Systems of Equations and Inequalities

7

Systems of equations can be used to determine the combinations of scoring plays for different sports, such as football.

© Robert Galbraith/Reuters/Corbis

SELECTED APPLICATIONS

Systems of equations and inequalities have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Break-Even Analysis, Exercises 61–64, page 504
- Data Analysis: Renewable Energy, Exercise 71, page 505
- Acid Mixture, Exercise 51, page 516
- Sports, Exercise 51, page 529
- Electrical Network, Exercise 65, page 530
- Thermodynamics, Exercise 57, page 540
- Data Analysis: Prescription Drugs, Exercise 77, page 550
- Investment Portfolio, Exercises 47 and 48, page 561
- Supply and Demand, Exercises 75 and 76, page 565

7.1 Linear and Nonlinear Systems of Equations

What you should learn

- Use the method of substitution to solve systems of linear equations in two variables.
- Use the method of substitution to solve systems of nonlinear equations in two variables.
- Use a graphical approach to solve systems of equations in two variables.
- Use systems of equations to model and solve real-life problems.

Why you should learn it

Graphs of systems of equations help you solve real-life problems. For instance, in Exercise 71 on page 505, you can use the graph of a system of equations to approximate when the consumption of wind energy exceeded the consumption of solar energy.

© ML Sinibaldi/Corbis

The Method of Substitution

Up to this point in the text, most problems have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such problems, you need to find solutions of a **system of equations.** Here is an example of a system of two equations in two unknowns.

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. Finding the set of all solutions is called **solving the system of equations.** For instance, the ordered pair $(2, 1)$ is a solution of this system. To check this, you can substitute 2 for x and 1 for y in *each* equation.

Check (2, 1) in Equation 1 and Equation 2:

$$2x + y = 5 \quad \text{Write Equation 1.}$$

$$2(2) + 1 \stackrel{?}{=} 5 \quad \text{Substitute 2 for } x \text{ and 1 for } y.$$

$$4 + 1 = 5 \quad \text{Solution checks in Equation 1. } \checkmark$$

$$3x - 2y = 4 \quad \text{Write Equation 2.}$$

$$3(2) - 2(1) \stackrel{?}{=} 4 \quad \text{Substitute 2 for } x \text{ and 1 for } y.$$

$$6 - 2 = 4 \quad \text{Solution checks in Equation 2. } \checkmark$$

In this chapter, you will study four ways to solve systems of equations, beginning with the **method of substitution.**

Method	*Section*	*Type of System*
1. Substitution	7.1	Linear or nonlinear, two variables
2. Graphical method	7.1	Linear or nonlinear, two variables
3. Elimination	7.2	Linear, two variables
4. Gaussian elimination	7.3	Linear, three or more variables

Method of Substitution

1. *Solve* one of the equations for one variable in terms of the other.
2. *Substitute* the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. *Solve* the equation obtained in Step 2.
4. *Back-substitute* the value obtained in Step 3 into the expression obtained in Step 1 to find the value of the other variable.
5. *Check* that the solution satisfies *each* of the original equations.

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

Exploration

Use a graphing utility to graph $y_1 = 4 - x$ and $y_2 = x - 2$ in the same viewing window. Use the *zoom* and *trace* features to find the coordinates of the point of intersection. What is the relationship between the point of intersection and the solution found in Example 1?

Example 1 Solving a System of Equations by Substitution

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

Solution

Begin by solving for y in Equation 1.

$$y = 4 - x \qquad \text{Solve for } y \text{ in Equation 1.}$$

Next, substitute this expression for y into Equation 2 and solve the resulting single-variable equation for x.

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Distributive Property} \\ 2x &= 6 && \text{Combine like terms.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for y by *back-substituting* $x = 3$ into the equation $y = 4 - x$, to obtain

$$\begin{aligned} y &= 4 - x && \text{Write revised Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1. && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair (3, 1). You can check this solution as follows.

Check

Substitute (3, 1) into Equation 1:

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\stackrel{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1. } \checkmark \end{aligned}$$

Substitute (3, 1) into Equation 2:

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\stackrel{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2. } \checkmark \end{aligned}$$

Because (3, 1) satisfies both equations in the system, it is a solution of the system of equations.

CHECKPOINT Now try Exercise 5.

STUDY TIP

Because many steps are required to solve a system of equations, it is very easy to make errors in arithmetic. So, you should always check your solution by substituting it into *each* equation in the original system.

The term *back-substitution* implies that you work *backwards*. First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

Example 2 Solving a System by Substitution

A total of \$12,000 is invested in two funds paying 5% and 3% simple interest. (Recall that the formula for simple interest is $I = Prt$, where P is the principal, r is the annual interest rate, and t is the time.) The yearly interest is \$500. How much is invested at each rate?

Solution

Verbal Model: 5% fund + 3% fund = Total investment

5% interest + 3% interest = Total interest

Labels:
Amount in 5% fund $= x$ (dollars)
Interest for 5% fund $= 0.05x$ (dollars)
Amount in 3% fund $= y$ (dollars)
Interest for 3% fund $= 0.03y$ (dollars)
Total investment $= 12{,}000$ (dollars)
Total interest $= 500$ (dollars)

System:
$$\begin{cases} x + y = 12{,}000 & \text{Equation 1} \\ 0.05x + 0.03y = 500 & \text{Equation 2} \end{cases}$$

To begin, it is convenient to multiply each side of Equation 2 by 100. This eliminates the need to work with decimals.

$$100(0.05x + 0.03y) = 100(500) \qquad \text{Multiply each side by 100.}$$
$$5x + 3y = 50{,}000 \qquad \text{Revised Equation 2}$$

To solve this system, you can solve for x in Equation 1.

$$x = 12{,}000 - y \qquad \text{Revised Equation 1}$$

Then, substitute this expression for x into revised Equation 2 and solve the resulting equation for y.

$$5x + 3y = 50{,}000 \qquad \text{Write revised Equation 2.}$$
$$5(12{,}000 - y) + 3y = 50{,}000 \qquad \text{Substitute } 12{,}000 - y \text{ for } x.$$
$$60{,}000 - 5y + 3y = 50{,}000 \qquad \text{Distributive Property}$$
$$-2y = -10{,}000 \qquad \text{Combine like terms.}$$
$$y = 5000 \qquad \text{Divide each side by } -2.$$

Next, back-substitute the value $y = 5000$ to solve for x.

$$x = 12{,}000 - y \qquad \text{Write revised Equation 1.}$$
$$x = 12{,}000 - 5000 \qquad \text{Substitute 5000 for } y.$$
$$x = 7000 \qquad \text{Simplify.}$$

The solution is (7000, 5000). So, \$7000 is invested at 5% and \$5000 is invested at 3%. Check this in the original system.

CHECKPOINT Now try Exercise 19.

STUDY TIP

When using the method of substitution, it does not matter which variable you choose to solve for first. Whether you solve for y first or x first, you will obtain the same solution. When making your choice, you should choose the variable and equation that are easier to work with. For instance, in Example 2, solving for x in Equation 1 is easier than solving for x in Equation 2.

Technology

One way to check the answers you obtain in this section is to use a graphing utility. For instance, enter the two equations in Example 2

$$y_1 = 12{,}000 - x$$
$$y_2 = \frac{500 - 0.05x}{0.03}$$

and find an appropriate viewing window that shows where the two lines intersect. Then use the *intersect* feature or the *zoom* and *trace* features to find the point of intersection. Does this point agree with the solution obtained at the right?

Nonlinear Systems of Equations

The equations in Examples 1 and 2 are linear. The method of substitution can also be used to solve systems in which one or both of the equations are nonlinear.

Example 3 Substitution: Two-Solution Case

Solve the system of equations.

$$\begin{cases} x^2 + 4x - y = 7 & \text{Equation 1} \\ 2x - y = -1 & \text{Equation 2} \end{cases}$$

Solution

Begin by solving for y in Equation 2 to obtain $y = 2x + 1$. Next, substitute this expression for y into Equation 1 and solve for x.

$$x^2 + 4x - (2x + 1) = 7 \quad \text{Substitute } 2x + 1 \text{ for } y \text{ into Equation 1.}$$

$$x^2 + 2x - 1 = 7 \quad \text{Simplify.}$$

$$x^2 + 2x - 8 = 0 \quad \text{Write in general form.}$$

$$(x + 4)(x - 2) = 0 \quad \text{Factor.}$$

$$x = -4, 2 \quad \text{Solve for } x.$$

Back-substituting these values of x to solve for the corresponding values of y produces the solutions $(-4, -7)$ and $(2, 5)$. Check these in the original system.

CHECKPOINT Now try Exercise 25.

Exploration

Use a graphing utility to graph the two equations in Example 3

$$y_1 = x^2 + 4x - 7$$

$$y_2 = 2x + 1$$

in the same viewing window. How many solutions do you think this system has? Repeat this experiment for the equations in Example 4. How many solutions does this system have? Explain your reasoning.

When using the method of substitution, you may encounter an equation that has no solution, as shown in Example 4.

Example 4 Substitution: No-Real-Solution Case

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \text{Equation 1} \\ x^2 + y = 3 & \text{Equation 2} \end{cases}$$

Solution

Begin by solving for y in Equation 1 to obtain $y = x + 4$. Next, substitute this expression for y into Equation 2 and solve for x.

$$x^2 + (x + 4) = 3 \quad \text{Substitute } x + 4 \text{ for } y \text{ into Equation 2.}$$

$$x^2 + x + 1 = 0 \quad \text{Simplify.}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2} \quad \text{Use the Quadratic Formula.}$$

Because the discriminant is negative, the equation $x^2 + x + 1 = 0$ has no (real) solution. So, the original system has no (real) solution.

CHECKPOINT Now try Exercise 27.

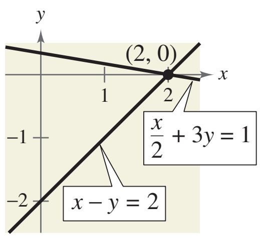
Technology

Most graphing utilities have built-in features that approximate the point(s) of intersection of two graphs. Typically, you must enter the equations of the graphs and visually locate a point of intersection before using the *intersect* feature.

Use this feature to find the points of intersection of the graphs in Figures 7.1 to 7.3. Be sure to adjust your viewing window so that you see all the points of intersection.

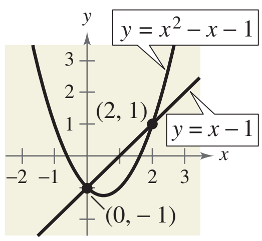
Graphical Approach to Finding Solutions

From Examples 2, 3, and 4, you can see that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. By using a **graphical method,** you can gain insight about the number of solutions and the location(s) of the solution(s) of a system of equations by graphing each of the equations in the same coordinate plane. The solutions of the system correspond to the **points of intersection** of the graphs. For instance, the two equations in Figure 7.1 graph as two lines with a *single point* of intersection; the two equations in Figure 7.2 graph as a parabola and a line with *two points* of intersection; and the two equations in Figure 7.3 graph as a line and a parabola that have *no points* of intersection.



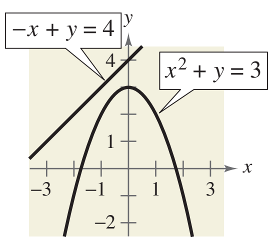
One intersection point

FIGURE 7.1



Two intersection points

FIGURE 7.2



No intersection points

FIGURE 7.3

Example 5 Solving a System of Equations Graphically

Solve the system of equations.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

Solution

Sketch the graphs of the two equations. From the graphs of these equations, it is clear that there is only one point of intersection and that $(1, 0)$ is the solution point (see Figure 7.4). You can confirm this by substituting 1 for x and 0 for y in *both* equations.

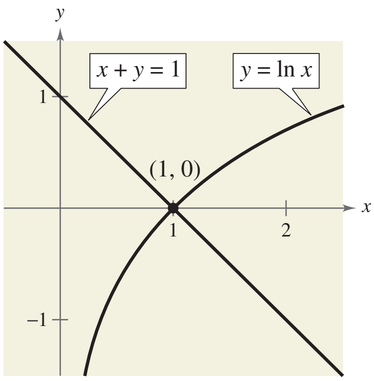


FIGURE 7.4

Check (1, 0) in Equation 1:

$y = \ln x$ Write Equation 1.

$0 = \ln 1$ Equation 1 checks. ✓

Check (1, 0) in Equation 2:

$x + y = 1$ Write Equation 2.

$1 + 0 = 1$ Equation 2 checks. ✓

CHECKPOINT Now try Exercise 33.

Example 5 shows the value of a graphical approach to solving systems of equations in two variables. Notice what would happen if you tried only the substitution method in Example 5. You would obtain the equation $x + \ln x = 1$. It would be difficult to solve this equation for x using standard algebraic techniques.

Applications

The total cost C of producing x units of a product typically has two components—the initial cost and the cost per unit. When enough units have been sold so that the total revenue R equals the total cost C, the sales are said to have reached the **break-even point.** You will find that the break-even point corresponds to the point of intersection of the cost and revenue curves.

Example 6 Break-Even Analysis

A shoe company invests \$300,000 in equipment to produce a new line of athletic footwear. Each pair of shoes costs \$5 to produce and is sold for \$60. How many pairs of shoes must be sold before the business breaks even?

Solution

The total cost of producing x units is

Total cost = Cost per unit · Number of units + Initial cost

$$C = 5x + 300{,}000. \qquad \text{Equation 1}$$

The revenue obtained by selling x units is

Total revenue = Price per unit · Number of units

$$R = 60x. \qquad \text{Equation 2}$$

Because the break-even point occurs when $R = C$, you have $C = 60x$, and the system of equations to solve is

$$\begin{cases} C = 5x + 300{,}000 \\ C = 60x \end{cases}.$$

Now you can solve by substitution.

$$60x = 5x + 300{,}000 \qquad \text{Substitute } 60x \text{ for } C \text{ in Equation 1.}$$

$$55x = 300{,}000 \qquad \text{Subtract } 5x \text{ from each side.}$$

$$x \approx 5455 \qquad \text{Divide each side by 55.}$$

So, the company must sell about 5455 pairs of shoes to break even. Note in Figure 7.5 that revenue less than the break-even point corresponds to an overall loss, whereas revenue greater than the break-even point corresponds to a profit.

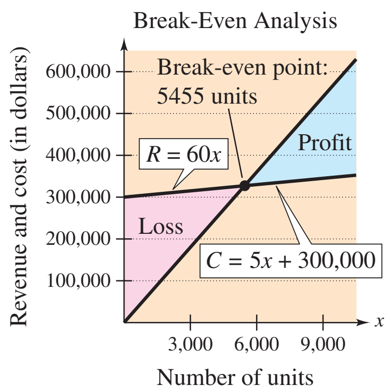


FIGURE 7.5

CHECKPOINT Now try Exercise 63.

Another way to view the solution in Example 6 is to consider the profit function

$$P = R - C.$$

The break-even point occurs when the profit is 0, which is the same as saying that $R = C$.

Example 7 Movie Ticket Sales

The weekly ticket sales for a new comedy movie decreased each week. At the same time, the weekly ticket sales for a new drama movie increased each week. Models that approximate the weekly ticket sales S (in millions of dollars) for each movie are

$$\begin{cases} S = 60 - 8x & \text{Comedy} \\ S = 10 + 4.5x & \text{Drama} \end{cases}$$

where x represents the number of weeks each movie was in theaters, with $x = 0$ corresponding to the ticket sales during the opening weekend. After how many weeks will the ticket sales for the two movies be equal?

Algebraic Solution

Because the second equation has already been solved for S in terms of x, substitute this value into the first equation and solve for x, as follows.

$$10 + 4.5x = 60 - 8x \quad \text{Substitute for } S \text{ in Equation 1.}$$

$$4.5x + 8x = 60 - 10 \quad \text{Add } 8x \text{ and } -10 \text{ to each side.}$$

$$12.5x = 50 \quad \text{Combine like terms.}$$

$$x = 4 \quad \text{Divide each side by 12.5.}$$

So, the weekly ticket sales for the two movies will be equal after 4 weeks.

CHECKPOINT Now try Exercise 65.

Numerical Solution

You can create a table of values for each model to determine when the ticket sales for the two movies will be equal.

Number of weeks, x	0	1	2	3	4	5	6
Sales, S (comedy)	60	52	44	36	28	20	12
Sales, S (drama)	10	14.5	19	23.5	28	32.5	37

So, from the table above, you can see that the weekly ticket sales for the two movies will be equal after 4 weeks.

WRITING ABOUT MATHEMATICS

Interpreting Points of Intersection You plan to rent a 14-foot truck for a two-day local move. At truck rental agency A, you can rent a truck for $29.95 per day plus $0.49 per mile. At agency B, you can rent a truck for $50 per day plus $0.25 per mile.

a. Write a total cost equation in terms of x and y for the total cost of renting the truck from each agency.

b. Use a graphing utility to graph the two equations in the same viewing window and find the point of intersection. Interpret the meaning of the point of intersection in the context of the problem.

c. Which agency should you choose if you plan to travel a total of 100 miles during the two-day move? Why?

d. How does the situation change if you plan to drive 200 miles during the two-day move?

7.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

VOCABULARY CHECK: Fill in the blanks.

1. A set of two or more equations in two or more variables is called a ________ of ________.
2. A ________ of a system of equations is an ordered pair that satisfies each equation in the system.
3. Finding the set of all solutions to a system of equations is called ________ the system of equations.
4. The first step in solving a system of equations by the method of ________ is to solve one of the equations for one variable in terms of the other variable.
5. Graphically, the solution of a system of two equations is the ________ of ________ of the graphs of the two equations.
6. In business applications, the point at which the revenue equals costs is called the ________ point.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, determine whether each ordered pair is a solution of the system of equations.

1. $\begin{cases} 4x - y = 1 \\ 6x + y = -6 \end{cases}$ (a) $(0, -3)$ (b) $(-1, -4)$ (c) $\left(-\frac{3}{2}, -2\right)$ (d) $\left(-\frac{1}{2}, -3\right)$

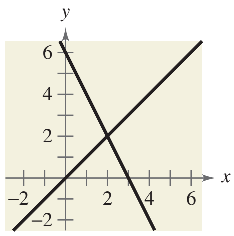
2. $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$ (a) $(2, -13)$ (b) $(2, -9)$ (c) $\left(-\frac{3}{2}, -\frac{31}{3}\right)$ (d) $\left(-\frac{7}{4}, -\frac{37}{4}\right)$

3. $\begin{cases} y = -2e^x \\ 3x - y = 2 \end{cases}$ (a) $(-2, 0)$ (b) $(0, -2)$ (c) $(0, -3)$ (d) $(-1, 2)$

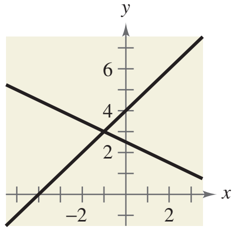
4. $\begin{cases} -\log x + 3 = y \\ \frac{1}{9}x + y = \frac{28}{9} \end{cases}$ (a) $\left(9, \frac{37}{9}\right)$ (b) $(10, 2)$ (c) $(1, 3)$ (d) $(2, 4)$

In Exercises 5–14, solve the system by the method of substitution. Check your solution graphically.

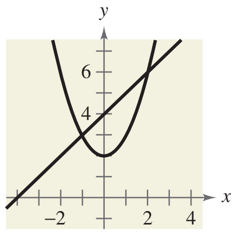
5. $\begin{cases} 2x + y = 6 \\ -x + y = 0 \end{cases}$



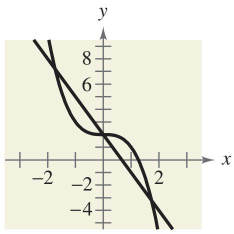
6. $\begin{cases} x - y = -4 \\ x + 2y = 5 \end{cases}$



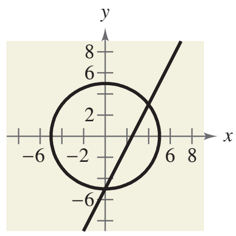
7. $\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$



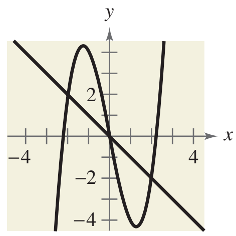
8. $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$



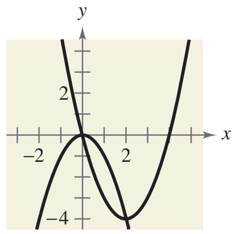
9. $\begin{cases} -2x + y = -5 \\ x^2 + y^2 = 25 \end{cases}$



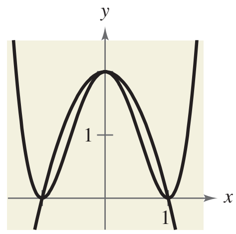
10. $\begin{cases} x + y = 0 \\ x^3 - 5x - y = 0 \end{cases}$



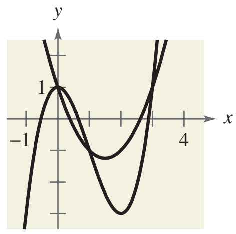
11. $\begin{cases} x^2 + y = 0 \\ x^2 - 4x - y = 0 \end{cases}$



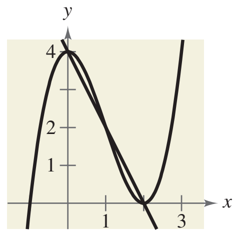
12. $\begin{cases} y = -2x^2 + 2 \\ y = 2(x^4 - 2x^2 + 1) \end{cases}$



13. $\begin{cases} y = x^3 - 3x^2 + 1 \\ y = x^2 - 3x + 1 \end{cases}$



14. $\begin{cases} y = x^3 - 3x^2 + 4 \\ y = -2x + 4 \end{cases}$



In Exercises 15–28, solve the system by the method of substitution.

15. $\begin{cases} x - y = 0 \\ 5x - 3y = 10 \end{cases}$

16. $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$

17. $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$

18. $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$

19. $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$

20. $\begin{cases} 0.5x + 3.2y = 9.0 \\ 0.2x - 1.6y = -3.6 \end{cases}$

21. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$

22. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{4}x - y = 4 \end{cases}$

23. $\begin{cases} 6x + 5y = -3 \\ -x - \frac{5}{6}y = -7 \end{cases}$

24. $\begin{cases} -\frac{2}{3}x + y = 2 \\ 2x - 3y = 6 \end{cases}$

25. $\begin{cases} x^2 - y = 0 \\ 2x + y = 0 \end{cases}$

26. $\begin{cases} x - 2y = 0 \\ 3x - y^2 = 0 \end{cases}$

27. $\begin{cases} x - y = -1 \\ x^2 - y = -4 \end{cases}$

28. $\begin{cases} y = -x \\ y = x^3 + 3x^2 + 2x \end{cases}$

In Exercises 29–42, solve the system graphically.

29. $\begin{cases} -x + 2y = 2 \\ 3x + y = 15 \end{cases}$

30. $\begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases}$

31. $\begin{cases} x - 3y = -2 \\ 5x + 3y = 17 \end{cases}$

32. $\begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases}$

33. $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

34. $\begin{cases} -x + y = 3 \\ x^2 - 6x - 27 + y^2 = 0 \end{cases}$

35. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$

36. $\begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases}$

37. $\begin{cases} 7x + 8y = 24 \\ x - 8y = 8 \end{cases}$

38. $\begin{cases} x - y = 0 \\ 5x - 2y = 6 \end{cases}$

39. $\begin{cases} 3x - 2y = 0 \\ x^2 - y^2 = 4 \end{cases}$

40. $\begin{cases} 2x - y + 3 = 0 \\ x^2 + y^2 - 4x = 0 \end{cases}$

41. $\begin{cases} x^2 + y^2 = 25 \\ 3x^2 - 16y = 0 \end{cases}$

42. $\begin{cases} x^2 + y^2 = 25 \\ (x - 8)^2 + y^2 = 41 \end{cases}$

In Exercises 43–48, use a graphing utility to solve the system of equations. Find the solution accurate to two decimal places.

43. $\begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases}$

44. $\begin{cases} y = -4e^{-x} \\ y + 3x + 8 = 0 \end{cases}$

45. $\begin{cases} x + 2y = 8 \\ y = \log_2 x \end{cases}$

46. $\begin{cases} y = -2 + \ln(x - 1) \\ 3y + 2x = 9 \end{cases}$

47. $\begin{cases} x^2 + y^2 = 169 \\ x^2 - 8y = 104 \end{cases}$

48. $\begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y = 2 \end{cases}$

In Exercises 49–60, solve the system graphically or algebraically. Explain your choice of method.

49. $\begin{cases} y = 2x \\ y = x^2 + 1 \end{cases}$

50. $\begin{cases} x + y = 4 \\ x^2 + y = 2 \end{cases}$

51. $\begin{cases} 3x - 7y + 6 = 0 \\ x^2 - y^2 = 4 \end{cases}$

52. $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

53. $\begin{cases} x - 2y = 4 \\ x^2 - y = 0 \end{cases}$

54. $\begin{cases} y = (x + 1)^3 \\ y = \sqrt{x - 1} \end{cases}$

55. $\begin{cases} y - e^{-x} = 1 \\ y - \ln x = 3 \end{cases}$

56. $\begin{cases} x^2 + y = 4 \\ e^x - y = 0 \end{cases}$

57. $\begin{cases} y = x^4 - 2x^2 + 1 \\ y = 1 - x^2 \end{cases}$

58. $\begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases}$

59. $\begin{cases} xy - 1 = 0 \\ 2x - 4y + 7 = 0 \end{cases}$

60. $\begin{cases} x - 2y = 1 \\ y = \sqrt{x - 1} \end{cases}$

Break-Even Analysis **In Exercises 61 and 62, find the sales necessary to break even $(R = C)$ for the cost C of producing x units and the revenue R obtained by selling x units. (Round to the nearest whole unit.)**

61. $C = 8650x + 250{,}000, \quad R = 9950x$

62. $C = 5.5\sqrt{x} + 10{,}000, \quad R = 3.29x$

63. ***Break-Even Analysis*** A small software company invests \$16,000 to produce a software package that will sell for \$55.95. Each unit can be produced for \$35.45.

(a) How many units must be sold to break even?

(b) How many units must be sold to make a profit of \$60,000?

64. ***Break-Even Analysis*** A small fast-food restaurant invests \$5000 to produce a new food item that will sell for \$3.49. Each item can be produced for \$2.16.

(a) How many items must be sold to break even?

(b) How many items must be sold to make a profit of \$8500?

65. ***DVD Rentals*** The weekly rentals for a newly released DVD of an animated film at a local video store decreased each week. At the same time, the weekly rentals for a newly released DVD of a horror film increased each week. Models that approximate the weekly rentals R for each DVD are

$$\begin{cases} R = 360 - 24x & \text{Animated film} \\ R = 24 + 18x & \text{Horror film} \end{cases}$$

where x represents the number of weeks each DVD was in the store, with $x = 1$ corresponding to the first week.

(a) After how many weeks will the rentals for the two movies be equal?

(b) Use a table to solve the system of equations numerically. Compare your result with that of part (a).

66. ***CD Sales*** The total weekly sales for a newly released rock CD increased each week. At the same time, the total weekly sales for a newly released rap CD decreased each week. Models that approximate the total weekly sales S (in thousands of units) for each CD are

$$\begin{cases} S = 25x + 100 & \text{Rock CD} \\ S = -50x + 475 & \text{Rap CD} \end{cases}$$

where x represents the number of weeks each CD was in stores, with $x = 0$ corresponding to the CD sales on the day each CD was first released in stores.

(a) After how many weeks will the sales for the two CDs be equal?

