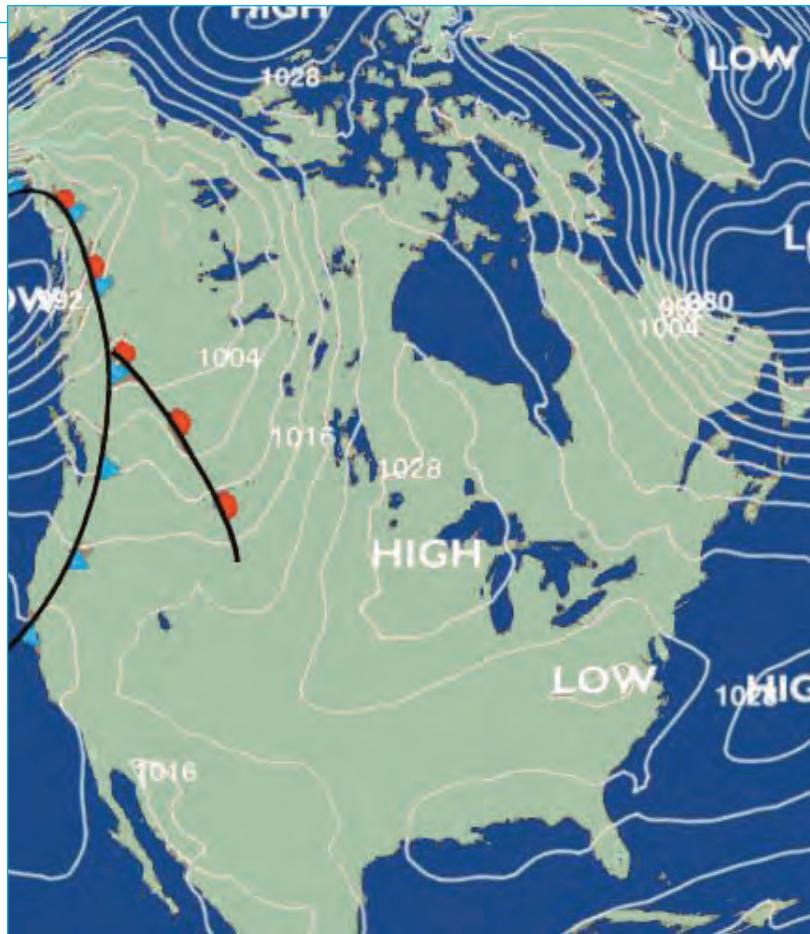


14

PARTIAL DERIVATIVES



Functions of two variables can be visualized by means of level curves, which connect points where the function takes on a given value. Atmospheric pressure at a given time is a function of longitude and latitude and is measured in millibars. Here the level curves are called isobars and those pictured join locations that had the same pressure on March 7, 2007. (The curves labeled 1028, for instance, connect points with pressure 1028 mb.) Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure, and are strongest where the isobars are tightly packed.

So far we have dealt with the calculus of functions of a single variable. But, in the real world, physical quantities often depend on two or more variables, so in this chapter we turn our attention to functions of several variables and extend the basic ideas of differential calculus to such functions.

14.1 FUNCTIONS OF SEVERAL VARIABLES

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

FUNCTIONS OF TWO VARIABLES

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y) . We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h . In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h , and we write $V(r, h) = \pi r^2 h$.

DEFINITION A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y) . The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

A function of two variables is just a function whose domain is a subset of \mathbb{R}^2 and whose range is a subset of \mathbb{R} . One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy -plane.

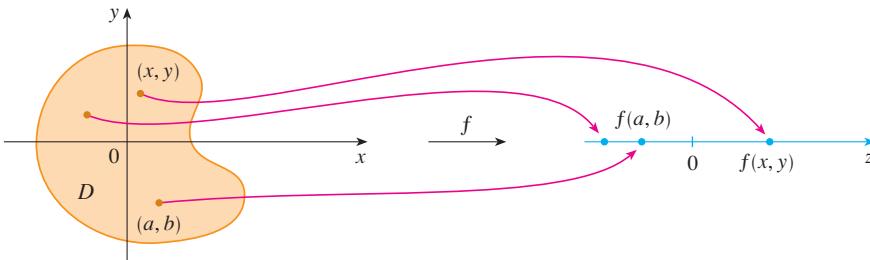


FIGURE 1

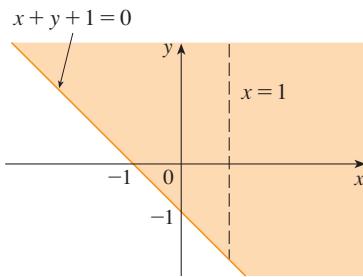
If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find the domain.

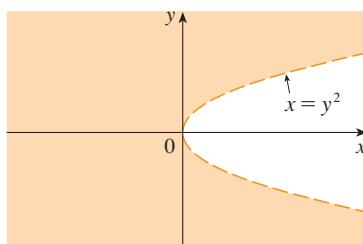
$$(a) f(x, y) = \frac{\sqrt{x+y+1}}{x-1} \quad (b) f(x, y) = x \ln(y^2 - x)$$

SOLUTION

$$(a) f(3, 2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$$

**FIGURE 2**

Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$

**FIGURE 3**

Domain of $f(x, y) = x \ln(y^2 - x)$

TABLE 1

Wind-chill index as a function of air temperature and wind speed

THE NEW WIND-CHILL INDEX

A new wind-chill index was introduced in November of 2001 and is more accurate than the old index at measuring how cold it feels when it's windy. The new index is based on a model of how fast a human face loses heat. It was developed through clinical trials in which volunteers were exposed to a variety of temperatures and wind speeds in a refrigerated wind tunnel.

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geq 0, x \neq 1\}$$

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain. (See Figure 2.)

(b)

$$f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$. (See Figure 3.) ■

Not all functions are given by explicit formulas. The function in the next example is described verbally and by numerical estimates of its values.

EXAMPLE 2 In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v . So W is a function of T and v , and we can write $W = f(T, v)$. Table 1 records values of W compiled by the NOAA National Weather Service of the US and the Meteorological Service of Canada.

Wind speed (km/h)

$T \backslash v$	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	-1	-1	-2	-2	-3
0	-2	-3	-4	-5	-6	-6	-7	-8	-9	-9	-10
-5	-7	-9	-11	-12	-12	-13	-14	-15	-16	-16	-17
-10	-13	-15	-17	-18	-19	-20	-21	-22	-23	-23	-24
-15	-19	-21	-23	-24	-25	-26	-27	-29	-30	-30	-31
-20	-24	-27	-29	-30	-32	-33	-34	-35	-36	-37	-38
-25	-30	-33	-35	-37	-38	-39	-41	-42	-43	-44	-45
-30	-36	-39	-41	-43	-44	-46	-48	-49	-50	-51	-52
-35	-41	-45	-48	-49	-51	-52	-54	-56	-57	-58	-60
-40	-47	-51	-54	-56	-57	-59	-61	-63	-64	-65	-67

For instance, the table shows that if the temperature is -5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about -15°C with no wind. So

$$f(-5, 50) = -15$$

EXAMPLE 3 In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They con-

TABLE 2

Year	<i>P</i>	<i>L</i>	<i>K</i>
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

sidered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

$$\boxed{1} \quad P(L, K) = bL^\alpha K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings). In Section 14.3 we will show how the form of Equation 1 follows from certain economic assumptions.

Cobb and Douglas used economic data published by the government to obtain Table 2. They took the year 1899 as a baseline, and P , L , and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

$$\boxed{2} \quad P(L, K) = 1.01L^{0.75}K^{0.25}$$

(See Exercise 75 for the details.)

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function (1) has subsequently been used in many settings, ranging from individual firms to global economic questions. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K) \mid L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative. ■

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

which is the disk with center $(0, 0)$ and radius 3. (See Figure 4.) The range of g is

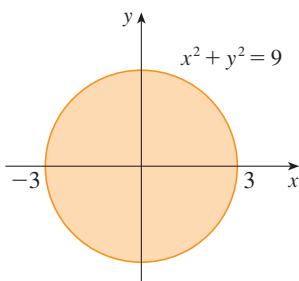
$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also

$$9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

So the range is

$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$

**FIGURE 4**

Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

GRAPHS

Another way of visualizing the behavior of a function of two variables is to consider its graph.

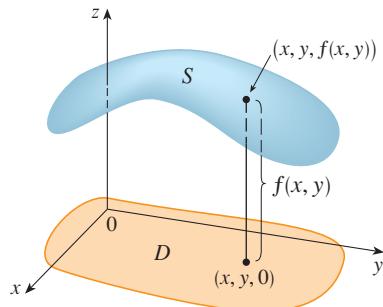


FIGURE 5

DEFINITION If f is a function of two variables with domain D , then the **graph** of f is the set of all points (x, y, z) in \mathbb{R}^3 such that $z = f(x, y)$ and (x, y) is in D .

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy -plane. (See Figure 5.)

EXAMPLE 5 Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

SOLUTION The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x -intercept. Similarly, the y -intercept is 3 and the z -intercept is 6. This helps us sketch the portion of the graph that lies in the first octant. (See Figure 6.)

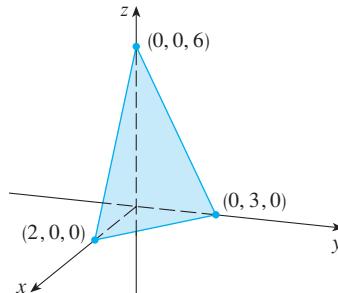


FIGURE 6

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \quad \text{or} \quad ax + by - z + c = 0$$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7). ■

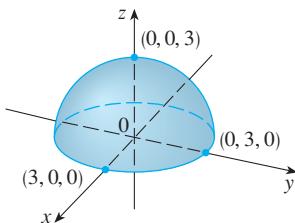


FIGURE 7

Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

NOTE An entire sphere can't be represented by a single function of x and y . As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

EXAMPLE 7 Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

SOLUTION Figure 8 shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as is to be expected.

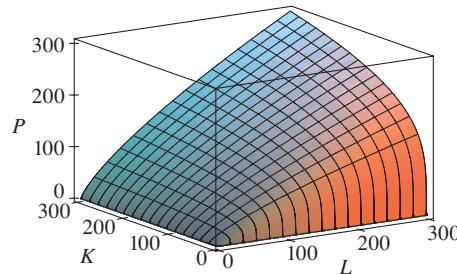


FIGURE 8



EXAMPLE 8 Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

SOLUTION Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y) , so the domain is \mathbb{R}^2 , the entire xy -plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geq 0$ and $y^2 \geq 0$, so $h(x, y) \geq 0$ for all x and y .]

The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 4 in Section 12.6. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

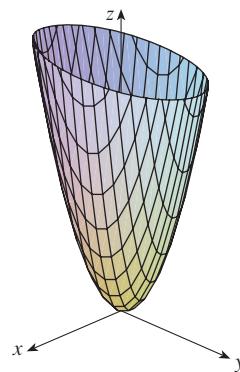


FIGURE 9
Graph of $h(x, y) = 4x^2 + y^2$



Computer programs are readily available for graphing functions of two variables. In most such programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal.

Figure 10 shows computer-generated graphs of several functions. Notice that we get an especially good picture of a function when rotation is used to give views from different vantage points. In parts (a) and (b) the graph of f is very flat and close to the xy -plane except near the origin; this is because $e^{-x^2-y^2}$ is very small when x or y is large.

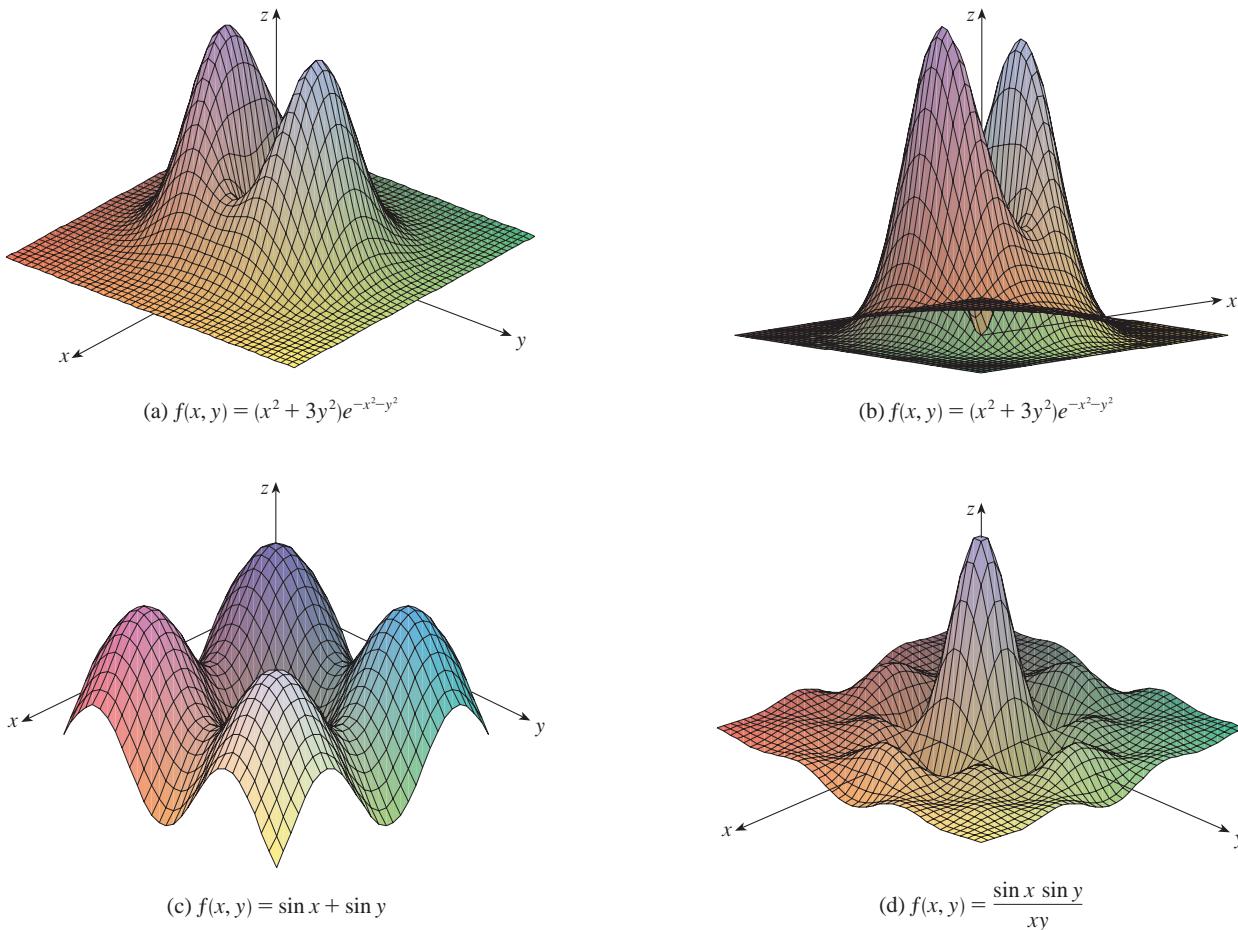


FIGURE 10

LEVEL CURVES

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

DEFINITION The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k . In other words, it shows where the graph of f has height k .

