NATURAL LOGARITHM

We first define $\ln x$ as an integral.

1 DEFINITION The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = 1/t$ from $t = 1$ to $t = x$. (See Figure 1.) For $x = 1$, we have

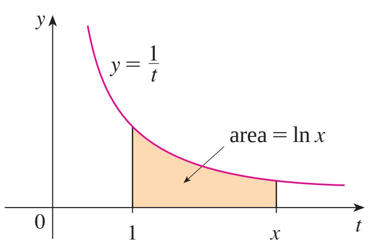


FIGURE 1

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

For $0 < x < 1$,
$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

and so $\ln x$ is the negative of the area shown in Figure 2.

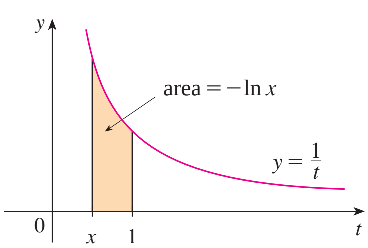


FIGURE 2

V EXAMPLE 1

(a) By comparing areas, show that $\frac{1}{2} < \ln 2 < \frac{3}{4}$.

(b) Use the Midpoint Rule with $n = 10$ to estimate the value of $\ln 2$.

SOLUTION

(a) We can interpret $\ln 2$ as the area under the curve $y = 1/t$ from 1 to 2. From Figure 3 we see that this area is larger than the area of rectangle $BCDE$ and smaller than the area of trapezoid $ABCD$. Thus we have

$$\tfrac{1}{2} \cdot 1 < \ln 2 < 1 \cdot \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right)$$

$$\tfrac{1}{2} < \ln 2 < \tfrac{3}{4}$$

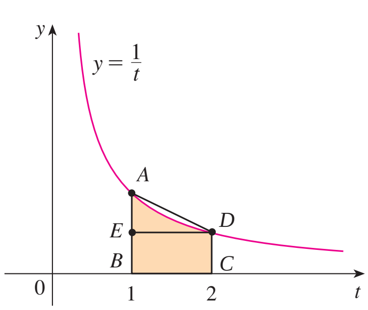


FIGURE 3

(b) If we use the Midpoint Rule with $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.1$, we get

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx (0.1)[f(1.05) + f(1.15) + \cdots + f(1.95)]$$

$$= (0.1)\left(\frac{1}{1.05} + \frac{1}{1.15} + \cdots + \frac{1}{1.95}\right) \approx 0.693$$ ■

Notice that the integral that defines $\ln x$ is exactly the type of integral discussed in Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). In fact, using that theorem, we have

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

and so

2
$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

We now use this differentiation rule to prove the following properties of the logarithm function.

3 LAWS OF LOGARITHMS If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

PROOF

1. Let $f(x) = \ln(ax)$, where a is a positive constant. Then, using Equation 2 and the Chain Rule, we have

$$f'(x) = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$$

Therefore $f(x)$ and $\ln x$ have the same derivative and so they must differ by a constant:

$$\ln(ax) = \ln x + C$$

Putting $x = 1$ in this equation, we get $\ln a = \ln 1 + C = 0 + C = C$. Thus

$$\ln(ax) = \ln x + \ln a$$

If we now replace the constant a by any number y, we have

$$\ln(xy) = \ln x + \ln y$$

2. Using Law 1 with $x = 1/y$, we have

$$\ln\frac{1}{y} + \ln y = \ln\left(\frac{1}{y}\cdot y\right) = \ln 1 = 0$$

and so
$$\ln\frac{1}{y} = -\ln y$$

Using Law 1 again, we have

$$\ln\left(\frac{x}{y}\right) = \ln\left(x\cdot\frac{1}{y}\right) = \ln x + \ln\frac{1}{y} = \ln x - \ln y$$

The proof of Law 3 is left as an exercise. ■

In order to graph $y = \ln x$, we first determine its limits:

4 $$\text{(a)} \lim_{x \to \infty} \ln x = \infty \qquad \text{(b)} \lim_{x \to 0^+} \ln x = -\infty$$

PROOF

(a) Using Law 3 with $x = 2$ and $r = n$ (where n is any positive integer), we have $\ln(2^n) = n \ln 2$. Now $\ln 2 > 0$, so this shows that $\ln(2^n) \to \infty$ as $n \to \infty$. But $\ln x$ is an increasing function since its derivative $1/x > 0$. Therefore $\ln x \to \infty$ as $x \to \infty$.

(b) If we let $t = 1/x$, then $t \to \infty$ as $x \to 0^+$. Thus, using (a), we have

$$\lim_{x \to 0^+} \ln x = \lim_{t \to \infty} \ln\left(\frac{1}{t}\right) = \lim_{t \to \infty} (-\ln t) = -\infty$$ ■

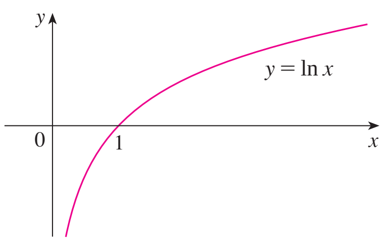


FIGURE 4

If $y = \ln x$, $x > 0$, then

$$\frac{dy}{dx} = \frac{1}{x} > 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$$

which shows that $\ln x$ is increasing and concave downward on $(0, \infty)$. Putting this information together with (4), we draw the graph of $y = \ln x$ in Figure 4.

Since $\ln 1 = 0$ and $\ln x$ is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where $\ln x$ takes on the value 1. (See Figure 5.) This important number is denoted by e.

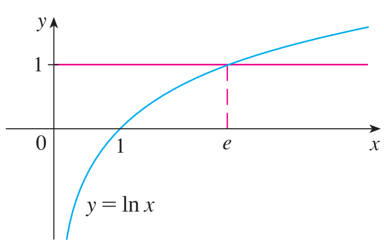


FIGURE 5

5 DEFINITION e is the number such that $\ln e = 1$.

We will show (in Theorem 19) that this definition is consistent with our previous definition of e.

THE NATURAL EXPONENTIAL FUNCTION

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function,

$f^{-1}(x) = y \iff f(y) = x$

6 $$\exp(x) = y \iff \ln y = x$$

and the cancellation equations are

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

7 $$\exp(\ln x) = x \qquad \text{and} \qquad \ln(\exp x) = x$$

In particular, we have

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$
$$\exp(1) = e \quad \text{since} \quad \ln e = 1$$

We obtain the graph of $y = \exp x$ by reflecting the graph of $y = \ln x$ about the line $y = x$.

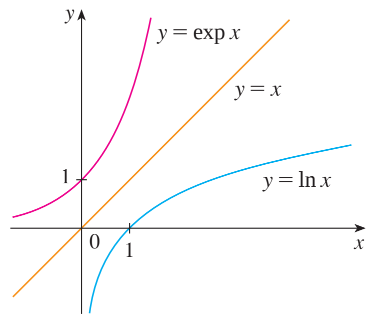


FIGURE 6

(See Figure 6.) The domain of exp is the range of ln, that is, $(-\infty, \infty)$; the range of exp is the domain of ln, that is, $(0, \infty)$.

If r is any rational number, then the third law of logarithms gives

$$\ln(e^r) = r \ln e = r$$

Therefore, by (6), $$\exp(r) = e^r$$

Thus $\exp(x) = e^x$ whenever x is a rational number. This leads us to define e^x, even for irrational values of x, by the equation

$$\boxed{e^x = \exp(x)}$$

In other words, for the reasons given, we define e^x to be the inverse of the function $\ln x$. In this notation (6) becomes

8 $$\boxed{e^x = y \iff \ln y = x}$$

and the cancellation equations (7) become

9 $$\boxed{e^{\ln x} = x \qquad x > 0}$$

10 $$\boxed{\ln(e^x) = x \qquad \text{for all } x}$$

The natural exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 7) and its properties (which follow from the fact that it is the inverse of the natural logarithmic function).

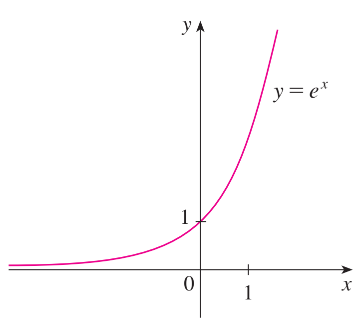


FIGURE 7
The natural exponential function

PROPERTIES OF THE EXPONENTIAL FUNCTION The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

We now verify that f has the other properties expected of an exponential function.

11 **LAWS OF EXPONENTS** If x and y are real numbers and r is rational, then

1. $e^{x+y} = e^x e^y$ **2.** $e^{x-y} = \dfrac{e^x}{e^y}$ **3.** $(e^x)^r = e^{rx}$

PROOF OF LAW 1 Using the first law of logarithms and Equation 10, we have

$$\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x + y = \ln(e^{x+y})$$

Since ln is a one-to-one function, it follows that $e^x e^y = e^{x+y}$.

Laws 2 and 3 are proved similarly (see Exercises 6 and 7). As we will soon see, Law 3 actually holds when r is any real number. ■

We now prove the differentiation formula for e^x.

12 $$\boxed{\frac{d}{dx}(e^x) = e^x}$$

PROOF The function $y = e^x$ is differentiable because it is the inverse function of $y = \ln x$, which we know is differentiable with nonzero derivative. To find its derivative, we use the inverse function method. Let $y = e^x$. Then $\ln y = x$ and, differentiating this latter equation implicitly with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x$$ ■

GENERAL EXPONENTIAL FUNCTIONS

If $a > 0$ and r is any rational number, then by (9) and (11),

$$a^r = (e^{\ln a})^r = e^{r \ln a}$$

Therefore, even for irrational numbers x, we *define*

13 $$\boxed{a^x = e^{x \ln a}}$$

Thus, for instance,

$$2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$$

The function $f(x) = a^x$ is called the **exponential function with base *a***. Notice that a^x is positive for all x because e^x is positive for all x.

Definition 13 allows us to extend one of the laws of logarithms. We already know that $\ln(a^r) = r \ln a$ when r is rational. But if we now let r be *any* real number we have, from Definition 13,

$$\ln a^r = \ln(e^{r \ln a}) = r \ln a$$

Thus

14 $$\ln a^r = r \ln a \qquad \text{for any real number } r$$

The general laws of exponents follow from Definition 13 together with the laws of exponents for e^x.

15 **LAWS OF EXPONENTS** If x and y are real numbers and $a, b > 0$, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = a^x/a^y$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

PROOF

1. Using Definition 13 and the laws of exponents for e^x, we have

$$\begin{aligned} a^{x+y} &= e^{(x+y)\ln a} = e^{x \ln a + y \ln a} \\ &= e^{x \ln a} e^{y \ln a} = a^x a^y \end{aligned}$$

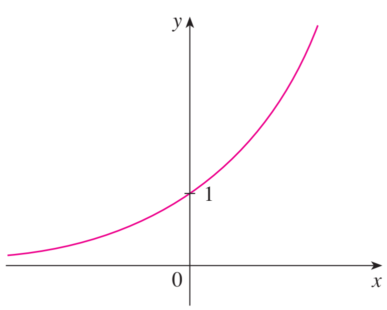
3. Using Equation 14 we obtain

$$(a^x)^y = e^{y \ln(a^x)} = e^{yx \ln a} = e^{xy \ln a} = a^{xy}$$

The remaining proofs are left as exercises. ■

The differentiation formula for exponential functions is also a consequence of Definition 13:

16
$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$



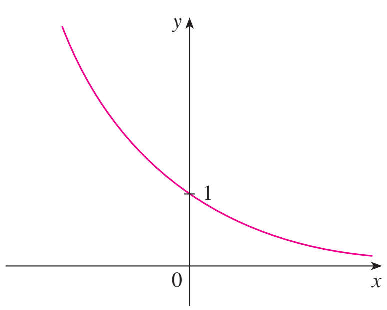
$\lim_{x \to -\infty} a^x = 0, \ \lim_{x \to \infty} a^x = \infty$

FIGURE 8 $y = a^x, \quad a > 1$

PROOF

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a} \frac{d}{dx}(x \ln a) = a^x \ln a$$ ■

If $a > 1$, then $\ln a > 0$, so $(d/dx)\, a^x = a^x \ln a > 0$, which shows that $y = a^x$ is increasing (see Figure 8). If $0 < a < 1$, then $\ln a < 0$ and so $y = a^x$ is decreasing (see Figure 9).



$\lim_{x \to -\infty} a^x = \infty, \ \lim_{x \to \infty} a^x = 0$

FIGURE 9 $y = a^x, \quad 0 < a < 1$

GENERAL LOGARITHMIC FUNCTIONS

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a one-to-one function. Its inverse function is called the **logarithmic function with base *a*** and is denoted by $\log_a$. Thus

17
$$\boxed{\log_a x = y \iff a^y = x}$$

In particular, we see that

$$\log_e x = \ln x$$

The laws of logarithms are similar to those for the natural logarithm and can be deduced from the laws of exponents (see Exercise 10).

To differentiate $y = \log_a x$, we write the equation as $a^y = x$. From Equation 14 we have $y \ln a = \ln x$, so

$$\log_a x = y = \frac{\ln x}{\ln a}$$

Since $\ln a$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

18 $$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

THE NUMBER e EXPRESSED AS A LIMIT

In this section we defined e as the number such that $\ln e = 1$. The next theorem shows that this is the same as the number e defined in Section 3.1 (see Equation 3.6.5).

19 $$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

PROOF Let $f(x) = \ln x$. Then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) = \lim_{x \to 0} \ln(1 + x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$ ■

G EXERCISES

1. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate $\ln 1.5$.

2. Refer to Example 1.

(a) Find the equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD.

(b) Use part (a) to show that $\ln 2 > 0.66$.

3. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n - 1}$$

4. (a) By comparing areas, show that $\ln 2 < 1 < \ln 3$.

(b) Deduce that $2 < e < 3$.

5. Prove the third law of logarithms. [*Hint:* Start by showing that both sides of the equation have the same derivative.]

6. Prove the second law of exponents for e^x [see (11)].

7. Prove the third law of exponents for e^x [see (11)].

8. Prove the second law of exponents [see (15)].

9. Prove the fourth law of exponents [see (15)].

10. Deduce the following laws of logarithms from (15):
(a) $\log_a(xy) = \log_a x + \log_a y$
(b) $\log_a(x/y) = \log_a x - \log_a y$
(c) $\log_a(x^y) = y \log_a x$

H COMPLEX NUMBERS

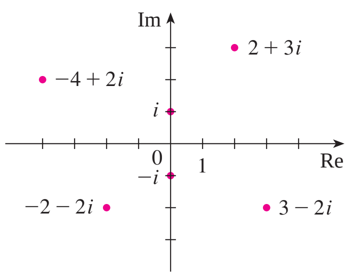


FIGURE 1
Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b. Thus the real part of $4 - 3i$ is 4 and the imaginary part is -3. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$; that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di) \\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

EXAMPLE 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i) \\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$ ■

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number $z = a + bi$, we define its **complex conjugate** to be $\bar{z} = a - bi$. To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

EXAMPLE 2 Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

SOLUTION We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29} i$$

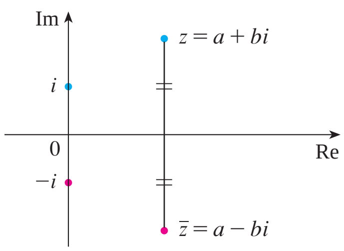


FIGURE 2

The geometric interpretation of the complex conjugate is shown in Figure 2: $\bar{z}$ is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

PROPERTIES OF CONJUGATES

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

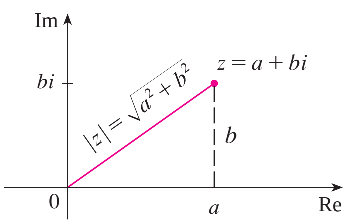


FIGURE 3

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2$$

and so

$$\boxed{z\bar{z} = |z|^2}$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3 Find the roots of the equation $x^2 + x + 1 = 0$.

SOLUTION Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b, and c are always complex conjugates. (If z is real, $\bar{z} = z$, so z is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

POLAR FORM

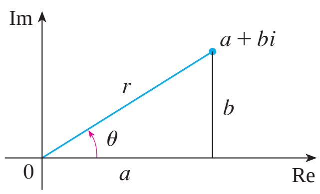


FIGURE 4

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geqslant 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

as in Figure 4. Therefore we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

Thus we can write any complex number z in the form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $\qquad r = |z| = \sqrt{a^2 + b^2} \qquad$ and $\qquad \tan\theta = \dfrac{b}{a}$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π.

EXAMPLE 4 Write the following numbers in polar form.

(a) $z = 1 + i$ $\qquad$ (b) $w = \sqrt{3} - i$

SOLUTION

(a) We have $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(b) Here we have $r = |w| = \sqrt{3 + 1} = 2$ and $\tan\theta = -1/\sqrt{3}$. Since w lies in the fourth quadrant, we take $\theta = -\pi/6$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

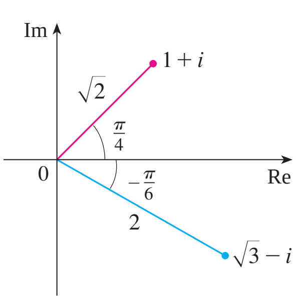


FIGURE 5

The numbers z and w are shown in Figure 5. ■

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

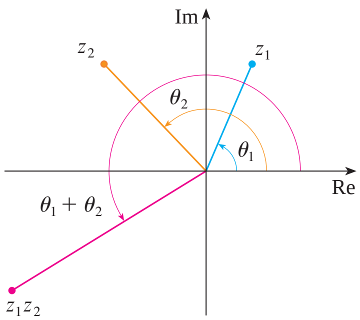


FIGURE 6

$$\boxed{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]} \tag{1}$$

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments.* (See Figure 6.)

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments.*

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0}$$

In particular, taking $z_1 = 1$ and $z_2 = z$ (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

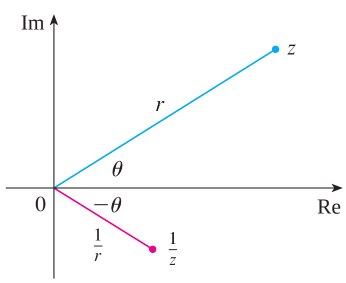


FIGURE 7

$$\boxed{\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).}$$

EXAMPLE 5 Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

SOLUTION From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and
$$\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

So, by Equation 1,

$$\begin{aligned} (1 + i)(\sqrt{3} - i) &= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right] \\ &= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \end{aligned}$$

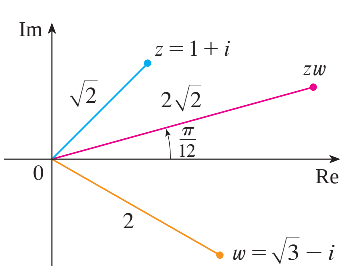


FIGURE 8

This is illustrated in Figure 8. ■

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos\theta + i\sin\theta)$$

then
$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and
$$z^3 = zz^2 = r^3(\cos 3\theta + i\sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

2 DE MOIVRE'S THEOREM If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This says that *to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n*.

EXAMPLE 6 Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An nth root of the complex number z is a complex number w such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos\phi + i\sin\phi) \qquad \text{and} \qquad z = r(\cos\theta + i\sin\theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

The equality of these two complex numbers shows that

$$s^n = r \qquad \text{or} \qquad s = r^{1/n}$$

and
$$\cos n\phi = \cos\theta \qquad \text{and} \qquad \sin n\phi = \sin\theta$$

From the fact that sine and cosine have period 2π it follows that

$$n\phi = \theta + 2k\pi \qquad \text{or} \qquad \phi = \frac{\theta + 2k\pi}{n}$$

Thus
$$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n - 1$, we have the following.

3 ROOTS OF A COMPLEX NUMBER Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Notice that each of the nth roots of z has modulus $|w_k| = r^{1/n}$. Thus all the nth roots of z lie on the circle of radius $r^{1/n}$ in the complex plane. Also, since the argument of each successive nth root exceeds the argument of the previous root by $2\pi/n$, we see that the nth roots of z are equally spaced on this circle.

EXAMPLE 7 Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

SOLUTION In trigonometric form, $z = 8(\cos\pi + i\sin\pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos\frac{\pi + 2k\pi}{6} + i\sin\frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of -8 by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_1 = 8^{1/6}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \sqrt{2}\,i$$

$$w_2 = 8^{1/6}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_3 = 8^{1/6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$w_4 = 8^{1/6}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt{2}\,i$$

$$w_5 = 8^{1/6}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

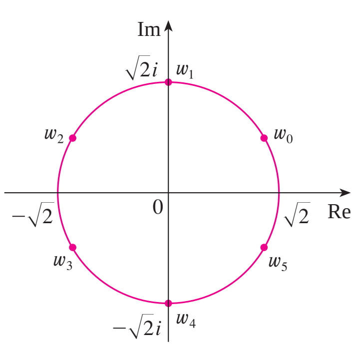


FIGURE 9
The six sixth roots of $z = -8$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9. ■

COMPLEX EXPONENTIALS

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 11 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (11.10.11) as our guide, we define

4 $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

5 $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots$$

we get
$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 11.10.16 and 11.10.15). The result is a famous formula called **Euler's formula**:

6 $$\boxed{e^{iy} = \cos y + i \sin y}$$

Combining Euler's formula with Equation 5, we get

7 $$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

EXAMPLE 8 Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

SOLUTION

(a) From Euler's equation (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

■ We could write the result of Example 8(a) as

$$e^{i\pi} + 1 = 0$$

This equation relates the five most famous numbers in all of mathematics: 0, 1, e, i, and π.

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$ ■

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos\theta + i \sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

H EXERCISES

1–14 Evaluate the expression and write your answer in the form $a + bi$.

1. $(5 - 6i) + (3 + 2i)$

2. $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right)$

3. $(2 + 5i)(4 - i)$

4. $(1 - 2i)(8 - 3i)$

5. $\overline{12 + 7i}$

6. $\overline{2i\left(\frac{1}{2} - i\right)}$

7. $\dfrac{1 + 4i}{3 + 2i}$

8. $\dfrac{3 + 2i}{1 - 4i}$

9. $\dfrac{1}{1 + i}$

10. $\dfrac{3}{4 - 3i}$

11. i^3

12. i^{100}

13. $\sqrt{-25}$

14. $\sqrt{-3}\sqrt{-12}$

15–17 Find the complex conjugate and the modulus of the number.

15. $12 - 5i$

16. $-1 + 2\sqrt{2}\, i$

17. $-4i$

18. Prove the following properties of complex numbers.
(a) $\overline{z + w} = \overline{z} + \overline{w}$
(b) $\overline{zw} = \overline{z}\,\overline{w}$
(c) $\overline{z^n} = \overline{z}^n$, where n is a positive integer
[*Hint:* Write $z = a + bi$, $w = c + di$.]

19–24 Find all solutions of the equation.

19. $4x^2 + 9 = 0$

20. $x^4 = 1$

21. $x^2 + 2x + 5 = 0$

22. $2x^2 - 2x + 1 = 0$

23. $z^2 + z + 2 = 0$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

25–28 Write the number in polar form with argument between 0 and 2π.

25. $-3 + 3i$

26. $1 - \sqrt{3}\, i$

27. $3 + 4i$

28. $8i$

29–32 Find polar forms for zw, z/w, and $1/z$ by first putting z and w into polar form.

29. $z = \sqrt{3} + i$, $\quad w = 1 + \sqrt{3}\, i$

30. $z = 4\sqrt{3} - 4i$, $\quad w = 8i$

31. $z = 2\sqrt{3} - 2i$, $\quad w = -1 + i$

32. $z = 4\left(\sqrt{3} + i\right)$, $\quad w = -3 - 3i$

33–36 Find the indicated power using De Moivre's Theorem.

33. $(1 + i)^{20}$

34. $\left(1 - \sqrt{3}\, i\right)^5$

35. $\left(2\sqrt{3} + 2i\right)^5$

36. $(1 - i)^8$

37–40 Find the indicated roots. Sketch the roots in the complex plane.

37. The eighth roots of 1

38. The fifth roots of 32

39. The cube roots of i

40. The cube roots of $1 + i$

41–46 Write the number in the form $a + bi$.

41. $e^{i\pi/2}$

42. $e^{2\pi i}$

43. $e^{i\pi/3}$

44. $e^{-i\pi}$

45. $e^{2+i\pi}$

46. $e^{\pi+i}$

47. Use De Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

48. Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

49. If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable x and the real and imaginary parts $f(x)$ and $g(x)$ are differentiable functions of x, then the derivative of u is defined to be $u'(x) = f'(x) + ig'(x)$. Use this together with Equation 7 to prove that if $F(x) = e^{rx}$, then $F'(x) = re^{rx}$ when $r = a + bi$ is a complex number.

50. (a) If u is a complex-valued function of a real variable, its indefinite integral $\int u(x)\,dx$ is an antiderivative of u. Evaluate

$$\int e^{(1+i)x}\,dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

$$\int e^x \cos x\,dx \qquad \text{and} \qquad \int e^x \sin x\,dx$$

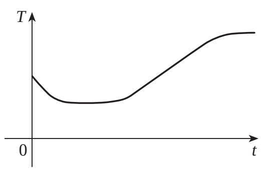
(c) Compare with the method used in Example 4 in Section 7.1.

I ANSWERS TO ODD-NUMBERED EXERCISES

CHAPTER 1

EXERCISES 1.1 ■ PAGE 20

1. (a) -2 (b) 2.8 (c) $-3, 1$ (d) $-2.5, 0.3$
(e) $[-3, 3], [-2, 3]$ (f) $[-1, 3]$
3. $[-85, 115]$ **5.** No
7. Yes, $[-3, 2], [-3, -2) \cup [-1, 3]$
9. Diet, exercise, or illness
11.



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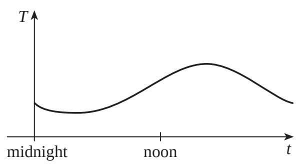

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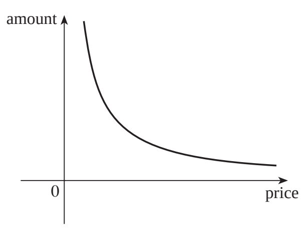
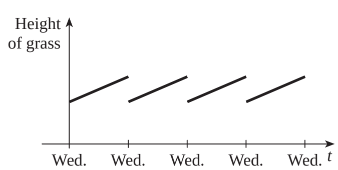
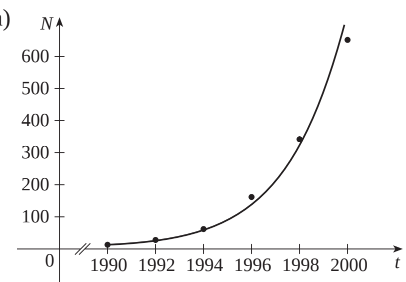


17.



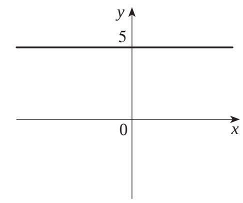
19. (a)



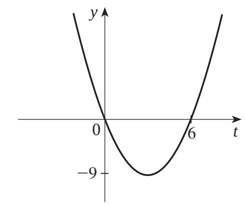
(b) In millions: 92; 485

21. $12, 16, 3a^2 - a + 2, 3a^2 + a + 2, 3a^2 + 5a + 4,$
$6a^2 - 2a + 4, 12a^2 - 2a + 2, 3a^4 - a^2 + 2,$
$9a^4 - 6a^3 + 13a^2 - 4a + 4, 3a^2 + 6ah + 3h^2 - a - h + 2$
23. $-3 - h$ **25.** $-1/(ax)$
27. $\{x \mid x \neq \frac{1}{3}\} = (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$
29. $[0, \infty)$ **31.** $(-\infty, 0) \cup (5, \infty)$

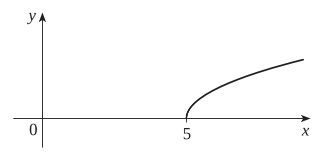
33. $(-\infty, \infty)$



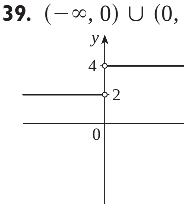
35. $(-\infty, \infty)$



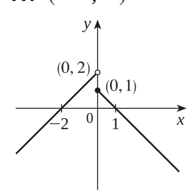
37. $[5, \infty)$



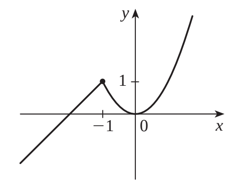
39. $(-\infty, 0) \cup (0, \infty)$



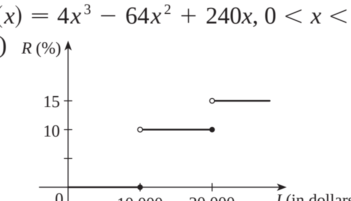
41. $(-\infty, \infty)$



43. $(-\infty, \infty)$

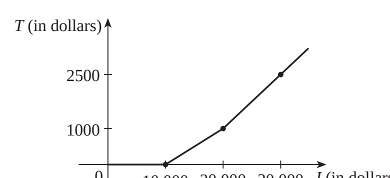


45. $f(x) = \frac{5}{2}x - \frac{11}{2}, 1 \leqslant x \leqslant 5$ **47.** $f(x) = 1 - \sqrt{-x}$
49. $f(x) = \begin{cases} -x + 3 & \text{if } 0 \leqslant x \leqslant 3 \\ 2x - 6 & \text{if } 3 < x \leqslant 5 \end{cases}$
51. $A(L) = 10L - L^2, 0 < L < 10$
53. $A(x) = \sqrt{3}x^2/4, x > 0$ **55.** $S(x) = x^2 + (8/x), x > 0$
57. $V(x) = 4x^3 - 64x^2 + 240x, 0 < x < 6$
59. (a)



(b) \$400, \$1900

(c)

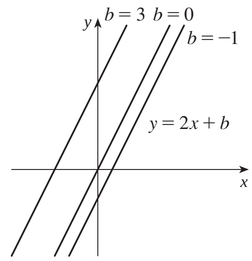


61. f is odd, g is even
63. (a) $(-5, 3)$ (b) $(-5, -3)$
65. Odd **67.** Neither **69.** Even

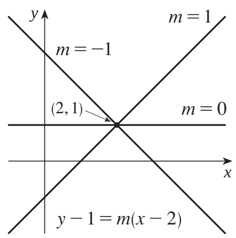
EXERCISES 1.2 ■ PAGE 34

1. (a) Root (b) Algebraic (c) Polynomial (degree 9)
(d) Rational (e) Trigonometric (f) Logarithmic
3. (a) h (b) f (c) g

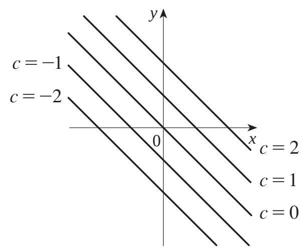
5. (a) $y = 2x + b$, where b is the y-intercept.



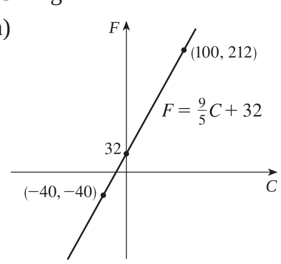
(b) $y = mx + 1 - 2m$, where m is the slope. See graph at right.
(c) $y = 2x - 3$



7. Their graphs have slope -1.

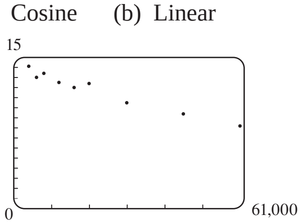


9. $f(x) = -3x(x + 1)(x - 2)$
11. (a) 8.34, change in mg for every 1 year change
(b) 8.34 mg
13. (a)



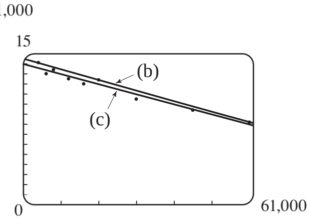
(b) $\frac{9}{5}$, change in °F for every 1°C change; 32, Fahrenheit temperature corresponding to 0°C

15. (a) $T = \frac{1}{6}N + \frac{307}{6}$ (b) $\frac{1}{6}$, change in °F for every chirp per minute change (c) 76°F
17. (a) $P = 0.434d + 15$ (b) 196 ft
19. (a) Cosine (b) Linear
21. (a)

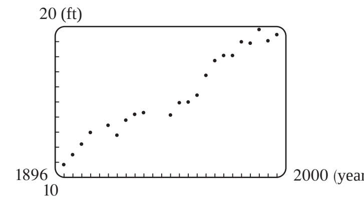


Linear model is appropriate

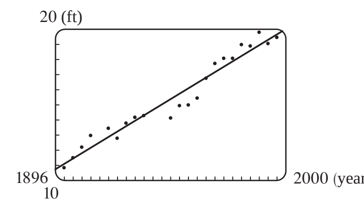
(b) $y = -0.000105x + 14.521$



(c) $y = -0.00009979x + 13.951$ [See graph in (b).]
(d) About 11.5 per 100 population (e) About 6% (f) No
23. (a)



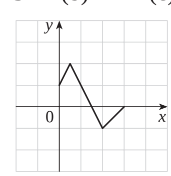
Linear model is appropriate
(b) $y = 0.08912x - 158.24$ (c) 20 ft (d) No



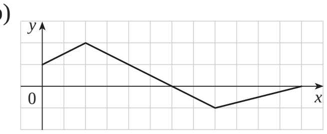
25. $y \approx 0.0012937x^3 - 7.06142x^2 + 12{,}823x - 7{,}743{,}770$; 1914 million

EXERCISES 1.3 ■ PAGE 43

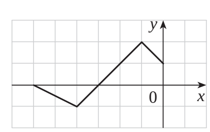
1. (a) $y = f(x) + 3$ (b) $y = f(x) - 3$ (c) $y = f(x - 3)$
(d) $y = f(x + 3)$ (e) $y = -f(x)$ (f) $y = f(-x)$
(g) $y = 3f(x)$ (h) $y = \frac{1}{3}f(x)$
3. (a) 3 (b) 1 (c) 4 (d) 5 (e) 2
5. (a)



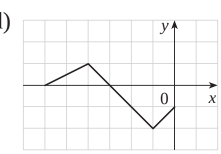
(b)



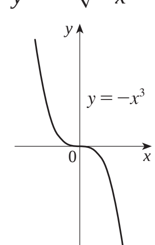
(c)



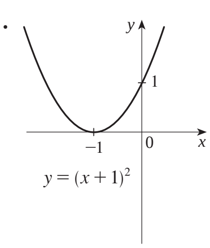
(d)



7. $y = -\sqrt{-x^2 - 5x - 4} - 1$
9.



11.



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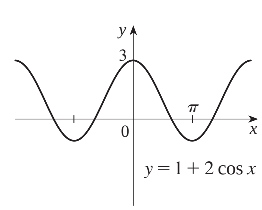

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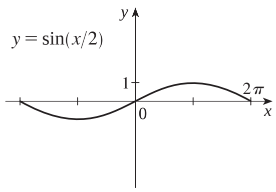

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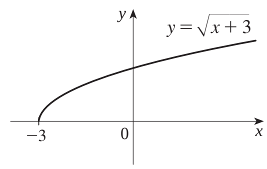

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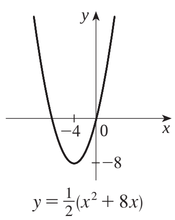

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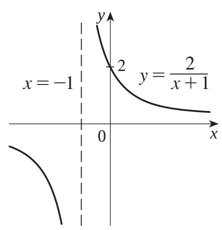
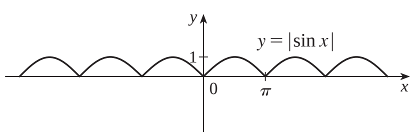


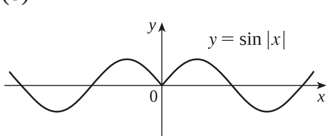
23.



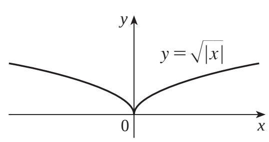
25. $L(t) = 12 + 2\sin\left[\dfrac{2\pi}{365}(t - 80)\right]$

27. (a) The portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

(b)



(c)



29. $(f + g)(x) = x^3 + 5x^2 - 1,\ (-\infty, \infty)$
$(f - g)(x) = x^3 - x^2 + 1,\ (-\infty, \infty)$
$(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2,\ (-\infty, \infty)$
$(f/g)(x) = (x^3 + 2x^2)/(3x^2 - 1),\ \{x \mid x \neq \pm 1/\sqrt{3}\}$

31. (a) $(f \circ g)(x) = 4x^2 + 4x,\ (-\infty, \infty)$
(b) $(g \circ f)(x) = 2x^2 - 1,\ (-\infty, \infty)$
(c) $(f \circ f)(x) = x^4 - 2x^2,\ (-\infty, \infty)$
(d) $(g \circ g)(x) = 4x + 3,\ (-\infty, \infty)$

33. (a) $(f \circ g)(x) = 1 - 3\cos x,\ (-\infty, \infty)$
(b) $(g \circ f)(x) = \cos(1 - 3x),\ (-\infty, \infty)$
(c) $(f \circ f)(x) = 9x - 2,\ (-\infty, \infty)$
(d) $(g \circ g)(x) = \cos(\cos x),\ (-\infty, \infty)$

35. (a) $(f \circ g)(x) = (2x^2 + 6x + 5)/[(x + 2)(x + 1)]$, $\{x \mid x \neq -2, -1\}$
(b) $(g \circ f)(x) = (x^2 + x + 1)/(x + 1)^2,\ \{x \mid x \neq -1, 0\}$
(c) $(f \circ f)(x) = (x^4 + 3x^2 + 1)/[x(x^2 + 1)],\ \{x \mid x \neq 0\}$
(d) $(g \circ g)(x) = (2x + 3)/(3x + 5),\ \{x \mid x \neq -2, -\frac{5}{3}\}$

37. $(f \circ g \circ h)(x) = 2x - 1$

39. $(f \circ g \circ h)(x) = \sqrt{x^6 + 4x^3 + 1}$

41. $g(x) = x^2 + 1,\ f(x) = x^{10}$

43. $g(x) = \sqrt[3]{x},\ f(x) = x/(1 + x)$

45. $g(t) = \cos t,\ f(t) = \sqrt{t}$

47. $h(x) = x^2,\ g(x) = 3^x,\ f(x) = 1 - x$

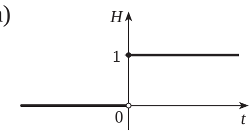
49. $h(x) = \sqrt{x},\ g(x) = \sec x,\ f(x) = x^4$

51. (a) 4 (b) 3 (c) 0 (d) Does not exist; $f(6) = 6$ is not in the domain of g. (e) 4 (f) -2

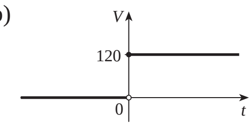
53. (a) $r(t) = 60t$ (b) $(A \circ r)(t) = 3600\pi t^2$; the area of the circle as a function of time

55. (a) $s = \sqrt{d^2 + 36}$ (b) $d = 30t$
(c) $s = \sqrt{900t^2 + 36}$; the distance between the lighthouse and the ship as a function of the time elapsed since noon

57. (a)

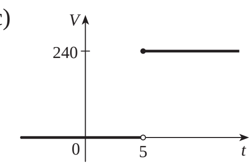


(b)



$V(t) = 120H(t)$

(c)



$V(t) = 240H(t - 5)$

59. Yes; $m_1 m_2$

61. (a) $f(x) = x^2 + 6$ (b) $g(x) = x^2 + x - 1$

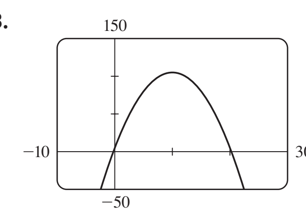
63. (a) Even; even (b) Odd; even

65. Yes

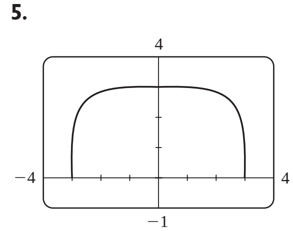
EXERCISES 1.4 ■ PAGE 51

1. (c)

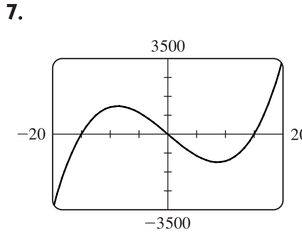
3.



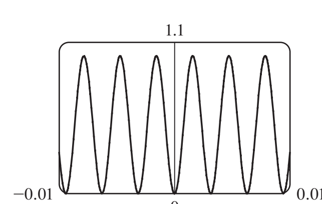
5.



7.



9.



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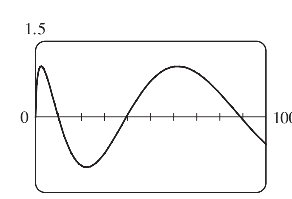

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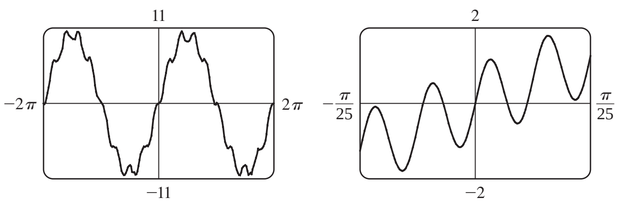

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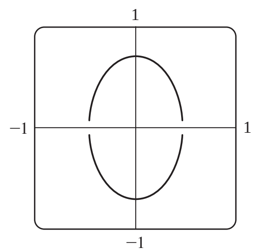
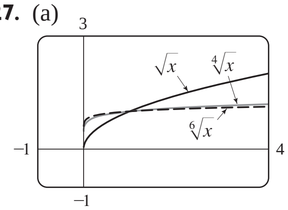


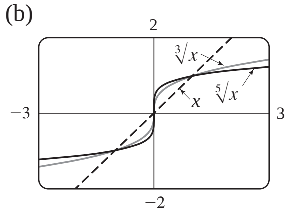
17. No **19.** 9.05 **21.** 0, 0.88 **23.** g

25. $-0.85 < x < 0.85$

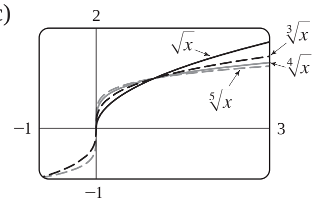
27. (a)



(b)

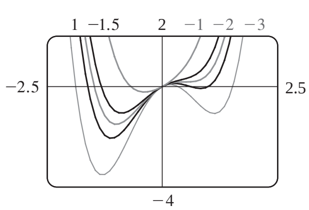


(c)



(d) Graphs of even roots are similar to $\sqrt{x}$, graphs of odd roots are similar to $\sqrt[3]{x}$. As n increases, the graph of $y = \sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

29.



If $c < -1.5$, the graph has three humps: two minimum points and a maximum point. These humps get flatter as c increases until at $c = -1.5$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

31. The hump gets larger and moves to the right.

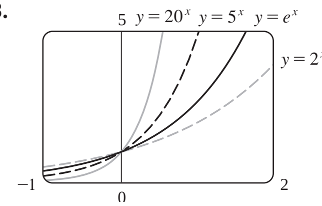
33. If $c < 0$, the loop is to the right of the origin; if $c > 0$, the loop is to the left. The closer c is to 0, the larger the loop.

EXERCISES 1.5 ▪ PAGE 58

1. (a) $f(x) = a^x$, $a > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$

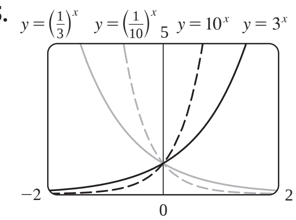
(d) See Figures 4(c), 4(b), and 4(a), respectively.

3.



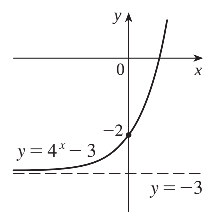
All approach 0 as $x \to -\infty$, all pass through (0, 1), and all are increasing. The larger the base, the faster the rate of increase.

5.

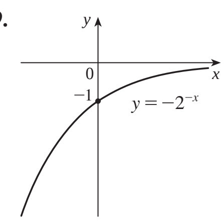


The functions with base greater than 1 are increasing and those with base less than 1 are decreasing. The latter are reflections of the former about the y-axis.

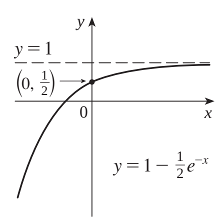
7.



9.



11.



13. (a) $y = e^x - 2$ (b) $y = e^{x-2}$ (c) $y = -e^x$

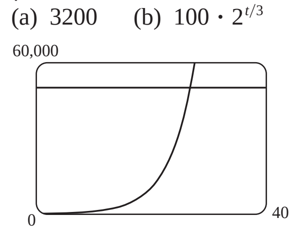
(d) $y = e^{-x}$ (e) $y = -e^{-x}$

15. (a) $(-\infty, \infty)$ (b) $(-\infty, 0) \cup (0, \infty)$

17. $f(x) = 3 \cdot 2^x$ **23.** At $x \approx 35.8$

25. (a) 3200 (b) $100 \cdot 2^{t/3}$ (c) 10,159

(d) $t \approx 26.9$ h



27. $y = ab^t$, where $a \approx 3.154832569 \times 10^{-12}$ and $b \approx 1.017764706$; 5498 million; 7417 million

EXERCISES 1.6 ▪ PAGE 70

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

3. No **5.** Yes **7.** No **9.** No **11.** Yes

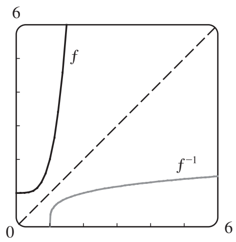
13. No **15.** 2 **17.** 0

19. $F = \frac{9}{5}C + 32$; the Fahrenheit temperature as a function of the Celsius temperature; $[-273.15, \infty)$

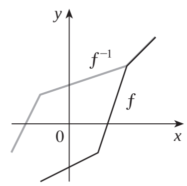
21. $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$, $x \geqslant 0$ **23.** $f^{-1}(x) = \sqrt[3]{\ln x}$

25. $y = e^x - 3$

27. $f^{-1}(x) = \sqrt[4]{x - 1}$



29.

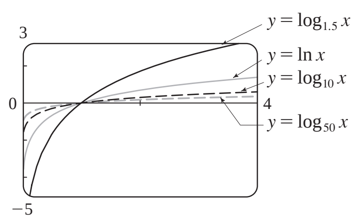


31. (a) It's defined as the inverse of the exponential function with base a, that is, $\log_a x = y \iff a^y = x$.
(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

33. (a) 3 (b) -3 **35.** (a) 3 (b) -2 **37.** ln 1215

39. $\ln \dfrac{(1 + x^2)\sqrt{x}}{\sin x}$

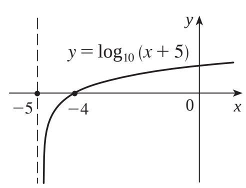
41.



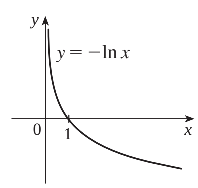
All graphs approach $-\infty$ as $x \to 0^+$, all pass through (1, 0), and all are increasing. The larger the base, the slower the rate of increase.

43. About 1,084,588 mi

45. (a)



(b)



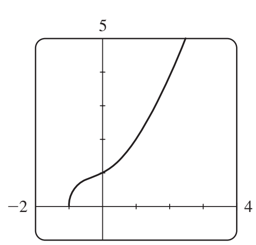
47. (a) $\sqrt{e}$ (b) $-\ln 5$

49. (a) $5 + \log_2 3$ or $5 + (\ln 3)/\ln 2$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$

51. (a) $x < \ln 10$ (b) $x > 1/e$

53. (a) $\left(-\infty, \frac{1}{2}\ln 3\right]$ (b) $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$, $\left[0, \sqrt{3}\right)$

55.



The graph passes the Horizontal Line Test.

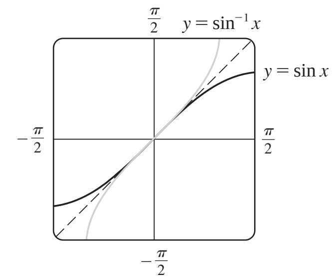
$f^{-1}(x) = -\left(\sqrt[3]{4}/6\right)\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$, where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$; two of the expressions are complex.

57. (a) $f^{-1}(n) = (3/\ln 2)\ln(n/100)$; the time elapsed when there are n bacteria (b) After about 26.9 hours

59. (a) $\pi/3$ (b) π **61.** (a) $\pi/4$ (b) $\pi/4$

63. (a) 10 (b) $\pi/3$ **67.** $x/\sqrt{1 + x^2}$

69.



The second graph is the reflection of the first graph about the line $y = x$.

71. (a) $\left[-\frac{2}{3}, 0\right]$ (b) $[-\pi/2, \pi/2]$

73. (a) $g^{-1}(x) = f^{-1}(x) - c$ (b) $h^{-1}(x) = (1/c)f^{-1}(x)$

CHAPTER 1 REVIEW ■ PAGE 73

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. False **13.** False

Exercises

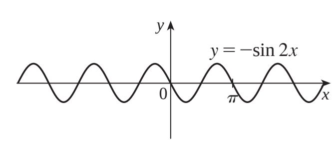
1. (a) 2.7 (b) 2.3, 5.6 (c) $[-6, 6]$ (d) $[-4, 4]$
(e) $[-4, 4]$ (f) No; it fails the Horizontal Line Test.
(g) Odd; its graph is symmetric about the origin.

3. $2a + h - 2$ **5.** $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$, $(-\infty, 0) \cup (0, \infty)$

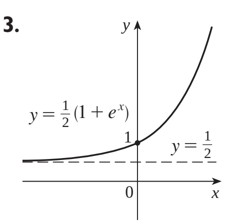
7. $(-6, \infty)$, $\mathbb{R}$

9. (a) Shift the graph 8 units upward.
(b) Shift the graph 8 units to the left.
(c) Stretch the graph vertically by a factor of 2, then shift it 1 unit upward.
(d) Shift the graph 2 units to the right and 2 units downward.
(e) Reflect the graph about the x-axis.
(f) Reflect the graph about the line $y = x$ (assuming f is one-to-one).

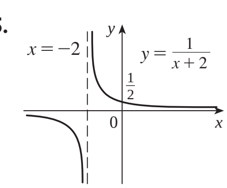
11.



13.



15.



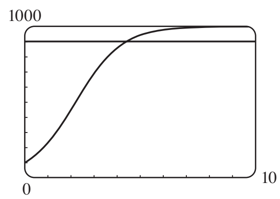
17. (a) Neither (b) Odd (c) Even (d) Neither

19. (a) $(f \circ g)(x) = \ln(x^2 - 9)$, $(-\infty, -3) \cup (3, \infty)$
(b) $(g \circ f)(x) = (\ln x)^2 - 9$, $(0, \infty)$
(c) $(f \circ f)(x) = \ln \ln x$, $(1, \infty)$
(d) $(g \circ g)(x) = (x^2 - 9)^2 - 9$, $(-\infty, \infty)$

21. $y = 0.2493x - 423.4818$; about 77.6 years

23. 1 **25.** (a) 9 (b) 2 (c) $1/\sqrt{3}$ (d) $\frac{3}{5}$

27. (a)

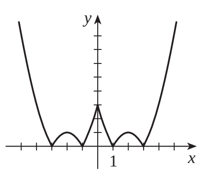


≈ 4.4 years

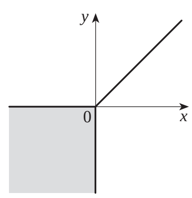
(b) $t = -\ln\left(\dfrac{1000 - P}{9P}\right)$; the time required for the population to reach a given number P.
(c) $\ln 81 \approx 4.4$ years

PRINCIPLES OF PROBLEM SOLVING ■ PAGE 81

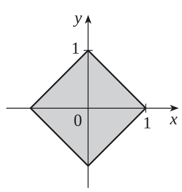
1. $a = 4\sqrt{h^2 - 16}/h$, where a is the length of the altitude and h is the length of the hypotenuse
3. $-\frac{7}{3}$, 9
5.