(b) Use a table to solve the system of equations numerically. Compare your result with that of part (a).

67. ***Choice of Two Jobs*** You are offered two jobs selling dental supplies. One company offers a straight commission of 6% of sales. The other company offers a salary of \$350 per week plus 3% of sales. How much would you have to sell in a week in order to make the straight commission offer better?

68. ***Supply and Demand*** The supply and demand curves for a business dealing with wheat are

Supply: $p = 1.45 + 0.00014x^2$

Demand: $p = (2.388 - 0.007x)^2$

where p is the price in dollars per bushel and x is the quantity in bushels per day. Use a graphing utility to graph the supply and demand equations and find the market equilibrium. (The market equilibrium is the point of intersection of the graphs for $x > 0$.)

69. ***Investment Portfolio*** A total of \$25,000 is invested in two funds paying 6% and 8.5% simple interest. (The 6% investment has a lower risk.) The investor wants a yearly interest income of \$2000 from the two investments.

(a) Write a system of equations in which one equation represents the total amount invested and the other equation represents the \$2000 required in interest. Let x and y represent the amounts invested at 6% and 8.5%, respectively.

(b) Use a graphing utility to graph the two equations in the same viewing window. As the amount invested at 6% increases, how does the amount invested at 8.5% change? How does the amount of interest income change? Explain.

(c) What amount should be invested at 6% to meet the requirement of \$2000 per year in interest?

70. ***Log Volume*** You are offered two different rules for estimating the number of board feet in a 16-foot log. (A board foot is a unit of measure for lumber equal to a board 1 foot square and 1 inch thick.) The first rule is the *Doyle Log Rule* and is modeled by

$$V_1 = (D - 4)^2, \quad 5 \le D \le 40$$

and the other is the *Scribner Log Rule* and is modeled by

$$V_2 = 0.79D^2 - 2D - 4, \quad 5 \le D \le 40$$

where D is the diameter (in inches) of the log and V is its volume (in board feet).

(a) Use a graphing utility to graph the two log rules in the same viewing window.

(b) For what diameter do the two scales agree?

(c) You are selling large logs by the board foot. Which scale would you use? Explain your reasoning.

Model It

71. ***Data Analysis: Renewable Energy*** The table shows the consumption C (in trillions of Btus) of solar energy and wind energy in the United States from 1998 to 2003. (Source: Energy Information Administration)

Year	Solar, C	Wind, C
1998	70	31
1999	69	46
2000	66	57
2001	65	68
2002	64	105
2003	63	108

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the solar energy consumption data and a linear model for the wind energy consumption data. Let t represent the year, with $t = 8$ corresponding to 1998.

(b) Use a graphing utility to graph the data and the two models in the same viewing window.

(c) Use the graph from part (b) to approximate the point of intersection of the graphs of the models. Interpret your answer in the context of the problem.

(d) Approximate the point of intersection of the graphs of the models algebraically.

(e) Compare your results from parts (c) and (d).

(f) Use your school's library, the Internet, or some other reference source to research the advantages and disadvantages of using renewable energy.

72. ***Data Analysis: Population*** The table shows the populations P (in thousands) of Alabama and Colorado from 1999 to 2003. (Source: U.S. Census Bureau)

Year	Alabama, P	Colorado, P
1999	4430	4226
2000	4447	4302
2001	4466	4429
2002	4479	4501
2003	4501	4551

(a) Use the *regression* feature of a graphing utility to find linear models for each set of data. Graph the models in the same viewing window. Let t represent the year, with $t = 9$ corresponding to 1999.

(b) Use your graph from part (a) to approximate when the population of Colorado exceeded the population of Alabama.

(c) Verify your answer from part (b) algebraically.

Geometry **In Exercises 73–76, find the dimensions of the rectangle meeting the specified conditions.**

73. The perimeter is 30 meters and the length is 3 meters greater than the width.

74. The perimeter is 280 centimeters and the width is 20 centimeters less than the length.

75. The perimeter is 42 inches and the width is three-fourths the length.

76. The perimeter is 210 feet and the length is $1\frac{1}{2}$ times the width.

77. ***Geometry*** What are the dimensions of a rectangular tract of land if its perimeter is 40 kilometers and its area is 96 square kilometers?

78. ***Geometry*** What are the dimensions of an isosceles right triangle with a two-inch hypotenuse and an area of 1 square inch?

Synthesis

True or False? **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

79. In order to solve a system of equations by substitution, you must always solve for y in one of the two equations and then back-substitute.

80. If a system consists of a parabola and a circle, then the system can have at most two solutions.

81. ***Writing*** List and explain the steps used to solve a system of equations by the method of substitution.

82. ***Think About It*** When solving a system of equations by substitution, how do you recognize that the system has no solution?

83. ***Exploration*** Find an equation of a line whose graph intersects the graph of the parabola $y = x^2$ at (a) two points, (b) one point, and (c) no points. (There is more than one correct answer.)

84. ***Conjecture*** Consider the system of equations

$$\begin{cases} y = b^x \\ y = x^b. \end{cases}$$

(a) Use a graphing utility to graph the system for $b = 1, 2, 3$, and 4.

(b) For a fixed even value of $b > 1$, make a conjecture about the number of points of intersection of the graphs in part (a).

Skills Review

In Exercises 85–90, find the general form of the equation of the line passing through the two points.

85. $(-2, 7), (5, 5)$

86. $(3.5, 4), (10, 6)$

87. $(6, 3), (10, 3)$

88. $(4, -2), (4, 5)$

89. $\left(\frac{3}{5}, 0\right), (4, 6)$

90. $\left(-\frac{7}{3}, 8\right), \left(\frac{5}{2}, \frac{1}{2}\right)$

In Exercises 91–94, find the domain of the function and identify any horizontal or vertical asymptotes.

91. $f(x) = \dfrac{5}{x - 6}$

92. $f(x) = \dfrac{2x - 7}{3x + 2}$

93. $f(x) = \dfrac{x^2 + 2}{x^2 - 16}$

94. $f(x) = 3 - \dfrac{2}{x^2}$

7.2 Two-Variable Linear Systems

What you should learn

- Use the method of elimination to solve systems of linear equations in two variables.
- Interpret graphically the numbers of solutions of systems of linear equations in two variables.
- Use systems of linear equations in two variables to model and solve real-life problems.

Why you should learn it

You can use systems of equations in two variables to model and solve real-life problems. For instance, in Exercise 63 on page 517, you will solve a system of equations to find a linear model that represents the relationship between wheat yield and amount of fertilizer applied.

The Method of Elimination

In Section 7.1, you studied two methods for solving a system of equations: substitution and graphing. Now you will study the **method of elimination.** The key step in this method is to obtain, for one of the variables, coefficients that differ only in sign so that *adding* the equations eliminates the variable.

$$\begin{array}{rll} 3x + 5y = & 7 & \text{Equation 1} \\ -3x - 2y = & -1 & \text{Equation 2} \\ \hline 3y = & 6 & \text{Add equations.} \end{array}$$

Note that by adding the two equations, you eliminate the x-terms and obtain a single equation in y. Solving this equation for y produces $y = 2$, which you can then back-substitute into one of the original equations to solve for x.

Example 1 Solving a System of Equations by Elimination

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 8 & \text{Equation 2} \end{cases}$$

Solution

Because the coefficients of y differ only in sign, you can eliminate the y-terms by adding the two equations.

$$\begin{array}{rll} 3x + 2y = & 4 & \text{Write Equation 1.} \\ 5x - 2y = & 8 & \text{Write Equation 2.} \\ \hline 8x \quad = & 12 & \text{Add equations.} \end{array}$$

So, $x = \frac{3}{2}$. By back-substituting this value into Equation 1, you can solve for y.

$$\begin{array}{rl} 3x + 2y = 4 & \text{Write Equation 1.} \\ 3\left(\frac{3}{2}\right) + 2y = 4 & \text{Substitute } \frac{3}{2} \text{ for } x. \\ \frac{9}{2} + 2y = 4 & \text{Simplify.} \\ y = -\frac{1}{4} & \text{Solve for } y. \end{array}$$

The solution is $\left(\frac{3}{2}, -\frac{1}{4}\right)$. Check this in the original system, as follows.

Check

$$\begin{array}{rl} 3\left(\frac{3}{2}\right) + 2\left(-\frac{1}{4}\right) \stackrel{?}{=} 4 & \text{Substitute into Equation 1.} \\ \frac{9}{2} - \frac{1}{2} = 4 & \text{Equation 1 checks. } \checkmark \\ 5\left(\frac{3}{2}\right) - 2\left(-\frac{1}{4}\right) \stackrel{?}{=} 8 & \text{Substitute into Equation 2.} \\ \frac{15}{2} + \frac{1}{2} = 8 & \text{Equation 2 checks. } \checkmark \end{array}$$

CHECKPOINT Now try Exercise 11.

Exploration

Use the method of substitution to solve the system in Example 1. Which method is easier?

Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y, perform the following steps.

1. *Obtain coefficients* for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. *Add* the equations to eliminate one variable, and solve the resulting equation.
3. *Back-substitute* the value obtained in Step 2 into either of the original equations and solve for the other variable.
4. *Check* your solution in both of the original equations.

Example 2 Solving a System of Equations by Elimination

Solve the system of linear equations.

$$\begin{cases} 2x - 3y = -7 & \text{Equation 1} \\ 3x + y = -5 & \text{Equation 2} \end{cases}$$

Solution

For this system, you can obtain coefficients that differ only in sign by multiplying Equation 2 by 3.

$$\begin{array}{lcll} 2x - 3y = -7 & \Rightarrow & 2x - 3y = -7 & \text{Write Equation 1.} \\ \underline{3x + y = -5} & \Rightarrow & \underline{9x + 3y = -15} & \text{Multiply Equation 2 by 3.} \\ & & 11x = -22 & \text{Add equations.} \end{array}$$

So, you can see that $x = -2$. By back-substituting this value of x into Equation 1, you can solve for y.

$$\begin{array}{rll} 2x - 3y = -7 & & \text{Write Equation 1.} \\ 2(-2) - 3y = -7 & & \text{Substitute } -2 \text{ for } x. \\ -3y = -3 & & \text{Combine like terms.} \\ y = 1 & & \text{Solve for } y. \end{array}$$

The solution is $(-2, 1)$. Check this in the original system, as follows.

Check

$$\begin{array}{rll} 2x - 3y = -7 & & \text{Write original Equation 1.} \\ 2(-2) - 3(1) \stackrel{?}{=} -7 & & \text{Substitute into Equation 1.} \\ -4 - 3 = -7 & & \text{Equation 1 checks. } \checkmark \\ 3x + y = -5 & & \text{Write original Equation 2.} \\ 3(-2) + 1 \stackrel{?}{=} -5 & & \text{Substitute into Equation 2.} \\ -6 + 1 = -5 & & \text{Equation 2 checks. } \checkmark \end{array}$$

CHECKPOINT Now try Exercise 13.

Exploration

Rewrite each system of equations in slope-intercept form and sketch the graph of each system. What is the relationship between the slopes of the two lines and the number of points of intersection?

a. $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$

b. $\begin{cases} 4x - 3y = 1 \\ -8x + 6y = -2 \end{cases}$

c. $\begin{cases} x + 2y = 3 \\ x + 2y = -8 \end{cases}$

In Example 2, the two systems of linear equations (the original system and the system obtained by multiplying by constants)

$$\begin{cases} 2x - 3y = -7 \\ 3x + y = -5 \end{cases} \quad \text{and} \quad \begin{cases} 2x - 3y = -7 \\ 9x + 3y = -15 \end{cases}$$

are called **equivalent systems** because they have precisely the same solution set. The operations that can be performed on a system of linear equations to produce an equivalent system are (1) interchanging any two equations, (2) multiplying an equation by a nonzero constant, and (3) adding a multiple of one equation to any other equation in the system.

Example 3 Solving the System of Equations by Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

Algebraic Solution

You can obtain coefficients that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{aligned} 5x + 3y &= 9 &\Rightarrow\quad 20x + 12y &= 36 && \text{Multiply Equation 1 by 4.} \\ 2x - 4y &= 14 &\Rightarrow\quad 6x - 12y &= 42 && \text{Multiply Equation 2 by 3.} \\ & & 26x &= 78 && \text{Add equations.} \end{aligned}$$

From this equation, you can see that $x = 3$. By back-substituting this value of x into Equation 2, you can solve for y.

$$\begin{aligned} 2x - 4y &= 14 && \text{Write Equation 2.} \\ 2(3) - 4y &= 14 && \text{Substitute 3 for } x. \\ -4y &= 8 && \text{Combine like terms.} \\ y &= -2 && \text{Solve for } y. \end{aligned}$$

The solution is $(3, -2)$. Check this in the original system.

Graphical Solution

Solve each equation for y. Then use a graphing utility to graph $y_1 = -\frac{5}{3}x + 3$ and $y_2 = \frac{1}{2}x - \frac{7}{2}$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the point of intersection of the graphs. From the graph in Figure 7.6, you can see that the point of intersection is $(3, -2)$. You can determine that this is the exact solution by checking $(3, -2)$ in both equations.

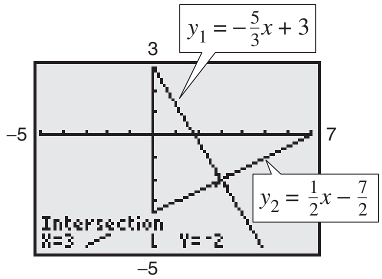


FIGURE 7.6

CHECKPOINT Now try Exercise 15.

You can check the solution from Example 3 as follows.

$$\begin{aligned} 5(3) + 3(-2) &\stackrel{?}{=} 9 && \text{Substitute 3 for } x \text{ and } -2 \text{ for } y \text{ in Equation 1.} \\ 15 - 6 &= 9 && \text{Equation 1 checks. } \checkmark \\ 2(3) - 4(-2) &\stackrel{?}{=} 14 && \text{Substitute 3 for } x \text{ and } -2 \text{ for } y \text{ in Equation 2.} \\ 6 + 8 &= 14 && \text{Equation 2 checks. } \checkmark \end{aligned}$$

Keep in mind that the terminology and methods discussed in this section apply only to systems of *linear* equations.

Graphical Interpretation of Solutions

It is possible for a *general* system of equations to have exactly one solution, two or more solutions, or no solution. If a system of *linear* equations has two different solutions, it must have an *infinite* number of solutions.

Graphical Interpretations of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following.

Number of Solutions	*Graphical Interpretation*	*Slopes of Lines*
1. Exactly one solution	The two lines intersect at one point.	The slopes of the two lines are not equal.
2. Infinitely many solutions	The two lines coincide (are identical).	The slopes of the two lines are equal.
3. No solution	The two lines are parallel.	The slopes of the two lines are equal.

A system of linear equations is **consistent** if it has at least one solution. A consistent system with exactly one solution is *independent*, whereas a consistent system with infinitely many solutions is *dependent*. A system is **inconsistent** if it has no solution.

Example 4 Recognizing Graphs of Linear Systems

Match each system of linear equations with its graph in Figure 7.7. Describe the number of solutions and state whether the system is consistent or inconsistent.

a. $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = 6 \end{cases}$ **b.** $\begin{cases} 2x - 3y = 3 \\ x + 2y = 5 \end{cases}$ **c.** $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = -6 \end{cases}$

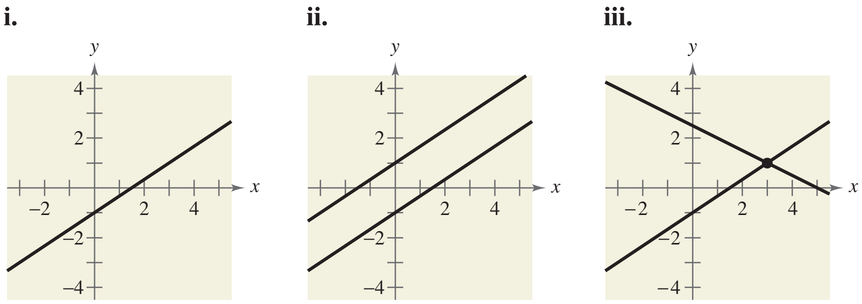


FIGURE 7.7

Solution

a. The graph of system (a) is a pair of parallel lines (ii). The lines have no point of intersection, so the system has no solution. The system is inconsistent.

b. The graph of system (b) is a pair of intersecting lines (iii). The lines have one point of intersection, so the system has exactly one solution. The system is consistent.

c. The graph of system (c) is a pair of lines that coincide (i). The lines have infinitely many points of intersection, so the system has infinitely many solutions. The system is consistent.

CHECKPOINT Now try Exercises 31–34.

STUDY TIP

A comparison of the slopes of two lines gives useful information about the number of solutions of the corresponding system of equations. To solve a system of equations graphically, it helps to begin by writing the equations in slope-intercept form. Try doing this for the systems in Example 4.

In Examples 5 and 6, note how you can use the method of elimination to determine that a system of linear equations has no solution or infinitely many solutions.

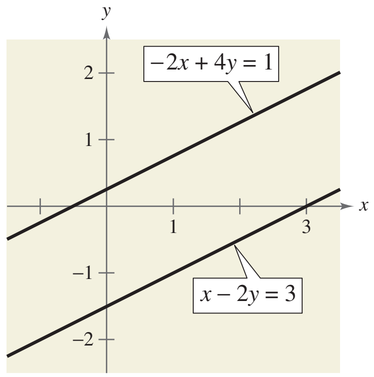


FIGURE 7.8

Example 5 No-Solution Case: Method of Elimination

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

Solution

To obtain coefficients that differ only in sign, multiply Equation 1 by 2.

$$\begin{array}{rcrl} x - 2y = 3 & \Rightarrow & 2x - 4y = 6 & \text{Multiply Equation 1 by 2.} \\ \underline{-2x + 4y = 1} & \Rightarrow & \underline{-2x + 4y = 1} & \text{Write Equation 2.} \\ & & 0 = 7 & \text{False statement} \end{array}$$

Because there are no values of x and y for which $0 = 7$, you can conclude that the system is inconsistent and has no solution. The lines corresponding to the two equations in this system are shown in Figure 7.8. Note that the two lines are parallel and therefore have no point of intersection.

CHECKPOINT Now try Exercise 19.

In Example 5, note that the occurrence of a false statement, such as $0 = 7$, indicates that the system has no solution. In the next example, note that the occurrence of a statement that is true for all values of the variables, such as $0 = 0$, indicates that the system has infinitely many solutions.

Example 6 Many-Solution Case: Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$$

Solution

To obtain coefficients that differ only in sign, multiply Equation 2 by $-\frac{1}{2}$.

$$\begin{array}{rcrl} 2x - y = 1 & \Rightarrow & 2x - y = 1 & \text{Write Equation 1.} \\ \underline{4x - 2y = 2} & \Rightarrow & \underline{-2x + y = -1} & \text{Multiply Equation 2 by } -\tfrac{1}{2}. \\ & & 0 = 0 & \text{Add equations.} \end{array}$$

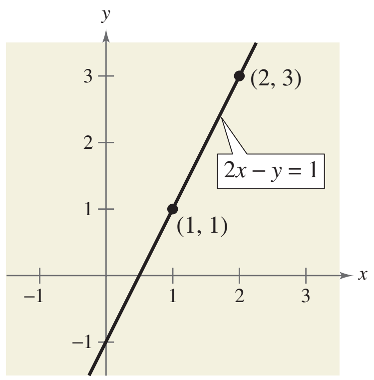


FIGURE 7.9

Because the two equations turn out to be equivalent (have the same solution set), you can conclude that the system has infinitely many solutions. The solution set consists of all points (x, y) lying on the line $2x - y = 1$, as shown in Figure 7.9. Letting $x = a$, where a is any real number, you can see that the solutions to the system are $(a, 2a - 1)$.

CHECKPOINT Now try Exercise 23.

Technology

The general solution of the linear system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

is $x = (ce - bf)/(ae - bd)$ and $y = (af - cd)/(ae - bd)$. If $ae - bd = 0$, the system does not have a unique solution. A graphing utility program (called Systems of Linear Equations) for solving such a system can be found at our website *college.hmco.com*. Try using the program for your graphing utility to solve the system in Example 7.

Example 7 illustrates a strategy for solving a system of linear equations that has decimal coefficients.

Example 7 A Linear System Having Decimal Coefficients

Solve the system of linear equations.

$$\begin{cases} 0.02x - 0.05y = -0.38 & \text{Equation 1} \\ 0.03x + 0.04y = 1.04 & \text{Equation 2} \end{cases}$$

Solution

Because the coefficients in this system have two decimal places, you can begin by multiplying each equation by 100. This produces a system in which the coefficients are all integers.

$$\begin{cases} 2x - 5y = -38 & \text{Revised Equation 1} \\ 3x + 4y = 104 & \text{Revised Equation 2} \end{cases}$$

Now, to obtain coefficients that differ only in sign, multiply Equation 1 by 3 and multiply Equation 2 by -2.

$$\begin{aligned} 2x - 5y &= -38 & \Longrightarrow\quad 6x - 15y &= -114 & &\text{Multiply Equation 1 by 3.} \\ 3x + 4y &= 104 & \Longrightarrow\quad -6x - 8y &= -208 & &\text{Multiply Equation 2 by } -2. \\ & & -23y &= -322 & &\text{Add equations.} \end{aligned}$$

So, you can conclude that

$$\begin{aligned} y &= \frac{-322}{-23} \\ &= 14. \end{aligned}$$

Back-substituting this value into revised Equation 2 produces the following.

$$\begin{aligned} 3x + 4y &= 104 & &\text{Write revised Equation 2.} \\ 3x + 4(14) &= 104 & &\text{Substitute 14 for } y. \\ 3x &= 48 & &\text{Combine like terms.} \\ x &= 16 & &\text{Solve for } x. \end{aligned}$$

The solution is (16, 14). Check this in the original system, as follows.

Check

$$\begin{aligned} 0.02x - 0.05y &= -0.38 & &\text{Write original Equation 1.} \\ 0.02(16) - 0.05(14) &\stackrel{?}{=} -0.38 & &\text{Substitute into Equation 1.} \\ 0.32 - 0.70 &= -0.38 & &\text{Equation 1 checks. } \checkmark \\ 0.03x + 0.04y &= 1.04 & &\text{Write original Equation 2.} \\ 0.03(16) + 0.04(14) &\stackrel{?}{=} 1.04 & &\text{Substitute into Equation 2.} \\ 0.48 + 0.56 &= 1.04 & &\text{Equation 2 checks. } \checkmark \end{aligned}$$

CHECKPOINT Now try Exercise 25.

Applications

At this point, you may be asking the question "How can I tell which application problems can be solved using a system of linear equations?" The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?
2. Are there two (or more) equations or conditions to be satisfied?

If one or both of these situations occur, the appropriate mathematical model for the problem may be a system of linear equations.

Example 8 An Application of a Linear System

An airplane flying into a headwind travels the 2000-mile flying distance between Chicopee, Massachusetts and Salt Lake City, Utah in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

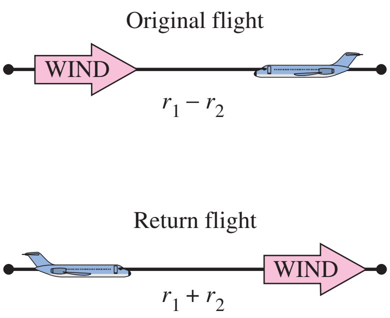


FIGURE 7.10

Solution

The two unknown quantities are the speeds of the wind and the plane. If r_1 is the speed of the plane and r_2 is the speed of the wind, then

$$r_1 - r_2 = \text{speed of the plane against the wind}$$

$$r_1 + r_2 = \text{speed of the plane with the wind}$$

as shown in Figure 7.10. Using the formula distance $=$ (rate)(time) for these two speeds, you obtain the following equations.

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as follows.

$$\begin{cases} 5000 = 11r_1 - 11r_2 & \text{Equation 1} \\ 500 = r_1 + r_2 & \text{Equation 2} \end{cases}$$

To solve this system by elimination, multiply Equation 2 by 11.

$$\begin{array}{lcll} 5000 = 11r_1 - 11r_2 & \Rightarrow & 5000 = 11r_1 - 11r_2 & \text{Write Equation 1.} \\ 500 = r_1 + r_2 & \Rightarrow & 5500 = 11r_1 + 11r_2 & \text{Multiply Equation 2 by 11.} \\ \hline & & 10{,}500 = 22r_1 & \text{Add equations.} \end{array}$$

So,

$$r_1 = \frac{10{,}500}{22} = \frac{5250}{11} \approx 477.27 \text{ miles per hour} \qquad \text{Speed of plane}$$

$$r_2 = 500 - \frac{5250}{11} = \frac{250}{11} \approx 22.73 \text{ miles per hour.} \qquad \text{Speed of wind}$$

Check this solution in the original statement of the problem.

CHECKPOINT Now try Exercise 43.

In a free market, the demands for many products are related to the prices of the products. As the prices decrease, the demands by consumers increase and the amounts that producers are able or willing to supply decrease.

Example 9 Finding the Equilibrium Point

The demand and supply functions for a new type of personal digital assistant are

$$\begin{cases} p = 150 - 0.00001x & \text{Demand equation} \\ p = 60 + 0.00002x & \text{Supply equation} \end{cases}$$

where p is the price in dollars and x represents the number of units. Find the equilibrium point for this market. The **equilibrium point** is the price p and number of units x that satisfy both the demand and supply equations.

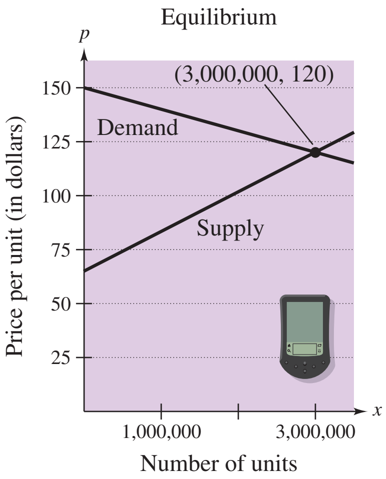


FIGURE 7.11

Solution

Because p is written in terms of x, begin by substituting the value of p given in the supply equation into the demand equation.

$$p = 150 - 0.00001x \qquad \text{Write demand equation.}$$

$$60 + 0.00002x = 150 - 0.00001x \qquad \text{Substitute } 60 + 0.00002x \text{ for } p.$$

$$0.00003x = 90 \qquad \text{Combine like terms.}$$

$$x = 3{,}000{,}000 \qquad \text{Solve for } x.$$

So, the equilibrium point occurs when the demand and supply are each 3 million units. (See Figure 7.11.) The price that corresponds to this x-value is obtained by back-substituting $x = 3{,}000{,}000$ into either of the original equations. For instance, back-substituting into the demand equation produces

$$\begin{aligned} p &= 150 - 0.00001(3{,}000{,}000) \\ &= 150 - 30 \\ &= \$120. \end{aligned}$$

The solution is (3,000,000, 120). You can check this as follows.

Check

Substitute (3,000,000, 120) into the demand equation.

$$p = 150 - 0.00001x \qquad \text{Write demand equation.}$$

$$120 \stackrel{?}{=} 150 - 0.00001(3{,}000{,}000) \qquad \text{Substitute 120 for } p \text{ and 3,000,000 for } x.$$

$$120 = 120 \qquad \text{Solution checks in demand equation. } \checkmark$$

Substitute (3,000,000, 120) into the supply equation.

$$p = 60 + 0.00002x \qquad \text{Write supply equation.}$$

$$120 \stackrel{?}{=} 60 + 0.00002(3{,}000{,}000) \qquad \text{Substitute 120 for } p \text{ and 3,000,000 for } x.$$

$$120 = 120 \qquad \text{Solution checks in supply equation. } \checkmark$$

CHECKPOINT Now try Exercise 45.