You can see from Figure 11 the relation between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy -plane. So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece

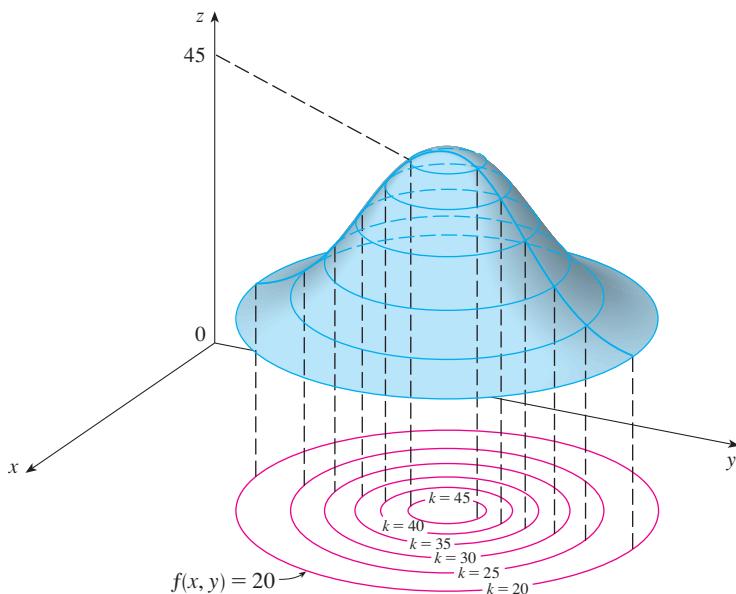


FIGURE 11

TEC Visual 14.1A animates Figure 11 by showing level curves being lifted up to graphs of functions.

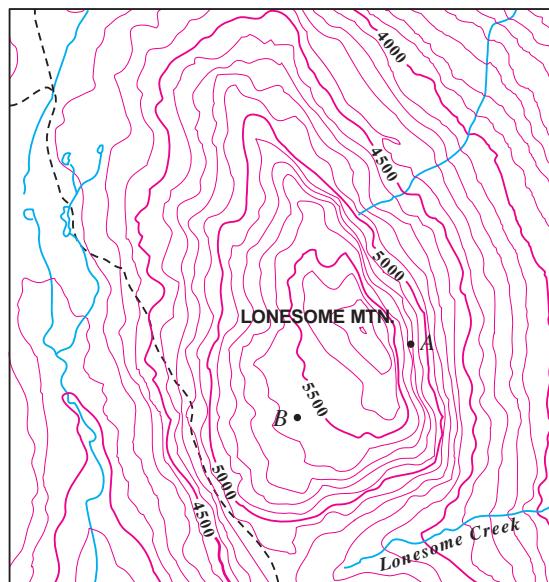


FIGURE 12

together a picture of the graph. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12. The level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend. Another common example is the temperature function introduced in the opening paragraph of this section. Here the level curves are called **isothermals** and join locations with the same temperature. Figure 13 shows a weather map of the world indicating the average January temperatures. The isothermals are the curves that separate the colored bands. The **isobars** in the atmospheric pressure map on page 854 provide another example of level curves.

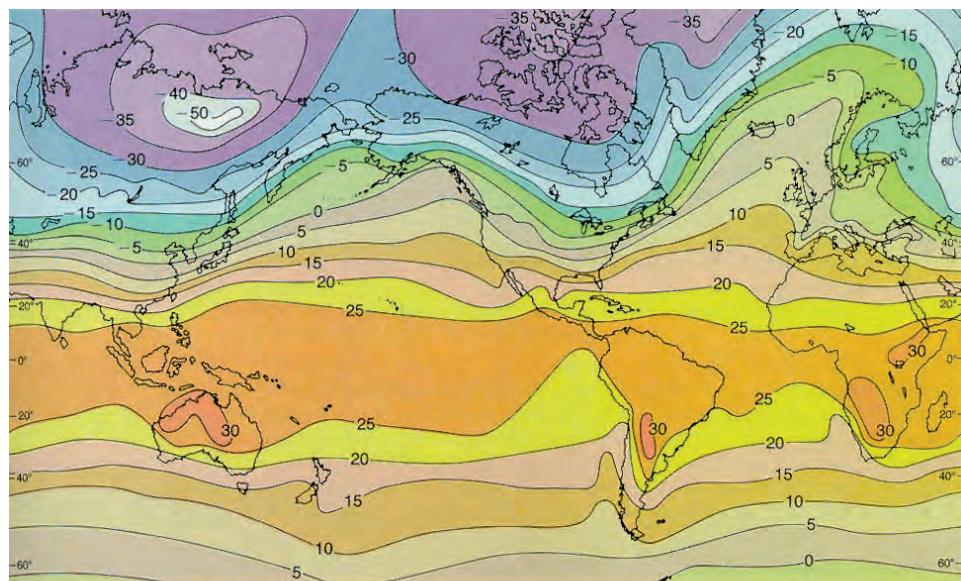


FIGURE 13

World mean sea-level temperatures in January in degrees Celsius

Tarbuck, *Atmosphere: Introduction to Meteorology*, 4th Edition, © 1989. Reprinted by permission of Pearson Education, Inc., Upper Saddle River, NJ.

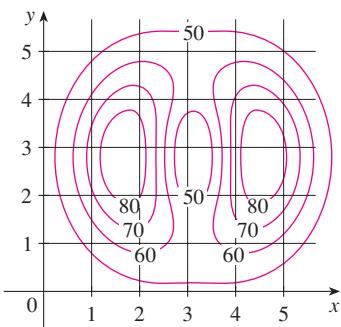


FIGURE 14

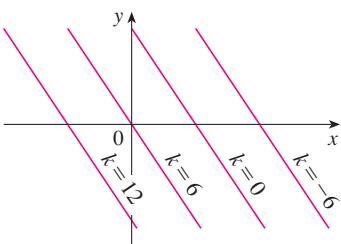


FIGURE 15

Contour map of
 $f(x, y) = 6 - 3x - 2y$

EXAMPLE 9 A contour map for a function f is shown in Figure 14. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

SOLUTION The point $(1, 3)$ lies partway between the level curves with z -values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that

$$f(4, 5) \approx 56$$

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 15. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6). ■

EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \quad \text{for } k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \quad \text{or} \quad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in Figure 16. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. (See TEC Visual 14.1A.)

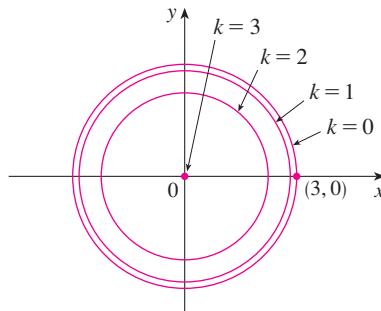


FIGURE 16

Contour map of $g(x, y) = \sqrt{9 - x^2 - y^2}$

EXAMPLE 12 Sketch some level curves of the function $h(x, y) = 4x^2 + y^2$.

SOLUTION The level curves are

$$4x^2 + y^2 = k \quad \text{or} \quad \frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which, for $k > 0$, describes a family of ellipses with semiaxes $\sqrt{k}/2$ and \sqrt{k} . Figure 17(a) shows a contour map of h drawn by a computer with level curves corresponding to $k = 0.25, 0.5, 0.75, \dots, 4$. Figure 17(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 17 how the graph of h is put together from the level curves.

TEC Visual 14.1B demonstrates the connection between surfaces and their contour maps.

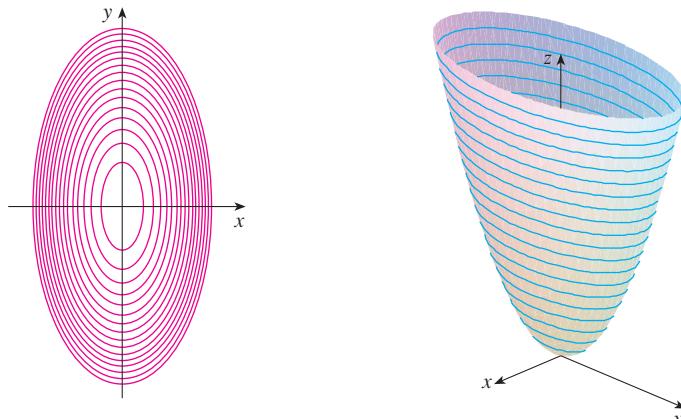


FIGURE 17

The graph of $h(x, y) = 4x^2 + y^2$ is formed by lifting the level curves.

(a) Contour map

(b) Horizontal traces are raised level curves

EXAMPLE 13 Plot level curves for the Cobb-Douglas production function of Example 3.

SOLUTION In Figure 18 we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

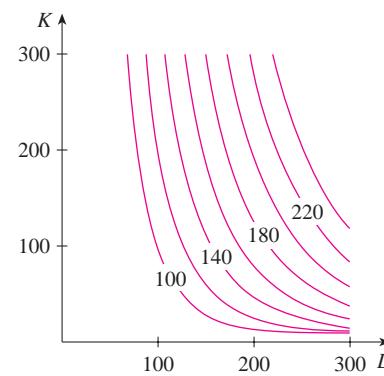
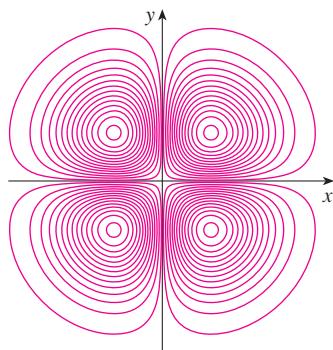
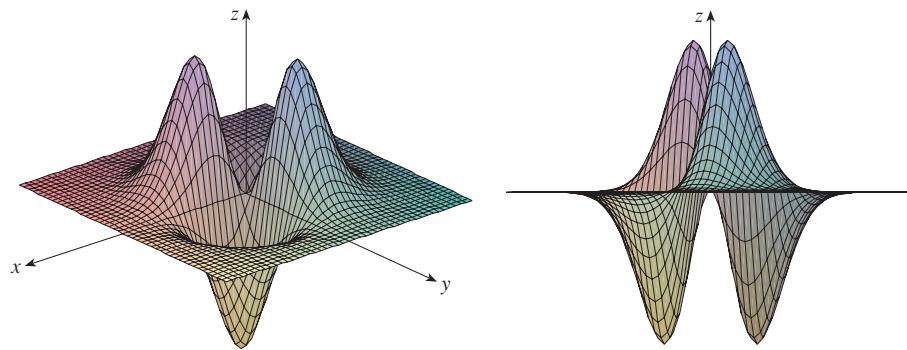
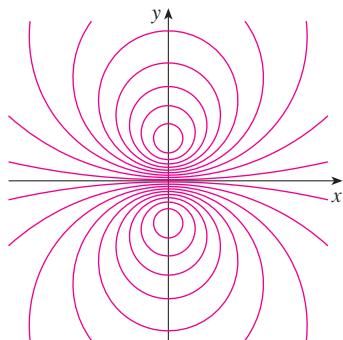
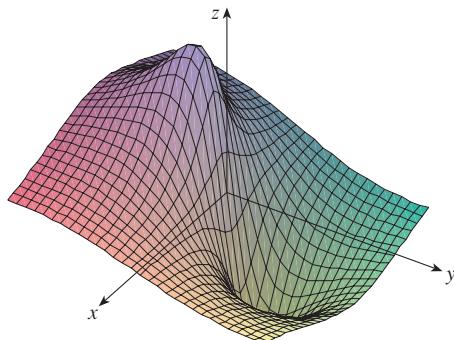


FIGURE 18

Level curves are labeled with the value of the production P . For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$. We see that, for a fixed value of P , as L increases K decreases, and vice versa.

For some purposes, a contour map is more useful than a graph. That is certainly true in Example 13. (Compare Figure 18 with Figure 8.) It is also true in estimating function values, as in Example 9.

Figure 19 shows some computer-generated level curves together with the corresponding computer-generated graphs. Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.

(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$ (b) Two views of $f(x, y) = -xye^{-x^2-y^2}$ (c) Level curves of $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$ (d) $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$

FUNCTIONS OF THREE OR MORE VARIABLES

A **function of three variables**, f , is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t , so we could write $T = f(x, y, t)$.