7.



9.



11. 5 **13.** $x \in \left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$
15. 40 mi/h **19.** $f_n(x) = x^{2^{n+1}}$

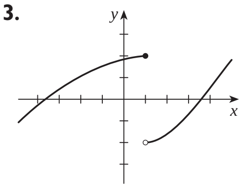
CHAPTER 2

EXERCISES 2.1 ■ PAGE 87

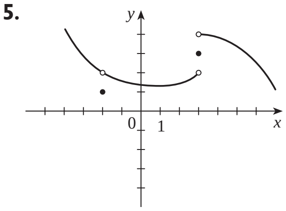
1. (a) -44.4, -38.8, -27.8, -22.2, $-16.\overline{6}$
(b) -33.3 (c) $-33\frac{1}{3}$
3. (a) (i) 0.333333 (ii) 0.263158 (iii) 0.251256
(iv) 0.250125 (v) 0.2 (vi) 0.238095 (vii) 0.248756
(viii) 0.249875 (b) $\frac{1}{4}$ (c) $y = \frac{1}{4}x + \frac{1}{4}$
5. (a) (i) -32 ft/s (ii) -25.6 ft/s (iii) -24.8 ft/s
(iv) -24.16 ft/s (b) -24 ft/s
7. (a) (i) 4.65 m/s (ii) 5.6 m/s (iii) 7.55 m/s
(iv) 7 m/s (b) 6.3 m/s
9. (a) 0, 1.7321, -1.0847, -2.7433, 4.3301, -2.8173, 0, -2.1651, -2.6061, -5, 3.4202; no (c) -31.4

EXERCISES 2.2 ■ PAGE 96

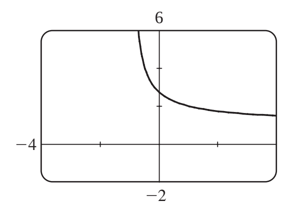
1. Yes
3. (a) $\lim_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
(b) $\lim_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.
5. (a) 2 (b) 3 (c) Does not exist (d) 4
(e) Does not exist
7. (a) -1 (b) -2 (c) Does not exist (d) 2 (e) 0
(f) Does not exist (g) 1 (h) 3
9. (a) $-\infty$ (b) ∞ (c) ∞ (d) $-\infty$ (e) ∞
(f) $x = -7, x = -3, x = 0, x = 6$
11. (a) 1 (b) 0 (c) Does not exist
13.



15.



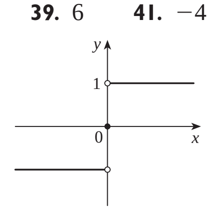
17. $\frac{2}{3}$ **19.** $\frac{1}{2}$ **21.** $\frac{1}{4}$ **23.** $\frac{3}{5}$ **25.** $-\infty$
27. ∞ **29.** $-\infty$ **31.** $-\infty$ **33.** $-\infty$; ∞
35. (a) 2.71828 (b)



37. (a) 0.998000, 0.638259, 0.358484, 0.158680, 0.038851, 0.008928, 0.001465; 0
(b) 0.000572, -0.000614, -0.000907, -0.000978, -0.000993, -0.001000; -0.001
39. No matter how many times we zoom in toward the origin, the graph appears to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.
41. $x \approx \pm 0.90, \pm 2.24$; $x = \pm\sin^{-1}(\pi/4), \pm(\pi - \sin^{-1}(\pi/4))$

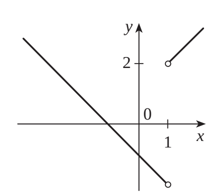
EXERCISES 2.3 ■ PAGE 106

1. (a) -6 (b) -8 (c) 2 (d) -6
(e) Does not exist (f) 0
3. 59 **5.** 390 **7.** $\frac{1}{8}$ **9.** 0 **11.** 5
13. Does not exist **15.** $\frac{6}{5}$ **17.** 8 **19.** $\frac{1}{12}$ **21.** 6
23. $\frac{1}{6}$ **25.** $-\frac{1}{16}$ **27.** $\frac{1}{128}$ **29.** $-\frac{1}{2}$ **31.** (a), (b) $\frac{2}{3}$
35. 7 **39.** 6 **41.** -4 **43.** Does not exist
45. (a)



(b) (i) 1
(ii) -1
(iii) Does not exist
(iv) 1

47. (a) (i) 2 (ii) -2 (b) No (c)



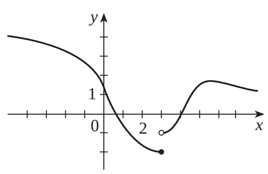
49. (a) (i) -2 (ii) Does not exist (iii) -3
(b) (i) $n-1$ (ii) n (c) a is not an integer.
55. 8 **61.** 15; -1

EXERCISES 2.4 ■ PAGE 117

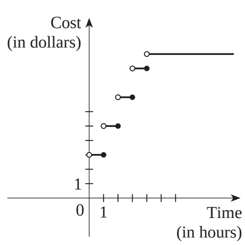
1. $\frac{4}{7}$ (or any smaller positive number)
3. 1.44 (or any smaller positive number)
5. 0.0906 (or any smaller positive number)
7. 0.11, 0.012 (or smaller positive numbers)
9. (a) 0.031 (b) 0.010
11. (a) $\sqrt{1000/\pi}$ cm (b) Within approximately 0.0445 cm
(c) Radius; area; $\sqrt{1000/\pi}$; 1000; 5; ≈ 0.0445
13. (a) 0.025 (b) 0.0025
35. (a) 0.093 (b) $\delta = (B^{2/3} - 12)/(6B^{1/3}) - 1$, where $B = 216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}$
41. Within 0.1

EXERCISES 2.5 ■ PAGE 128

1. $\lim_{x\to 4} f(x) = f(4)$
3. (a) -4 (removable), -2 (jump), 2 (jump), 4 (infinite)
(b) -4, neither; -2, left; 2, right; 4, right
5.

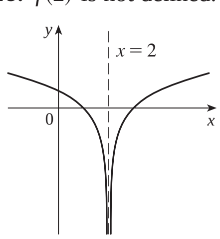


7. (a)

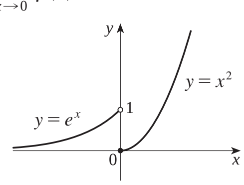


(b) Discontinuous at $t = 1, 2, 3, 4$

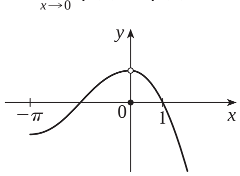
9. 6
15. $f(2)$ is not defined.



17. $\lim_{x\to 0} f(x)$ does not exist.



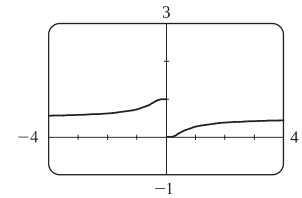
19. $\lim_{x\to 0} f(x) \neq f(0)$



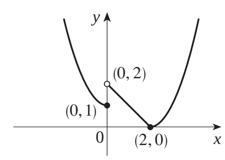
21. $\{x \mid x \neq -3, -2\}$

23. $[\frac{1}{2}, \infty)$ **25.** $(-\infty, \infty)$ **27.** $(-\infty, -1) \cup (1, \infty)$

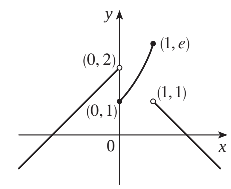
29. $x = 0$



31. $\frac{7}{3}$ **33.** 1
37. 0, left



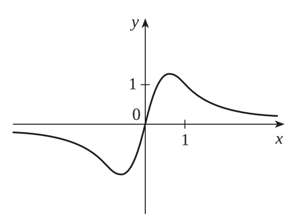
39. 0, right; 1, left



41. $\frac{2}{3}$ **43.** (a) $g(x) = x^3 + x^2 + x + 1$ (b) $g(x) = x^2 + x$
51. (b) (0.86, 0.87) **53.** (b) 70.347
59. None **61.** Yes

EXERCISES 2.6 ■ PAGE 140

1. (a) As x becomes large, $f(x)$ approaches 5.
(b) As x becomes large negative, $f(x)$ approaches 3.
3. (a) ∞ (b) ∞ (c) $-\infty$ (d) 1 (e) 2
(f) $x = -1, x = 2, y = 1, y = 2$
5.



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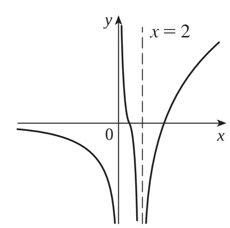
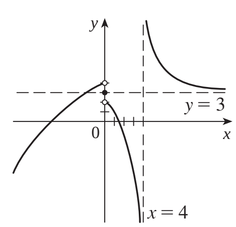
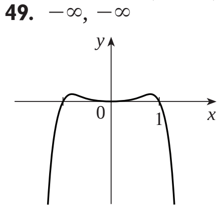


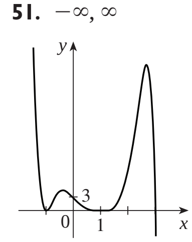
9.



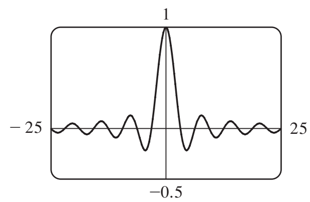
11. 0 **13.** $\frac{3}{2}$ **15.** 0 **17.** $-\frac{1}{2}$ **19.** $\frac{1}{2}$ **21.** 2
23. 3 **25.** $\frac{1}{6}$ **27.** $\frac{1}{2}(a - b)$ **29.** ∞ **31.** $-\infty$
33. $-\frac{1}{2}$ **35.** 0 **37.** (a), (b) $-\frac{1}{2}$ **39.** $y = 2; x = 2$
41. $y = 2; x = -2, x = 1$ **43.** $x = 5$ **45.** $y = 3$
47. $f(x) = \dfrac{2 - x}{x^2(x - 3)}$
49. $-\infty, -\infty$



51. $-\infty, \infty$

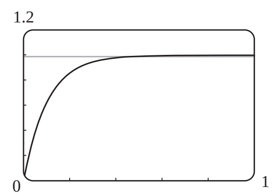


53. (a) 0 (b) An infinite number of times



55. (a) 0 (b) $\pm\infty$ **57.** 5

59. (a) v^* (b)



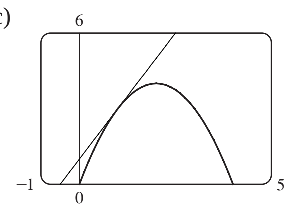
≈ 0.47 s

61. $N \geqslant 15$ **63.** $N \leqslant -6$, $N \leqslant -22$ **65.** (a) $x > 100$

EXERCISES 2.7 ■ PAGE 150

1. (a) $\dfrac{f(x) - f(3)}{x - 3}$ (b) $\displaystyle\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$

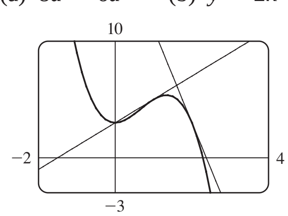
3. (a) 2 (b) $y = 2x + 1$ (c)



5. $y = -x + 5$ **7.** $y = \frac{1}{2}x + \frac{1}{2}$

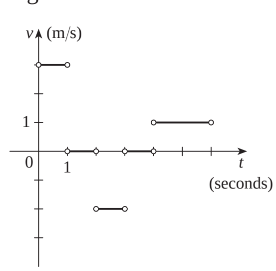
9. (a) $8a - 6a^2$ (b) $y = 2x + 3$, $y = -8x + 19$

(c)



11. (a) Right: $0 < t < 1$ and $4 < t < 6$; left: $2 < t < 3$; standing still: $1 < t < 2$ and $3 < t < 4$

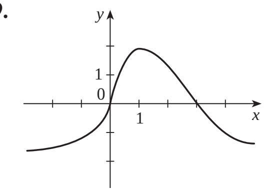
(b)



13. -24 ft/s **15.** $-2/a^3$ m/s; -2 m/s; $-\frac{1}{4}$ m/s; $-\frac{2}{27}$ m/s

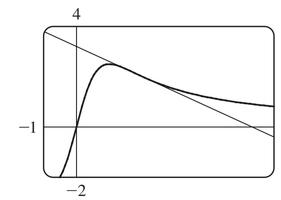
17. $g'(0)$, 0, $g'(4)$, $g'(2)$, $g'(-2)$

19.



21. 7; $y = 7x - 12$

23. (a) $-\frac{3}{5}$; $y = -\frac{3}{5}x + \frac{16}{5}$ (b)



25. $-2 + 8a$ **27.** $\dfrac{5}{(a + 3)^2}$ **29.** $\dfrac{-1}{2(a + 2)^{3/2}}$

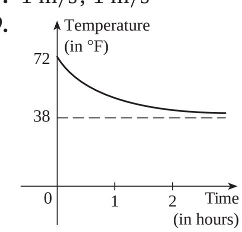
31. $f(x) = x^{10}$, $a = 1$ or $f(x) = (1 + x)^{10}$, $a = 0$

33. $f(x) = 2^x$, $a = 5$

35. $f(x) = \cos x$, $a = \pi$ or $f(x) = \cos(\pi + x)$, $a = 0$

37. 1 m/s; 1 m/s

39.



Greater (in magnitude)

41. (a) (i) 11 percent/year (ii) 13 percent/year
(iii) 16 percent/year
(b) 14.5 percent/year (c) 15 percent/year

43. (a) (i) \$20.25/unit (ii) \$20.05/unit (b) \$20/unit

45. (a) The rate at which the cost is changing per ounce of gold produced; dollars per ounce
(b) When the 800th ounce of gold is produced, the cost of production is \$17/oz.
(c) Decrease in the short term; increase in the long term

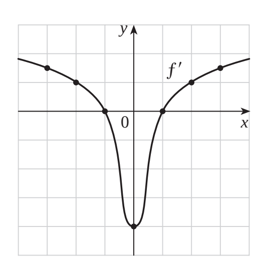
47. The rate at which the temperature is changing at 10:00 AM; 4°F/h

49. (a) The rate at which the oxygen solubility changes with respect to the water temperature; (mg/L)/°C
(b) $S'(16) \approx -0.25$; as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.

51. Does not exist

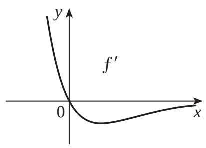
EXERCISES 2.8 ■ PAGE 162

1. (a) 1.5
(b) 1
(c) 0
(d) -4
(e) 0
(f) 1
(g) 1.5

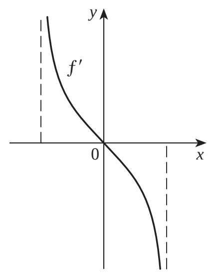


3. (a) II (b) IV (c) I (d) III

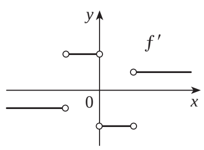
5.



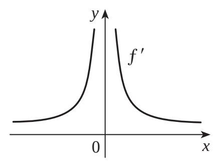
7.



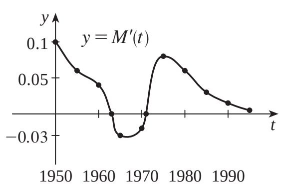
9.



11.

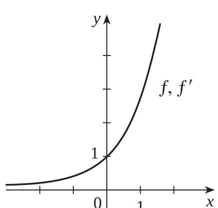


13.



1963 to 1971

15.



$f'(x) = e^x$

17. (a) 0, 1, 2, 4 (b) −1, −2, −4 (c) $f'(x) = 2x$

19. $f'(x) = \frac{1}{2}, \mathbb{R}, \mathbb{R}$ **21.** $f'(t) = 5 - 18t, \mathbb{R}, \mathbb{R}$

23. $f'(x) = 3x^2 - 3, \mathbb{R}, \mathbb{R}$

25. $g'(x) = 1/\sqrt{1 + 2x}, \left[-\frac{1}{2}, \infty\right), \left(-\frac{1}{2}, \infty\right)$

27. $G'(t) = \dfrac{4}{(t+1)^2}, (-\infty, -1) \cup (-1, \infty), (-\infty, -1) \cup (-1, \infty)$

29. $f'(x) = 4x^3, \mathbb{R}, \mathbb{R}$ **31.** (a) $f'(x) = 4x^3 + 2$

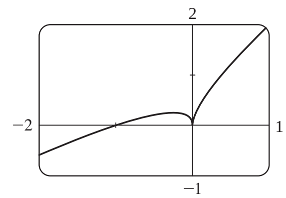
33. (a) The rate at which the unemployment rate is changing, in percent unemployed per year
(b)

t	$U'(t)$	t	$U'(t)$
1993	−0.80	1998	−0.35
1994	−0.65	1999	−0.25
1995	−0.35	2000	0.25
1996	−0.35	2001	0.90
1997	−0.45	2002	1.10

35. −4 (corner); 0 (discontinuity)

37. −1 (vertical tangent); 4 (corner)

39.

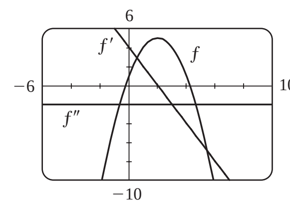


Differentiable at −1; not differentiable at 0

41. $a = f, b = f', c = f''$

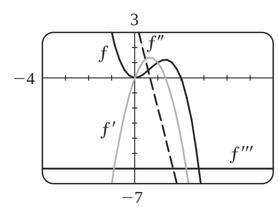
43. a = acceleration, b = velocity, c = position

45.



$f'(x) = 4 - 2x$,
$f''(x) = -2$

47.

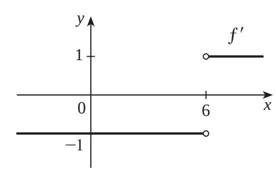


$f'(x) = 4x - 3x^2$,
$f''(x) = 4 - 6x$,
$f'''(x) = -6$,
$f^{(4)}(x) = 0$

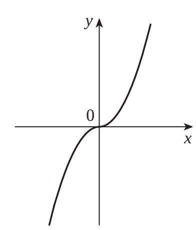
49. (a) $\frac{1}{3}a^{-2/3}$

51. $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

or $f'(x) = \dfrac{x - 6}{|x - 6|}$



53. (a)



(b) All x
(c) $f'(x) = 2|x|$

57. 63°

CHAPTER 2 REVIEW ■ PAGE 166

True-False Quiz

1. False **3.** True **5.** False **7.** True **9.** True
11. False **13.** True **15.** True **17.** False **19.** False

Exercises

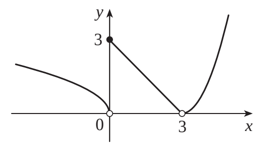
1. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2
(v) ∞ (vi) −∞ (vii) 4 (viii) −1
(b) $y = 4, y = -1$ (c) $x = 0, x = 2$ (d) −3, 0, 2, 4

3. 1 **5.** $\frac{3}{2}$ **7.** 3 **9.** ∞ **11.** $\frac{4}{7}$ **13.** $\frac{1}{2}$

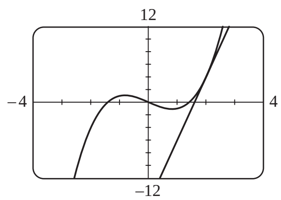
15. −∞ **17.** 2 **19.** $\pi/2$ **21.** $x = 0, y = 0$ **23.** 1

29. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 0 (v) 0 (vi) 0

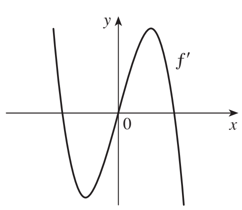
(b) At 0 and 3 (c)



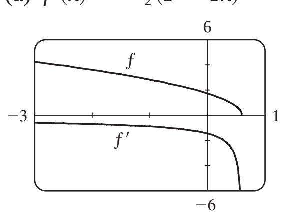
31. $\mathbb{R}$ **35.** (a) -8 (b) $y = -8x + 17$
37. (a) (i) 3 m/s (ii) 2.75 m/s (iii) 2.625 m/s
(iv) 2.525 m/s (b) 2.5 m/s
39. (a) 10 (b) $y = 10x - 16$
(c)



41. (a) The rate at which the cost changes with respect to the interest rate; dollars/(percent per year)
(b) As the interest rate increases past 10%, the cost is increasing at a rate of \$1200/(percent per year).
(c) Always positive
43.



45. (a) $f'(x) = -\frac{5}{2}(3 - 5x)^{-1/2}$ (b) $\left(-\infty, \frac{3}{5}\right], \left(-\infty, \frac{3}{5}\right)$
(c)



47. -4 (discontinuity), -1 (corner), 2 (discontinuity), 5 (vertical tangent)
49. The rate at which the total value of US currency in circulation is changing in billions of dollars per year; \$22.2 billion/year
51. 0

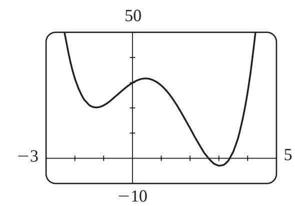
PROBLEMS PLUS ■ PAGE 170

1. $\frac{2}{3}$ **3.** -4 **5.** 1 **7.** $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$
9. $\frac{3}{4}$ **11.** (b) Yes (c) Yes; no
13. (a) 0 (b) 1 (c) $f'(x) = x^2 + 1$

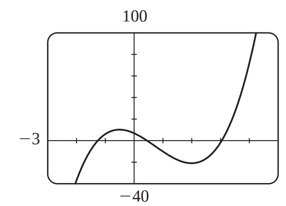
CHAPTER 3

EXERCISES 3.1 ■ PAGE 180

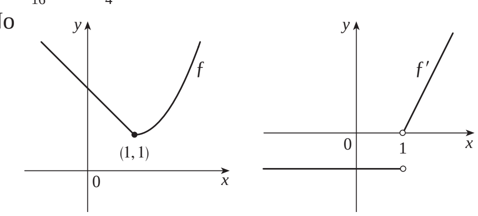
1. (a) See Definition of the Number e (page 179).
(b) 0.99, 1.03; $2.7 < e < 2.8$
3. $f'(x) = 0$ **5.** $f'(t) = -\frac{2}{3}$ **7.** $f'(x) = 3x^2 - 4$
9. $f'(t) = t^3$ **11.** $y' = -\frac{2}{5}x^{-7/5}$ **13.** $V'(r) = 4\pi r^2$
15. $A'(s) = 60/s^6$ **17.** $G'(x) = 1/\left(2\sqrt{x}\right) - 2e^x$
19. $F'(x) = \frac{5}{32}x^4$ **21.** $y' = 2ax + b$
23. $y' = \frac{3}{2}\sqrt{x} + \left(2/\sqrt{x}\right) - 3/\left(2x\sqrt{x}\right)$
25. $y' = 0$ **27.** $H'(x) = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$
29. $u' = \frac{1}{5}t^{-4/5} + 10t^{3/2}$ **31.** $z' = -10A/y^{11} + Be^y$
33. $y = \frac{1}{4}x + \frac{3}{4}$
35. Tangent: $y = 2x + 2$; normal: $y = -\frac{1}{2}x + 2$
37. $y = 3x - 1$ **39.** $e^x - 5$ **41.** $45x^{14} - 15x^2$
43. (a)



(c) $4x^3 - 9x^2 - 12x + 7$



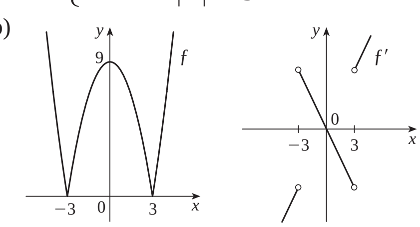
45. $f'(x) = 4x^3 - 9x^2 + 16$, $f''(x) = 12x^2 - 18x$
47. $f'(x) = 2 - \frac{15}{4}x^{-1/4}$, $f''(x) = \frac{15}{16}x^{-5/4}$
49. (a) $v(t) = 3t^2 - 3$, $a(t) = 6t$ (b) $12\ \mathrm{m/s^2}$
(c) $a(1) = 6\ \mathrm{m/s^2}$ **51.** $(-2, 21)$, $(1, -6)$
55. $y = 12x - 15$, $y = 12x + 17$ **57.** $y = \frac{1}{3}x - \frac{1}{3}$
59. $(\pm 2, 4)$ **63.** $P(x) = x^2 - x + 3$
65. $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$
67. No



69. (a) Not differentiable at 3 or -3

$$f'(x) = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

(b)



71. $y = 2x^2 - x$ **73.** $a = -\frac{1}{2}$, $b = 2$ **75.** $m = 4$, $b = -4$
77. 1000 **79.** 3; 1

EXERCISES 3.2 ■ PAGE 187

1. $y' = 5x^4 + 3x^2 + 2x$
3. $f'(x) = e^x(x^3 + 3x^2 + 2x + 2)$
5. $y' = (x - 2)e^x/x^3$ **7.** $g'(x) = 5/(2x + 1)^2$
9. $V'(x) = 14x^6 - 4x^3 - 6$
11. $F'(y) = 5 + 14/y^2 + 9/y^4$
13. $y' = \dfrac{x^2(3 - x^2)}{(1 - x^2)^2}$ **15.** $y' = \dfrac{2t(-t^4 - 4t^2 + 7)}{(t^4 - 3t^2 + 1)^2}$

17. $y' = (r^2 - 2)e^r$ **19.** $y' = 2v - 1/\sqrt{v}$

21. $f'(t) = \dfrac{4 + t^{1/2}}{(2 + \sqrt{t})^2}$ **23.** $f'(x) = -ACe^x/(B + Ce^x)^2$

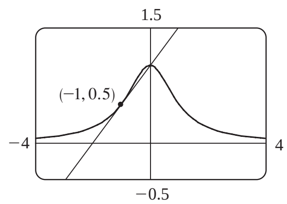
25. $f'(x) = 2cx/(x^2 + c)^2$

27. $(x^4 + 4x^3)e^x$; $(x^4 + 8x^3 + 12x^2)e^x$

29. $\dfrac{2x^2 + 2x}{(1 + 2x)^2}$; $\dfrac{2}{(1 + 2x)^3}$

31. $y = \frac{1}{2}x + \frac{1}{2}$ **33.** $y = 2x$; $y = -\frac{1}{2}x$

35. (a) $y = \frac{1}{2}x + 1$ (b)



37. (a) $e^x(x - 3)/x^4$ **39.** xe^x, $(x + 1)e^x$

41. $\frac{1}{4}$ **43.** (a) -16 (b) $-\frac{20}{9}$ (c) 20

45. 7 **47.** (a) 0 (b) $-\frac{2}{3}$

49. (a) $y' = xg'(x) + g(x)$ (b) $y' = [g(x) - xg'(x)]/[g(x)]^2$
(c) $y' = [xg'(x) - g(x)]/x^2$

51. Two, $\left(-2 \pm \sqrt{3}, \left(1 \mp \sqrt{3}\right)/2\right)$

53. \$1.627 billion/year **55.** (c) $3e^{3x}$

57. $f'(x) = (x^2 + 2x)e^x$, $f''(x) = (x^2 + 4x + 2)e^x$,
$f'''(x) = (x^2 + 6x + 6)e^x$, $f^{(4)}(x) = (x^2 + 8x + 12)e^x$,
$f^{(5)}(x) = (x^2 + 10x + 20)e^x$; $f^{(n)}(x) = [x^2 + 2nx + n(n - 1)]e^x$

EXERCISES 3.3 ■ PAGE 195

1. $f'(x) = 6x + 2\sin x$ **3.** $f'(x) = \cos x - \frac{1}{2}\csc^2 x$

5. $g'(t) = 3t^2 \cos t - t^3 \sin t$

7. $h'(\theta) = -\csc\theta\cot\theta + e^\theta(\cot\theta - \csc^2\theta)$

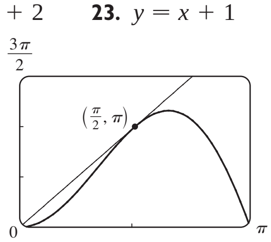
9. $y' = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$ **11.** $f'(\theta) = \dfrac{\sec\theta\tan\theta}{(1 + \sec\theta)^2}$

13. $y' = (x\cos x - 2\sin x)/x^3$

15. $f'(x) = e^x \csc x\,(-x\cot x + x + 1)$

21. $y = 2\sqrt{3}x - \frac{2}{3}\sqrt{3}\pi + 2$ **23.** $y = x + 1$

25. (a) $y = 2x$ (b)



27. (a) $\sec x\tan x - 1$

29. $\theta\cos\theta + \sin\theta$; $2\cos\theta - \theta\sin\theta$

31. (a) $f'(x) = (1 + \tan x)/\sec x$ (b) $f'(x) = \cos x + \sin x$

33. $(2n + 1)\pi \pm \frac{1}{3}\pi$, n an integer

35. (a) $v(t) = 8\cos t$, $a(t) = -8\sin t$
(b) $4\sqrt{3}$, -4, $-4\sqrt{3}$; to the left

37. 5 ft/rad **39.** 3 **41.** 3 **43.** sin 1

45. $\frac{1}{2}$ **47.** $-\sqrt{2}$

49. (a) $\sec^2 x = 1/\cos^2 x$ (b) $\sec x\tan x = (\sin x)/\cos^2 x$
(c) $\cos x - \sin x = (\cot x - 1)/\csc x$

51. 1

EXERCISES 3.4 ■ PAGE 203

1. $4\cos 4x$ **3.** $-20x(1 - x^2)^9$ **5.** $e^{\sqrt{x}}/\left(2\sqrt{x}\right)$

7. $F'(x) = 10x(x^4 + 3x^2 - 2)^4(2x^2 + 3)$

9. $F'(x) = \dfrac{2 + 3x^2}{4(1 + 2x + x^3)^{3/4}}$ **11.** $g'(t) = -\dfrac{12t^3}{(t^4 + 1)^4}$

13. $y' = -3x^2\sin(a^3 + x^3)$ **15.** $y' = e^{-kx}(-kx + 1)$

17. $g'(x) = 4(1 + 4x)^4(3 + x - x^2)^7(17 + 9x - 21x^2)$

19. $y' = 8(2x - 5)^3(8x^2 - 5)^{-4}(-4x^2 + 30x - 5)$

21. $y' = \dfrac{-12x(x^2 + 1)^2}{(x^2 - 1)^4}$ **23.** $y' = (\cos x - x\sin x)e^{x\cos x}$

25. $F'(z) = 1/[(z - 1)^{1/2}(z + 1)^{3/2}]$

27. $y' = (r^2 + 1)^{-3/2}$ **29.** $y' = 2\cos(\tan 2x)\sec^2(2x)$

31. $y' = 2^{\sin\pi x}(\pi\ln 2)\cos\pi x$ **33.** $y' = 4\sec^2 x\tan x$

35. $y' = \dfrac{4e^{2x}}{(1 + e^{2x})^2}\sin\dfrac{1 - e^{2x}}{1 + e^{2x}}$

37. $y' = -2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$

39. $f'(t) = \sec^2(e^t)e^t + e^{\tan t}\sec^2 t$

41. $f'(t) = 4\sin(e^{\sin^2 t})\cos(e^{\sin^2 t})\,e^{\sin^2 t}\sin t\cos t$

43. $g'(x) = 2r^2p(\ln a)(2ra^{rx} + n)^{p-1}a^{rx}$

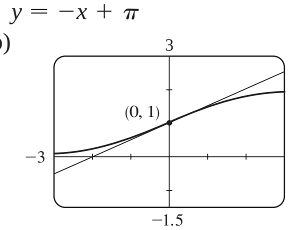
45. $y' = \dfrac{-\pi\cos(\tan\pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}$

47. $h'(x) = x/\sqrt{x^2 + 1}$, $h''(x) = 1/(x^2 + 1)^{3/2}$

49. $e^{\alpha x}(\beta\cos\beta x + \alpha\sin\beta x)$; $e^{\alpha x}[(\alpha^2 - \beta^2)\sin\beta x + 2\alpha\beta\cos\beta x]$

51. $y = 20x + 1$ **53.** $y = -x + \pi$

55. (a) $y = \frac{1}{2}x + 1$ (b)



57. (a) $f'(x) = (2 - 2x^2)/\sqrt{2 - x^2}$

59. $((\pi/2) + 2n\pi, 3)$, $((3\pi/2) + 2n\pi, -1)$, n an integer

61. 24 **63.** (a) 30 (b) 36

65. (a) $\frac{3}{4}$ (b) Does not exist (c) -2

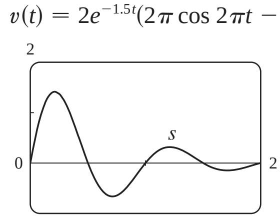
67. (a) $F'(x) = e^x f'(e^x)$ (b) $G'(x) = e^{f(x)}f'(x)$

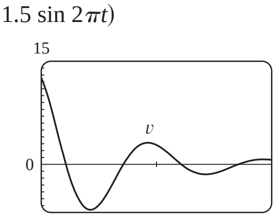
69. 120 **71.** 96 **75.** $-2^{50}\cos 2x$

77. $v(t) = \frac{5}{2}\pi\cos(10\pi t)$ cm/s

79. (a) $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$ (b) 0.16

81. $v(t) = 2e^{-1.5t}(2\pi\cos 2\pi t - 1.5\sin 2\pi t)$

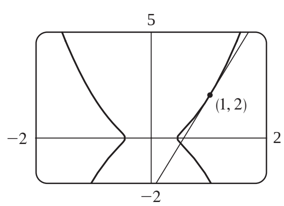




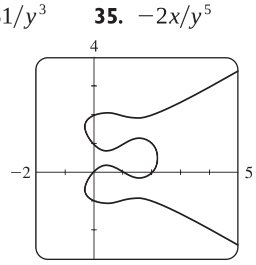
83. dv/dt is the rate of change of velocity with respect to time; dv/ds is the rate of change of velocity with respect to displacement
85. (a) $y = ab^t$ where $a \approx 100.01244$ and $b \approx 0.000045146$
(b) $-670.63\ \mu\text{A}$
87. (b) The factored form
91. (b) $-n \cos^{n-1}x \sin[(n+1)x]$

EXERCISES 3.5 ▪ PAGE 213

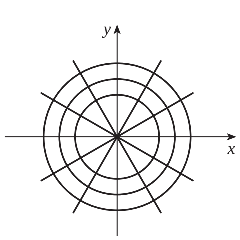
1. (a) $y' = -(y + 2 + 6x)/x$
(b) $y = (4/x) - 2 - 3x$, $y' = -(4/x^2) - 3$
3. (a) $y' = -y^2/x^2$ (b) $y = x/(x-1)$, $y' = -1/(x-1)^2$
5. $y' = -x^2/y^2$
7. $y' = \dfrac{2x + y}{2y - x}$ **9.** $y' = \dfrac{3y^2 - 5x^4 - 4x^3y}{x^4 + 3y^2 - 6xy}$
11. $y' = \dfrac{-2xy^2 - \sin y}{2x^2y + x \cos y}$ **13.** $y' = \tan x \tan y$
15. $y' = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$ **17.** $y' = \dfrac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$
19. $y' = \dfrac{e^y \sin x + y \cos(xy)}{e^y \cos x - x \cos(xy)}$ **21.** $-\frac{16}{13}$
23. $x' = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$ **25.** $y = -x + 2$
27. $y = x + \frac{1}{2}$ **29.** $y = -\frac{9}{13}x + \frac{40}{13}$
31. (a) $y = \frac{9}{2}x - \frac{5}{2}$ (b)



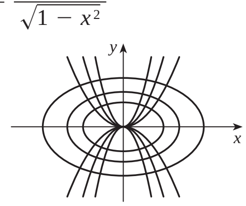
33. $-81/y^3$ **35.** $-2x/y^5$
37. (a)



Eight; $x \approx 0.42, 1.58$
(b) $y = -x + 1$, $y = \frac{1}{3}x + 2$ (c) $1 \mp \frac{1}{3}\sqrt{3}$
39. $\left(\pm\frac{5}{4}\sqrt{3}, \pm\frac{5}{4}\right)$ **41.** $(x_0x/a^2) - (y_0y/b^2) = 1$
45. $y' = \dfrac{1}{2\sqrt{x}(1+x)}$ **47.** $y' = \dfrac{1}{\sqrt{-x^2 - x}}$
49. $G'(x) = -1 - \dfrac{x \arccos x}{\sqrt{1 - x^2}}$ **51.** $h'(t) = 0$
53. $y' = -2e^{2x}/\sqrt{1 - e^{4x}}$ **55.** $1 - \dfrac{x \arcsin x}{\sqrt{1 - x^2}}$
59.



61.



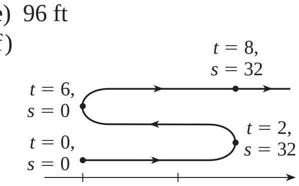
63. $\left(\pm\sqrt{3}, 0\right)$ **65.** $(-1, -1)$, $(1, 1)$ **67.** (b) $\frac{3}{2}$ **69.** 2

EXERCISES 3.6 ▪ PAGE 220

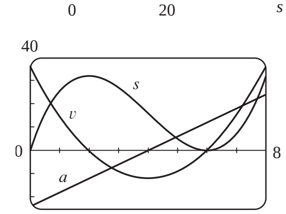
1. The differentiation formula is simplest.
3. $f'(x) = \dfrac{\cos(\ln x)}{x}$ **5.** $f'(x) = \dfrac{3}{(3x-1)\ln 2}$
7. $f'(x) = \dfrac{1}{5x\sqrt[5]{(\ln x)^4}}$ **9.** $f'(x) = \dfrac{\sin x}{x} + \cos x \ln(5x)$
11. $F'(t) = \dfrac{6}{2t+1} - \dfrac{12}{3t-1}$ **13.** $g'(x) = \dfrac{2x^2 - 1}{x(x^2 - 1)}$
15. $f'(u) = \dfrac{1 + \ln 2}{u[1 + \ln(2u)]^2}$ **17.** $y' = \dfrac{10x + 1}{5x^2 + x - 2}$
19. $y' = \dfrac{-x}{1 + x}$ **21.** $y' = \dfrac{1}{\ln 10} + \log_{10} x$
23. $y' = x + 2x\ln(2x)$; $y'' = 3 + 2\ln(2x)$
25. $y' = \dfrac{1}{\sqrt{1 + x^2}}$; $y'' = \dfrac{-x}{(1 + x^2)^{3/2}}$
27. $f'(x) = \dfrac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1 - \ln(x-1)]^2}$;
$(1, 1+e) \cup (1+e, \infty)$
29. $f'(x) = \dfrac{2(x-1)}{x(x-2)}$; $(-\infty, 0) \cup (2, \infty)$
31. 1 **33.** $y = 3x - 2$ **35.** $\cos x + 1/x$
37. $y' = (2x+1)^5(x^4 - 3)^6\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4 - 3}\right)$
39. $y' = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$
41. $y' = x^x(1 + \ln x)$
43. $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)$
45. $y' = (\cos x)^x(-x\tan x + \ln\cos x)$
47. $y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right)$
49. $y' = \dfrac{2x}{x^2 + y^2 - 2y}$ **51.** $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(x-1)^n}$

EXERCISES 3.7 ▪ PAGE 230

1. (a) $3t^2 - 24t + 36$ (b) -9 ft/s (c) $t = 2, 6$
(d) $0 \leqslant t < 2$, $t > 6$
(e) 96 ft
(f)

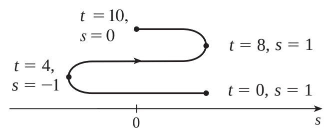


(g) $6t - 24$; $-6\ \text{m/s}^2$
(h)

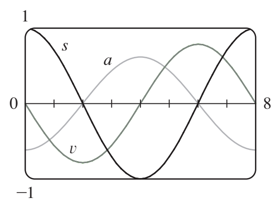


(i) Speeding up when $2 < t < 4$ or $t > 6$; slowing down when $0 \leqslant t < 2$ or $4 < t < 6$

3. (a) $-\frac{\pi}{4}\sin\left(\frac{\pi t}{4}\right)$ (b) $-\frac{1}{8}\pi\sqrt{2}$ ft/s (c) $t = 0, 4, 8$
(d) $4 < t < 8$ (e) 4 ft
(f)



(g) $-\frac{1}{16}\pi^2\cos(\pi t/4)$; $\frac{1}{32}\pi^2\sqrt{2}$ ft/s^2
(h)



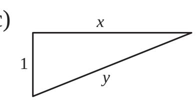
(i) Speeding up when $0 < t < 2$, $4 < t < 6$;
slowing down when $2 < t < 4$, $6 < t < 8$
5. (a) Speeding up when $0 < t < 1$ or $2 < t < 3$;
slowing down when $1 < t < 2$
(b) Speeding up when $1 < t < 2$ or $3 < t < 4$;
slowing down when $0 < t < 1$ or $2 < t < 3$
7. (a) $t = 4$ s
(b) $t = 1.5$ s; the velocity has an absolute minimum.
9. (a) 5.02 m/s (b) $\sqrt{17}$ m/s
11. (a) 30 mm^2/mm; the rate at which the area is increasing with respect to side length as x reaches 15 mm
(b) $\Delta A \approx 2x\,\Delta x$
13. (a) (i) 5π (ii) 4.5π (iii) 4.1π
(b) 4π (c) $\Delta A \approx 2\pi r\,\Delta r$
15. (a) 8π ft^2/ft (b) 16π ft^2/ft (c) 24π ft^2/ft
The rate increases as the radius increases.
17. (a) 6 kg/m (b) 12 kg/m (c) 18 kg/m
At the right end; at the left end
19. (a) 4.75 A (b) 5 A; $t = \frac{2}{3}$ s
21. (a) $dV/dP = -C/P^2$ (b) At the beginning
23. $400(3^t)\ln 3$; ≈ 6850 bacteria/h
25. (a) 16 million/year; 78.5 million/year
(b) $P(t) = at^3 + bt^2 + ct + d$, where $a \approx 0.00129371$, $b \approx -7.061422$, $c \approx 12{,}822.979$, $d \approx -7{,}743{,}770$
(c) $P'(t) = 3at^2 + 2bt + c$
(d) 14.48 million/year; 75.29 million/year (smaller)
(e) 81.62 million/year
27. (a) 0.926 cm/s; 0.694 cm/s; 0
(b) 0; -92.6 (cm/s)/cm; -185.2 (cm/s)/cm
(c) At the center; at the edge
29. (a) $C'(x) = 12 - 0.2x + 0.0015x^2$
(b) \$32/yard; the cost of producing the 201st yard
(c) \$32.20
31. (a) $[xp'(x) - p(x)]/x^2$; the average productivity increases as new workers are added.
33. -0.2436 K/min
35. (a) 0 and 0 (b) $C = 0$
(c) (0, 0), (500, 50); it is possible for the species to coexist.

EXERCISES 3.8 ■ PAGE 239

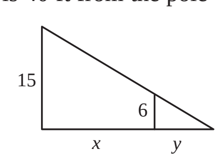
1. About 235
3. (a) $100(4.2)^t$ (b) ≈ 7409 (c) $\approx 10{,}632$ bacteria/h
(d) $(\ln 100)/(\ln 4.2) \approx 3.2$ h
5. (a) 1508 million, 1871 million (b) 2161 million
(c) 3972 million; wars in the first half of century, increased life expectancy in second half
7. (a) $Ce^{-0.0005t}$ (b) $-2000\ln 0.9 \approx 211$ s
9. (a) $100 \times 2^{-t/30}$ mg (b) ≈ 9.92 mg (c) ≈ 199.3 years
11. ≈ 2500 years **13.** (a) $\approx 137°$F (b) ≈ 116 min
15. (a) $13.\overline{3}$°C (b) ≈ 67.74 min
17. (a) ≈ 64.5 kPa (b) ≈ 39.9 kPa
19. (a) (i) \$3828.84 (ii) \$3840.25 (iii) \$3850.08
(iv) \$3851.61 (v) \$3852.01 (vi) \$3852.08
(b) $dA/dt = 0.05A$, $A(0) = 3000$

EXERCISES 3.9 ■ PAGE 245

1. $dV/dt = 3x^2\,dx/dt$ **3.** 48 cm^2/s **5.** $3/(25\pi)$ m/min
7. 70 **9.** $\pm\frac{46}{13}$
11. (a) The plane's altitude is 1 mi and its speed is 500 mi/h.
(b) The rate at which the distance from the plane to the station is increasing when the plane is 2 mi from the station
(c)



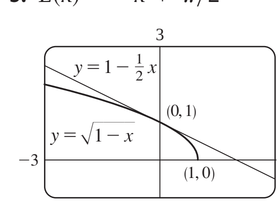
(d) $y^2 = x^2 + 1$
(e) $250\sqrt{3}$ mi/h
13. (a) The height of the pole (15 ft), the height of the man (6 ft), and the speed of the man (5 ft/s)
(b) The rate at which the tip of the man's shadow is moving when he is 40 ft from the pole
(c)



(d) $\frac{15}{6} = \frac{x+y}{y}$ (e) $\frac{25}{3}$ ft/s
15. 65 mi/h **17.** $837/\sqrt{8674} \approx 8.99$ ft/s
19. -1.6 cm/min **21.** $\frac{720}{13} \approx 55.4$ km/h
23. $(10{,}000 + 800{,}000\pi/9) \approx 2.89 \times 10^5$ cm^3/min
25. $\frac{10}{3}$ cm/min **27.** $6/(5\pi) \approx 0.38$ ft/min **29.** 0.3 m^2/s
31. 80 cm^3/min **33.** $\frac{107}{810} \approx 0.132\ \Omega$/s **35.** 0.396 m/min
37. (a) 360 ft/s (b) 0.096 rad/s **39.** $\frac{10}{9}\pi$ km/min
41. $1650/\sqrt{31} \approx 296$ km/h **43.** $\frac{7}{4}\sqrt{15} \approx 6.78$ m/s

EXERCISES 3.10 ■ PAGE 252

1. $L(x) = -10x - 6$ **3.** $L(x) = -x + \pi/2$
5. $\sqrt{1-x} \approx 1 - \frac{1}{2}x$;
$\sqrt{0.9} \approx 0.95$,
$\sqrt{0.99} \approx 0.995$



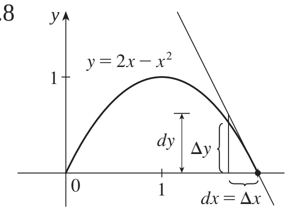
7. $-1.204 < x < 0.706$ **9.** $-0.045 < x < 0.055$
11. (a) $dy = 2x(x\cos 2x + \sin 2x)\,dx$ (b) $dy = \frac{t}{1+t^2}\,dt$

13. (a) $dy = \dfrac{-2}{(u-1)^2}\,du$ (b) $dy = -\dfrac{6r^2}{(1+r^3)^3}\,dr$

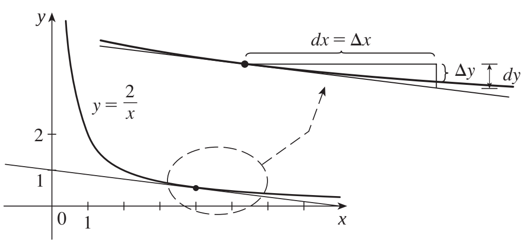
15. (a) $dy = \frac{1}{10}\,e^{x/10}\,dx$ (b) 0.01; 0.0101

17. (a) $dy = \sec^2 x\,dx$ (b) -0.2

19. $\Delta y = 0.64$, $dy = 0.8$



21. $\Delta y = -0.1$, $dy = -0.125$



23. 32.08 **25.** 4.02 **27.** $1 - \pi/90 \approx 0.965$

33. (a) 270 cm^3, 0.01, 1% (b) 36 cm^2, $0.00\overline{6}$, $0.\overline{6}\%$

35. (a) $84/\pi \approx 27\text{ cm}^2$; $\frac{1}{84} \approx 0.012$
(b) $1764/\pi^2 \approx 179\text{ cm}^3$; $\frac{1}{56} \approx 0.018$

37. (a) $2\pi rh\,\Delta r$ (b) $\pi(\Delta r)^2 h$

43. (a) 4.8, 5.2 (b) Too large

EXERCISES 3.11 ■ PAGE 259

1. (a) 0 (b) 1 **3.** (a) $\frac{3}{4}$ (b) $\frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

5. (a) 1 (b) 0

21. $\operatorname{sech} x = \frac{3}{5}$, $\sinh x = \frac{4}{3}$, $\operatorname{csch} x = \frac{3}{4}$, $\tanh x = \frac{4}{5}$, $\coth x = \frac{5}{4}$

23. (a) 1 (b) -1 (c) ∞ (d) $-\infty$ (e) 0 (f) 1
(g) ∞ (h) $-\infty$ (i) 0

31. $f'(x) = x\cosh x$ **33.** $h'(x) = \tanh x$

35. $y' = 3e^{\cosh 3x}\sinh 3x$ **37.** $f'(t) = -2e^t\operatorname{sech}^2(e^t)\tanh(e^t)$

39. $y' = \dfrac{\operatorname{sech}^2 x}{1 + \tanh^2 x}$ **41.** $G'(x) = \dfrac{-2\sinh x}{(1+\cosh x)^2}$

43. $y' = \dfrac{1}{2\sqrt{x}\,(1-x)}$ **45.** $y' = \sinh^{-1}(x/3)$

47. $y' = \dfrac{-1}{x\sqrt{x^2+1}}$

51. (a) 0.3572 (b) 70.34°

53. (b) $y = 2\sinh 3x - 4\cosh 3x$

55. $\left(\ln\left(1+\sqrt{2}\right), \sqrt{2}\right)$

CHAPTER 3 REVIEW ■ PAGE 261

True-False Quiz

1. True **3.** True **5.** False **7.** False **9.** True
11. True

Exercises

1. $6x(x^4 - 3x^2 + 5)^2(2x^2 - 3)$ **3.** $\dfrac{1}{2\sqrt{x}} - \dfrac{4}{3\sqrt[3]{x^7}}$

5. $\dfrac{2(2x^2+1)}{\sqrt{x^2+1}}$ **7.** $2\cos 2\theta\, e^{\sin 2\theta}$

9. $\dfrac{t^2+1}{(1-t^2)^2}$ **11.** $\dfrac{\cos\sqrt{x} - \sqrt{x}\sin\sqrt{x}}{2\sqrt{x}}$

13. $-\dfrac{e^{1/x}(1+2x)}{x^4}$ **15.** $\dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

17. $\dfrac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1+\tan 2\theta)^2}$ **19.** $(1+c^2)e^{cx}\sin x$

21. $3^{x\ln x}(\ln 3)(1 + \ln x)$ **23.** $-(x-1)^{-2}$

25. $\dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$ **27.** $\dfrac{2}{(1+2x)\ln 5}$

29. $\cot x - \sin x\cos x$ **31.** $\dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$

33. $5\sec 5x$ **35.** $-6x\csc^2(3x^2+5)$

37. $\cos\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\dfrac{3x^2}{2\sqrt{1+x^3}}$

39. $2\cos\theta\tan(\sin\theta)\sec^2(\sin\theta)$

41. $\dfrac{(x-2)^4(3x^2-55x-52)}{2\sqrt{x+1}\,(x+3)^8}$ **43.** $2x^2\cosh(x^2) + \sinh(x^2)$

45. $3\tanh 3x$ **47.** $\dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

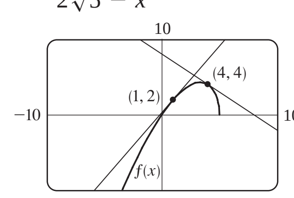
49. $\dfrac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)e^{\sqrt{\tan 3x}}\sec^2(3x)}{2\sqrt{\tan 3x}}$ **51.** $-\frac{4}{27}$ **53.** $-5x^4/y^{11}$

57. $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$ **59.** $y = 2x + 1$

61. $y = -x + 2$; $y = x + 2$

63. (a) $\dfrac{10-3x}{2\sqrt{5-x}}$ (b) $y = \frac{7}{4}x + \frac{1}{4}$, $y = -x + 8$

(c)



65. $\left(\pi/4, \sqrt{2}\right)$, $\left(5\pi/4, -\sqrt{2}\right)$ **69.** (a) 2 (b) 44

71. $2xg(x) + x^2g'(x)$ **73.** $2g(x)g'(x)$ **75.** $g'(e^x)e^x$

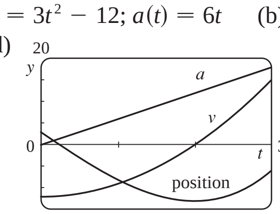
77. $g'(x)/g(x)$ **79.** $\dfrac{f'(x)[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2}$

81. $f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$ **83.** $(-3, 0)$

85. $y = -\frac{2}{3}x^2 + \frac{14}{3}x$

87. $v(t) = -Ae^{-ct}[c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)]$,
$a(t) = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)]$

89. (a) $v(t) = 3t^2 - 12$; $a(t) = 6t$ (b) $t > 2$; $0 \leqslant t < 2$
(c) 23 (d)



(e) $t > 2$; $0 < t < 2$

91. 4 kg/m **93.** (a) $200(3.24)^t$ (b) $\approx 22{,}040$
(c) $\approx 25{,}910$ bacteria/h (d) $(\ln 50)/(\ln 3.24) \approx 3.33$ h
95. (a) $C_0 e^{-kt}$ (b) ≈ 100 h **97.** $\frac{4}{3}$ cm²/min
99. 13 ft/s **101.** 400 ft/h
103. (a) $L(x) = 1 + x$; $\sqrt[3]{1 + 3x} \approx 1 + x$; $\sqrt[3]{1.03} \approx 1.01$
(b) $-0.23 < x < 0.40$
105. $12 + \frac{3}{2}\pi \approx 16.7$ cm² **107.** $\frac{1}{32}$ **109.** $\frac{1}{4}$ **111.** $\frac{1}{8}x^2$

PROBLEMS PLUS ▪ PAGE 266

1. $\left(\pm\frac{1}{2}\sqrt{3}, \frac{1}{4}\right)$ **9.** $\left(0, \frac{5}{4}\right)$
11. (a) $4\pi\sqrt{3}/\sqrt{11}$ rad/s (b) $40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm
(c) $-480\pi \sin\theta \left(1 + \cos\theta/\sqrt{8 + \cos^2\theta}\right)$ cm/s
15. $x_T \in (3, \infty)$, $y_T \in (2, \infty)$, $x_N \in \left(0, \frac{5}{3}\right)$, $y_N \in \left(-\frac{5}{2}, 0\right)$
17. (b) (i) 53° (or 127°) (ii) 63° (or 117°)
19. R approaches the midpoint of the radius AO.
21. $-\sin a$ **23.** $2\sqrt{e}$ **27.** $(1, -2), (-1, 0)$
29. $\sqrt{29}/58$ **31.** $2 + \frac{375}{128}\pi \approx 11.204$ cm³/min

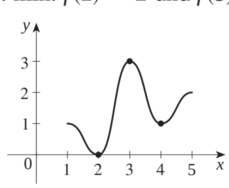
CHAPTER 4

EXERCISES 4.1 ▪ PAGE 277

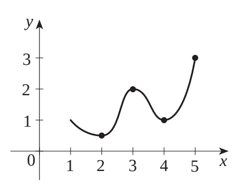
Abbreviations: abs., absolute; loc., local; max., maximum; min., minimum

1. Absolute minimum: smallest function value on the entire domain of the function; local minimum at c: smallest function value when x is near c
3. Abs. max. at s, abs. min. at r, loc. max. at c, loc. min. at b and r
5. Abs. max. $f(4) = 5$, loc. max. $f(4) = 5$ and $f(6) = 4$, loc. min. $f(2) = 2$ and $f(5) = 3$

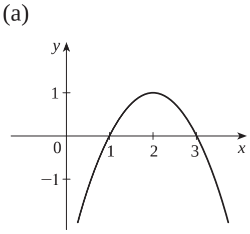
7.