7.2 Exercises

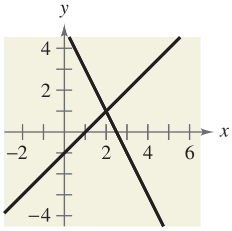
VOCABULARY CHECK: Fill in the blanks.

1. The first step in solving a system of equations by the method of ________ is to obtain coefficients for x (or y) that differ only in sign.
2. Two systems of equations that have the same solution set are called ________ systems.
3. A system of linear equations that has at least one solution is called ________, whereas a system of linear equations that has no solution is called ________.
4. In business applications, the ________ ________ is defined as the price p and the number of units x that satisfy both the demand and supply equations.

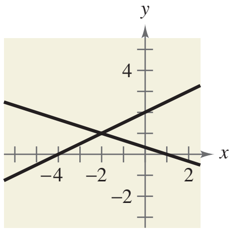
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, solve the system by the method of elimination. Label each line with its equation. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

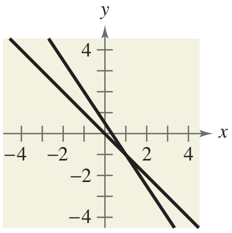
1. $\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$



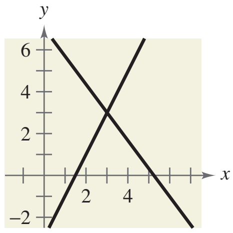
2. $\begin{cases} x + 3y = 1 \\ -x + 2y = 4 \end{cases}$



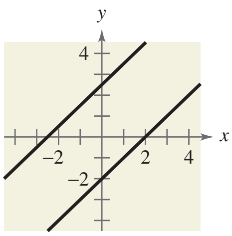
3. $\begin{cases} x + y = 0 \\ 3x + 2y = 1 \end{cases}$



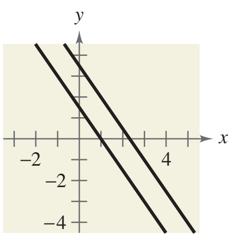
4. $\begin{cases} 2x - y = 3 \\ 4x + 3y = 21 \end{cases}$



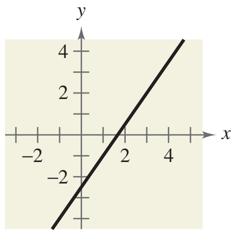
5. $\begin{cases} x - y = 2 \\ -2x + 2y = 5 \end{cases}$



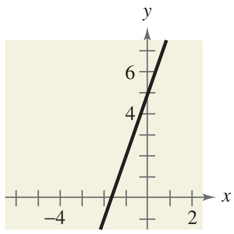
6. $\begin{cases} 3x + 2y = 3 \\ 6x + 4y = 14 \end{cases}$



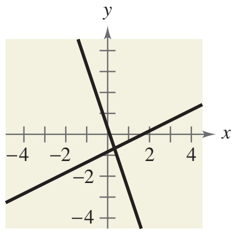
7. $\begin{cases} 3x - 2y = 5 \\ -6x + 4y = -10 \end{cases}$



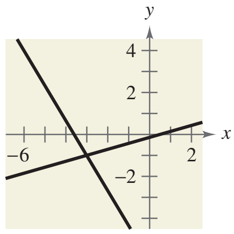
8. $\begin{cases} 9x - 3y = -15 \\ -3x + y = 5 \end{cases}$



9. $\begin{cases} 9x + 3y = 1 \\ 3x - 6y = 5 \end{cases}$



10. $\begin{cases} 5x + 3y = -18 \\ 2x - 6y = 1 \end{cases}$



In Exercises 11–30, solve the system by the method of elimination and check any solutions algebraically.

11. $\begin{cases} x + 2y = 4 \\ x - 2y = 1 \end{cases}$

12. $\begin{cases} 3x - 5y = 2 \\ 2x + 5y = 13 \end{cases}$

13. $\begin{cases} 2x + 3y = 18 \\ 5x - y = 11 \end{cases}$

14. $\begin{cases} x + 7y = 12 \\ 3x - 5y = 10 \end{cases}$

15. $\begin{cases} 3x + 2y = 10 \\ 2x + 5y = 3 \end{cases}$

16. $\begin{cases} 2r + 4s = 5 \\ 16r + 50s = 55 \end{cases}$

17. $\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$

18. $\begin{cases} 3x + 11y = 4 \\ -2x - 5y = 9 \end{cases}$

19. $\begin{cases} \frac{9}{5}x + \frac{6}{5}y = 4 \\ 9x + 6y = 3 \end{cases}$

20. $\begin{cases} \frac{3}{4}x + y = \frac{1}{8} \\ \frac{9}{4}x + 3y = \frac{3}{8} \end{cases}$

21. $\begin{cases} \dfrac{x}{4} + \dfrac{y}{6} = 1 \\ x - y = 3 \end{cases}$

22. $\begin{cases} \dfrac{2}{3}x + \dfrac{1}{6}y = \dfrac{2}{3} \\ 4x + y = 4 \end{cases}$

23. $\begin{cases} -5x + 6y = -3 \\ 20x - 24y = 12 \end{cases}$

24. $\begin{cases} 7x + 8y = 6 \\ -14x - 16y = -12 \end{cases}$

25. $\begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$

26. $\begin{cases} 0.2x - 0.5y = -27.8 \\ 0.3x + 0.4y = 68.7 \end{cases}$

27. $\begin{cases} 4b + 3m = 3 \\ 3b + 11m = 13 \end{cases}$

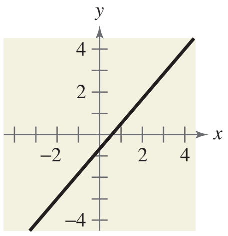
28. $\begin{cases} 2x + 5y = 8 \\ 5x + 8y = 10 \end{cases}$

29. $\begin{cases} \dfrac{x+3}{4} + \dfrac{y-1}{3} = 1 \\ 2x - y = 12 \end{cases}$

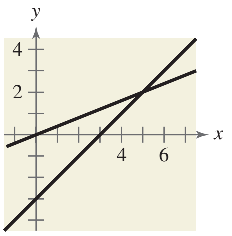
30. $\begin{cases} \dfrac{x-1}{2} + \dfrac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$

In Exercises 31–34, match the system of linear equations with its graph. Describe the number of solutions and state whether the system is consistent or inconsistent. [The graphs are labeled (a), (b), (c) and (d).]

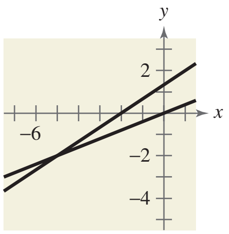
(a)



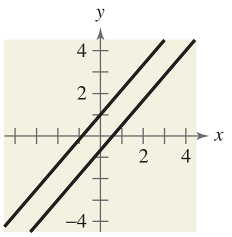
(b)



(c)



(d)



31. $\begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$

32. $\begin{cases} -7x + 6y = -4 \\ 14x - 12y = 8 \end{cases}$

33. $\begin{cases} 2x - 5y = 0 \\ 2x - 3y = -4 \end{cases}$

34. $\begin{cases} 7x - 6y = -6 \\ -7x + 6y = -4 \end{cases}$

In Exercises 35–42, use any method to solve the system.

35. $\begin{cases} 3x - 5y = 7 \\ 2x + y = 9 \end{cases}$

36. $\begin{cases} -x + 3y = 17 \\ 4x + 3y = 7 \end{cases}$

37. $\begin{cases} y = 2x - 5 \\ y = 5x - 11 \end{cases}$

38. $\begin{cases} 7x + 3y = 16 \\ y = x + 2 \end{cases}$

39. $\begin{cases} x - 5y = 21 \\ 6x + 5y = 21 \end{cases}$

40. $\begin{cases} y = -3x - 8 \\ y = 15 - 2x \end{cases}$

41. $\begin{cases} -2x + 8y = 19 \\ y = x - 3 \end{cases}$

42. $\begin{cases} 4x - 3y = 6 \\ -5x + 7y = -1 \end{cases}$

43. ***Airplane Speed*** An airplane flying into a headwind travels the 1800-mile flying distance between Pittsburgh, Pennsylvania and Phoenix, Arizona in 3 hours and 36 minutes. On the return flight, the distance is traveled in 3 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

44. ***Airplane Speed*** Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane if 2 hours after the first plane departs the planes are 3200 kilometers apart.

Supply and Demand **In Exercises 45–48, find the equilibrium point of the demand and supply equations. The equilibrium point is the price p and number of units x that satisfy both the demand and supply equations.**

	Demand	*Supply*
45.	$p = 50 - 0.5x$	$p = 0.125x$
46.	$p = 100 - 0.05x$	$p = 25 + 0.1x$
47.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
48.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

49. ***Nutrition*** Two cheeseburgers and one small order of French fries from a fast-food restaurant contain a total of 850 calories. Three cheeseburgers and two small orders of French fries contain a total of 1390 calories. Find the caloric content of each item.

50. ***Nutrition*** One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 185 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 452 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

51. ***Acid Mixture*** Ten liters of a 30% acid solution is obtained by mixing a 20% solution with a 50% solution.

(a) Write a system of equations in which one equation represents the amount of final mixture required and the other represents the percent of acid in the final mixture. Let x and y represent the amounts of the 20% and 50% solutions, respectively.

(b) Use a graphing utility to graph the two equations in part (a) in the same viewing window. As the amount of the 20% solution increases, how does the amount of the 50% solution change?

(c) How much of each solution is required to obtain the specified concentration of the final mixture?

52. ***Fuel Mixture*** Five hundred gallons of 89 octane gasoline is obtained by mixing 87 octane gasoline with 92 octane gasoline.

(a) Write a system of equations in which one equation represents the amount of final mixture required and the other represents the amounts of 87 and 92 octane gasolines in the final mixture. Let x and y represent the numbers of gallons of 87 octane and 92 octane gasolines, respectively.

(b) Use a graphing utility to graph the two equations in part (a) in the same viewing window. As the amount of 87 octane gasoline increases, how does the amount of 92 octane gasoline change?

(c) How much of each type of gasoline is required to obtain the 500 gallons of 89 octane gasoline?

53. ***Investment Portfolio*** A total of \$12,000 is invested in two corporate bonds that pay 7.5% and 9% simple interest. The investor wants an annual interest income of \$990 from the investments. What amount should be invested in the 7.5% bond?

54. ***Investment Portfolio*** A total of \$32,000 is invested in two municipal bonds that pay 5.75% and 6.25% simple interest. The investor wants an annual interest income of \$1900 from the investments. What amount should be invested in the 5.75% bond?

55. ***Ticket Sales*** At a local high school city championship basketball game, 1435 tickets were sold. A student admission ticket cost \$1.50 and an adult admission ticket cost \$5.00. The sum of all the total ticket receipts for the basketball game were \$3552.50. How many of each type of ticket were sold?

56. ***Consumer Awareness*** A department store held a sale to sell all of the 214 winter jackets that remained after the season ended. Until noon, each jacket in the store was priced at \$31.95. At noon, the price of the jackets was further reduced to \$18.95. After the last jacket was sold, total receipts for the clearance sale were \$5108.30. How many jackets were sold before noon and how many were sold after noon?

Fitting a Line to Data **In Exercises 57–62, find the least squares regression line $y = ax + b$ for the points**

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

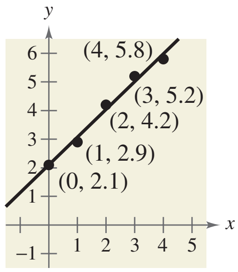
by solving the system for a and b.

$$nb + \left(\sum_{i=1}^{n} x_i\right)a = \left(\sum_{i=1}^{n} y_i\right)$$

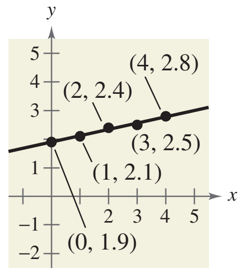
$$\left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \left(\sum_{i=1}^{n} x_i y_i\right)$$

Then use a graphing utility to confirm the result. (If you are unfamiliar with summation notation, look at the discussion in Section 9.1 or in Appendix B at the website for this text at *college.hmco.com*.)

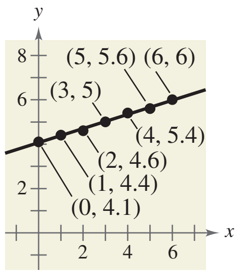
57. $\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$



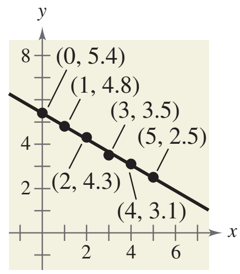
58. $\begin{cases} 5b + 10a = 11.7 \\ 10b + 30a = 25.6 \end{cases}$



59. $\begin{cases} 7b + 21a = 35.1 \\ 21b + 91a = 114.2 \end{cases}$



60. $\begin{cases} 6b + 15a = 23.6 \\ 15b + 55a = 48.8 \end{cases}$



61. (0, 4), (1, 3), (1, 1), (2, 0)

62. (1, 0), (2, 0), (3, 0), (3, 1), (4, 1), (4, 2), (5, 2), (6, 2)

63. ***Data Analysis*** A farmer used four test plots to determine the relationship between wheat yield y (in bushels per acre) and the amount of fertilizer x (in hundreds of pounds per acre). The results are shown in the table.

Fertilizer, x	Yield, y
1.0	32
1.5	41
2.0	48
2.5	53

(a) Use the technique demonstrated in Exercises 57–62 to set up a system of equations for the data and to find the least squares regression line $y = ax + b$.

(b) Use the linear model to predict the yield for a fertilizer application of 160 pounds per acre.

Model It

64. ***Data Analysis*** The table shows the average room rates y for a hotel room in the United States for the years 1995 through 2001. (Source: American Hotel & Motel Association)

Year	Average room rate, y
1995	\$66.65
1996	\$70.93
1997	\$75.31
1998	\$78.62
1999	\$81.33
2000	\$85.89
2001	\$88.27

(a) Use the technique demonstrated in Exercises 57–62 to set up a system of equations for the data and to find the least squares regression line $y = at + b$. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. How does this model compare with the model obtained in part (a)?

(c) Use the linear model to create a table of estimated values of y. Compare the estimated values with the actual data.

(d) Use the linear model to predict the average room rate in 2002. The actual average room rate in 2002 was \$83.54. How does this value compare with your prediction?

(e) Use the linear model to predict when the average room rate will be \$100.00. Using your result from part (d), do you think this prediction is accurate?

Synthesis

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

65. If two lines do not have exactly one point of intersection, then they must be parallel.

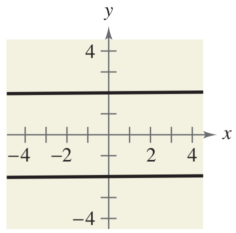
66. Solving a system of equations graphically will always give an exact solution.

67. ***Writing*** Briefly explain whether or not it is possible for a consistent system of linear equations to have exactly two solutions.

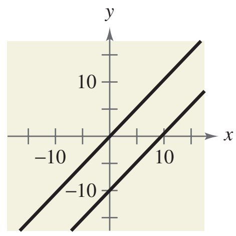
68. ***Think About It*** Give examples of a system of linear equations that has (a) no solution and (b) an infinite number of solutions.

Think About It **In Exercises 69 and 70, the graphs of the two equations appear to be parallel. Yet, when the system is solved algebraically, you find that the system does have a solution. Find the solution and explain why it does not appear on the portion of the graph that is shown.**

69. $\begin{cases} 100y - x = 200 \\ 99y - x = -198 \end{cases}$



70. $\begin{cases} 21x - 20y = 0 \\ 13x - 12y = 120 \end{cases}$



In Exercises 71 and 72, find the value of k such that the system of linear equations is inconsistent.

71. $\begin{cases} 4x - 8y = -3 \\ 2x + ky = 16 \end{cases}$

72. $\begin{cases} 15x + 3y = 6 \\ -10x + ky = 9 \end{cases}$

Skills Review

In Exercises 73–80, solve the inequality and graph the solution on the real number line.

73. $-11 - 6x \geq 33$

74. $2(x - 3) > -5x + 1$

75. $8x - 15 \leq -4(2x - 1)$

76. $-6 \leq 3x - 10 < 6$

77. $|x - 8| < 10$

78. $|x + 10| \geq -3$

79. $2x^2 + 3x - 35 < 0$

80. $3x^2 + 12x > 0$

In Exercises 81–84, write the expression as the logarithm of a single quantity.

81. $\ln x + \ln 6$

82. $\ln x - 5\ln(x + 3)$

83. $\log_9 12 - \log_9 x$

84. $\frac{1}{4}\log_6 3x$

In Exercises 85 and 86, solve the system by the method of substitution.

85. $\begin{cases} 2x - y = 4 \\ -4x + 2y = -12 \end{cases}$

86. $\begin{cases} 30x - 40y - 33 = 0 \\ 10x + 20y - 21 = 0 \end{cases}$

87. **Make a Decision** To work an extended application analyzing the average undergraduate tuition, room, and board charges at private colleges in the United States from 1985 to 2003, visit this text's website at *college.hmco.com*. *(Data Source: U.S. Dept. of Education)*

7.3 Multivariable Linear Systems

What you should learn

- Use back-substitution to solve linear systems in row-echelon form.
- Use Gaussian elimination to solve systems of linear equations.
- Solve nonsquare systems of linear equations.
- Use systems of linear equations in three or more variables to model and solve real-life problems.

Why you should learn it

Systems of linear equations in three or more variables can be used to model and solve real-life problems. For instance, in Exercise 71 on page 531, a system of linear equations can be used to analyze the reproduction rates of deer in a wildlife preserve.

Jeanne Drake/Tony Stone Images

Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. In fact, this method easily adapts to computer use for solving linear systems with dozens of variables.

When elimination is used to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. To see how this works, consider the following two systems of linear equations.

System of Three Linear Equations in Three Variables: (See Example 3.)

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Equivalent System in Row-Echelon Form: (See Example 1.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

The second system is said to be in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the system in row-echelon form, using back-substitution.

Example 1 Using Back-Substitution in Row-Echelon Form

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 3z = 5 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

Solution

From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$y + 3(2) = 5 \qquad \text{Substitute 2 for } z.$$

$$y = -1. \qquad \text{Solve for } y.$$

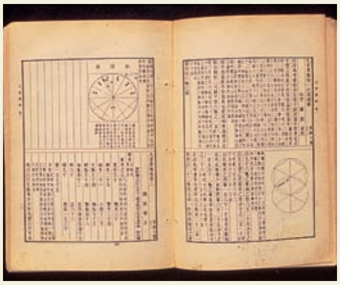
Finally, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \qquad \text{Substitute } -1 \text{ for } y \text{ and 2 for } z.$$

$$x = 1. \qquad \text{Solve for } x.$$

The solution is $x = 1$, $y = -1$, and $z = 2$, which can be written as the **ordered triple** $(1, -1, 2)$. Check this in the original system of equations.

CHECKPOINT Now try Exercise 5.



Christopher Lui/China Stock

Historical Note

One of the most influential Chinese mathematics books was the *Chui-chang suan-shu* or *Nine Chapters on the Mathematical Art* (written in approximately 250 B.C.). Chapter Eight of the *Nine Chapters* contained solutions of systems of linear equations using positive and negative numbers. One such system was as follows.

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 34 \\ x + 2y + 3z = 26 \end{cases}$$

This system was solved using column operations on a matrix. Matrices (plural for matrix) will be discussed in the next chapter.

Gaussian Elimination

Two systems of equations are *equivalent* if they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form by using the following operations.

Operations That Produce Equivalent Systems

Each of the following **row operations** on a system of linear equations produces an *equivalent* system of linear equations.

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one of the equations to another equation to replace the latter equation.

To see how this is done, take another look at the method of elimination, as applied to a system of two linear equations.

Example 2 Using Gaussian Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} 3x - 2y = -1 & \text{Equation 1} \\ x - y = 0 & \text{Equation 2} \end{cases}$$

Solution

There are two strategies that seem reasonable: eliminate the variable x or eliminate the variable y. The following steps show how to use the first strategy.

$$\begin{cases} x - y = 0 \\ 3x - 2y = -1 \end{cases}$$ Interchange the two equations in the system.

$$\begin{cases} -3x + 3y = 0 \\ 3x - 2y = -1 \end{cases}$$ Multiply the first equation by -3.

$$\begin{array}{r} -3x + 3y = 0 \\ \underline{3x - 2y = -1} \\ y = -1 \end{array}$$ Add the multiple of the first equation to the second equation to obtain a new second equation.

$$\begin{cases} x - y = 0 \\ y = -1 \end{cases}$$ New system in row-echelon form

Now, using back-substitution, you can determine that the solution is $y = -1$ and $x = -1$, which can be written as the ordered pair $(-1, -1)$. Check this solution in the original system of equations.

CHECKPOINT Now try Exercise 13.

STUDY TIP

As demonstrated in the first step in the solution of Example 2, interchanging rows is an easy way of obtaining a leading coefficient of 1.

As shown in Example 2, rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, each of which is obtained by using one of the three basic row operations listed on the previous page. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

STUDY TIP

Arithmetic errors are often made when performing elementary row operations. You should note the operation performed in each step so that you can go back and check your work.

Example 3 Using Gaussian Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y = -4 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

Solution

Because the leading coefficient of the first equation is 1, you can begin by saving the x at the upper left and eliminating the other x-terms from the first column.

$$\begin{array}{rll} x - 2y + 3z = 9 & & \text{Write Equation 1.} \\ -x + 3y \quad = -4 & & \text{Write Equation 2.} \\ \hline y + 3z = 5 & & \text{Add Equation 1 to Equation 2.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{array}{rll} -2x + 4y - 6z = -18 & & \text{Multiply Equation 1 by } -2. \\ 2x - 5y + 5z = 17 & & \text{Write Equation 3.} \\ \hline -y - z = -1 & & \text{Add revised Equation 1 to Equation 3.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that all but the first x have been eliminated from the first column, go to work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system that was solved in Example 1, and, as in that example, you can conclude that the solution is

$$x = 1, \qquad y = -1, \qquad \text{and} \qquad z = 2.$$

CHECKPOINT Now try Exercise 15.

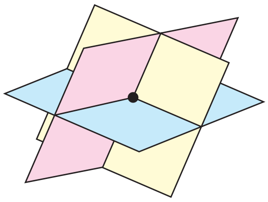


FIGURE 7.12 *Solution: one point*

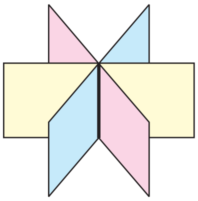


FIGURE 7.13 *Solution: one line*

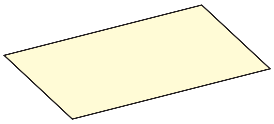


FIGURE 7.14 *Solution: one plane*

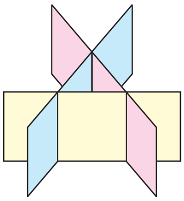


FIGURE 7.15 *Solution: none*

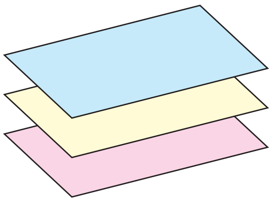


FIGURE 7.16 *Solution: none*

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process you obtain a false statement such as $0 = -2$.

Example 4 An Inconsistent System

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding -1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, you can conclude that this system is inconsistent and so has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

CHECKPOINT Now try Exercise 19.

As with a system of linear equations in two variables, the solution(s) of a system of linear equations in more than two variables must fall into one of three categories.

The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. There is exactly one solution.

2. There are infinitely many solutions.

3. There is no solution.

In Section 7.2, you learned that a system of two linear equations in two variables can be represented graphically as a pair of lines that are intersecting, coincident, or parallel. A system of three linear equations in three variables has a similar graphical representation—it can be represented as three planes in space that intersect in one point (exactly one solution) [see Figure 7.12], intersect in a line or a plane (infinitely many solutions) [see Figures 7.13 and 7.14], or have no points common to all three planes (no solution) [see Figures 7.15 and 7.16].

Example 5 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

Adding -3 times the second equation to the third equation produces a new third equation.

This result means that Equation 3 depends on Equations 1 and 2 in the sense that it gives no additional information about the variables. Because $0 = 0$ is a true statement, you can conclude that this system will have infinitely many solutions. However, it is incorrect to say simply that the solution is "infinite." You must also specify the correct form of the solution. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}.$$

In the last equation, solve for y in terms of z to obtain $y = z$. Back-substituting for y in the first equation produces $x = 2z - 1$. Finally, letting $z = a$, where a is a real number, the solutions to the given system are all of the form $x = 2a - 1$, $y = a$, and $z = a$. So, every ordered triple of the form

$(2a - 1, a, a)$, $\quad a$ is a real number

is a solution of the system.

CHECKPOINT Now try Exercise 23.

STUDY TIP

In Example 5, x and y are solved in terms of the third variable z. To write the correct form of the solution to the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Then solve for x and y. The solution can then be written in terms of a, which is not one of the variables of the system.

In Example 5, there are other ways to write the same infinite set of solutions. For instance, letting $x = b$, the solutions could have been written as

$\left(b, \frac{1}{2}(b + 1), \frac{1}{2}(b + 1)\right)$, $\quad b$ is a real number.

To convince yourself that this description produces the same set of solutions, consider the following.

Substitution	*Solution*	
$a = 0$	$(2(0) - 1, 0, 0) = (-1, 0, 0)$	Same solution
$b = -1$	$\left(-1, \frac{1}{2}(-1 + 1), \frac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$	
$a = 1$	$(2(1) - 1, 1, 1) = (1, 1, 1)$	Same solution
$b = 1$	$\left(1, \frac{1}{2}(1 + 1), \frac{1}{2}(1 + 1)\right) = (1, 1, 1)$	
$a = 2$	$(2(2) - 1, 2, 2) = (3, 2, 2)$	Same solution
$b = 3$	$\left(3, \frac{1}{2}(3 + 1), \frac{1}{2}(3 + 1)\right) = (3, 2, 2)$	

STUDY TIP

When comparing descriptions of an infinite solution set, keep in mind that there is more than one way to describe the set.

Nonsquare Systems

So far, each system of linear equations you have looked at has been *square*, which means that the number of equations is equal to the number of variables. In a **nonsquare** system, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

Example 6 A System with Fewer Equations than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 \\ 2x - y - z = 1 \end{cases}$$ Equation 1, Equation 2

Solution

Begin by rewriting the system in row-echelon form.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solve for y in terms of z, to obtain

$$y = z - 1.$$

By back-substituting into Equation 1, you can solve for x, as follows.

$$x - 2y + z = 2$$ Write Equation 1.

$$x - 2(z - 1) + z = 2$$ Substitute for y in Equation 1.

$$x - 2z + 2 + z = 2$$ Distributive Property

$$x = z$$ Solve for x.

Finally, by letting $z = a$, where a is a real number, you have the solution

$$x = a, \qquad y = a - 1, \qquad \text{and} \qquad z = a.$$

So, every ordered triple of the form

$$(a, a - 1, a), \qquad a \text{ is a real number}$$

is a solution of the system. Because there were originally three variables and only two equations, the system cannot have a unique solution.

CHECKPOINT Now try Exercise 27.

In Example 6, try choosing some values of a to obtain different solutions of the system, such as $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 2, 3)$. Then check each of the solutions in the original system to verify that they are solutions of the original system.