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$. ■

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

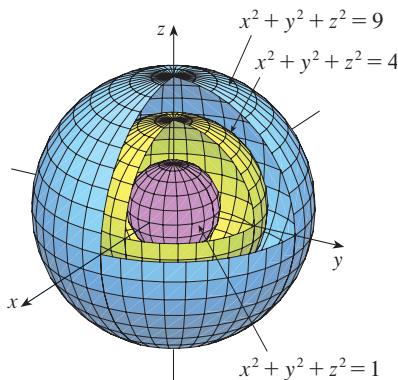


FIGURE 20

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius \sqrt{k} . (See Figure 20.) Thus, as (x, y, z) varies over any sphere with center O , the value of $f(x, y, z)$ remains fixed. ■

Functions of any number of variables can be considered. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \dots, x_n)$ to an n -tuple (x_1, x_2, \dots, x_n) of real numbers. We denote by \mathbb{R}^n the set of all such n -tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the i th ingredient, and x_i units of the i th ingredient are used, then the total cost C of the ingredients is a function of the n variables x_1, x_2, \dots, x_n :

$$3 \quad C = f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of \mathbb{R}^n . Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \dots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \dots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors \mathbf{c} and \mathbf{x} in V_n .

In view of the one-to-one correspondence between points (x_1, x_2, \dots, x_n) in \mathbb{R}^n and their position vectors $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ in V_n , we have three ways of looking at a function f defined on a subset of \mathbb{R}^n :

1. As a function of n real variables x_1, x_2, \dots, x_n
2. As a function of a single point variable (x_1, x_2, \dots, x_n)
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$

We will see that all three points of view are useful.

14.1 EXERCISES

1. In Example 2 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1.
 - (a) What is the value of $f(-15, 40)$? What is its meaning?
 - (b) Describe in words the meaning of the question "For what value of v is $f(-20, v) = -30$?" Then answer the question.
 - (c) Describe in words the meaning of the question "For what value of T is $f(T, 20) = -49$?" Then answer the question.
 - (d) What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
 - (e) What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

2. The *temperature-humidity index I* (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h , so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

TABLE 3 Apparent temperature as a function of temperature and humidity

		Relative humidity (%)					
		20	30	40	50	60	70
Actual temperature (°F)	20	77	78	79	81	82	83
	80	82	84	86	88	90	93
	85	87	90	93	96	100	106
	90	93	96	101	107	114	124
	95	99	104	110	120	132	144
	100	104	110	120	132	144	

- (a) What is the value of $f(95, 70)$? What is its meaning?
 (b) For what value of h is $f(90, h) = 100$?
 (c) For what value of T is $f(T, 50) = 88$?
 (d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h .
3. Verify for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

discussed in Example 3 that the production will be doubled if both the amount of labor and the amount of capital are doubled. Determine whether this is also true for the general production function

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

4. The wind-chill index W discussed in Example 2 has been modeled by the following function:

$$W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

Check to see how closely this model agrees with the values in Table 1 for a few values of T and v .

5. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.
 (a) What is the value of $f(40, 15)$? What is its meaning?
 (b) What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
 (c) What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

TABLE 4

Duration (hours)

$v \setminus t$	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

6. Let $f(x, y) = \ln(x + y - 1)$.
- Evaluate $f(1, 1)$.
 - Evaluate $f(e, 1)$.
 - Find and sketch the domain of f .
 - Find the range of f .
7. Let $f(x, y) = x^2e^{3xy}$.
- Evaluate $f(2, 0)$.
 - Find the domain of f .
 - Find the range of f .
8. Find and sketch the domain of the function $f(x, y) = \sqrt{1 + x - y^2}$. What is the range of f ?
9. Let $f(x, y, z) = e^{\sqrt{z-x^2-y^2}}$.
- Evaluate $f(2, -1, 6)$.
 - Find the domain of f .
 - Find the range of f .
10. Let $g(x, y, z) = \ln(25 - x^2 - y^2 - z^2)$.
- Evaluate $g(2, -2, 4)$.
 - Find the domain of g .
 - Find the range of g .
- 11–20 Find and sketch the domain of the function.
11. $f(x, y) = \sqrt{x + y}$
12. $f(x, y) = \sqrt{xy}$
13. $f(x, y) = \ln(9 - x^2 - 9y^2)$
14. $f(x, y) = \sqrt{y - x} \ln(y + x)$
15. $f(x, y) = \sqrt{1 - x^2} - \sqrt{1 - y^2}$
16. $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$
17. $f(x, y) = \frac{\sqrt{y - x^2}}{1 - x^2}$
18. $f(x, y) = \arcsin(x^2 + y^2 - 2)$
19. $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$
20. $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

21–29 Sketch the graph of the function.

21. $f(x, y) = 3$

22. $f(x, y) = y$

23. $f(x, y) = 10 - 4x - 5y$

24. $f(x, y) = \cos x$

25. $f(x, y) = y^2 + 1$

26. $f(x, y) = 3 - x^2 - y^2$

27. $f(x, y) = 4x^2 + y^2 + 1$

28. $f(x, y) = \sqrt{16 - x^2 - 16y^2}$

29. $f(x, y) = \sqrt{x^2 + y^2}$

30. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = |x| + |y|$

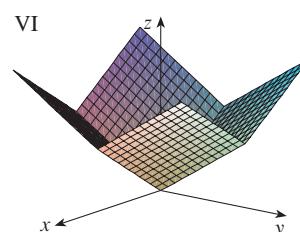
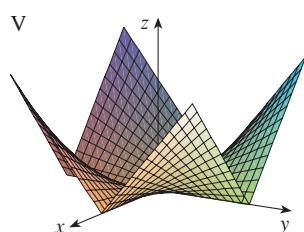
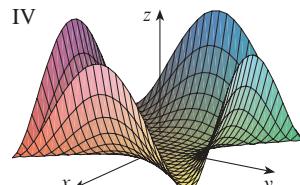
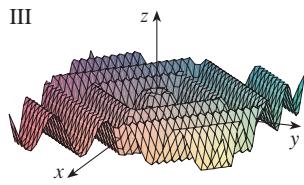
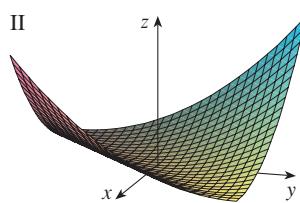
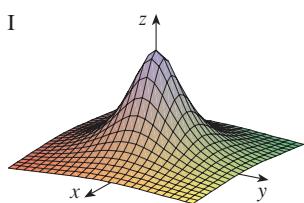
(b) $f(x, y) = |xy|$

(c) $f(x, y) = \frac{1}{1 + x^2 + y^2}$

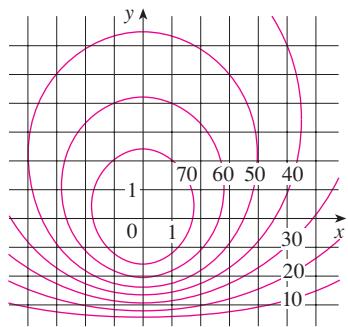
(d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^2$

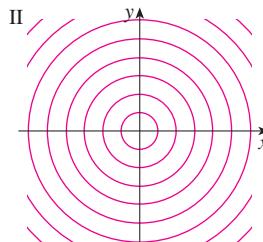
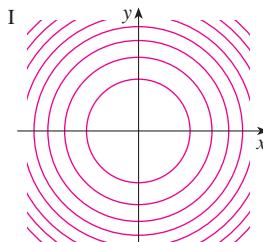
(f) $f(x, y) = \sin(|x| + |y|)$



31. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

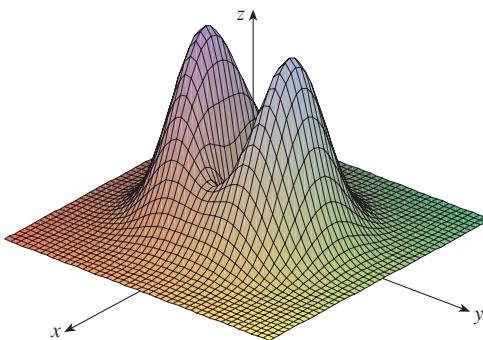


32. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?

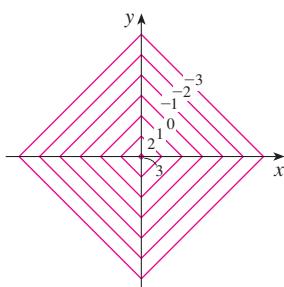
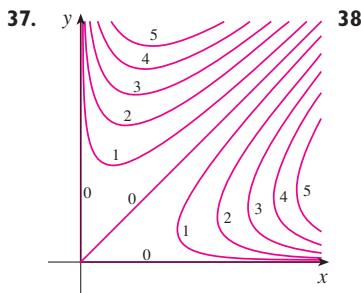
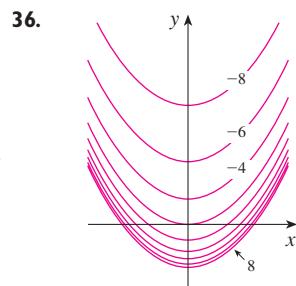
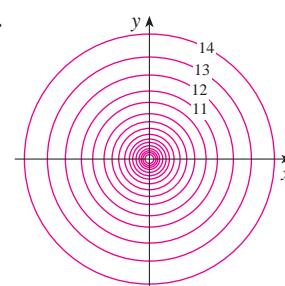


33. Locate the points A and B in the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A ? Near B ?

34. Make a rough sketch of a contour map for the function whose graph is shown.



35–38 A contour map of a function is shown. Use it to make a rough sketch of the graph of f .



39–46 Draw a contour map of the function showing several level curves.

39. $f(x, y) = (y - 2x)^2$

40. $f(x, y) = x^3 - y$

41. $f(x, y) = y - \ln x$

42. $f(x, y) = e^{y/x}$

43. $f(x, y) = ye^x$

44. $f(x, y) = y \sec x$

45. $f(x, y) = \sqrt{y^2 - x^2}$

46. $f(x, y) = y/(x^2 + y^2)$

47–48 Sketch both a contour map and a graph of the function and compare them.

47. $f(x, y) = x^2 + 9y^2$

48. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

49. A thin metal plate, located in the xy -plane, has temperature $T(x, y)$ at the point (x, y) . The level curves of T are called *isothermals* because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = 100/(1 + x^2 + 2y^2)$$

50. If $V(x, y)$ is the electric potential at a point (x, y) in the xy -plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

51–54 Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

51. $f(x, y) = e^{-x^2} + e^{-2y^2}$

52. $f(x, y) = (1 - 3x^2 + y^2)e^{1-x^2-y^2}$

53. $f(x, y) = xy^2 - x^3$ (monkey saddle)

54. $f(x, y) = xy^3 - yx^3$ (dog saddle)

55–60 Match the function (a) with its graph (labeled A–F on page 869) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

55. $z = \sin(xy)$

56. $z = e^x \cos y$

57. $z = \sin(x - y)$

58. $z = \sin x - \sin y$

59. $z = (1 - x^2)(1 - y^2)$

60. $z = \frac{x - y}{1 + x^2 + y^2}$

61–64 Describe the level surfaces of the function.

61. $f(x, y, z) = x + 3y + 5z$

62. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

63. $f(x, y, z) = x^2 - y^2 + z^2$

64. $f(x, y, z) = x^2 - y^2$

65–66 Describe how the graph of g is obtained from the graph of f .

65. (a) $g(x, y) = f(x, y) + 2$ (b) $g(x, y) = 2f(x, y)$
 (c) $g(x, y) = -f(x, y)$ (d) $g(x, y) = 2 - f(x, y)$

66. (a) $g(x, y) = f(x - 2, y)$ (b) $g(x, y) = f(x, y + 2)$
 (c) $g(x, y) = f(x + 3, y - 4)$

67–68 Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the “peaks and valleys.” Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be “local maximum points”? What about “local minimum points”?

67. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

68. $f(x, y) = xye^{-x^2-y^2}$

69–70 Use a computer to graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

69. $f(x, y) = \frac{x + y}{x^2 + y^2}$ 70. $f(x, y) = \frac{xy}{x^2 + y^2}$

71. Use a computer to investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c ?

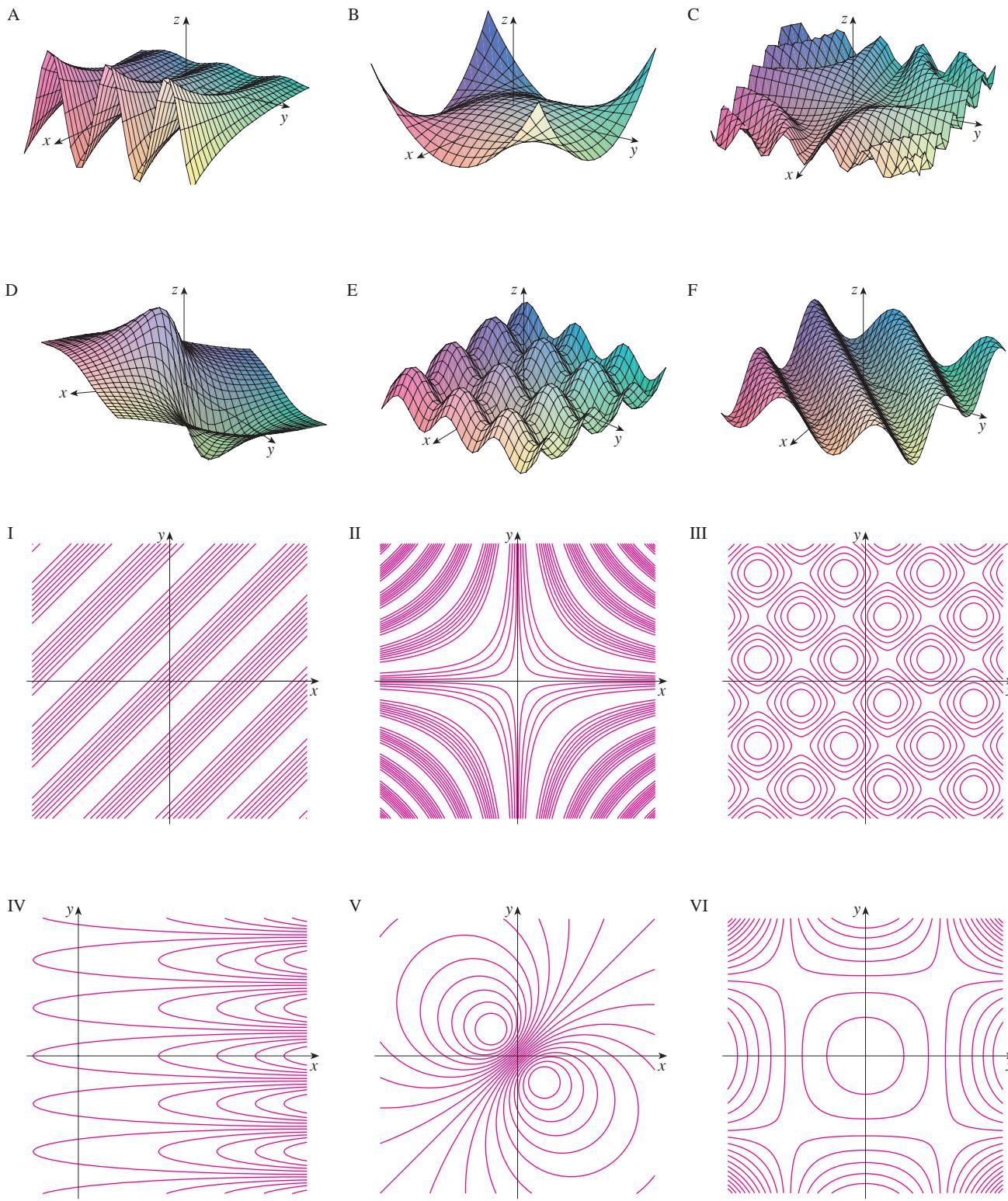
72. Use a computer to investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b ?