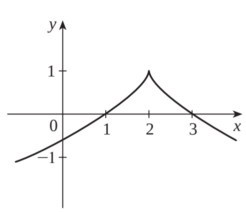
9.



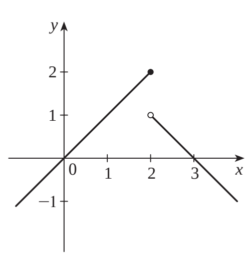
11. (a)



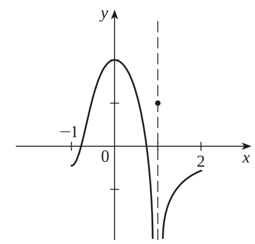
(b)



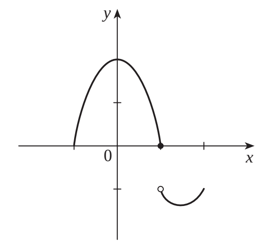
(c)



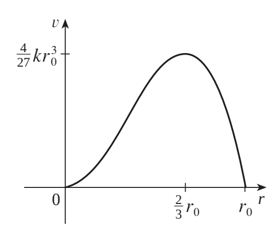
13. (a)



(b)

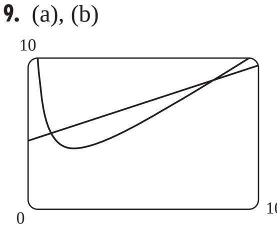


15. Abs. max. $f(1) = 5$ **17.** None
19. Abs. min. $f(0) = 0$
21. Abs. max. $f(-3) = 9$, abs. and loc. min. $f(0) = 0$
23. Abs. max. $f(2) = \ln 2$
25. Abs. max. $f(0) = 1$ **27.** Abs. max. $f(3) = 2$
29. $-\frac{2}{5}$ **31.** $-4, 2$ **33.** $0, \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$ **35.** $0, 2$
37. $0, \frac{4}{9}$ **39.** $0, \frac{8}{7}, 4$ **41.** $n\pi$ (n an integer) **43.** $0, \frac{2}{3}$
45. 10 **47.** $f(0) = 5$, $f(2) = -7$
49. $f(-1) = 8$, $f(2) = -19$
51. $f(3) = 66$, $f(\pm 1) = 2$ **53.** $f(1) = \frac{1}{2}$, $f(0) = 0$
55. $f\left(\sqrt{2}\right) = 2$, $f(-1) = -\sqrt{3}$
57. $f(\pi/6) = \frac{3}{2}\sqrt{3}$, $f(\pi/2) = 0$
59. $f(2) = 2/\sqrt{e}$, $f(-1) = -1/\sqrt[8]{e}$
61. $f(1) = \ln 3$, $f\left(-\frac{1}{2}\right) = \ln \frac{3}{4}$
63. $f\left(\dfrac{a}{a + b}\right) = \dfrac{a^a b^b}{(a + b)^{a+b}}$
65. (a) 2.19, 1.81 (b) $\frac{6}{25}\sqrt{\frac{3}{5}} + 2$, $-\frac{6}{25}\sqrt{\frac{3}{5}} + 2$
67. (a) 0.32, 0.00 (b) $\frac{3}{16}\sqrt{3}, 0$ **69.** ≈ 3.9665°C
71. Cheapest, $t \approx 0.855$ (June 1994); most expensive, $t \approx 4.618$ (March 1998)
73. (a) $r = \frac{2}{3}r_0$ (b) $v = \frac{4}{27}kr_0^3$
(c)

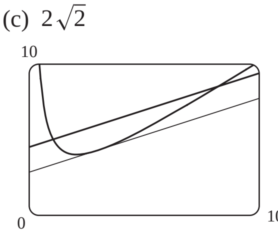


EXERCISES 4.2 ▪ PAGE 285

1. 2 **3.** $\frac{9}{4}$ **5.** f is not differentiable on $(-1, 1)$
7. 0.8, 3.2, 4.4, 6.1
9. (a), (b)



(c) $2\sqrt{2}$



11. 0 **13.** $-\frac{1}{2}\ln\left[\frac{1}{6}\left(1 - e^{-6}\right)\right]$ **15.** f is not continous at 3
23. 16 **25.** No **31.** No

EXERCISES 4.3 ■ PAGE 295

Abbreviations: inc., increasing; dec., decreasing; CD, concave downward; CU, concave upward; HA, horizontal asymptote; VA, vertical asymptote; IP, inflection point(s)

1. (a) (1, 3), (4, 6) (b) (0, 1), (3, 4) (c) (0, 2)
(d) (2, 4), (4, 6) (e) (2, 3)

3. (a) I/D Test (b) Concavity Test
(c) Find points at which the concavity changes.

5. (a) Inc. on (1, 5); dec. on (0, 1) and (5, 6)
(b) Loc. max. at $x = 5$, loc. min. at $x = 1$

7. $x = 1, 7$

9. (a) Inc. on $(-\infty, 3)$, $(2, \infty)$; dec. on $(-3, 2)$
(b) Loc. max. $f(-3) = 81$; loc. min. $f(2) = -44$
(c) CU on $(-\frac{1}{2}, \infty)$; CD on $\left(-\infty, -\frac{1}{2}\right)$; IP $\left(-\frac{1}{2}, \frac{37}{2}\right)$

11. (a) Inc. on $(-1, 0)$, $(1, \infty)$; dec. on $(-\infty, -1)$, $(0, 1)$
(b) Loc. max. $f(0) = 3$; loc. min. $f(\pm 1) = 2$
(c) CU on $\left(-\infty, -\sqrt{3}/3\right)$, $\left(\sqrt{3}/3, \infty\right)$;
CD on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$; IP $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$

13. (a) Inc. on $(0, \pi/4)$, $(5\pi/4, 2\pi)$; dec. on $(\pi/4, 5\pi/4)$
(b) Loc. max. $f(\pi/4) = \sqrt{2}$; loc. min. $f(5\pi/4) = -\sqrt{2}$
(c) CU on $(3\pi/4, 7\pi/4)$; CD on $(0, 3\pi/4)$, $(7\pi/4, 2\pi)$;
IP $(3\pi/4, 0)$, $(7\pi/4, 0)$

15. (a) Inc. on $\left(-\frac{1}{3}\ln 2, \infty\right)$; dec. on $\left(-\infty, -\frac{1}{3}\ln 2\right)$
(b) Loc. min. $f\left(-\frac{1}{3}\ln 2\right) = 2^{-2/3} + 2^{1/3}$ (c) CU on $(-\infty, \infty)$

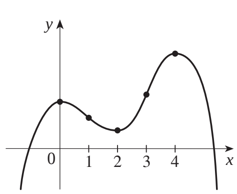
17. (a) Inc. on $(0, e^2)$; dec. on (e^2, ∞)
(b) Loc. max. $f(e^2) = 2/e$
(c) CU on $(e^{8/3}, \infty)$; CD on $(0, e^{8/3})$; IP $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$

19. Loc. max. $f(-1) = 7$, loc. min. $f(1) = -1$

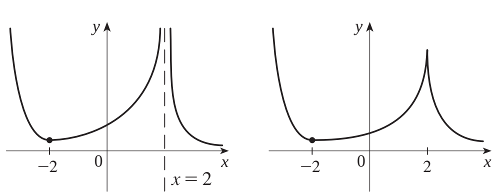
21. Loc. max. $f\left(\frac{3}{4}\right) = \frac{5}{4}$

23. (a) f has a local maximum at 2.
(b) f has a horizontal tangent at 6.

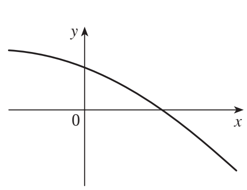
25.



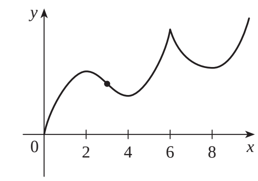
27.



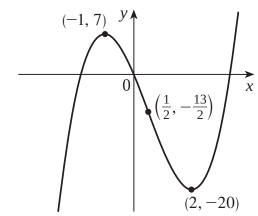
29.



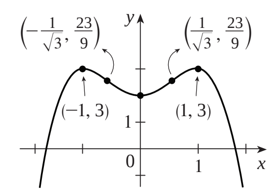
31. (a) Inc. on (0, 2), (4, 6), $(8, \infty)$;
dec. on (2, 4), (6, 8)
(b) Loc. max. at $x = 2, 6$;
loc. min. at $x = 4, 8$
(c) CU on (3, 6), $(6, \infty)$;
CD on (0, 3)
(d) 3 (e) See graph at right.



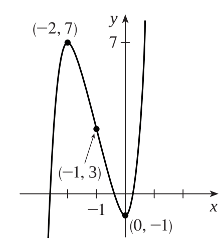
33. (a) Inc. on $(-\infty, -1)$, $(2, \infty)$;
dec. on $(-1, 2)$
(b) Loc. max. $f(-1) = 7$;
loc. min. $f(2) = -20$
(c) CU on $\left(\frac{1}{2}, \infty\right)$; CD on $\left(-\infty, \frac{1}{2}\right)$;
IP $\left(\frac{1}{2}, -\frac{13}{2}\right)$
(d) See graph at right.



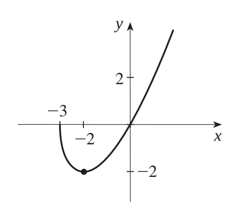
35. (a) Inc. on $(-\infty, -1)$, $(0, 1)$;
dec. on $(-1, 0)$, $(1, \infty)$
(b) Loc. max. $f(-1) = 3$, $f(1) = 3$;
loc. min. $f(0) = 2$
(c) CU on $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$;
CD on $\left(-\infty, -1/\sqrt{3}\right)$, $\left(1/\sqrt{3}, \infty\right)$;
IP $\left(\pm 1/\sqrt{3}, \frac{23}{9}\right)$
(d) See graph at right.



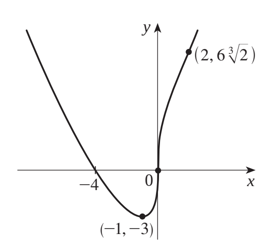
37. (a) Inc. on $(-\infty, -2)$, $(0, \infty)$;
dec. on $(-2, 0)$
(b) Loc. max. $h(-2) = 7$;
loc. min. $h(0) = -1$
(c) CU on $(-1, \infty)$;
CD on $(-\infty, -1)$; IP $(-1, 3)$
(d) See graph at right.



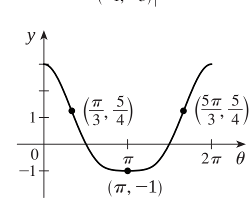
39. (a) Inc. on $(-2, \infty)$;
dec. on $(-3, -2)$
(b) Loc. min. $A(-2) = -2$
(c) CU on $(-3, \infty)$
(d) See graph at right.



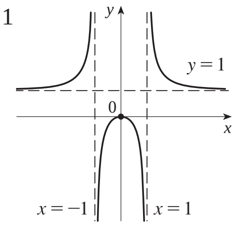
41. (a) Inc. on $(-1, \infty)$;
dec. on $(-\infty, -1)$
(b) Loc. min. $C(-1) = -3$
(c) CU on $(-\infty, 0)$, $(2, \infty)$;
CD on (0, 2);
IPs (0, 0), $\left(2, 6\sqrt[3]{2}\right)$
(d) See graph at right.



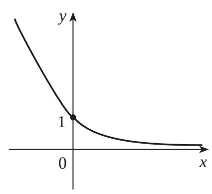
43. (a) Inc. on $(\pi, 2\pi)$;
dec. on $(0, \pi)$
(b) Loc. min. $f(\pi) = -1$
(c) CU on $(\pi/3, 5\pi/3)$;
CD on $(0, \pi/3)$, $(5\pi/3, 2\pi)$;
IP $\left(\pi/3, \frac{5}{4}\right)$, $\left(5\pi/3, \frac{5}{4}\right)$
(d) See graph at right.



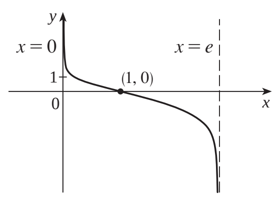
45. (a) HA $y = 1$, VA $x = -1$, $x = 1$
(b) Inc. on $(-\infty, -1)$, $(-1, 0)$;
dec. on $(0, 1)$, $(1, \infty)$
(c) Loc. max. $f(0) = 0$
(d) CU on $(-\infty, -1)$, $(1, \infty)$;
CD on $(-1, 1)$
(e) See graph at right.



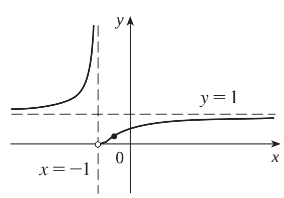
47. (a) HA $y = 0$
(b) Dec. on $(-\infty, \infty)$
(c) None
(d) CU on $(-\infty, \infty)$
(e) See graph at right.



49. (a) VA $x = 0$, $x = e$
(b) Dec. on $(0, e)$
(c) None
(d) CU on $(0, 1)$; CD on $(1, e)$;
IP $(1, 0)$
(e) See graph at right.



51. (a) HA $y = 1$, VA $x = -1$
(b) Inc. on $(-\infty, -1)$, $(-1, \infty)$
(c) None
(d) CU on $(-\infty, -1)$, $\left(-1, -\frac{1}{2}\right)$;
CD on $\left(-\frac{1}{2}, \infty\right)$; IP $\left(-\frac{1}{2}, 1/e^2\right)$
(e) See graph at right.



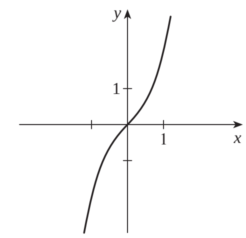
53. $(3, \infty)$
55. (a) Loc. and abs. max. $f(1) = \sqrt{2}$, no min.
(b) $\frac{1}{4}\left(3 - \sqrt{17}\right)$
57. (b) CU on $(0.94, 2.57)$, $(3.71, 5.35)$;
CD on $(0, 0.94)$, $(2.57, 3.71)$, $(5.35, 2\pi)$;
IP $(0.94, 0.44)$, $(2.57, -0.63)$, $(3.71, -0.63)$, $(5.35, 0.44)$
59. CU on $(-\infty, -0.6)$, $(0.0, \infty)$; CD on $(-0.6, 0.0)$
61. (a) The rate of increase is initially very small, increases to a maximum at $t \approx 8$ h, then decreases toward 0.
(b) When $t = 8$ (c) CU on $(0, 8)$; CD on $(8, 18)$ (d) $(8, 350)$
63. $K(3) - K(2)$; CD
65. 28.57 min, when the rate of increase of drug level in the bloodstream is greatest; 85.71 min, when rate of decrease is greatest
67. $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$

EXERCISES 4.4 ■ PAGE 304

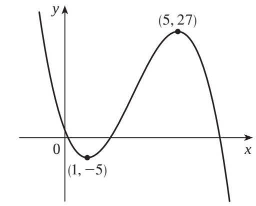
1. (a) Indeterminate (b) 0 (c) 0
(d) ∞, $-\infty$, or does not exist (e) Indeterminate
3. (a) $-\infty$ (b) Indeterminate (c) ∞
5. 2 **7.** $\frac{9}{5}$ **9.** $-\infty$ **11.** ∞ **13.** p/q
15. 0 **17.** $-\infty$ **19.** ∞ **21.** $\frac{1}{2}$ **23.** 1
25. $\ln\frac{5}{3}$ **27.** 1 **29.** $\frac{1}{2}$ **31.** 0 **33.** $-1/\pi^2$
35. $\frac{1}{2}a(a - 1)$ **37.** $\frac{1}{24}$ **39.** π **41.** 3 **43.** 0
45. $-2/\pi$ **47.** $\frac{1}{2}$ **49.** $\frac{1}{2}$ **51.** ∞ **53.** 1
55. e^{-2} **57.** e^3 **59.** 1 **61.** e^4
63. $1/\sqrt{e}$ **65.** e^2 **67.** $\frac{1}{4}$ **71.** 1 **77.** $\frac{16}{9}a$ **79.** 56
83. (a) 0

EXERCISES 4.5 ■ PAGE 314

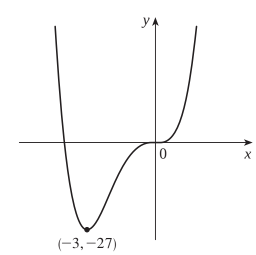
1. A. $\mathbb{R}$ B. y-int. 0; x-int. 0
C. About $(0, 0)$ D. None
E. Inc. on $(-\infty, \infty)$ F. None
G. CU on $(0, \infty)$; CD on $(-\infty, 0)$;
IP $(0, 0)$
H. See graph at right.



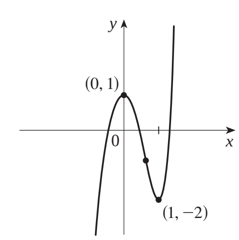
3. A. $\mathbb{R}$ B. y-int. 2; x-int. 2, $\frac{1}{2}\left(7 \pm 3\sqrt{5}\right)$
C. None D. None
E. Inc. on $(1, 5)$;
dec. on $(-\infty, 1)$, $(5, \infty)$
F. Loc. min. $f(1) = -5$;
loc. max. $f(5) = 27$
G. CU on $(-\infty, 3)$;
CD on $(3, \infty)$; IP $(3, 11)$
H. See graph at right.



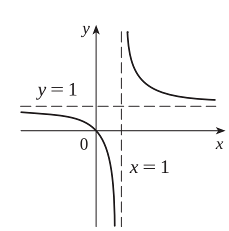
5. A. $\mathbb{R}$ B. y-int. 0; x-int. -4, 0
C. None D. None
E. Inc. on $(-3, \infty)$;
dec. on $(-\infty, -3)$
F. Loc. min. $f(-3) = -27$
G. CU on $(-\infty, -2)$, $(0, \infty)$;
CD on $(-2, 0)$; IP $(0, 0)$, $(-2, -16)$
H. See graph at right.



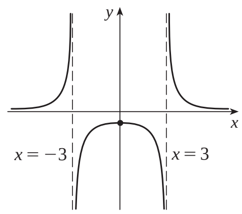
7. A. $\mathbb{R}$ B. y-int. 1
C. None D. None
E. Inc. on $(-\infty, 0)$, $(1, \infty)$;
dec. on $(0, 1)$
F. Loc. max. $f(0) = 1$;
loc. min. $f(1) = -2$
G. CU on $\left(1/\sqrt[3]{4}, \infty\right)$;
CD on $\left(-\infty, 1/\sqrt[3]{4}\right)$;
IP $\left(1/\sqrt[3]{4}, 1 - 9/\left(2\sqrt[3]{16}\right)\right)$
H. See graph at right.



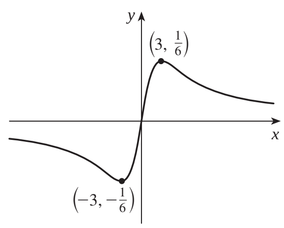
9. A. $\{x \mid x \neq 1\}$ B. y-int. 0; x-int. 0
C. None D. VA $x = 1$, HA $y = 1$
E. Dec. on $(-\infty, 1)$, $(1, \infty)$
F. None
G. CU on $(1, \infty)$; CD on $(-\infty, 1)$
H. See graph at right.



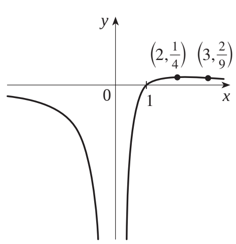
11. A. $\{x \mid x \neq \pm 3\}$ B. y-int. $-\frac{1}{9}$
C. About y-axis D. VA $x = \pm 3$, HA $y = 0$
E. Inc. on $(-\infty, -3)$, $(-3, 0)$;
dec. on $(0, 3)$, $(3, \infty)$
F. Loc. max. $f(0) = -\frac{1}{9}$
G. CU on $(-\infty, -3)$, $(3, \infty)$;
CD on $(-3, 3)$
H. See graph at right.



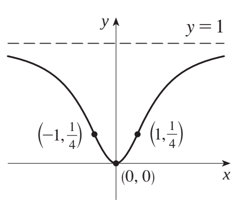
13. A. $\mathbb{R}$ B. y-int. 0; x-int. 0
C. About $(0, 0)$ D. HA $y = 0$
E. Inc. on $(-3, 3)$;
dec. on $(-\infty, -3)$, $(3, \infty)$
F. Loc. min. $f(-3) = -\frac{1}{6}$;
loc. max. $f(3) = \frac{1}{6}$;
G. CU on $(-3\sqrt{3}, 0)$, $(3\sqrt{3}, \infty)$;
CD on $(-\infty, -3\sqrt{3})$, $(0, 3\sqrt{3})$;
IP $(0, 0)$, $(\pm 3\sqrt{3}, \pm\sqrt{3}/12)$
H. See graph at right.



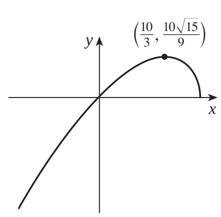
15. A. $(-\infty, 0) \cup (0, \infty)$ B. x-int. 1
C. None D. HA $y = 0$; VA $x = 0$
E. Inc. on $(0, 2)$;
dec. on $(-\infty, 0)$, $(2, \infty)$
F. Loc. max. $f(2) = \frac{1}{4}$
G. CU on $(3, \infty)$;
CD on $(-\infty, 0)$, $(0, 3)$; IP $(3, \frac{2}{9})$
H. See graph at right



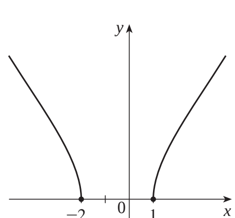
17. A. $\mathbb{R}$ B. y-int. 0, x-int. 0
C. About y-axis D. HA $y = 1$
E. Inc. on $(0, \infty)$; dec. on $(-\infty, 0)$
F. Loc. min. $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IP $(\pm 1, \frac{1}{4})$
H. See graph at right



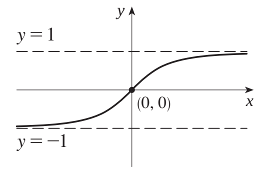
19. A. $(-\infty, 5]$ B. y-int. 0; x-int. 0, 5
C. None D. None
E. Inc. on $(-\infty, \frac{10}{3})$; dec. on $(\frac{10}{3}, 5)$
F. Loc. max. $f(\frac{10}{3}) = \frac{10}{9}\sqrt{15}$
G. CD on $(-\infty, 5)$
H. See graph at right.



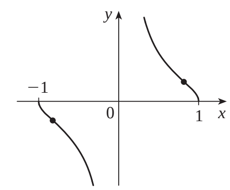
21. A. $(-\infty, -2) \cup (1, \infty)$
B. x-int. -2, 1
C. None D. None
E. Inc. on $(1, \infty)$; dec. on $(-\infty, -2)$
F. None
G. CD on $(-\infty, -2)$, $(1, \infty)$
H. See graph at right.



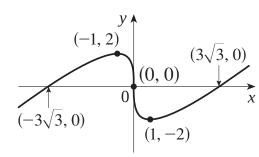
23. A. $\mathbb{R}$ B. y-int. 0; x-int. 0
C. About the origin
D. HA $y = \pm 1$
E. Inc. on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 0)$
H. See graph at right.



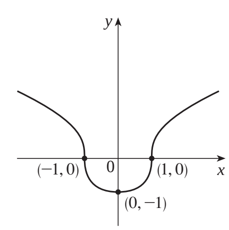
25. A. $\{x \mid |x| \leqslant 1, x \neq 0\} = [-1, 0) \cup (0, 1]$
B. x-int. ± 1 C. About $(0, 0)$
D. VA $x = 0$
E. Dec. on $(-1, 0)$, $(0, 1)$
F. None
G. CU on $(-1, -\sqrt{2/3})$, $(0, \sqrt{2/3})$;
CD on $(-\sqrt{2/3}, 0)$, $(\sqrt{2/3}, 1)$;
IP $(\pm\sqrt{2/3}, \pm 1/\sqrt{2})$
H. See graph at right.



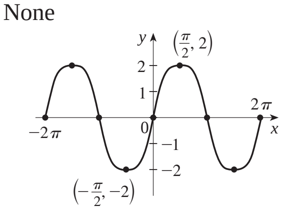
27. A. $\mathbb{R}$ B. y-int. 0; x-int. 0, $\pm 3\sqrt{3}$ C. About the origin
D. None E. Inc. on $(-\infty, -1)$, $(1, \infty)$; dec. on $(-1, 1)$
F. Loc. max. $f(-1) = 2$;
loc. min. $f(1) = -2$
G. CU on $(0, \infty)$; CD on $(-\infty, 0)$;
IP $(0, 0)$
H. See graph at right.



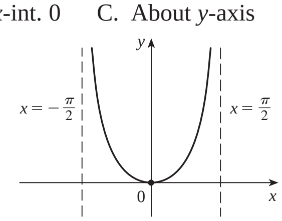
29. A. $\mathbb{R}$ B. y-int. -1; x-int. ± 1
C. About y-axis D. None
E. Inc. on $(0, \infty)$; dec. on $(-\infty, 0)$
F. Loc. min. $f(0) = -1$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$;
IP $(\pm 1, 0)$
H. See graph at right.



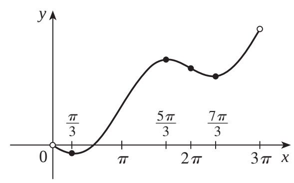
31. A. $\mathbb{R}$ B. y-int. 0; x-int. $n\pi$ (n an integer)
C. About the origin, period 2π D. None
E. Inc. on $(2n\pi - \pi/2, 2n\pi + \pi/2)$;
dec. on $(2n\pi + \pi/2, 2n\pi + 3\pi/2)$
F. Loc. max. $f(2n\pi + \pi/2) = 2$;
loc. min. $f(2n\pi + 3\pi/2) = -2$
G. CU on $((2n - 1)\pi, 2n\pi)$;
CD on $(2n\pi, (2n + 1)\pi)$; IP $(n\pi, 0)$
H. See graph at right.



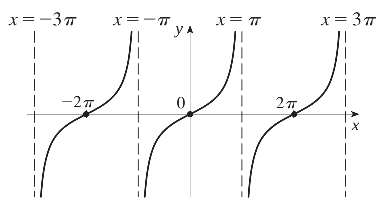
33. A. $(-\pi/2, \pi/2)$ B. y-int. 0; x-int. 0 C. About y-axis
D. VA $x = \pm\pi/2$
E. Inc. on $(0, \pi/2)$;
dec. on $(-\pi/2, 0)$
F. Loc. min. $f(0) = 0$
G. CU on $(-\pi/2, \pi/2)$
H. See graph at right.



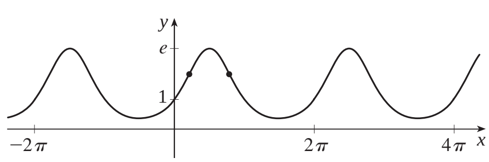
35. A. $(0, 3\pi)$ C. None D. None
E. Inc. on $(\pi/3, 5\pi/3)$, $(7\pi/3, 3\pi)$;
dec. on $(0, \pi/3)$, $(5\pi/3, 7\pi/3)$
F. Loc. min. $f(\pi/3) = (\pi/6) - \frac{1}{2}\sqrt{3}$, $f(7\pi/3) = (7\pi/6) - \frac{1}{2}\sqrt{3}$;
loc. max. $f(5\pi/3) = (5\pi/6) + \frac{1}{2}\sqrt{3}$
G. CU on $(0, \pi)$, $(2\pi, 3\pi)$;
CD on $(\pi, 2\pi)$;
IP $(\pi, \pi/2)$, $(2\pi, \pi)$
H. See graph at right.



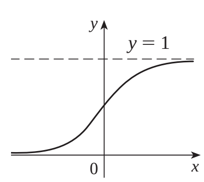
37. A. All reals except $(2n + 1)\pi$ (n an integer)
B. y-int. 0; x-int. $2n\pi$
C. About the origin, period 2π
D. VA $x = (2n + 1)\pi$
E. Inc. on $((2n - 1)\pi, (2n + 1)\pi)$ F. None
G. CU on $(2n\pi, (2n + 1)\pi)$; CD on $((2n - 1)\pi, 2n\pi)$;
IP $(2n\pi, 0)$
H.



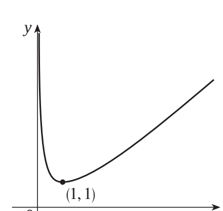
39. A. $\mathbb{R}$ B. y-int. 1 C. Period 2π D. None
Answers for E–G are for the interval $[0, 2\pi]$.
E. Inc. on $(0, \pi/2)$, $(3\pi/2, 2\pi)$; dec. on $(\pi/2, 3\pi/2)$
F. Loc. max. $f(\pi/2) = e$; loc. min. $f(3\pi/2) = e^{-1}$
G. CU on $(0, \alpha)$, $(\beta, 2\pi)$ where $\alpha = \sin^{-1}\left(\frac{1}{2}\left(-1 + \sqrt{5}\right)\right)$,
$\beta = \pi - \alpha$; CD on (α, β); IP when $x = \alpha, \beta$
H.



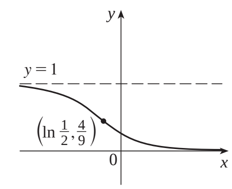
41. A. $\mathbb{R}$ B. y-int. $\frac{1}{2}$ C. None
D. HA $y = 0$, $y = 1$
E. Inc. on $\mathbb{R}$ F. None
G. CU on $(-\infty, 0)$; CD on $(0, \infty)$;
IP $\left(0, \frac{1}{2}\right)$ H. See graph at right.



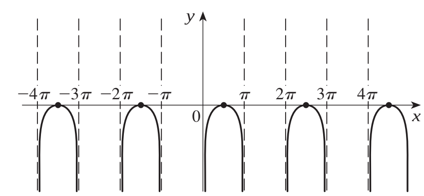
43. A. $(0, \infty)$ B. None
C. None D. VA $x = 0$
E. Inc. on $(1, \infty)$; dec. on $(0, 1)$
F. Loc. min. $f(1) = 1$
G. CU on $(0, \infty)$
H. See graph at right.



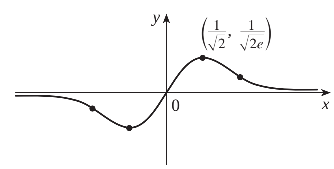
45. A. $\mathbb{R}$ B. y-int. $\frac{1}{4}$ C. None
D. HA $y = 0$, $y = 1$
E. Dec. on $\mathbb{R}$ F. None
G. CU on $\left(\ln \frac{1}{2}, \infty\right)$; CD on $\left(-\infty, \ln \frac{1}{2}\right)$;
IP $\left(\ln \frac{1}{2}, \frac{4}{9}\right)$
H. See graph at right.



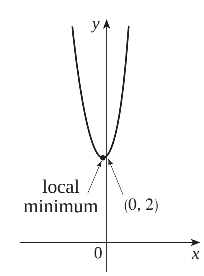
47. A. All x in $(2n\pi, (2n + 1)\pi)$ (n an integer)
B. x-int. $\pi/2 + 2n\pi$ C. Period 2π D. VA $x = n\pi$
E. Inc. on $(2n\pi, \pi/2 + 2n\pi)$; dec. on $(\pi/2 + 2n\pi, (2n + 1)\pi)$
F. Loc. max. $f(\pi/2 + 2n\pi) = 0$ G. CD on $(2n\pi, (2n + 1)\pi)$
H.



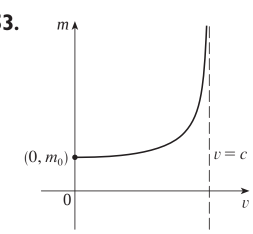
49. A. $\mathbb{R}$ B. y-int. 0; x-int. 0 C. About $(0, 0)$ D. HA $y = 0$
E. Inc. on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$; dec. on $\left(-\infty, -1/\sqrt{2}\right)$, $\left(1/\sqrt{2}, \infty\right)$
F. Loc. min. $f\left(-1/\sqrt{2}\right) = -1/\sqrt{2e}$; loc. max. $f\left(1/\sqrt{2}\right) = 1/\sqrt{2e}$
G. CU on $\left(-\sqrt{3/2}, 0\right)$, $\left(\sqrt{3/2}, \infty\right)$; CD on $\left(-\infty, -\sqrt{3/2}\right)$, $\left(0, \sqrt{3/2}\right)$;
IP $\left(\pm\sqrt{3/2}, \pm\sqrt{3/2}e^{-3/2}\right)$, $(0, 0)$
H.



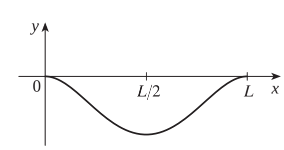
51. A. $\mathbb{R}$ B. y-int. 2
C. None D. None
E. Inc. on $\left(\frac{1}{5}\ln\frac{2}{3}, \infty\right)$; dec. on $\left(-\infty, \frac{1}{5}\ln\frac{2}{3}\right)$
F. Loc. min. $f\left(\frac{1}{5}\ln\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5}$
G. CU on $(-\infty, \infty)$
H. See graph at right.



53.

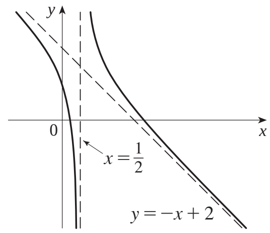


55.

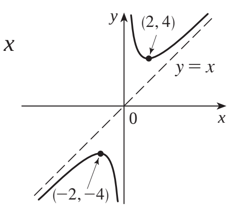


57. $y = x - 1$ **59.** $y = 2x - 2$

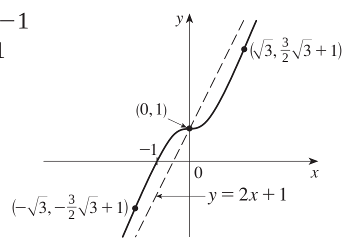
61. A. $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$
B. y-int. 1; x-int. $\frac{1}{4}\left(5 \pm \sqrt{17}\right)$
C. None
D. VA $x = \frac{1}{2}$; SA $y = -x + 2$
E. Dec. on $\left(-\infty, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \infty\right)$
F. None
G. CU on $\left(\frac{1}{2}, \infty\right)$; CD on $\left(-\infty, \frac{1}{2}\right)$
H. See graph at right



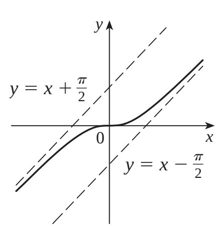
63. A. $\{x \mid x \neq 0\}$ B. None
C. About (0, 0) D. VA $x = 0$; SA $y = x$
E. Inc. on $(-\infty, -2)$, $(2, \infty)$;
dec. on $(-2, 0)$, $(0, 2)$
F. Loc. max. $f(-2) = -4$;
loc. min. $f(2) = 4$
G. CU on $(0, \infty)$; CD on $(-\infty, 0)$
H. See graph at right.



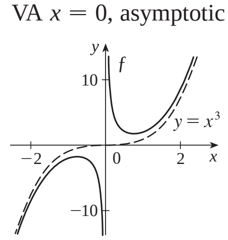
65. A. $\mathbb{R}$ B. y-int. 1; x-int. -1
C. None D. SA $y = 2x + 1$
E. Inc. on $(-\infty, \infty)$ F. None
G. CU on $\left(-\infty, -\sqrt{3}\right)$,
$\left(0, \sqrt{3}\right)$;
CD on $\left(-\sqrt{3}, 0\right)$, $\left(\sqrt{3}, \infty\right)$;
IP $\left(\pm\sqrt{3}, 1 \pm \frac{3}{2}\sqrt{3}\right)$, (0, 1)
H. See graph at right.



67.

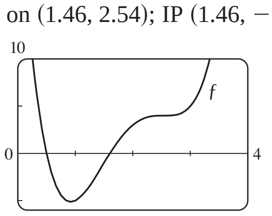


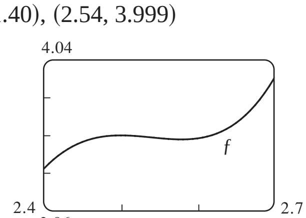
71. VA $x = 0$, asymptotic to $y = x^3$



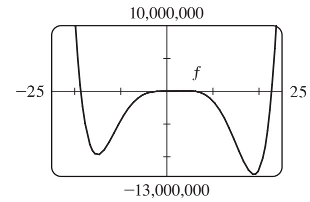
EXERCISES 4.6 ■ PAGE 320

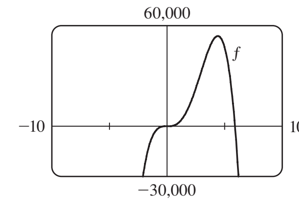
1. Inc. on (0.92, 2.5), $(2.58, \infty)$; dec. on $(-\infty, 0.92)$, (2.5, 2.58);
loc. max. $f(2.5) \approx 4$; loc. min. $f(0.92) \approx -5.12$, $f(2.58) \approx 3.998$;
CU on $(-\infty, 1.46)$, $(2.54, \infty)$;
CD on (1.46, 2.54); IP (1.46, -1.40), (2.54, 3.999)



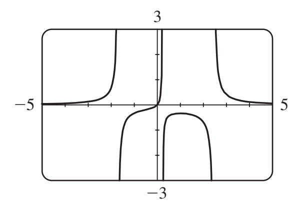


3. Inc. on $(-15, 4.40)$, $(18.93, \infty)$;
dec. on $(-\infty, -15)$, (4.40, 18.93);
loc. max. $f(4.40) \approx 53{,}800$; loc. min. $f(-15) \approx -9{,}700{,}000$,
$f(18.93) \approx -12{,}700{,}000$; CU on $(-\infty, -11.34)$, (0, 2.92),
$(15.08, \infty)$; CD on $(-11.34, 0)$, (2.92, 15.08);
IP (0, 0), $\approx (-11.34, -6{,}250{,}000)$, (2.92, 31,800),
$(15.08, -8{,}150{,}000)$

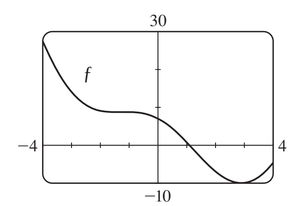


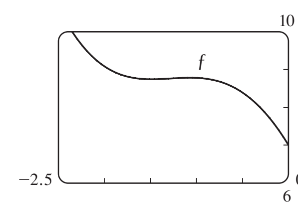


5. Inc. on $(-\infty, -1.7)$, $(-1.7, 0.24)$, (0.24, 1);
dec. on (1, 2.46), $(2.46, \infty)$; loc. max. $f(1) = -\frac{1}{3}$;
CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, $(2.46, \infty)$;
CD on $(-1.7, -0.506)$, (0.24, 2.46); IP $(-0.506, -0.192)$

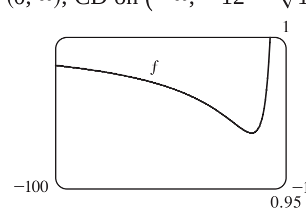


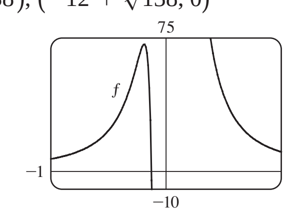
7. Inc. on $(-1.49, -1.07)$, (2.89, 4); dec. on $(-4, -1.49)$,
$(-1.07, 2.89)$; loc. max. $f(-1.07) \approx 8.79$; loc. min.
$f(-1.49) \approx 8.75$, $f(2.89) \approx -9.99$; CU on $(-4, -1.28)$,
(1.28, 4); CD on $(-1.28, 1.28)$; IP $(-1.28, 8.77)$, $(1.28, -1.48)$



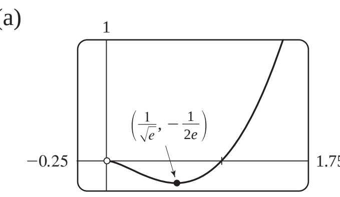


9. Inc. on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$; dec. on $\left(-\infty, -8 - \sqrt{61}\right)$,
$\left(-8 + \sqrt{61}, 0\right)$, $(0, \infty)$; CU on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$,
$(0, \infty)$; CD on $\left(-\infty, -12 - \sqrt{138}\right)$, $\left(-12 + \sqrt{138}, 0\right)$



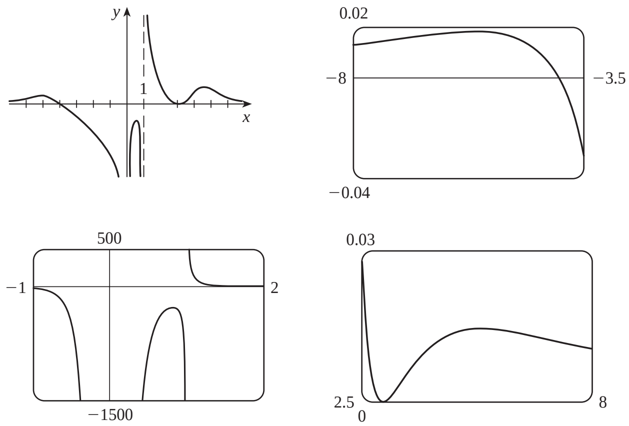


11. (a)



(b) $\lim_{x \to 0^+} f(x) = 0$
(c) Loc. min. $f\left(1/\sqrt{e}\right) = -1/(2e)$;
CD on $(0, e^{-3/2})$; CU on $(e^{-3/2}, \infty)$

13. Loc. max. $f(-5.6) \approx 0.018$, $f(0.82) \approx -281.5$, $f(5.2) \approx 0.0145$; loc. min. $f(3) = 0$

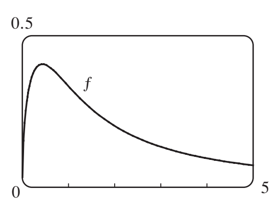


15. $f'(x) = -\dfrac{x(x+1)^2(x^3 + 18x^2 - 44x - 16)}{(x-2)^3(x-4)^5}$

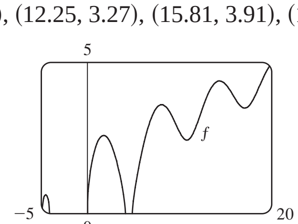
$f''(x) = 2\dfrac{(x+1)(x^6 + 36x^5 + 6x^4 - 628x^3 + 684x^2 + 672x + 64)}{(x-2)^4(x-4)^6}$

CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, $(4, \infty)$;
CD on $(-\infty, -35.3)$, $(-5.0, -1)$, $(-0.5, -0.1)$;
IP $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, $(-0.1, 0.0000066)$

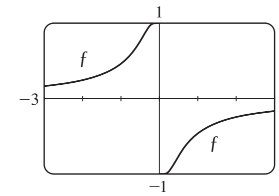
17. Inc. on $(0, 0.43)$; dec. on $(0.43, \infty)$; loc. max. $f(0.43) \approx 0.41$;
CU on $(0.94, \infty)$; CD on $(0, 0.94)$;
IP $(0.94, 0.34)$



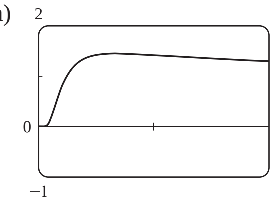
19. Inc. on $(-4.91, -4.51)$, $(0, 1.77)$, $(4.91, 8.06)$, $(10.79, 14.34)$, $(17.08, 20)$;
dec. on $(-4.51, -4.10)$, $(1.77, 4.10)$, $(8.06, 10.79)$, $(14.34, 17.08)$;
loc. max. $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, $f(14.34) \approx 4.39$;
loc. min. $f(10.79) \approx 2.43$, $f(17.08) \approx 3.49$; CU on $(9.60, 12.25)$, $(15.81, 18.65)$;
CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, $(12.25, 15.81)$, $(18.65, 20)$;
IPs at $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, $(18.65, 4.20)$



21. Inc. on $(-\infty, 0)$, $(0, \infty)$;
CU on $(-\infty, -0.4)$, $(0, 0.4)$;
CD on $(-0.4, 0)$, $(0.4, \infty)$;
IP $(\mp 0.4, \pm 0.8)$

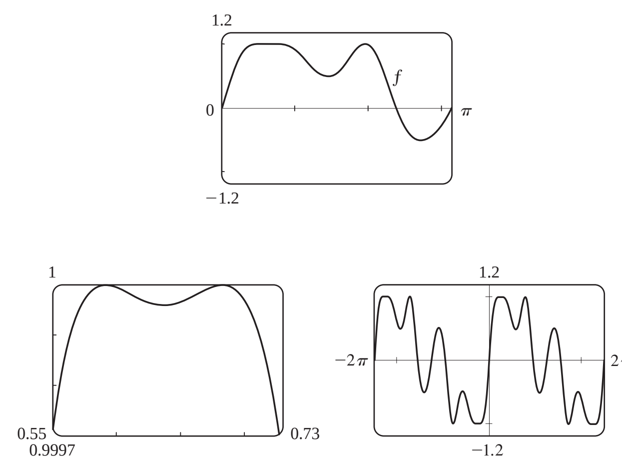


23. (a)

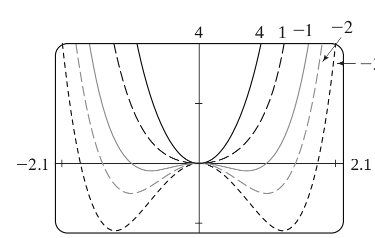


(b) $\lim_{x \to 0^+} x^{1/x} = 0$, $\lim_{x \to \infty} x^{1/x} = 1$
(c) Loc. max. $f(e) = e^{1/e}$ (d) IP at $x \approx 0.58, 4.37$

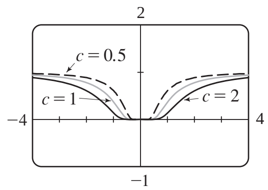
25. Max. $f(0.59) \approx 1$, $f(0.68) \approx 1$, $f(1.96) \approx 1$;
min. $f(0.64) \approx 0.99996$, $f(1.46) \approx 0.49$, $f(2.73) \approx -0.51$;
IP $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, $(2.28, 0.34)$

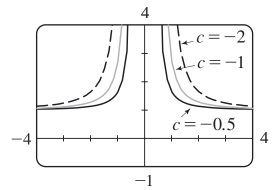


27. For $c \geqslant 0$, there is no IP and only one extreme point, the origin. For $c < 0$, there is a maximum point at the origin, two minimum points, and two IPs, which move downward and away from the origin as $c \to -\infty$.