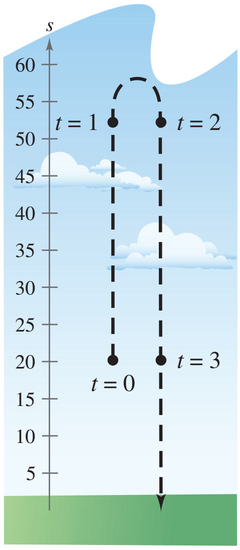


FIGURE 7.17

Applications

Example 7 Vertical Motion

The height at time t of an object that is moving in a (vertical) line with constant acceleration a is given by the **position equation**

$$s = \frac{1}{2}at^2 + v_0t + s_0.$$

The height s is measured in feet, the acceleration a is measured in feet per second squared, t is measured in seconds, v_0 is the initial velocity (at $t = 0$), and s_0 is the initial height. Find the values of a, v_0, and s_0 if $s = 52$ at $t = 1$, $s = 52$ at $t = 2$, and $s = 20$ at $t = 3$, and interpret the result. (See Figure 7.17.)

Solution

By substituting the three values of t and s into the position equation, you can obtain three linear equations in a, v_0, and s_0.

When $t = 1$: $\frac{1}{2}a(1)^2 + v_0(1) + s_0 = 52 \quad \Rightarrow \quad a + 2v_0 + 2s_0 = 104$

When $t = 2$: $\frac{1}{2}a(2)^2 + v_0(2) + s_0 = 52 \quad \Rightarrow \quad 2a + 2v_0 + s_0 = 52$

When $t = 3$: $\frac{1}{2}a(3)^2 + v_0(3) + s_0 = 20 \quad \Rightarrow \quad 9a + 6v_0 + 2s_0 = 40$

This produces the following system of linear equations.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ 2a + 2v_0 + s_0 = 52 \\ 9a + 6v_0 + 2s_0 = 40 \end{cases}$$

Now solve the system using Gaussian elimination.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad -2v_0 - 3s_0 = -156 \\ 9a + 6v_0 + 2s_0 = 40 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad -2v_0 - 3s_0 = -156 \\ \quad -12v_0 - 16s_0 = -896 \end{cases}$$

Adding -9 times the first equation to the third equation produces a new third equation.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad -2v_0 - 3s_0 = -156 \\ \quad 2s_0 = 40 \end{cases}$$

Adding -6 times the second equation to the third equation produces a new third equation.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad v_0 + \frac{3}{2}s_0 = 78 \\ \quad s_0 = 20 \end{cases}$$

Multiplying the second equation by $-\frac{1}{2}$ produces a new second equation and multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

So, the solution of this system is $a = -32$, $v_0 = 48$, and $s_0 = 20$. This solution results in a position equation of $s = -16t^2 + 48t + 20$ and implies that the object was thrown upward at a velocity of 48 feet per second from a height of 20 feet.

CHECKPOINT Now try Exercise 39.

Example 8 Data Analysis: Curve-Fitting

Find a quadratic equation

$$y = ax^2 + bx + c$$

whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

Solution

Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can write the following.

When $x = -1$, $y = 3$: $\quad a(-1)^2 + b(-1) + c = 3$

When $x = 1$, $y = 1$: $\quad a(1)^2 + b(1) + c = 1$

When $x = 2$, $y = 6$: $\quad a(2)^2 + b(2) + c = 6$

This produces the following system of linear equations.

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

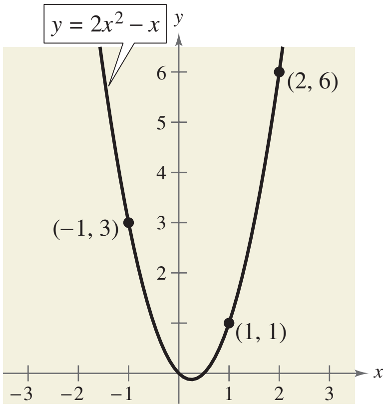


FIGURE 7.18

The solution of this system is $a = 2$, $b = -1$, and $c = 0$. So, the equation of the parabola is $y = 2x^2 - x$, as shown in Figure 7.18.

CHECKPOINT Now try Exercise 43.

Example 9 Investment Analysis

An inheritance of \$12,000 was invested among three funds: a money-market fund that paid 5% annually, municipal bonds that paid 6% annually, and mutual funds that paid 12% annually. The amount invested in mutual funds was \$4000 more than the amount invested in municipal bonds. The total interest earned during the first year was \$1120. How much was invested in each type of fund?

Solution

Let x, y, and z represent the amounts invested in the money-market fund, municipal bonds, and mutual funds, respectively. From the given information, you can write the following equations.

$$\begin{cases} x + y + z = 12{,}000 & \text{Equation 1} \\ z = y + 4000 & \text{Equation 2} \\ 0.05x + 0.06y + 0.12z = 1120 & \text{Equation 3} \end{cases}$$

Rewriting this system in standard form without decimals produces the following.

$$\begin{cases} x + y + z = 12{,}000 & \text{Equation 1} \\ -y + z = 4{,}000 & \text{Equation 2} \\ 5x + 6y + 12z = 112{,}000 & \text{Equation 3} \end{cases}$$

Using Gaussian elimination to solve this system yields $x = 2000$, $y = 3000$, and $z = 7000$. So, \$2000 was invested in the money-market fund, \$3000 was invested in municipal bonds, and \$7000 was invested in mutual funds.

CHECKPOINT Now try Exercise 53.

7.3 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A system of equations that is in ________ form has a "stair-step" pattern with leading coefficients of 1.
2. A solution to a system of three linear equations in three unknowns can be written as an ________ ________, which has the form (x, y, z).
3. The process used to write a system of linear equations in row-echelon form is called ________ elimination.
4. Interchanging two equations of a system of linear equations is a ________ ________ that produces an equivalent system.
5. A system of equations is called ________ if the number of equations differs from the number of variables in the system.
6. The equation $s = \frac{1}{2}at^2 + v_0t + s_0$ is called the ________ equation, and it models the height s of an object at time t that is moving in a vertical line with a constant acceleration a.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, determine whether each ordered triple is a solution of the system of equations.

1. $\begin{cases} 3x - y + z = 1 \\ 2x - 3z = -14 \\ 5y + 2z = 8 \end{cases}$

(a) $(2, 0, -3)$ (b) $(-2, 0, 8)$

(c) $(0, -1, 3)$ (d) $(-1, 0, 4)$

2. $\begin{cases} 3x + 4y - z = 17 \\ 5x - y + 2z = -2 \\ 2x - 3y + 7z = -21 \end{cases}$

(a) $(3, -1, 2)$ (b) $(1, 3, -2)$

(c) $(4, 1, -3)$ (d) $(1, -2, 2)$

3. $\begin{cases} 4x + y - z = 0 \\ -8x - 6y + z = -\frac{7}{4} \\ 3x - y = -\frac{9}{4} \end{cases}$

(a) $\left(\frac{1}{2}, -\frac{3}{4}, -\frac{7}{4}\right)$ (b) $\left(-\frac{3}{2}, \frac{5}{4}, -\frac{5}{4}\right)$

(c) $\left(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{4}\right)$ (d) $\left(-\frac{1}{2}, \frac{1}{6}, -\frac{3}{4}\right)$

4. $\begin{cases} -4x - y - 8z = -6 \\ y + z = 0 \\ 4x - 7y = 6 \end{cases}$

(a) $(-2, -2, 2)$ (b) $\left(-\frac{33}{2}, -10, 10\right)$

(c) $\left(\frac{1}{8}, -\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{11}{2}, -4, 4\right)$

In Exercises 5–10, use back-substitution to solve the system of linear equations.

5. $\begin{cases} 2x - y + 5z = 24 \\ y + 2z = 6 \\ z = 4 \end{cases}$

6. $\begin{cases} 4x - 3y - 2z = 21 \\ 6y - 5z = -8 \\ z = -2 \end{cases}$

7. $\begin{cases} 2x + y - 3z = 10 \\ y + z = 12 \\ z = 2 \end{cases}$

8. $\begin{cases} x - y + 2z = 22 \\ 3y - 8z = -9 \\ z = -3 \end{cases}$

9. $\begin{cases} 4x - 2y + z = 8 \\ -y + z = 4 \\ z = 2 \end{cases}$

10. $\begin{cases} 5x - 8z = 22 \\ 3y - 5z = 10 \\ z = -4 \end{cases}$

In Exercises 11 and 12, perform the row operation and write the equivalent system.

11. Add Equation 1 to Equation 2.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

12. Add -2 times Equation 1 to Equation 3.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

In Exercises 13–38, solve the system of linear equations and check any solution algebraically.

13. $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x - z = 0 \end{cases}$

14. $\begin{cases} x + y + z = 3 \\ x - 2y + 4z = 5 \\ 3y + 4z = 5 \end{cases}$

15. $\begin{cases} 2x + 2z = 2 \\ 5x + 3y = 4 \\ 3y - 4z = 4 \end{cases}$

16. $\begin{cases} 2x + 4y + z = 1 \\ x - 2y - 3z = 2 \\ x + y - z = -1 \end{cases}$

17. $\begin{cases} 6y + 4z = -12 \\ 3x + 3y = 9 \\ 2x - 3z = 10 \end{cases}$

18. $\begin{cases} 2x + 4y - z = 7 \\ 2x - 4y + 2z = -6 \\ x + 4y + z = 0 \end{cases}$

19. $\begin{cases} 2x + y - z = 7 \\ x - 2y + 2z = -9 \\ 3x - y + z = 5 \end{cases}$

20. $\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$

21. $\begin{cases} 3x - 5y + 5z = 1 \\ 5x - 2y + 3z = 0 \\ 7x - y + 3z = 0 \end{cases}$

22. $\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 8z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$

23. $\begin{cases} x + 2y - 7z = -4 \\ 2x + y + z = 13 \\ 3x + 9y - 36z = -33 \end{cases}$

24. $\begin{cases} 2x + y - 3z = 4 \\ 4x + 2z = 10 \\ -2x + 3y - 13z = -8 \end{cases}$

25. $\begin{cases} 3x - 3y + 6z = 6 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$

26. $\begin{cases} x + 2z = 5 \\ 3x - y - z = 1 \\ 6x - y + 5z = 16 \end{cases}$

27. $\begin{cases} x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$

28. $\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$

29. $\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$

30. $\begin{cases} 2x + 3y + 3z = 7 \\ 4x + 18y + 15z = 44 \end{cases}$

31. $\begin{cases} x + 3w = 4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$

32. $\begin{cases} x + y + z + w = 6 \\ 2x + 3y - w = 0 \\ -3x + 4y + z + 2w = 4 \\ x + 2y - z + w = 0 \end{cases}$

33. $\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$

34. $\begin{cases} 2x - 2y - 6z = -4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = -3 \end{cases}$

35. $\begin{cases} 2x + 3y = 0 \\ 4x + 3y - z = 0 \\ 8x + 3y + 3z = 0 \end{cases}$

36. $\begin{cases} 4x + 3y + 17z = 0 \\ 5x + 4y + 22z = 0 \\ 4x + 2y + 19z = 0 \end{cases}$

37. $\begin{cases} 12x + 5y + z = 0 \\ 23x + 4y - z = 0 \end{cases}$

38. $\begin{cases} 2x - y - z = 0 \\ -2x + 6y + 4z = 2 \end{cases}$

Vertical Motion **In Exercises 39–42, an object moving vertically is at the given heights at the specified times. Find the position equation $s = \frac{1}{2}at^2 + v_0t + s_0$ for the object.**

39. At $t = 1$ second, $s = 128$ feet
At $t = 2$ seconds, $s = 80$ feet
At $t = 3$ seconds, $s = 0$ feet

40. At $t = 1$ second, $s = 48$ feet
At $t = 2$ seconds, $s = 64$ feet
At $t = 3$ seconds, $s = 48$ feet

41. At $t = 1$ second, $s = 452$ feet
At $t = 2$ seconds, $s = 372$ feet
At $t = 3$ seconds, $s = 260$ feet

42. At $t = 1$ second, $s = 132$ feet
At $t = 2$ seconds, $s = 100$ feet
At $t = 3$ seconds, $s = 36$ feet

In Exercises 43–46, find the equation of the parabola

$$y = ax^2 + bx + c$$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.

43. (0, 0), (2, −2), (4, 0) **44.** (0, 3), (1, 4), (2, 3)

45. (2, 0), (3, −1), (4, 0) **46.** (1, 3), (2, 2), (3, −3)

In Exercises 47–50, find the equation of the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the circle.

47. (0, 0), (2, 2), (4, 0)

48. (0, 0), (0, 6), (3, 3)

49. (−3, −1), (2, 4), (−6, 8)

50. (0, 0), (0, −2), (3, 0)

51. ***Sports*** In Super Bowl I, on January 15, 1967, the Green Bay Packers defeated the Kansas City Chiefs by a score of 35 to 10. The total points scored came from 13 different scoring plays, which were a combination of touchdowns, extra-point kicks, and field goals, worth 6, 1, and 3 points respectively. The same number of touchdowns and extra point kicks were scored. There were six times as many touchdowns as field goals. How many touchdowns, extra-point kicks, and field goals were scored during the game? (Source: SuperBowl.com)

52. ***Sports*** In the 2004 Women's NCAA Final Four Championship game, the University of Connecticut Huskies defeated the University of Tennessee Lady Volunteers by a score of 70 to 61. The Huskies won by scoring a combination of two-point baskets, three-point baskets, and one-point free throws. The number of two-point baskets was two more than the number of free throws. The number of free throws was one more than two times the number of three-point baskets. What combination of scoring accounted for the Huskies' 70 points? (Source: National Collegiate Athletic Association)

53. ***Finance*** A small corporation borrowed \$775,000 to expand its clothing line. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if the annual interest owed was \$67,500 and the amount borrowed at 8% was four times the amount borrowed at 10%?

54. ***Finance*** A small corporation borrowed \$800,000 to expand its line of toys. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if the annual interest owed was \$67,000 and the amount borrowed at 8% was five times the amount borrowed at 10%?

Investment Portfolio **In Exercises 55 and 56, consider an investor with a portfolio totaling \$500,000 that is invested in certificates of deposit, municipal bonds, blue-chip stocks, and growth or speculative stocks. How much is invested in each type of investment?**

55. The certificates of deposit pay 10% annually, and the municipal bonds pay 8% annually. Over a five-year period, the investor expects the blue-chip stocks to return 12% annually and the growth stocks to return 13% annually. The investor wants a combined annual return of 10% and also wants to have only one-fourth of the portfolio invested in stocks.

56. The certificates of deposit pay 9% annually, and the municipal bonds pay 5% annually. Over a five-year period, the investor expects the blue-chip stocks to return 12% annually and the growth stocks to return 14% annually. The investor wants a combined annual return of 10% and also wants to have only one-fourth of the portfolio invested in stocks.

57. ***Agriculture*** A mixture of 5 pounds of fertilizer A, 13 pounds of fertilizer B, and 4 pounds of fertilizer C provides the optimal nutrients for a plant. Commercial brand X contains equal parts of fertilizer B and fertilizer C. Commercial brand Y contains one part of fertilizer A and two parts of fertilizer B. Commercial brand Z contains two parts of fertilizer A, five parts of fertilizer B, and two parts of fertilizer C. How much of each fertilizer brand is needed to obtain the desired mixture?

58. ***Agriculture*** A mixture of 12 liters of chemical A, 16 liters of chemical B, and 26 liters of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains only chemicals A and B in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

59. ***Coffee Mixture*** A coffee manufacturer sells a 10-pound package of coffee that consists of three flavors of coffee. Vanilla-flavored coffee costs \$2 per pound, hazelnut-flavored coffee costs \$2.50 per pound, and mocha-flavored coffee costs \$3 per pound. The package contains the same amount of hazelnut coffee as mocha coffee. The cost of the 10-pound package is \$26. How many pounds of each type of coffee are in the package?

60. ***Floral Arrangements*** A florist is creating 10 centerpieces for a wedding. The florist can use roses that cost \$2.50 each, lilies that cost \$4 each, and irises that cost \$2 each to make the bouquets. The customer has a budget of \$300 and wants each bouquet to contain 12 flowers, with twice as many roses used as the other two types of flowers combined. How many of each type of flower should be in each centerpiece?

61. ***Advertising*** A health insurance company advertises on television, radio, and in the local newspaper. The marketing department has an advertising budget of \$42,000 per month. A television ad costs \$1000, a radio ad costs \$200, and a newspaper ad costs \$500. The department wants to run 60 ads per month, and have as many television ads as radio and newspaper ads combined. How many of each type of ad can the department run each month?

62. ***Radio*** You work as a disc jockey at your college radio station. You are supposed to play 32 songs within two hours. You are to choose the songs from the latest rock, dance, and pop albums. You want to play twice as many rock songs as pop songs and four more pop songs than dance songs. How many of each type of song will you play?

63. ***Acid Mixture*** A chemist needs 10 liters of a 25% acid solution. The solution is to be mixed from three solutions whose concentrations are 10%, 20%, and 50%. How many liters of each solution will satisfy each condition?

(a) Use 2 liters of the 50% solution.

(b) Use as little as possible of the 50% solution.

(c) Use as much as possible of the 50% solution.

64. ***Acid Mixture*** A chemist needs 12 gallons of a 20% acid solution. The solution is to be mixed from three solutions whose concentrations are 10%, 15%, and 25%. How many gallons of each solution will satisfy each condition?

(a) Use 4 gallons of the 25% solution.

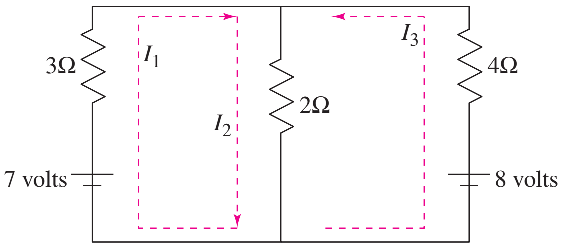
(b) Use as little as possible of the 25% solution.

(c) Use as much as possible of the 25% solution.

65. ***Electrical Network*** Applying Kirchhoff's Laws to the electrical network in the figure, the currents I_1, I_2, and I_3 are the solution of the system

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 = 7 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

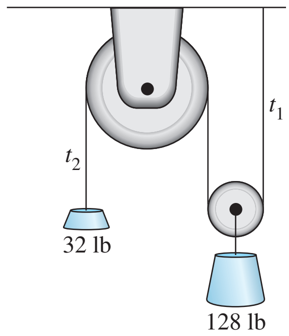
find the currents.



66. ***Pulley System*** A system of pulleys is loaded with 128-pound and 32-pound weights (see figure). The tensions t_1 and t_2 in the ropes and the acceleration a of the 32-pound weight are found by solving the system of equations

$$\begin{cases} t_1 - 2t_2 = 0 \\ t_1 - 2a = 128 \\ t_2 + a = 32 \end{cases}$$

where t_1 and t_2 are measured in pounds and a is measured in feet per second squared.



(a) Solve this system.

(b) The 32-pound weight in the pulley system is replaced by a 64-pound weight. The new pulley system will be modeled by the following system of equations.

$$\begin{cases} t_1 - 2t_2 = 0 \\ t_1 - 2a = 128 \\ t_2 + a = 64 \end{cases}$$

Solve this system and use your answer for the acceleration to describe what (if anything) is happening in the pulley system.

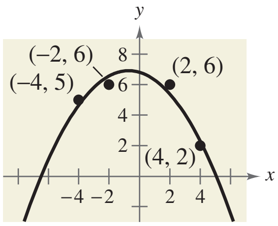
Fitting a Parabola **In Exercises 67–70, find the least squares regression parabola $y = ax^2 + bx + c$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by solving the following system of linear equations for a, b, and c. Then use the *regression* feature of a graphing utility to confirm the result. (If you are unfamiliar with summation notation, look at the discussion in Section 9.1 or in Appendix B at the website for this text at *college.hmco.com*.)**

$$nc + \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} y_i$$

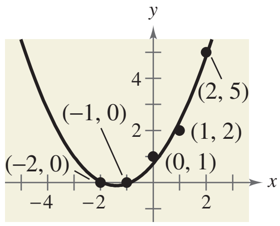
$$\left(\sum_{i=1}^{n} x_i\right)c + \left(\sum_{i=1}^{n} x_i^2\right)b + \left(\sum_{i=1}^{n} x_i^3\right)a = \sum_{i=1}^{n} x_i y_i$$

$$\left(\sum_{i=1}^{n} x_i^2\right)c + \left(\sum_{i=1}^{n} x_i^3\right)b + \left(\sum_{i=1}^{n} x_i^4\right)a = \sum_{i=1}^{n} x_i^2 y_i$$

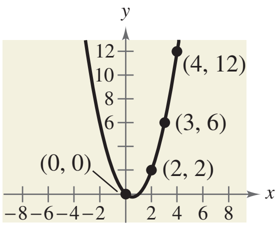
67.



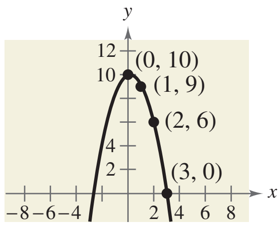
68.



69.



70.



Model It

71. ***Data Analysis: Wildlife*** A wildlife management team studied the reproduction rates of deer in three tracts of a wildlife preserve. Each tract contained 5 acres. In each tract, the number of females x, and the percent of females y that had offspring the following year, were recorded. The results are shown in the table.

Number, x	Percent, y
100	75
120	68
140	55

(a) Use the technique demonstrated in Exercises 67–70 to set up a system of equations for the data and to find a least squares regression parabola that models the data.

(b) Use a graphing utility to graph the parabola and the data in the same viewing window.

(c) Use the model to create a table of estimated values of y. Compare the estimated values with the actual data.

(d) Use the model to estimate the percent of females that had offspring when there were 170 females.

(e) Use the model to estimate the number of females when 40% of the females had offspring.

72. ***Data Analysis: Stopping Distance*** In testing a new automobile braking system, the speed x (in miles per hour) and the stopping distance y (in feet) were recorded in the table.

Speed, x	Stopping distance, y
30	55
40	105
50	188

(a) Use the technique demonstrated in Exercises 67–70 to set up a system of equations for the data and to find a least squares regression parabola that models the data.

(b) Graph the parabola and the data on the same set of axes.

(c) Use the model to estimate the stopping distance when the speed is 70 miles per hour.

73. ***Sports*** In Super Bowl XXXVIII, on February 1, 2004, the New England Patriots beat the Carolina Panthers by a score of 32 to 29. The total points scored came from 16 different scoring plays, which were a combination of touchdowns, extra-point kicks, two-point conversions, and field goals, worth 6, 1, 2, and 3 points, respectively. There were four times as many touchdowns as field goals and two times as many field goals as two-point conversions. How many touchdowns, extra-point kicks, two-point conversions, and field goals were scored during the game? (Source: SuperBowl.com)

74. ***Sports*** In the 2005 Orange Bowl, the University of Southern California won the National Championship by defeating the University of Oklahoma by a score of 55 to 19. The total points scored came from 22 different scoring plays, which were a combination of touchdowns, extra-point kicks, field goals and safeties, worth 6, 1, 3, and 2 points respectively. The same number of touchdowns and extra-point kicks were scored, and there were three times as many field goals as safeties. How many touchdowns, extra-point kicks, field goals, and safeties were scored? (Source: ESPN.com)

Advanced Applications **In Exercises 75–78, find values of x, y, and λ that satisfy the system. These systems arise in certain optimization problems in calculus, and λ is called a Lagrange multiplier.**

75. $\begin{cases} y + \lambda = 0 \\ x + \lambda = 0 \\ x + y - 10 = 0 \end{cases}$

76. $\begin{cases} 2x + \lambda = 0 \\ 2y + \lambda = 0 \\ x + y - 4 = 0 \end{cases}$

77. $\begin{cases} 2x - 2x\lambda = 0 \\ -2y + \lambda = 0 \\ y - x^2 = 0 \end{cases}$

78. $\begin{cases} 2 + 2y + 2\lambda = 0 \\ 2x + 1 + \lambda = 0 \\ 2x + y - 100 = 0 \end{cases}$

Synthesis

True or False? **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

79. The system

$$\begin{cases} x + 3y - 6z = -16 \\ 2y - z = -1 \\ z = 3 \end{cases}$$

is in row-echelon form.

80. If a system of three linear equations is inconsistent, then its graph has no points common to all three equations.

81. ***Think About It*** Are the following two systems of equations equivalent? Give reasons for your answer.

$$\begin{cases} x + 3y - z = 6 \\ 2x - y + 2z = 1 \\ 3x + 2y - z = 2 \end{cases} \qquad \begin{cases} x + 3y - z = 6 \\ -7y + 4z = 1 \\ -7y - 4z = -16 \end{cases}$$

82. ***Writing*** When using Gaussian elimination to solve a system of linear equations, explain how you can recognize that the system has no solution. Give an example that illustrates your answer.

In Exercises 83–86, find two systems of linear equations that have the ordered triple as a solution. (There are many correct answers.)

83. $(4, -1, 2)$

84. $(-5, -2, 1)$

85. $\left(3, -\frac{1}{2}, \frac{7}{4}\right)$

86. $\left(-\frac{3}{2}, 4, -7\right)$

Skills Review

In Exercises 87–90, solve the percent problem.

87. What is $7\frac{1}{2}\%$ of 85?

88. 225 is what percent of 150?

89. 0.5% of what number is 400?

90. 48% of what number is 132?

In Exercises 91–96, perform the operation and write the result in standard form.

91. $(7 - i) + (4 + 2i)$

92. $(-6 + 3i) - (1 + 6i)$

93. $(4 - i)(5 + 2i)$

94. $(1 + 2i)(3 - 4i)$

95. $\dfrac{i}{1 + i} + \dfrac{6}{1 - i}$

96. $\dfrac{i}{4 + i} - \dfrac{2i}{8 - 3i}$

In Exercises 97–100, (a) determine the real zeros of f and (b) sketch the graph of f.

97. $f(x) = x^3 + x^2 - 12x$

98. $f(x) = -8x^4 + 32x^2$

99. $f(x) = 2x^3 + 5x^2 - 21x - 36$

100. $f(x) = 6x^3 - 29x^2 - 6x + 5$

In Exercises 101–104, use a graphing utility to construct a table of values for the equation. Then sketch the graph of the equation by hand.

101. $y = 4^{x-4} - 5$

102. $y = \left(\frac{5}{2}\right)^{-x+1} - 4$

103. $y = 1.9^{-0.8x} + 3$

104. $y = 3.5^{-x+2} + 6$

In Exercises 105 and 106, solve the system by elimination.

105. $\begin{cases} 2x + y = 120 \\ x + 2y = 120 \end{cases}$

106. $\begin{cases} 6x - 5y = 3 \\ 10x - 12y = 5 \end{cases}$

107. Make a Decision To work an extended application analyzing the earnings per share for Wal-Mart Stores, Inc. from 1988 to 2003, visit this text's website at *college.hmco.com*. *(Data Source: Wal-Mart Stores, Inc.)*

7.4 Partial Fractions

What you should learn

- Recognize partial fraction decompositions of rational expressions.
- Find partial fraction decompositions of rational expressions.

Why you should learn it

Partial fractions can help you analyze the behavior of a rational function. For instance, in Exercise 57 on page 540, you can analyze the exhaust temperatures of a diesel engine using partial fractions.