73. Use a computer to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

Graphs and Contour Maps for Exercises 55–60



74. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = e^{\sqrt{x^2 + y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2}$$

$$f(x, y) = \sin(\sqrt{x^2 + y^2})$$

and

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g(\sqrt{x^2 + y^2})$$

obtained from the graph of g ?

75. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^\alpha K^{1-\alpha}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 3) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then use a graphing calculator or computer to find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

14.2 LIMITS AND CONTINUITY

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

TABLE 1 Values of $f(x, y)$

$x \setminus y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

TABLE 2 Values of $g(x, y)$

$x \setminus y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.) It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and} \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ does not exist}$$

In general, we use the notation

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f . In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b) , but not equal to (a, b) . A more precise definition follows.

1 DEFINITION Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) . Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y) \in D \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \text{ then } |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = L \quad \text{and} \quad f(x, y) \rightarrow L \text{ as } (x, y) \rightarrow (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L , and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b) . Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L , then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

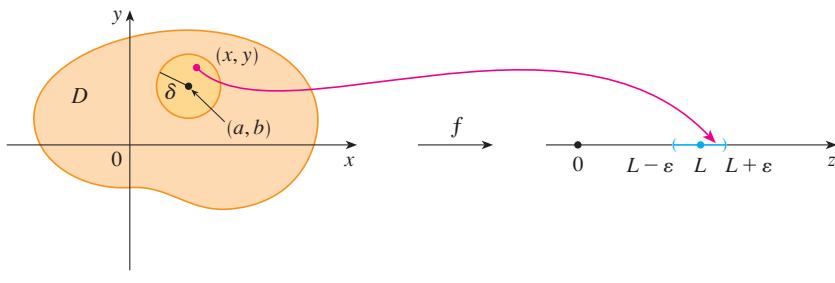


FIGURE 1

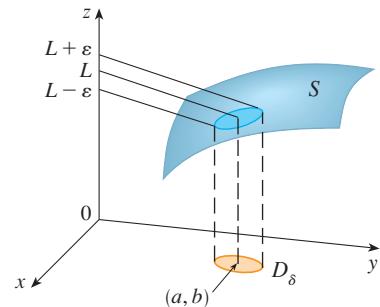


FIGURE 2

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f . If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

For functions of a single variable, when we let x approach a , there are only two possible directions of approach, from the left or from the right. We recall from Chapter 2 that if $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f .

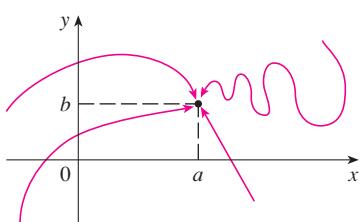


FIGURE 3

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b) . It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b) . Thus if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x -axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \rightarrow 1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y -axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \rightarrow -1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.) Since f has two different limits along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.) █

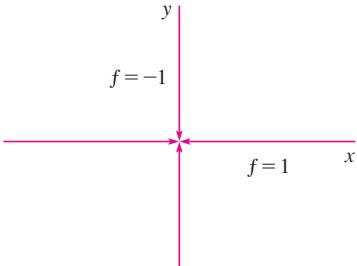


FIGURE 4

EXAMPLE 2 If $f(x, y) = \frac{xy}{x^2 + y^2}$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along $y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. █

FIGURE 5

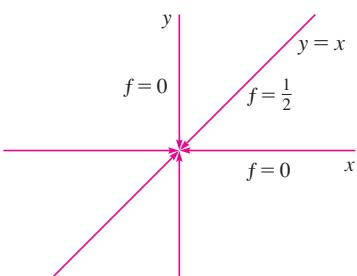


Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

TEC In Visual 14.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

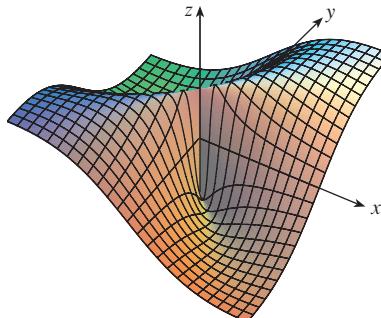


FIGURE 6

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

V EXAMPLE 3 If $f(x, y) = \frac{xy^2}{x^2 + y^4}$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \rightarrow (0, 0)$ along any nonvertical line through the origin. Then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

So $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along $y = mx$

Thus f has the same limiting value along every nonvertical line through the origin. But that does not show that the given limit is 0, for if we now let $(x, y) \rightarrow (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist. ■

Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

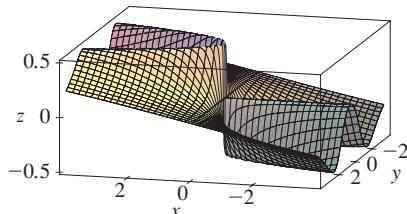


FIGURE 7

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables: The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\boxed{2} \quad \lim_{(x, y) \rightarrow (a, b)} x = a \quad \lim_{(x, y) \rightarrow (a, b)} y = b \quad \lim_{(x, y) \rightarrow (a, b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE 4 Find $\lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas

$y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{that is, } \text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $x^2/(x^2 + y^2) \leq 1$ and therefore

$$\boxed{3} \quad \frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0$$

- Another way to do Example 4 is to use the Squeeze Theorem instead of Definition 1. From (2) it follows that

$$\lim_{(x,y) \rightarrow (0,0)} 3|y| = 0$$

and so the first inequality in (3) shows that the given limit is 0.

CONTINUITY

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \rightarrow a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

4 DEFINITION A function f of two variables is called **continuous at (a, b)** if

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

We say f is **continuous on D** if f is continuous at every point (a, b) in D .

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use this fact to give examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in (2) show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f , g , and h by multiplication and addition, it follows that *all polynomials are continuous on \mathbb{R}^2* . Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

EXAMPLE 5 Evaluate $\lim_{(x, y) \rightarrow (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x, y) \rightarrow (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

EXAMPLE 6 Where is the function $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$.

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x, y) \rightarrow (0, 0)} g(x, y)$ does not exist (see Example 1).

Figure 8 shows the graph of the continuous function in Example 8.

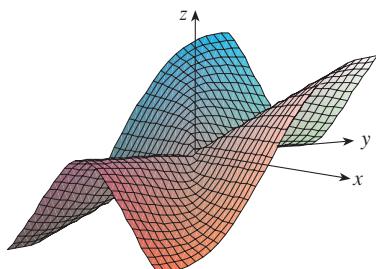


FIGURE 8

EXAMPLE 8 Let

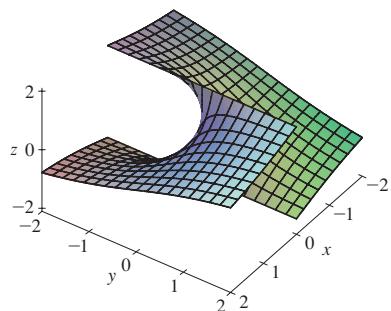
$$f(x, y) = \begin{cases} \frac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on \mathbb{R}^2 .

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f , then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

**FIGURE 9**

The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

EXAMPLE 9 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 9 shows the break in the graph of h above the y -axis. ■

FUNCTIONS OF THREE OR MORE VARIABLES

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f . Because the distance between two points (x, y, z) and (a, b, c) in \mathbb{R}^3 is given by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, we can write the precise definition as follows: For every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$

$$\text{then } |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in \mathbb{R}^3 except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

5 If f is defined on a subset D of \mathbb{R}^n , then $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \mathbf{x} \in D \text{ and } 0 < |\mathbf{x} - \mathbf{a}| < \delta \text{ then } |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (5) is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$, $\mathbf{a} = \langle a, b \rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x-a)^2 + (y-b)^2}$, so (5) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (5) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 EXERCISES

- Suppose that $\lim_{(x,y) \rightarrow (3,1)} f(x,y) = 6$. What can you say about the value of $f(3,1)$? What if f is continuous?
- Explain why each function is continuous or discontinuous.
 - The outdoor temperature as a function of longitude, latitude, and time
 - Elevation (height above sea level) as a function of longitude, latitude, and time
 - The cost of a taxi ride as a function of distance traveled and time

3–4 Use a table of numerical values of $f(x,y)$ for (x,y) near the origin to make a conjecture about the value of the limit of $f(x,y)$ as $(x,y) \rightarrow (0,0)$. Then explain why your guess is correct.

$$3. f(x,y) = \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$$

$$4. f(x,y) = \frac{2xy}{x^2 + 2y^2}$$

5–22 Find the limit, if it exists, or show that the limit does not exist.

$$5. \lim_{(x,y) \rightarrow (1,2)} (5x^3 - x^2y^2)$$

$$6. \lim_{(x,y) \rightarrow (1,-1)} e^{-xy} \cos(x+y)$$

$$7. \lim_{(x,y) \rightarrow (2,1)} \frac{4 - xy}{x^2 + 3y^2}$$

$$8. \lim_{(x,y) \rightarrow (1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$$

$$9. \lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{x^4 + 3y^4}$$

$$10. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$$

$$11. \lim_{(x,y) \rightarrow (0,0)} \frac{xy \cos y}{3x^2 + y^2}$$

$$12. \lim_{(x,y) \rightarrow (0,0)} \frac{6x^3y}{2x^4 + y^4}$$

$$13. \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

$$14. \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2}$$

$$15. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2ye^y}{x^4 + 4y^2}$$

$$16. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

$$17. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

$$18. \lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^2 + y^8}$$

$$19. \lim_{(x,y,z) \rightarrow (3,0,1)} e^{-xy} \sin(\pi z/2)$$

$$20. \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$$

$$21. \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$$

$$22. \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2}$$

23–24 Use a computer graph of the function to explain why the limit does not exist.

$$23. \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$$

$$24. \lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6}$$

25–26 Find $h(x,y) = g(f(x,y))$ and the set on which h is continuous.

$$25. g(t) = t^2 + \sqrt{t}, \quad f(x,y) = 2x + 3y - 6$$

$$26. g(t) = t + \ln t, \quad f(x,y) = \frac{1 - xy}{1 + x^2y^2}$$

27–28 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

$$27. f(x,y) = e^{1/(x-y)}$$

$$28. f(x,y) = \frac{1}{1 - x^2 - y^2}$$

29–38 Determine the set of points at which the function is continuous.

$$29. F(x,y) = \frac{\sin(xy)}{e^x - y^2}$$

$$30. F(x,y) = \frac{x - y}{1 + x^2 + y^2}$$

$$31. F(x,y) = \arctan(x + \sqrt{y})$$

$$32. F(x,y) = e^{x^2y} + \sqrt{x + y^2}$$

$$33. G(x,y) = \ln(x^2 + y^2 - 4)$$

$$34. G(x,y) = \tan^{-1}((x+y)^{-2})$$

$$35. f(x,y,z) = \frac{\sqrt{y}}{x^2 - y^2 + z^2}$$

$$36. f(x,y,z) = \sqrt{x + y + z}$$

$$37. f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

$$38. f(x,y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x,y) with $r \geq 0$, note that $r \rightarrow 0^+$ as $(x,y) \rightarrow (0,0)$.]

$$39. \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$$

$$40. \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2)$$

$$41. \lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$$

42. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed that $f(x, y) \rightarrow 1$ as $(x, y) \rightarrow (0, 0)$ on the basis of numerical evidence. Use polar coordinates to confirm the value of the limit. Then graph the function.

43. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \frac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

44. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \quad \text{or} \quad y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

- (a) Show that $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $a < 4$.
 (b) Despite part (a), show that f is discontinuous at $(0, 0)$.
 (c) Show that f is discontinuous on two entire curves.

45. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on \mathbb{R}^n . [Hint: Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

46. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on \mathbb{R}^n .

14.3 PARTIAL DERIVATIVES

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H . So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

TABLE I

Heat index I as a function of temperature and humidity

		Relative humidity (%)								
		50	55	60	65	70	75	80	85	90
T \ H		90	98	100	103	106	109	112	115	119
Actual temperature (°F)	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of $H = 70\%$, we are considering the heat index as a function of the single variable T for a fixed value of H . Let's write $g(T) = f(T, 70)$. Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%. The derivative of g when $T = 96^\circ\text{F}$ is the rate of change of I with respect to T when $T = 96^\circ\text{F}$:

$$g'(96) = \lim_{h \rightarrow 0} \frac{g(96 + h) - g(96)}{h} = \lim_{h \rightarrow 0} \frac{f(96 + h, 70) - f(96, 70)}{h}$$

We can approximate $g'(96)$ using the values in Table 1 by taking $h = 2$ and -2 :

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98, 70) - f(96, 70)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94, 70) - f(96, 70)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we can say that the derivative $g'(96)$ is approximately 3.75. This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises!