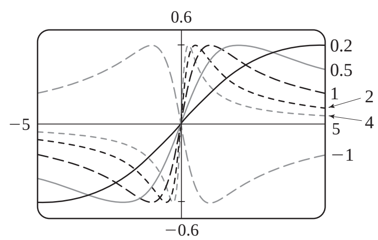


29. There is no maximum or minimum, regardless of the value of c. For $c < 0$, there is a vertical asymptote at $x = 0$, $\lim_{x\to 0} f(x) = \infty$, and $\lim_{x\to\pm\infty} f(x) = 1$.
$c = 0$ is a transitional value at which $f(x) = 1$ for $x \neq 0$.
For $c > 0$, $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to\pm\infty} f(x) = 1$, and there are two IPs, which move away from the y-axis as $c \to \infty$.

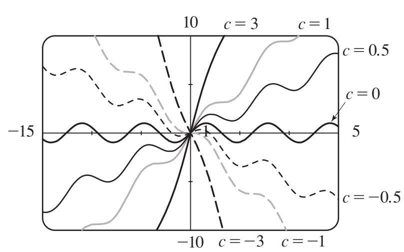




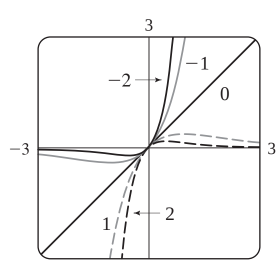
31. For $c > 0$, the maximum and minimum values are always $\pm\frac{1}{2}$, but the extreme points and IPs move closer to the y-axis as c increases. $c = 0$ is a transitional value: when c is replaced by $-c$, the curve is reflected in the x-axis.



33. For $|c| < 1$, the graph has local maximum and minimum values; for $|c| \geqslant 1$ it does not. The function increases for $c \geqslant 1$ and decreases for $c \leqslant -1$. As c changes, the IPs move vertically but not horizontally.

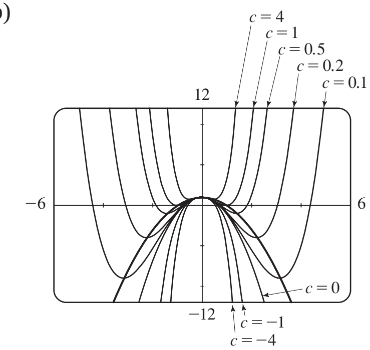


35.



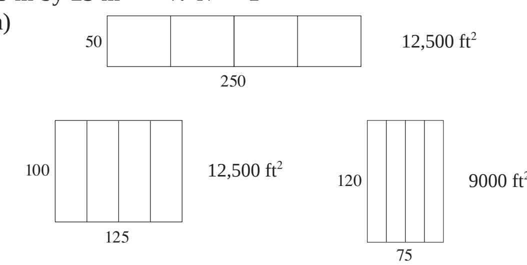
For $c > 0$, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.
For $c < 0$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.
As $|c|$ increases, the maximum and minimum points and the IPs get closer to the origin.

37. (a) Positive (b)

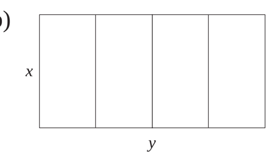


EXERCISES 4.7 ■ PAGE 328

1. (a) 11, 12 (b) 11.5, 11.5 **3.** 10, 10
5. 25 m by 25 m **7.** $N = 1$
9. (a)



(b)



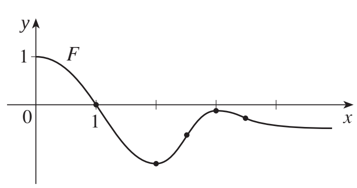
(c) $A = xy$ (d) $5x + 2y = 750$ (e) $A(x) = 375x - \frac{5}{2}x^2$
(f) $14{,}062.5\ \text{ft}^2$
11. 1000 ft by 1500 ft **13.** $4000\ \text{cm}^3$ **15.** \$191.28
17. $\left(-\frac{28}{17}, \frac{7}{17}\right)$ **19.** $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$ **21.** Square, side $\sqrt{2}\,r$
23. $L/2$, $\sqrt{3}\,L/4$ **25.** Base $\sqrt{3}\,r$, height $3r/2$
27. $4\pi r^3/(3\sqrt{3})$ **29.** $\pi r^2(1 + \sqrt{5})$ **31.** 24 cm, 36 cm
33. (a) Use all of the wire for the square
(b) $40\sqrt{3}/(9 + 4\sqrt{3})$ m for the square
35. Height $=$ radius $= \sqrt[3]{V/\pi}$ cm **37.** $V = 2\pi R^3/(9\sqrt{3})$
41. $E^2/(4r)$
43. (a) $\frac{3}{2}S^2 \csc\theta\,(\csc\theta - \sqrt{3}\cot\theta)$ (b) $\cos^{-1}(1/\sqrt{3}) \approx 55°$
(c) $6s\left[h + s/(2\sqrt{2})\right]$
45. Row directly to B **47.** ≈ 4.85 km east of the refinery
49. $10\sqrt[3]{3}/(1 + \sqrt[3]{3})$ ft from the stronger source
51. $(a^{2/3} + b^{2/3})^{3/2}$
53. (b) (i) \$342,491; \$342/unit; \$390/unit (ii) 400
(iii) \$320/unit
55. (a) $p(x) = 19 - \frac{1}{3000}x$ (b) \$9.50
57. (a) $p(x) = 550 - \frac{1}{10}x$ (b) \$175 (c) \$100
61. 9.35 m **65.** $x = 6$ in. **67.** $\pi/6$
69. At a distance $5 - 2\sqrt{5}$ from A **71.** $\frac{1}{2}(L + W)^2$
73. (a) About 5.1 km from B (b) C is close to B; C is close to D; $W/L = \sqrt{25 + x^2}/x$, where $x = |BC|$ (c) ≈ 1.07; no such value (d) $\sqrt{41}/4 \approx 1.6$

EXERCISES 4.8 ■ PAGE 338

1. (a) $x_2 \approx 2.3$, $x_3 \approx 3$ (b) No **3.** $\frac{4}{5}$ **5.** 1.1797
7. 1.1785 **9.** -1.25 **11.** 1.82056420 **13.** 1.217562
15. 0.876726 **17.** -0.724492, 1.220744
19. 1.412391, 3.057104 **21.** 0.641714
23. -1.93822883, -1.21997997, 1.13929375, 2.98984102
25. -1.97806681, -0.82646233
27. 0.21916368, 1.08422462 **29.** (b) 31.622777
35. (a) -1.293227, -0.441731, 0.507854 (b) -2.0212
37. (0.904557, 1.855277) **39.** (0.410245, 0.347810)
41. 0.76286%

EXERCISES 4.9 ■ PAGE 345

1. $F(x) = \frac{1}{2}x^2 - 3x + C$ **3.** $F(x) = \frac{1}{2}x + \frac{1}{4}x^3 - \frac{1}{5}x^4 + C$
5. $F(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$ **7.** $F(x) = 4x^{5/4} - 4x^{7/4} + C$
9. $F(x) = 4x^{3/2} - \frac{6}{7}x^{7/6} + C$
11. $F(x) = \begin{cases} -5/(4x^8) + C_1 & \text{if } x < 0 \\ -5/(4x^8) + C_2 & \text{if } x > 0 \end{cases}$
13. $F(u) = \frac{1}{3}u^3 - 6u^{-1/2} + C$
15. $G(\theta) = \sin\theta + 5\cos\theta + C$
17. $F(x) = 5e^x - 3\sinh x + C$
19. $F(x) = \frac{1}{2}x^2 - \ln|x| - 1/x^2 + C$
21. $F(x) = x^5 - \frac{1}{3}x^6 + 4$ **23.** $x^3 + x^4 + Cx + D$
25. $\frac{3}{20}x^{8/3} + Cx + D$ **27.** $e^t + \frac{1}{2}Ct^2 + Dt + E$
29. $x - 3x^2 + 8$ **31.** $4x^{3/2} + 2x^{5/2} + 4$
33. $2\sin t + \tan t + 4 - 2\sqrt{3}$
35. $\frac{3}{2}x^{2/3} - \frac{1}{2}$ if $x > 0$; $\frac{3}{2}x^{2/3} - \frac{5}{2}$ if $x < 0$
37. $2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}$
39. $-\sin\theta - \cos\theta + 5\theta + 4$ **41.** $x^2 - 2x^3 + 9x + 9$
43. $x^2 - \cos x - \frac{1}{2}\pi x$ **45.** $-\ln x + (\ln 2)x - \ln 2$
47. 10 **49.** b
51.



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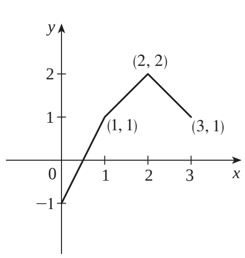

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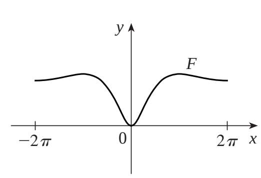


57. $s(t) = 1 - \cos t - \sin t$ **59.** $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$
61. $s(t) = -10\sin t - 3\cos t + (6/\pi)t + 3$
63. (a) $s(t) = 450 - 4.9t^2$ (b) $\sqrt{450/4.9} \approx 9.58$ s
(c) $-9.8\sqrt{450/4.9} \approx -93.9$ m/s (d) About 9.09 s
67. 225 ft **69.** \$742.08 **71.** $\frac{130}{11} \approx 11.8$ s
73. $\frac{88}{15} \approx 5.87$ ft/s^2 **75.** 62,500 km/h^2 ≈ 4.82 m/s^2
77. (a) 22.9125 mi (b) 21.675 mi (c) 30 min 33 s
(d) 55.425 mi

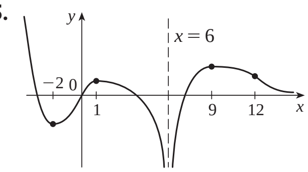
CHAPTER 4 REVIEW ■ PAGE 347

True-False Quiz

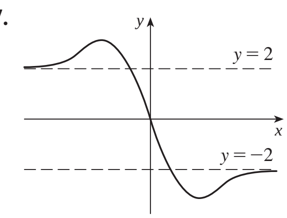
1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True **19.** True

Exercises

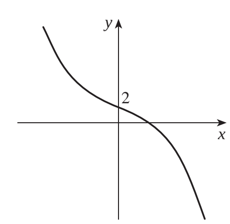
1. Abs. max. $f(4) = 5$, abs. and loc. min. $f(3) = 1$;
loc. min. $f(3) = 1$
3. Abs. max. $f(2) = \frac{2}{5}$, abs. and loc. min. $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$
5. Abs. max. $f(\pi) = \pi$; abs. min. $f(0) = 0$; loc. max.
$f(\pi/3) = (\pi/3) + \frac{1}{2}\sqrt{3}$; loc. min. $f(2\pi/3) = (2\pi/3) - \frac{1}{2}\sqrt{3}$
7. π **9.** 8 **11.** 0 **13.** $\frac{1}{2}$
15.



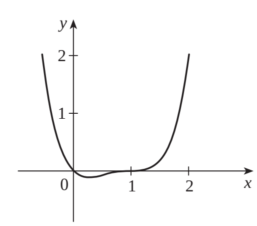
17.



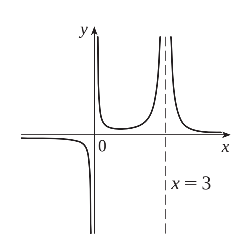
19. A. $\mathbb{R}$ B. y-int. 2
C. None D. None
E. Dec. on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 2)$
H. See graph at right.



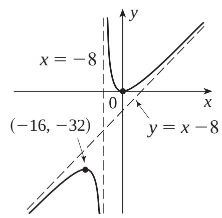
21. A. $\mathbb{R}$ B. y-int. 0; x-int. 0, 1
C. None D. None
E. Inc. on $\left(\frac{1}{4}, \infty\right)$, dec. on $\left(-\infty, \frac{1}{4}\right)$
F. Loc. min. $f\left(\frac{1}{4}\right) = -\frac{27}{256}$
G. CU on $\left(-\infty, \frac{1}{2}\right)$, $(1, \infty)$;
CD on $\left(\frac{1}{2}, 1\right)$; IP $\left(\frac{1}{2}, -\frac{1}{16}\right)$, $(1, 0)$
H. See graph at right.



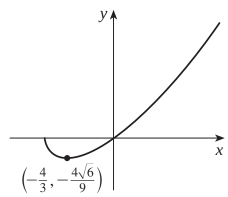
23. A. $\{x \mid x \neq 0, 3\}$
B. None C. None
D. HA $y = 0$; VA $x = 0$, $x = 3$
E. Inc. on $(1, 3)$; dec. on $(-\infty, 0)$,
$(0, 1)$, $(3, \infty)$
F. Loc. min. $f(1) = \frac{1}{4}$
G. CU on $(0, 3)$, $(3, \infty)$; CD on $(-\infty, 0)$
H. See graph at right.



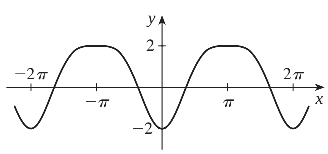
25. A. $\{x \mid x \neq -8\}$
B. y-int. 0, x-int. 0 C. None
D. VA $x = -8$; SA $y = x - 8$
E. Inc. on $(-\infty, -16)$, $(0, \infty)$;
dec. on $(-16, -8)$, $(-8, 0)$
F. Loc. max. $f(-16) = -32$;
loc. min. $f(0) = 0$
G. CU on $(-8, \infty)$; CD on $(-\infty, -8)$
H. See graph at right.



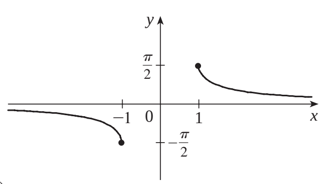
27. A. $[-2, \infty)$
B. y-int. 0; x-int. -2, 0
C. None D. None
E. Inc. on $\left(-\frac{4}{3}, \infty\right)$, dec. on $\left(-2, -\frac{4}{3}\right)$
F. Loc. min. $f\left(-\frac{4}{3}\right) = -\frac{4}{9}\sqrt{6}$
G. CU on $(-2, \infty)$
H. See graph at right.



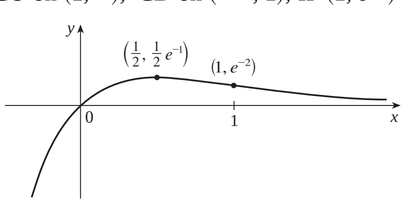
29. A. $\mathbb{R}$ B. y-int. -2
C. About y-axis, period 2π D. None
E. Inc. on $(2n\pi, (2n + 1)\pi)$, n an integer; dec. on $((2n - 1)\pi, 2n\pi)$
F. Loc. max. $f((2n + 1)\pi) = 2$; loc. min. $f(2n\pi) = -2$
G. CU on $(2n\pi - (\pi/3), 2n\pi + (\pi/3))$;
CD on $(2n\pi + (\pi/3), 2n\pi + (5\pi/3))$; IP $\left(2n\pi \pm (\pi/3), -\frac{1}{4}\right)$
H.



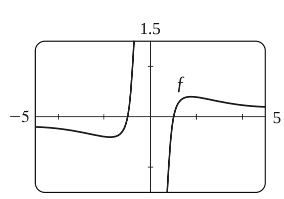
31. A. $\{x \mid |x| \geqslant 1\}$
B. None C. About (0, 0)
D. HA $y = 0$
E. Dec. on $(-\infty, -1)$, $(1, \infty)$
F. None
G. CU on $(1, \infty)$; CD on $(-\infty, -1)$
H. See graph at right.



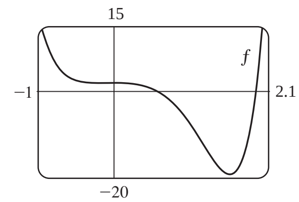
33. A. $\mathbb{R}$ B. y-int. 0, x-int. 0 C. None D. HA $y = 0$
E. Inc. on $\left(-\infty, \frac{1}{2}\right)$, dec. on $\left(\frac{1}{2}, \infty\right)$ F. Loc. max. $f\left(\frac{1}{2}\right) = 1/(2e)$
G. CU on $(1, \infty)$; CD on $(-\infty, 1)$; IP $(1, e^{-2})$
H.

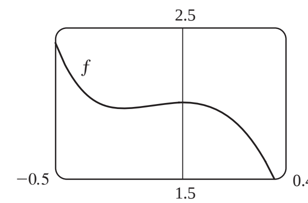


35. Inc. on $\left(-\sqrt{3}, 0\right)$, $\left(0, \sqrt{3}\right)$;
dec. on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3}, \infty\right)$;
loc. max. $f\left(\sqrt{3}\right) = \frac{2}{9}\sqrt{3}$,
loc. min. $f\left(-\sqrt{3}\right) = -\frac{2}{9}\sqrt{3}$;
CU on $\left(-\sqrt{6}, 0\right)$, $\left(\sqrt{6}, \infty\right)$;
CD on $\left(-\infty, -\sqrt{6}\right)$, $\left(0, \sqrt{6}\right)$;
IP $\left(\sqrt{6}, \frac{5}{36}\sqrt{6}\right)$, $\left(-\sqrt{6}, -\frac{5}{36}\sqrt{6}\right)$

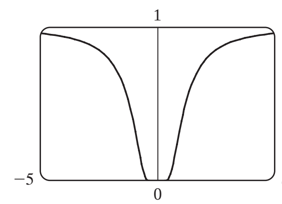


37. Inc. on $(-0.23, 0)$, $(1.62, \infty)$; dec. on $(-\infty, -0.23)$, $(0, 1.62)$;
loc. max. $f(0) = 2$; loc. min. $f(-0.23) \approx 1.96$, $f(1.62) \approx -19.2$;
CU on $(-\infty, -0.12)$, $(1.24, \infty)$;
CD on $(-0.12, 1.24)$; IP $(-0.12, 1.98)$, $(1.24, -12.1)$



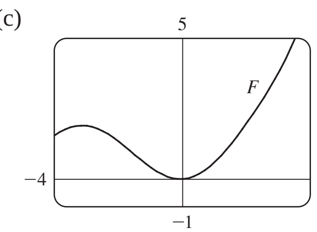


39.



$(\pm 0.82, 0.22)$; $\left(\pm\sqrt{2/3}, e^{-3/2}\right)$

41. -2.96, -0.18, 3.01; -1.57, 1.57; -2.16, -0.75, 0.46, 2.21
43. For $C > -1$, f is periodic with period 2π and has local maxima at $2n\pi + \pi/2$, n an integer. For $C \leqslant -1$, f has no graph. For $-1 < C \leqslant 1$, f has vertical asymptotes. For $C > 1$, f is continuous on $\mathbb{R}$. As C increases, f moves upward and its oscillations become less pronounced.
49. (a) 0 (b) CU on $\mathbb{R}$ **53.** $3\sqrt{3}r^2$
55. $4/\sqrt{3}$ cm from D **57.** $L = C$ **59.** \$11.50
61. 1.297383 **63.** 1.16718557
65. $f(x) = \sin x - \sin^{-1}x + C$
67. $f(x) = \frac{2}{5}x^{5/2} + \frac{3}{5}x^{5/3} + C$
69. $f(t) = t^2 + 3\cos t + 2$
71. $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$
73. $s(t) = t^2 - \tan^{-1}t + 1$
75. (b) $0.1e^x - \cos x + 0.9$ (c)



77. No
79. (b) About 8.5 in. by 2 in. (c) $20/\sqrt{3}$ in., $20\sqrt{2/3}$ in.

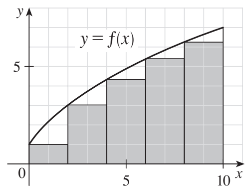
PROBLEMS PLUS ■ PAGE 352

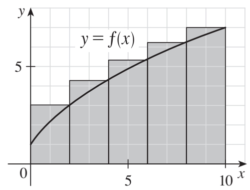
5. 24 **7.** $(-2, 4)$, $(2, -4)$ **11.** $-3.5 < a < -2.5$
13. $(m/2, m^2/4)$ **15.** $a \leqslant e^{1/e}$
19. (a) $T_1 = D/c_1$, $T_2 = (2h \sec\theta)/c_1 + (D - 2h\tan\theta)/c_2$, $T_3 = \sqrt{4h^2 + D^2}/c_1$
(c) $c_1 \approx 3.85$ km/s, $c_2 \approx 7.66$ km/s, $h \approx 0.42$ km
23. $3/\left(\sqrt[3]{2} - 1\right) \approx 11\frac{1}{2}$ h

CHAPTER 5

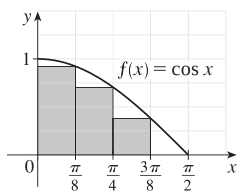
EXERCISES 5.1 ▪ PAGE 364

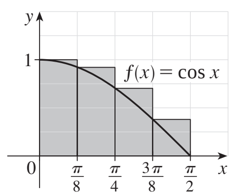
1. (a) 40, 52 (b) 43.2, 49.2



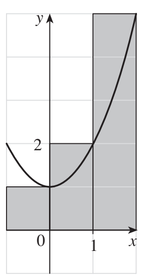


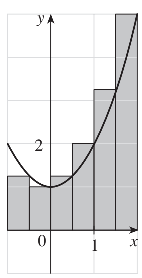
3. (a) 0.7908, underestimate (b) 1.1835, overestimate

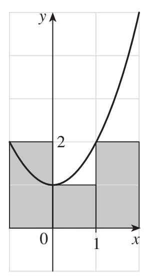


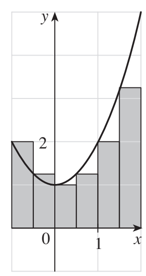


5. (a) 8, 6.875 (b) 5, 5.375

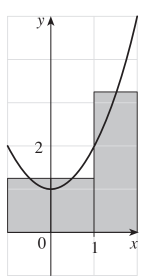


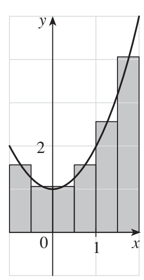






(c) 5.75, 5.9375





(d) M_6

7. 0.2533, 0.2170, 0.2101, 0.2050; 0.2

9. (a) Left: 0.8100, 0.7937, 0.7904; right: 0.7600, 0.7770, 0.7804

11. 34.7 ft, 44.8 ft **13.** 63.2 L, 70 L **15.** 155 ft

17. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + 15i/n} \cdot (15/n)$ **19.** $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i\pi}{2n} \cos \frac{i\pi}{2n}\right) \frac{\pi}{2n}$

21. The region under the graph of $y = \tan x$ from 0 to $\pi/4$

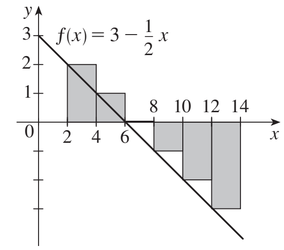
23. (a) $\lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5$ (b) $\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$ (c) $\frac{32}{3}$

25. $\sin b$, 1

EXERCISES 5.2 ▪ PAGE 376

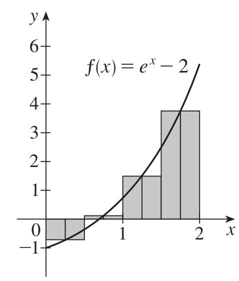
1. -6

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the net area of the rectangles with respect to the x-axis.



3. 2.322986

The Riemann sum represents the sum of the areas of the three rectangles above the x-axis minus the area of the rectangle below the x-axis.



5. (a) 4 (b) 6 (c) 10 **7.** -475, -85 **9.** 124.1644

11. 0.3084 **13.** 0.30843908, 0.30981629, 0.31015563

15.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

17. $\int_2^6 x \ln(1 + x^2)\, dx$ **19.** $\int_1^8 \sqrt{2x + x^2}\, dx$ **21.** 42

23. $\frac{4}{3}$ **25.** 3.75 **29.** $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2 + 4i/n}{1 + (2 + 4i/n)^5} \cdot \frac{4}{n}$

31. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\sin \frac{5\pi i}{n}\right) \frac{\pi}{n} = \frac{2}{5}$

33. (a) 4 (b) 10 (c) -3 (d) 2 **35.** $-\frac{3}{4}$

37. $3 + \frac{9}{4}\pi$ **39.** 2.5 **41.** 0 **43.** 3 **45.** $e^5 - e^3$

47. $\int_{-1}^{5} f(x)\, dx$ **49.** 122

51. $2m \leqslant \int_0^2 f(x)\, dx \leqslant 2M$ by Comparison Property 8

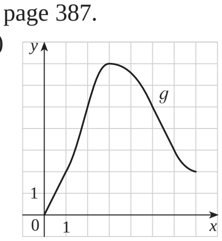
55. $3 \leqslant \int_1^4 \sqrt{x}\, dx \leqslant 6$ **57.** $\frac{\pi}{12} \leqslant \int_{\pi/4}^{\pi/3} \tan x\, dx \leqslant \frac{\pi}{12}\sqrt{3}$

59. $0 \leqslant \int_0^2 xe^{-x}\, dx \leqslant 2/e$ **69.** $\int_0^1 x^4\, dx$ **71.** $\frac{1}{2}$

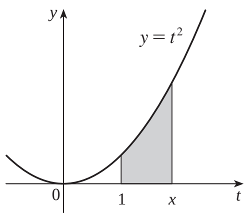
EXERCISES 5.3 ▪ PAGE 387

1. One process undoes what the other one does. See the Fundamental Theorem of Calculus, page 387.

3. (a) 0, 2, 5, 7, 3
(b) (0, 3)
(c) $x = 3$
(d)



5.



(a), (b) x^2

7. $g'(x) = 1/(x^3 + 1)$

9. $g'(y) = y^2 \sin y$ **11.** $F'(x) = -\sqrt{1 + \sec x}$

13. $h'(x) = -\dfrac{\arctan(1/x)}{x^2}$ **15.** $y' = \sqrt{\tan x + \sqrt{\tan x}}\,\sec^2 x$

17. $y' = \dfrac{3(1 - 3x)^3}{1 + (1 - 3x)^2}$ **19.** $\frac{3}{4}$ **21.** 63

23. $\frac{5}{9}$ **25.** $\frac{7}{8}$ **27.** $\frac{156}{7}$ **29.** $\frac{40}{3}$ **31.** 1 **33.** $\frac{49}{3}$

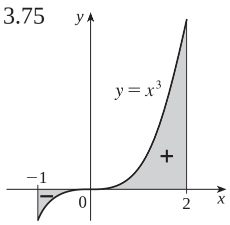
35. ln 3 **37.** π **39.** $e^2 - 1$ **41.** 0

43. The function $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied.

45. The function $f(\theta) = \sec\theta \tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied.

47. $\frac{243}{4}$ **49.** 2

51. 3.75

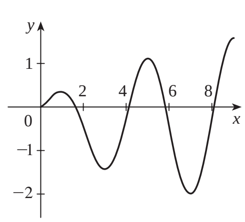


53. $g'(x) = \dfrac{-2(4x^2 - 1)}{4x^2 + 1} + \dfrac{3(9x^2 - 1)}{9x^2 + 1}$

55. $y' = 3x^{7/2}\sin(x^3) - \dfrac{\sin\sqrt{x}}{2\sqrt[4]{x}}$ **57.** $\sqrt{257}$ **59.** 29

61. (a) $-2\sqrt{n}$, $\sqrt{4n - 2}$, n an integer > 0
(b) $(0, 1)$, $\left(-\sqrt{4n - 1}, -\sqrt{4n - 3}\right)$, and $\left(\sqrt{4n - 1}, \sqrt{4n + 1}\right)$, n an integer > 0 (c) 0.74

63. (a) Loc. max. at 1 and 5; loc. min. at 3 and 7
(b) $x = 9$
(c) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$, $(8, 9)$
(d) See graph at right.



65. $\frac{1}{4}$ **73.** $f(x) = x^{3/2}$, $a = 9$

75. (b) Average expenditure over $[0, t]$; minimize average expenditure

EXERCISES 5.4 ■ PAGE 397

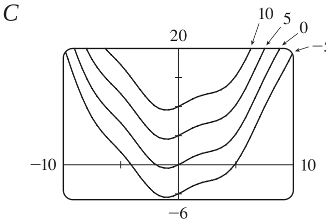
5. $\frac{1}{3}x^3 - (1/x) + C$ **7.** $\frac{1}{5}x^5 - \frac{1}{8}x^4 + \frac{1}{8}x^2 - 2x + C$

9. $2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$ **11.** $\frac{1}{3}x^3 - 4\sqrt{x} + C$

13. $-\cos x + \cosh x + C$ **15.** $\frac{1}{2}\theta^2 + \csc\theta + C$

17. $\tan\alpha + C$

19. $\sin x + \frac{1}{4}x^2 + C$



21. 18 **23.** $-2 + 1/e$ **25.** 52

27. $\frac{256}{15}$ **29.** $-\frac{63}{4}$ **31.** $\frac{55}{63}$ **33.** $2\sqrt{5}$ **35.** 8

37. $1 + \pi/4$ **39.** $\frac{256}{5}$ **41.** $\pi/6$ **43.** -3.5

45. 0, 1.32; 0.84 **47.** $\frac{4}{3}$

49. The increase in the child's weight (in pounds) between the ages of 5 and 10

51. Number of gallons of oil leaked in the first 2 hours

53. Increase in revenue when production is increased from 1000 to 5000 units

55. Newton-meters (or joules) **57.** (a) $-\frac{3}{2}$ m (b) $\frac{41}{6}$ m

59. (a) $v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s (b) $416\frac{2}{3}$ m

61. $46\frac{2}{3}$ kg **63.** 1.4 mi **65.** \$58,000

67. (b) At most 40%; $\frac{5}{36}$

EXERCISES 5.5 ■ PAGE 406

1. $-e^{-x} + C$ **3.** $\frac{2}{9}(x^3 + 1)^{3/2} + C$ **5.** $-\frac{1}{4}\cos^4\theta + C$

7. $-\frac{1}{2}\cos(x^2) + C$ **9.** $\frac{1}{63}(3x - 2)^{21} + C$

11. $\frac{1}{3}(2x + x^2)^{3/2} + C$ **13.** $-\frac{1}{3}\ln|5 - 3x| + C$

15. $-(1/\pi)\cos\pi t + C$ **17.** $\frac{2}{3}\sqrt{3ax + bx^3} + C$

19. $\frac{1}{3}(\ln x)^3 + C$ **21.** $2\sin\sqrt{t} + C$ **23.** $\frac{1}{7}\sin^7\theta + C$

25. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **27.** $\frac{1}{2}(1 + z^3)^{2/3} + C$ **29.** $e^{\tan x} + C$

31. $-1/(\sin x) + C$ **33.** $-\frac{2}{3}(\cot x)^{3/2} + C$

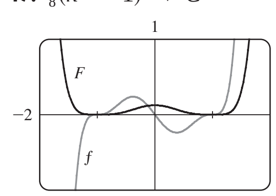
35. $-\ln(1 + \cos^2 x) + C$ **37.** $\ln|\sin x| + C$

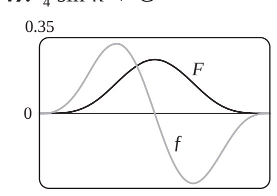
39. $\frac{1}{3}\sec^3 x + C$ **41.** $\ln|\sin^{-1}x| + C$

43. $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$

45. $\frac{4}{7}(x + 2)^{7/4} - \frac{8}{3}(x + 2)^{3/4} + C$

47. $\frac{1}{8}(x^2 - 1)^4 + C$ **49.** $\frac{1}{4}\sin^4 x + C$





51. 0 **53.** $\frac{182}{9}$ **55.** 4

57. 0 **59.** $e - \sqrt{e}$ **61.** 3 **63.** $\frac{1}{3}\left(2\sqrt{2} - 1\right)a^3$

65. $\frac{16}{15}$ **67.** 2 **69.** $\ln(e + 1)$ **71.** $\sqrt{3} - \frac{1}{3}$

73. 6π **75.** All three areas are equal. **77.** ≈ 4512 L

79. $\dfrac{5}{4\pi}\left(1 - \cos\dfrac{2\pi t}{5}\right)$ L **81.** 5 **87.** $\pi^2/4$

CHAPTER 5 REVIEW ■ PAGE 409

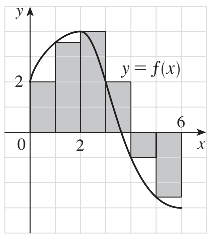
True-False Quiz

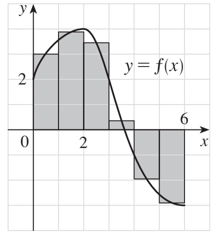
1. True **3.** True **5.** False **7.** True **9.** True

11. False **13.** False **15.** False

Exercises

1. (a) 8 (b) 5.7





3. $\frac{1}{2} + \pi/4$ **5.** 3 **7.** f is c, f' is b, $\int_0^x f(t)\,dt$ is a
9. 37 **11.** $\frac{9}{10}$ **13.** -76 **15.** $\frac{21}{4}$ **17.** Does not exist
19. $\frac{1}{3}\sin 1$ **21.** 0 **23.** $-(1/x) - 2\ln|x| + x + C$
25. $\sqrt{x^2 + 4x} + C$ **27.** $[1/(2\pi)]\sin^2 \pi t + C$
29. $2e^{\sqrt{x}} + C$ **31.** $-\frac{1}{2}[\ln(\cos x)]^2 + C$
33. $\frac{1}{4}\ln(1 + x^4) + C$ **35.** $\ln|1 + \sec\theta| + C$ **37.** $\frac{23}{3}$
39. $2\sqrt{1 + \sin x} + C$ **41.** $\frac{64}{5}$ **43.** $F'(x) = x^2/(1 + x^3)$
45. $g'(x) = 4x^3\cos(x^8)$ **47.** $y' = \left(2e^x - e^{\sqrt{x}}\right)/(2x)$
49. $4 \leqslant \int_1^3 \sqrt{x^2 + 3}\,dx \leqslant 4\sqrt{3}$ **55.** 0.280981
57. Number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2008
59. 72,400 **61.** 3 **63.** $c \approx 1.62$
65. $f(x) = e^{2x}(1 + 2x)/(1 - e^{-x})$ **71.** $\frac{2}{3}$

PROBLEMS PLUS ■ PAGE 413

1. $\pi/2$ **3.** $f(x) = \frac{1}{2}x$ **5.** -1 **7.** e^{-2} **9.** $[-1, 2]$
11. (a) $\frac{1}{2}(n - 1)n$ (b) $\frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$
17. $2\left(\sqrt{2} - 1\right)$

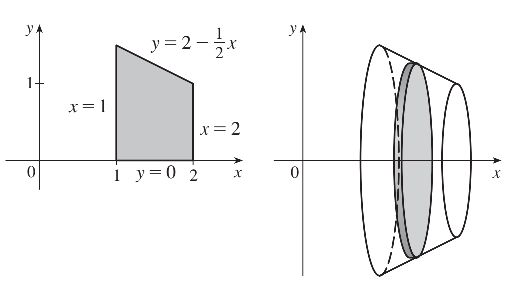
CHAPTER 6

EXERCISES 6.1 ■ PAGE 420

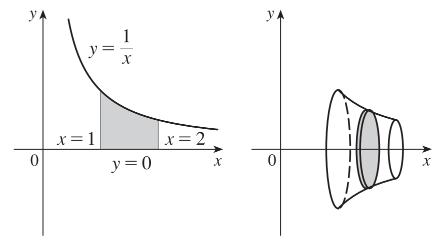
1. $\frac{32}{3}$ **3.** $e - (1/e) + \frac{10}{3}$ **5.** 19.5 **7.** $\frac{1}{6}$ **9.** $\ln 2 - \frac{1}{2}$
11. $\frac{1}{3}$ **13.** 72 **15.** $2 - 2\ln 2$ **17.** $\frac{59}{12}$ **19.** $\frac{32}{3}$
21. $\frac{8}{3}$ **23.** $\frac{1}{2}$ **25.** $\pi - \frac{2}{3}$ **27.** $\ln 2$ **29.** 6.5
31. $\frac{3}{2}\sqrt{3} - 1$ **33.** 0.6407 **35.** 0, 0.90; 0.04 **37.** 8.38
39. $12\sqrt{6} - 9$ **41.** $117\frac{1}{3}$ ft **43.** 4232 cm^2
45. (a) Car A (b) The distance by which A is ahead of B after 1 minute (c) Car A (d) $t \approx 2.2$ min
47. $\frac{24}{5}\sqrt{3}$ **49.** $4^{2/3}$ **51.** ± 6
53. $0 < m < 1$; $m - \ln m - 1$

EXERCISES 6.2 ■ PAGE 430

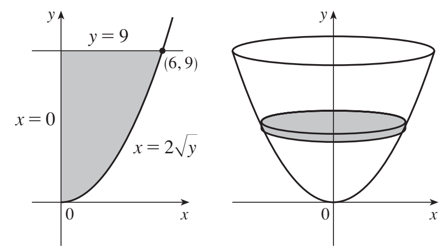
1. $19\pi/12$



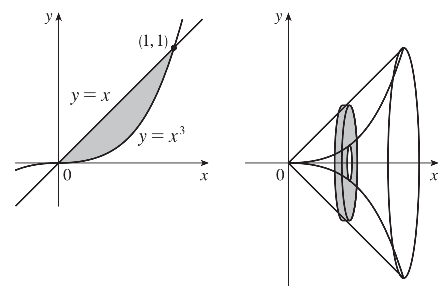
3. $\pi/2$



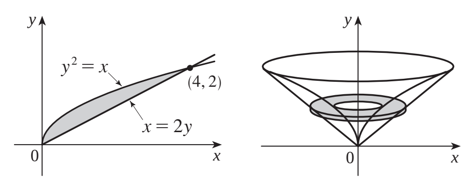
5. 162π



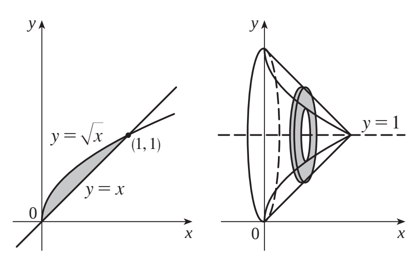
7. $4\pi/21$



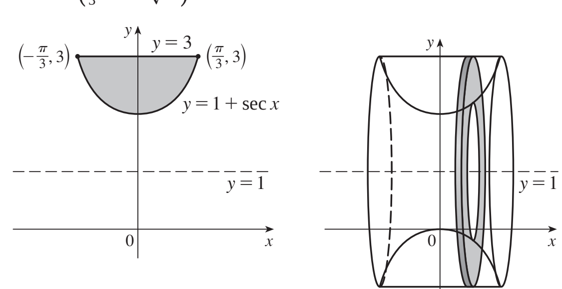
9. $64\pi/15$



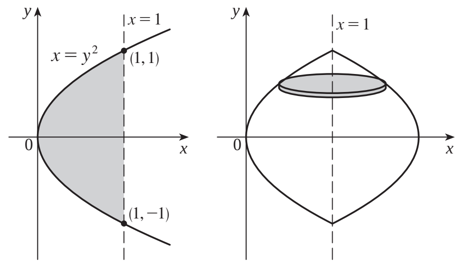
11. $\pi/6$



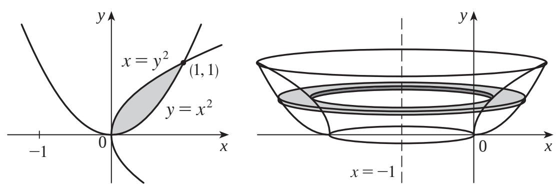
13. $2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$



15. $16\pi/15$



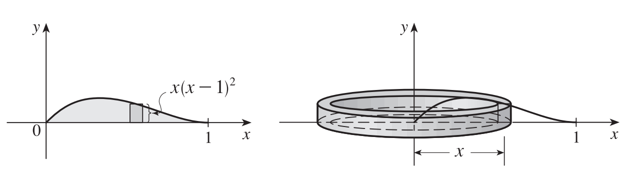
17. $29\pi/30$



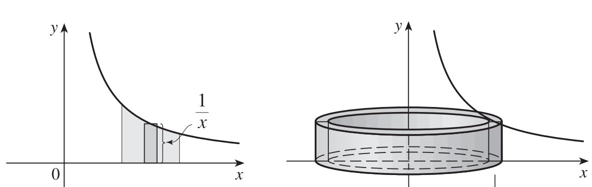
19. $\pi/7$ **21.** $\pi/10$ **23.** $\pi/2$ **25.** $7\pi/15$
27. $5\pi/14$ **29.** $13\pi/30$ **31.** $\pi\int_0^{\pi/4}(1-\tan^3 x)^2\,dx$
33. $\pi\int_0^{\pi}[1^2-(1-\sin x)^2]\,dx$
35. $\pi\int_{-2\sqrt{2}}^{2\sqrt{2}}\left[5^2-\left(\sqrt{1+y^2}+2\right)^2\right]dy$
37. $-1.288, 0.884; 23.780$ **39.** $\frac{11}{8}\pi^2$
41. Solid obtained by rotating the region $0 \leqslant y \leqslant \cos x$, $0 \leqslant x \leqslant \pi/2$ about the x-axis
43. Solid obtained by rotating the region above the x-axis bounded by $x = y^2$ and $x = y^4$ about the y-axis
45. 1110 cm^3 **47.** (a) 196 (b) 838 **49.** $\frac{1}{3}\pi r^2 h$
51. $\pi h^2\left(r-\frac{1}{3}h\right)$ **53.** $\frac{2}{3}b^2 h$ **55.** 10 cm^3 **57.** 24
59. $\frac{1}{3}$ **61.** $\frac{8}{15}$
63. (a) $8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$ (b) $2\pi^2 r^2 R$
65. (b) $\pi r^2 h$ **67.** $\frac{5}{12}\pi r^3$ **69.** $8\int_0^r\sqrt{R^2-y^2}\sqrt{r^2-y^2}\,dy$

EXERCISES 6.3 ■ PAGE 436

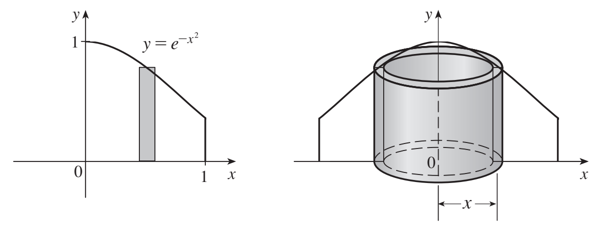
1. Circumference $= 2\pi x$, height $= x(x-1)^2$; $\pi/15$



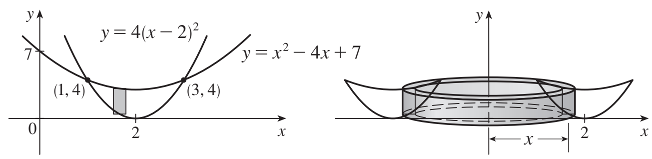
3. 2π



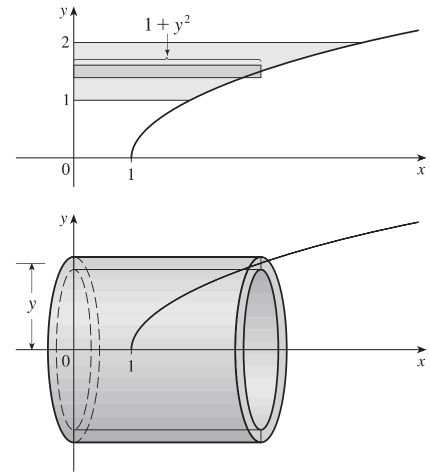
5. $\pi(1-1/e)$



7. 16π



9. $21\pi/2$



11. $768\pi/7$ **13.** $16\pi/3$ **15.** $7\pi/15$ **17.** $8\pi/3$
19. $5\pi/14$ **21.** $\int_1^2 2\pi x\ln x\,dx$
23. $\int_0^1 2\pi(x+1)[\sin(\pi x/2)-x^4]\,dx$
25. $\int_0^{\pi}2\pi(4-y)\sqrt{\sin y}\,dy$ **27.** 3.68
29. Solid obtained by rotating the region $0 \leqslant y \leqslant x^4$, $0 \leqslant x \leqslant 3$ about the y-axis
31. Solid obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$, or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$
33. 0.13 **35.** $\frac{1}{32}\pi^3$ **37.** 8π **39.** $2\pi(12-4\ln 4)$
41. $\frac{4}{3}\pi$ **43.** $\frac{4}{3}\pi r^3$ **45.** $\frac{1}{3}\pi r^2 h$

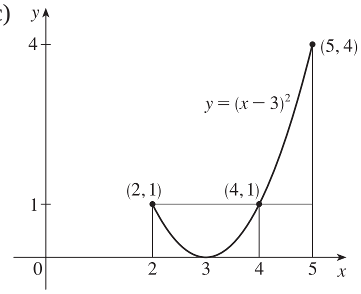
EXERCISES 6.4 ■ PAGE 441

1. 588 J **3.** 9 ft-lb **5.** 180 J **7.** $\frac{15}{4}$ ft-lb
9. (a) $\frac{25}{24} \approx 1.04$ J (b) 10.8 cm **11.** $W_2 = 3W_1$
13. (a) 625 ft-lb (b) $\frac{1875}{4}$ ft-lb **15.** 650,000 ft-lb
17. 3857 J **19.** 2450 J **21.** $\approx 1.06 \times 10^6$ J
23. $\approx 1.04 \times 10^5$ ft-lb **25.** 2.0 m **29.** $Gm_1m_2\left(\frac{1}{a}-\frac{1}{b}\right)$

EXERCISES 6.5 ■ PAGE 445

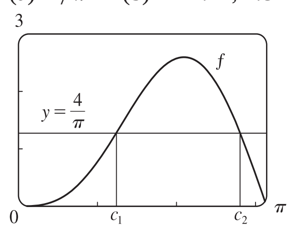
1. $\frac{8}{3}$ **3.** $\frac{45}{28}$ **5.** $\frac{1}{10}(1 - e^{-25})$ **7.** $2/(5\pi)$

9. (a) 1 (b) 2, 4 (c)



11. (a) $4/\pi$ (b) $\approx 1.24, 2.81$

(c)



15. $38\frac{1}{3}$ **17.** $(50 + 28/\pi)°\text{F} \approx 59°\text{F}$ **19.** 6 kg/m

21. $5/(4\pi) \approx 0.4$ L

CHAPTER 6 REVIEW ■ PAGE 446

Exercises

1. $\frac{8}{3}$ **3.** $\frac{7}{12}$ **5.** $\frac{4}{3} + 4/\pi$ **7.** $64\pi/15$ **9.** $1656\pi/5$

11. $\frac{4}{3}\pi(2ah + h^2)^{3/2}$ **13.** $\int_{-\pi/3}^{\pi/3} 2\pi(\pi/2 - x)\left(\cos^2 x - \frac{1}{4}\right) dx$

15. (a) $2\pi/15$ (b) $\pi/6$ (c) $8\pi/15$

17. (a) 0.38 (b) 0.87

19. Solid obtained by rotating the region $0 \leqslant y \leqslant \cos x$, $0 \leqslant x \leqslant \pi/2$ about the y-axis

21. Solid obtained by rotating the region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant 2 - \sin x$ about the x-axis

23. 36 **25.** $\frac{125}{3}\sqrt{3}\ \text{m}^3$ **27.** 3.2 J

29. (a) $8000\pi/3 \approx 8378$ ft-lb (b) 2.1 ft **31.** $f(x)$

PROBLEMS PLUS ■ PAGE 448

1. (a) $f(t) = 3t^2$ (b) $f(x) = \sqrt{2x/\pi}$ **3.** $\frac{32}{27}$

5. (b) 0.2261 (c) 0.6736 m

(d) (i) $1/(105\pi) \approx 0.003$ in/s (ii) $370\pi/3$ s ≈ 6.5 min

9. $y = \frac{32}{9}x^2$

11. (a) $V = \int_0^h \pi[f(y)]^2\,dy$ (c) $f(y) = \sqrt{kA/(\pi C)}\,y^{1/4}$

Advantage: the markings on the container are equally spaced.