Introduction

In this section, you will learn to write a rational expression as the sum of two or more simpler rational expressions. For example, the rational expression

$$\frac{x+7}{x^2-x-6}$$

can be written as the sum of two fractions with first-degree denominators. That is,

$$\frac{x+7}{x^2-x-6} = \overbrace{\underbrace{\frac{2}{x-3}}_{\text{Partial fraction}} + \underbrace{\frac{-1}{x+2}}_{\text{Partial fraction}}}^{\text{Partial fraction decomposition of } \frac{x+7}{x^2-x-6}}.$$

Each fraction on the right side of the equation is a **partial fraction,** and together they make up the **partial fraction decomposition** of the left side.

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. *Divide if improper:* If $N(x)/D(x)$ is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 below to the proper rational expression $N_1(x)/D(x)$. Note that $N_1(x)$ is the remainder from the division of $N(x)$ by $D(x)$.

2. *Factor the denominator:* Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where (ax^2+bx+c) is irreducible.

3. *Linear factors:* For *each* factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. *Quadratic factors:* For *each* factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

STUDY TIP

Section A.4, shows you how to combine expressions such as

$$\frac{1}{x-2} + \frac{-1}{x+3} = \frac{5}{(x-2)(x+3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x-2)(x+3)} = \frac{?}{x-2} + \frac{?}{x+3}$$

Partial Fraction Decomposition

Algebraic techniques for determining the constants in the numerators of partial fractions are demonstrated in the examples that follow. Note that the techniques vary slightly, depending on the type of factors of the denominator: linear or quadratic, distinct or repeated.

Example 1 Distinct Linear Factors

Write the partial fraction decomposition of $\dfrac{x+7}{x^2-x-6}$.

Solution

The expression is proper, so be sure to factor the denominator. Because $x^2 - x - 6 = (x-3)(x+2)$, you should include one partial fraction with a constant numerator for each linear factor of the denominator. Write the form of the decomposition as follows.

$$\frac{x+7}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} \qquad \text{Write form of decomposition.}$$

Multiplying each side of this equation by the least common denominator, $(x-3)(x+2)$, leads to the **basic equation**

$$x + 7 = A(x+2) + B(x-3). \qquad \text{Basic equation}$$

Because this equation is true for all x, you can substitute any *convenient* values of x that will help determine the constants A and B. Values of x that are especially convenient are ones that make the factors $(x+2)$ and $(x-3)$ equal to zero. For instance, let $x = -2$. Then

$$-2 + 7 = A(-2+2) + B(-2-3) \qquad \text{Substitute } -2 \text{ for } x.$$

$$5 = A(0) + B(-5)$$

$$5 = -5B$$

$$-1 = B.$$

To solve for A, let $x = 3$ and obtain

$$3 + 7 = A(3+2) + B(3-3) \qquad \text{Substitute 3 for } x.$$

$$10 = A(5) + B(0)$$

$$10 = 5A$$

$$2 = A.$$

So, the partial fraction decomposition is

$$\frac{x+7}{x^2-x-6} = \frac{2}{x-3} + \frac{-1}{x+2}$$

Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

CHECKPOINT Now try Exercise 15.

Technology

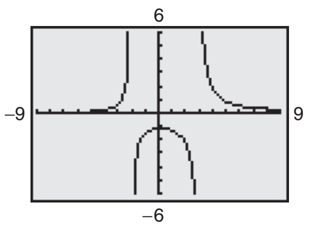
You can use a graphing utility to check *graphically* the decomposition found in Example 1. To do this, graph

$$y_1 = \frac{x+7}{x^2-x-6}$$

and

$$y_2 = \frac{2}{x-3} + \frac{-1}{x+2}$$

in the same viewing window. The graphs should be identical, as shown below.



The next example shows how to find the partial fraction decomposition of a rational expression whose denominator has a *repeated* linear factor.

Example 2 Repeated Linear Factors

Write the partial fraction decomposition of $\dfrac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x}$.

Solution

This rational expression is improper, so you should begin by dividing the numerator by the denominator to obtain

$$x + \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Because the denominator of the remainder factors as

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x + 1)^2$$

you should include one partial fraction with a constant numerator for each power of x and $(x + 1)$ and write the form of the decomposition as follows.

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$ Write form of decomposition.

Multiplying by the LCD, $x(x + 1)^2$, leads to the basic equation

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx.$$ Basic equation

Letting $x = -1$ eliminates the A- and B-terms and yields

$$5(-1)^2 + 20(-1) + 6 = A(-1 + 1)^2 + B(-1)(-1 + 1) + C(-1)$$
$$5 - 20 + 6 = 0 + 0 - C$$
$$C = 9.$$

Letting $x = 0$ eliminates the B- and C-terms and yields

$$5(0)^2 + 20(0) + 6 = A(0 + 1)^2 + B(0)(0 + 1) + C(0)$$
$$6 = A(1) + 0 + 0$$
$$6 = A.$$

At this point, you have exhausted the most convenient choices for x, so to find the value of B, use *any other value* for x along with the known values of A and C. So, using $x = 1$, $A = 6$, and $C = 9$,

$$5(1)^2 + 20(1) + 6 = 6(1 + 1)^2 + B(1)(1 + 1) + 9(1)$$
$$31 = 6(4) + 2B + 9$$
$$-2 = 2B$$
$$-1 = B.$$

So, the partial fraction decomposition is

$$\frac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x} = x + \frac{6}{x} + \frac{-1}{x + 1} + \frac{9}{(x + 1)^2}.$$

CHECKPOINT Now try Exercise 27.

The procedure used to solve for the constants in Examples 1 and 2 works well when the factors of the denominator are linear. However, when the denominator contains irreducible quadratic factors, you should use a different procedure, which involves writing the right side of the basic equation in polynomial form and *equating the coefficients* of like terms. Then you can use a system of equations to solve for the coefficients.

The Granger Collection

Historical Note
John Bernoulli (1667–1748), a Swiss mathematician, introduced the method of partial fractions and was instrumental in the early development of calculus. Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

Example 3 Distinct Linear and Quadratic Factors

Write the partial fraction decomposition of

$$\frac{3x^2 + 4x + 4}{x^3 + 4x}.$$

Solution

This expression is proper, so factor the denominator. Because the denominator factors as

$$x^3 + 4x = x(x^2 + 4)$$

you should include one partial fraction with a constant numerator and one partial fraction with a linear numerator and write the form of the decomposition as follows.

$$\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$ Write form of decomposition.

Multiplying by the LCD, $x(x^2 + 4)$, yields the basic equation

$$3x^2 + 4x + 4 = A(x^2 + 4) + (Bx + C)x.$$ Basic equation

Expanding this basic equation and collecting like terms produces

$$\begin{aligned} 3x^2 + 4x + 4 &= Ax^2 + 4A + Bx^2 + Cx \\ &= (A + B)x^2 + Cx + 4A. \end{aligned}$$ Polynomial form

Finally, because two polynomials are equal if and only if the coefficients of like terms are equal, you can equate the coefficients of like terms on opposite sides of the equation.

$$3x^2 + 4x + 4 = (A + B)x^2 + Cx + 4A$$ Equate coefficients of like terms.

You can now write the following system of linear equations.

$$\begin{cases} A + B \quad\quad = 3 & \text{Equation 1} \\ \quad\quad\quad C = 4 & \text{Equation 2} \\ 4A \quad\quad\quad = 4 & \text{Equation 3} \end{cases}$$

From this system you can see that $A = 1$ and $C = 4$. Moreover, substituting $A = 1$ into Equation 1 yields

$$1 + B = 3 \implies B = 2.$$

So, the partial fraction decomposition is

$$\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{2x + 4}{x^2 + 4}.$$

CHECKPOINT Now try Exercise 29.

The next example shows how to find the partial fraction decomposition of a rational expression whose denominator has a *repeated* quadratic factor.

Example 4 Repeated Quadratic Factors

Write the partial fraction decomposition of $\dfrac{8x^3 + 13x}{(x^2 + 2)^2}$.

Solution

You need to include one partial fraction with a linear numerator for each power of $(x^2 + 2)$.

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$ Write form of decomposition.

Multiplying by the LCD, $(x^2 + 2)^2$, yields the basic equation

$$\begin{aligned} 8x^3 + 13x &= (Ax + B)(x^2 + 2) + Cx + D && \text{Basic equation} \\ &= Ax^3 + 2Ax + Bx^2 + 2B + Cx + D \\ &= Ax^3 + Bx^2 + (2A + C)x + (2B + D). && \text{Polynomial form} \end{aligned}$$

Equating coefficients of like terms on opposite sides of the equation

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$$

produces the following system of linear equations.

$$\begin{cases} A \qquad\qquad\qquad = 8 & \text{Equation 1} \\ \qquad B \qquad\qquad = 0 & \text{Equation 2} \\ 2A + \qquad C \qquad = 13 & \text{Equation 3} \\ \qquad 2B + \qquad D = 0 & \text{Equation 4} \end{cases}$$

Finally, use the values $A = 8$ and $B = 0$ to obtain the following.

$$\begin{aligned} 2(8) + C &= 13 && \text{Substitute 8 for } A \text{ in Equation 3.} \\ C &= -3 \end{aligned}$$

$$\begin{aligned} 2(0) + D &= 0 && \text{Substitute 0 for } B \text{ in Equation 4.} \\ D &= 0 \end{aligned}$$

So, using $A = 8$, $B = 0$, $C = -3$, and $D = 0$, the partial fraction decomposition is

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}.$$

Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

CHECKPOINT Now try Exercise 49.

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the *zeros* of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in Step 1 to rewrite the basic equation. Then substitute *other* convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like terms to obtain equations involving A, B, C, and so on.
4. Use a system of linear equations to solve for $A, B, C, \ldots$.

Keep in mind that for *improper* rational expressions such as

$$\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}$$

you must first divide before applying partial fraction decomposition.

Writing about Mathematics

Error Analysis You are tutoring a student in algebra. In trying to find a partial fraction decomposition, the student writes the following.

$$\frac{x^2 + 1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}$$

$$\frac{x^2 + 1}{x(x - 1)} = \frac{A(x - 1)}{x(x - 1)} + \frac{Bx}{x(x - 1)}$$

$$x^2 + 1 = A(x - 1) + Bx \qquad \text{Basic equation}$$

By substituting $x = 0$ and $x = 1$ into the basic equation, the student concludes that $A = -1$ and $B = 2$. However, in checking this solution, the student obtains the following.

$$\begin{aligned}\frac{-1}{x} + \frac{2}{x - 1} &= \frac{(-1)(x - 1) + 2(x)}{x(x - 1)}\\ &= \frac{x + 1}{x(x - 1)}\\ &\neq \frac{x^2 + 1}{x(x + 1)}\end{aligned}$$

What has gone wrong?

7.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The process of writing a rational expression as the sum or difference of two or more simpler rational expressions is called ________ ________ ________.
2. If the degree of the numerator of a rational expression is greater than or equal to the degree of the denominator, then the fraction is called ________.
3. In order to find the partial fraction decomposition of a rational expression, the denominator must be completely factored into ________ factors of the form $(px + q)^m$ and ________ factors of the form $(ax^2 + bx + c)^n$, which are ________ over the rationals.
4. The ________ ________ is derived after multiplying each side of the partial fraction decomposition form by the least common denominator.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, match the rational expression with the form of its decomposition. [The decompositions are labeled (a), (b), (c), and (d).]

(a) $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$ (b) $\dfrac{A}{x} + \dfrac{B}{x-4}$

(c) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}$ (d) $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$

1. $\dfrac{3x-1}{x(x-4)}$ **2.** $\dfrac{3x-1}{x^2(x-4)}$

3. $\dfrac{3x-1}{x(x^2+4)}$ **4.** $\dfrac{3x-1}{x(x^2-4)}$

In Exercises 5–14, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

5. $\dfrac{7}{x^2 - 14x}$ **6.** $\dfrac{x-2}{x^2+4x+3}$

7. $\dfrac{12}{x^3 - 10x^2}$ **8.** $\dfrac{x^2-3x+2}{4x^3+11x^2}$

9. $\dfrac{4x^2+3}{(x-5)^3}$ **10.** $\dfrac{6x+5}{(x+2)^4}$

11. $\dfrac{2x-3}{x^3+10x}$ **12.** $\dfrac{x-6}{2x^3+8x}$

13. $\dfrac{x-1}{x(x^2+1)^2}$ **14.** $\dfrac{x+4}{x^2(3x-1)^2}$

In Exercises 15–38, write the partial fraction decomposition of the rational expression. Check your result algebraically.

15. $\dfrac{1}{x^2-1}$ **16.** $\dfrac{1}{4x^2-9}$

17. $\dfrac{1}{x^2+x}$ **18.** $\dfrac{3}{x^2-3x}$

19. $\dfrac{1}{2x^2+x}$ **20.** $\dfrac{5}{x^2+x-6}$

21. $\dfrac{3}{x^2+x-2}$ **22.** $\dfrac{x+1}{x^2+4x+3}$

23. $\dfrac{x^2+12x+12}{x^3-4x}$ **24.** $\dfrac{x+2}{x(x-4)}$

25. $\dfrac{4x^2+2x-1}{x^2(x+1)}$ **26.** $\dfrac{2x-3}{(x-1)^2}$

27. $\dfrac{3x}{(x-3)^2}$ **28.** $\dfrac{6x^2+1}{x^2(x-1)^2}$

29. $\dfrac{x^2-1}{x(x^2+1)}$ **30.** $\dfrac{x}{(x-1)(x^2+x+1)}$

31. $\dfrac{x}{x^3-x^2-2x+2}$ **32.** $\dfrac{x+6}{x^3-3x^2-4x+12}$

33. $\dfrac{x^2}{x^4-2x^2-8}$ **34.** $\dfrac{2x^2+x+8}{(x^2+4)^2}$

35. $\dfrac{x}{16x^4-1}$ **36.** $\dfrac{x+1}{x^3+x}$

37. $\dfrac{x^2+5}{(x+1)(x^2-2x+3)}$ **38.** $\dfrac{x^2-4x+7}{(x+1)(x^2-2x+3)}$

In Exercises 39–44, write the partial fraction decomposition of the improper rational expression.

39. $\dfrac{x^2-x}{x^2+x+1}$ **40.** $\dfrac{x^2-4x}{x^2+x+6}$

41. $\dfrac{2x^3-x^2+x+5}{x^2+3x+2}$ **42.** $\dfrac{x^3+2x^2-x+1}{x^2+3x-4}$

43. $\dfrac{x^4}{(x-1)^3}$ **44.** $\dfrac{16x^4}{(2x-1)^3}$

In Exercises 45–52, write the partial fraction decomposition of the rational expression. Use a graphing utility to check your result graphically.

45. $\dfrac{5 - x}{2x^2 + x - 1}$ **46.** $\dfrac{3x^2 - 7x - 2}{x^3 - x}$

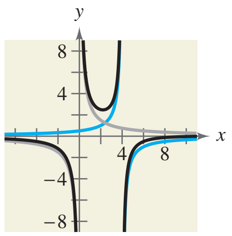
47. $\dfrac{x - 1}{x^3 + x^2}$ **48.** $\dfrac{4x^2 - 1}{2x(x + 1)^2}$

49. $\dfrac{x^2 + x + 2}{(x^2 + 2)^2}$ **50.** $\dfrac{x^3}{(x + 2)^2(x - 2)^2}$

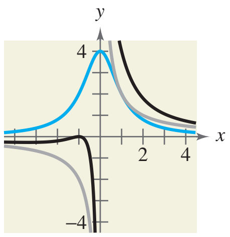
51. $\dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}$ **52.** $\dfrac{x^3 - x + 3}{x^2 + x - 2}$

Graphical Analysis **In Exercises 53–56, (a) write the partial fraction decomposition of the rational function, (b) identify the graph of the rational function and the graph of each term of its decomposition, and (c) state any relationship between the vertical asymptotes of the graph of the rational function and the vertical asymptotes of the graphs of the terms of the decomposition. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

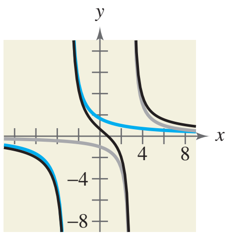
53. $y = \dfrac{x - 12}{x(x - 4)}$



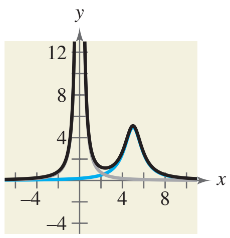
54. $y = \dfrac{2(x + 1)^2}{x(x^2 + 1)}$



55. $y = \dfrac{2(4x - 3)}{x^2 - 9}$



56. $y = \dfrac{2(4x^2 - 15x + 39)}{x^2(x^2 - 10x + 26)}$



Model It

57. ***Thermodynamics*** The magnitude of the range R of exhaust temperatures (in degrees Fahrenheit) in an experimental diesel engine is approximated by the model

$$R = \frac{2000(4 - 3x)}{(11 - 7x)(7 - 4x)}, \quad 0 < x \leq 1$$

Model It (continued)

where x is the relative load (in foot-pounds).

(a) Write the partial fraction decomposition of the equation.

(b) The decomposition in part (a) is the difference of two fractions. The absolute values of the terms give the expected maximum and minimum temperatures of the exhaust gases for different loads.

Ymax = |1st term| Ymin = |2nd term|

Write the equations for Ymax and Ymin.

(c) Use a graphing utility to graph each equation from part (b) in the same viewing window.

(d) Determine the expected maximum and minimum temperatures for a relative load of 0.5.

Synthesis

58. ***Writing*** Describe two ways of solving for the constants in a partial fraction decomposition.

True or False? **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

59. For the rational expression $\dfrac{x}{(x + 10)(x - 10)^2}$ the partial fraction decomposition is of the form $\dfrac{A}{x + 10} + \dfrac{B}{(x - 10)^2}$.

60. When writing the partial fraction decomposition of the expression $\dfrac{x^3 + x - 2}{x^2 - 5x - 14}$ the first step is to factor the denominator.

In Exercises 61–64, write the partial fraction decomposition of the rational expression. Check your result algebraically. Then assign a value to the constant *a* to check the result graphically.

61. $\dfrac{1}{a^2 - x^2}$ **62.** $\dfrac{1}{x(x + a)}$

63. $\dfrac{1}{y(a - y)}$ **64.** $\dfrac{1}{(x + 1)(a - x)}$

Skills Review

In Exercises 65–70, sketch the graph of the function.

65. $f(x) = x^2 - 9x + 18$ **66.** $f(x) = 2x^2 - 9x - 5$

67. $f(x) = -x^2(x - 3)$ **68.** $f(x) = \frac{1}{2}x^3 - 1$

69. $f(x) = \dfrac{x^2 + x - 6}{x + 5}$ **70.** $f(x) = \dfrac{3x - 1}{x^2 + 4x - 12}$

7.5 Systems of Inequalities

What you should learn

- Sketch the graphs of inequalities in two variables.
- Solve systems of inequalities.
- Use systems of inequalities in two variables to model and solve real-life problems.

Why you should learn it

You can use systems of inequalities in two variables to model and solve real-life problems. For instance, in Exercise 77 on page 550, you will use a system of inequalities to analyze the retail sales of prescription drugs.

Jon Feingersh/Masterfile

The Graph of an Inequality

The statements $3x - 2y < 6$ and $2x^2 + 3y^2 \geq 6$ are inequalities in two variables. An ordered pair (a, b) is a **solution of an inequality** in x and y if the inequality is true when a and b are substituted for x and y, respectively. The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the *corresponding equation*. The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true.

1. *All* points in the region are solutions of the inequality.
2. *No* point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by simply testing *one* point in the region.

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign, and sketch the graph of the resulting equation. (Use a dashed line for $<$ or $>$ and a solid line for $\leq$ or $\geq$.)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, shade the entire region to denote that every point in the region satisfies the inequality.

Example 1 Sketching the Graph of an Inequality

To sketch the graph of $y \geq x^2 - 1$, begin by graphing the corresponding equation $y = x^2 - 1$, which is a parabola, as shown in Figure 7.19. By testing a point *above* the parabola $(0, 0)$ and a point *below* the parabola $(0, -2)$, you can see that the points that satisfy the inequality are those lying above (or on) the parabola.

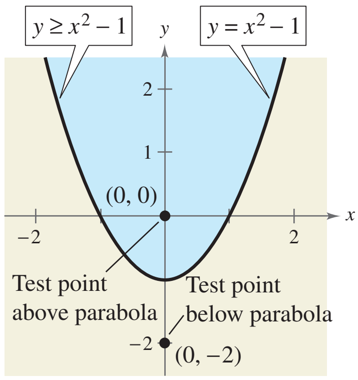


FIGURE 7.19

CHECKPOINT Now try Exercise 1.

STUDY TIP

Note that when sketching the graph of an inequality in two variables, a dashed line means all points on the line or curve *are not* solutions of the inequality. A solid line means all points on the line or curve *are* solutions of the inequality.

The inequality in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve **linear inequalities** such as $ax + by < c$ (a and b are not both zero). The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$.

Example 2 Sketching the Graph of a Linear Inequality

Sketch the graph of each linear inequality.

a. $x > -2$ **b.** $y \leq 3$

Solution

a. The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of this line, as shown in Figure 7.20.

b. The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \leq 3$ are those lying below (or on) this line, as shown in Figure 7.21.

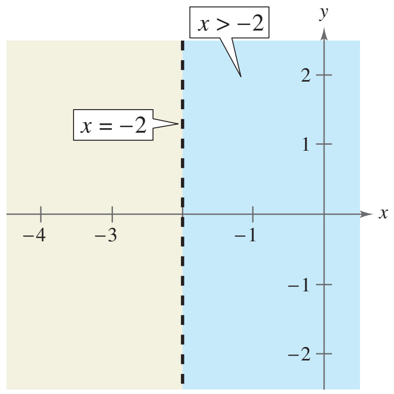


FIGURE 7.20

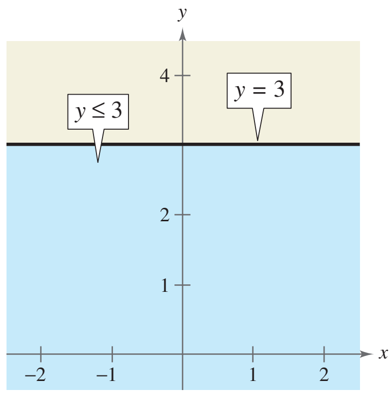
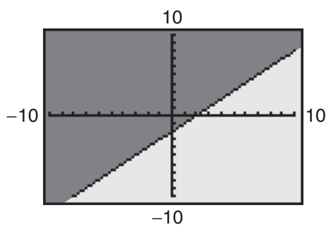


FIGURE 7.21

CHECKPOINT Now try Exercise 3.

Technology

A graphing utility can be used to graph an inequality or a system of inequalities. For instance, to graph $y \geq x - 2$, enter $y = x - 2$ and use the *shade* feature of the graphing utility to shade the correct part of the graph. You should obtain the graph below. Consult the user's guide for your graphing utility for specific keystrokes.



Example 3 Sketching the Graph of a Linear Inequality

Sketch the graph of $x - y < 2$.

Solution

The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 7.22. Because the origin $(0, 0)$ satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

CHECKPOINT Now try Exercise 9.

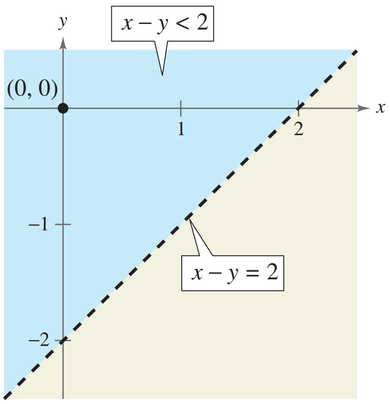


FIGURE 7.22

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing $x - y < 2$ in the form

$$y > x - 2$$

you can see that the solution points lie *above* the line $x - y = 2$ (or $y = x - 2$), as shown in Figure 7.22.

Systems of Inequalities

Many practical problems in business, science, and engineering involve systems of linear inequalities. A **solution** of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. This region represents the **solution set** of the system. For systems of *linear inequalities*, it is helpful to find the vertices of the solution region.

Example 4 Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system.

$$\begin{cases} x - y < 2 & \text{Inequality 1} \\ x > -2 & \text{Inequality 2} \\ y \le 3 & \text{Inequality 3} \end{cases}$$

Solution

The graphs of these inequalities are shown in Figures 7.22, 7.20, and 7.21, respectively, on page 542. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown in Figure 7.23. To find the vertices of the region, solve the three systems of corresponding equations obtained by taking *pairs* of equations representing the boundaries of the individual regions.

STUDY TIP

Using different colored pencils to shade the solution of each inequality in a system will make identifying the solution of the system of inequalities easier.

Vertex A: $(-2, -4)$
$$\begin{cases} x - y = 2 \\ x = -2 \end{cases}$$

Vertex B: $(5, 3)$
$$\begin{cases} x - y = 2 \\ y = 3 \end{cases}$$

Vertex C: $(-2, 3)$
$$\begin{cases} x = -2 \\ y = 3 \end{cases}$$

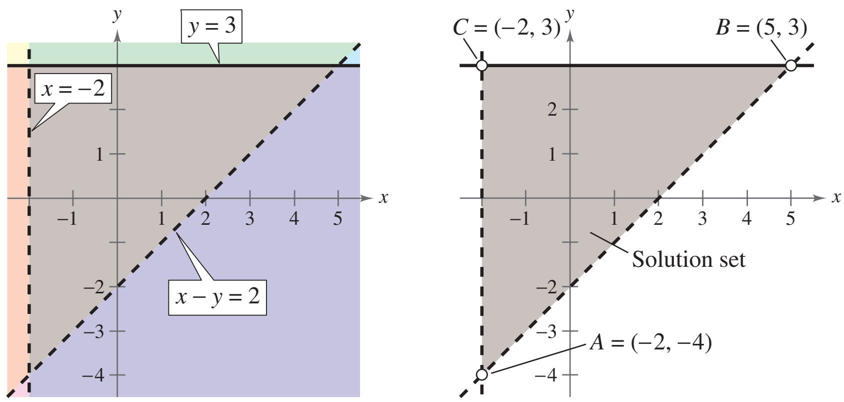


FIGURE 7.23

Note in Figure 7.23 that the vertices of the region are represented by open dots. This means that the vertices *are not* solutions of the system of inequalities.

CHECKPOINT Now try Exercise 35.

For the triangular region shown in Figure 7.23, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 7.24. To keep track of which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.

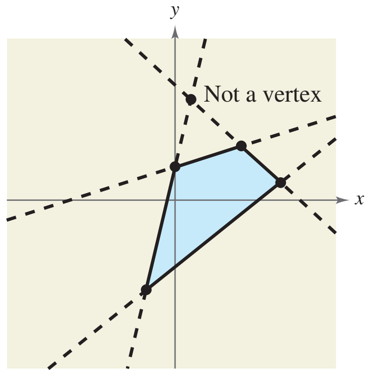


FIGURE 7.24

Example 5 Solving a System of Inequalities

Sketch the region containing all points that satisfy the system of inequalities.

$$\begin{cases} x^2 - y \le 1 \\ -x + y \le 1 \end{cases}$$ Inequality 1 / Inequality 2

Solution

As shown in Figure 7.25, the points that satisfy the inequality

$x^2 - y \le 1$ Inequality 1

are the points lying above (or on) the parabola given by

$y = x^2 - 1.$ Parabola

The points satisfying the inequality

$-x + y \le 1$ Inequality 2

are the points lying below (or on) the line given by

$y = x + 1.$ Line

To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

$$\begin{cases} x^2 - y = 1 \\ -x + y = 1 \end{cases}$$

Using the method of substitution, you can find the solutions to be $(-1, 0)$ and $(2, 3)$. So, the region containing all points that satisfy the system is indicated by the shaded region in Figure 7.25.