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of $T = 96^{\circ}\text{F}$. The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96^{\circ}\text{F}$. The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h \rightarrow 0} \frac{G(70 + h) - G(70)}{h} = \lim_{h \rightarrow 0} \frac{f(96, 70 + h) - f(96, 70)}{h}$$

By taking $h = 5$ and -5 , we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

In general, if f is a function of two variables x and y , suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x , namely, $g(x) = f(x, b)$. If g has a derivative at a , then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

1

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

2

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of f with respect to y at (a, b)** , denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96^\circ\text{F}$ and $H = 70\%$ as follows:

$$f_T(96, 70) \approx 3.75 \quad f_H(96, 70) \approx 0.9$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$. But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

NOTATIONS FOR PARTIAL DERIVATIVES If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following rule.

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$

1. To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x , we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y , we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$



INTERPRETATIONS OF PARTIAL DERIVATIVES

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S . By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S . (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2 . Both of the curves C_1 and C_2 pass through the point P . (See Figure 1.)

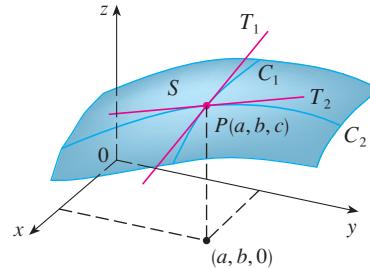


FIGURE 1

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2 .

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z / \partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z / \partial y$ represents the rate of change of z with respect to y when x is fixed.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

SOLUTION We have

$$f_x(x, y) = -2x \quad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \quad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

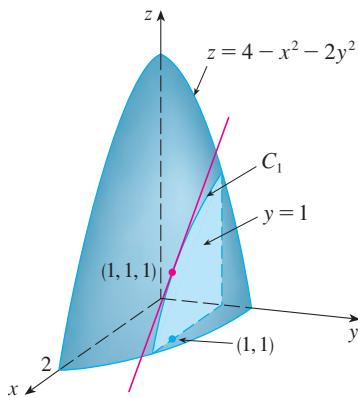


FIGURE 2

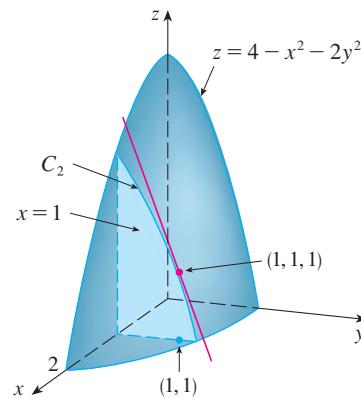


FIGURE 3

Figure 4 is a computer-drawn counterpart to Figure 2. Part (a) shows the plane $y = 1$ intersecting the surface to form the curve C_1 and part (b) shows C_1 and T_1 . [We have used the vector equations $\mathbf{r}(t) = \langle t, 1, 2 - t^2 \rangle$ for C_1 and $\mathbf{r}(t) = \langle 1 + t, 1, 1 - 2t \rangle$ for T_1 .] Similarly, Figure 5 corresponds to Figure 3.

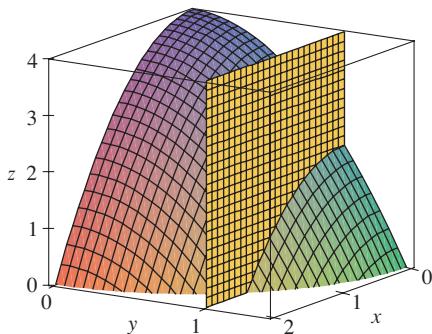
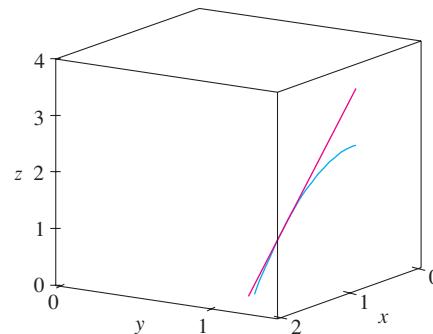


FIGURE 4

(a)



(b)

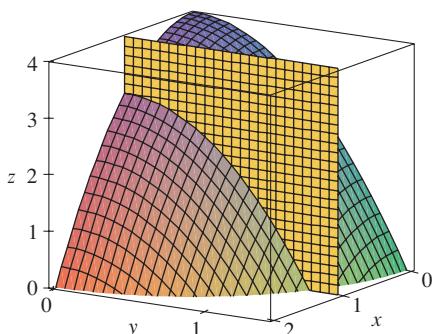
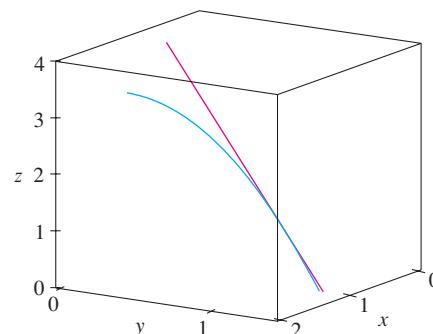


FIGURE 5



V EXAMPLE 3 If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

- Some computer algebra systems can plot surfaces defined by implicit equations in three variables. Figure 6 shows such a plot of the surface defined by the equation in Example 4.

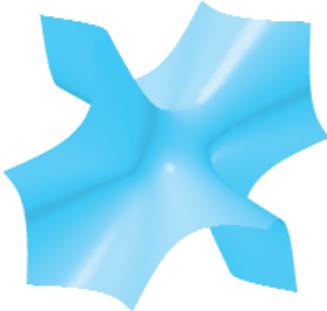


FIGURE 6

V EXAMPLE 4 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

SOLUTION To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x , being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Solving this equation for $\frac{\partial z}{\partial x}$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

FUNCTIONS OF MORE THAN TWO VARIABLES

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x , y , and z , then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x . If $w = f(x, y, z)$, then $f_x = \frac{\partial w}{\partial x}$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \dots, x_n)$, its partial derivative with respect to the i th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

EXAMPLE 5 Find f_x , f_y , and f_z if $f(x, y, z) = e^{xy} \ln z$.

SOLUTION Holding y and z constant and differentiating with respect to x , we have

$$f_x = ye^{xy} \ln z$$

$$\text{Similarly, } f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z} \quad \blacksquare$$

HIGHER DERIVATIVES

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f . If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f / \partial y \partial x$) means that we first differentiate with respect to x and then with respect to y , whereas in computing f_{yx} the order is reversed.

EXAMPLE 6 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

SOLUTION In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \quad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x} (3x^2 + 2xy^3) = 6x + 2y^3 \quad f_{xy} = \frac{\partial}{\partial y} (3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x} (3x^2y^2 - 4y) = 6xy^2 \quad f_{yy} = \frac{\partial}{\partial y} (3x^2y^2 - 4y) = 6x^2y - 4 \quad \blacksquare$$

Figure 7 shows the graph of the function f in Example 6 and the graphs of its first- and second-order partial derivatives for $-2 \leq x \leq 2$, $-2 \leq y \leq 2$. Notice that these graphs are consistent with our interpretations of f_x and f_y as slopes of tangent lines to traces of the graph of f . For instance, the graph of f decreases if we start at $(0, -2)$ and move in the positive x -direction. This is reflected in the negative values of f_x . You should compare the graphs of f_{xy} and f_{yy} with the graph of f_y to see the relationships.

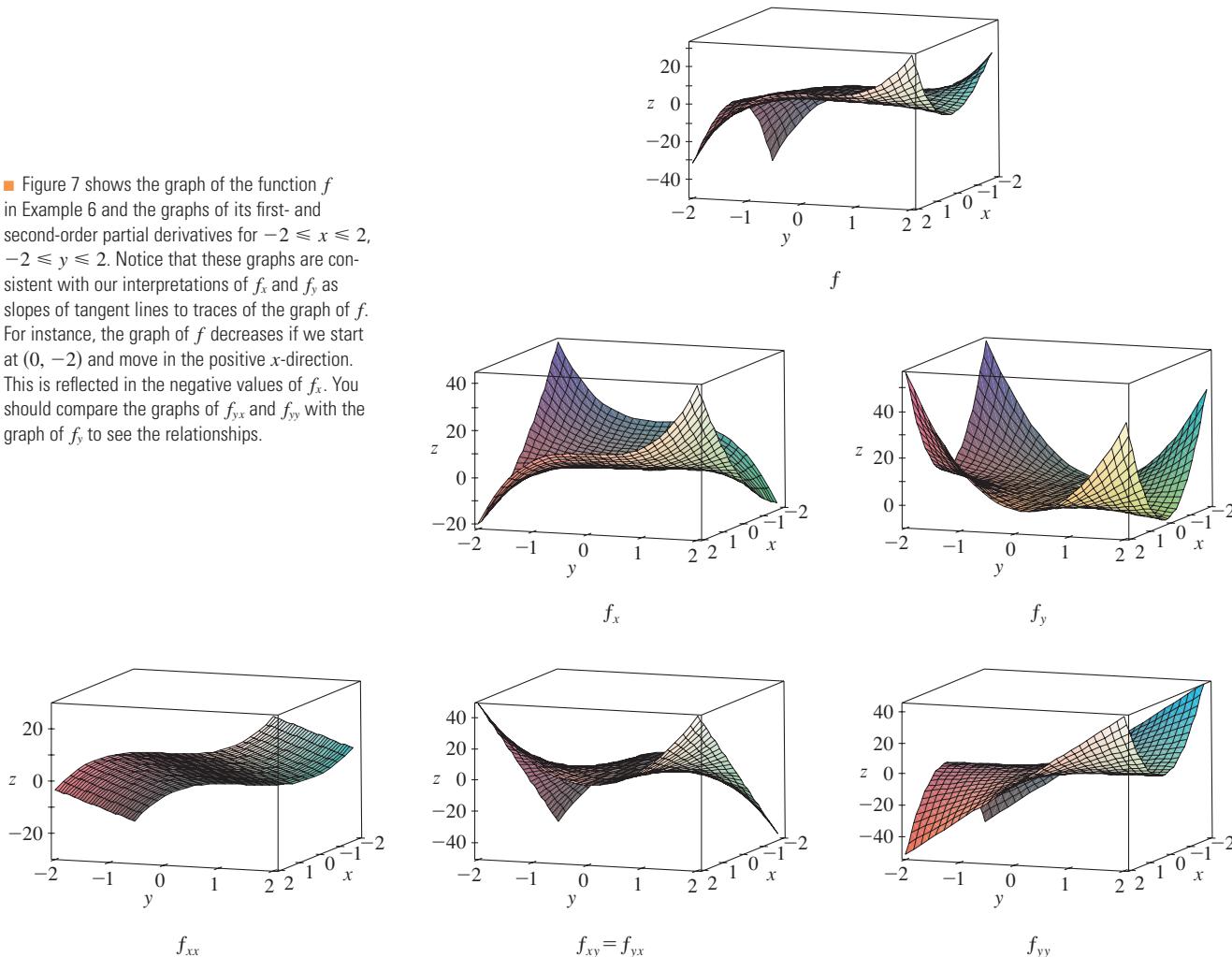


FIGURE 7

Notice that $f_{xy} = f_{yx}$ in Example 6. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice. The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$. The proof is given in Appendix F.

■ Alexis Clairaut was a child prodigy in mathematics: he read l'Hospital's textbook on calculus when he was ten and presented a paper on geometry to the French Academy of Sciences when he was 13. At the age of 18, Clairaut published *Recherches sur les courbes à double courbure*, which was the first systematic treatise on three-dimensional analytic geometry and included the calculus of space curves.

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

V EXAMPLE 7 Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

SOLUTION

$$f_x = 3 \cos(3x + yz)$$

$$f_{xx} = -9 \sin(3x + yz)$$

$$f_{xxy} = -9z \cos(3x + yz)$$

$$f_{xxyz} = -9 \cos(3x + yz) + 9yz \sin(3x + yz)$$



PARTIAL DIFFERENTIAL EQUATIONS

Partial derivatives occur in *partial differential equations* that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

EXAMPLE 8 Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

SOLUTION

$$u_x = e^x \sin y \quad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \quad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.



The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string. For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 8), then $u(x, t)$ satisfies the wave equation. Here the constant a depends on the density of the string and on the tension in the string.

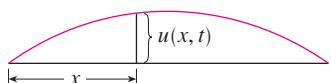


FIGURE 8

EXAMPLE 9 Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

SOLUTION

$$u_x = \cos(x - at) \quad u_{xx} = -\sin(x - at)$$

$$u_t = -a \cos(x - at) \quad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation.



THE COBB-DOUGLAS PRODUCTION FUNCTION

In Example 3 in Section 14.1 we described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of the amount of labor L and the capital investment K . Here we use partial derivatives to show how the particular form of their model follows from certain assumptions they made about the economy.

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\partial P / \partial L$ is the rate at which production changes with respect to the amount of labor. Economists call it the marginal production with respect to labor or the *marginal productivity of labor*. Likewise, the partial derivative $\partial P / \partial K$ is the rate of change of production with respect to capital and is called the *marginal productivity of capital*. In these terms, the assumptions made by Cobb and Douglas can be stated as follows.

- (i) If either labor or capital vanishes, then so will production.
- (ii) The marginal productivity of labor is proportional to the amount of production per unit of labor.
- (iii) The marginal productivity of capital is proportional to the amount of production per unit of capital.

Because the production per unit of labor is P/L , assumption (ii) says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α . If we keep K constant ($K = K_0$), then this partial differential equation becomes an ordinary differential equation:

$$5 \quad \frac{dP}{dL} = \alpha \frac{P}{L}$$

If we solve this separable differential equation by the methods of Section 9.3 (see also Exercise 79), we get

$$6 \quad P(L, K_0) = C_1(K_0)L^\alpha$$

Notice that we have written the constant C_1 as a function of K_0 because it could depend on the value of K_0 .