13. $b = 2a$ **15.** $B = 16A$

CHAPTER 7

EXERCISES 7.1 ■ PAGE 457

1. $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$ **3.** $\frac{1}{5}x \sin 5x + \frac{1}{25}\cos 5x + C$

5. $2(r - 2)e^{r/2} + C$

7. $-\dfrac{1}{\pi}x^2 \cos \pi x + \dfrac{2}{\pi^2}x \sin \pi x + \dfrac{2}{\pi^3}\cos \pi x + C$

9. $\frac{1}{2}(2x + 1)\ln(2x + 1) - x + C$

11. $t \arctan 4t - \frac{1}{8}\ln(1 + 16t^2) + C$

13. $\frac{1}{2}t \tan 2t - \frac{1}{4}\ln|\sec 2t| + C$

15. $x(\ln x)^2 - 2x \ln x + 2x + C$

17. $\frac{1}{13}e^{2\theta}(2 \sin 3\theta - 3 \cos 3\theta) + C$

19. $\pi/3$ **21.** $1 - 1/e$ **23.** $\frac{1}{2} - \frac{1}{2}\ln 2$ **25.** $\frac{1}{4} - \frac{3}{4}e^{-2}$

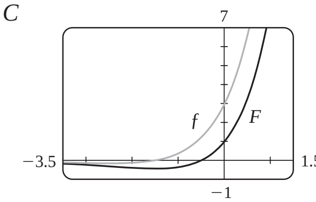
27. $\frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$ **29.** $\sin x\,(\ln \sin x - 1) + C$

31. $\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$

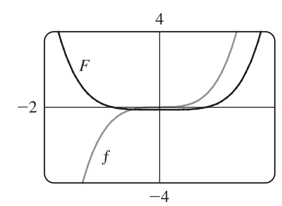
33. $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ **35.** $-\frac{1}{2} - \pi/4$

37. $\frac{1}{2}(x^2 - 1)\ln(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$

39. $(2x + 1)e^x + C$



41. $\frac{1}{3}x^2(1 + x^2)^{3/2} - \frac{2}{15}(1 + x^2)^{5/2} + C$



43. (b) $-\frac{1}{4}\cos x \sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$

45. (b) $\frac{2}{3}, \frac{8}{15}$ **51.** $x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$

53. $\frac{25}{4} - \frac{75}{4}e^{-2}$ **55.** 1.0475, 2.8731; 2.1828 **57.** $4 - 8/\pi$

59. $2\pi e$ **61.** $\frac{9}{2}\ln 3 - \frac{13}{9}$ **63.** $2 - e^{-t}(t^2 + 2t + 2)$ m

65. 2

EXERCISES 7.2 ■ PAGE 465

1. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$ **3.** $-\frac{11}{384}$

5. $\dfrac{1}{3\pi}\sin^3(\pi x) - \dfrac{2}{5\pi}\sin^5(\pi x) + \dfrac{1}{7\pi}\sin^7(\pi x) + C$

7. $\pi/4$ **9.** $3\pi/8$ **11.** $\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C$

13. $\pi/16$ **15.** $\frac{2}{45}\sqrt{\sin\alpha}\,(45 - 18\sin^2\alpha + 15\sin^4\alpha) + C$

17. $\frac{1}{2}\cos^2 x - \ln|\cos x| + C$ **19.** $\ln|\sin x| + 2\sin x + C$

21. $\frac{1}{2}\tan^2 x + C$ **23.** $\tan x - x + C$

25. $\frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C$ **27.** $\frac{117}{8}$

29. $\frac{1}{3}\sec^3 x - \sec x + C$

31. $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$

33. $\frac{1}{6}\tan^6\theta + \frac{1}{4}\tan^4\theta + C$

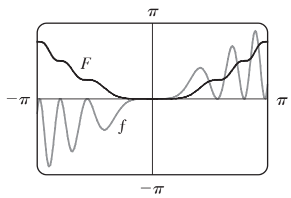
35. $x \sec x - \ln|\sec x + \tan x| + C$ **37.** $\sqrt{3} - \frac{1}{3}\pi$

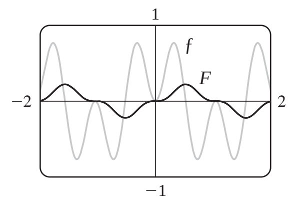
39. $\frac{1}{3}\csc^3\alpha - \frac{1}{5}\csc^5\alpha + C$ **41.** $\ln|\csc x - \cot x| + C$

43. $-\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$ **45.** $\frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$

47. $\frac{1}{2}\sin 2x + C$ **49.** $\frac{1}{10}\tan^5(t^2) + C$

51. $\frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C$ **53.** $\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$





55. 0 **57.** 1 **59.** 0 **61.** $\pi^2/4$ **63.** $\pi\left(2\sqrt{2} - \frac{5}{2}\right)$

65. $s = (1 - \cos^3\omega t)/(3\omega)$

EXERCISES 7.3 ■ PAGE 472

1. $\sqrt{x^2 - 9}/(9x) + C$ **3.** $\frac{1}{3}(x^2 - 18)\sqrt{x^2 + 9} + C$

5. $\pi/24 + \sqrt{3}/8 - \frac{1}{4}$ **7.** $-\sqrt{25 - x^2}/(25x) + C$

9. $\ln\left(\sqrt{x^2 + 16} + x\right) + C$ **11.** $\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1 - 4x^2} + C$

13. $\frac{1}{6}\sec^{-1}(x/3) - \sqrt{x^2 - 9}/(2x^2) + C$

15. $\frac{1}{16}\pi a^4$ **17.** $\sqrt{x^2 - 7} + C$

19. $\ln\left|\left(\sqrt{1 + x^2} - 1\right)/x\right| + \sqrt{1 + x^2} + C$ **21.** $\frac{9}{500}\pi$

23. $\frac{9}{2}\sin^{-1}((x - 2)/3) + \frac{1}{2}(x - 2)\sqrt{5 + 4x - x^2} + C$

25. $\sqrt{x^2 + x + 1} - \frac{1}{2}\ln\left(\sqrt{x^2 + x + 1} + x + \frac{1}{2}\right) + C$

27. $\frac{1}{2}(x + 1)\sqrt{x^2 + 2x} - \frac{1}{2}\ln\left|x + 1 + \sqrt{x^2 + 2x}\right| + C$

29. $\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1 - x^4} + C$

33. $\frac{1}{6}\left(\sqrt{48} - \sec^{-1} 7\right)$ **37.** 0.81, 2; 2.10

41. $r\sqrt{R^2 - r^2} + \pi r^2/2 - R^2\arcsin(r/R)$ **43.** $2\pi^2 Rr^2$

EXERCISES 7.4 ■ PAGE 481

1. (a) $\dfrac{A}{x + 3} + \dfrac{B}{3x + 1}$ (b) $\dfrac{A}{x} + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2}$

3. (a) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx + E}{x^2 + 4}$

(b) $\dfrac{A}{x + 3} + \dfrac{B}{(x + 3)^2} + \dfrac{C}{x - 3} + \dfrac{D}{(x - 3)^2}$

5. (a) $1 + \dfrac{A}{x - 1} + \dfrac{B}{x + 1} + \dfrac{Cx + D}{x^2 + 1}$

(b) $\dfrac{At + B}{t^2 + 1} + \dfrac{Ct + D}{t^2 + 4} + \dfrac{Et + F}{(t^2 + 4)^2}$

7. $x + 6\ln|x - 6| + C$

9. $2\ln|x + 5| - \ln|x - 2| + C$ **11.** $\frac{1}{2}\ln\frac{3}{2}$

13. $a\ln|x - b| + C$ **15.** $\frac{7}{6} + \ln\frac{2}{3}$

17. $\frac{27}{5}\ln 2 - \frac{9}{5}\ln 3$ $\left(\text{or } \frac{9}{5}\ln\frac{8}{3}\right)$

19. $-\dfrac{1}{36}\ln|x + 5| + \dfrac{1}{6}\dfrac{1}{x + 5} + \dfrac{1}{36}\ln|x - 1| + C$

21. $\frac{1}{2}x^2 - 2\ln(x^2 + 4) + 2\tan^{-1}(x/2) + C$

23. $2\ln|x| + (1/x) + 3\ln|x + 2| + C$

25. $\ln|x - 1| - \frac{1}{2}\ln(x^2 + 9) - \frac{1}{3}\tan^{-1}(x/3) + C$

27. $\frac{1}{2}\ln(x^2 + 1) + \left(1/\sqrt{2}\right)\tan^{-1}\left(x/\sqrt{2}\right) + C$

29. $\frac{1}{2}\ln(x^2 + 2x + 5) + \frac{3}{2}\tan^{-1}\left(\dfrac{x + 1}{2}\right) + C$

31. $\frac{1}{3}\ln|x - 1| - \frac{1}{6}\ln(x^2 + x + 1) - \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x + 1}{\sqrt{3}} + C$

33. $\frac{1}{4}\ln\frac{8}{3}$ **35.** $\frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2 + 4) + \dfrac{1}{8(x^2 + 4)} + C$

37. $\frac{7}{8}\sqrt{2}\tan^{-1}\left(\dfrac{x - 2}{\sqrt{2}}\right) + \dfrac{3x - 8}{4(x^2 - 4x + 6)} + C$

39. $\ln\left|\dfrac{\sqrt{x + 1} - 1}{\sqrt{x + 1} + 1}\right| + C$

41. $2 + \ln\frac{25}{9}$ **43.** $\frac{3}{10}(x^2 + 1)^{5/3} - \frac{3}{4}(x^2 + 1)^{2/3} + C$

45. $2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$

47. $\ln\left[\dfrac{(e^x + 2)^2}{e^x + 1}\right] + C$

49. $\ln|\tan t + 1| - \ln|\tan t + 2| + C$

51. $\left(x - \frac{1}{2}\right)\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}\left(\dfrac{2x - 1}{\sqrt{7}}\right) + C$

53. $-\frac{1}{2}\ln 3 \approx -0.55$

55. $\frac{1}{2}\ln\left|\dfrac{x - 2}{x}\right| + C$ **59.** $\frac{1}{5}\ln\left|\dfrac{2\tan(x/2) - 1}{\tan(x/2) + 2}\right| + C$

61. $4\ln\frac{2}{3} + 2$ **63.** $-1 + \frac{11}{3}\ln 2$

65. $t = -\ln P - \frac{1}{9}\ln(0.9P + 900) + C$, where $C \approx 10.23$

67. (a) $\dfrac{24{,}110}{4879}\dfrac{1}{5x + 2} - \dfrac{668}{323}\dfrac{1}{2x + 1} - \dfrac{9438}{80{,}155}\dfrac{1}{3x - 7} + \dfrac{1}{260{,}015}\dfrac{22{,}098x + 48{,}935}{x^2 + x + 5}$

(b) $\dfrac{4822}{4879}\ln|5x + 2| - \dfrac{334}{323}\ln|2x + 1| - \dfrac{3146}{80{,}155}\ln|3x - 7| + \dfrac{11{,}049}{260{,}015}\ln(x^2 + x + 5) + \dfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\dfrac{2x + 1}{\sqrt{19}} + C$

The CAS omits the absolute value signs and the constant of integration.

EXERCISES 7.5 ■ PAGE 488

1. $\sin x + \frac{1}{3}\sin^3 x + C$

3. $\sin x + \ln|\csc x - \cot x| + C$

5. $4 - \ln 9$ **7.** $e^{\pi/4} - e^{-\pi/4}$

9. $\frac{243}{5}\ln 3 - \frac{242}{25}$ **11.** $\frac{1}{2}\ln(x^2 - 4x + 5) + \tan^{-1}(x - 2) + C$

13. $\frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C$ $\left(\text{or } \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C\right)$

15. $x/\sqrt{1 - x^2} + C$

17. $\frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$ $\left(\text{or } \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C\right)$

19. $e^{e^x} + C$ **21.** $(x + 1)\arctan\sqrt{x} - \sqrt{x} + C$

23. $\frac{4097}{45}$ **25.** $3x + \frac{23}{3}\ln|x - 4| - \frac{5}{3}\ln|x + 2| + C$

27. $x - \ln(1 + e^x) + C$ **29.** $15 + 7\ln\frac{2}{7}$

31. $\sin^{-1}x - \sqrt{1 - x^2} + C$

33. $2\sin^{-1}\left(\dfrac{x + 1}{2}\right) + \dfrac{x + 1}{2}\sqrt{3 - 2x - x^2} + C$

35. 0 **37.** $\pi/8 - \frac{1}{4}$ **39.** $\ln|\sec\theta - 1| - \ln|\sec\theta| + C$

41. $\theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$ **43.** $\frac{2}{3}(1 + e^x)^{3/2} + C$

45. $-\frac{1}{3}(x^3 + 1)e^{-x^3} + C$

47. $\ln|x - 1| - 3(x - 1)^{-1} - \frac{3}{2}(x - 1)^{-2} - \frac{1}{3}(x - 1)^{-3} + C$

49. $\ln\left|\dfrac{\sqrt{4x + 1} - 1}{\sqrt{4x + 1} + 1}\right| + C$ **51.** $-\ln\left|\dfrac{\sqrt{4x^2 + 1} + 1}{2x}\right| + C$

53. $\dfrac{1}{m}x^2\cosh(mx) - \dfrac{2}{m^2}x\sinh(mx) + \dfrac{2}{m^3}\cosh(mx) + C$

55. $2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

57. $\frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + C$

59. $\sin(\sin x) - \frac{1}{3}\sin^3(\sin x) + C$ **61.** $2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$

63. $-\tan^{-1}(\cos^2 x) + C$ **65.** $\frac{2}{3}[(x+1)^{3/2} - x^{3/2}] + C$

67. $\sqrt{2} - 2/\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{2}\right)$

69. $e^x - \ln(1+e^x) + C$

71. $-\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + C$

73. $\frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{8}\tan^{-1}(x/2) + C$

75. $2(x-2)\sqrt{1+e^x} + 2\ln\dfrac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C$

77. $\frac{2}{3}\tan^{-1}(x^{3/2}) + C$

79. $\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C$ **81.** $xe^{x^2} + C$

EXERCISES 7.6 ■ PAGE 493

1. $(-1/x)\sqrt{7-2x^2} - \sqrt{2}\sin^{-1}\left(\sqrt{2}x/\sqrt{7}\right) + C$

3. $\dfrac{1}{2\pi}\sec(\pi x)\tan(\pi x) + \dfrac{1}{2\pi}\ln|\sec(\pi x) + \tan(\pi x)| + C$

5. $\pi/4$ **7.** $\dfrac{1}{2\pi}\tan^2(\pi x) + \dfrac{1}{\pi}\ln|\cos(\pi x)| + C$

9. $-\sqrt{4x^2+9}/(9x) + C$ **11.** $e - 2$

13. $-\frac{1}{2}\tan^2(1/z) - \ln|\cos(1/z)| + C$

15. $\frac{1}{2}(e^{2x}+1)\arctan(e^x) - \frac{1}{2}e^x + C$

17. $\dfrac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\dfrac{2y-1}{\sqrt{7}}\right)$
$- \frac{1}{12}(6+4y-4y^2)^{3/2} + C$

19. $\frac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C$

21. $\dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C$

23. $\frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C$

25. $\frac{1}{2}(\ln x)\sqrt{4+(\ln x)^2} + 2\ln\left[\ln x + \sqrt{4+(\ln x)^2}\right] + C$

27. $\sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C$

29. $\frac{1}{5}\ln|x^5 + \sqrt{x^{10}-2}| + C$ **31.** $2\pi^2$

35. $\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x + C$

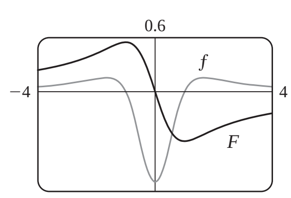
37. $\frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4} + x\right) + C$

39. $\frac{1}{10}(1+2x)^{5/2} - \frac{1}{6}(1+2x)^{3/2} + C$

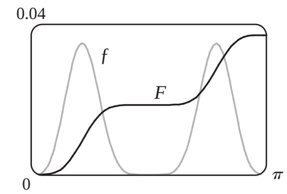
41. $-\ln|\cos x| - \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

43. (a) $-\ln\left|\dfrac{1+\sqrt{1-x^2}}{x}\right| + C$;
both have domain $(-1, 0) \cup (0, 1)$

45. $F(x) = \frac{1}{2}\ln(x^2 - x + 1) - \frac{1}{2}\ln(x^2 + x + 1)$;
max. at -1, min. at 1; IP at -1.7, 0, and 1.7



47. $F(x) = -\frac{1}{10}\sin^3 x\cos^7 x - \frac{3}{80}\sin x\cos^7 x + \frac{1}{160}\sin x\cos^5 x$
$+ \frac{1}{128}\sin x\cos^3 x + \frac{3}{256}\sin x\cos x + \frac{3}{256}x$;
max. at π, min. at 0; IP at 0.7, $\pi/2$, and 2.5



EXERCISES 7.7 ■ PAGE 505

1. (a) $L_2 = 6$, $R_2 = 12$, $M_2 \approx 9.6$
(b) L_2 is an underestimate, R_2 and M_2 are overestimates.
(c) $T_2 = 9 < I$ (d) $L_n < T_n < I < M_n < R_n$

3. (a) $T_4 \approx 0.895759$ (underestimate)
(b) $M_4 \approx 0.908907$ (overestimate)
$T_4 < I < M_4$

5. (a) 5.932957, $E_M \approx -0.063353$
(b) 5.869247, $E_S \approx 0.000357$

7. (a) 2.413790 (b) 2.411453 (c) 2.412232

9. (a) 0.146879 (b) 0.147391 (c) 0.147219

11. (a) 0.451948 (b) 0.451991 (c) 0.451976

13. (a) 4.513618 (b) 4.748256 (c) 4.675111

15. (a) -0.495333 (b) -0.543321 (c) -0.526123

17. (a) 1.064275 (b) 1.067416 (c) 1.074915

19. (a) $T_8 \approx 0.902333$, $M_8 \approx 0.905620$
(b) $|E_T| \leqslant 0.0078$, $|E_M| \leqslant 0.0039$
(c) $n = 71$ for T_n, $n = 50$ for M_n

21. (a) $T_{10} \approx 1.983524$, $E_T \approx 0.016476$;
$M_{10} \approx 2.008248$, $E_M \approx -0.008248$;
$S_{10} \approx 2.000110$, $E_S \approx -0.000110$
(b) $|E_T| \leqslant 0.025839$, $|E_M| \leqslant 0.012919$, $|E_S| \leqslant 0.000170$
(c) $n = 509$ for T_n, $n = 360$ for M_n, $n = 22$ for S_n

23. (a) 2.8 (b) 7.954926518 (c) 0.2894
(d) 7.954926521 (e) The actual error is much smaller.
(f) 10.9 (g) 7.953789422 (h) 0.0593
(i) The actual error is smaller. (j) $n \geqslant 50$

25.

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	-0.286599	-0.014771	0.007379
10	0.132218	-0.139610	-0.003696	0.001848
20	0.067033	-0.068881	-0.000924	0.000462

Observations are the same as after Example 1.

27.

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

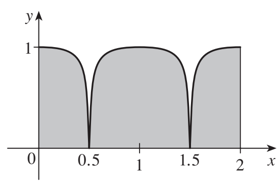
Observations are the same as after Example 1.

29. (a) 19.8 (b) 20.6 (c) $20.5\overline{3}$

31. (a) 23.44 (b) 0.3413 **33.** $37.7\overline{3}$ ft/s

35. 10,177 megawatt-hours **37.** 828 **39.** 6.0 **41.** 59.4

43.



EXERCISES 7.8 ▪ PAGE 515

Abbreviations: C, convergent; D, divergent

1. (a) Infinite interval (b) Infinite discontinuity
(c) Infinite discontinuity (d) Infinite interval

3. $\frac{1}{2} - 1/(2t^2)$; 0.495, 0.49995, 0.4999995; 0.5

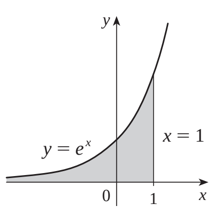
5. $\frac{1}{12}$ **7.** D **9.** $2e^{-2}$ **11.** D **13.** 0 **15.** D

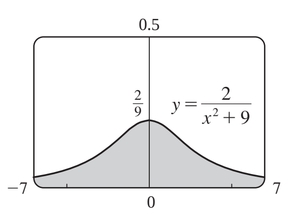
17. D **19.** $\frac{1}{25}$ **21.** D **23.** $\pi/9$

25. $\frac{1}{2}$ **27.** D **29.** $\frac{32}{3}$ **31.** D **33.** $\frac{75}{4}$

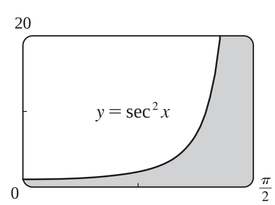
35. D **37.** $-2/e$ **39.** $\frac{8}{3}\ln 2 - \frac{8}{9}$

41. e **43.** $2\pi/3$





45. Infinite area

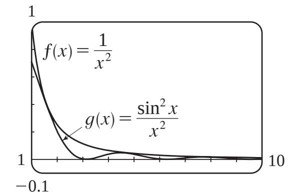


47. (a)

t	$\int_1^t [(\sin^2 x)/x^2]\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1,000	0.672957
10,000	0.673407

It appears that the integral is convergent.

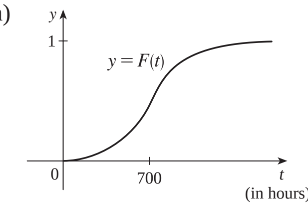
(c)



49. C **51.** D **53.** D **55.** π **57.** $p < 1$, $1/(1-p)$

59. $p > -1$, $-1/(p+1)^2$ **65.** $\sqrt{2GM/R}$

67. (a)



(b) The rate at which the fraction $F(t)$ increases as t increases
(c) 1; all bulbs burn out eventually

69. 1000

71. (a) $F(s) = 1/s$, $s > 0$ (b) $F(s) = 1/(s-1)$, $s > 1$
(c) $F(s) = 1/s^2$, $s > 0$

77. $C = 1$; $\ln 2$ **79.** No

CHAPTER 7 REVIEW ▪ PAGE 518

True-False Quiz

1. False **3.** False **5.** False **7.** False

9. (a) True (b) False **11.** False **13.** False

Exercises

1. $5 + 10\ln\frac{2}{3}$ **3.** $\ln 2$ **5.** $\frac{2}{15}$

7. $-\cos(\ln t) + C$ **9.** $\frac{64}{5}\ln 4 - \frac{124}{25}$

11. $\sqrt{3} - \frac{1}{3}\pi$ **13.** $3e^{\sqrt[3]{x}}\left(\sqrt[3]{x^2} - 2\sqrt[3]{x} + 2\right) + C$

15. $-\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x+2| + C$

17. $x\sec x - \ln|\sec x + \tan x| + C$

19. $\frac{1}{18}\ln(9x^2 + 6x + 5) + \frac{1}{9}\tan^{-1}\left[\frac{1}{2}(3x+1)\right] + C$

21. $\ln\left|x - 2 + \sqrt{x^2 - 4x}\right| + C$

23. $\ln\left|\dfrac{\sqrt{x^2+1} - 1}{x}\right| + C$

25. $\frac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(x/\sqrt{2}\right) + C$

27. $\frac{2}{5}$ **29.** 0 **31.** $6 - \frac{3}{2}\pi$

33. $\dfrac{x}{\sqrt{4 - x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$

35. $4\sqrt{1 + \sqrt{x}} + C$ **37.** $\frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$

39. $\frac{1}{8}e - \frac{1}{4}$ **41.** $\frac{1}{36}$ **43.** D

45. $4\ln 4 - 8$ **47.** $-\frac{4}{3}$ **49.** $\pi/4$

51. $(x+1)\ln(x^2 + 2x + 2) + 2\arctan(x+1) - 2x + C$

53. 0

55. $\frac{1}{4}(2x-1)\sqrt{4x^2 - 4x - 3} -$
$\ln\left|2x - 1 + \sqrt{4x^2 - 4x - 3}\right| + C$

57. $\frac{1}{2}\sin x\sqrt{4+\sin^2 x}+2\ln\left(\sin x+\sqrt{4+\sin^2 x}\right)+C$
61. No
63. (a) 1.925444 (b) 1.920915 (c) 1.922470
65. (a) 0.01348, $n \geqslant 368$ (b) 0.00674, $n \geqslant 260$
67. 8.6 mi
69. (a) 3.8 (b) 1.7867, 0.000646 (c) $n \geqslant 30$
71. C **73.** 2 **75.** $\frac{3}{16}\pi^2$

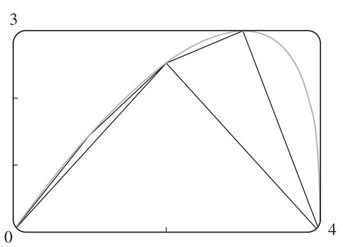
PROBLEMS PLUS ■ PAGE 521

1. About 1.85 inches from the center **3.** 0
7. $f(\pi) = -\pi/2$ **11.** $(b^b a^{-a})^{1/(b-a)} e^{-1}$
13. $2 - \sin^{-1}\left(2/\sqrt{5}\right)$

CHAPTER 8

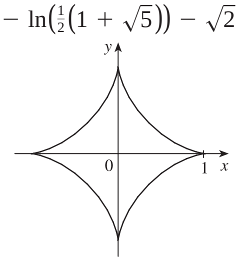
EXERCISES 8.1 ■ PAGE 530

1. $4\sqrt{5}$ **3.** $\int_0^{2\pi}\sqrt{1+\sin^2 x}\,dx$ **5.** $\int_1^4 \sqrt{9y^4+6y^2+2}\,dy$
7. $\frac{2}{243}\left(82\sqrt{82}-1\right)$ **9.** $\frac{1261}{240}$ **11.** $\frac{32}{3}$
13. $\ln\left(\sqrt{2}+1\right)$ **15.** $\ln 3 - \frac{1}{2}$
17. $\sqrt{1+e^2}-\sqrt{2}+\ln\left(\sqrt{1+e^2}-1\right)-1-\ln\left(\sqrt{2}-1\right)$
19. $\sqrt{2}+\ln\left(1+\sqrt{2}\right)$ **21.** $\frac{46}{3}$ **23.** 5.115840
25. 1.569619
27. (a), (b)



$L_1 = 4$,
$L_2 \approx 6.43$,
$L_4 \approx 7.50$

(c) $\int_0^4 \sqrt{1+[4(3-x)/(3(4-x)^{2/3})]^2}\,dx$ (d) 7.7988
29. $\sqrt{5}-\ln\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)-\sqrt{2}+\ln\left(1+\sqrt{2}\right)$
31. 6



33. $s(x) = \frac{2}{27}\left[(1+9x)^{3/2}-10\sqrt{10}\right]$ **35.** $2\sqrt{2}\left(\sqrt{1+x}-1\right)$
37. 209.1 m **39.** 29.36 in. **41.** 12.4

EXERCISES 8.2 ■ PAGE 537

1. (a) $\int_0^1 2\pi x^4\sqrt{1+16x^6}\,dx$ (b) $\int_0^1 2\pi x\sqrt{1+16x^6}\,dx$
3. (a) $\displaystyle\int_0^1 2\pi\tan^{-1}x\sqrt{1+\frac{1}{(1+x^2)^2}}\,dx$
(b) $\displaystyle\int_0^1 2\pi x\sqrt{1+\frac{1}{(1+x^2)^2}}\,dx$
5. $\frac{1}{27}\pi\left(145\sqrt{145}-1\right)$ **7.** $\frac{98}{3}\pi$
9. $2\sqrt{1+\pi^2}+(2/\pi)\ln\left(\pi+\sqrt{1+\pi^2}\right)$ **11.** $\frac{21}{2}\pi$
13. $\frac{1}{27}\pi\left(145\sqrt{145}-10\sqrt{10}\right)$ **15.** πa^2
17. 9.023754 **19.** 13.527296
21. $\frac{1}{4}\pi\left[4\ln\left(\sqrt{17}+4\right)-4\ln\left(\sqrt{2}+1\right)-\sqrt{17}+4\sqrt{2}\right]$
23. $\frac{1}{6}\pi\left[\ln\left(\sqrt{10}+3\right)+3\sqrt{10}\right]$
27. (a) $\frac{1}{3}\pi a^2$ (b) $\frac{56}{45}\pi\sqrt{3}a^2$
29. (a) $2\pi\left[b^2+\dfrac{a^2 b\sin^{-1}\left(\sqrt{a^2-b^2}/a\right)}{\sqrt{a^2-b^2}}\right]$
(b) $2\pi\left[a^2+\dfrac{ab^2\sin^{-1}\left(\sqrt{b^2-a^2}/b\right)}{\sqrt{b^2-a^2}}\right]$
31. $\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx$ **33.** $4\pi^2 r^2$

EXERCISES 8.3 ■ PAGE 547

1. (a) 187.5 lb/ft^2 (b) 1875 lb (c) 562.5 lb
3. 6000 lb **5.** 6.7×10^4 N **7.** 9.8×10^3 N
9. 1.2×10^4 lb **11.** $\frac{2}{3}\delta ah$ **13.** 5.27×10^5 N
15. (a) 314 N (b) 353 N
17. (a) 5.63×10^3 lb (b) 5.06×10^4 lb
(c) 4.88×10^4 lb (d) 3.03×10^5 lb
19. 2.5×10^5 N **21.** 230; $\frac{23}{7}$ **23.** 10; 1; $\left(\frac{1}{21}, \frac{10}{21}\right)$
25. (0, 1.6) **27.** $\left(\dfrac{1}{e-1}, \dfrac{e+1}{4}\right)$ **29.** $\left(\frac{9}{20}, \frac{9}{20}\right)$
31. $\left(\dfrac{\pi\sqrt{2}-4}{4\left(\sqrt{2}-1\right)}, \dfrac{1}{4\left(\sqrt{2}-1\right)}\right)$ **33.** (2, 0)
35. 60; 160; $\left(\frac{8}{3}, 1\right)$ **37.** (0.781, 1.330) **41.** $\left(0, \frac{1}{12}\right)$
45. $\frac{1}{3}\pi r^2 h$

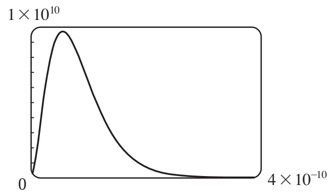
EXERCISES 8.4 ■ PAGE 553

1. \$38,000 **3.** \$43,866,933.33 **5.** \$407.25
7. \$12,000 **9.** 3727; \$37,753
11. $\frac{2}{3}\left(16\sqrt{2}-8\right) \approx$ \$9.75 million **13.** $\dfrac{(1-k)(b^{2-k}-a^{2-k})}{(2-k)(b^{1-k}-a^{1-k})}$
15. 1.19×10^{-4} cm^3/s
17. 6.60 L/min **19.** 5.77 L/min

EXERCISES 8.5 ■ PAGE 560

1. (a) The probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles
(b) The probability that a randomly chosen tire will have a lifetime of at least 25,000 miles
3. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) $1 - \frac{3}{8}\sqrt{3} \approx 0.35$
5. (a) $1/\pi$ (b) $\frac{1}{2}$
7. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) 5
11. (a) $e^{-4/2.5} \approx 0.20$ (b) $1 - e^{-2/2.5} \approx 0.55$ (c) If you aren't served within 10 minutes, you get a free hamburger.
13. $\approx 44\%$
15. (a) 0.0668 (b) $\approx 5.21\%$
17. ≈ 0.9545

19. (b) 0; a_0 (c)



(d) $1 - 41e^{-8} \approx 0.986$ (e) $\frac{3}{2}a_0$

CHAPTER 8 REVIEW ■ PAGE 562

Exercises

1. $\frac{15}{2}$ **3.** (a) $\frac{21}{16}$ (b) $\frac{41}{10}\pi$ **5.** 3.292287 **7.** $\frac{124}{5}$
9. ≈ 458 lb **11.** $\left(\frac{8}{5}, 1\right)$ **13.** $\left(2, \frac{2}{3}\right)$ **15.** $2\pi^2$
17. \$7166.67
19. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) ≈ 0.3455 (c) 5, yes
21. (a) $1 - e^{-3/8} \approx 0.31$ (b) $e^{-5/4} \approx 0.29$
(c) $8 \ln 2 \approx 5.55$ min

PROBLEMS PLUS ■ PAGE 564

1. $\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$
3. (a) $2\pi r(r \pm d)$ (b) $\approx 3.36 \times 10^6$ mi^2
(d) $\approx 7.84 \times 10^7$ mi^2
5. (a) $P(z) = P_0 + g\int_0^z \rho(x)\,dx$
(b) $(P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\,dx$
7. Height $\sqrt{2}\,b$, volume $\left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$ **9.** 0.14 m
11. $2/\pi$, $1/\pi$

CHAPTER 9

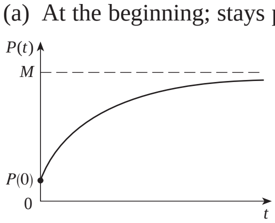
EXERCISES 9.1 ■ PAGE 571

3. (a) $\frac{1}{2}$, -1 **5.** (d)

7. (a) It must be either 0 or decreasing
(c) $y = 0$ (d) $y = 1/(x + 2)$

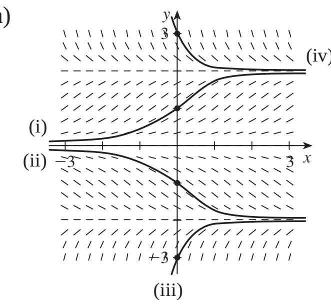
9. (a) $0 < P < 4200$ (b) $P > 4200$
(c) $P = 0$, $P = 4200$

13. (a) At the beginning; stays positive, but decreases
(c)



EXERCISES 9.2 ■ PAGE 578

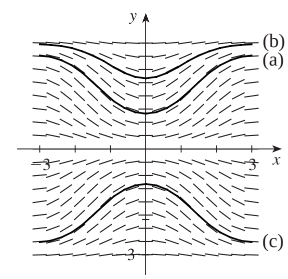
1. (a)



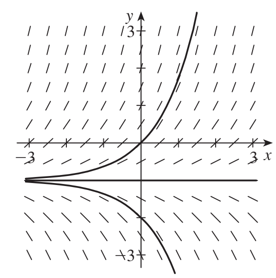
(b) $y = 0$,
$y = 2$,
$y = -2$

3. III **5.** IV

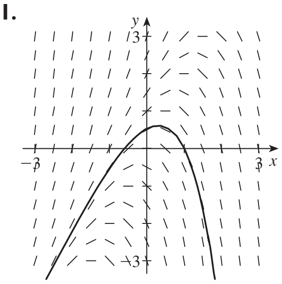
7.



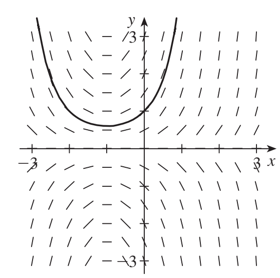
9.



11.



13.



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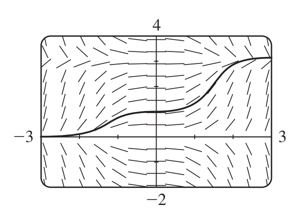
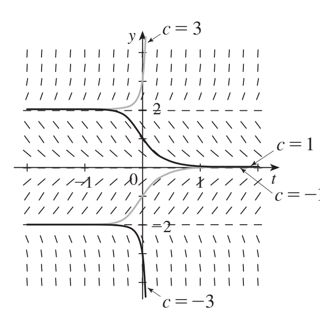
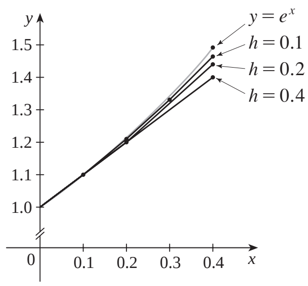


17.



$-2 \leqslant c \leqslant 2$; $-2, 0, 2$

19. (a) (i) 1.4 (ii) 1.44 (iii) 1.4641

(b) 

Underestimates

(c) (i) 0.0918 (ii) 0.0518 (iii) 0.0277

It appears that the error is also halved (approximately).

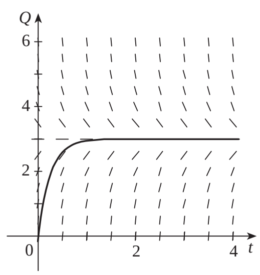
21. $-1, -3, -6.5, -12.25$ **23.** 1.7616

25. (a) (i) 3 (ii) 2.3928 (iii) 2.3701 (iv) 2.3681

(c) (i) -0.6321 (ii) -0.0249 (iii) -0.0022 (iv) -0.0002

It appears that the error is also divided by 10 (approximately).

27. (a), (d)



(b) 3

(c) Yes; $Q = 3$

(e) 2.77 C

EXERCISES 9.3 ■ PAGE 586

1. $y = Kx$ **3.** $y = K\sqrt{x^2 + 1}$

5. $y + \ln|\sec y| = \frac{1}{3}x^3 + x + C$

7. $y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$ **9.** $u = Ae^{2t + t^2/2} - 1$

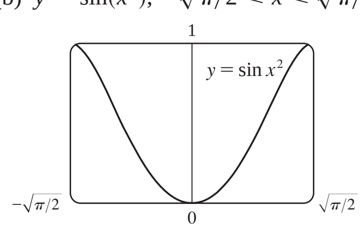
11. $y = -\sqrt{x^2 + 9}$ **13.** $\cos x + x \sin x = y^2 + \frac{1}{3}e^{3y} + \frac{2}{3}$

15. $u = -\sqrt{t^2 + \tan t + 25}$ **17.** $y = \dfrac{4a}{\sqrt{3}} \sin x - a$

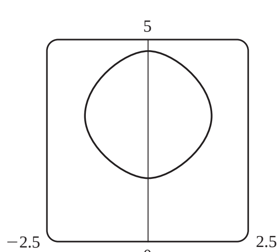
19. $y = e^{x^2/2}$ **21.** $y = Ke^x - x - 1$

23. (a) $\sin^{-1} y = x^2 + C$

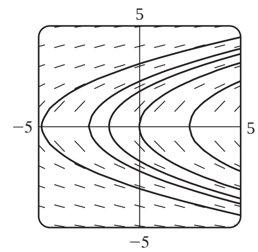
(b) $y = \sin(x^2)$, $-\sqrt{\pi/2} \leqslant x \leqslant \sqrt{\pi/2}$ (c) No



25. $\cos y = \cos x - 1$

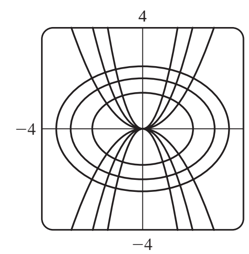


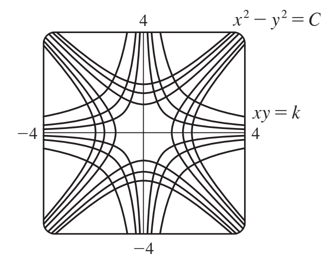
27. (a), (c)



(b) $y = \pm\sqrt{2(x + C)}$

29. $y = Cx^2$ **31.** $x^2 - y^2 = C$





33. $Q(t) = 3 - 3e^{-4t}$; 3 **35.** $P(t) = M - Me^{-kt}$; M

37. (a) $x = a - \dfrac{4}{(kt + 2/\sqrt{a})^2}$

(b) $t = \dfrac{2}{k\sqrt{a - b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a - b}} - \tan^{-1}\sqrt{\dfrac{b - x}{a - b}}\right)$

39. (a) $C(t) = (C_0 - r/k)e^{-kt} + r/k$

(b) r/k; the concentration approaches r/k regardless of the value of C_0

41. (a) $15e^{-t/100}$ kg (b) $15e^{-0.2} \approx 12.3$ kg

43. About 4.9% **45.** g/k

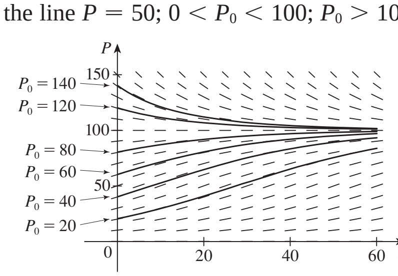
47. (a) $dA/dt = k\sqrt{A}\,(M - A)$ (b) $A(t) = M\left(\dfrac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$,

where $C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$ and $A_0 = A(0)$

EXERCISES 9.4 ■ PAGE 598

1. (a) 100; 0.05 (b) Where P is close to 0 or 100; on the line $P = 50$; $0 < P_0 < 100$; $P_0 > 100$

(c)



Solutions approach 100; some increase and some decrease, some have an inflection point but others don't; solutions with $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$

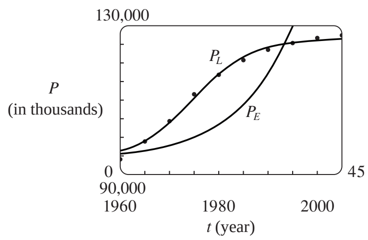
(d) $P = 0$, $P = 100$; other solutions move away from $P = 0$ and toward $P = 100$

3. (a) 3.23×10^7 kg (b) ≈ 1.55 years

5. (a) $dP/dt = \frac{1}{265}P(1 - P/100)$, P in billions
(b) 5.49 billion (c) In billions: 7.81, 27.72
(d) In billions: 5.48, 7.61, 22.41

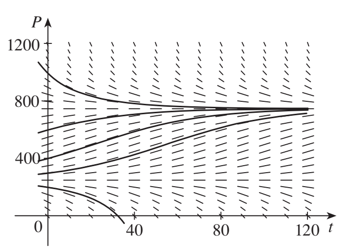
7. (a) $dy/dt = ky(1 - y)$ (b) $y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$
(c) 3:36 PM

11. $P_E(t) = 1578.3(1.0933)^t + 94{,}000$;
$P_L(t) = \dfrac{32{,}658.5}{1 + 12.75e^{-0.1706t}} + 94{,}000$



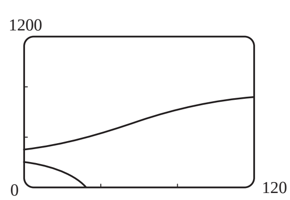
13. (a) $P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$ (b) $m < kP_0$
(c) $m = kP_0$, $m > kP_0$ (d) Declining

15. (a) Fish are caught at a rate of 15 per week.
(b) See part (d) (c) $P = 250$, $P = 750$
(d)

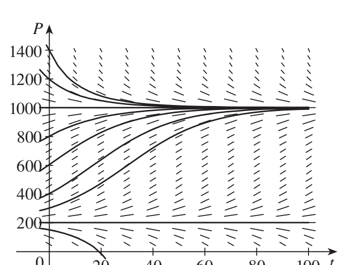


$0 < P_0 < 250$: $P \to 0$;
$P_0 = 250$: $P \to 250$;
$P_0 > 250$: $P \to 750$

(e) $P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$
where $k = \frac{1}{11}, -\frac{1}{9}$



17. (b)



$0 < P_0 < 200$: $P \to 0$;
$P_0 = 200$: $P \to 200$;
$P_0 > 200$: $P \to 1000$

(c) $P(t) = \dfrac{m(K - P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}$

19. (a) $P(t) = P_0 e^{(k/r)[\sin(rt - \phi) + \sin\phi]}$ (b) Does not exist

EXERCISES 9.5 ■ PAGE 606

1. Yes **3.** No **5.** $y = \frac{2}{3}e^x + Ce^{-2x}$

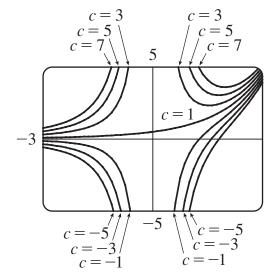
7. $y = x^2 \ln|x| + Cx^2$ **9.** $y = \frac{2}{3}\sqrt{x} + C/x$

11. $y = \dfrac{\int \sin(x^2)\,dx + C}{\sin x}$ **13.** $u = \dfrac{t^2 + 2t + 2C}{2(t + 1)}$

15. $y = -x - 1 + 3e^x$ **17.** $v = t^3e^{t^2} + 5e^{t^2}$

19. $y = -x\cos x - x$

21. $y = \dfrac{(x - 1)e^x + C}{x^2}$

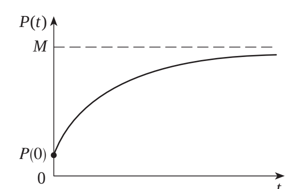


25. $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$

27. (a) $I(t) = 4 - 4e^{-5t}$ (b) $4 - 4e^{-1/2} \approx 1.57$ A

29. $Q(t) = 3(1 - e^{-4t})$, $I(t) = 12e^{-4t}$

31. $P(t) = M + Ce^{-kt}$



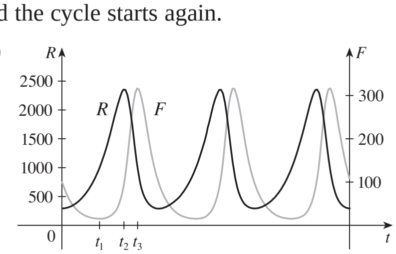
33. $y = \frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}$; 0.2275 kg/L

35. (b) mg/c (c) $(mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

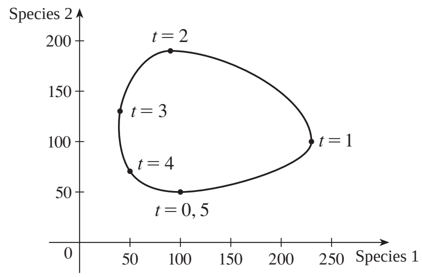
EXERCISES 9.6 ■ PAGE 612

1. (a) x = predators, y = prey; growth is restricted only by predators, which feed only on prey.
(b) x = prey, y = predators; growth is restricted by carrying capacity and by predators, which feed only on prey.

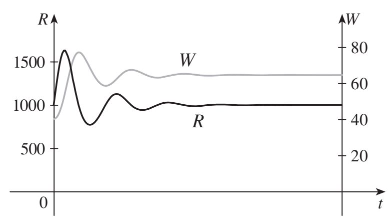
3. (a) The rabbit population starts at about 300, increases to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.
(b)



5.



9. (a) Population stabilizes at 5000.
(b) (i) $W = 0$, $R = 0$: Zero populations
(ii) $W = 0$, $R = 5000$: In the absence of wolves, the rabbit population is always 5000.
(iii) $W = 64$, $R = 1000$: Both populations are stable.
(c) The populations stabilize at 1000 rabbits and 64 wolves.
(d)



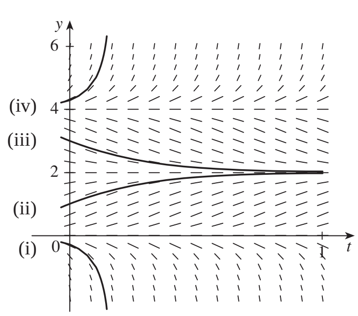
CHAPTER 9 REVIEW ■ PAGE 615

True-False Quiz

1. True **3.** False **5.** True **7.** True

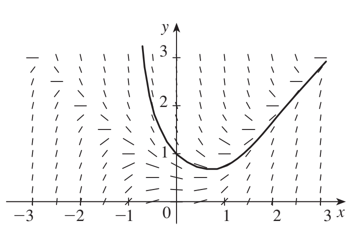
Exercises

1. (a)



(b) $0 \leqslant c \leqslant 4$;
$y = 0$, $y = 2$, $y = 4$

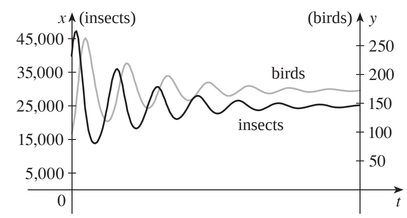
3. (a)



$y(0.3) \approx 0.8$

(b) 0.75676
(c) $y = x$ and $y = -x$; there is a local maximum or minimum
5. $y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$ **7.** $y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$
9. $r(t) = 5e^{t-t^2}$ **11.** $y = \frac{1}{2}x(\ln x)^2 + 2x$ **13.** $x = C - \frac{1}{2}y^2$

15. (a) $P(t) = \dfrac{2000}{1 + 19e^{-0.1t}}$; ≈ 560 (b) $t = -10 \ln \frac{2}{57} \approx 33.5$
17. (a) $L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$ (b) $L(t) = 53 - 43e^{-0.2t}$
19. 15 days **21.** $k \ln h + h = (-R/V)t + C$
23. (a) Stabilizes at 200,000
(b) (i) $x = 0$, $y = 0$: Zero populations
(ii) $x = 200{,}000$, $y = 0$: In the absence of birds, the insect population is always 200,000.
(iii) $x = 25{,}000$, $y = 175$: Both populations are stable.
(c) The populations stabilize at 25,000 insects and 175 birds.
(d)



25. (a) $y = (1/k)\cosh kx + a - 1/k$ or
$y = (1/k)\cosh kx - (1/k)\cosh kb + h$ (b) $(2/k)\sinh kb$

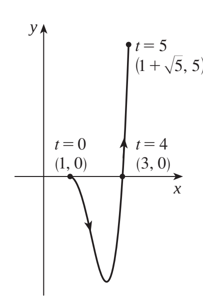
PROBLEMS PLUS ■ PAGE 618

1. $f(x) = \pm 10e^x$ **5.** $y = x^{1/n}$ **7.** 20°C
9. (b) $f(x) = \dfrac{x^2 - L^2}{4L} - \frac{1}{2}L \ln\left(\dfrac{x}{L}\right)$ (c) No
11. (a) 9.8 h (b) $31{,}900\pi \approx 100{,}000 \text{ ft}^2$; $6283 \text{ ft}^2/\text{h}$
(c) 5.1 h
13. $x^2 + (y - 6)^2 = 25$

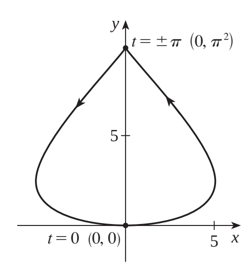
CHAPTER 10

EXERCISES 10.1 ■ PAGE 626

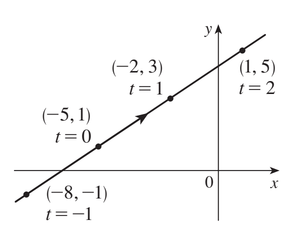
1.



3.

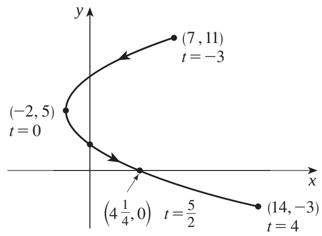


5. (a)



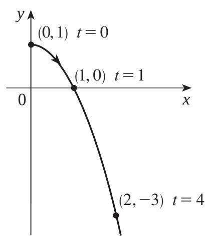
(b) $y = \frac{2}{3}x + \frac{13}{3}$

7. (a)



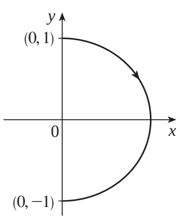
(b) $x = \frac{1}{4}(y-5)^2 - 2$, $-3 \leqslant y \leqslant 11$

9. (a)

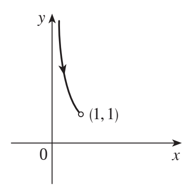


(b) $y = 1 - x^2$, $x \geqslant 0$

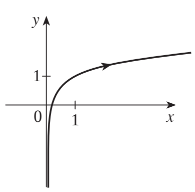
11. (a) $x^2 + y^2 = 1$, $x \geqslant 0$
(b)



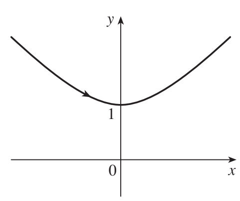
13. (a) $y = 1/x$, $y > 1$
(b)



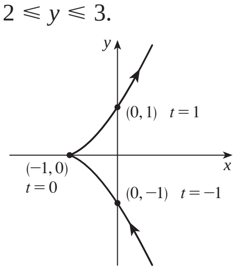
15. (a) $y = \frac{1}{2}\ln x + 1$
(b)



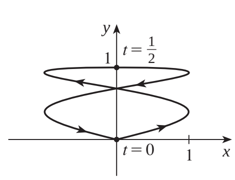
17. (a) $y^2 - x^2 = 1$, $y \geqslant 1$
(b)



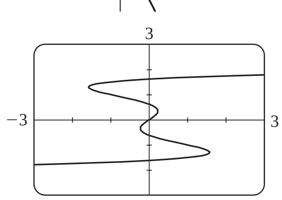
19. Moves counterclockwise along the circle $(x-3)^2 + (y-1)^2 = 4$ from (3, 3) to (3, −1)
21. Moves 3 times clockwise around the ellipse $(x^2/25) + (y^2/4) = 1$, starting and ending at (0, −2)
23. It is contained in the rectangle described by $1 \leqslant x \leqslant 4$ and $2 \leqslant y \leqslant 3$.
25.



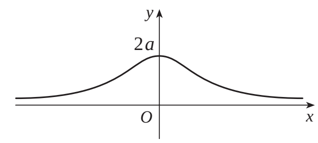
27.



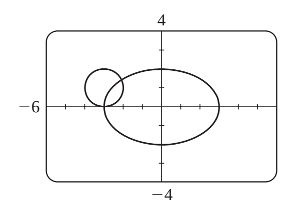
29.



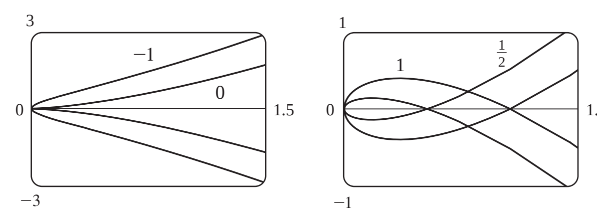
31. (b) $x = -2 + 5t$, $y = 7 - 8t$, $0 \leqslant t \leqslant 1$
33. (a) $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \leqslant t \leqslant 2\pi$
(b) $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \leqslant t \leqslant 6\pi$
(c) $x = 2\cos t$, $y = 1 + 2\sin t$, $\pi/2 \leqslant t \leqslant 3\pi/2$
37. The curve $y = x^{2/3}$ is generated in (a). In (b), only the portion with $x \geqslant 0$ is generated, and in (c) we get only the portion with $x > 0$.
41. $x = a\cos\theta$, $y = b\sin\theta$; $(x^2/a^2) + (y^2/b^2) = 1$, ellipse
43.



45. (a) Two points of intersection



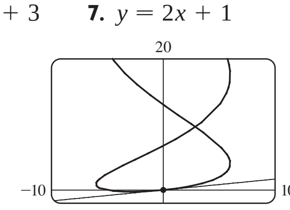
(b) One collision point at (−3, 0) when $t = 3\pi/2$
(c) There are still two intersection points, but no collision point.
47. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases.