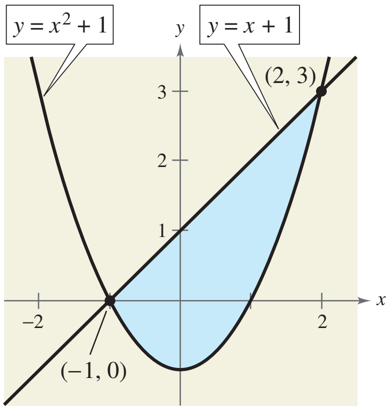


FIGURE 7.25

CHECKPOINT Now try Exercise 37.

When solving a system of inequalities, you should be aware that the system might have no solution *or* it might be represented by an unbounded region in the plane. These two possibilities are shown in Examples 6 and 7.

Example 6 A System with No Solution

Sketch the solution set of the system of inequalities.

$$\begin{cases} x + y > 3 & \text{Inequality 1} \\ x + y < -1 & \text{Inequality 2} \end{cases}$$

Solution

From the way the system is written, it is clear that the system has no solution, because the quantity $(x + y)$ cannot be both less than -1 and greater than 3. Graphically, the inequality $x + y > 3$ is represented by the half-plane lying above the line $x + y = 3$, and the inequality $x + y < -1$ is represented by the half-plane lying below the line $x + y = -1$, as shown in Figure 7.26. These two half-planes have no points in common. So, the system of inequalities has no solution.

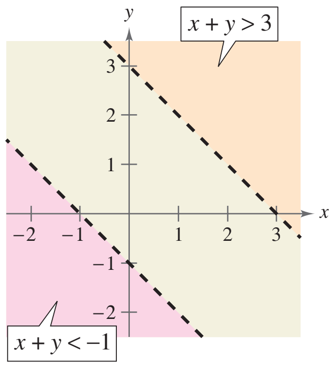


FIGURE 7.26

CHECKPOINT Now try Exercise 39.

Example 7 An Unbounded Solution Set

Sketch the solution set of the system of inequalities.

$$\begin{cases} x + y < 3 & \text{Inequality 1} \\ x + 2y > 3 & \text{Inequality 2} \end{cases}$$

Solution

The graph of the inequality $x + y < 3$ is the half-plane that lies below the line $x + y = 3$, as shown in Figure 7.27. The graph of the inequality $x + 2y > 3$ is the half-plane that lies above the line $x + 2y = 3$. The intersection of these two half-planes is an *infinite wedge* that has a vertex at $(3, 0)$. So, the solution set of the system of inequalities is unbounded.

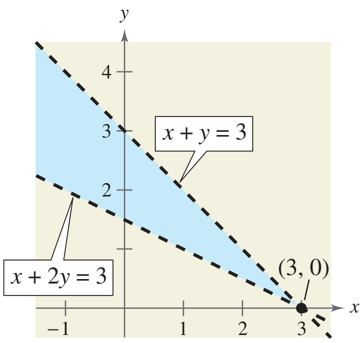


FIGURE 7.27

CHECKPOINT Now try Exercise 41.

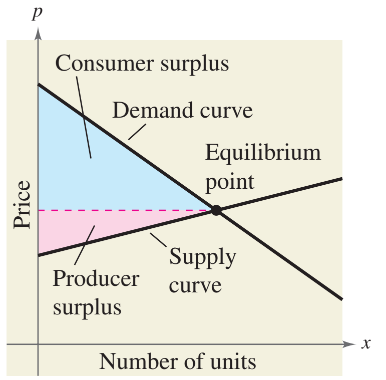


FIGURE 7.28

Applications

Example 9 in Section 7.2 discussed the *equilibrium point* for a system of demand and supply functions. The next example discusses two related concepts that economists call **consumer surplus** and **producer surplus.** As shown in Figure 7.28, the consumer surplus is defined as the area of the region that lies *below* the demand curve, *above* the horizontal line passing through the equilibrium point, and to the right of the p-axis. Similarly, the producer surplus is defined as the area of the region that lies *above* the supply curve, *below* the horizontal line passing through the equilibrium point, and to the right of the p-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay *above what they actually paid*, whereas the producer surplus is a measure of the amount that producers would have been willing to receive *below what they actually received*.

Example 8 Consumer Surplus and Producer Surplus

The demand and supply functions for a new type of personal digital assistant are given by

$$\begin{cases} p = 150 - 0.00001x & \text{Demand equation} \\ p = 60 + 0.00002x & \text{Supply equation} \end{cases}$$

where p is the price (in dollars) and x represents the number of units. Find the consumer surplus and producer surplus for these two equations.

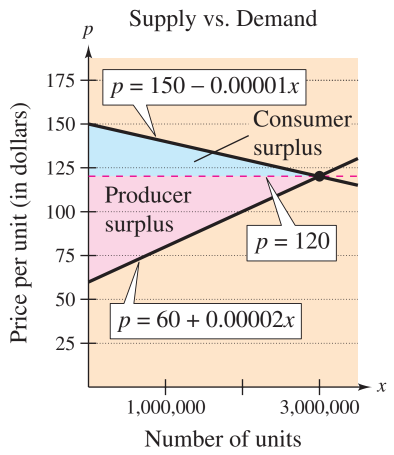


FIGURE 7.29

Solution

Begin by finding the equilibrium point (when supply and demand are equal) by solving the equation

$$60 + 0.00002x = 150 - 0.00001x.$$

In Example 9 in Section 7.2, you saw that the solution is $x = 3{,}000{,}000$ units, which corresponds to an equilibrium price of $p = \$120$. So, the consumer surplus and producer surplus are the areas of the following triangular regions.

Consumer Surplus

$$\begin{cases} p \le 150 - 0.00001x \\ p \ge 120 \\ x \ge 0 \end{cases}$$

Producer Surplus

$$\begin{cases} p \ge 60 + 0.00002x \\ p \le 120 \\ x \ge 0 \end{cases}$$

In Figure 7.29, you can see that the consumer and producer surpluses are defined as the areas of the shaded triangles.

$$\text{Consumer surplus} = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(3{,}000{,}000)(30) = \$45{,}000{,}000$$

$$\text{Producer surplus} = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(3{,}000{,}000)(60) = \$90{,}000{,}000$$

CHECKPOINT Now try Exercise 65.

Example 9 **Nutrition**

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes how many cups of each drink should be consumed each day to meet or exceed the minimum daily requirements for calories and vitamins.

Solution

Begin by letting x and y represent the following.

$x =$ number of cups of dietary drink X

$y =$ number of cups of dietary drink Y

To meet or exceed the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{cases} 60x + 60y \geq 300 & \text{Calories} \\ 12x + 6y \geq 36 & \text{Vitamin A} \\ 10x + 30y \geq 90 & \text{Vitamin C} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of inequalities is shown in Figure 7.30. (More is said about this application in Example 6 in Section 7.6.)

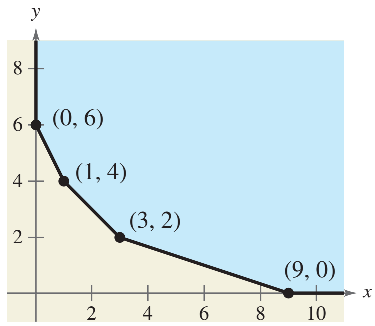


FIGURE 7.30

CHECKPOINT Now try Exercise 69.

WRITING ABOUT MATHEMATICS

Creating a System of Inequalities Plot the points (0, 0), (4, 0), (3, 2), and (0, 2) in a coordinate plane. Draw the quadrilateral that has these four points as its vertices. Write a system of linear inequalities that has the quadrilateral as its solution. Explain how you found the system of inequalities.

7.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. An ordered pair (a, b) is a ________ of an inequality in x and y if the inequality is true when a and b are substituted for x and y, respectively.
2. The ________ of an inequality is the collection of all solutions of the inequality.
3. The graph of a ________ inequality is a half-plane lying on one side of the line $ax + by = c$.
4. A ________ of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system.
5. The area of the region that lies below the demand curve, above the horizontal line passing through the equilibrium point, to the right of the p-axis is called the ________ ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

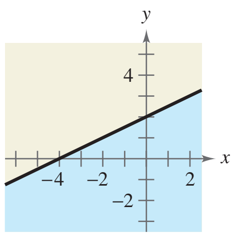
In Exercises 1–14, sketch the graph of the inequality.

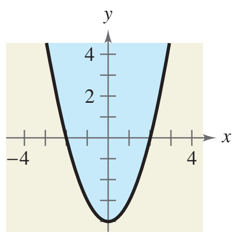
1. $y < 2 - x^2$
2. $y^2 - x < 0$
3. $x \geq 2$
4. $x \leq 4$
5. $y \geq -1$
6. $y \leq 3$
7. $y < 2 - x$
8. $y > 2x - 4$
9. $2y - x \geq 4$
10. $5x + 3y \geq -15$
11. $(x + 1)^2 + (y - 2)^2 < 9$
12. $(x - 1)^2 + (y - 4)^2 > 9$
13. $y \leq \dfrac{1}{1 + x^2}$
14. $y > \dfrac{-15}{x^2 + x + 4}$

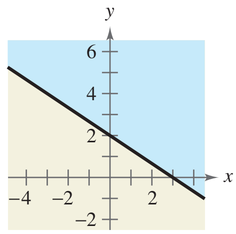
In Exercises 15–26, use a graphing utility to graph the inequality. Shade the region representing the solution.

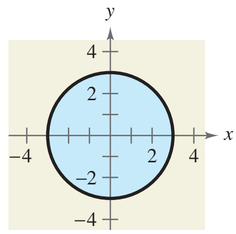
15. $y < \ln x$
16. $y \geq 6 - \ln(x + 5)$
17. $y < 3^{-x-4}$
18. $y \leq 2^{2x-0.5} - 7$
19. $y \geq \frac{2}{3}x - 1$
20. $y \leq 6 - \frac{3}{2}x$
21. $y < -3.8x + 1.1$
22. $y \geq -20.74 + 2.66x$
23. $x^2 + 5y - 10 \leq 0$
24. $2x^2 - y - 3 > 0$
25. $\frac{5}{2}y - 3x^2 - 6 \geq 0$
26. $-\frac{1}{10}x^2 - \frac{3}{8}y < -\frac{1}{4}$

In Exercises 27–30, write an inequality for the shaded region shown in the figure.

27.


28.


29.


30.


In Exercises 31–34, determine whether each ordered pair is a solution of the system of linear inequalities.

31. $\begin{cases} x \geq -4 \\ y > -3 \\ y \leq -8x - 3 \end{cases}$ (a) $(0, 0)$ (b) $(-1, -3)$ (c) $(-4, 0)$ (d) $(-3, 11)$

32. $\begin{cases} -2x + 5y \geq 3 \\ y < 4 \\ -4x + 2y < 7 \end{cases}$ (a) $(0, 2)$ (b) $(-6, 4)$ (c) $(-8, -2)$ (d) $(-3, 2)$

33. $\begin{cases} 3x + y > 1 \\ -y - \frac{1}{2}x^2 \leq -4 \\ -15x + 4y > 0 \end{cases}$ (a) $(0, 10)$ (b) $(0, -1)$ (c) $(2, 9)$ (d) $(-1, 6)$

34. $\begin{cases} x^2 + y^2 \geq 36 \\ -3x + y \leq 10 \\ \frac{2}{3}x - y \geq 5 \end{cases}$ (a) $(-1, 7)$ (b) $(-5, 1)$ (c) $(6, 0)$ (d) $(4, -8)$

In Exercises 35–48, sketch the graph and label the vertices of the solution set of the system of inequalities.

35. $\begin{cases} x + y \leq 1 \\ -x + y \leq 1 \\ y \geq 0 \end{cases}$

36. $\begin{cases} 3x + 2y < 6 \\ x > 0 \\ y > 0 \end{cases}$

37. $\begin{cases} x^2 + y \leq 5 \\ x \geq -1 \\ y \geq 0 \end{cases}$

38. $\begin{cases} 2x^2 + y \geq 2 \\ x \leq 2 \\ y \leq 1 \end{cases}$

39. $\begin{cases} 2x + y > 2 \\ 6x + 3y < 2 \end{cases}$

40. $\begin{cases} x - 7y > -36 \\ 5x + 2y > 5 \\ 6x - 5y > 6 \end{cases}$

41. $\begin{cases} -3x + 2y < 6 \\ x - 4y > -2 \\ 2x + y < 3 \end{cases}$

42. $\begin{cases} x - 2y < -6 \\ 5x - 3y > -9 \end{cases}$

43. $\begin{cases} x > y^2 \\ x < y + 2 \end{cases}$

44. $\begin{cases} x - y^2 > 0 \\ x - y > 2 \end{cases}$

45. $\begin{cases} x^2 + y^2 \le 9 \\ x^2 + y^2 \ge 1 \end{cases}$

46. $\begin{cases} x^2 + y^2 \le 25 \\ 4x - 3y \le 0 \end{cases}$

47. $\begin{cases} 3x + 4 \ge y^2 \\ x - y < 0 \end{cases}$

48. $\begin{cases} x < 2y - y^2 \\ 0 < x + y \end{cases}$

In Exercises 49–54, use a graphing utility to graph the inequalities. Shade the region representing the solution set of the system.

49. $\begin{cases} y \le \sqrt{3x} + 1 \\ y \ge x^2 + 1 \end{cases}$

50. $\begin{cases} y < -x^2 + 2x + 3 \\ y > x^2 - 4x + 3 \end{cases}$

51. $\begin{cases} y < x^3 - 2x + 1 \\ y > -2x \\ x \le 1 \end{cases}$

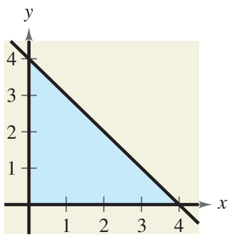
52. $\begin{cases} y \ge x^4 - 2x^2 + 1 \\ y \le 1 - x^2 \end{cases}$

53. $\begin{cases} x^2 y \ge 1 \\ 0 < x \le 4 \\ y \le 4 \end{cases}$

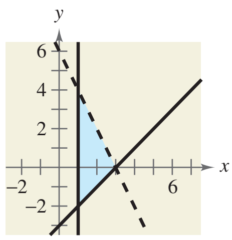
54. $\begin{cases} y \le e^{-x^2/2} \\ y \ge 0 \\ -2 \le x \le 2 \end{cases}$

In Exercises 55–64, derive a set of inequalities to describe the region.

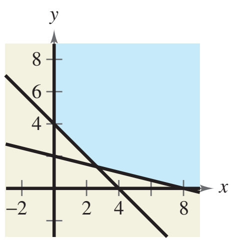
55.



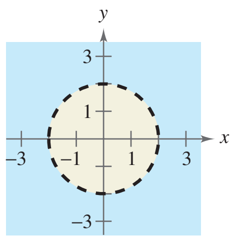
56.



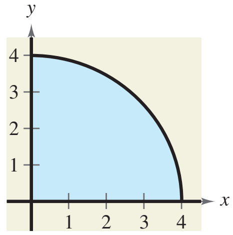
57.



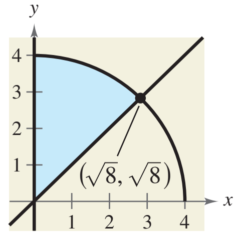
58.



59.



60.



61. Rectangle: vertices at $(2, 1)$, $(5, 1)$, $(5, 7)$, $(2, 7)$

62. Parallelogram: vertices at $(0, 0)$, $(4, 0)$, $(1, 4)$, $(5, 4)$

63. Triangle: vertices at $(0, 0)$, $(5, 0)$, $(2, 3)$

64. Triangle: vertices at $(-1, 0)$, $(1, 0)$, $(0, 1)$

Supply and Demand **In Exercises 65–68, (a) graph the systems representing the consumer surplus and producer surplus for the supply and demand equations and (b) find the consumer surplus and producer surplus.**

	Demand	*Supply*
65.	$p = 50 - 0.5x$	$p = 0.125x$
66.	$p = 100 - 0.05x$	$p = 25 + 0.1x$
67.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
68.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

69. ***Production*** A furniture company can sell all the tables and chairs it produces. Each table requires 1 hour in the assembly center and $1\frac{1}{3}$ hours in the finishing center. Each chair requires $1\frac{1}{2}$ hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. The company's assembly center is available 12 hours per day, and its finishing center is available 15 hours per day. Find and graph a system of inequalities describing all possible production levels.

70. ***Inventory*** A store sells two models of computers. Because of the demand, the store stocks at least twice as many units of model A as of model B. The costs to the store for the two models are \$800 and \$1200, respectively. The management does not want more than \$20,000 in computer inventory at any one time, and it wants at least four model A computers and two model B computers in inventory at all times. Find and graph a system of inequalities describing all possible inventory levels.

71. ***Investment Analysis*** A person plans to invest up to \$20,000 in two different interest-bearing accounts. Each account is to contain at least \$5000. Moreover, the amount in one account should be at least twice the amount in the other account. Find and graph a system of inequalities to describe the various amounts that can be deposited in each account.

72. ***Ticket Sales*** For a concert event, there are \$30 reserved seat tickets and \$20 general admission tickets. There are 2000 reserved seats available, and fire regulations limit the number of paid ticket holders to 3000. The promoter must take in at least \$75,000 in ticket sales. Find and graph a system of inequalities describing the different numbers of tickets that can be sold.

73. ***Shipping*** A warehouse supervisor is told to ship at least 50 packages of gravel that weigh 55 pounds each and at least 40 bags of stone that weigh 70 pounds each. The maximum weight capacity in the truck he is loading is 7500 pounds. Find and graph a system of inequalities describing the numbers of bags of stone and gravel that he can send.

74. ***Truck Scheduling*** A small company that manufactures two models of exercise machines has an order for 15 units of the standard model and 16 units of the deluxe model. The company has trucks of two different sizes that can haul the products, as shown in the table.

Truck	Standard	Deluxe
Large	6	3
Medium	4	6

Find and graph a system of inequalities describing the numbers of trucks of each size that are needed to deliver the order.

75. ***Nutrition*** A dietitian is asked to design a special dietary supplement using two different foods. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B. The minimum daily requirements of the diet are 300 units of calcium, 150 units of iron, and 200 units of vitamin B.

(a) Write a system of inequalities describing the different amounts of food X and food Y that can be used.

(b) Sketch a graph of the region corresponding to the system in part (a).

(c) Find two solutions of the system and interpret their meanings in the context of the problem.

76. ***Health*** A person's maximum heart rate is $220 - x$, where x is the person's age in years for $20 \le x \le 70$. When a person exercises, it is recommended that the person strive for a heart rate that is at least 50% of the maximum and at most 75% of the maximum. (Source: American Heart Association)

(a) Write a system of inequalities that describes the exercise target heart rate region.

(b) Sketch a graph of the region in part (a).

(c) Find two solutions to the system and interpret their meanings in the context of the problem.

Model It

77. ***Data Analysis: Prescription Drugs*** The table shows the retail sales y (in billions of dollars) of prescription drugs in the United States from 1999 to 2003. (Source: National Association of Chain Drug Stores)

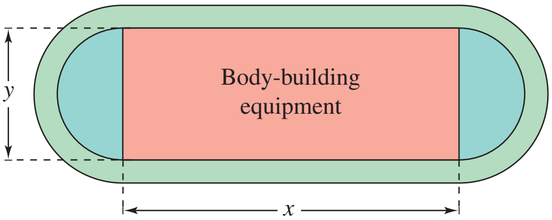
Year	Retail sales, y
1999	125.8
2000	145.6
2001	164.1
2002	182.7
2003	203.1

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(b) The total retail sales of prescription drugs in the United States during this five-year period can be approximated by finding the area of the trapezoid bounded by the linear model you found in part (a) and the lines $y = 0$, $t = 8.5$, and $t = 13.5$. Use a graphing utility to graph this region.

(c) Use the formula for the area of a trapezoid to approximate the total retail sales of prescription drugs.

78. ***Physical Fitness Facility*** An indoor running track is to be constructed with a space for body-building equipment inside the track (see figure). The track must be at least 125 meters long, and the body-building space must have an area of at least 500 square meters.



(a) Find a system of inequalities describing the requirements of the facility.

(b) Graph the system from part (a).

Synthesis

True or False? **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

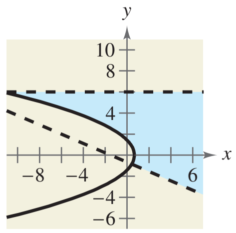
79. The area of the figure defined by the system

$$\begin{cases} x \geq -3 \\ x \leq 6 \\ y \leq 5 \\ y \geq -6 \end{cases}$$

is 99 square units.

80. The graph below shows the solution of the system

$$\begin{cases} y \leq 6 \\ -4x - 9y > 6. \\ 3x + y^2 \geq 2 \end{cases}$$



81. ***Writing*** Explain the difference between the graphs of the inequality $x \leq 4$ on the real number line and on the rectangular coordinate system.

82. ***Think About It*** After graphing the boundary of an inequality in x and y, how do you decide on which side of the boundary the solution set of the inequality lies?

83. ***Graphical Reasoning*** Two concentric circles have radii x and y, where $y > x$. The area between the circles must be at least 10 square units.

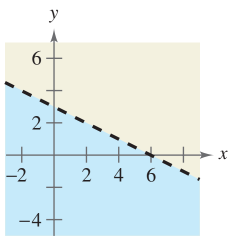
(a) Find a system of inequalities describing the constraints on the circles.

(b) Use a graphing utility to graph the system of inequalities in part (a). Graph the line $y = x$ in the same viewing window.

(c) Identify the graph of the line in relation to the boundary of the inequality. Explain its meaning in the context of the problem.

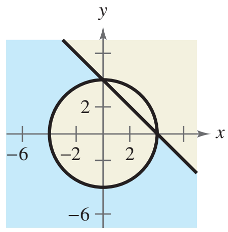
84. The graph of the solution of the inequality $x + 2y < 6$ is shown in the figure. Describe how the solution set would change for each of the following.

(a) $x + 2y \leq 6$ (b) $x + 2y > 6$

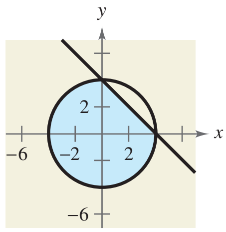


In Exercises 85–88, match the system of inequalities with the graph of its solution. [The graphs are labeled (a), (b), (c), and (d).]

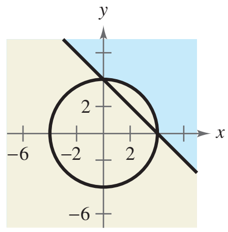
(a)



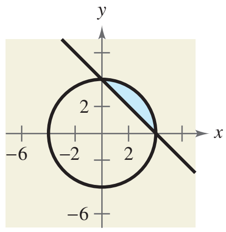
(b)



(c)



(d)



85. $\begin{cases} x^2 + y^2 \leq 16 \\ x + y \geq 4 \end{cases}$

86. $\begin{cases} x^2 + y^2 \leq 16 \\ x + y \leq 4 \end{cases}$

87. $\begin{cases} x^2 + y^2 \geq 16 \\ x + y \geq 4 \end{cases}$

88. $\begin{cases} x^2 + y^2 \geq 16 \\ x + y \leq 4 \end{cases}$

Skills Review

In Exercises 89–94, find the equation of the line passing through the two points.

89. $(-2, 6), (4, -4)$

90. $(-8, 0), (3, -1)$

91. $\left(\frac{3}{4}, -2\right), \left(-\frac{7}{2}, 5\right)$

92. $\left(-\frac{1}{2}, 0\right), \left(\frac{11}{2}, 12\right)$

93. $(3.4, -5.2), (-2.6, 0.8)$

94. $(-4.1, -3.8), (2.9, 8.2)$

95. ***Data Analysis: Cell Phone Bills*** The average monthly cell phone bills y (in dollars) in the United States from 1998 to 2003, where t is the year, are shown as data points (t, y). (Source: Cellular Telecommunications & Internet Association)

(1998, 39.43), (1999, 41.24), (2000, 45.27)
(2001, 47.37), (2002, 48.40), (2003, 49.91)

(a) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, and an exponential model for the data. Let $t = 8$ correspond to 1998.

(b) Use a graphing utility to plot the data and the models in the same viewing window.

(c) Which model is the best fit for the data?

(d) Use the model from part (c) to predict the average monthly cell phone bill in 2008.

7.6 Linear Programming

What you should learn

- Solve linear programming problems.
- Use linear programming to model and solve real-life problems.

Why you should learn it

Linear programming is often useful in making real-life economic decisions. For example, Exercise 44 on page 560 shows how you can determine the optimal cost of a blend of gasoline and compare it with the national average.

Tim Boyle/Getty Images

Linear Programming: A Graphical Approach

Many applications in business and economics involve a process called **optimization,** in which you are asked to find the minimum or maximum of a quantity. In this section, you will study an optimization strategy called **linear programming.**

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints.** The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions.** For example, suppose you are asked to maximize the value of

$z = ax + by$ Objective function

subject to a set of constraints that determines the shaded region in Figure 7.31.

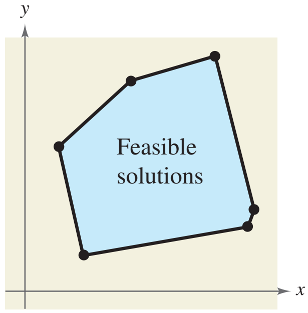


FIGURE 7.31

Because every point in the shaded region satisfies each constraint, it is not clear how you should find the point that yields a maximum value of z. Fortunately, it can be shown that if there is an optimal solution, it must occur at one of the vertices. This means that *you can find the maximum value of z by testing z at each of the vertices*.

Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions. If there is more than one solution, at least one of them must occur at such a vertex. In either case, the value of the objective function is unique.

Some guidelines for solving a linear programming problem in two variables are listed at the top of the next page.

Solving a Linear Programming Problem

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and a maximum value will exist. (For an unbounded region, *if* an optimal solution exists, it will occur at a vertex.)

Example 1 Solving a Linear Programming Problem

Find the maximum value of

$$z = 3x + 2y \qquad \text{Objective function}$$

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\} \qquad \text{Constraints}$$

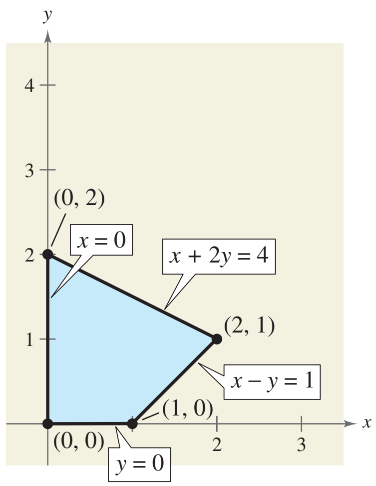


FIGURE 7.32

Solution

The constraints form the region shown in Figure 7.32. At the four vertices of this region, the objective function has the following values.

At $(0, 0)$: $z = 3(0) + 2(0) = 0$
At $(1, 0)$: $z = 3(1) + 2(0) = 3$
At $(2, 1)$: $z = 3(2) + 2(1) = 8$ Maximum value of z
At $(0, 2)$: $z = 3(0) + 2(2) = 4$

So, the maximum value of z is 8, and this occurs when $x = 2$ and $y = 1$.

CHECKPOINT Now try Exercise 5.

In Example 1, try testing some of the *interior* points in the region. You will see that the corresponding values of z are less than 8. Here are some examples.