Similarly, assumption (iii) says that

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

and we can solve this differential equation to get

$$7 \quad P(L_0, K) = C_2(L_0)K^\beta$$

Comparing Equations 6 and 7, we have

$$8 \quad P(L, K) = bL^\alpha K^\beta$$

where b is a constant that is independent of both L and K . Assumption (i) shows that $\alpha > 0$ and $\beta > 0$.

Notice from Equation 8 that if labor and capital are both increased by a factor m , then

$$P(mL, mK) = b(mL)^\alpha(mK)^\beta = m^{\alpha+\beta}bL^\alpha K^\beta = m^{\alpha+\beta}P(L, K)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of m . That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

This is the Cobb-Douglas production function that we discussed in Section 14.1.

14.3 EXERCISES

1. The temperature T at a location in the Northern Hemisphere depends on the longitude x , latitude y , and time t , so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.
- What are the meanings of the partial derivatives $\partial T / \partial x$, $\partial T / \partial y$, and $\partial T / \partial t$?
 - Honolulu has longitude 158° W and latitude 21° N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.
2. At the beginning of this section we discussed the function $I = f(T, H)$, where I is the heat index, T is the temperature, and H is the relative humidity. Use Table 1 to estimate $f_T(92, 60)$ and $f_H(92, 60)$. What are the practical interpretations of these values?
3. The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v , so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1.

Wind speed (km/h)

Actual temperature ($^{\circ}\text{C}$)	Wind speed (km/h)					
	20	30	40	50	60	70
-10	-18	-20	-21	-22	-23	-23
-15	-24	-26	-27	-29	-30	-30
-20	-30	-33	-34	-35	-36	-37
-25	-37	-39	-41	-42	-43	-44

- (a) Estimate the values of $f_T(-15, 30)$ and $f_v(-15, 30)$. What are the practical interpretations of these values?

- (b) In general, what can you say about the signs of $\partial W / \partial T$ and $\partial W / \partial v$?
- (c) What appears to be the value of the following limit?

$$\lim_{v \rightarrow \infty} \frac{\partial W}{\partial v}$$

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

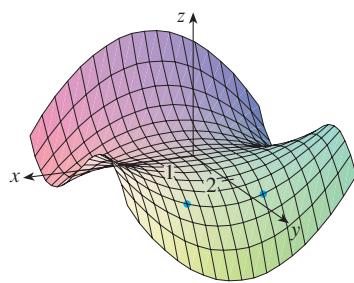
Duration (hours)

$v \setminus t$	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

- (a) What are the meanings of the partial derivatives $\partial h / \partial v$ and $\partial h / \partial t$?
- (b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?
- (c) What appears to be the value of the following limit?

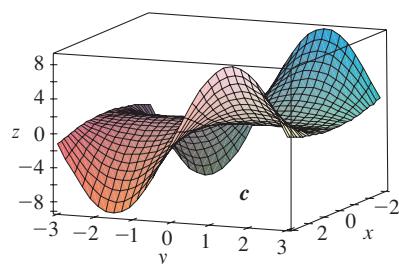
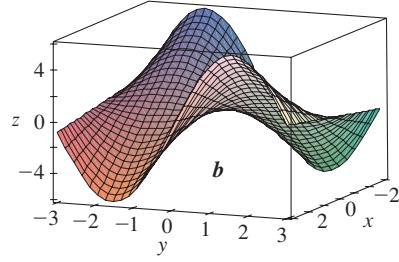
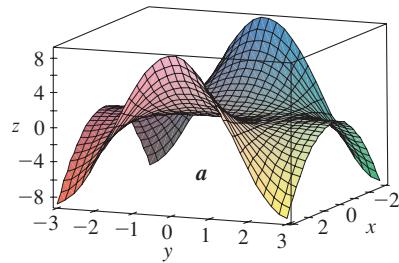
$$\lim_{t \rightarrow \infty} \frac{\partial h}{\partial t}$$

- 5–8** Determine the signs of the partial derivatives for the function f whose graph is shown.

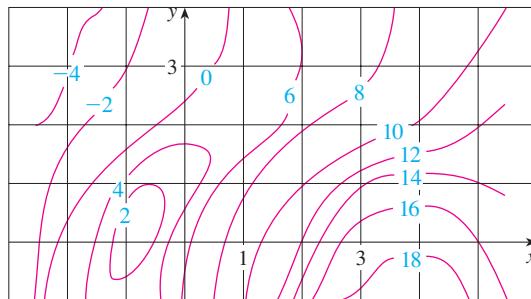


- 5.** (a) $f_x(1, 2)$ (b) $f_y(1, 2)$
6. (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$
7. (a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$
8. (a) $f_{xy}(1, 2)$ (b) $f_{yx}(-1, 2)$

- 9.** The following surfaces, labeled a , b , and c , are graphs of a function f and its partial derivatives f_x and f_y . Identify each surface and give reasons for your choices.



- 10.** A contour map is given for a function f . Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



- 11.** If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.
12. If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

- 13–14** Find f_x and f_y and graph f , f_x , and f_y with domains and viewpoints that enable you to see the relationships between them.
13. $f(x, y) = x^2 + y^2 + x^2y$ **14.** $f(x, y) = xe^{-x^2-y^2}$

- 15–38** Find the first partial derivatives of the function.

- 15.** $f(x, y) = y^5 - 3xy$ **16.** $f(x, y) = x^4y^3 + 8x^2y$
17. $f(x, t) = e^{-t}\cos \pi x$ **18.** $f(x, t) = \sqrt{x} \ln t$
19. $z = (2x + 3y)^{10}$ **20.** $z = \tan xy$
21. $f(x, y) = \frac{x - y}{x + y}$ **22.** $f(x, y) = x^y$
23. $w = \sin \alpha \cos \beta$ **24.** $w = e^v/(u + v^2)$
25. $f(r, s) = r \ln(r^2 + s^2)$ **26.** $f(x, t) = \arctan(x\sqrt{t})$
27. $u = te^{w/t}$ **28.** $f(x, y) = \int_y^x \cos(t^2) dt$
29. $f(x, y, z) = xz - 5x^2y^3z^4$ **30.** $f(x, y, z) = x \sin(y - z)$
31. $w = \ln(x + 2y + 3z)$ **32.** $w = ze^{xyz}$
33. $u = xy \sin^{-1}(yz)$ **34.** $u = x^{y/z}$
35. $f(x, y, z, t) = xyz^2 \tan(yt)$ **36.** $f(x, y, z, t) = \frac{xy^2}{t + 2z}$
37. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
38. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

- 39–42** Find the indicated partial derivatives.

- 39.** $f(x, y) = \ln(x + \sqrt{x^2 + y^2}); f_x(3, 4)$
40. $f(x, y) = \arctan(y/x); f_x(2, 3)$
41. $f(x, y, z) = \frac{y}{x + y + z}; f_y(2, 1, -1)$

42. $f(x, y, z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}; f_z(0, 0, \pi/4)$

43–44 Use the definition of partial derivatives as limits (4) to find $f_x(x, y)$ and $f_y(x, y)$.

43. $f(x, y) = xy^2 - x^3y$

44. $f(x, y) = \frac{x}{x+y^2}$

45–48 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

45. $x^2 + y^2 + z^2 = 3xyz$

46. $yz = \ln(x+z)$

47. $x - z = \arctan(yz)$

48. $\sin(xyz) = x + 2y + 3z$

49–50 Find $\partial z/\partial x$ and $\partial z/\partial y$.

49. (a) $z = f(x) + g(y)$

(b) $z = f(x+y)$

50. (a) $z = f(x)g(y)$

(b) $z = f(xy)$

(c) $z = f(x/y)$

51–56 Find all the second partial derivatives.

51. $f(x, y) = x^3y^5 + 2x^4y$

52. $f(x, y) = \sin^2(mx+ny)$

53. $w = \sqrt{u^2 + v^2}$

54. $v = \frac{xy}{x-y}$

55. $z = \arctan \frac{x+y}{1-xy}$

56. $v = e^{xe^y}$

57–60 Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

57. $u = x \sin(x+2y)$

58. $u = x^4y^2 - 2xy^5$

59. $u = \ln \sqrt{x^2 + y^2}$

60. $u = xye^y$

61–68 Find the indicated partial derivative.

61. $f(x, y) = 3xy^4 + x^3y^2; f_{xxy}, f_{yyy}$

62. $f(x, t) = x^2e^{-ct}; f_{tt}, f_{txx}$

63. $f(x, y, z) = \cos(4x+3y+2z); f_{xyz}, f_{yyz}$

64. $f(r, s, t) = r \ln(rs^2t^3); f_{rss}, f_{rst}$

65. $u = e^{r\theta} \sin \theta; \frac{\partial^3 u}{\partial r^2 \partial \theta}$

66. $z = u\sqrt{v-w}; \frac{\partial^3 z}{\partial u \partial v \partial w}$

67. $w = \frac{x}{y+2z}; \frac{\partial^3 w}{\partial z \partial y \partial x}, \frac{\partial^3 w}{\partial x^2 \partial y}$

68. $u = x^a y^b z^c; \frac{\partial^6 u}{\partial x \partial y^2 \partial z^3}$

69. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$.

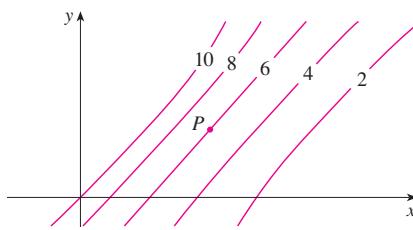
$x \setminus y$	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

70. Level curves are shown for a function f . Determine whether the following partial derivatives are positive or negative at the point P .

(a) f_x
(d) f_{xy}

(b) f_y
(e) f_{yy}

(c) f_{xx}



71. Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the heat conduction equation $u_t = \alpha^2 u_{xx}$.

72. Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

(a) $u = x^2 + y^2$
(b) $u = x^2 - y^2$
(c) $u = x^3 + 3xy^2$
(d) $u = \ln \sqrt{x^2 + y^2}$
(e) $u = \sin x \cosh y + \cos x \sinh y$
(f) $u = e^{-x} \cos y - e^{-y} \cos x$

73. Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

74. Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.

(a) $u = \sin(kx) \sin(akt)$
(b) $u = t/(a^2 t^2 - x^2)$
(c) $u = (x-at)^6 + (x+at)^6$
(d) $u = \sin(x-at) + \ln(x+at)$

75. If f and g are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x+at) + g(x-at)$$

is a solution of the wave equation given in Exercise 74.

76. If $u = e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}$, where $a_1^2 + a_2^2 + \dots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = u$$

77. Verify that the function $z = \ln(e^x + e^y)$ is a solution of the differential equations

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

and

$$\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0$$

78. Show that the Cobb-Douglas production function $P = bL^\alpha K^\beta$ satisfies the equation

$$L \frac{\partial P}{\partial L} + K \frac{\partial P}{\partial K} = (\alpha + \beta)P$$

79. Show that the Cobb-Douglas production function satisfies $P(L, K_0) = C_1(K_0)L^\alpha$ by solving the differential equation

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

(See Equation 5.)

80. The temperature at a point (x, y) on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where T is measured in $^{\circ}\text{C}$ and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in (a) the x -direction and (b) the y -direction.

81. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\partial R/\partial R_1$.

82. The gas law for a fixed mass m of an ideal gas at absolute temperature T , pressure P , and volume V is $PV = mRT$, where R is the gas constant. Show that

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$$

83. For the ideal gas of Exercise 82, show that

$$T \frac{\partial P}{\partial T} \frac{\partial V}{\partial T} = mR$$

84. The wind-chill index is modeled by the function

$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature ($^{\circ}\text{C}$) and v is the wind speed (km/h). When $T = -15^{\circ}\text{C}$ and $v = 30 \text{ km/h}$, by how much would you expect the apparent temperature W to drop if the actual temperature decreases by 1°C ? What if the wind speed increases by 1 km/h ?

85. The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m} \frac{\partial^2 K}{\partial v^2} = K$$

86. If a, b, c are the sides of a triangle and A, B, C are the opposite angles, find $\partial A/\partial a, \partial A/\partial b, \partial A/\partial c$ by implicit differentiation of the Law of Cosines.

87. You are told that there is a function f whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

88. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

89. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

90. In a study of frost penetration it was found that the temperature T at time t (measured in days) at a depth x (measured in feet) can be modeled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and λ is a positive constant.

- (a) Find $\partial T/\partial x$. What is its physical significance?
- (b) Find $\partial T/\partial t$. What is its physical significance?
- (c) Show that T satisfies the heat equation $T_t = kT_{xx}$ for a certain constant k .
- (d) If $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$, use a computer to graph $T(x, t)$.
- (e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

91. Use Clairaut's Theorem to show that if the third-order partial derivatives of f are continuous, then

$$f_{xyy} = f_{yx} = f_{yyx}$$

92. (a) How many n th-order partial derivatives does a function of two variables have?
 (b) If these partial derivatives are all continuous, how many of them can be distinct?
 (c) Answer the question in part (a) for a function of three variables.

93. If $f(x, y) = x(x^2 + y^2)^{-3/2}e^{\sin(x^2y)}$, find $f_x(1, 0)$.

[Hint: Instead of finding $f_x(x, y)$ first, note that it's easier to use Equation 1 or Equation 2.]

94. If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

95. Let

$$f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

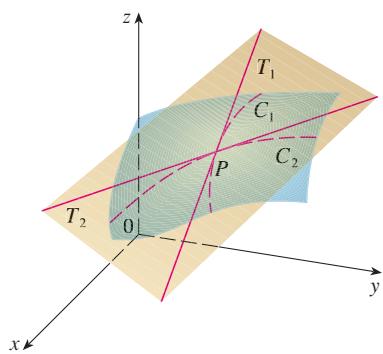
- (a) Use a computer to graph f .
- (b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.
- (c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.
- (d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.
- (e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

14.4

TANGENT PLANES AND LINEAR APPROXIMATIONS

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. (See Section 3.10.) Here we develop similar ideas in three dimensions. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables. We also extend the idea of a differential to functions of two or more variables.