49. As n increases, the number of oscillations increases; a and b determine the width and height.

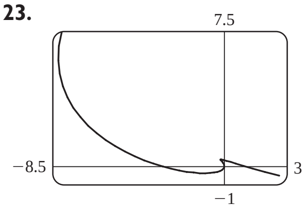
EXERCISES 10.2 ▪ PAGE 636

1. $\dfrac{2t+1}{t\cos t + \sin t}$ **3.** $y = -x$
5. $y = -(2/e)x + 3$ **7.** $y = 2x + 1$
9. $y = \frac{1}{6}x$

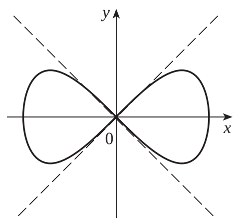


11. $1 + \frac{3}{2}t$, $3/(4t)$, $t > 0$
13. $-e^{-t}$, $e^{-t}/(1 - e^t)$, $t < 0$
15. $-\frac{3}{2}\tan t$, $-\frac{3}{4}\sec^3 t$, $\pi/2 < t < 3\pi/2$
17. Horizontal at (6, ±16), vertical at (10, 0)
19. Horizontal at $(\pm\sqrt{2}, \pm 1)$ (four points), vertical at (±2, 0)
21. (0.6, 2); $(5 \cdot 6^{-6/5}, e^{6^{-1/5}})$

23.



25. $y = x, y = -x$



27. (a) $d \sin \theta/(r - d \cos \theta)$ **29.** $\left(\frac{16}{27}, \frac{29}{9}\right)$, $(-2, -4)$

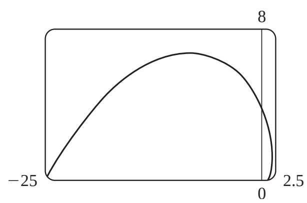
31. πab **33.** $3 - e$ **35.** $2\pi r^2 + \pi d^2$

37. $\int_1^2 \sqrt{1 + 4t^2}\, dt \approx 3.1678$

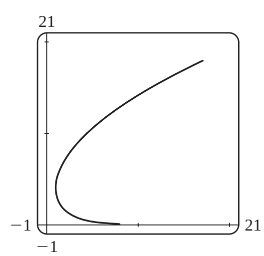
39. $\int_0^{2\pi} \sqrt{3 - 2 \sin t - 2 \cos t}\, dt \approx 10.0367$ **41.** $4\sqrt{2} - 2$

43. $-\sqrt{10}/3 + \ln\left(3 + \sqrt{10}\right) + \sqrt{2} - \ln\left(1 + \sqrt{2}\right)$

45. $\sqrt{2}\,(e^{\pi} - 1)$

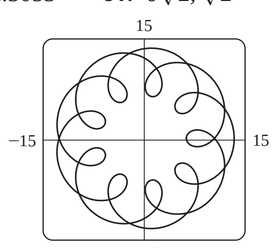


47. $e^3 + 11 - e^{-8}$



49. 612.3053 **51.** $6\sqrt{2}, \sqrt{2}$

55. (a) $t \in [0, 4\pi]$



(b) ≈ 294

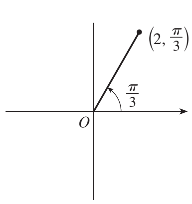
57. $\int_0^1 2\pi(t^2 + 1)e^t\sqrt{e^{2t}(t + 1)^2(t^2 + 2t + 2)}\, dt \approx 103.5999$

59. $\frac{2}{1215}\pi\left(247\sqrt{13} + 64\right)$ **61.** $\frac{6}{5}\pi a^2$ **63.** 59.101

65. $\frac{24}{5}\pi\left(949\sqrt{26} + 1\right)$ **71.** $\frac{1}{4}$

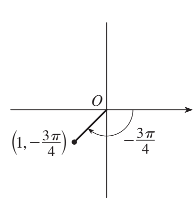
EXERCISES 10.3 ■ PAGE 647

1. (a)



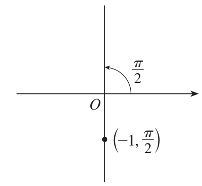
$(2, 7\pi/3), (-2, 4\pi/3)$

(b)



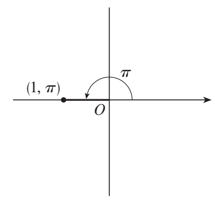
$(1, 5\pi/4), (-1, \pi/4)$

(c)



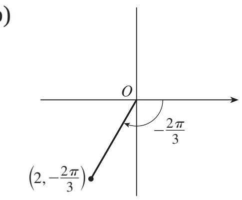
$(1, 3\pi/2), (-1, 5\pi/2)$

3. (a)



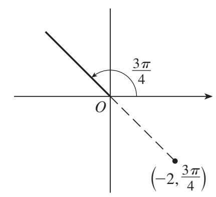
$(-1, 0)$

(b)



$\left(-1, -\sqrt{3}\right)$

(c)

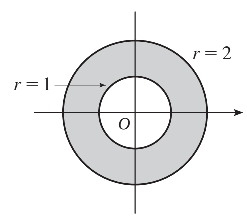


$\left(\sqrt{2}, -\sqrt{2}\right)$

5. (a) (i) $\left(2\sqrt{2}, 7\pi/4\right)$ (ii) $\left(-2\sqrt{2}, 3\pi/4\right)$

(b) (i) $(2, 2\pi/3)$ (ii) $(-2, 5\pi/3)$

7.



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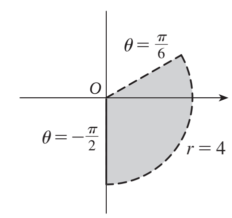
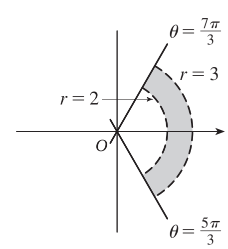


11.



13. $2\sqrt{3}$ **15.** Circle, center O, radius 2

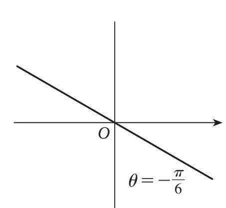
17. Circle, center $\left(0, \frac{3}{2}\right)$, radius $\frac{3}{2}$

19. Horizontal line, 1 unit above the x-axis

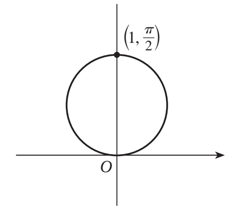
21. $r = 3 \sec \theta$ **23.** $r = -\cot \theta \csc \theta$ **25.** $r = 2c \cos \theta$

27. (a) $\theta = \pi/6$ (b) $x = 3$

29.



31.



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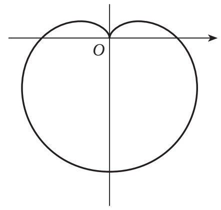

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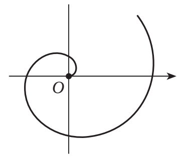

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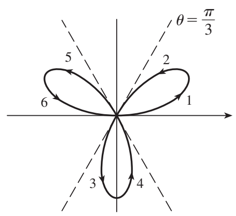

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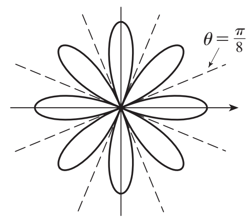

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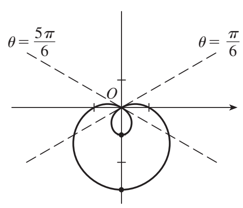

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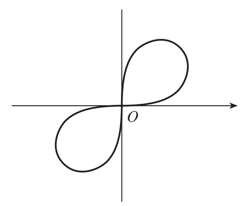

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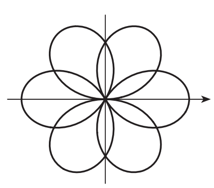

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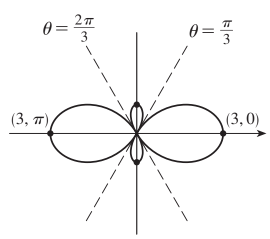

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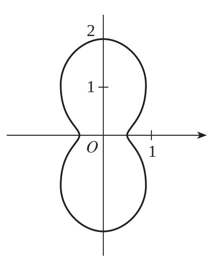

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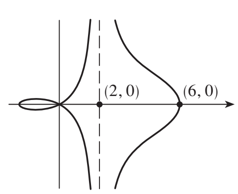

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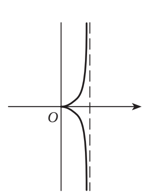


55. (a) For $c < -1$, the inner loop begins at $\theta = \sin^{-1}(-1/c)$ and ends at $\theta = \pi - \sin^{-1}(-1/c)$; for $c > 1$, it begins at $\theta = \pi + \sin^{-1}(1/c)$ and ends at $\theta = 2\pi - \sin^{-1}(1/c)$.

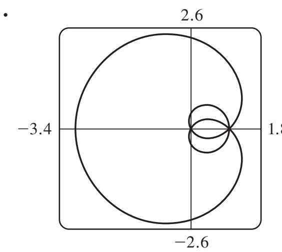
57. $\sqrt{3}$ **59.** $-\pi$ **61.** 1

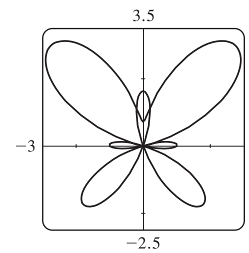
63. Horizontal at $(3/\sqrt{2}, \pi/4)$, $(-3/\sqrt{2}, 3\pi/4)$; vertical at $(3, 0)$, $(0, \pi/2)$

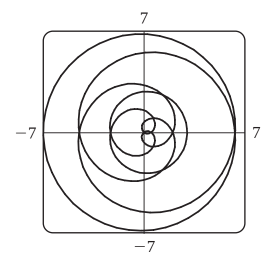
65. Horizontal at $\left(\frac{3}{2}, \pi/3\right)$, $(0, \pi)$ [the pole], and $\left(\frac{3}{2}, 5\pi/3\right)$; vertical at $(2, 0)$, $\left(\frac{1}{2}, 2\pi/3\right)$, $\left(\frac{1}{2}, 4\pi/3\right)$

67. Horizontal at $(3, \pi/2)$, $(1, 3\pi/2)$; vertical at $\left(\frac{3}{2} + \frac{1}{2}\sqrt{3}, \alpha\right)$, $\left(\frac{3}{2} + \frac{1}{2}\sqrt{3}, \pi - \alpha\right)$ where $\alpha = \sin^{-1}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)$

69. Center $(b/2, a/2)$, radius $\sqrt{a^2 + b^2}/2$

71.


73.


75.


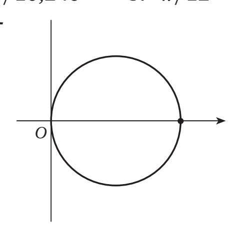
77. By counterclockwise rotation through angle $\pi/6$, $\pi/3$, or α about the origin

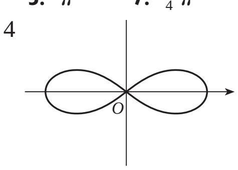
79. (a) A rose with n loops if n is odd and $2n$ loops if n is even
(b) Number of loops is always $2n$

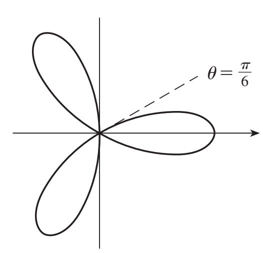
81. For $0 < a < 1$, the curve is an oval, which develops a dimple as $a \to 1^-$. When $a > 1$, the curve splits into two parts, one of which has a loop.

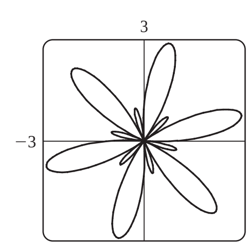
EXERCISES 10.4 ■ PAGE 653

1. $\pi^5/10{,}240$ **3.** $\pi/12 + \frac{1}{8}\sqrt{3}$ **5.** π^2 **7.** $\frac{41}{4}\pi$

9. $\frac{9}{4}\pi$


11. 4


13. π


15. 3π


17. $\frac{1}{8}\pi$ **19.** $\frac{9}{20}\pi$ **21.** $\pi - \frac{3}{2}\sqrt{3}$ **23.** $\frac{1}{3}\pi + \frac{1}{2}\sqrt{3}$

25. $4\sqrt{3} - \frac{4}{3}\pi$ **27.** π **29.** $\frac{5}{24}\pi - \frac{1}{4}\sqrt{3}$ **31.** $\frac{1}{2}\pi - 1$

33. $\frac{1}{8}\pi - \frac{1}{4}$ **35.** $\frac{1}{4}\left(\pi + 3\sqrt{3}\right)$

37. $\left(\frac{3}{2}, \pi/6\right)$, $\left(\frac{3}{2}, 5\pi/6\right)$, and the pole

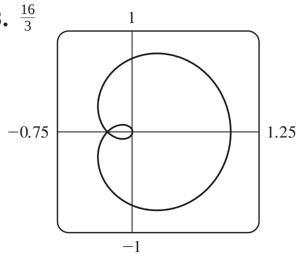
39. $(1, \theta)$ where $\theta = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$ and $(-1, \theta)$ where $\theta = 7\pi/12, 11\pi/12, 19\pi/12, 23\pi/12$

41. $\left(\frac{1}{2}\sqrt{3}, \pi/3\right), \left(\frac{1}{2}\sqrt{3}, 2\pi/3\right)$, and the pole

43. Intersection at $\theta \approx 0.89, 2.25$; area ≈ 3.46 **45.** π

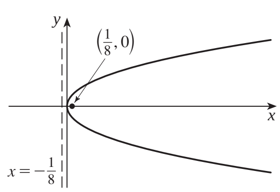
47. $\frac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$ **49.** 29.0653 **51.** 9.6884

53. $\frac{16}{3}$ **55.** (b) $2\pi(2 - \sqrt{2})$

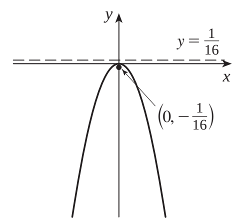


EXERCISES 10.5 ■ PAGE 660

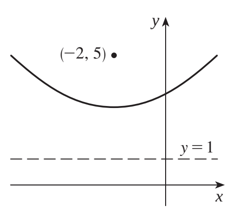
1. $(0, 0), \left(\frac{1}{8}, 0\right), x = -\frac{1}{8}$



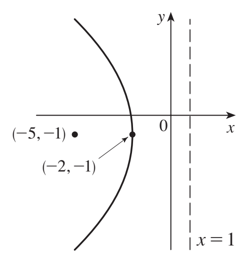
3. $(0, 0), \left(0, -\frac{1}{16}\right), y = \frac{1}{16}$



5. $(-2, 3), (-2, 5), y = 1$

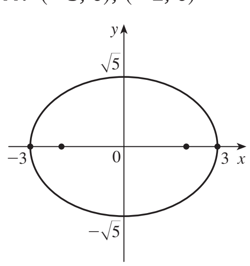


7. $(-2, -1), (-5, -1), x = 1$

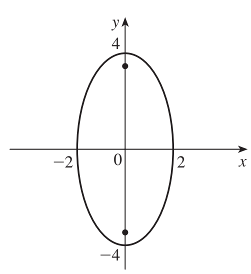


9. $x = -y^2$, focus $\left(-\frac{1}{4}, 0\right)$, directrix $x = \frac{1}{4}$

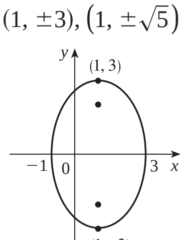
11. $(\pm 3, 0), (\pm 2, 0)$



13. $(0, \pm 4), (0, \pm 2\sqrt{3})$

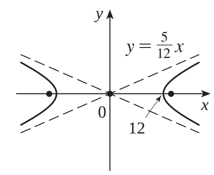


15. $(1, \pm 3), (1, \pm\sqrt{5})$

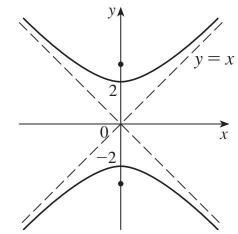


17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, foci $(0, \pm\sqrt{5})$

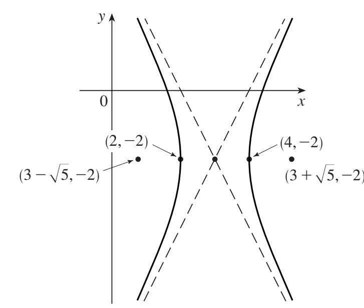
19. $(\pm 12, 0), (\pm 13, 0)$, $y = \pm\frac{5}{12}x$



21. $(0, \pm 2), (0, \pm 2\sqrt{2})$, $y = \pm x$



23. $(4, -2), (2, -2)$; $(3 \pm \sqrt{5}, -2)$; $y + 2 = \pm 2(x - 3)$



25. Parabola, $(0, -1), \left(0, -\frac{3}{4}\right)$

27. Ellipse, $(\pm\sqrt{2}, 1), (\pm 1, 1)$

29. Hyperbola, $(0, 1), (0, -3)$; $(0, -1 \pm \sqrt{5})$ **31.** $x^2 = -8y$

33. $y^2 = -12(x + 1)$ **35.** $y - 3 = 2(x - 2)^2$

37. $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$ **39.** $\dfrac{x^2}{12} + \dfrac{(y - 4)^2}{16} = 1$

41. $\dfrac{(x + 1)^2}{12} + \dfrac{(y - 4)^2}{16} = 1$ **43.** $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

45. $\dfrac{(y - 1)^2}{25} - \dfrac{(x + 3)^2}{39} = 1$ **47.** $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$

49. $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$

51. (a) $\dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$ (b) ≈ 248 mi

55. (a) Ellipse (b) Hyperbola (c) No curve

59. 9.69 **61.** $\dfrac{b^2c}{a} + ab\ln\left(\dfrac{a}{b + c}\right)$ where $c^2 = a^2 + b^2$

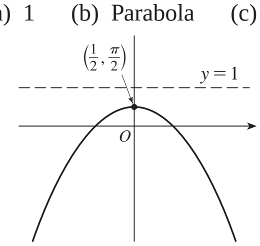
EXERCISES 10.6 ■ PAGE 668

1. $r = \dfrac{42}{4 + 7\sin\theta}$ **3.** $r = \dfrac{15}{4 - 3\cos\theta}$

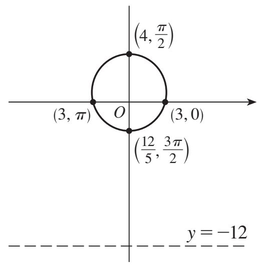
5. $r = \dfrac{8}{1 - \sin\theta}$ **7.** $r = \dfrac{4}{2 + \cos\theta}$

9. (a) 1 (b) Parabola (c) $y = 1$

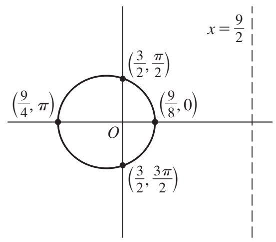
(d)



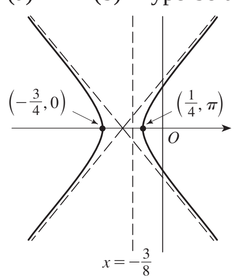
11. (a) $\frac{1}{4}$ (b) Ellipse (c) $y = -12$
(d)



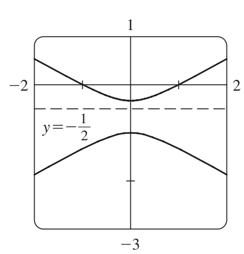
13. (a) $\frac{1}{3}$ (b) Ellipse (c) $x = \frac{9}{2}$
(d)



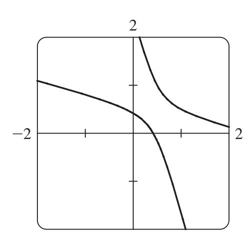
15. (a) 2 (b) Hyperbola (c) $x = -\frac{3}{8}$
(d)



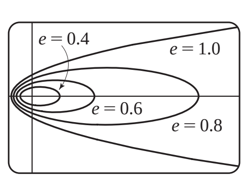
17. (a) $2, y = -\frac{1}{2}$



(b) $r = \dfrac{1}{1 - 2\sin(\theta - 3\pi/4)}$



19. The ellipse is nearly circular when e is close to 0 and becomes more elongated as $e \to 1^-$. At $e = 1$, the curve becomes a parabola.



25. $r = \dfrac{2.26 \times 10^8}{1 + 0.093\cos\theta}$

27. 35.64 AU **29.** 7.0×10^7 km **31.** 3.6×10^8 km

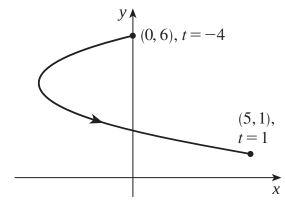
CHAPTER 10 REVIEW ■ PAGE 669

True-False Quiz

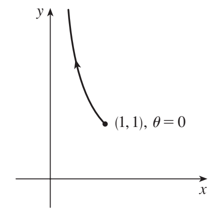
1. False **3.** False **5.** True **7.** False **9.** True

Exercises

1. $x = y^2 - 8y + 12$

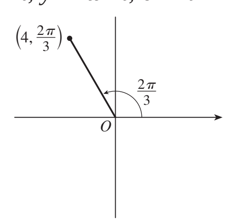


3. $y = 1/x$



5. $x = t, y = \sqrt{t}$; $x = t^4, y = t^2$; $x = \tan^2 t, y = \tan t, 0 \leqslant t < \pi/2$

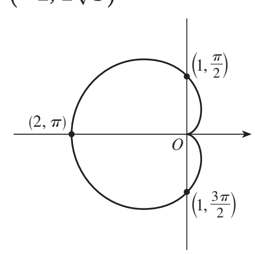
7. (a)



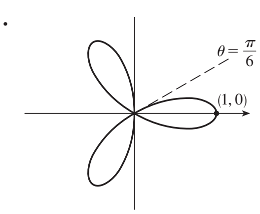
(b) $(3\sqrt{2}, 3\pi/4)$, $(-3\sqrt{2}, 7\pi/4)$

$(-2, 2\sqrt{3})$

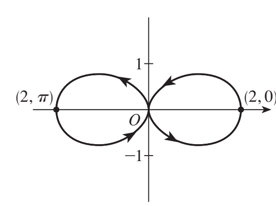
9.



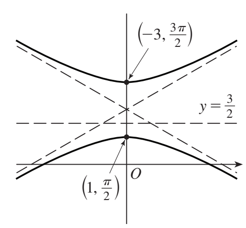
11.



13.

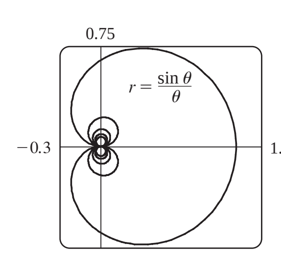


15.



17. $r = \dfrac{2}{\cos\theta + \sin\theta}$

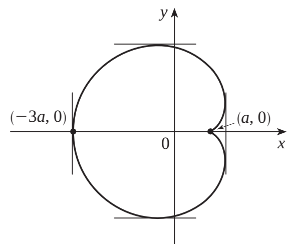
19.



21. 2 **23.** -1

25. $\dfrac{1+\sin t}{1+\cos t}, \dfrac{1+\cos t+\sin t}{(1+\cos t)^3}$ **27.** $\left(\frac{11}{8}, \frac{3}{4}\right)$

29. Vertical tangent at $\left(\frac{3}{2}a, \pm\frac{1}{2}\sqrt{3}\,a\right)$, $(-3a, 0)$; horizontal tangent at $(a, 0)$, $\left(-\frac{1}{2}a, \pm\frac{3}{2}\sqrt{3}\,a\right)$



31. 18 **33.** $(2, \pm\pi/3)$ **35.** $\frac{1}{2}(\pi - 1)$

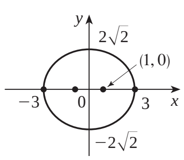
37. $2\left(5\sqrt{5} - 1\right)$

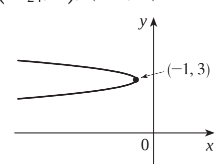
39. $\dfrac{2\sqrt{\pi^2+1} - \sqrt{4\pi^2+1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$

41. $471{,}295\pi/1024$

43. All curves have the vertical asymptote $x = 1$. For $c < -1$, the curve bulges to the right. At $c = -1$, the curve is the line $x = 1$. For $-1 < c < 0$, it bulges to the left. At $c = 0$ there is a cusp at $(0, 0)$. For $c > 0$, there is a loop.

45. $(\pm 1, 0)$, $(\pm 3, 0)$ **47.** $\left(-\frac{25}{24}, 3\right)$, $(-1, 3)$





49. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **51.** $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$

53. $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$ **55.** $r = \dfrac{4}{3 + \cos\theta}$

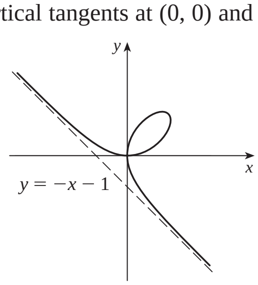
57. $x = a(\cot\theta + \sin\theta\cos\theta)$, $y = a(1 + \sin^2\theta)$

PROBLEMS PLUS ■ PAGE 672

1. $\ln(\pi/2)$

3. $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$

5. (a) At $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$
(b) Horizontal tangents at $(0, 0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$; vertical tangents at $(0, 0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$
(d)



(g) $\frac{3}{2}$

CHAPTER 11

EXERCISES 11.1 ■ PAGE 684

Abbreviations: C, convergent; D, divergent

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large.
(c) The terms a_n become large as n becomes large.

3. 0.8, 0.96, 0.992, 0.9984, 0.99968 **5.** $-3, \frac{3}{2}, -\frac{1}{2}, \frac{1}{8}, -\frac{1}{40}$

7. 3, 5, 9, 17, 33 **9.** $a_n = 1/(2n - 1)$ **11.** $a_n = 5n - 3$

13. $a_n = \left(-\frac{2}{3}\right)^{n-1}$ **15.** $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$; yes; $\frac{1}{2}$

17. 1 **19.** 5 **21.** 1 **23.** 1 **25.** 0 **27.** D

29. 0 **31.** 0 **33.** 0 **35.** 0 **37.** 1 **39.** e^2

41. ln 2 **43.** D **45.** D **47.** 1 **49.** $\frac{1}{2}$

51. D **53.** 0

55. (a) 1060, 1123.60, 1191.02, 1262.48, 1338.23 (b) D

57. $-1 < r < 1$

59. Convergent by the Monotonic Sequence Theorem; $5 \leqslant L < 8$

61. Decreasing; yes **63.** Not monotonic; no

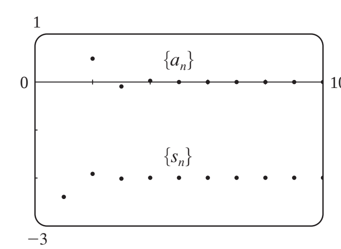
65. Decreasing; yes **67.** 2 **69.** $\frac{1}{2}\left(3 + \sqrt{5}\right)$

71. (b) $\frac{1}{2}\left(1 + \sqrt{5}\right)$ **73.** (a) 0 (b) 9, 11

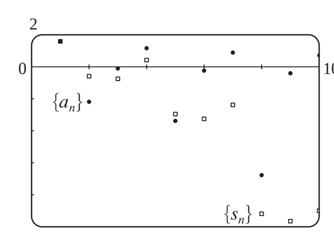
EXERCISES 11.2 ■ PAGE 694

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

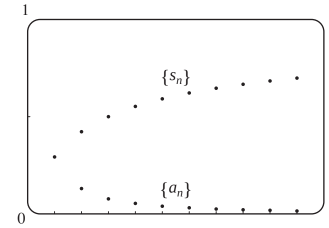
3. -2.40000, -1.92000, -2.01600, -1.99680, -2.00064, -1.99987, -2.00003, -1.99999, -2.00000, -2.00000; convergent, sum $= -2$



5. 1.55741, -0.62763, -0.77018, 0.38764, -2.99287, -3.28388, -2.41243, -9.21214, -9.66446, -9.01610; divergent



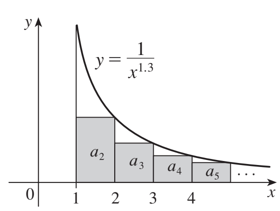
7. 0.29289, 0.42265, 0.50000, 0.55279, 0.59175, 0.62204, 0.64645, 0.66667, 0.68377, 0.69849; convergent, sum $= 1$



9. (a) C (b) D **11.** 9 **13.** D **15.** 60 **17.** $\frac{1}{7}$
19. D **21.** D **23.** D **25.** $\frac{5}{2}$ **27.** D **29.** D
31. D **33.** $e/(e-1)$ **35.** $\frac{3}{2}$ **37.** $\frac{11}{6}$ **39.** $e-1$
41. $\frac{2}{9}$ **43.** 1138/333 **45.** 5063/3300
47. $-3 < x < 3$; $\dfrac{x}{3-x}$ **49.** $-\frac{1}{4} < x < \frac{1}{4}$; $\dfrac{1}{1-4x}$
51. All x; $\dfrac{2}{2-\cos x}$ **53.** 1
55. $a_1 = 0$, $a_n = \dfrac{2}{n(n+1)}$ for $n > 1$, sum $= 1$
57. (a) $S_n = \dfrac{D(1-c^n)}{1-c}$ (b) 5 **59.** $\frac{1}{2}(\sqrt{3}-1)$
63. $\dfrac{1}{n(n+1)}$ **65.** The series is divergent.
71. $\{s_n\}$ is bounded and increasing.
73. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$
75. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$; $\dfrac{(n+1)!-1}{(n+1)!}$ (c) 1

EXERCISES 11.3 ■ PAGE 703

1. C



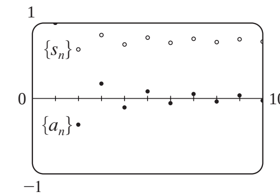
3. D **5.** C **7.** C **9.** D **11.** C **13.** D **15.** C
17. C **19.** C **21.** D **23.** C **25.** C **27.** $p > 1$
29. $p < -1$ **31.** $(1, \infty)$
33. (a) 1.54977, error $\leqslant 0.1$ (b) 1.64522, error $\leqslant 0.005$
(c) $n > 1000$
35. 0.00145 **41.** $b < 1/e$

EXERCISES 11.4 ■ PAGE 709

1. (a) Nothing (b) C **3.** C **5.** D **7.** C **9.** C
11. C **13.** C **15.** C **17.** D **19.** D **21.** C
23. C **25.** D **27.** C **29.** C **31.** D
33. 1.249, error < 0.1 **35.** 0.76352, error < 0.001
45. Yes

EXERCISES 11.5 ■ PAGE 713

1. (a) A series whose terms are alternately positive and negative (b) $0 < b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$, where $b_n = |a_n|$ (c) $|R_n| \leqslant b_{n+1}$
3. C **5.** C **7.** D **9.** C **11.** C **13.** D
15. C **17.** C **19.** D
21. 1.0000, 0.6464, 0.8389, 0.7139, 0.8033, 0.7353, 0.7893, 0.7451, 0.7821, 0.7505; error < 0.0275



23. 5 **25.** 4 **27.** 0.9721 **29.** 0.0676
31. An underestimate **33.** p is not a negative integer
35. $\{b_n\}$ is not decreasing

EXERCISES 11.6 ■ PAGE 719

Abbreviations: AC, absolutely convergent;
CC, conditionally convergent

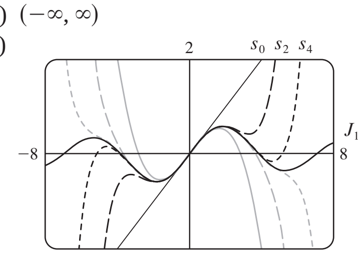
1. (a) D (b) C (c) May converge or diverge
3. AC **5.** CC **7.** AC **9.** D **11.** AC **13.** AC
15. AC **17.** CC **19.** AC **21.** AC **23.** D
25. AC **27.** D **29.** D **31.** (a) and (d)
35. (a) $\frac{661}{960} \approx 0.68854$, error < 0.00521
(b) $n \geqslant 11$, 0.693109

EXERCISES 11.7 ■ PAGE 722

1. C **3.** D **5.** C **7.** D **9.** C **11.** C **13.** C
15. C **17.** D **19.** C **21.** C **23.** D **25.** C
27. C **29.** C **31.** D **33.** C **35.** C **37.** C

EXERCISES 11.8 ■ PAGE 727

1. A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, where x is a variable and a and the c_n's are constants
3. 1, $[-1, 1)$ **5.** 1, $[-1, 1]$ **7.** ∞, $(-\infty, \infty)$
9. 2, $(-2, 2)$ **11.** $\frac{1}{2}$, $\left(-\frac{1}{2}, \frac{1}{2}\right]$ **13.** 4, $(-4, 4]$
15. 1, $[1, 3]$ **17.** $\frac{1}{3}$, $\left[-\frac{13}{3}, -\frac{11}{3}\right)$ **19.** ∞, $(-\infty, \infty)$
21. b, $(a-b, a+b)$ **23.** 0, $\left\{\frac{1}{2}\right\}$ **25.** $\frac{1}{4}$, $\left[-\frac{1}{2}, 0\right]$
27. ∞, $(-\infty, \infty)$ **29.** (a) Yes (b) No **31.** k^k **33.** No
35. (a) $(-\infty, \infty)$
(b), (c)



37. $(-1, 1)$, $f(x) = (1+2x)/(1-x^2)$ **41.** 2

EXERCISES 11.9 ■ PAGE 733

1. 10 **3.** $\sum_{n=0}^{\infty} (-1)^n x^n$, $(-1, 1)$ **5.** $2\sum_{n=0}^{\infty} \dfrac{1}{3^{n+1}} x^n$, $(-3, 3)$

7. $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{9^{n+1}} x^{2n+1}$, $(-3, 3)$ **9.** $1 + 2\sum_{n=1}^{\infty} x^n$, $(-1, 1)$

11. $\displaystyle\sum_{n=0}^{\infty}\left[(-1)^{n+1}-\frac{1}{2^{n+1}}\right]x^n,\ (-1, 1)$

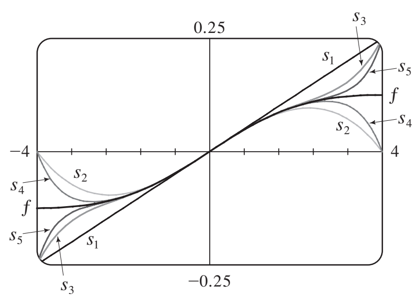
13. (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n,\ R=1$

(b) $\displaystyle\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n,\ R=1$

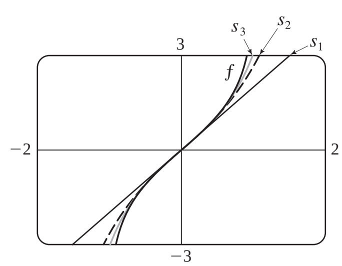
(c) $\displaystyle\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)x^n,\ R=1$

15. $\displaystyle\ln 5-\sum_{n=1}^{\infty}\frac{x^n}{n5^n},\ R=5$ **17.** $\displaystyle\sum_{n=3}^{\infty}\frac{n-2}{2^{n-1}}x^n,\ R=2$

19. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{16^{n+1}}x^{2n+1},\ R=4$



21. $\displaystyle\sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1},\ R=1$



23. $\displaystyle C+\sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2},\ R=1$

25. $\displaystyle C+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{4n^2-1},\ R=1$

27. 0.199989 **29.** 0.000983 **31.** 0.09531

33. (b) 0.920 **37.** $[-1, 1], [-1, 1), (-1, 1)$

EXERCISES 11.10 ■ PAGE 746

1. $b_8=f^{(8)}(5)/8!$ **3.** $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n,\ R=1$

5. $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n,\ R=1$

7. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}x^{2n+1},\ R=\infty$

9. $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n,\ R=\infty$ **11.** $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},\ R=\infty$

13. $-1-2(x-1)+3(x-1)^2+4(x-1)^3+(x-1)^4$, $R=\infty$

15. $\displaystyle\sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n,\ R=\infty$

17. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}(x-\pi)^{2n},\ R=\infty$

19. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n,\ R=9$

25. $\displaystyle 1+\frac{x}{2}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n,\ R=1$

27. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)}{2^{n+4}}x^n,\ R=2$

29. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}x^{2n+1},\ R=\infty$

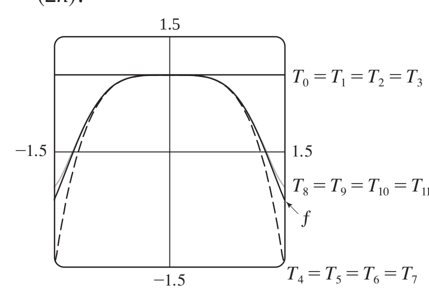
31. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+1}{n!}x^n,\ R=\infty$

33. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n}(2n)!}x^{4n+1},\ R=\infty$

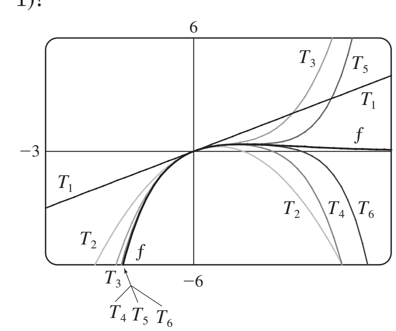
35. $\displaystyle\frac{1}{2}x+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1},\ R=2$

37. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}}{(2n)!}x^{2n},\ R=\infty$

39. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{4n},\ R=\infty$



41. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^n,\ R=\infty$



43. 0.81873

45. (a) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}x^{2n}$

(b) $\displaystyle x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n n!}x^{2n+1}$

47. $C + \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{6n+2}}{(6n+2)(2n)!}$, $R = \infty$

49. $C + \sum_{n=1}^{\infty} (-1)^n \dfrac{1}{2n\,(2n)!} x^{2n}$, $R = \infty$ **51.** 0.440

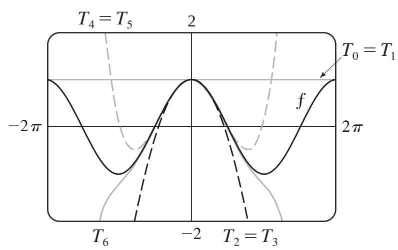
53. 0.40102 **55.** $\frac{1}{3}$ **57.** $\frac{1}{120}$ **59.** $1 - \frac{3}{2}x^2 + \frac{25}{24}x^4$

61. $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$ **63.** e^{-x^4}

65. $1/\sqrt{2}$ **67.** $e^3 - 1$

EXERCISES 11.11 ▪ PAGE 755

1. (a) $T_0(x) = 1 = T_1(x)$, $T_2(x) = 1 - \frac{1}{2}x^2 = T_3(x)$,
$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 = T_5(x)$,
$T_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

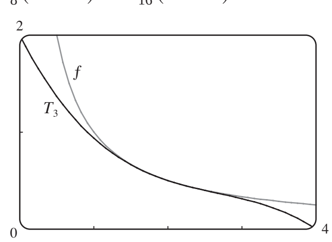


(b)

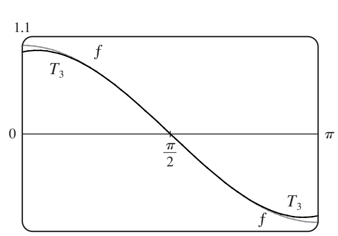
x	f	$T_0 = T_1$	$T_2 = T_3$	$T_4 = T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	−0.2337	0.0200	−0.0009
π	−1	1	−3.9348	0.1239	−1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

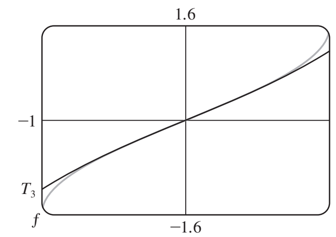
3. $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$



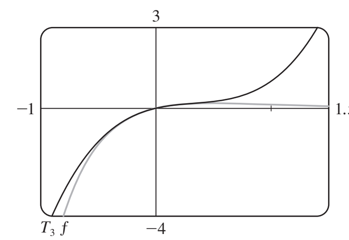
5. $-\left(x - \dfrac{\pi}{2}\right) + \dfrac{1}{6}\left(x - \dfrac{\pi}{2}\right)^3$



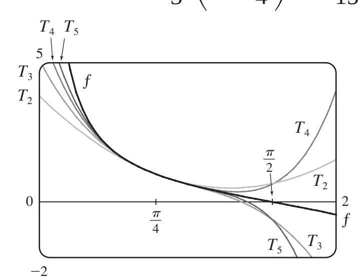
7. $x + \frac{1}{6}x^3$



9. $x - 2x^2 + 2x^3$



11. $T_5(x) = 1 - 2\left(x - \dfrac{\pi}{4}\right) + 2\left(x - \dfrac{\pi}{4}\right)^2 - \dfrac{8}{3}\left(x - \dfrac{\pi}{4}\right)^3 + \dfrac{10}{3}\left(x - \dfrac{\pi}{4}\right)^4 - \dfrac{64}{15}\left(x - \dfrac{\pi}{4}\right)^5$



13. (a) $2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$ (b) 1.5625×10^{-5}

15. (a) $1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3$ (b) 0.000097

17. (a) $1 + \frac{1}{2}x^2$ (b) 0.0015 **19.** (a) $1 + x^2$ (b) 0.00006

21. (a) $x^2 - \frac{1}{6}x^4$ (b) 0.042 **23.** 0.17365 **25.** Four

27. $-1.037 < x < 1.037$ **29.** $-0.86 < x < 0.86$

31. 21 m, no **37.** (c) They differ by about 8×10^{-9} km.

CHAPTER 11 REVIEW ▪ PAGE 759

True-False Quiz

1. False **3.** True **5.** False **7.** False
9. False **11.** True **13.** True **15.** False
17. True **19.** True

Exercises

1. $\frac{1}{2}$ **3.** D **5.** 0 **7.** e^{12} **9.** 2 **11.** C
13. C **15.** D **17.** C **19.** C **21.** C **23.** CC
25. AC **27.** $\frac{1}{11}$ **29.** $\pi/4$ **31.** e^{-e} **35.** 0.9721

37. 0.18976224, error $< 6.4 \times 10^{-7}$
41. $4, [-6, 2)$ **43.** $0.5, [2.5, 3.5)$

45. $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n}+\frac{\sqrt{3}}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}\right]$

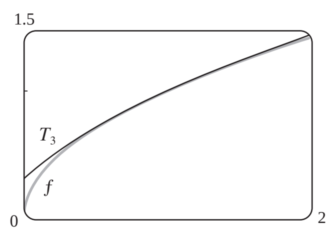
47. $\sum_{n=0}^{\infty}(-1)^n x^{n+2}, R = 1$ **49.** $-\sum_{n=1}^{\infty}\frac{x^n}{n}, R = 1$

51. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+4}}{(2n+1)!}, R = \infty$

53. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{n!\,2^{6n+1}}x^n, R = 16$

55. $C + \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n \cdot n!}$

57. (a) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
(b)



(c) 0.000006

59. $-\frac{1}{6}$

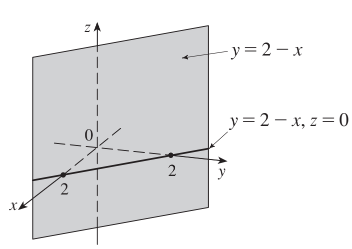
PROBLEMS PLUS ■ PAGE 762

1. $15!/5! = 10{,}897{,}286{,}400$
3. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq k\pi$, k an integer
5. (a) $s_n = 3 \cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $\frac{2}{5}\sqrt{3}$
9. $(-1, 1)$, $\frac{x^3 + 4x^2 + x}{(1-x)^4}$ **11.** $\ln \frac{1}{2}$
13. (a) $\frac{250}{101}\pi(e^{-(n-1)\pi/5} - e^{-n\pi/5})$ (b) $\frac{250}{101}\pi$

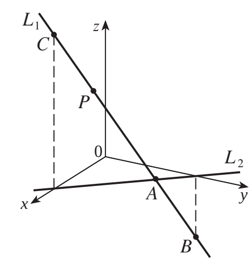
CHAPTER 12

EXERCISES 12.1 ■ PAGE 769

1. $(4, 0, -3)$ **3.** Q; R
5. A vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$ (see graph at right)



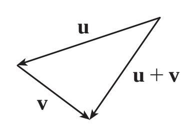
7. $|PQ| = 6$, $|QR| = 2\sqrt{10}$, $|RP| = 6$; isosceles triangle
9. (a) No (b) Yes
11. $(x-1)^2 + (y+4)^4 + (z-3)^2 = 25$;
$(x-1)^2 + (z-3)^2 = 9$, $y = 0$ (a circle)
13. $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$
15. $(3, -2, 1)$, 5
17. $(2, 0, -6)$, $9/\sqrt{2}$ **19.** (b) $\frac{5}{2}, \frac{1}{2}\sqrt{94}, \frac{1}{2}\sqrt{85}$
21. (a) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 36$
(b) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 4$
(c) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 9$
23. A plane parallel to the xz-plane and 4 units to the left of it
25. A half-space consisting of all points in front of the plane $x = 3$
27. All points on or between the horizontal planes $z = 0$ and $z = 6$
29. All points on or inside a sphere with radius $\sqrt{3}$ and center O
31. All points on or inside a circular cylinder of radius 3 with axis the y-axis
33. $0 < x < 5$ **35.** $r^2 < x^2 + y^2 + z^2 < R^2$
37. (a) $(2, 1, 4)$ (b)



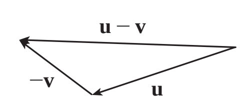
39. $14x - 6y - 10z = 9$, a plane perpendicular to AB

EXERCISES 12.2 ■ PAGE 777

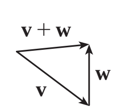
1. (a) Scalar (b) Vector (c) Vector (d) Scalar
3. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, $\overrightarrow{EA} = \overrightarrow{CE}$
5. (a)



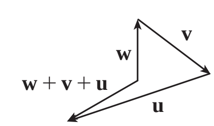
(b)



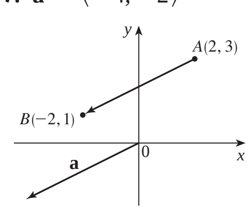
(c)



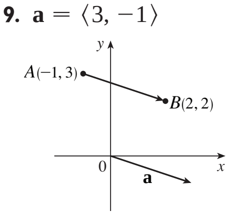
(d)



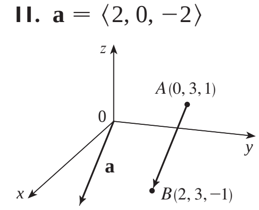
7. $\mathbf{a} = \langle -4, -2 \rangle$



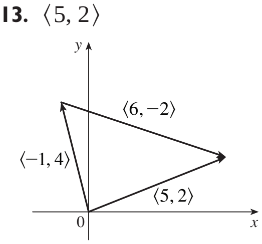
9. $\mathbf{a} = \langle 3, -1 \rangle$



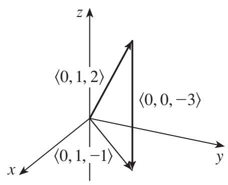
11. $\mathbf{a} = \langle 2, 0, -2 \rangle$



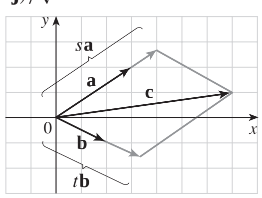
13. $\langle 5, 2 \rangle$



15. $\langle 0, 1, -1 \rangle$



17. $\langle 2, -18 \rangle$, $\langle 1, -42 \rangle$, 13, 10
19. $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $-4\,\mathbf{i} + \mathbf{j} + 9\,\mathbf{k}$, $\sqrt{14}$, $\sqrt{82}$
21. $-\dfrac{3}{\sqrt{58}}\,\mathbf{i} + \dfrac{7}{\sqrt{58}}\,\mathbf{j}$ **23.** $\frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$
25. $\langle 2, 2\sqrt{3} \rangle$ **27.** ≈ 45.96 ft/s, ≈ 38.57 ft/s
29. $100\sqrt{7} \approx 264.6$ N, $\approx 139.1°$
31. $\sqrt{493} \approx 22.2$ mi/h, N8°W
33. $\mathbf{T}_1 \approx -196\,\mathbf{i} + 3.92\,\mathbf{j}$, $\mathbf{T}_2 \approx 196\,\mathbf{i} + 3.92\,\mathbf{j}$
35. $\pm(\mathbf{i} + 4\,\mathbf{j})/\sqrt{17}$ **37.** $\mathbf{0}$
39. (a), (b)



(d) $s = \frac{9}{7}$, $t = \frac{11}{7}$

41. A sphere with radius 1, centered at (x_0, y_0, z_0)

EXERCISES 12.3 ■ PAGE 784

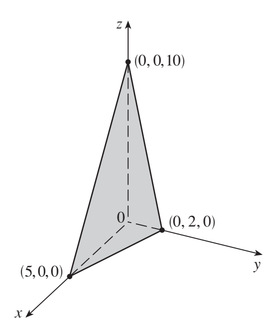
1. (b), (c), (d) are meaningful
3. 14 **5.** 19 **7.** 32 **9.** -15
11. $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}$, $\mathbf{u} \cdot \mathbf{w} = -\frac{1}{2}$
15. $\cos^{-1}\left(\dfrac{9 - 4\sqrt{7}}{20}\right) \approx 95°$ **17.** $\cos^{-1}\left(\dfrac{5}{\sqrt{1015}}\right) \approx 81°$
19. $\cos^{-1}\left(\dfrac{-1}{2\sqrt{7}}\right) \approx 101°$ **21.** 45°, 45°, 90°
23. (a) Neither (b) Orthogonal
(c) Orthogonal (d) Parallel
25. Yes **27.** $(\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$ $[\text{or } (-\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}]$
29. $\dfrac{3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}}$; 65°, 56°, 45°
31. $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$; 73°, 65°, 149°
33. $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$; 55°, 55°, 55°
35. 3, $\langle \frac{9}{5}, -\frac{12}{5} \rangle$ **37.** $\frac{9}{7}$, $\langle \frac{27}{49}, \frac{54}{49}, -\frac{18}{49} \rangle$
39. $1/\sqrt{21}$, $\frac{2}{21}\,\mathbf{i} - \frac{1}{21}\,\mathbf{j} + \frac{4}{21}\,\mathbf{k}$
43. $\langle 0, 0, -2\sqrt{10} \rangle$ or any vector of the form
$\langle s, t, 3s - 2\sqrt{10} \rangle$, $s, t \in \mathbb{R}$
45. 144 J **47.** $2400\cos(40°) \approx 1839$ ft-lb **49.** $\frac{13}{5}$
51. $\cos^{-1}(1/\sqrt{3}) \approx 55°$

EXERCISES 12.4 ■ PAGE 792

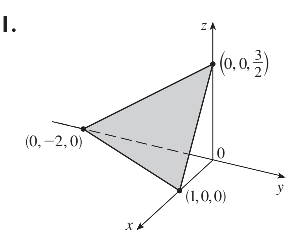
1. $16\,\mathbf{i} + 48\,\mathbf{k}$ **3.** $15\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k}$ **5.** $\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}$
7. $t^4\mathbf{i} - 2t^3\mathbf{j} + t^2\mathbf{k}$ **9.** $\mathbf{0}$ **11.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$
13. (a) Scalar (b) Meaningless (c) Vector
(d) Meaningless (e) Meaningless (f) Scalar
15. 24; into the page **17.** $\langle 5, -3, 1 \rangle$, $\langle -5, 3, -1 \rangle$
19. $\langle -2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6} \rangle$, $\langle 2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6} \rangle$
27. 16 **29.** (a) $\langle 6, 3, 2 \rangle$ (b) $\frac{7}{2}$
31. (a) $\langle 13, -14, 5 \rangle$ (b) $\frac{1}{2}\sqrt{390}$
33. 82 **35.** 3 **39.** $10.8 \sin 80° \approx 10.6$ N·m
41. ≈ 417 N **43.** (b) $\sqrt{97/3}$
49. (a) No (b) No (c) Yes

EXERCISES 12.5 ■ PAGE 802

1. (a) True (b) False (c) True (d) False (e) False
(f) True (g) False (h) True (i) True (j) False
(k) True
3. $\mathbf{r} = (2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}) + t(3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})$;
$x = 2 + 3t$, $y = 2.4 + 2t$, $z = 3.5 - t$
5. $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\,\mathbf{j} + \mathbf{k})$;
$x = 1 + t$, $y = 3t$, $z = 6 + t$
7. $x = 1 - 5t$, $y = 3$, $z = 2 - 2t$; $\dfrac{x - 1}{-5} = \dfrac{z - 2}{-2}$, $y = 3$
9. $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$,
$z = -3 - 4t$;
$(x - 2)/2 = 2y - 2 = (z + 3)/(-4)$
11. $x = 1 + t$, $y = -1 + 2t$, $z = 1 + t$;
$x - 1 = (y + 1)/2 = z - 1$
13. Yes
15. (a) $(x - 1)/(-1) = (y + 5)/2 = (z - 6)/(-3)$
(b) $(-1, -1, 0)$, $\left(-\frac{3}{2}, 0, -\frac{3}{2}\right)$, $(0, -3, 3)$
17. $\mathbf{r}(t) = (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + t(2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k})$, $0 \leqslant t \leqslant 1$
19. Parallel **21.** Skew
23. $-2x + y + 5z = 1$ **25.** $x + y - z = -1$
27. $2x - y + 3z = 0$ **29.** $3x - 7z = -9$
31. $x + y + z = 2$ **33.** $-13x + 17y + 7z = -42$
35. $33x + 10y + 4z = 190$ **37.** $x - 2y + 4z = -1$
39.