At $(1, 1)$: $z = 3(1) + 2(1) = 5$ At $\left(\frac{1}{2}, \frac{3}{2}\right)$: $z = 3\left(\frac{1}{2}\right) + 2\left(\frac{3}{2}\right) = \frac{9}{2}$

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in slope-intercept form

$$y = -\frac{3}{2}x + \frac{z}{2} \qquad \text{Family of lines}$$

where $z/2$ is the y-intercept of the objective function. This equation represents a family of lines, each of slope $-\frac{3}{2}$. Of these infinitely many lines, you want the one that has the largest z-value while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-\frac{3}{2}$, you want the one that has the largest y-intercept *and* intersects the given region, as shown in Figure 7.33. From the graph you can see that such a line will pass through one (or more) of the vertices of the region.

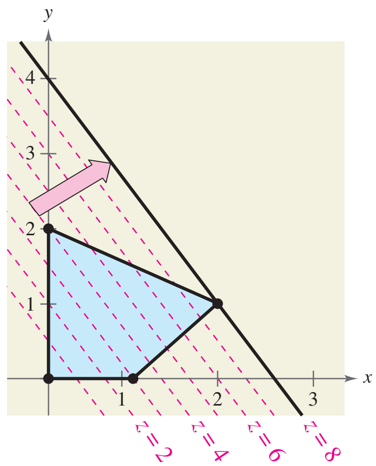


FIGURE 7.33

The next example shows that the same basic procedure can be used to solve a problem in which the objective function is to be *minimized*.

Example 2 Minimizing an Objective Function

Find the minimum value of

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27 \end{aligned}\right\} \qquad \text{Constraints}$$

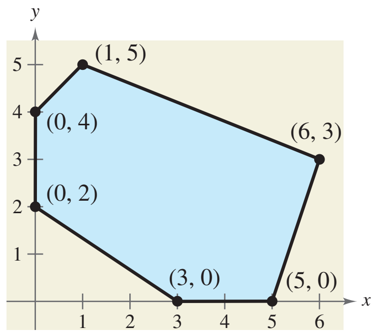


FIGURE 7.34

Solution

The region bounded by the constraints is shown in Figure 7.34. By testing the objective function at each vertex, you obtain the following.

At $(0, 2)$: $z = 5(0) + 7(2) = 14$ Minimum value of z

At $(0, 4)$: $z = 5(0) + 7(4) = 28$

At $(1, 5)$: $z = 5(1) + 7(5) = 40$

At $(6, 3)$: $z = 5(6) + 7(3) = 51$

At $(5, 0)$: $z = 5(5) + 7(0) = 25$

At $(3, 0)$: $z = 5(3) + 7(0) = 15$

So, the minimum value of z is 14, and this occurs when $x = 0$ and $y = 2$.

CHECKPOINT Now try Exercise 13.

Edward W. Souza/News Service/Stanford University

Historical Note

George Dantzig (1914–) was the first to propose the simplex method, or linear programming, in 1947. This technique defined the steps needed to find the optimal solution to a complex multivariable problem.

Example 3 Maximizing an Objective Function

Find the maximum value of

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27 \end{aligned}\right\} \qquad \text{Constraints}$$

Solution

This linear programming problem is identical to that given in Example 2 above, *except* that the objective function is maximized instead of minimized. Using the values of z at the vertices shown above, you can conclude that the maximum value of z is

$$z = 5(6) + 7(3) = 51$$

and occurs when $x = 6$ and $y = 3$.

CHECKPOINT Now try Exercise 15.

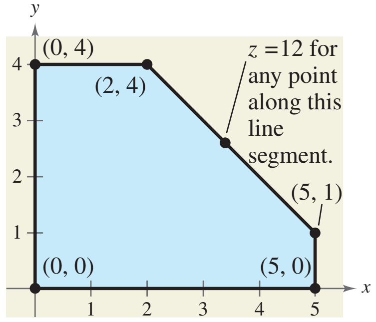


FIGURE 7.35

It is possible for the maximum (or minimum) value in a linear programming problem to occur at *two* different vertices. For instance, at the vertices of the region shown in Figure 7.35, the objective function

$$z = 2x + 2y \qquad \text{Objective function}$$

has the following values.

$$\begin{aligned} &\text{At } (0, 0): && z = 2(0) + 2(0) = 0 \\ &\text{At } (0, 4): && z = 2(0) + 2(4) = 8 \\ &\text{At } (2, 4): && z = 2(2) + 2(4) = 12 && \text{Maximum value of } z \\ &\text{At } (5, 1): && z = 2(5) + 2(1) = 12 && \text{Maximum value of } z \\ &\text{At } (5, 0): && z = 2(5) + 2(0) = 10 \end{aligned}$$

In this case, you can conclude that the objective function has a maximum value not only at the vertices (2, 4) and (5, 1); it also has a maximum value (of 12) at *any point on the line segment connecting these two vertices*. Note that the objective function in slope-intercept form $y = -x + \frac{1}{2}z$ has the same slope as the line through the vertices (2, 4) and (5, 1).

Some linear programming problems have no optimal solutions. This can occur if the region determined by the constraints is *unbounded*. Example 4 illustrates such a problem.

Example 4 An Unbounded Region

Find the maximum value of

$$z = 4x + 2y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} x + 2y &\geq 4 \\ 3x + y &\geq 7 \\ -x + 2y &\leq 7 \end{aligned}\right\} \qquad \text{Constraints}$$

Solution

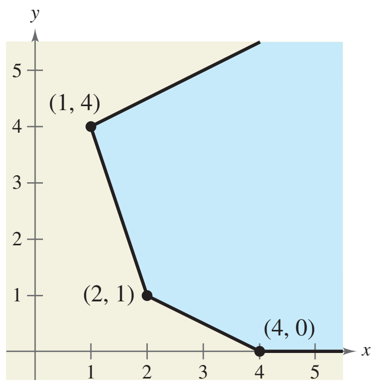


FIGURE 7.36

The region determined by the constraints is shown in Figure 7.36. For this unbounded region, there is no maximum value of z. To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. Substituting this point into the objective function, you get

$$z = 4(x) + 2(0) = 4x.$$

By choosing x to be large, you can obtain values of z that are as large as you want. So, there is no maximum value of z. However, there *is* a minimum value of z.

$$\begin{aligned} &\text{At } (1, 4): && z = 4(1) + 2(4) = 12 \\ &\text{At } (2, 1): && z = 4(2) + 2(1) = 10 && \text{Minimum value of } z \\ &\text{At } (4, 0): && z = 4(4) + 2(0) = 16 \end{aligned}$$

So, the minimum value of z is 10, and this occurs when $x = 2$ and $y = 1$.

CHECKPOINT Now try Exercise 17.

Applications

Example 5 shows how linear programming can be used to find the maximum profit in a business application.

Example 5 Optimal Profit

A candy manufacturer wants to maximize the profit for two types of boxed chocolates. A box of chocolate covered creams yields a profit of \$1.50 per box, and a box of chocolate covered nuts yields a profit of \$2.00 per box. Market tests and available resources have indicated the following constraints.

1. The combined production level should not exceed 1200 boxes per month.
2. The demand for a box of chocolate covered nuts is no more than half the demand for a box of chocolate covered creams.
3. The production level for chocolate covered creams should be less than or equal to 600 boxes plus three times the production level for chocolate covered nuts.

Solution

Let x be the number of boxes of chocolate covered creams and let y be the number of boxes of chocolate covered nuts. So, the objective function (for the combined profit) is given by

$$P = 1.5x + 2y.$$ Objective function

The three constraints translate into the following linear inequalities.

1. $x + y \le 1200 \quad \Rightarrow \quad x + y \le 1200$

2. $y \le \frac{1}{2}x \quad \Rightarrow \quad -x + 2y \le 0$

3. $x \le 600 + 3y \quad \Rightarrow \quad x - 3y \le 600$

Because neither x nor y can be negative, you also have the two additional constraints of $x \ge 0$ and $y \ge 0$. Figure 7.37 shows the region determined by the constraints. To find the maximum profit, test the values of P at the vertices of the region.

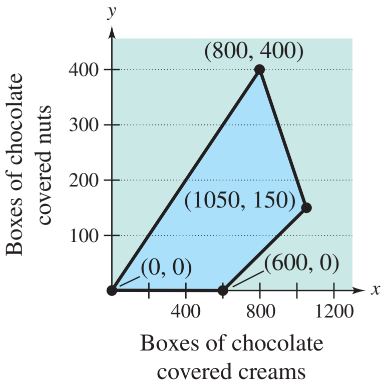


FIGURE 7.37

$$\begin{aligned}
\text{At } (0, 0)\text{:} \quad & P = 1.5(0) + 2(0) = 0 \\
\text{At } (800, 400)\text{:} \quad & P = 1.5(800) + 2(400) = 2000 \qquad \text{Maximum profit} \\
\text{At } (1050, 150)\text{:} \quad & P = 1.5(1050) + 2(150) = 1875 \\
\text{At } (600, 0)\text{:} \quad & P = 1.5(600) + 2(0) = 900
\end{aligned}$$

So, the maximum profit is \$2000, and it occurs when the monthly production consists of 800 boxes of chocolate covered creams and 400 boxes of chocolate covered nuts.

CHECKPOINT Now try Exercise 39.

In Example 5, if the manufacturer improved the production of chocolate covered creams so that they yielded a profit of \$2.50 per unit, the maximum profit could then be found using the objective function $P = 2.5x + 2y$. By testing the values of P at the vertices of the region, you would find that the maximum profit was \$2925 and that it occurred when $x = 1050$ and $y = 150$.

Example 6 Optimal Cost

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of dietary drink X costs $0.12 and provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y costs $0.15 and provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. How many cups of each drink should be consumed each day to obtain an optimal cost and still meet the daily requirements?

Solution

As in Example 9 in Section 7.5, let x be the number of cups of dietary drink X and let y be the number of cups of dietary drink Y.

$$\left.\begin{array}{lrl} \text{For calories:} & 60x + 60y \geq & 300 \\ \text{For vitamin A:} & 12x + 6y \geq & 36 \\ \text{For vitamin C:} & 10x + 30y \geq & 90 \\ & x \geq & 0 \\ & y \geq & 0 \end{array}\right\} \quad \text{Constraints}$$

The cost C is given by $C = 0.12x + 0.15y$. Objective function

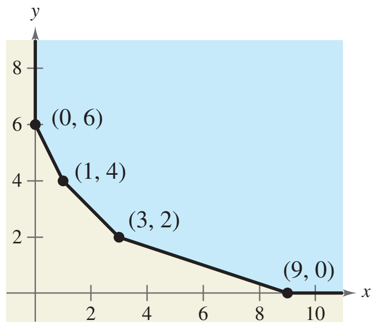


FIGURE 7.38

The graph of the region corresponding to the constraints is shown in Figure 7.38. Because you want to incur as little cost as possible, you want to determine the *minimum* cost. To determine the minimum cost, test C at each vertex of the region.

At $(0, 6)$: $C = 0.12(0) + 0.15(6) = 0.90$

At $(1, 4)$: $C = 0.12(1) + 0.15(4) = 0.72$

At $(3, 2)$: $C = 0.12(3) + 0.15(2) = 0.66$ Minimum value of C

At $(9, 0)$: $C = 0.12(9) + 0.15(0) = 1.08$

So, the minimum cost is $0.66 per day, and this occurs when 3 cups of drink X and 2 cups of drink Y are consumed each day.

CHECKPOINT Now try Exercise 43.

WRITING ABOUT MATHEMATICS

Creating a Linear Programming Problem Sketch the region determined by the following constraints.

$$\left.\begin{array}{r} x + 2y \leq 8 \\ x + y \leq 5 \\ x \geq 0 \\ y \geq 0 \end{array}\right\} \quad \text{Constraints}$$

Find, if possible, an objective function of the form $z = ax + by$ that has a maximum at each indicated vertex of the region.

a. $(0, 4)$ **b.** $(2, 3)$ **c.** $(5, 0)$ **d.** $(0, 0)$

Explain how you found each objective function.

7.6 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. In the process called ________, you are asked to find the maximum or minimum value of a quantity.
2. One type of optimization strategy is called ________ ________.
3. The ________ function of a linear programming problem gives the quantity that is to be maximized or minimized.
4. The ________ of a linear programming problem determine the set of ________ ________.
5. If a linear programming problem has a solution, it must occur at a ________ of the set of feasible solutions.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–12, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.)

1. Objective function:

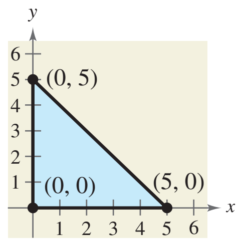
$z = 4x + 3y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \leq 5$



2. Objective function:

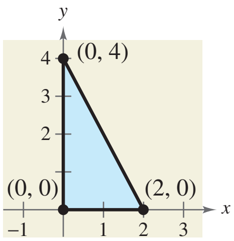
$z = 2x + 8y$

Constraints:

$x \geq 0$

$y \geq 0$

$2x + y \leq 4$



3. Objective function:

$z = 3x + 8y$

Constraints:
(See Exercise 1.)

4. Objective function:

$z = 7x + 3y$

Constraints:
(See Exercise 2.)

5. Objective function:

$z = 3x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 3y \leq 15$

$4x + y \leq 16$

6. Objective function:

$z = 4x + 5y$

Constraints:

$x \geq 0$

$2x + 3y \geq 6$

$3x - y \leq 9$

$x + 4y \leq 16$

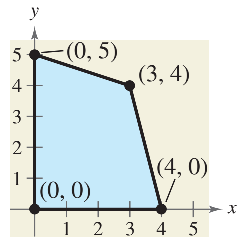


FIGURE FOR 5

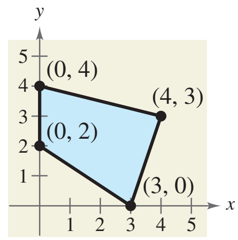


FIGURE FOR 6

7. Objective function:

$z = 5x + 0.5y$

Constraints:
(See Exercise 5.)

8. Objective function:

$z = 2x + y$

Constraints:
(See Exercise 6.)

9. Objective function:

$z = 10x + 7y$

Constraints:

$0 \leq x \leq 60$

$0 \leq y \leq 45$

$5x + 6y \leq 420$

10. Objective function:

$z = 25x + 35y$

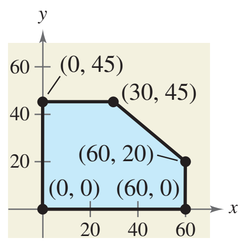
Constraints:

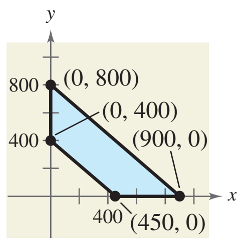
$x \geq 0$

$y \geq 0$

$8x + 9y \leq 7200$

$8x + 9y \geq 3600$





11. Objective function:

$z = 25x + 30y$

Constraints:
(See Exercise 9.)

12. Objective function:

$z = 15x + 20y$

Constraints:
(See Exercise 10.)

In Exercises 13–20, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints.

13. Objective function:
$z = 6x + 10y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 5y &\leq 10 \end{aligned}$$

14. Objective function:
$z = 7x + 8y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + \tfrac{1}{2}y &\leq 4 \end{aligned}$$

15. Objective function:
$z = 9x + 24y$
Constraints:
(See Exercise 13.)

16. Objective function:
$z = 7x + 2y$
Constraints:
(See Exercise 14.)

17. Objective function:
$z = 4x + 5y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\geq 8 \\ 3x + 5y &\geq 30 \end{aligned}$$

18. Objective function:
$z = 4x + 5y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 2y &\leq 10 \\ x + 2y &\leq 6 \end{aligned}$$

19. Objective function:
$z = 2x + 7y$
Constraints:
(See Exercise 17.)

20. Objective function:
$z = 2x - y$
Constraints:
(See Exercise 18.)

In Exercises 21–24, use a graphing utility to graph the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints.

21. Objective function:
$z = 4x + y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 40 \\ 2x + 3y &\geq 72 \end{aligned}$$

22. Objective function:
$z = x$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 3y &\leq 60 \\ 2x + y &\leq 28 \\ 4x + y &\leq 48 \end{aligned}$$

23. Objective function:
$z = x + 4y$
Constraints:
(See Exercise 21.)

24. Objective function:
$z = y$
Constraints:
(See Exercise 22.)

In Exercises 25–28, find the maximum value of the objective function and where it occurs, subject to the constraints $x \geq 0$, $y \geq 0$, $3x + y \leq 15$, and $4x + 3y \leq 30$.

25. $z = 2x + y$

26. $z = 5x + y$

27. $z = x + y$

28. $z = 3x + y$

In Exercises 29–32, find the maximum value of the objective function and where it occurs, subject to the constraints $x \geq 0$, $y \geq 0$, $x + 4y \leq 20$, $x + y \leq 18$, and $2x + 2y \leq 21$.

29. $z = x + 5y$

30. $z = 2x + 4y$

31. $z = 4x + 5y$

32. $z = 4x + y$

In Exercises 33–38, the linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. Find the maximum value of the objective function and where it occurs.

33. Objective function:
$z = 2.5x + y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 3x + 5y &\leq 15 \\ 5x + 2y &\leq 10 \end{aligned}$$

34. Objective function:
$z = x + y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 1 \\ -x + 2y &\leq 4 \end{aligned}$$

35. Objective function:
$z = -x + 2y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 10 \\ x + y &\leq 7 \end{aligned}$$

36. Objective function:
$z = x + y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 0 \\ -3x + y &\geq 3 \end{aligned}$$

37. Objective function:
$z = 3x + 4y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 1 \\ 2x + y &\leq 4 \end{aligned}$$

38. Objective function:
$z = x + 2y$
Constraints:
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ 2x + y &\leq 4 \end{aligned}$$

39. ***Optimal Profit*** A manufacturer produces two models of bicycles. The times (in hours) required for assembling, painting, and packaging each model are shown in the table.

Process	Hours, model A	Hours, model B
Assembling	2	2.5
Painting	4	1
Packaging	1	0.75

The total times available for assembling, painting, and packaging are 4000 hours, 4800 hours, and 1500 hours, respectively. The profits per unit are \$45 for model A and \$50 for model B. What is the optimal production level for each model? What is the optimal profit?

40. ***Optimal Profit*** A manufacturer produces two models of bicycles. The times (in hours) required for assembling, painting, and packaging each model are shown in the table.

Process	Hours, model A	Hours, model B
Assembling	2.5	3
Painting	2	1
Packaging	0.75	1.25

The total times available for assembling, painting, and packaging are 4000 hours, 2500 hours, and 1500 hours, respectively. The profits per unit are \$50 for model A and \$52 for model B. What is the optimal production level for each model? What is the optimal profit?

41. ***Optimal Profit*** A merchant plans to sell two models of MP3 players at costs of \$250 and \$300. The \$250 model yields a profit of \$25 per unit and the \$300 model yields a profit of \$40 per unit. The merchant estimates that the total monthly demand will not exceed 250 units. The merchant does not want to invest more than \$65,000 in inventory for these products. What is the optimal inventory level for each model? What is the optimal profit?

42. ***Optimal Profit*** A fruit grower has 150 acres of land available to raise two crops, A and B. It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days available for picking. The profit is \$140 per acre for crop A and \$235 per acre for crop B. What is the optimal acreage for each fruit? What is the optimal profit?

43. ***Optimal Cost*** A farming cooperative mixes two brands of cattle feed. Brand X costs \$25 per bag and contains two units of nutritional element A, two units of element B, and two units of element C. Brand Y costs \$20 per bag and contains one unit of nutritional element A, nine units of element B, and three units of element C. The minimum requirements of nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively. What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

Model It

44. ***Optimal Cost*** According to AAA (Automobile Association of America), on January 24, 2005, the national average price per gallon for regular unleaded (87-octane) gasoline was \$1.84, and the price for premium unleaded (93-octane) gasoline was \$2.03.

(a) Write an objective function that models the cost of the blend of mid-grade unleaded gasoline (89-octane).

(b) Determine the constraints for the objective function in part (a).

(c) Sketch a graph of the region determined by the constraints from part (b).

(d) Determine the blend of regular and premium unleaded gasoline that results in an optimal cost of mid-grade unleaded gasoline.

(e) What is the optimal cost?

(f) Is the cost lower than the national average of \$1.96 per gallon for mid-grade unleaded gasoline?

45. ***Optimal Revenue*** An accounting firm has 900 hours of staff time and 155 hours of reviewing time available each week. The firm charges \$2500 for an audit and \$350 for a tax return. Each audit requires 75 hours of staff time and 10 hours of review time. Each tax return requires 12.5 hours of staff time and 2.5 hours of review time. What numbers of audits and tax returns will yield an optimal revenue? What is the optimal revenue?

46. ***Optimal Revenue*** The accounting firm in Exercise 45 lowers its charge for an audit to \$2000. What numbers of audits and tax returns will yield an optimal revenue? What is the optimal revenue?

47. ***Investment Portfolio*** An investor has up to \$250,000 to invest in two types of investments. Type A pays 8% annually and type B pays 10% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-fourth of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

48. ***Investment Portfolio*** An investor has up to \$450,000 to invest in two types of investments. Type A pays 6% annually and type B pays 10% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-half of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

Synthesis

True or False? **In Exercises 49 and 50, determine whether the statement is true or false. Justify your answer.**

49. If an objective function has a maximum value at the vertices $(4, 7)$ and $(8, 3)$, you can conclude that it also has a maximum value at the points $(4.5, 6.5)$ and $(7.8, 3.2)$.

50. When solving a linear programming problem, if the objective function has a maximum value at more than one vertex, you can assume that there are an infinite number of points that will produce the maximum value.

In Exercises 51 and 52, determine values of t such that the objective function has maximum values at the indicated vertices.

51. Objective function: $z = 3x + ty$

Constraints:

$$x \geq 0$$
$$y \geq 0$$
$$x + 3y \leq 15$$
$$4x + y \leq 16$$

(a) $(0, 5)$

(b) $(3, 4)$

52. Objective function: $z = 3x + ty$

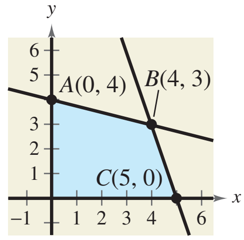
Constraints:

$$x \geq 0$$
$$y \geq 0$$
$$x + 2y \leq 4$$
$$x - y \leq 1$$

(a) $(2, 1)$

(b) $(0, 2)$

Think About It **In Exercises 53–56, find an objective function that has a maximum or minimum value at the indicated vertex of the constraint region shown below. (There are many correct answers.)**



53. The maximum occurs at vertex A.

54. The maximum occurs at vertex B.

55. The maximum occurs at vertex C.

56. The minimum occurs at vertex C.

Skills Review

In Exercises 57–60, simplify the complex fraction.

57. $\dfrac{\left(\dfrac{9}{x}\right)}{\left(\dfrac{6}{x} + 2\right)}$

58. $\dfrac{\left(1 + \dfrac{2}{x}\right)}{\left(x - \dfrac{4}{x}\right)}$

59. $\dfrac{\left(\dfrac{4}{x^2 - 9} + \dfrac{2}{x - 2}\right)}{\left(\dfrac{1}{x + 3} + \dfrac{1}{x - 3}\right)}$

60. $\dfrac{\left(\dfrac{1}{x + 1} + \dfrac{1}{2}\right)}{\left(\dfrac{3}{2x^2 + 4x + 2}\right)}$

In Exercises 61–66, solve the equation algebraically. Round the result to three decimal places.

61. $e^{2x} + 2e^{x} - 15 = 0$

62. $e^{2x} - 10e^{x} + 24 = 0$

63. $8(62 - e^{x/4}) = 192$

64. $\dfrac{150}{e^{-x} - 4} = 75$

65. $7 \ln 3x = 12$

66. $\ln(x + 9)^2 = 2$

In Exercises 67 and 68, solve the system of linear equations and check any solution algebraically.

67. $\begin{cases} -x - 2y + 3z = -23 \\ 2x + 6y - z = 17 \\ 5y + z = 8 \end{cases}$

68. $\begin{cases} 7x - 3y + 5z = -28 \\ 4x + 4z = -16 \\ 7x + 2y - z = 0 \end{cases}$

7 Chapter Summary

What did you learn?

Section 7.1	Review Exercises
☐ Use the method of substitution to solve systems of linear equations in two variables *(p. 496)*.	1–4
☐ Use the method of substitution to solve systems of nonlinear equations in two variables *(p. 499)*.	5–8
☐ Use a graphical approach to solve systems of equations in two variables *(p. 500)*.	9–14
☐ Use systems of equations to model and solve real-life problems *(p. 501)*.	15–18
Section 7.2	
☐ Use the method of elimination to solve systems of linear equations in two variables *(p.507)*.	19–26
☐ Interpret graphically the numbers of solutions of systems of linear equations in two variables *(p. 510)*.	27–30
☐ Use systems of linear equations in two variables to model and solve real-life problems *(p. 513)*.	31, 32
Section 7.3	
☐ Use back-substitution to solve linear systems in row-echelon form *(p. 519)*.	33, 34
☐ Use Gaussian elimination to solve systems of linear equations *(p. 520)*.	35–38
☐ Solve nonsquare systems of linear equations *(p. 524)*.	39, 40
☐ Use systems of linear equations in three or more variables to model and solve real-life problems *(p. 525)*.	41–48
Section 7.4	
☐ Recognize partial fraction decompositions of rational expressions *(p. 533)*.	49–52
☐ Find partial fraction decompositions of rational expressions *(p. 534)*.	53–60
Section 7.5	
☐ Sketch the graphs of inequalities in two variables *(p. 541)*.	61–64
☐ Solve systems of inequalities *(p. 543)*.	65–72
☐ Use systems of inequalities in two variables to model and solve real-life problems *(p. 546)*.	73–76
Section 7.6	
☐ Solve linear programming problems *(p. 552)*.	77–82
☐ Use linear programming to model and solve real-life problems *(p. 556)*.	83–86

7 Review Exercises

7.1 **In Exercises 1–8, solve the system by the method of substitution.**

1. $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$

2. $\begin{cases} 2x - 3y = 3 \\ x - y = 0 \end{cases}$

3. $\begin{cases} 0.5x + y = 0.75 \\ 1.25x - 4.5y = -2.5 \end{cases}$

4. $\begin{cases} -x + \frac{2}{5}y = \frac{3}{5} \\ -x + \frac{1}{5}y = -\frac{4}{5} \end{cases}$

5. $\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \end{cases}$

6. $\begin{cases} x^2 + y^2 = 169 \\ 3x + 2y = 39 \end{cases}$

7. $\begin{cases} y = 2x^2 \\ y = x^4 - 2x^2 \end{cases}$

8. $\begin{cases} x = y + 3 \\ x = y^2 + 1 \end{cases}$

In Exercises 9–12, solve the system graphically.

9. $\begin{cases} 2x - y = 10 \\ x + 5y = -6 \end{cases}$

10. $\begin{cases} 8x - 3y = -3 \\ 2x + 5y = 28 \end{cases}$

11. $\begin{cases} y = 2x^2 - 4x + 1 \\ y = x^2 - 4x + 3 \end{cases}$

12. $\begin{cases} y^2 - 2y + x = 0 \\ x + y = 0 \end{cases}$

In Exercises 13 and 14, use a graphing utility to solve the system of equations. Find the solution accurate to two decimal places.

13. $\begin{cases} y = -2e^{-x} \\ 2e^x + y = 0 \end{cases}$

14. $\begin{cases} y = \ln(x - 1) - 3 \\ y = 4 - \frac{1}{2}x \end{cases}$

15. ***Break-Even Analysis*** You set up a scrapbook business and make an initial investment of \$50,000. The unit cost of a scrapbook kit is \$12 and the selling price is \$25. How many kits must you sell to break even?

16. ***Choice of Two Jobs*** You are offered two sales jobs at a pharmaceutical company. One company offers an annual salary of \$35,000 plus a year-end bonus of 1.5% of your total sales. The other company offers an annual salary of \$32,000 plus a year-end bonus of 2% of your total sales. What amount of sales will make the second offer better? Explain.

17. ***Geometry*** The perimeter of a rectangle is 480 meters and its length is 150% of its width. Find the dimensions of the rectangle.

18. ***Geometry*** The perimeter of a rectangle is 68 feet and its width is $\frac{8}{9}$ times its length. Find the dimensions of the rectangle.

7.2 **In Exercises 19–26, solve the system by the method of elimination.**

19. $\begin{cases} 2x - y = 2 \\ 6x + 8y = 39 \end{cases}$

20. $\begin{cases} 40x + 30y = 24 \\ 20x - 50y = -14 \end{cases}$

21. $\begin{cases} 0.2x + 0.3y = 0.14 \\ 0.4x + 0.5y = 0.20 \end{cases}$

22. $\begin{cases} 12x + 42y = -17 \\ 30x - 18y = 19 \end{cases}$

23. $\begin{cases} 3x - 2y = 0 \\ 3x + 2(y + 5) = 10 \end{cases}$

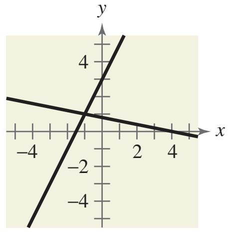
24. $\begin{cases} 7x + 12y = 63 \\ 2x + 3(y + 2) = 21 \end{cases}$

25. $\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$

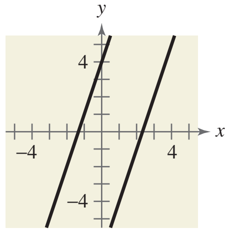
26. $\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$

In Exercises 27–30, match the system of linear equations with its graph. Describe the number of solutions and state whether the system is consistent or inconsistent. [The graphs are labeled (a), (b), (c), and (d).]

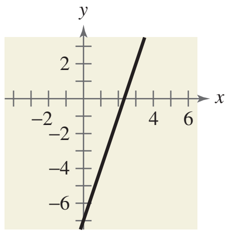
(a)



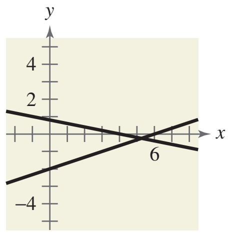
(b)



(c)



(d)



27. $\begin{cases} x + 5y = 4 \\ x - 3y = 6 \end{cases}$

28. $\begin{cases} -3x + y = -7 \\ 9x - 3y = 21 \end{cases}$

29. $\begin{cases} 3x - y = 7 \\ -6x + 2y = 8 \end{cases}$

30. $\begin{cases} 2x - y = -3 \\ x + 5y = 4 \end{cases}$

Supply and Demand **In Exercises 31 and 32, find the equilibrium point of demand and supply equations.**

	Demand	*Supply*
31.	$p = 37 - 0.0002x$	$p = 22 + 0.00001x$
32.	$p = 120 - 0.0001x$	$p = 45 + 0.0002x$

7.3 **In Exercises 33 and 34, use back-substitution to solve the system of linear equations.**

33. $\begin{cases} x - 4y + 3z = 3 \\ -y + z = -1 \\ z = -5 \end{cases}$

34. $\begin{cases} x - 7y + 8z = 85 \\ y - 9z = -35 \\ z = 3 \end{cases}$

In Exercises 35–38, use Gaussian elimination to solve the system of equations.

35. $\begin{cases} x + 2y + 6z = 4 \\ -3x + 2y - z = -4 \\ 4x + 2z = 16 \end{cases}$

36. $\begin{cases} x + 3y - z = 13 \\ 2x - 5z = 23 \\ 4x - y - 2z = 14 \end{cases}$

37. $\begin{cases} x - 2y + z = -6 \\ 2x - 3y = -7 \\ -x + 3y - 3z = 11 \end{cases}$

38. $\begin{cases} 2x + 6z = -9 \\ 3x - 2y + 11z = -16 \\ 3x - y + 7z = -11 \end{cases}$

In Exercises 39 and 40, solve the nonsquare system of equations.

39. $\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$

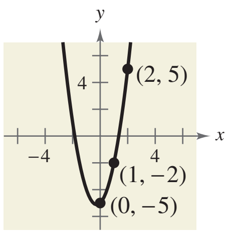
40. $\begin{cases} 2x + 5y - 19z = 34 \\ 3x + 8y - 31z = 54 \end{cases}$

In Exercises 41 and 42, find the equation of the parabola

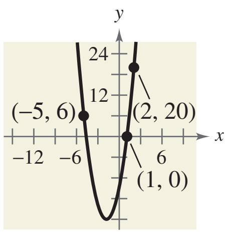
$$y = ax^2 + bx + c$$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.

41.



42.

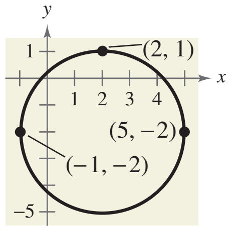


In Exercises 43 and 44, find the equation of the circle

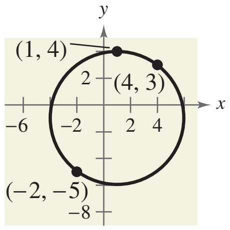
$$x^2 + y^2 + Dx + Ey + F = 0$$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the circle.

43.



44.



45. ***Data Analysis: Online Shopping*** The table shows the projected numbers y (in millions) of people shopping online in the United States from 2003 to 2005. (Source: eMarketer)

Year	Online shoppers, y
2003	101.7
2004	108.4
2005	121.1

(a) Use the technique demonstrated in Exercises 67–70 in Section 7.3 to set up a system of equations for the data and to find a least squares regression parabola that models the data. Let x represent the year, with $x = 3$ corresponding to 2003.

(b) Use a graphing utility to graph the parabola and the data in the same viewing window. How well does the model fit the data?

(c) Use the model to estimate the number of online shoppers in 2008. Does your answer seem reasonable?

46. ***Agriculture*** A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

47. ***Investment Analysis*** An inheritance of \$40,000 was divided among three investments yielding \$3500 in interest per year. The interest rates for the three investments were 7%, 9%, and 11%. Find the amount placed in each investment if the second and third were \$3000 and \$5000 less than the first, respectively.

48. ***Vertical Motion*** An object moving vertically is at the given heights at the specified times. Find the position equation $s = \frac{1}{2}at^2 + v_0t + s_0$ for the object.

(a) At $t = 1$ second, $s = 134$ feet

At $t = 2$ seconds, $s = 86$ feet

At $t = 3$ seconds, $s = 6$ feet

(b) At $t = 1$ second, $s = 184$ feet

At $t = 2$ seconds, $s = 116$ feet

At $t = 3$ seconds, $s = 16$ feet

7.4 **In Exercises 49–52, write the form of the partial fraction decomposition for the rational expression. Do not solve for the constants.**

49. $\dfrac{3}{x^2 + 20x}$

50. $\dfrac{x - 8}{x^2 - 3x - 28}$

51. $\dfrac{3x - 4}{x^3 - 5x^2}$

52. $\dfrac{x - 2}{x(x^2 + 2)^2}$

In Exercises 53–60, write the partial fraction decomposition of the rational expression.

53. $\dfrac{4 - x}{x^2 + 6x + 8}$

54. $\dfrac{-x}{x^2 + 3x + 2}$

55. $\dfrac{x^2}{x^2 + 2x - 15}$

56. $\dfrac{9}{x^2 - 9}$

57. $\dfrac{x^2 + 2x}{x^3 - x^2 + x - 1}$

58. $\dfrac{4x}{3(x - 1)^2}$

59. $\dfrac{3x^2 + 4x}{(x^2 + 1)^2}$

60. $\dfrac{4x^2}{(x - 1)(x^2 + 1)}$

7.5 **In Exercises 61–64, sketch the graph of the inequality.**

61. $y \le 5 - \frac{1}{2}x$

62. $3y - x \ge 7$

63. $y - 4x^2 > -1$

64. $y \le \dfrac{3}{x^2 + 2}$

In Exercises 65–72, sketch the graph and label the vertices of the solution set of the system of inequalities.

65. $\begin{cases} x + 2y \le 160 \\ 3x + y \le 180 \\ x \ge 0 \\ y \ge 0 \end{cases}$

66. $\begin{cases} 2x + 3y \le 24 \\ 2x + y \le 16 \\ x \ge 0 \\ y \ge 0 \end{cases}$

67. $\begin{cases} 3x + 2y \ge 24 \\ x + 2y \ge 12 \\ 2 \le x \le 15 \\ y \le 15 \end{cases}$

68. $\begin{cases} 2x + y \ge 16 \\ x + 3y \ge 18 \\ 0 \le x \le 25 \\ 0 \le y \le 25 \end{cases}$

69. $\begin{cases} y < x + 1 \\ y > x^2 - 1 \end{cases}$

70. $\begin{cases} y \le 6 - 2x - x^2 \\ y \ge x + 6 \end{cases}$

71. $\begin{cases} 2x - 3y \ge 0 \\ 2x - y \le 8 \\ y \ge 0 \end{cases}$

72. $\begin{cases} x^2 + y^2 \le 9 \\ (x - 3)^2 + y^2 \le 9 \end{cases}$

73. ***Inventory Costs*** A warehouse operator has 24,000 square feet of floor space in which to store two products. Each unit of product I requires 20 square feet of floor space and costs \$12 per day to store. Each unit of product II requires 30 square feet of floor space and costs \$8 per day to store. The total storage cost per day cannot exceed \$12,400. Find and graph a system of inequalities describing all possible inventory levels.

74. ***Nutrition*** A dietitian is asked to design a special dietary supplement using two different foods. Each ounce of food X contains 12 units of calcium, 10 units of iron, and 20 units of vitamin B. Each ounce of food Y contains 15 units of calcium, 20 units of iron, and 12 units of vitamin B. The minimum daily requirements of the diet are 300 units of calcium, 280 units of iron, and 300 units of vitamin B.

(a) Write a system of inequalities describing the different amounts of food X and food Y that can be used.

(b) Sketch a graph of the region in part (a).

(c) Find two solutions to the system and interpret their meanings in the context of the problem.

Supply and Demand **In Exercises 75 and 76, (a) graph the systems representing the consumer surplus and producer surplus for the supply and demand equations and (b) find the consumer surplus and producer surplus.**

	Demand	*Supply*
75.	$p = 160 - 0.0001x$	$p = 70 + 0.0002x$
76.	$p = 130 - 0.0002x$	$p = 30 + 0.0003x$

7.6 **In Exercises 77–82, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the indicated restraints.**

77. Objective function:

$z = 3x + 4y$

Constraints:

$x \ge 0$

$y \ge 0$

$2x + 5y \le 50$

$4x + y \le 28$

78. Objective function:

$z = 10x + 7y$

Constraints:

$x \ge 0$

$y \ge 0$

$2x + y \ge 100$

$x + y \ge 75$

79. Objective function:

$z = 1.75x + 2.25y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + y &\geq 25 \\ 3x + 2y &\geq 45 \end{aligned}$$

80. Objective function:

$z = 50x + 70y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 1500 \\ 5x + 2y &\leq 3500 \end{aligned}$$

81. Objective function:

$z = 5x + 11y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 3y &\leq 12 \\ 3x + 2y &\leq 15 \end{aligned}$$

82. Objective function:

$z = -2x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\geq 7 \\ 5x + 2y &\geq 20 \end{aligned}$$

83. ***Optimal Revenue*** A student is working part time as a hairdresser to pay college expenses. The student may work no more than 24 hours per week. Haircuts cost \$25 and require an average of 20 minutes, and permanents cost \$70 and require an average of 1 hour and 10 minutes. What combination of haircuts and/or permanents will yield an optimal revenue? What is the optimal revenue?

84. ***Optimal Profit*** A shoe manufacturer produces a walking shoe and a running shoe yielding profits of \$18 and \$24, respectively. Each shoe must go through three processes, for which the required times per unit are shown in the table.

	Process I	Process II	Process III
Hours for walking shoe	4	1	1
Hours for running shoe	2	2	1
Hours available per day	24	9	8

What is the optimal production level for each type of shoe? What is the optimal profit?

85. ***Optimal Cost*** A pet supply company mixes two brands of dry dog food. Brand X costs \$15 per bag and contains eight units of nutritional element A, one unit of nutritional element B, and two units of nutritional element C. Brand Y costs \$30 per bag and contains two units of nutritional element A, one unit of nutritional element B, and seven units of nutritional element C. Each bag of mixed dog food must contain at least 16 units, 5 units, and 20 units of nutritional elements A, B, and C, respectively. Find the numbers of bags of brands X and Y that should be mixed to produce a mixture meeting the minimum nutritional requirements and having an optimal cost. What is the optimal cost?

86. ***Optimal Cost*** Regular unleaded gasoline and premium unleaded gasoline have octane ratings of 87 and 93, respectively. For the week of January 3, 2005, regular unleaded gasoline in Houston, Texas averaged \$1.63 per gallon. For the same week, premium unleaded gasoline averaged \$1.83 per gallon. Determine the blend of regular and premium unleaded gasoline that results in an optimal cost of mid-grade unleaded (89-octane) gasoline. What is the optimal cost? (Source: Energy Information Administration)

Synthesis

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

87. The system

$$\begin{cases} y \leq 5 \\ y \geq -2 \\ y \geq \frac{7}{2}x - 9 \\ y \geq -\frac{7}{2}x + 26 \end{cases}$$

represents the region covered by an isosceles trapezoid.

88. It is possible for an objective function of a linear programming problem to have exactly 10 maximum value points.

In Exercises 89–92, find a system of linear equations having the ordered pair as a solution. (There are many correct answers.)

89. $(-6, 8)$

90. $(5, -4)$

91. $\left(\frac{4}{3}, 3\right)$

92. $\left(-1, \frac{9}{4}\right)$

In Exercises 93–96, find a system of linear equations having the ordered triple as a solution. (There are many answers.)

93. $(4, -1, 3)$

94. $(-3, 5, 6)$

95. $\left(5, \frac{3}{2}, 2\right)$

96. $\left(\frac{3}{4}, -2, 8\right)$

97. ***Writing*** Explain what is meant by an inconsistent system of linear equations.

98. How can you tell graphically that a system of linear equations in two variables has no solution? Give an example.

99. ***Writing*** Write a brief paragraph describing any advantages of substitution over the graphical method of solving a system of equations.

7 Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, solve the system by the method of substitution.

1. $\begin{cases} x - y = -7 \\ 4x + 5y = 8 \end{cases}$ **2.** $\begin{cases} y = x - 1 \\ y = (x - 1)^3 \end{cases}$ **3.** $\begin{cases} 2x - y^2 = 0 \\ x - y = 4 \end{cases}$

In Exercises 4–6, solve the system graphically.

4. $\begin{cases} 2x - 3y = 0 \\ 2x + 3y = 12 \end{cases}$ **5.** $\begin{cases} y = 9 - x^2 \\ y = x + 3 \end{cases}$ **6.** $\begin{cases} y - \ln x = 12 \\ 7x - 2y + 11 = -6 \end{cases}$

In Exercises 7–10, solve the linear system by the method of elimination.

7. $\begin{cases} 2x + 3y = 17 \\ 5x - 4y = -15 \end{cases}$ **8.** $\begin{cases} 2.5x - y = 6 \\ 3x + 4y = 2 \end{cases}$

9. $\begin{cases} x - 2y + 3z = 11 \\ 2x - z = 3 \\ 3y + z = -8 \end{cases}$ **10.** $\begin{cases} 3x + 2y + z = 17 \\ -x + y + z = 4 \\ x - y - z = 3 \end{cases}$

In Exercises 11–14, write the partial fraction decomposition of the rational expression.

11. $\dfrac{2x + 5}{x^2 - x - 2}$ **12.** $\dfrac{3x^2 - 2x + 4}{x^2(2 - x)}$ **13.** $\dfrac{x^2 + 5}{x^3 - x}$ **14.** $\dfrac{x^2 - 4}{x^3 + 2x}$

In Exercises 15–17, sketch the graph and label the vertices of the solution of the system of inequalities.

15. $\begin{cases} 2x + y \le 4 \\ 2x - y \ge 0 \\ x \ge 0 \end{cases}$ **16.** $\begin{cases} y < -x^2 + x + 4 \\ y > 4x \end{cases}$ **17.** $\begin{cases} x^2 + y^2 \le 16 \\ x \ge 1 \\ y \ge -3 \end{cases}$

18. Find the maximum and minimum values of the objective function $z = 20x + 12y$ and where they occur, subject to the following constraints.

$$\left.\begin{array}{r} x \ge 0 \\ y \ge 0 \\ x + 4y \le 32 \\ 3x + 2y \le 36 \end{array}\right\} \quad \text{Constraints}$$

19. A total of \$50,000 is invested in two funds paying 8% and 8.5% simple interest. The yearly interest is \$4150. How much is invested at each rate?

20. Find the equation of the parabola $y = ax^2 + bx + c$ passing through the points $(0, 6)$, $(-2, 2)$, and $\left(3, \frac{9}{2}\right)$.

21. A manufacturer produces two types of television stands. The amounts (in hours) of time for assembling, staining, and packaging the two models are shown in the table at the left. The total amounts of time available for assembling, staining, and packaging are 4000, 8950, and 2650 hours, respectively. The profits per unit are \$30 (model I) and \$40 (model II). What is the optimal inventory level for each model? What is the optimal profit?

	Model I	Model II
Assembling	0.5	0.75
Staining	2.0	1.5
Packaging	0.5	0.5

TABLE FOR 21

Proofs in Mathematics

An **indirect proof** can be useful in proving statements of the form "p implies q." Recall that the conditional statement $p \rightarrow q$ is false only when p is true and q is false. To prove a conditional statement indirectly, assume that p is true and q is false. If this assumption leads to an impossibility, then you have proved that the conditional statement is true. An indirect proof is also called a **proof by contradiction.**

You can use an indirect proof to prove the following conditional statement,

"If a is a positive integer and a^2 is divisible by 2, then a is divisible by 2,"

as follows. First, assume that p, "a is a positive integer and a^2 is divisible by 2," is true and q, "a is divisible by 2," is false. This means that a is not divisible by 2. If so, a is odd and can be written as $a = 2n + 1$, where n is an integer.

$$a = 2n + 1 \qquad \text{Definition of an odd integer}$$

$$a^2 = 4n^2 + 4n + 1 \qquad \text{Square each side.}$$

$$a^2 = 2(2n^2 + 2n) + 1 \qquad \text{Distributive Property}$$

So, by the definition of an odd integer, a^2 is odd. This contradicts the assumption, and you can conclude that a is divisible by 2.

Example Using an Indirect Proof

Use an indirect proof to prove that $\sqrt{2}$ is an irrational number.

Solution

Begin by assuming that $\sqrt{2}$ is *not* an irrational number. Then $\sqrt{2}$ can be written as the quotient of two integers a and $b(b \neq 0)$ that have no common factors.

$$\sqrt{2} = \frac{a}{b} \qquad \text{Assume that } \sqrt{2} \text{ is a rational number.}$$

$$2 = \frac{a^2}{b^2} \qquad \text{Square each side.}$$

$$2b^2 = a^2 \qquad \text{Multiply each side by } b^2.$$

This implies that 2 is a factor of a^2. So, 2 is also a factor of a, and a can be written as $2c$, where c is an integer.

$$2b^2 = (2c)^2 \qquad \text{Substitute } 2c \text{ for } a.$$

$$2b^2 = 4c^2 \qquad \text{Simplify.}$$

$$b^2 = 2c^2 \qquad \text{Divide each side by 2.}$$

This implies that 2 is a factor of b^2 and also a factor of b. So, 2 is a factor of both a and b. This contradicts the assumption that a and b have no common factors. So, you can conclude that $\sqrt{2}$ is an irrational number.

P.S. Problem Solving

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. A theorem from geometry states that if a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle. Show that this theorem is true for the circle

$$x^2 + y^2 = 100$$

and the triangle formed by the lines

$$y = 0,\ y = \tfrac{1}{2}x + 5, \text{ and } y = -2x + 20.$$

2. Find k_1 and k_2 such that the system of equations has an infinite number of solutions.

$$\begin{cases} 3x - 5y = 8 \\ 2x + k_1 y = k_2 \end{cases}$$

3. Consider the following system of linear equations in x and y.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

Under what conditions will the system have exactly one solution?

4. Graph the lines determined by each system of linear equations. Then use Gaussian elimination to solve each system. At each step of the elimination process, graph the corresponding lines. What do you observe?

(a) $\begin{cases} x - 4y = -3 \\ 5x - 6y = 13 \end{cases}$

(b) $\begin{cases} 2x - 3y = 7 \\ -4x + 6y = -14 \end{cases}$

5. A system of two equations in two unknowns is solved and has a finite number of solutions. Determine the maximum number of solutions of the system satisfying each condition.

(a) Both equations are linear.

(b) One equation is linear and the other is quadratic.

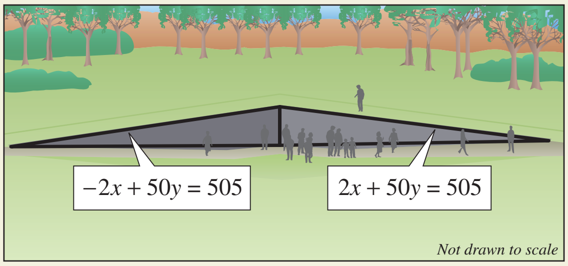
(c) Both equations are quadratic.

6. In the 2004 presidential election, approximately 118.304 million voters divided their votes among three presidential candidates. George W. Bush received 3,320,000 votes more than John Kerry. Ralph Nader received 0.3% of the votes. Write and solve a system of equations to find the total number of votes cast for each candidate. Let B represent the total votes cast for Bush, K the total votes cast for Kerry, and N the total votes cast for Nader. (Source: CNN.com)

7. The Vietnam Veterans Memorial (or "The Wall") in Washington, D.C. was designed by Maya Ying Lin when she was a student at Yale University. This monument has two vertical, triangular sections of black granite with a common side (see figure). The bottom of each section is level with the ground. The tops of the two sections can be approximately modeled by the equations

$$-2x + 50y = 505 \quad \text{and} \quad 2x + 50y = 505$$

when the x-axis is superimposed at the base of the wall. Each unit in the coordinate system represents 1 foot. How high is the memorial at the point where the two sections meet? How long is each section?



8. Weights of atoms and molecules are measured in atomic mass units (u). A molecule of C_2H_6 (ethane) is made up of two carbon atoms and six hydrogen atoms and weighs 30.07 u. A molecule of C_3H_8 (propane) is made up of three carbon atoms and eight hydrogen atoms and weighs 44.097 u. Find the weights of a carbon atom and a hydrogen atom.

9. To connect a DVD player to a television set, a cable with special connectors is required at both ends. You buy a six-foot cable for \$15.50 and a three-foot cable for \$10.25. Assuming that the cost of a cable is the sum of the cost of the two connectors and the cost of the cable itself, what is the cost of a four-foot cable? Explain your reasoning.

10. A hotel 35 miles from an airport runs a shuttle service to and from the airport. The 9:00 A.M. bus leaves for the airport traveling at 30 miles per hour. The 9:15 A.M. bus leaves for the airport traveling at 40 miles per hour. Write a system of linear equations that represents distance as a function of time for each bus. Graph and solve the system. How far from the airport will the 9:15 A.M. bus catch up to the 9:00 A.M. bus?

11. Solve each system of equations by letting $X = 1/x$, $Y = 1/y$, and $Z = 1/z$.

(a) $$\begin{cases} \dfrac{12}{x} - \dfrac{12}{y} = 7 \\ \dfrac{3}{x} + \dfrac{4}{y} = 0 \end{cases}$$

(b) $$\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} - \dfrac{3}{z} = 4 \\ \dfrac{4}{x} \qquad + \dfrac{2}{z} = 10 \\ -\dfrac{2}{x} + \dfrac{3}{y} - \dfrac{13}{z} = -8 \end{cases}$$

12. What values should be given to a, b, and c so that the linear system shown has $(-1, 2, -3)$ as its only solution?

$$\begin{cases} x + 2y - 3z = a & \text{Equation 1} \\ -x - y + z = b & \text{Equation 2} \\ 2x + 3y - 2z = c & \text{Equation 3} \end{cases}$$

13. The following system has one solution: $x = 1$, $y = -1$, and $z = 2$.

$$\begin{cases} 4x - 2y + 5z = 16 \\ x + y \qquad = 0 \\ -x - 3y + 2z = 6 \end{cases}$$

Solve the system given by (a) Equation 1 and Equation 2, (b) Equation 1 and Equation 3, and (c) Equation 2 and Equation 3. (d) How many solutions does each of these systems have?

14. Solve the system of linear equations algebraically.

$$\begin{cases} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 6 \\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 = 14 \\ \quad - x_2 - x_3 - x_4 - 3x_5 = -3 \\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 = 10 \\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 = 13 \end{cases}$$

15. Each day, an average adult moose can process about 32 kilograms of terrestrial vegetation (twigs and leaves) and aquatic vegetation. From this food, it needs to obtain about 1.9 grams of sodium and 11,000 calories of energy. Aquatic vegetation has about 0.15 gram of sodium per kilogram and about 193 calories of energy per kilogram, whereas terrestrial vegetation has minimal sodium and about four times more energy than aquatic vegetation. Write and graph a system of inequalities that describes the amounts t and a of terrestrial and aquatic vegetation, respectively, for the daily diet of an average adult moose. (Source: Biology by Numbers)

16. For a healthy person who is 4 feet 10 inches tall, the recommended minimum weight is about 91 pounds and increases by about 3.7 pounds for each additional inch of height. The recommended maximum weight is about 119 pounds and increases by about 4.8 pounds for each additional inch of height. (Source: Dietary Guidelines Advisory Committee)

(a) Let x be the number of inches by which a person's height exceeds 4 feet 10 inches and let y be the person's weight in pounds. Write a system of inequalities that describes the possible values of x and y for a healthy person.

(b) Use a graphing utility to graph the system of inequalities from part (a).

(c) What is the recommended weight range for someone 6 feet tall?

17. The cholesterol in human blood is necessary, but too much cholesterol can lead to health problems. A blood cholesterol test gives three readings: LDL ("bad") cholesterol, HDL ("good") cholesterol, and total cholesterol (LDL + HDL). It is recommended that your LDL cholesterol level be less than 130 milligrams per deciliter, your HDL cholesterol level be at least 35 milligrams per deciliter, and your total cholesterol level be no more than 200 milligrams per deciliter. (Source: WebMD, Inc.)

(a) Write a system of linear inequalities for the recommended cholesterol levels. Let x represent HDL cholesterol and let y represent LDL cholesterol.

(b) Graph the system of inequalities from part (a). Label any vertices of the solution region.

(c) Are the following cholesterol levels within recommendations? Explain your reasoning.

LDL: 120 milligrams per deciliter

HDL: 90 milligrams per deciliter

Total: 210 milligrams per deciliter

(d) Give an example of cholesterol levels in which the LDL cholesterol level is too high but the HDL and total cholesterol levels are acceptable.

(e) Another recommendation is that the ratio of total cholesterol to HDL cholesterol be less than 4. Find a point in your solution region from part (b) that meets this recommendation, and explain why it meets the recommendation.