41.



43. $(2, 3, 5)$ **45.** $(2, 3, 1)$ **47.** $1, 0, -1$
49. Perpendicular **51.** Neither, $\approx 70.5°$ **53.** Parallel
55. (a) $x = 1$, $y = -t$, $z = t$ (b) $\cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8°$
57. $x = 1$, $y - 2 = -z$
59. $x + 2y + z = 5$ **61.** $(x/a) + (y/b) + (z/c) = 1$

63. $x = 3t,\ y = 1 - t,\ z = 2 - 2t$

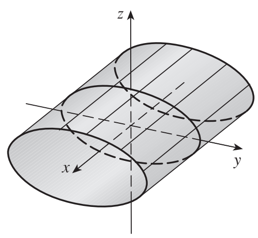
65. P_1 and P_3 are parallel, P_2 and P_4 are identical

67. $\sqrt{61/14}$ **69.** $\frac{18}{7}$ **71.** $5/(2\sqrt{14})$ **75.** $1/\sqrt{6}$

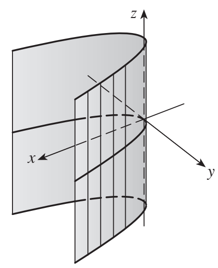
EXERCISES 12.6 ■ PAGE 810

1. (a) Parabola
(b) Parabolic cylinder with rulings parallel to the z-axis
(c) Parabolic cylinder with rulings parallel to the x-axis

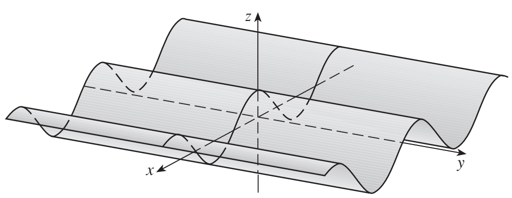
3. Elliptic cylinder



5. Parabolic cylinder

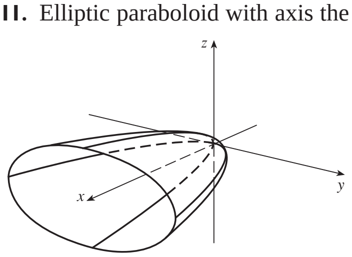


7. Cylindrical surface

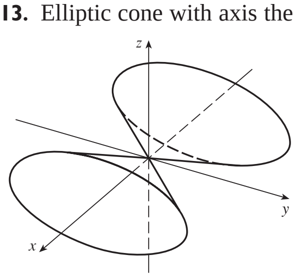


9. (a) $x = k,\ y^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$y = k,\ x^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$z = k,\ x^2 + z^2 = 1 + k^2$, circle
(b) The hyperboloid is rotated so that it has axis the y-axis
(c) The hyperboloid is shifted one unit in the negative y-direction

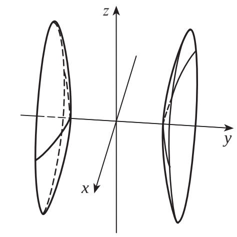
11. Elliptic paraboloid with axis the x-axis



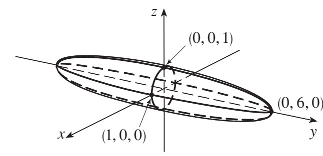
13. Elliptic cone with axis the x-axis



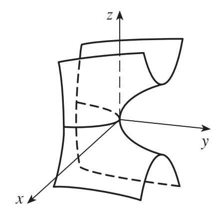
15. Hyperboloid of two sheets



17. Ellipsoid

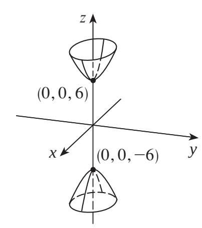


19. Hyperbolic paraboloid



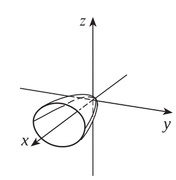
21. VII **23.** II **25.** VI **27.** VIII

29. $-\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{36} = 1$
Hyperboloid of two sheets with axis the z-axis



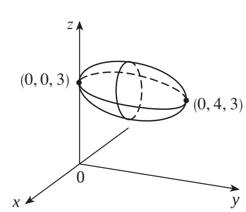
31. $\dfrac{x}{6} = \dfrac{y^2}{3} + \dfrac{z^2}{2}$

Elliptic paraboloid with vertex (0, 0, 0) and axis the x-axis

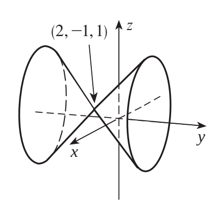


33. $x^2 + \dfrac{(y-2)^2}{4} + (z-3)^2 = 1$

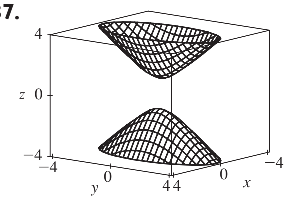
Ellipsoid with center (2, 2, 3)



35. $(y+1)^2 = (x-2)^2 + (z-1)^2$
Circular cone with vertex $(2, -1, 1)$ and axis parallel to the y-axis



37.



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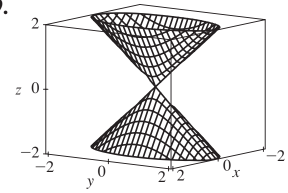

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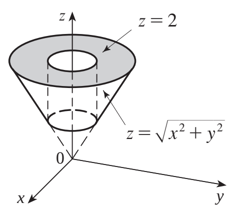
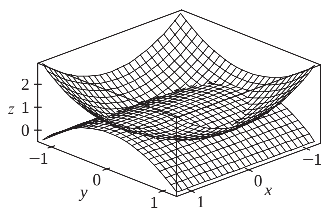


43. $y = x^2 + z^2$ **45.** $-4x = y^2 + z^2$, paraboloid

47. (a) $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} + \dfrac{z^2}{(6356.523)^2} = 1$
(b) Circle (c) Ellipse

51.



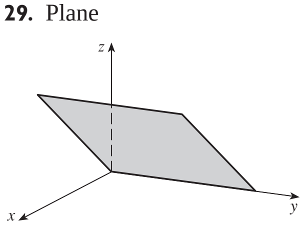
CHAPTER 12 REVIEW ■ PAGE 812

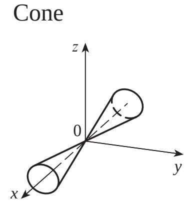
True-False Quiz

1. True **3.** True **5.** True **7.** True **9.** True
11. False **13.** False **15.** False **17.** True

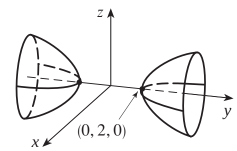
Exercises

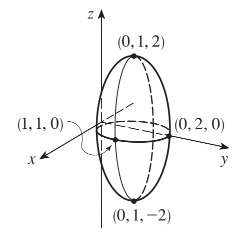
1. (a) $(x + 1)^2 + (y - 2)^2 + (z - 1)^2 = 69$
(b) $(y - 2)^2 + (z - 1)^2 = 68$, $x = 0$
(c) Center $(4, -1, -3)$, radius 5
3. $\mathbf{u} \cdot \mathbf{v} = 3\sqrt{2}$; $|\mathbf{u} \times \mathbf{v}| = 3\sqrt{2}$; out of the page
5. $-2, -4$ **7.** (a) 2 (b) -2 (c) -2 (d) 0
9. $\cos^{-1}\left(\frac{1}{3}\right) \approx 71°$ **11.** (a) $\langle 4, -3, 4 \rangle$ (b) $\sqrt{41}/2$
13. 166 N, 114 N
15. $x = 4 - 3t$, $y = -1 + 2t$, $z = 2 + 3t$
17. $x = -2 + 2t$, $y = 2 - t$, $z = 4 + 5t$
19. $-4x + 3y + z = -14$ **21.** $(1, 4, 4)$
23. Skew **25.** $x + y + z = 4$
27. $22/\sqrt{26}$
29. Plane **31.** Cone





33. Hyperboloid of two sheets **35.** Ellipsoid





37. $4x^2 + y^2 + z^2 = 16$

PROBLEMS PLUS ■ PAGE 815

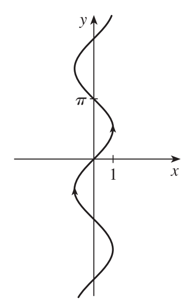
1. $(\sqrt{3} - 1.5)$ m
3. (a) $(x + 1)/(-2c) = (y - c)/(c^2 - 1) = (z - c)/(c^2 + 1)$
(b) $x^2 + y^2 = t^2 + 1$, $z = t$ (c) $4\pi/3$

CHAPTER 13

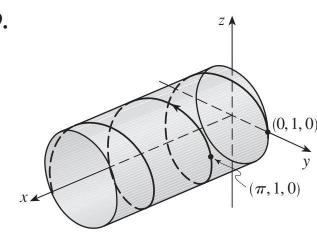
EXERCISES 13.1 ■ PAGE 822

1. $(-1, 2]$ **3.** $\langle 1, 0, 0 \rangle$ **5.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

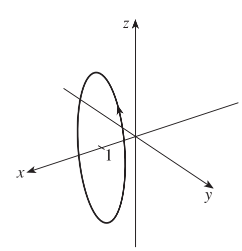
7.



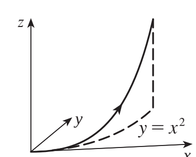
9.



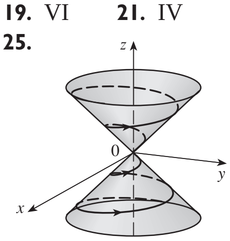
11.



13.

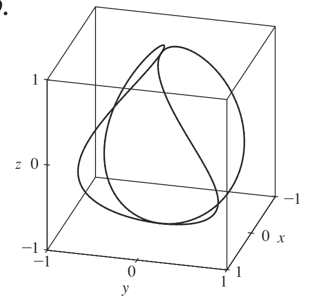


15. $\mathbf{r}(t) = \langle t, 2t, 3t \rangle$, $0 \leqslant t \leqslant 1$;
$x = t$, $y = 2t$, $z = 3t$, $0 \leqslant t \leqslant 1$
17. $\mathbf{r}(t) = \langle 3t + 1, 2t - 1, 5t + 2 \rangle$, $0 \leqslant t \leqslant 1$;
$x = 3t + 1$, $y = 2t - 1$, $z = 5t + 2$, $0 \leqslant t \leqslant 1$
19. VI **21.** IV **23.** V
25.

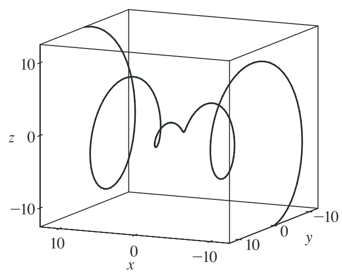


27. $(0, 0, 0)$, $(1, 0, 1)$

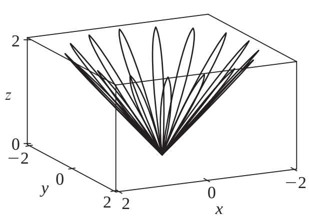
29.



31.



33.

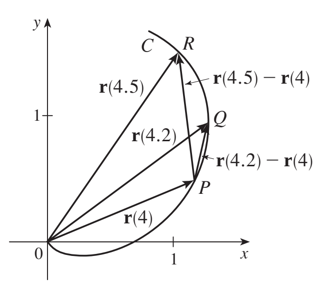


37. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$

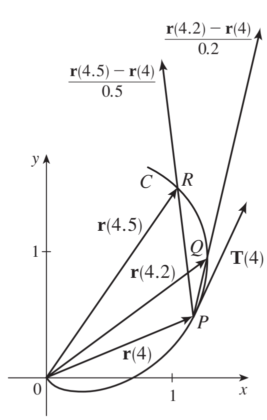
39. $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t$ **41.** Yes

EXERCISES 13.2 ▪ PAGE 828

1. (a)

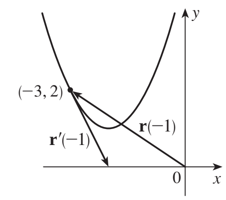


(b), (d)

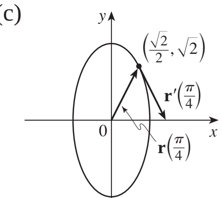


(c) $\mathbf{r}'(4) = \lim_{h\to 0} \dfrac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$; $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$

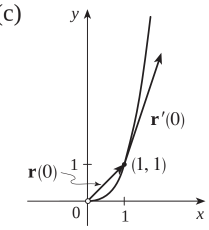
3. (a), (c) (b) $\mathbf{r}'(t) = \langle 1, 2t\rangle$



5. (a), (c)



7. (a), (c)



(b) $\mathbf{r}'(t) = \cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$ (b) $\mathbf{r}'(t) = e^t\,\mathbf{i} + 3e^{3t}\,\mathbf{j}$

9. $\mathbf{r}'(t) = \langle t\cos t + \sin t,\ 2t,\ \cos 2t - 2t\sin 2t\rangle$

11. $\mathbf{r}'(t) = 4e^{4t}\,\mathbf{k}$ **13.** $\mathbf{r}'(t) = 2te^{t^2}\,\mathbf{i} + [3/(1+3t)]\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{b} + 2t\,\mathbf{c}$ **17.** $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$ **19.** $\frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$

21. $\langle 1, 2t, 3t^2\rangle$, $\langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}\rangle$, $\langle 0, 2, 6t\rangle$, $\langle 6t^2, -6t, 2\rangle$

23. $x = 3 + t,\ y = 2t,\ z = 2 + 4t$

25. $x = 1 - t,\ y = t,\ z = 1 - t$

27. $x = t,\ y = 1 - t,\ z = 2t$

29. $x = -\pi - t,\ y = \pi + t,\ z = -\pi t$

31. 66° **33.** $4\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$ **35.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

37. $e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}$

39. $t^2\,\mathbf{i} + t^3\,\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$

45. $2t\cos t + 2\sin t - 2\cos t\sin t$

EXERCISES 13.3 ▪ PAGE 836

1. $20\sqrt{29}$ **3.** $e - e^{-1}$ **5.** $\frac{1}{27}(13^{3/2} - 8)$ **7.** 15.3841

9. 1.2780 **11.** 42

13. $\mathbf{r}(t(s)) = \dfrac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \dfrac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \dfrac{4}{\sqrt{29}}s\right)\mathbf{k}$

15. $(3\sin 1, 4, 3\cos 1)$

17. (a) $\langle (2/\sqrt{29})\cos t,\ 5/\sqrt{29},\ (-2/\sqrt{29})\sin t\rangle$, $\langle -\sin t, 0, -\cos t\rangle$ (b) $\frac{2}{29}$

19. (a) $\dfrac{1}{e^{2t}+1}\langle \sqrt{2}e^t, e^{2t}, -1\rangle$, $\dfrac{1}{e^{2t}+1}\langle 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t\rangle$

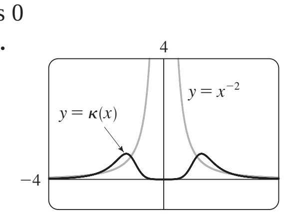
(b) $\sqrt{2}e^{2t}/(e^{2t}+1)^2$

21. $2/(4t^2+1)^{3/2}$ **23.** $\frac{4}{25}$ **25.** $\frac{1}{7}\sqrt{\frac{19}{14}}$

27. $2/(4x^2 - 8x + 5)^{3/2}$ **29.** $15\sqrt{x}/(1 + 100x^3)^{3/2}$

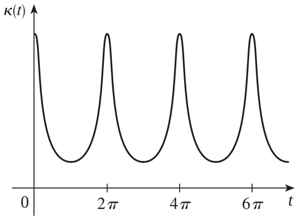
31. $\left(-\frac{1}{2}\ln 2, 1/\sqrt{2}\right)$; approaches 0

33. (a) P (b) 1.3, 0.7 **35.**



37. a is $y = f(x)$, b is $y = \kappa(x)$

39. $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$

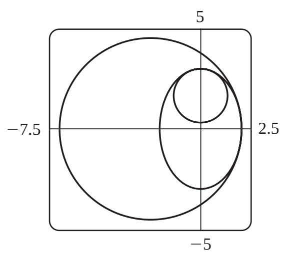


integer multiples of 2π

41. $1/(\sqrt{2}e^t)$ **43.** $\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$, $\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$, $\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$

45. $y = 6x + \pi$, $x + 6y = 6\pi$

47. $(x + \frac{5}{2})^2 + y^2 = \frac{81}{4}$, $x^2 + (y - \frac{5}{3})^2 = \frac{16}{9}$



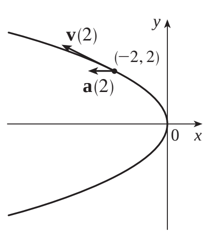
49. $(-1, -3, 1)$ **57.** $2/(t^4 + 4t^2 + 1)$

59. 2.07×10^{10} Å $\approx$ 2 m

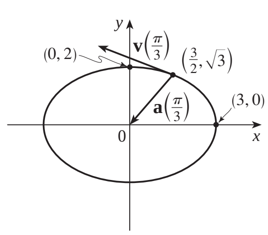
EXERCISES 13.4 ■ PAGE 846

1. (a) $1.8\mathbf{i} - 3.8\mathbf{j} - 0.7\mathbf{k}$, $2.0\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k}$, $2.8\mathbf{i} + 1.8\mathbf{j} - 0.3\mathbf{k}$, $2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$
(b) $2.4\mathbf{i} - 0.8\mathbf{j} - 0.5\mathbf{k}$, 2.58

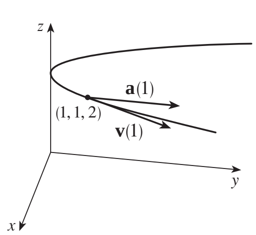
3. $\mathbf{v}(t) = \langle -t, 1 \rangle$
$\mathbf{a}(t) = \langle -1, 0 \rangle$
$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$



5. $\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$
$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{5\sin^2 t + 4}$



7. $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$
$\mathbf{a}(t) = 2\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$



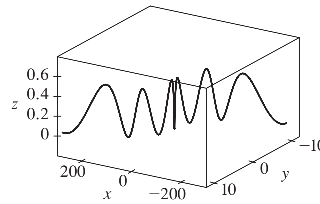
9. $\langle 2t, 3t^2, 2t \rangle$, $\langle 2, 6t, 2 \rangle$, $|t|\sqrt{9t^2 + 8}$

11. $\sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $e^t + e^{-t}$

13. $e^t[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + (t + 1)\mathbf{k}]$, $e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t + 2)\mathbf{k}]$, $e^t\sqrt{t^2 + 2t + 3}$

15. $\mathbf{v}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}(t) = (\frac{1}{2}t^2 + 1)\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$

17. (a) $\mathbf{r}(t) = (\frac{1}{3}t^3 + t)\,\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + (\frac{1}{4} - \frac{1}{4}\cos 2t)\,\mathbf{k}$
(b)



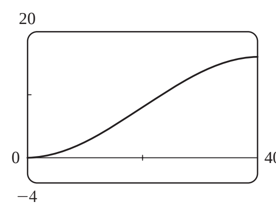
19. $t = 4$ **21.** $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{25t^2 + 2}$

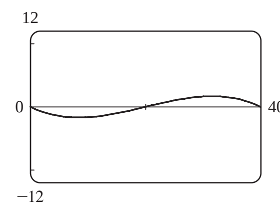
23. (a) $\approx$22 km (b) $\approx$3.2 km (c) 500 m/s

25. 30 m/s **27.** $\approx 10.2°$, $\approx 79.8°$

29. $13.0° < \theta < 36.0°$, $55.4° < \theta < 85.5°$

31. (a) 16 m (b) $\approx 23.6°$ upstream





33. $6t$, 6 **35.** 0, 1 **37.** $e^t - e^{-t}$, $\sqrt{2}$

39. 4.5 cm/s², 9.0 cm/s² **41.** $t = 1$

CHAPTER 13 REVIEW ■ PAGE 850

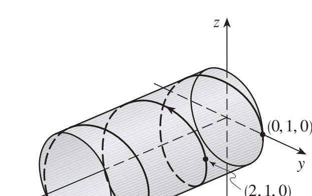
True-False Quiz

1. True **3.** False **5.** False **7.** True

9. False **11.** True

Exercises

1. (a)



(b) $\mathbf{r}'(t) = \mathbf{i} - \pi\sin\pi t\,\mathbf{j} + \pi\cos\pi t\,\mathbf{k}$, $\mathbf{r}''(t) = -\pi^2\cos\pi t\,\mathbf{j} - \pi^2\sin\pi t\,\mathbf{k}$

3. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5 - 4\cos t)\mathbf{k}$, $0 \leqslant t \leqslant 2\pi$

5. $\frac{1}{3}\mathbf{i} - (2/\pi^2)\mathbf{j} + (2/\pi)\mathbf{k}$ **7.** 86.631 **9.** $\pi/2$

11. (a) $\langle t^2, t, 1 \rangle / \sqrt{t^4 + t^2 + 1}$
(b) $\langle 2t, 1 - t^4, -2t^3 - t \rangle / \sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}$
(c) $\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}/(t^4 + t^2 + 1)^2$

13. $12/17^{3/2}$ **15.** $x - 2y + 2\pi = 0$

17. $\mathbf{v}(t) = (1 + \ln t)\,\mathbf{i} + \mathbf{j} - e^{-t}\,\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\mathbf{a}(t) = (1/t)\,\mathbf{i} + e^{-t}\,\mathbf{k}$

19. (a) About 3.8 ft above the ground, 60.8 ft from the athlete
(b) ≈ 21.4 ft (c) ≈ 64.2 ft from the athlete
21. (c) $-2e^{-t}\mathbf{v}_d + e^{-t}\mathbf{R}$

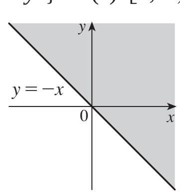
PROBLEMS PLUS ▪ PAGE 852

1. (a) $\mathbf{v} = \omega R(-\sin \omega t\,\mathbf{i} + \cos \omega t\,\mathbf{j})$ (c) $\mathbf{a} = -\omega^2\mathbf{r}$
3. (a) $90°$, $v_0^2/(2g)$
5. (a) ≈ 0.94 ft to the right of the table's edge, ≈ 15 ft/s
(b) $\approx 7.6°$ (c) ≈ 2.13 ft to the right of the table's edge
7. $56°$

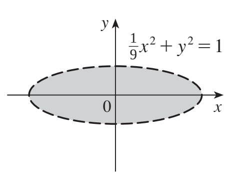
CHAPTER 14

EXERCISES 14.1 ▪ PAGE 865

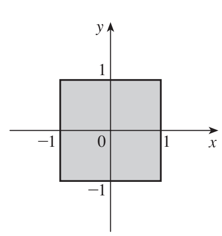
1. (a) -27; a temperature of $-15°$C with wind blowing at 40 km/h feels equivalent to about $-27°$C without wind.
(b) When the temperature is $-20°$C, what wind speed gives a wind chill of $-30°$C? 20 km/h
(c) With a wind speed of 20 km/h, what temperature gives a wind chill of $-49°$C? $-35°$C
(d) A function of wind speed that gives wind-chill values when the temperature is $-5°$C
(e) A function of temperature that gives wind-chill values when the wind speed is 50 km/h
3. Yes
5. (a) 25; a 40-knot wind blowing in the open sea for 15 h will create waves about 25 ft high.
(b) $f(30, t)$ is a function of t giving the wave heights produced by 30-knot winds blowing for t hours.
(c) $f(v, 30)$ is a function of v giving the wave heights produced by winds of speed v blowing for 30 hours.
7. (a) 4 (b) $\mathbb{R}^2$ (c) $[0, \infty)$
9. (a) e (b) $\{(x, y, z) \mid z \geqslant x^2 + y^2\}$ (c) $[1, \infty)$
11. $\{(x, y) \mid y \geqslant -x\}$



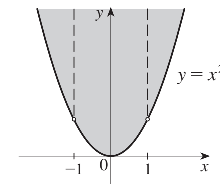
13. $\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\}$



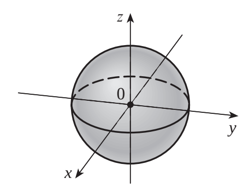
15. $\{(x, y) \mid -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$



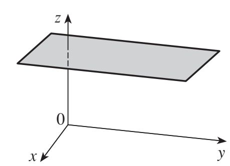
17. $\{(x, y) \mid y \geqslant x^2, x \neq \pm 1\}$



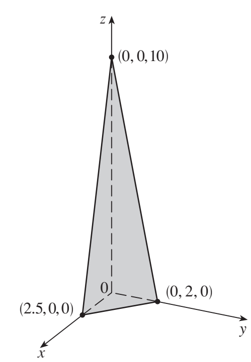
19. $\{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$



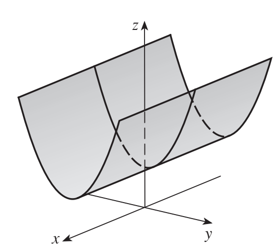
21. $z = 3$, horizontal plane



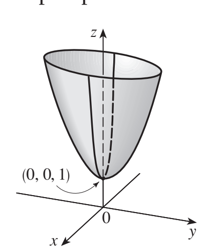
23. $4x + 5y + z = 10$, plane



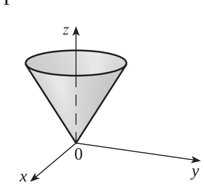
25. $z = y^2 + 1$, parabolic cylinder



27. $z = 4x^2 + y^2 + 1$
elliptic paraboloid

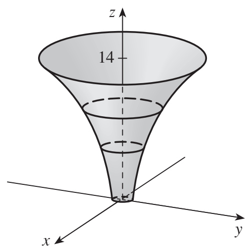


29. $z = \sqrt{x^2 + y^2}$,
top half of cone

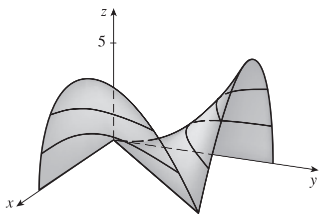


31. ≈ 56, ≈ 35 **33.** Steep; nearly flat

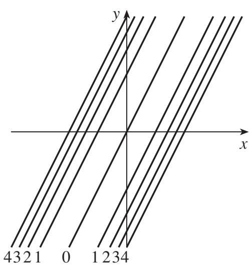
35.



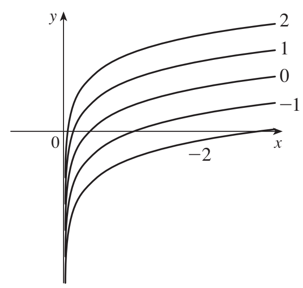
37.



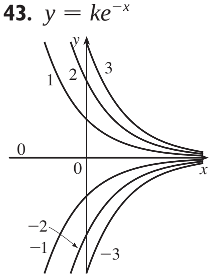
39. $(y - 2x)^2 = k$



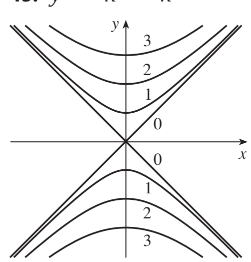
41. $y = \ln x + k$



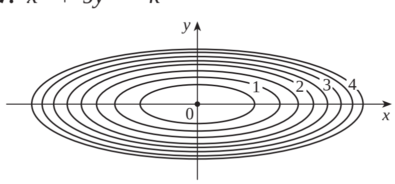
43. $y = ke^{-x}$

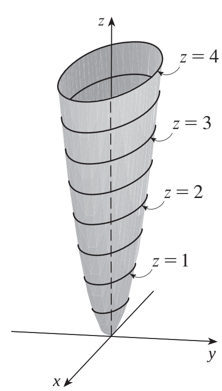


45. $y^2 - x^2 = k^2$

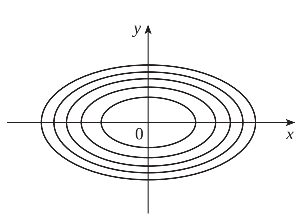


47. $x^2 + 9y^2 = k$

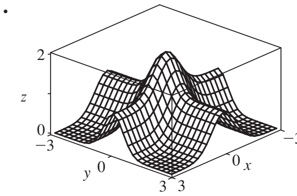




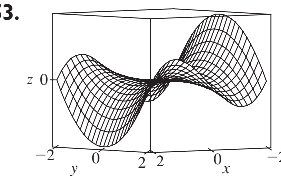
49.



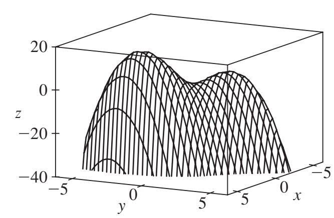
51.



53.

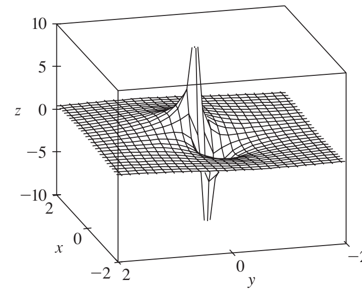


55. (a) C (b) II **57.** (a) F (b) I
59. (a) B (b) VI
61. Family of parallel planes
63. Family of hyperboloids of one or two sheets with axis the *y*-axis
65. (a) Shift the graph of f upward 2 units
(b) Stretch the graph of f vertically by a factor of 2
(c) Reflect the graph of f about the xy-plane
(d) Reflect the graph of f about the xy-plane and then shift it upward 2 units
67.



f appears to have a maximum value of about 15. There are two local maximum points but no local minimum point.

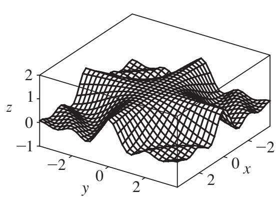
69.



The function values approach 0 as x, y become large; as (x, y) approaches the origin, f approaches $\pm\infty$ or 0, depending on the direction of approach.
71. If $c = 0$, the graph is a cylindrical surface. For $c > 0$, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For $c < 0$, the level curves are hyperbolas. The graph curves upward in the y-direction and downward, approaching the xy-plane, in the x-direction giving a saddle-shaped appearance near (0, 0, 1).
73. $c = -2, 0, 2$ **75.** (b) $y = 0.75x + 0.01$

EXERCISES 14.2 ■ PAGE 877

1. Nothing; if f is continuous, $f(3, 1) = 6$ **3.** $-\frac{5}{2}$
5. 1 **7.** $\frac{2}{7}$ **9.** Does not exist **11.** Does not exist
13. 0 **15.** Does not exist **17.** 2 **19.** 1
21. Does not exist
23. The graph shows that the function approaches different numbers along different lines.
25. $h(x, y) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$; $\{(x, y) \mid 2x + 3y \geqslant 6\}$
27. Along the line $y = x$ **29.** $\{(x, y) \mid y \neq \pm e^{x/2}\}$
31. $\{(x, y) \mid y \geqslant 0\}$ **33.** $\{(x, y) \mid x^2 + y^2 > 4\}$
35. $\left\{(x, y, z) \mid y \geqslant 0, y \neq \sqrt{x^2 + z^2}\right\}$
37. $\{(x, y) \mid (x, y) \neq (0, 0)\}$ **39.** 0 **41.** -1
43.

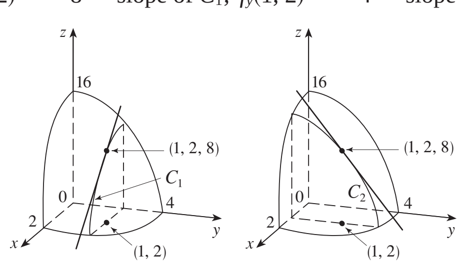


f is continuous on $\mathbb{R}^2$

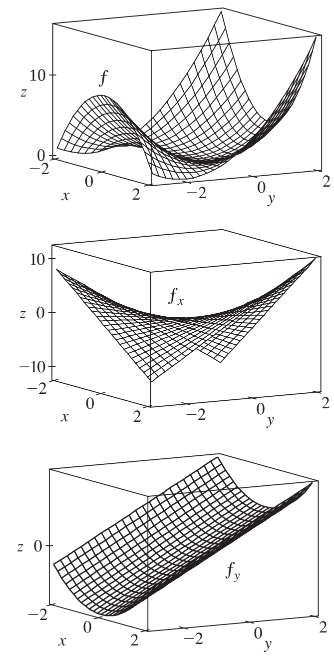
EXERCISES 14.3 ■ PAGE 888

1. (a) The rate of change of temperature as longitude varies, with latitude and time fixed; the rate of change as only latitude varies; the rate of change as only time varies.
(b) Positive, negative, positive

3. (a) $f_T(-15, 30) \approx 1.3$; for a temperature of $-15°C$ and wind speed of 30 km/h, the wind-chill index rises by 1.3°C for each degree the temperature increases. $f_v(-15, 30) \approx -0.15$; for a temperature of $-15°C$ and wind speed of 30 km/h, the wind-chill index decreases by 0.15°C for each km/h the wind speed increases.
(b) Positive, negative (c) 0
5. (a) Positive (b) Negative
7. (a) Positive (b) Negative
9. $c = f$, $b = f_x$, $a = f_y$
11. $f_x(1, 2) = -8 =$ slope of C_1, $f_y(1, 2) = -4 =$ slope of C_2

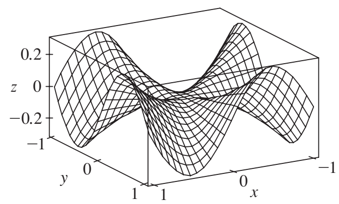


13. $f_x = 2x + 2xy$, $f_y = 2y + x^2$



15. $f_x(x, y) = -3y$, $f_y(x, y) = 5y^4 - 3x$
17. $f_x(x, t) = -\pi e^{-t} \sin \pi x$, $f_t(x, t) = -e^{-t} \cos \pi x$
19. $\partial z/\partial x = 20(2x + 3y)^9$, $\partial z/\partial y = 30(2x + 3y)^9$
21. $f_x(x, y) = 2y/(x + y)^2$, $f_y(x, y) = -2x/(x + y)^2$
23. $\partial w/\partial \alpha = \cos \alpha \cos \beta$, $\partial w/\partial \beta = -\sin \alpha \sin \beta$
25. $f_r(r, s) = \dfrac{2r^2}{r^2 + s^2} + \ln(r^2 + s^2)$, $f_s(r, s) = \dfrac{2rs}{r^2 + s^2}$
27. $\partial u/\partial t = e^{w/t}(1 - w/t)$, $\partial u/\partial w = e^{w/t}$
29. $f_x = z - 10xy^3z^4$, $f_y = -15x^2y^2z^4$, $f_z = x - 20x^2y^3z^3$
31. $\partial w/\partial x = 1/(x + 2y + 3z)$, $\partial w/\partial y = 2/(x + 2y + 3z)$, $\partial w/\partial z = 3/(x + 2y + 3z)$
33. $\partial u/\partial x = y \sin^{-1}(yz)$, $\partial u/\partial y = x \sin^{-1}(yz) + xyz/\sqrt{1 - y^2z^2}$, $\partial u/\partial z = xy^2/\sqrt{1 - y^2z^2}$
35. $f_x = yz^2 \tan(yt)$, $f_y = xyz^2t \sec^2(yt) + xz^2 \tan(yt)$, $f_z = 2xyz \tan(yt)$, $f_t = xy^2z^2 \sec^2(yt)$
37. $\partial u/\partial x_i = x_i/\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
39. $\frac{1}{5}$ **41.** $\frac{1}{4}$
43. $f_x(x, y) = y^2 - 3x^2y$, $f_y(x, y) = 2xy - x^3$
45. $\dfrac{\partial z}{\partial x} = \dfrac{3yz - 2x}{2z - 3xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{3xz - 2y}{2z - 3xy}$
47. $\dfrac{\partial z}{\partial x} = \dfrac{1 + y^2z^2}{1 + y + y^2z^2}$, $\dfrac{\partial z}{\partial y} = \dfrac{-z}{1 + y + y^2z^2}$
49. (a) $f'(x)$, $g'(y)$ (b) $f'(x + y)$, $f'(x + y)$
51. $f_{xx} = 6xy^5 + 24x^2y$, $f_{xy} = 15x^2y^4 + 8x^3 = f_{yx}$, $f_{yy} = 20x^3y^3$
53. $w_{uu} = v^2/(u^2 + v^2)^{3/2}$, $w_{uv} = -uv/(u^2 + v^2)^{3/2} = w_{vu}$, $w_{vv} = u^2/(u^2 + v^2)^{3/2}$
55. $z_{xx} = -2x/(1 + x^2)^2$, $z_{xy} = 0 = z_{yx}$, $z_{yy} = -2y/(1 + y^2)^2$

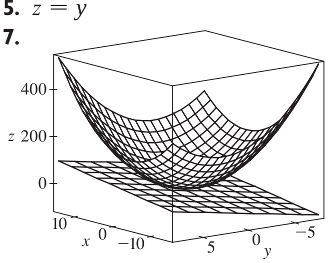
61. $12xy$, $72xy$
63. $24\sin(4x + 3y + 2z)$, $12\sin(4x + 3y + 2z)$
65. $\theta e^{r\theta}(2\sin\theta + \theta\cos\theta + r\theta\sin\theta)$ **67.** $4/(y + 2z)^3$, 0
69. ≈ 12.2, ≈ 16.8, ≈ 23.25 **81.** R^2/R_1^2
87. No **89.** $x = 1 + t$, $y = 2$, $z = 2 - 2t$
93. -2
95. (a)



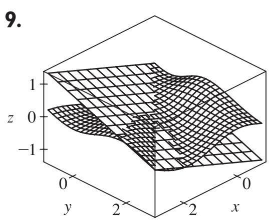
(b) $f_x(x, y) = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$, $f_y(x, y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$
(c) 0, 0 (e) No, since f_{xy} and f_{yx} are not continuous.

EXERCISES 14.4 ▪ PAGE 899

1. $z = -8x - 2y$
3. $x + y - 2z = 0$
5. $z = y$
7.



9.



11. $2x + \frac{1}{4}y - 1$ **13.** $\frac{1}{9}x - \frac{2}{9}y + \frac{2}{3}$ **15.** $1 - \pi y$
19. $-\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3}$; 2.846 **21.** $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$; 6.9914
23. $4T + H - 329$; 129°F
25. $dz = 3x^2\ln(y^2)\,dx + (2x^3/y)\,dy$
27. $dm = 5p^4q^3\,dp + 3p^5q^2\,dq$
29. $dR = \beta^2\cos\gamma\,d\alpha + 2\alpha\beta\cos\gamma\,d\beta - \alpha\beta^2\sin\gamma\,d\gamma$
31. $\Delta z = 0.9225$, $dz = 0.9$ **33.** 5.4 cm^2 **35.** 16 cm^3
37. 150 **39.** $\frac{1}{17} \approx 0.059\ \Omega$ **41.** 2.3%
43. $\varepsilon_1 = \Delta x$, $\varepsilon_2 = \Delta y$

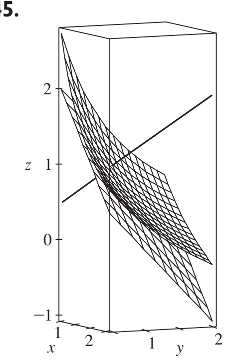
EXERCISES 14.5 ▪ PAGE 907

1. $(2x + y)\cos t + (2y + x)e^t$
3. $[(x/t) - y\sin t]/\sqrt{1 + x^2 + y^2}$
5. $e^{y/z}[2t - (x/z) - (2xy/z^2)]$
7. $\partial z/\partial s = 2xy^3\cos t + 3x^2y^2\sin t$,
$\partial z/\partial t = -2sxy^3\sin t + 3sx^2y^2\cos t$
9. $\partial z/\partial s = t^2\cos\theta\cos\phi - 2st\sin\theta\sin\phi$,
$\partial z/\partial t = 2st\cos\theta\cos\phi - s^2\sin\theta\sin\phi$
11. $\dfrac{\partial z}{\partial s} = e^r\left(t\cos\theta - \dfrac{s}{\sqrt{s^2 + t^2}}\sin\theta\right)$,
$\dfrac{\partial z}{\partial t} = e^r\left(s\cos\theta - \dfrac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$

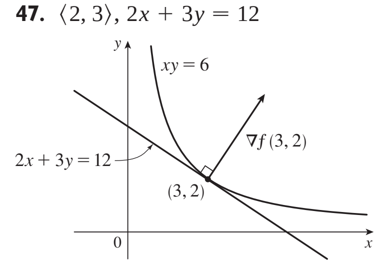
13. 62 **15.** 7, 2
17. $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}$, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}$,
$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$
19. $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial x}$,
$\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial y}$
21. 85, 178, 54 **23.** $\frac{9}{7}$, $\frac{9}{7}$ **25.** 36, 24, 30
27. $\dfrac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}$ **29.** $\dfrac{\sin(x - y) + e^y}{\sin(x - y) - xe^y}$
31. $\dfrac{3yz - 2x}{2z - 3xy}$, $\dfrac{3xz - 2y}{2z - 3xy}$
33. $\dfrac{1 + y^2z^2}{1 + y + y^2z^2}$, $-\dfrac{z}{1 + y + y^2z^2}$
35. 2°C/s **37.** ≈ -0.33 m/s per minute
39. (a) $6\text{ m}^3/\text{s}$ (b) $10\text{ m}^2/\text{s}$ (c) 0 m/s
41. ≈ -0.27 L/s **43.** $-1/(12\sqrt{3})$ rad/s
45. (a) $\partial z/\partial r = (\partial z/\partial x)\cos\theta + (\partial z/\partial y)\sin\theta$,
$\partial z/\partial\theta = -(\partial z/\partial x)r\sin\theta + (\partial z/\partial y)r\cos\theta$
51. $4rs\,\partial^2 z/\partial x^2 + (4r^2 + 4s^2)\partial^2 z/\partial x\,\partial y + 4rs\,\partial^2 z/\partial y^2 + 2\,\partial z/\partial y$

EXERCISES 14.6 ▪ PAGE 920

1. ≈ -0.08 mb/km **3.** ≈ 0.778 **5.** $2 + \sqrt{3}/2$
7. (a) $\nabla f(x, y) = \langle 2\cos(2x + 3y), 3\cos(2x + 3y)\rangle$
(b) $\langle 2, 3\rangle$ (c) $\sqrt{3} - \frac{3}{2}$
9. (a) $\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\rangle$ (b) $\langle 1, 12, 0\rangle$ (c) $-\frac{22}{3}$
11. 23/10 **13.** $-8/\sqrt{10}$ **15.** $4/\sqrt{30}$ **17.** $9/(2\sqrt{5})$
19. 2/5 **21.** $4\sqrt{2}$, $\langle -1, 1\rangle$ **23.** 1, $\langle 0, 1\rangle$
25. 1, $\langle 3, 6, -2\rangle$ **27.** (b) $\langle -12, 92\rangle$
29. All points on the line $y = x + 1$
31. (a) $-40/(3\sqrt{3})$
33. (a) $32/\sqrt{3}$ (b) $\langle 38, 6, 12\rangle$ (c) $2\sqrt{406}$ **35.** $\frac{327}{13}$
39. (a) $x + y + z = 11$ (b) $x - 3 = y - 3 = z - 5$
41. (a) $4x - 5y - z = 4$ (b) $\dfrac{x - 2}{4} = \dfrac{y - 1}{-5} = \dfrac{z + 1}{-1}$
43. (a) $x + y - z = 1$ (b) $x - 1 = y = -z$
45.



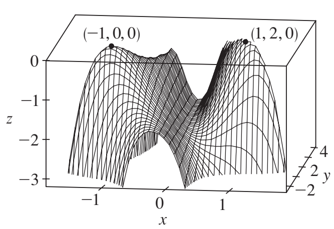
47. $\langle 2, 3\rangle$, $2x + 3y = 12$



53. No **59.** $x = -1 - 10t$, $y = 1 - 16t$, $z = 2 - 12t$
63. If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $af_x + bf_y$ and $cf_x + df_y$ are known, so we solve linear equations for f_x and f_y.

EXERCISES 14.7 ■ PAGE 930

1. (a) f has a local minimum at (1, 1).
(b) f has a saddle point at (1, 1).
3. Local minimum at (1, 1), saddle point at (0, 0)
5. Maximum $f\left(-1, \frac{1}{2}\right) = 11$
7. Minima $f(1, 1) = 0$, $f(-1, -1) = 0$, saddle point at (0, 0)
9. Saddle points at $(1, -1)$, $(-1, 1)$
11. Minimum $f(2, 1) = -8$, saddle point at (0, 0)
13. None **15.** Minimum $f(0, 0) = 0$, saddle points at $(\pm 1, 0)$
17. Minima $f(0, 1) = f(\pi, -1) = f(2\pi, 1) = -1$,
saddle points at $(\pi/2, 0)$, $(3\pi/2, 0)$
21. Minima $f(1, \pm 1) = 3$, $f(-1, \pm 1) = 3$
23. Maximum $f(\pi/3, \pi/3) = 3\sqrt{3}/2$,
minimum $f(5\pi/3, 5\pi/3) = -3\sqrt{3}/2$, saddle point at (π, π)
25. Minima $f(-1.714, 0) \approx -9.200$, $f(1.402, 0) \approx 0.242$,
saddle point $(0.312, 0)$, lowest point $(-1.714, 0, -9.200)$
27. Maxima $f(-1.267, 0) \approx 1.310$, $f(1.629, \pm 1.063) \approx 8.105$,
saddle points $(-0.259, 0)$, $(1.526, 0)$,
highest points $(1.629, \pm 1.063, 8.105)$
29. Maximum $f(2, 0) = 9$, minimum $f(0, 3) = -14$
31. Maximum $f(\pm 1, 1) = 7$, minimum $f(0, 0) = 4$
33. Maximum $f(3, 0) = 83$, minimum $f(1, 1) = 0$
35. Maximum $f(1, 0) = 2$, minimum $f(-1, 0) = -2$
37.



39. $\sqrt{3}$ **41.** $(2, 1, \sqrt{5})$, $(2, 1, -\sqrt{5})$ **43.** $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$
45. $8r^3/(3\sqrt{3})$
47. $\frac{4}{3}$ **49.** Cube, edge length $c/12$
51. Square base of side 40 cm, height 20 cm **53.** $L^3/(3\sqrt{3})$

EXERCISES 14.8 ■ PAGE 940

1. $\approx$59, 30
3. No maximum, minima $f(1, 1) = f(-1, -1) = 2$
5. Maxima $f(\pm 2, 1) = 4$, minima $f(\pm 2, -1) = -4$
7. Maximum $f(1, 3, 5) = 70$, minimum $f(-1, -3, -5) = -70$
9. Maximum $2/\sqrt{3}$, minimum $-2/\sqrt{3}$
11. Maximum $\sqrt{3}$, minimum 1
13. Maximum $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$,
minimum $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$
15. Maximum $f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$,
minimum $f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$
17. Maximum $\frac{3}{2}$, minimum $\frac{1}{2}$
19. Maxima $f\left(\pm 1/\sqrt{2}, \mp 1/(2\sqrt{2})\right) = e^{1/4}$,
minima $f\left(\pm 1/\sqrt{2}, \pm 1/(2\sqrt{2})\right) = e^{-1/4}$
27–37. See Exercises 39–49 in Section 14.7.
39. $L^3/(3\sqrt{3})$
41. Nearest $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, farthest $(-1, -1, 2)$
43. Maximum $\approx$9.7938, minimum ≈ -5.3506
45. (a) c/n (b) When $x_1 = x_2 = \cdots = x_n$

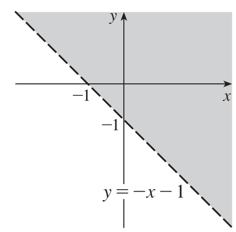
CHAPTER 14 REVIEW ■ PAGE 944

True-False Quiz

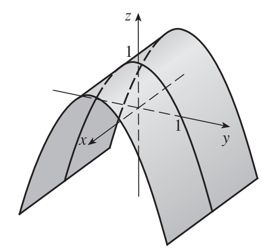
1. True **3.** False **5.** False **7.** True **9.** False
11. True

Exercises

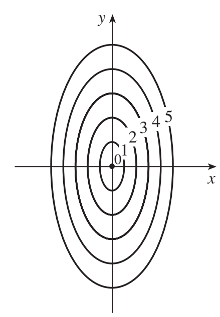
1. $\{(x, y) \mid y > -x - 1\}$



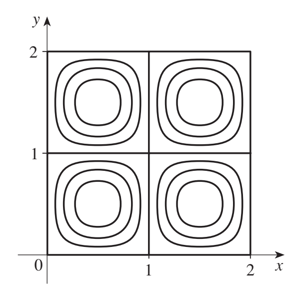
3.



5.



7.



9. $\frac{2}{3}$
11. (a) $\approx$3.5°C/m, -3.0°C/m (b) $\approx$0.35°C/m by Equation 14.6.9 (Definition 14.6.2 gives $\approx$1.1°C/m.)
(c) -0.25
13. $f_x = 1/\sqrt{2x + y^2}$, $f_y = y/\sqrt{2x + y^2}$
15. $g_u = \tan^{-1} v$, $g_v = u/(1 + v^2)$
17. $T_p = \ln(q + e^r)$, $T_q = p/(q + e^r)$, $T_r = pe^r/(q + e^r)$
19. $f_{xx} = 24x$, $f_{xy} = -2y = f_{yx}$, $f_{yy} = -2x$
21. $f_{xx} = k(k-1)x^{k-2}y^l z^m$, $f_{xy} = klx^{k-1}y^{l-1}z^m = f_{yx}$,
$f_{xz} = kmx^{k-1}y^l z^{m-1} = f_{zx}$, $f_{yy} = l(l-1)x^k y^{l-2}z^m$,
$f_{yz} = lmx^k y^{l-1}z^{m-1} = f_{zy}$, $f_{zz} = m(m-1)x^k y^l z^{m-2}$
25. (a) $z = 8x + 4y + 1$ (b) $\dfrac{x-1}{8} = \dfrac{y+2}{4} = 1 - z$
27. (a) $2x - 2y - 3z = 3$ (b) $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$
29. (a) $4x - y - 2z = 6$
(b) $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$
31. $\left(2, \frac{1}{2}, -1\right)$, $\left(-2, -\frac{1}{2}, 1\right)$
33. $60x + \frac{24}{5}y + \frac{32}{5}z - 120$; 38.656
35. $2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$
37. -47, 108 **43.** $ze^{x\sqrt{y}}\left\langle z\sqrt{y}, xz/(2\sqrt{y}), 2\right\rangle$ **45.** $\frac{43}{5}$
47. $\sqrt{145}/2$, $\left\langle 4, \frac{9}{2}\right\rangle$ **49.** $\approx \frac{5}{8}$ knot/mi

51. Minimum $f(-4, 1) = -11$
53. Maximum $f(1, 1) = 1$; saddle points $(0, 0)$, $(0, 3)$, $(3, 0)$
55. Maximum $f(1, 2) = 4$, minimum $f(2, 4) = -64$
57. Maximum $f(-1, 0) = 2$, minima $f(1, \pm 1) = -3$, saddle points $(-1, \pm 1)$, $(1, 0)$
59. Maximum $f\left(\pm\sqrt{2/3}, 1/\sqrt{3}\right) = 2/\left(3\sqrt{3}\right)$, minimum $f\left(\pm\sqrt{2/3}, -1/\sqrt{3}\right) = -2/\left(3\sqrt{3}\right)$
61. Maximum 1, minimum -1
63. $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$, $\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$
65. $P\left(2 - \sqrt{3}\right)$, $P\left(3 - \sqrt{3}\right)/6$, $P\left(2\sqrt{3} - 3\right)/3$

PROBLEMS PLUS ▪ PAGE 948

1. L^2W^2, $\frac{1}{4}L^2W^2$ **3.** (a) $x = w/3$, base $= w/3$ (b) Yes
7. $\sqrt{6}/2$, $3\sqrt{2}/2$

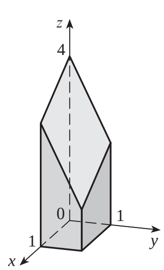
CHAPTER 15

EXERCISES 15.1 ▪ PAGE 958

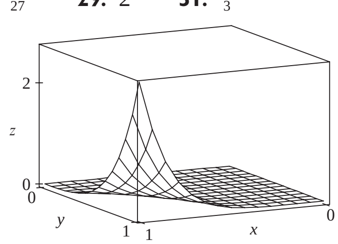
1. (a) 288 (b) 144
3. (a) $\pi^2/2 \approx 4.935$ (b) 0
5. (a) -6 (b) -3.5
7. $U < V < L$
9. (a) ≈ 248 (b) 15.5
11. 60 **13.** 3
15. 1.141606, 1.143191, 1.143535, 1.143617, 1.143637, 1.143642

EXERCISES 15.2 ▪ PAGE 964

1. $500y^3$, $3x^2$ **3.** 10 **5.** 2 **7.** 261,632/45 **9.** $\frac{21}{2}\ln 2$
11. 0 **13.** π **15.** $\frac{21}{2}$ **17.** $9\ln 2$
19. $\frac{1}{2}\left(\sqrt{3} - 1\right) - \frac{1}{12}\pi$ **21.** $\frac{1}{2}(e^2 - 3)$
23.



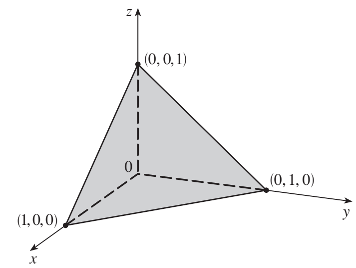
25. 47.5 **27.** $\frac{166}{27}$ **29.** 2 **31.** $\frac{64}{3}$
33. $21e - 57$



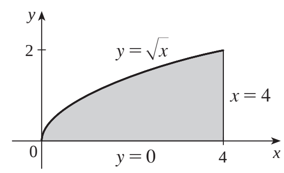
35. $\frac{5}{6}$
37. Fubini's Theorem does not apply. The integrand has an infinite discontinuity at the origin.

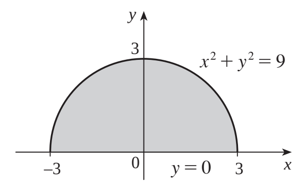
EXERCISES 15.3 ▪ PAGE 972

1. 32 **3.** $\frac{3}{10}$ **5.** $e - 1$ **7.** $\frac{4}{3}$ **9.** π **11.** $\frac{1}{2}e^{16} - \frac{17}{2}$
13. $\frac{1}{2}(1 - \cos 1)$ **15.** $\frac{147}{20}$ **17.** 0 **19.** $\frac{7}{18}$ **21.** $\frac{31}{8}$
23. 6 **25.** $\frac{128}{15}$ **27.** $\frac{1}{3}$ **29.** 0, 1.213, 0.713 **31.** $\frac{64}{3}$
33.

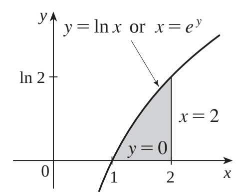


35. 13,984,735,616/14,549,535 **37.** $\pi/2$
39. $\int_0^2 \int_{y^2}^4 f(x, y)\,dx\,dy$ **41.** $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x, y)\,dy\,dx$





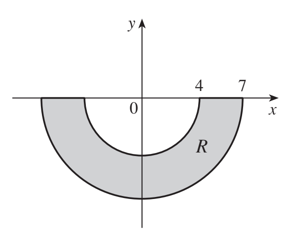
43. $\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\,dx\,dy$



45. $\frac{1}{6}(e^9 - 1)$ **47.** $\frac{1}{3}\ln 9$ **49.** $\frac{1}{3}\left(2\sqrt{2} - 1\right)$ **51.** 1
53. $(\pi/16)e^{-1/16} \leqslant \iint_Q e^{-(x^2+y^2)^2}\,dA \leqslant \pi/16$ **55.** $\frac{3}{4}$
59. 8π **61.** $2\pi/3$

EXERCISES 15.4 ▪ PAGE 978

1. $\int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ **3.** $\int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\,dy\,dx$
5. $33\pi/2$



7. 0 **9.** $\frac{1}{2}\pi \sin 9$ **11.** $(\pi/2)(1 - e^{-4})$ **13.** $\frac{3}{64}\pi^2$
15. $\pi/12$ **17.** $\frac{1}{8}(\pi - 2)$ **19.** $\frac{16}{3}\pi$ **21.** $\frac{4}{3}\pi$
23. $\frac{4}{3}\pi a^3$ **25.** $(2\pi/3)\left[1 - \left(1/\sqrt{2}\right)\right]$
27. $(8\pi/3)\left(64 - 24\sqrt{3}\right)$
29. $\frac{1}{2}\pi(1 - \cos 9)$ **31.** $2\sqrt{2}/3$
33. 1800π ft^3 **35.** $\frac{15}{16}$ **37.** (a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/2$

EXERCISES 15.5 ■ PAGE 988

1. $\frac{64}{3}$ C **3.** $\frac{4}{3}, \left(\frac{4}{3}, 0\right)$ **5.** $6, \left(\frac{3}{4}, \frac{3}{2}\right)$

7. $\frac{1}{4}(e^2 - 1), \left(\dfrac{e^2 + 1}{2(e^2 - 1)}, \dfrac{4(e^3 - 1)}{9(e^2 - 1)}\right)$

9. $L/4, (L/2, 16/(9\pi))$ **11.** $\left(\frac{3}{8}, 3\pi/16\right)$ **13.** $(0, 45/(14\pi))$

15. $(2a/5, 2a/5)$ if vertex is $(0, 0)$ and sides are along positive axes

17. $\frac{1}{16}(e^4 - 1), \frac{1}{8}(e^2 - 1), \frac{1}{16}(e^4 + 2e^2 - 3)$

19. $7ka^6/180, 7ka^6/180, 7ka^6/90$ if vertex is $(0, 0)$ and sides are along positive axes

21. $m = \pi^2/8, (\bar{x}, \bar{y}) = \left(\dfrac{2\pi}{3} - \dfrac{1}{\pi}, \dfrac{16}{9\pi}\right), I_x = 3\pi^2/64,$
$I_y = \frac{1}{16}(\pi^4 - 3\pi^2), I_0 = \pi^4/16 - 9\pi^2/64$

23. $\rho bh^3/3, \rho b^3h/3; b/\sqrt{3}, h/\sqrt{3}$

25. $\rho a^4\pi/16, \rho a^4\pi/16; a/2, a/2$

27. (a) $\frac{1}{2}$ (b) 0.375 (c) $\frac{5}{48} \approx 0.1042$

29. (b) (i) $e^{-0.2} \approx 0.8187$
(ii) $1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$ (c) 2, 5

31. (a) ≈ 0.500 (b) ≈ 0.632

33. (a) $\iint_D (k/20)\left[20 - \sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$, where D is the disk with radius 10 mi centered at the center of the city
(b) $200\pi k/3 \approx 209k, 200\left(\pi/2 - \frac{8}{9}\right)k \approx 136k$, on the edge

EXERCISES 15.6 ■ PAGE 998

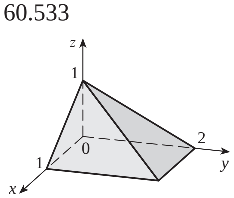
1. $\frac{27}{4}$ **3.** 1 **5.** $\frac{1}{3}(e^3 - 1)$ **7.** $-\frac{1}{3}$ **9.** 4 **11.** $\frac{65}{28}$

13. $8/(3e)$ **15.** $\frac{1}{60}$ **17.** $16\pi/3$ **19.** $\frac{16}{3}$ **21.** 36π

23. (a) $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$ (b) $\frac{1}{4}\pi - \frac{1}{3}$

25. 60.533

27.



29. $\int_{-2}^{2} \int_0^{4-x^2} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dy\, dx$
$= \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dx\, dy$
$= \int_{-1}^{1} \int_0^{4-4z^2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dy\, dz$
$= \int_0^4 \int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dz\, dy$
$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{4-x^2-4z^2} f(x, y, z)\, dy\, dz\, dx$
$= \int_{-1}^{1} \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_0^{4-x^2-4z^2} f(x, y, z)\, dy\, dx\, dz$

31. $\int_{-2}^{2} \int_{x^2}^{4} \int_0^{2-y/2} f(x, y, z)\, dz\, dy\, dx$
$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{2-y/2} f(x, y, z)\, dz\, dx\, dy$
$= \int_0^2 \int_0^{4-2z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$
$= \int_0^4 \int_0^{2-y/2} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dz\, dy$
$= \int_{-2}^{2} \int_0^{2-x^2/2} \int_{x^2}^{4-2z} f(x, y, z)\, dy\, dz\, dx$
$= \int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} f(x, y, z)\, dy\, dx\, dz$

33. $\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$
$= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, dz\, dx\, dy$
$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z)\, dx\, dy\, dz$
$= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dx\, dz\, dy$
$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx$
$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz$

35. $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy = \int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx$
$= \int_0^1 \int_z^1 \int_y^1 f(x, y, z)\, dx\, dy\, dz = \int_0^1 \int_0^y \int_y^1 f(x, y, z)\, dx\, dz\, dy$
$= \int_0^1 \int_0^x \int_z^x f(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_z^1 \int_z^x f(x, y, z)\, dy\, dx\, dz$

37. $\frac{79}{30}, \left(\frac{358}{553}, \frac{33}{79}, \frac{571}{553}\right)$ **39.** $a^5, (7a/12, 7a/12, 7a/12)$

41. $I_x = I_y = I_z = \frac{2}{3}kL^5$ **43.** $\frac{1}{2}\pi kha^4$

45. (a) $m = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2 + y^2}\, dz\, dy\, dx$
(b) $(\bar{x}, \bar{y}, \bar{z})$, where
$\bar{x} = (1/m) \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} x\sqrt{x^2 + y^2}\, dz\, dy\, dx$
$\bar{y} = (1/m) \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} y\sqrt{x^2 + y^2}\, dz\, dy\, dx$
$\bar{z} = (1/m) \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} z\sqrt{x^2 + y^2}\, dz\, dy\, dx$
(c) $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} (x^2 + y^2)^{3/2}\, dz\, dy\, dx$

47. (a) $\frac{3}{32}\pi + \frac{11}{24}$
(b) $(\bar{x}, \bar{y}, \bar{z}) = \left(\dfrac{28}{9\pi + 44}, \dfrac{30\pi + 128}{45\pi + 220}, \dfrac{45\pi + 208}{135\pi + 660}\right)$
(c) $\frac{1}{240}(68 + 15\pi)$

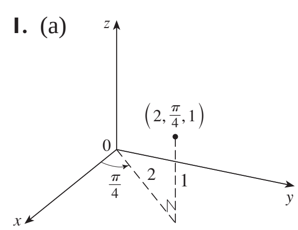
49. (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$

51. $L^3/8$

53. The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$

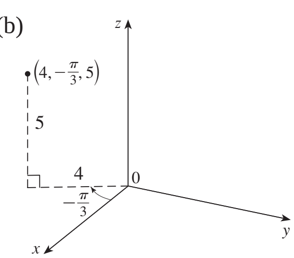
EXERCISES 15.7 ■ PAGE 1004

1. (a)



$\left(\sqrt{2}, \sqrt{2}, 1\right)$

(b)



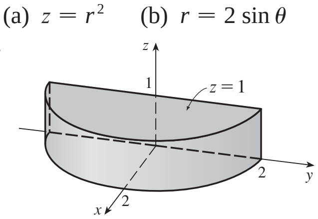
$\left(2, -2\sqrt{3}, 5\right)$

3. (a) $\left(\sqrt{2}, 7\pi/4, 4\right)$ (b) $\left(2, 4\pi/3, 2\right)$

5. Vertical half-plane through the z-axis **7.** Circular paraboloid

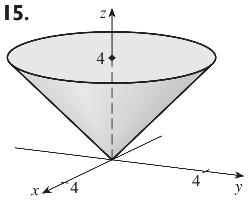
9. (a) $z = r^2$ (b) $r = 2 \sin \theta$

11.



13. Cylindrical coordinates: $6 \leqslant r \leqslant 7$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant 20$

15.



$64\pi/3$

17. 384π **19.** $\pi(e^6 - e - 5)$ **21.** $2\pi/5$

23. (a) 162π (b) $(0, 0, 15)$

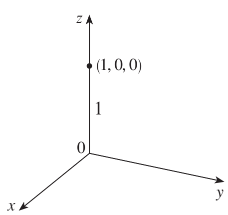
25. $\pi K a^2/8$, $(0, 0, 2a/3)$ **27.** 0

29. (a) $\iiint_C h(P)g(P)\,dV$, where C is the cone

(b) $\approx 3.1 \times 10^{19}$ ft-lb

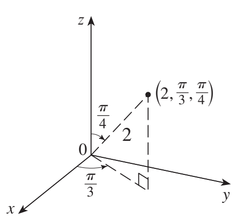
EXERCISES 15.8 ■ PAGE 1010

1. (a)



$(0, 0, 1)$

(b)



$\left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{6}, \sqrt{2}\right)$

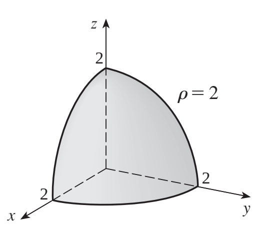
3. (a) $(4, \pi/3, \pi/6)$ (b) $\left(\sqrt{2}, 3\pi/2, 3\pi/4\right)$

5. Half-cone

7. Sphere, radius $\frac{1}{2}$, center $\left(0, \frac{1}{2}, 0\right)$

9. (a) $\cos^2\phi = \sin^2\phi$ (b) $\rho^2(\sin^2\phi\cos^2\theta + \cos^2\phi) = 9$

11.



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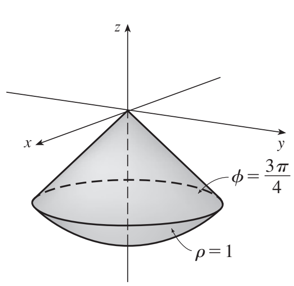
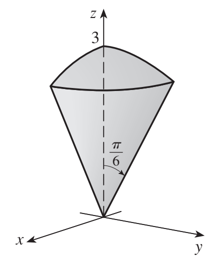


15. $0 \leqslant \phi \leqslant \pi/4$, $0 \leqslant \rho \leqslant \cos\phi$

17.



$(9\pi/4)\left(2 - \sqrt{3}\right)$

19. $\int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$

21. $312{,}500\pi/7$ **23.** $15\pi/16$ **25.** $1562\pi/15$

27. $\left(\sqrt{3} - 1\right)\pi a^3/3$ **29.** (a) 10π (b) $(0, 0, 2.1)$

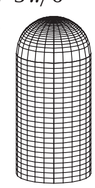
31. $\left(0, \frac{525}{296}, 0\right)$

33. (a) $\left(0, 0, \frac{3}{8}a\right)$ (b) $4K\pi a^5/15$

35. $(2\pi/3)\left[1 - \left(1/\sqrt{2}\right)\right]$, $\left(0, 0, 3/\left[8\left(2 - \sqrt{2}\right)\right]\right)$

37. $5\pi/6$ **39.** $\left(4\sqrt{2} - 5\right)/15$

41.



43. $136\pi/99$

EXERCISES 15.9 ■ PAGE 1020

1. 16 **3.** $\sin^2\theta - \cos^2\theta$ **5.** 0

7. The parallelogram with vertices $(0, 0)$, $(6, 3)$, $(12, 1)$, $(6, -2)$

9. The region bounded by the line $y = 1$, the y-axis, and $y = \sqrt{x}$

11. -3 **13.** 6π **15.** $2 \ln 3$

17. (a) $\frac{4}{3}\pi abc$ (b) 1.083×10^{12} km^3

19. $\frac{8}{5} \ln 8$ **21.** $\frac{3}{2} \sin 1$ **23.** $e - e^{-1}$

CHAPTER 15 REVIEW ■ PAGE 1021

True-False Quiz

1. True **3.** True **5.** True **7.** False

Exercises

1. ≈ 64.0 **3.** $4e^2 - 4e + 3$ **5.** $\frac{1}{2} \sin 1$ **7.** $\frac{2}{3}$

9. $\int_0^{\pi} \int_2^4 f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

11. The region inside the loop of the four-leaved rose $r = \sin 2\theta$ in the first quadrant

13. $\frac{1}{2} \sin 1$ **15.** $\frac{1}{2}e^6 - \frac{7}{2}$ **17.** $\frac{1}{4} \ln 2$ **19.** 8

21. $81\pi/5$ **23.** 40.5 **25.** $\pi/96$ **27.** $\frac{64}{15}$ **29.** 176

31. $\frac{2}{3}$ **33.** $2ma^3/9$

35. (a) $\frac{1}{4}$ (b) $\left(\frac{1}{3}, \frac{8}{15}\right)$

(c) $I_x = \frac{1}{12}$, $I_y = \frac{1}{24}$; $\bar{\bar{y}} = 1/\sqrt{3}$, $\bar{\bar{x}} = 1/\sqrt{6}$

37. $(0, 0, h/4)$

39. 97.2 **41.** 0.0512

43. (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

45. $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\,dx\,dy\,dz$ **47.** $-\ln 2$ **49.** 0

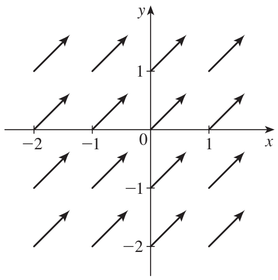
PROBLEMS PLUS ■ PAGE 1024

1. 30 **3.** $\frac{1}{2} \sin 1$ **7.** (b) 0.90

CHAPTER 16

EXERCISES 16.1 ■ PAGE 1032

1.



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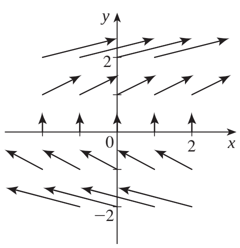

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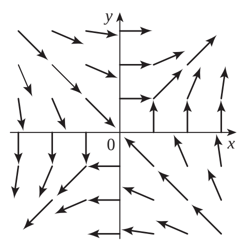

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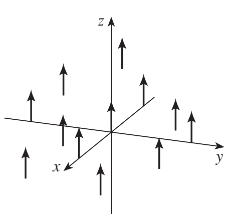

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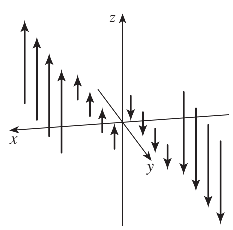
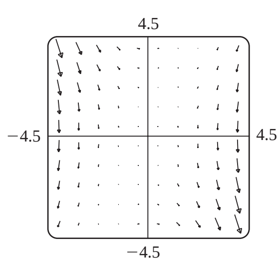


11. II **13.** I **15.** IV **17.** III

19.

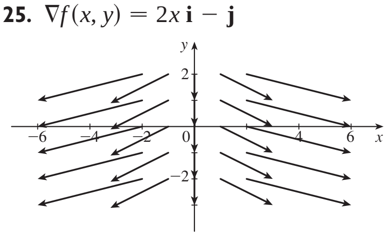


The line $y = 2x$

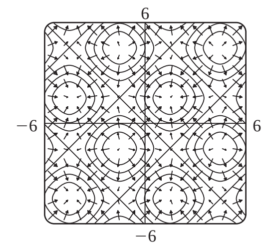
21. $\nabla f(x, y) = (xy + 1)e^{xy}\,\mathbf{i} + x^2 e^{xy}\,\mathbf{j}$

23. $\nabla f(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$

25. $\nabla f(x, y) = 2x\,\mathbf{i} - \mathbf{j}$

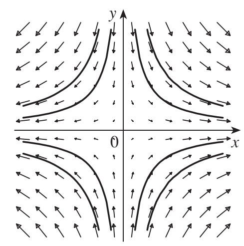


27.



29. III **31.** II **33.** (2.04, 1.03)

35. (a)



$y = C/x$

(b) $y = 1/x,\ x > 0$

EXERCISES 16.2 ■ PAGE 1043

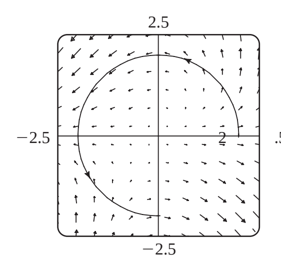
1. $\frac{1}{54}(145^{3/2} - 1)$ **3.** 1638.4 **5.** $\frac{243}{8}$ **7.** $\frac{17}{3}$ **9.** $\sqrt{5}\,\pi$

11. $\frac{1}{12}\sqrt{14}(e^6 - 1)$ **13.** $\frac{1}{5}$ **15.** $\frac{97}{3}$

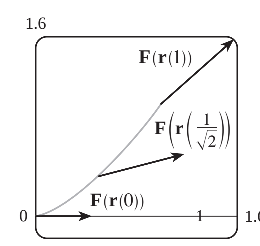
17. (a) Positive (b) Negative

19. 45 **21.** $\frac{6}{5} - \cos 1 - \sin 1$ **23.** 1.9633 **25.** 15.0074

27. $3\pi + \frac{2}{3}$



29. (a) $\frac{11}{8} - 1/e$ (b)



31. $\frac{172{,}704}{5{,}632{,}705}\sqrt{2}(1 - e^{-14\pi})$ **33.** $2\pi k$, $(4/\pi, 0)$

35. (a) $\bar{x} = (1/m)\int_C x\rho(x, y, z)\,ds$,
$\bar{y} = (1/m)\int_C y\rho(x, y, z)\,ds$,
$\bar{z} = (1/m)\int_C z\rho(x, y, z)\,ds$, where $m = \int_C \rho(x, y, z)\,ds$

(b) $(0, 0, 3\pi)$

37. $I_x = k\left(\frac{1}{2}\pi - \frac{4}{3}\right)$, $I_y = k\left(\frac{1}{2}\pi - \frac{2}{3}\right)$

39. $2\pi^2$ **41.** 26 **43.** 1.67×10^4 ft-lb **45.** (b) Yes

47. ≈ 22 J

EXERCISES 16.3 ■ PAGE 1053

1. 40 **3.** $f(x, y) = x^2 - 3xy + 2y^2 - 8y + K$
5. $f(x, y) = e^x \sin y + K$ **7.** $f(x, y) = ye^x + x \sin y + K$
9. $f(x, y) = x \ln y + x^2y^3 + K$
11. (b) 16 **13.** (a) $f(x, y) = \frac{1}{2}x^2y^2$ (b) 2
15. (a) $f(x, y, z) = xyz + z^2$ (b) 77
17. (a) $f(x, y, z) = xy^2 \cos z$ (b) 0
19. 2 **21.** 30 **23.** No **25.** Conservative
29. (a) Yes (b) Yes (c) Yes
31. (a) Yes (b) Yes (c) No

EXERCISES 16.4 ■ PAGE 1060

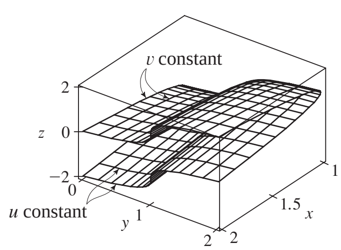
1. 8π **3.** $\frac{2}{3}$ **5.** 12 **7.** $\frac{1}{3}$ **9.** -24π **11.** $\frac{4}{3} - 2\pi$
13. $\frac{625}{2}\pi$ **15.** $-8e + 48e^{-1}$ **17.** $-\frac{1}{12}$ **19.** 3π **21.** (c) $\frac{9}{2}$
23. $(4a/3\pi, 4a/3\pi)$ if the region is the portion of the disk $x^2 + y^2 = a^2$ in the first quadrant

EXERCISES 16.5 ■ PAGE 1068

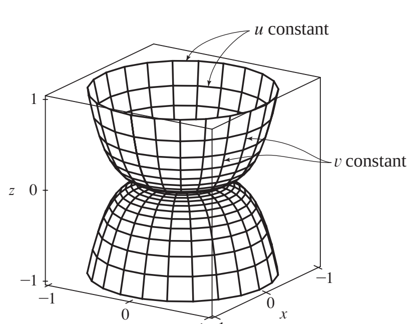
1. (a) $-x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$ (b) yz
3. (a) $(x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$ (b) $z - 1/(2\sqrt{z})$
5. (a) $\mathbf{0}$ (b) $2/\sqrt{x^2 + y^2 + z^2}$
7. (a) $\langle 1/y, -1/x, 1/x \rangle$ (b) $1/x + 1/y + 1/z$
9. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$
11. (a) Zero (b) curl $\mathbf{F}$ points in the negative z-direction
13. $f(x, y, z) = xy^2z^3 + K$ **15.** $f(x, y, z) = x^2y + y^2z + K$
17. Not conservative **19.** No

EXERCISES 16.6 ■ PAGE 1078

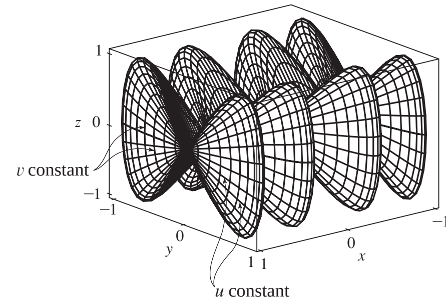
1. P: no; Q: yes
3. Plane through $(0, 3, 1)$ containing vectors $\langle 1, 0, 4 \rangle$, $\langle 1, -1, 5 \rangle$
5. Hyperbolic paraboloid
7.



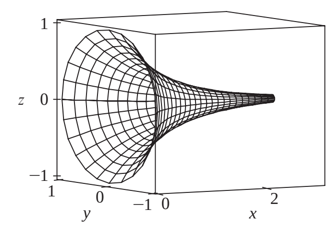
9.



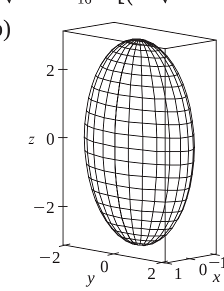
11.



13. IV **15.** II **17.** III
19. $x = 1 + u + v, y = 2 + u - v, z = -3 - u + v$
21. $x = x, z = z, y = \sqrt{1 - x^2 + z^2}$
23. $x = 2 \sin \phi \cos \theta, y = 2 \sin \phi \sin \theta$,
$z = 2 \cos \phi, 0 \leqslant \phi \leqslant \pi/4, 0 \leqslant \theta \leqslant 2\pi$
$\left[\text{or } x = x, y = y, z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 \leqslant 2\right]$
25. $x = x, y = 4 \cos \theta, z = 4 \sin \theta, 0 \leqslant x \leqslant 5, 0 \leqslant \theta \leqslant 2\pi$
29. $x = x, y = e^{-x} \cos \theta$,
$z = e^{-x} \sin \theta, 0 \leqslant x \leqslant 3$,
$0 \leqslant \theta \leqslant 2\pi$



31. (a) Direction reverses (b) Number of coils doubles
33. $3x - y + 3z = 3$ **35.** $-x + 2z = 1$ **37.** $3\sqrt{14}$
39. $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1)$ **41.** $(2\pi/3)(2\sqrt{2} - 1)$
43. $(\pi/6)(17\sqrt{17} - 5\sqrt{5})$
45. $\frac{1}{2}\sqrt{21} + \frac{17}{4}\left[\ln(2 + \sqrt{21}) - \ln\sqrt{17}\right]$ **47.** 4
49. 13.9783
51. (a) 24.2055 (b) 24.2476
53. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$
55. (b)



(c) $\int_0^{2\pi}\int_0^{\pi}\sqrt{36 \sin^4 u \cos^2 v + 9 \sin^4 u \sin^2 v + 4 \cos^2 u \sin^2 u}\; du\, dv$
57. 4π **59.** $2a^2(\pi - 2)$

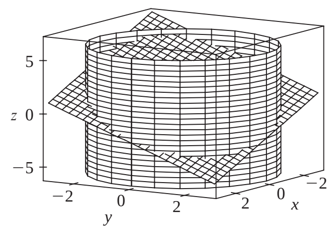
EXERCISES 16.7 ■ PAGE 1091

1. 49.09 **3.** 900π **5.** $171\sqrt{14}$ **7.** $\sqrt{3}/24$
9. $5\sqrt{5}/48 + 1/240$ **11.** $364\sqrt{2}\pi/3$
13. $(\pi/60)(391\sqrt{17} + 1)$ **15.** 16π **17.** 12

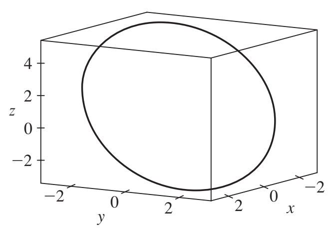
19. $\frac{713}{180}$ **21.** $-\frac{1}{6}$ **23.** $-\frac{4}{3}\pi$ **25.** 0 **27.** 48
29. $2\pi + \frac{8}{3}$ **31.** 0.1642 **33.** 3.4895
35. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D [P(\partial h/\partial x) - Q + R(\partial h/\partial z)]\, dA$, where D = projection of S on xz-plane
37. $(0, 0, a/2)$
39. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\, dS$ (b) $4329\sqrt{2}\,\pi/5$
41. 0 kg/s **43.** $\frac{8}{3}\pi a^3 \varepsilon_0$ **45.** 1248π

EXERCISES 16.8 ■ PAGE 1097

3. 0 **5.** 0 **7.** -1 **9.** 80π
11. (a) $81\pi/2$ (b)



(c) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3(\cos t + \sin t)$, $0 \leqslant t \leqslant 2\pi$



17. 3

EXERCISES 16.9 ■ PAGE 1103

5. 2 **7.** $9\pi/2$
9. 0 **11.** $32\pi/3$ **13.** 0
15. $341\sqrt{2}/60 + \frac{81}{20}\arcsin\left(\sqrt{3}/3\right)$ **17.** $13\pi/20$
19. Negative at P_1, positive at P_2
21. div $\mathbf{F} > 0$ in quadrants I, II; div $\mathbf{F} < 0$ in quadrants III, IV

CHAPTER 16 REVIEW ■ PAGE 1106

True-False Quiz

1. False **3.** True **5.** False **7.** True

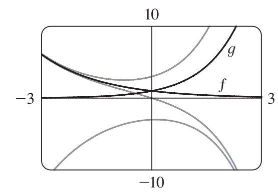
Exercises

1. (a) Negative (b) Positive **3.** $6\sqrt{10}$ **5.** $\frac{4}{15}$
7. $\frac{110}{3}$ **9.** $\frac{11}{12} - 4/e$ **11.** $f(x, y) = e^y + xe^{xy}$ **13.** 0
17. -8π **25.** $\frac{1}{6}\left(27 - 5\sqrt{5}\right)$
27. $(\pi/60)\left(391\sqrt{17} + 1\right)$ **29.** $-64\pi/3$
33. $-\frac{1}{2}$ **37.** -4 **39.** 21

CHAPTER 17

EXERCISES 17.1 ■ PAGE 1117

1. $y = c_1 e^{3x} + c_2 e^{-2x}$ **3.** $y = c_1 \cos 4x + c_2 \sin 4x$
5. $y = c_1 e^{2x/3} + c_2 x e^{2x/3}$ **7.** $y = c_1 + c_2 e^{x/2}$
9. $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$
11. $y = c_1 e^{(\sqrt{3}-1)t/2} + c_2 e^{-(\sqrt{3}+1)t/2}$
13. $P = e^{-t}\left[c_1 \cos\left(\frac{1}{10}t\right) + c_2 \sin\left(\frac{1}{10}t\right)\right]$
15.

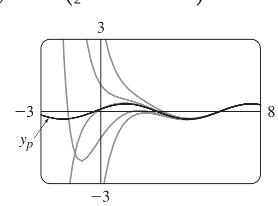


All solutions approach either 0 or $\pm\infty$ as $x \to \pm\infty$.

17. $y = 2e^{-3x/2} + e^{-x}$ **19.** $y = e^{x/2} - 2xe^{x/2}$
21. $y = 3\cos 4x - \sin 4x$ **23.** $y = e^{-x}(2\cos x + 3\sin x)$
25. $y = 3\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right)$ **27.** $y = \dfrac{e^{x+3}}{e^3 - 1} + \dfrac{e^{2x}}{1 - e^3}$
29. No solution
31. $y = e^{-2x}(2\cos 3x - e^{\pi}\sin 3x)$
33. (b) $\lambda = n^2\pi^2/L^2$, n a positive integer; $y = C\sin(n\pi x/L)$

EXERCISES 17.2 ■ PAGE 1124

1. $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$
3. $y = c_1 + c_2 e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$
5. $y = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}$
7. $y = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$
9. $y = e^x\left(\frac{1}{2}x^2 - x + 2\right)$
11.



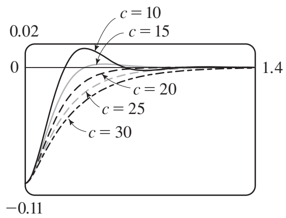
The solutions are all asymptotic to $y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x$ as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.

13. $y_p = Ae^{2x} + (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$
15. $y_p = Ax + (Bx + C)e^{9x}$
17. $y_p = xe^{-x}[(Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x]$
19. $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right) - \frac{1}{3}\cos x$
21. $y = c_1 e^x + c_2 x e^x + e^{2x}$
23. $y = c_1 \sin x + c_2 \cos x + \sin x \ln(\sec x + \tan x) - 1$
25. $y = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$
27. $y = e^x\left[c_1 + c_2 x - \frac{1}{2}\ln(1 + x^2) + x\tan^{-1}x\right]$

EXERCISES 17.3 ■ PAGE 1132

1. $x = 0.35\cos\left(2\sqrt{5}\,t\right)$ **3.** $x = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$ **5.** $\frac{49}{12}$ kg

7.



13. $Q(t) = (-e^{-10t}/250)(6\cos 20t + 3\sin 20t) + \frac{3}{125}$,
$I(t) = \frac{3}{5}e^{-10t}\sin 20t$

15. $Q(t) = e^{-10t}\left[\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right]$
$-\frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$

EXERCISES 17.4 ■ PAGE 1137

1. $c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$ **3.** $c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 e^{x^3/3}$

5. $c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$

7. $c_0 + c_1 \sum_{n=1}^{\infty} \frac{x^n}{n} = c_0 - c_1 \ln(1-x)$ for $|x| < 1$

9. $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$

11. $x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!} x^{3n+1}$

CHAPTER 17 REVIEW ■ PAGE 1138

True-False Quiz

1. True **3.** True

Exercises

1. $y = c_1 e^{5x} + c_2 e^{-3x}$ **3.** $y = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$
5. $y = e^{2x}(c_1 \cos x + c_2 \sin x + 1)$
7. $y = c_1 e^x + c_2 x e^x - \frac{1}{2}\cos x - \frac{1}{2}(x+1)\sin x$
9. $y = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$
11. $y = 5 - 2e^{-6(x-1)}$ **13.** $y = (e^{4x} - e^x)/3$
15. $\sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$
17. $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$
19. (c) $2\pi/k \approx 85$ min (d) $\approx 17{,}600$ mi/h

APPENDIXES

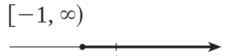
EXERCISES A ■ PAGE A9

1. 18 **3.** π **5.** $5 - \sqrt{5}$ **7.** $2 - x$

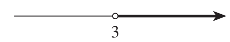
9. $|x+1| = \begin{cases} x+1 & \text{for } x \geqslant -1 \\ -x-1 & \text{for } x < -1 \end{cases}$ **11.** $x^2 + 1$

13. $(-2, \infty)$

15. $[-1, \infty)$

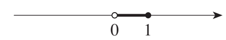


17. $(3, \infty)$

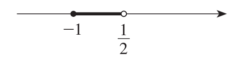


19. $(2, 6)$

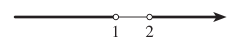
21. $(0, 1]$



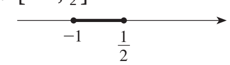
23. $\left[-1, \frac{1}{2}\right)$



25. $(-\infty, 1) \cup (2, \infty)$

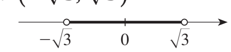


27. $\left[-1, \frac{1}{2}\right]$

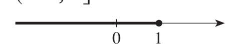


29. $(-\infty, \infty)$

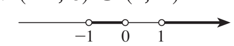
31. $(-\sqrt{3}, \sqrt{3})$



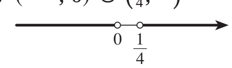
33. $(-\infty, 1]$



35. $(-1, 0) \cup (1, \infty)$



37. $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$

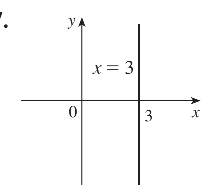


39. $10 \leqslant C \leqslant 35$ **41.** (a) $T = 20 - 10h$, $0 \leqslant h \leqslant 12$
(b) $-30°\text{C} \leqslant T \leqslant 20°\text{C}$ **43.** $\pm\frac{3}{2}$ **45.** $2, -\frac{4}{3}$
47. $(-3, 3)$ **49.** $(3, 5)$ **51.** $(-\infty, -7] \cup [-3, \infty)$
53. $[1.3, 1.7]$ **55.** $[-4, -1] \cup [1, 4]$
57. $x \geqslant (a+b)c/(ab)$ **59.** $x > (c-b)/a$

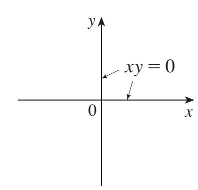
EXERCISES B ■ PAGE A15

1. 5 **3.** $\sqrt{74}$ **5.** $2\sqrt{37}$ **7.** 2 **9.** $-\frac{9}{2}$

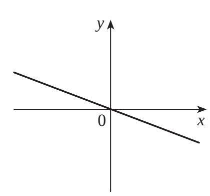
17.

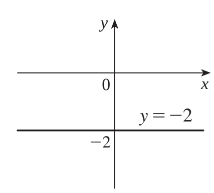


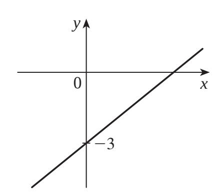
19.



21. $y = 6x - 15$ **23.** $2x - 3y + 19 = 0$
25. $5x + y = 11$ **27.** $y = 3x - 2$ **29.** $y = 3x - 3$
31. $y = 5$ **33.** $x + 2y + 11 = 0$ **35.** $5x - 2y + 1 = 0$
37. $m = -\frac{1}{3}$, $b = 0$ **39.** $m = 0$, $b = -2$ **41.** $m = \frac{3}{4}$, $b = -3$







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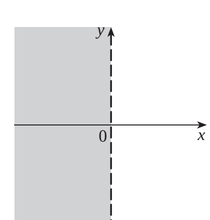

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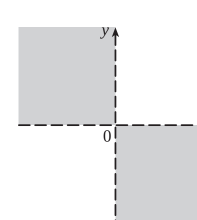

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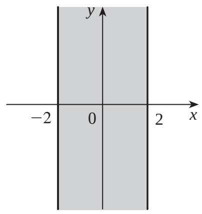

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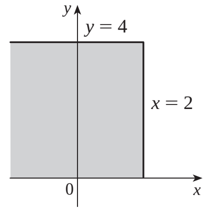

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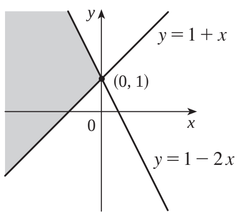
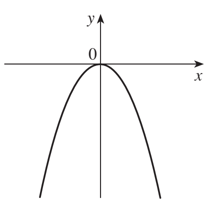


53. $(0, -4)$ **55.** (a) $(4, 9)$ (b) $(3.5, -3)$ **57.** $(1, -2)$
59. $y = x - 3$ **61.** (b) $4x - 3y - 24 = 0$

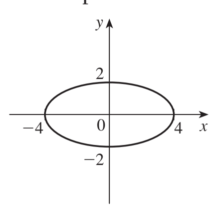
EXERCISES C ▪ PAGE A23

1. $(x - 3)^2 + (y + 1)^2 = 25$ **3.** $x^2 + y^2 = 65$
5. $(2, -5)$, 4 **7.** $\left(-\frac{1}{2}, 0\right), \frac{1}{2}$ **9.** $\left(\frac{1}{4}, -\frac{1}{4}\right), \sqrt{10}/4$

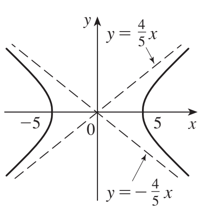
11. Parabola



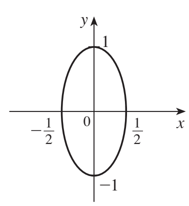
13. Ellipse



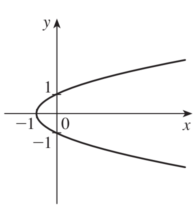
15. Hyperbola



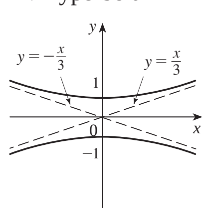
17. Ellipse



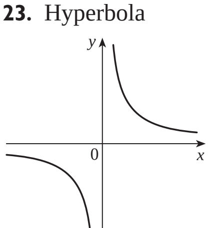
19. Parabola



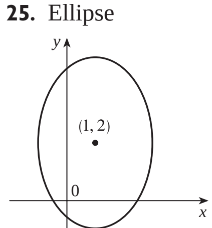
21. Hyperbola



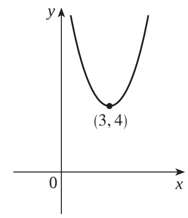
23. Hyperbola



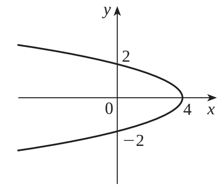
25. Ellipse



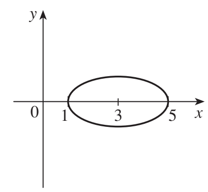
27. Parabola



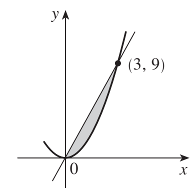
29. Parabola



31. Ellipse

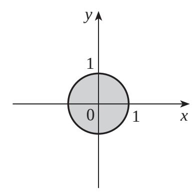


33.

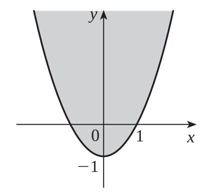


35. $y = x^2 - 2x$

37.



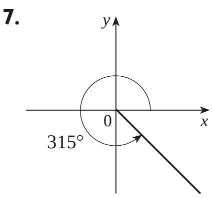
39.



EXERCISES D ▪ PAGE A32

1. $7\pi/6$ **3.** $\pi/20$ **5.** 5π **7.** $720°$ **9.** $75°$
11. $-67.5°$ **13.** 3π cm **15.** $\frac{2}{3}$ rad $= (120/\pi)°$

17.



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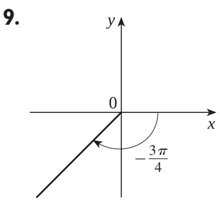

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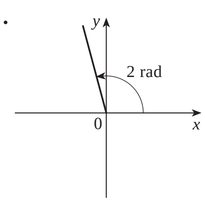


23. $\sin(3\pi/4) = 1/\sqrt{2}$, $\cos(3\pi/4) = -1/\sqrt{2}$, $\tan(3\pi/4) = -1$, $\csc(3\pi/4) = \sqrt{2}$, $\sec(3\pi/4) = -\sqrt{2}$, $\cot(3\pi/4) = -1$

25. $\sin(9\pi/2) = 1$, $\cos(9\pi/2) = 0$, $\csc(9\pi/2) = 1$, $\cot(9\pi/2) = 0$, $\tan(9\pi/2)$ and $\sec(9\pi/2)$ undefined

27. $\sin(5\pi/6) = \frac{1}{2}$, $\cos(5\pi/6) = -\sqrt{3}/2$, $\tan(5\pi/6) = -1/\sqrt{3}$, $\csc(5\pi/6) = 2$, $\sec(5\pi/6) = -2/\sqrt{3}$, $\cot(5\pi/6) = -\sqrt{3}$

29. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

31. $\sin\phi = \sqrt{5}/3$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\sqrt{5}/2$, $\csc\phi = 3/\sqrt{5}$, $\cot\phi = -2/\sqrt{5}$

33. $\sin \beta = -1/\sqrt{10}$, $\cos \beta = -3/\sqrt{10}$, $\tan \beta = \frac{1}{3}$, $\csc \beta = -\sqrt{10}$, $\sec \beta = -\sqrt{10}/3$

35. 5.73576 cm **37.** 24.62147 cm **59.** $\frac{1}{15}(4 + 6\sqrt{2})$

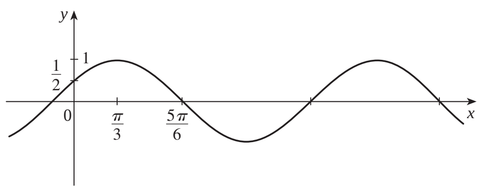
61. $\frac{1}{15}(3 + 8\sqrt{2})$ **63.** $\frac{24}{25}$ **65.** $\pi/3, 5\pi/3$

67. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ **69.** $\pi/6, \pi/2, 5\pi/6, 3\pi/2$

71. $0, \pi, 2\pi$ **73.** $0 \leqslant x \leqslant \pi/6$ and $5\pi/6 \leqslant x \leqslant 2\pi$

75. $0 \leqslant x < \pi/4$, $3\pi/4 < x < 5\pi/4$, $7\pi/4 < x \leqslant 2\pi$

77.



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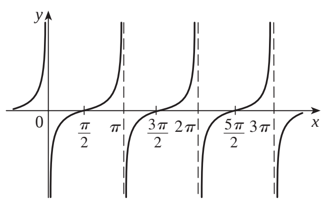
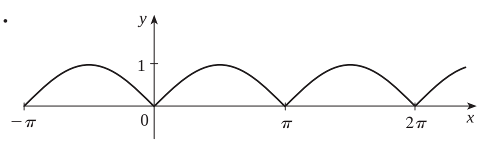


81.



89. 14.34457 cm^2

EXERCISES E ■ PAGE A38

1. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$ **3.** $3^4 + 3^5 + 3^6$

5. $-1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$ **7.** $1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

9. $1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$ **11.** $\sum_{i=1}^{10} i$

13. $\sum_{i=1}^{19} \frac{i}{i+1}$ **15.** $\sum_{i=1}^{n} 2i$ **17.** $\sum_{i=0}^{5} 2^i$ **19.** $\sum_{i=1}^{n} x^i$

21. 80 **23.** 3276 **25.** 0 **27.** 61 **29.** $n(n + 1)$

31. $n(n^2 + 6n + 17)/3$ **33.** $n(n^2 + 6n + 11)/3$

35. $n(n^3 + 2n^2 - n - 10)/4$

41. (a) n^4 (b) $5^{100} - 1$ (c) $\frac{97}{300}$ (d) $a_n - a_0$

43. $\frac{1}{3}$ **45.** 14 **49.** $2^{n+1} + n^2 + n - 2$

EXERCISES G ■ PAGE A56

1. (b) 0.405

EXERCISES H ■ PAGE A64

1. $8 - 4i$ **3.** $13 + 18i$ **5.** $12 - 7i$ **7.** $\frac{11}{13} + \frac{10}{13}i$

9. $\frac{1}{2} - \frac{1}{2}i$ **11.** $-i$ **13.** $5i$ **15.** $12 + 5i$, 13

17. $4i$, 4 **19.** $\pm\frac{3}{2}i$ **21.** $-1 \pm 2i$

23. $-\frac{1}{2} \pm (\sqrt{7}/2)i$ **25.** $3\sqrt{2}\,[\cos(3\pi/4) + i\sin(3\pi/4)]$

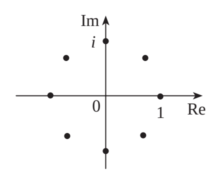
27. $5\{\cos[\tan^{-1}(\frac{4}{3})] + i\sin[\tan^{-1}(\frac{4}{3})]\}$

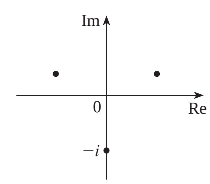
29. $4[\cos(\pi/2) + i\sin(\pi/2)]$, $\cos(-\pi/6) + i\sin(-\pi/6)$, $\frac{1}{2}[\cos(-\pi/6) + i\sin(-\pi/6)]$

31. $4\sqrt{2}\,[\cos(7\pi/12) + i\sin(7\pi/12)]$, $(2\sqrt{2})[\cos(13\pi/12) + i\sin(13\pi/12)]$, $\frac{1}{4}[\cos(\pi/6) + i\sin(\pi/6)]$

33. -1024 **35.** $-512\sqrt{3} + 512i$

37. ± 1, $\pm i$, $(1/\sqrt{2})(\pm 1 \pm i)$ **39.** $\pm(\sqrt{3}/2) + \frac{1}{2}i$, $-i$





41. i **43.** $\frac{1}{2} + (\sqrt{3}/2)i$ **45.** $-e^2$

47. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$