TANGENT PLANES

**FIGURE 1**

The tangent plane contains the tangent lines T_1 and T_2 .

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S . As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S . Then the point P lies on both C_1 and C_2 . Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P . Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2 . (See Figure 1.)

We will see in Section 14.6 that if C is any other curve that lies on the surface S and passes through P , then its tangent line at P also lies in the tangent plane. Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P . The tangent plane at P is the plane that most closely approximates the surface S near the point P .

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

1

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P , then its intersection with the plane $y = y_0$ must be the tangent line T_1 . Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \quad y = y_0$$

and we recognize these as the equations (in point-slope form) of a line with slope a . But from Section 14.3 we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2 , so $b = f_y(x_0, y_0)$.

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

■ Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \quad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \quad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$



TEC Visual 14.4 shows an animation of Figures 2 and 3.

Figure 2(a) shows the elliptic paraboloid and its tangent plane at $(1, 1, 3)$ that we found in Example 1. In parts (b) and (c) we zoom in toward the point $(1, 1, 3)$ by restricting the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.

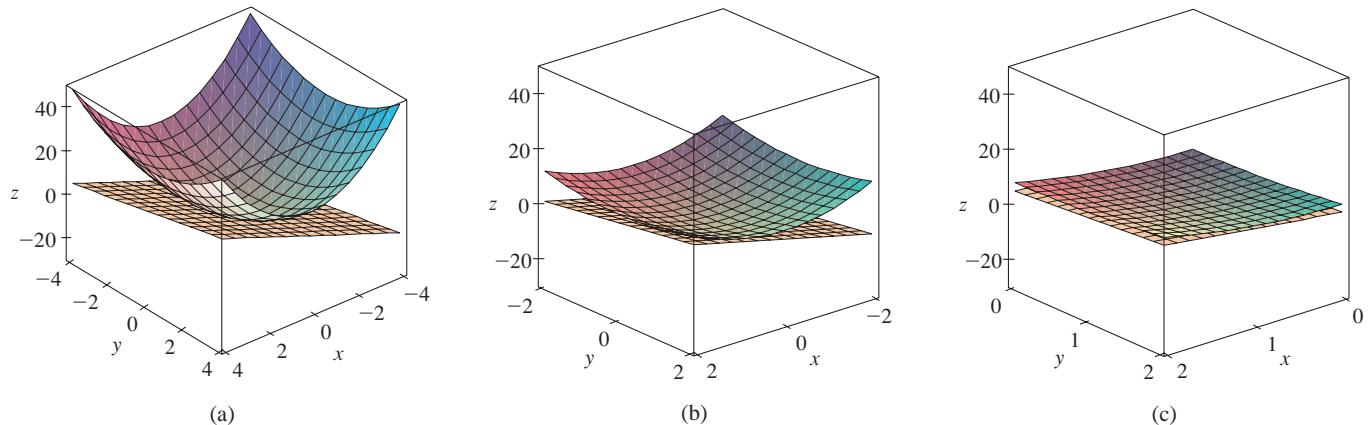


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

In Figure 3 we corroborate this impression by zooming in toward the point $(1, 1)$ on a contour map of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

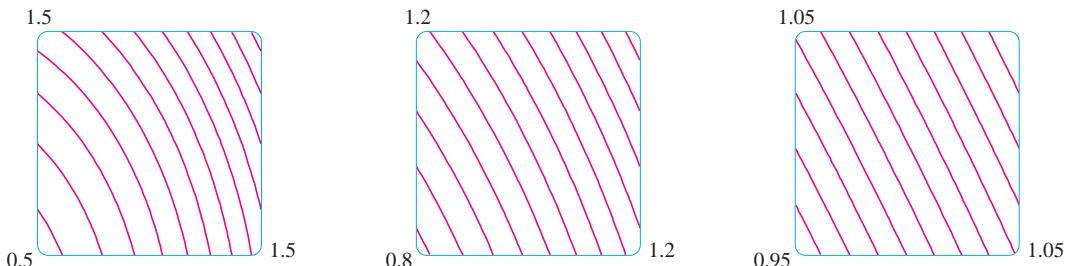


FIGURE 3
Zooming in toward $(1, 1)$ on a contour map of $f(x, y) = 2x^2 + y^2$

LINEAR APPROXIMATIONS

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$. Therefore, in view of the visual evidence in Figures 2 and 3, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near $(1, 1)$. The function L is called the *linearization* of f at $(1, 1)$ and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at $(1, 1)$.

For instance, at the point $(1.1, 0.95)$ the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$. But if we take a point farther away from $(1, 1)$, such as $(2, 3)$, we no longer get a good approximation. In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

In general, we know from (2) that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$3 \quad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$4 \quad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b) .

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

You can verify (see Exercise 46) that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous. The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$. So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

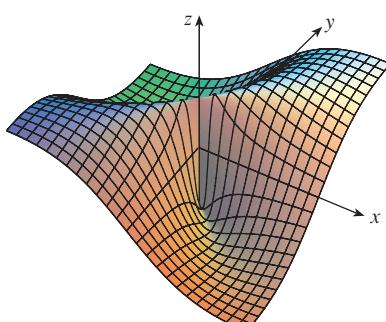


FIGURE 4

$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ if } (x, y) \neq (0, 0), \\ f(0, 0) = 0$$

In Chapter 3 we showed that if f is differentiable at a , then

■ This is Equation 3.4.7.

$$5 \quad \Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$6 \quad \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. By analogy with (5) we define the differentiability of a function of two variables as follows.

7 DEFINITION If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b) . In other words, the tangent plane approximates the graph of f well near the point of tangency.

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

■ Theorem 8 is proved in Appendix F.

8 THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

V EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \quad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \quad f_y(1, 0) = 1$$

■ Figure 5 shows the graphs of the function f and its linearization L in Example 2.

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(x - 1) + 1 \cdot y = x + y \end{aligned}$$

The corresponding linear approximation is

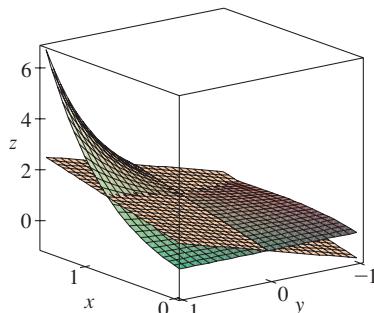
$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$.

FIGURE 5



EXAMPLE 3 At the beginning of Section 14.3 we discussed the heat index (perceived temperature) I as a function of the actual temperature T and the relative humidity H and gave the following table of values from the National Weather Service.

		Relative humidity (%)									
		50	55	60	65	70	75	80	85	90	
$T \backslash H$		90	96	98	100	103	106	109	112	115	119
92		100	103	105	108	112	115	119	123	128	
94		104	107	111	114	118	122	127	132	137	
96		109	113	116	121	125	130	135	141	146	
98		114	118	123	127	133	138	144	150	157	
100		119	124	129	135	141	147	154	161	168	

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96°F and H is near 70% . Use it to estimate the heat index when the temperature is 97°F and the relative humidity is 72% .

SOLUTION We read from the table that $f(96, 70) = 125$. In Section 14.3 we used the tabular values to estimate that $f_T(96, 70) \approx 3.75$ and $f_H(96, 70) \approx 0.9$. (See pages 878–79.) So the linear approximation is

$$\begin{aligned}f(T, H) &\approx f(96, 70) + f_T(96, 70)(T - 96) + f_H(96, 70)(H - 70) \\&\approx 125 + 3.75(T - 96) + 0.9(H - 70)\end{aligned}$$

In particular,

$$f(97, 72) \approx 125 + 3.75(1) + 0.9(2) = 130.55$$

Therefore, when $T = 97^{\circ}\text{F}$ and $H = 72\%$, the heat index is

$$I \approx 131^{\circ}\text{F}$$



DIFFERENTIALS

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined as

9

$$dy = f'(x) dx$$

(See Section 3.10.) Figure 6 shows the relationship between the increment Δy and the differential dy : Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz , also called the **total differential**, is defined by

10

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

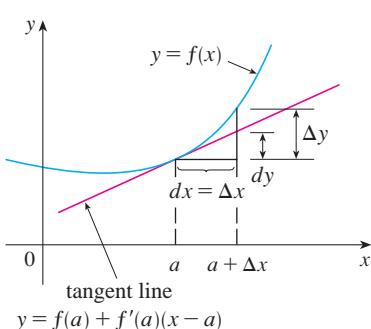


FIGURE 6

(Compare with Equation 9.) Sometimes the notation df is used in place of dz .

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation (4) can be written as

$$f(x, y) \approx f(a, b) + dz$$

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz : dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

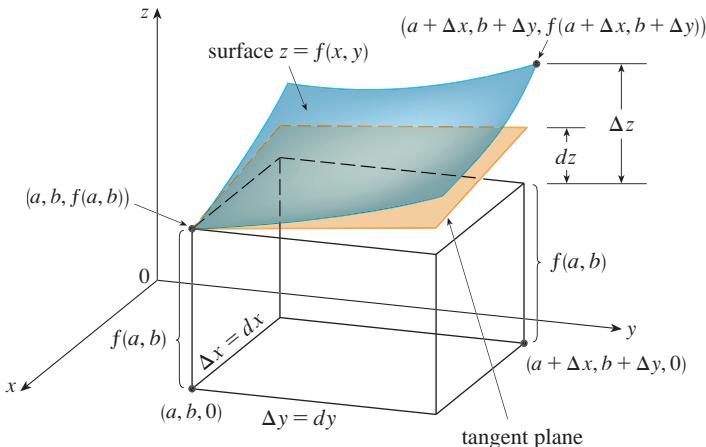


FIGURE 7

EXAMPLE 4

- If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz .
- If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz .

SOLUTION

- Definition 10 gives

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x + 3y) dx + (3x - 2y) dy$$

- Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3) \\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2] \\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

EXAMPLE 5 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in

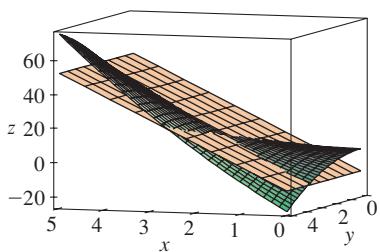


FIGURE 8

each. Use differentials to estimate the maximum error in the calculated volume of the cone.

SOLUTION The volume V of a cone with base radius r and height h is $V = \pi r^2 h / 3$. So the differential of V is

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = \frac{2\pi rh}{3} dr + \frac{\pi r^2}{3} dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leq 0.1$, $|\Delta h| \leq 0.1$. To find the largest error in the volume we take the largest error in the measurement of r and of h . Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3} (0.1) + \frac{100\pi}{3} (0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about $20\pi \text{ cm}^3 \approx 63 \text{ cm}^3$.

FUNCTIONS OF THREE OR MORE VARIABLES

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx , dy , and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

EXAMPLE 6 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

SOLUTION If the dimensions of the box are x , y , and z , its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = yz dx + xz dy + xy dz$$

We are given that $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$. To find the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

14.4 EXERCISES

- 1–6** Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 4x^2 - y^2 + 2y, \quad (-1, 2, 4)$
2. $z = 3(x - 1)^2 + 2(y + 3)^2 + 7, \quad (2, -2, 12)$
3. $z = \sqrt{xy}, \quad (1, 1, 1)$
4. $z = y \ln x, \quad (1, 4, 0)$
5. $z = y \cos(x - y), \quad (2, 2, 2)$
6. $z = e^{x^2-y^2}, \quad (1, -1, 1)$

7–8 Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

7. $z = x^2 + xy + 3y^2, \quad (1, 1, 5)$
8. $z = \arctan(xy^2), \quad (1, 1, \pi/4)$

CAS **9–10** Draw the graph of f and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

9. $f(x, y) = \frac{xy \sin(x - y)}{1 + x^2 + y^2}, \quad (1, 1, 0)$
10. $f(x, y) = e^{-xy/10}(\sqrt{x} + \sqrt{y} + \sqrt{xy}), \quad (1, 1, 3e^{-0.1})$

- 11–16** Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = x\sqrt{y}, \quad (1, 4)$
12. $f(x, y) = x^3y^4, \quad (1, 1)$
13. $f(x, y) = \frac{x}{x + y}, \quad (2, 1)$
14. $f(x, y) = \sqrt{x + e^{4y}}, \quad (3, 0)$
15. $f(x, y) = e^{-xy} \cos y, \quad (\pi, 0)$
16. $f(x, y) = \sin(2x + 3y), \quad (-3, 2)$

- 17–18** Verify the linear approximation at $(0, 0)$.

$$17. \frac{2x + 3}{4y + 1} \approx 3 + 2x - 12y \quad 18. \sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$$

- 19.** Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $f(1.95, 1.08)$.

- 20.** Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing f and the tangent plane.

- 21.** Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

- 22.** The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

		Duration (hours)						
v	t	5	10	15	20	30	40	50
20	5	7	8	8	9	9	9	9
30	9	13	16	17	18	19	19	19
40	14	21	25	28	31	33	33	33
50	19	29	36	40	45	48	50	50
60	24	37	47	54	62	67	69	69

Use the table to find a linear approximation to the wave height function when v is near 40 knots and t is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

- 23.** Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94°F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95°F and the relative humidity is 78%.

- 24.** The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v , so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1.

		Wind speed (km/h)					
T	v	20	30	40	50	60	70
-10	-18	-20	-21	-22	-23	-23	-23
-15	-24	-26	-27	-29	-30	-30	-30
-20	-30	-33	-34	-35	-36	-37	-37
-25	-37	-39	-41	-42	-43	-44	-44

Use the table to find a linear approximation to the wind-chill

index function when T is near -15°C and v is near 50 km/h. Then estimate the wind-chill index when the temperature is -17°C and the wind speed is 55 km/h.

25–30 Find the differential of the function.

25. $z = x^3 \ln(y^2)$

26. $v = y \cos xy$

27. $m = p^5 q^3$

28. $T = \frac{v}{1 + uvw}$

29. $R = \alpha \beta^2 \cos \gamma$

30. $w = xye^{xz}$

31. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz .

32. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz .

33. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

34. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

35. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

36. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

37. A boundary stripe 3 in. wide is painted around a rectangle whose dimensions are 100 ft by 200 ft. Use differentials to approximate the number of square feet of paint in the stripe.

38. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

39. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3 , then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25 \Omega$, $R_2 = 40 \Omega$, and $R_3 = 50 \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R .

40. Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding.

41. A model for the surface area of a human body is given by $S = 0.1091w^{0.425}h^{0.725}$, where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

42. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S but you know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle$$

$$\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle$$

both lie on S . Find an equation of the tangent plane at P .

43–44 Show that the function is differentiable by finding values of ε_1 and ε_2 that satisfy Definition 7.

43. $f(x, y) = x^2 + y^2$

44. $f(x, y) = xy - 5y^2$

45. Prove that if f is a function of two variables that is differentiable at (a, b) , then f is continuous at (a, b) .

Hint: Show that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

46. (a) The function

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [Hint: Use the result of Exercise 45.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.

14.5**THE CHAIN RULE**

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\boxed{1} \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version (Theorem 2) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t . This means that z is indirectly a function of t , $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t . We assume that f is differentiable (Definition 14.4.7). Recall that this is the case when f_x and f_y are continuous (Theorem 14.4.8).

2 THE CHAIN RULE (CASE I) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

PROOF A change of Δt in t produces changes of Δx in x and Δy in y . These, in turn, produce a change of Δz in z , and from Definition 14.4.7 we have

$$\Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$. [If the functions ε_1 and ε_2 are not defined at $(0, 0)$, we can define them to be 0 there.] Dividing both sides of this equation by Δt , we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \rightarrow 0$, then $\Delta x = g(t + \Delta t) - g(t) \rightarrow 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \rightarrow 0$. This, in turn, means that $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$, so

$$\begin{aligned} \frac{dz}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} \\ &= \frac{\partial f}{\partial x} \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \rightarrow 0} \varepsilon_1 \right) \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \rightarrow 0} \varepsilon_2 \right) \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \end{aligned}$$
■

Since we often write $\partial z / \partial x$ in place of $\partial f / \partial x$, we can rewrite the Chain Rule in the form

- Notice the similarity to the definition of the differential:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}}$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (2xy + 3y^4)(2 \cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

It's not necessary to substitute the expressions for x and y in terms of t . We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left. \frac{dz}{dt} \right|_{t=0} = (0 + 3)(2 \cos 0) + (0 + 0)(-\sin 0) = 6 \quad \blacksquare$$

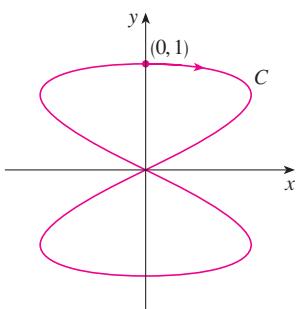


FIGURE 1

The curve $x = \sin 2t$, $y = \cos t$

The derivative in Example 1 can be interpreted as the rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$. (See Figure 1.) In particular, when $t = 0$, the point (x, y) is $(0, 1)$ and $dz/dt = 6$ is the rate of increase as we move along the curve C through $(0, 1)$. If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y) , then the composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C and the derivative dz/dt represents the rate at which the temperature changes along C .

V EXAMPLE 2 The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

SOLUTION If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31 \frac{T}{V}$$

the Chain Rule gives

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial T} \frac{dT}{dt} + \frac{\partial P}{\partial V} \frac{dV}{dt} = \frac{8.31}{V} \frac{dT}{dt} - \frac{8.31T}{V^2} \frac{dV}{dt} \\ &= \frac{8.31}{100} (0.1) - \frac{8.31(300)}{100^2} (0.2) = -0.04155\end{aligned}$$

The pressure is decreasing at a rate of about 0.042 kPa/s. ■

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t : $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we

wish to find $\partial z / \partial s$ and $\partial z / \partial t$. Recall that in computing $\partial z / \partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t . Therefore we can apply Theorem 2 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

A similar argument holds for $\partial z / \partial s$ and so we have proved the following version of the Chain Rule.

3 THE CHAIN RULE (CASE 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z / \partial s$ and $\partial z / \partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t) \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t) \end{aligned}$$

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable. Notice that Theorem 3 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y . Then we draw branches from x and y to the independent variables s and t . On each branch we write the corresponding partial derivative. To find $\partial z / \partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

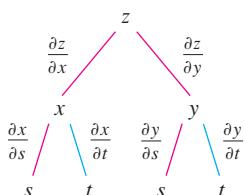


FIGURE 2

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

Similarly, we find $\partial z / \partial t$ by using the paths from z to t .

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables x_1, \dots, x_n , each of which is, in turn, a function of m independent variables t_1, \dots, t_m . Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

4 THE CHAIN RULE (GENERAL VERSION) Suppose that u is a differentiable function of the n variables x_1, x_2, \dots, x_n and each x_j is a differentiable function of the m variables t_1, t_2, \dots, t_m . Then u is a function of t_1, t_2, \dots, t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \dots, m$.

V EXAMPLE 4 Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

SOLUTION We apply Theorem 4 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u , then the partial derivative for that branch is $\partial y / \partial u$. With the aid of the tree diagram, we can now write the required expressions:

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial v}\end{aligned}$$

■

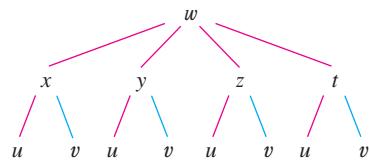


FIGURE 3

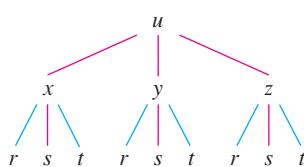


FIGURE 4

V EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s \sin t$, find the value of $\partial u / \partial s$ when $r = 2$, $s = 1$, $t = 0$.

SOLUTION With the help of the tree diagram in Figure 4, we have

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s} \\ &= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)\end{aligned}$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

■

EXAMPLE 6 If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0$$

SOLUTION Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\begin{aligned}\frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s) \\ \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)\end{aligned}$$

Therefore

$$t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = \left(2st \frac{\partial f}{\partial x} - 2st \frac{\partial f}{\partial y}\right) + \left(-2st \frac{\partial f}{\partial x} + 2st \frac{\partial f}{\partial y}\right) = 0$$

■

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z / \partial r$ and (b) $\partial^2 z / \partial r^2$.

SOLUTION

(a) The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x} (2r) + \frac{\partial z}{\partial y} (2s)$$

(b) Applying the Product Rule to the expression in part (a), we get

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left(2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y} \right) \\ [5] &= 2 \frac{\partial z}{\partial x} + 2r \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x} \right) + 2s \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y} \right) \end{aligned}$$

But, using the Chain Rule again (see Figure 5), we have

$$\begin{aligned} \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x} \right) &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2} (2r) + \frac{\partial^2 z}{\partial y \partial x} (2s) \\ \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y} \right) &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x \partial y} (2r) + \frac{\partial^2 z}{\partial y^2} (2s) \end{aligned}$$

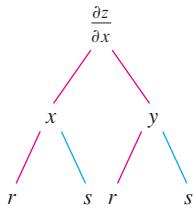


FIGURE 5

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= 2 \frac{\partial z}{\partial x} + 2r \left(2r \frac{\partial^2 z}{\partial x^2} + 2s \frac{\partial^2 z}{\partial y \partial x} \right) + 2s \left(2r \frac{\partial^2 z}{\partial x \partial y} + 2s \frac{\partial^2 z}{\partial y^2} \right) \\ &= 2 \frac{\partial z}{\partial x} + 4r^2 \frac{\partial^2 z}{\partial x^2} + 8rs \frac{\partial^2 z}{\partial x \partial y} + 4s^2 \frac{\partial^2 z}{\partial y^2} \end{aligned}$$

IMPLICIT DIFFERENTIATION

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 3.5 and 14.3. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x , that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f . If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x . Since both x and y are functions of x , we obtain

$$\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F / \partial y \neq 0$ we solve for dy/dx and obtain

[6]

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x . The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: It states that if F is defined on a disk containing (a, b) , where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

- The solution to Example 8 should be compared to the one in Example 2 in Section 3.5.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$
■

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f . If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

If $\partial F / \partial z \neq 0$, we solve for $\partial z / \partial x$ and obtain the first formula in Equations 7. The formula for $\partial z / \partial y$ is obtained in a similar manner.

7

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c) , where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x , F_y , and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (7).

EXAMPLE 9 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}\end{aligned}$$

The solution to Example 9 should be compared to the one in Example 4 in Section 14.3.

14.5 EXERCISES

1–6 Use the Chain Rule to find dz/dt or dw/dt .

1. $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$

2. $z = \cos(x + 4y)$, $x = 5t^4$, $y = 1/t$

3. $z = \sqrt{1 + x^2 + y^2}$, $x = \ln t$, $y = \cos t$

4. $z = \tan^{-1}(y/x)$, $x = e^t$, $y = 1 - e^{-t}$

5. $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$

6. $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = x^2y^3$, $x = s \cos t$, $y = s \sin t$

8. $z = \arcsin(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$

9. $z = \sin \theta \cos \phi$, $\theta = st^2$, $\phi = s^2t$

10. $z = e^{x+2y}$, $x = s/t$, $y = t/s$

11. $z = e^r \cos \theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$

12. $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

13. If $z = f(x, y)$, where f is differentiable, and

$$x = g(t) \quad y = h(t)$$

$$g(3) = 2 \quad h(3) = 7$$

$$g'(3) = 5 \quad h'(3) = -4$$

$$f_x(2, 7) = 6 \quad f_y(2, 7) = -8$$

find dz/dt when $t = 3$.

14. Let $W(s, t) = F(u(s, t), v(s, t))$, where F , u , and v are differentiable, and

$$u(1, 0) = 2 \quad v(1, 0) = 3$$

$$u_s(1, 0) = -2 \quad v_s(1, 0) = 5$$

$$u_t(1, 0) = 6 \quad v_t(1, 0) = 4$$

$$F_u(2, 3) = -1 \quad F_v(2, 3) = 10$$

Find $W_s(1, 0)$ and $W_t(1, 0)$.

15. Suppose f is a differentiable function of x and y , and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

16. Suppose f is a differentiable function of x and y , and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 15 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

17–20 Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

17. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$

18. $R = f(x, y, z, t)$, where $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, $t = t(u, v, w)$

19. $w = f(r, s, t)$, where $r = r(x, y)$, $s = s(x, y)$, $t = t(x, y)$

20. $t = f(u, v, w)$, where $u = u(p, q, r, s)$, $v = v(p, q, r, s)$, $w = w(p, q, r, s)$

21–26 Use the Chain Rule to find the indicated partial derivatives.

21. $z = x^2 + xy^3$, $x = uv^2 + w^3$, $y = u + ve^w$;

$$\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w} \text{ when } u = 2, v = 1, w = 0$$

22. $u = \sqrt{r^2 + s^2}$, $r = y + x \cos t$, $s = x + y \sin t$;

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t} \text{ when } x = 1, y = 2, t = 0$$

23. $R = \ln(u^2 + v^2 + w^2)$,

$$u = x + 2y, \quad v = 2x - y, \quad w = 2xy;$$

$$\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y} \text{ when } x = y = 1$$

24. $M = xe^{y-z^2}$, $x = 2uv$, $y = u - v$, $z = u + v$;

$$\frac{\partial M}{\partial u}, \frac{\partial M}{\partial v} \text{ when } u = 3, v = -1$$

25. $u = x^2 + yz$, $x = pr \cos \theta$, $y = pr \sin \theta$, $z = p + r$;

$$\frac{\partial u}{\partial p}, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta} \text{ when } p = 2, r = 3, \theta = 0$$

26. $Y = w \tan^{-1}(uv)$, $u = r + s$, $v = s + t$, $w = t + r$;

$$\frac{\partial Y}{\partial r}, \frac{\partial Y}{\partial s}, \frac{\partial Y}{\partial t} \text{ when } r = 1, s = 0, t = 1$$

27–30 Use Equation 6 to find dy/dx .

27. $\sqrt{xy} = 1 + x^2y$

28. $y^5 + x^2y^3 = 1 + ye^{x^2}$

29. $\cos(x - y) = xe^y$

30. $\sin x + \cos y = \sin x \cos y$

31–34 Use Equations 7 to find $\partial z/\partial x$ and $\partial z/\partial y$.

31. $x^2 + y^2 + z^2 = 3xyz$

32. $xyz = \cos(x + y + z)$

33. $x - z = \arctan(yz)$

34. $yz = \ln(x + z)$

35. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

36. Wheat production W in a given year depends on the average temperature T and the annual rainfall R . Scientists estimate that the average temperature is rising at a rate of $0.15^\circ\text{C}/\text{year}$ and rainfall is decreasing at a rate of 0.1 cm/year . They also estimate that, at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

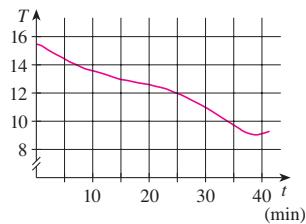
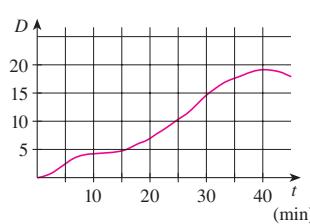
(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production, dW/dt .

37. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), and D is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?



38. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s . At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in. ?

39. The length ℓ , width w , and height h of a box change with time. At a certain instant the dimensions are $\ell = 1 \text{ m}$ and $w = h = 2 \text{ m}$, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s . At that instant find the rates at which the following quantities are changing.

- (a) The volume
- (b) The surface area
- (c) The length of a diagonal

40. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400 \Omega$, $I = 0.08 \text{ A}$, $dV/dt = -0.01 \text{ V/s}$, and $dR/dt = 0.03 \Omega/\text{s}$.

41. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s . Use the equation in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K .

42. Car A is traveling north on Highway 16 and car B is traveling west on Highway 83. Each car is approaching the intersection of these highways. At a certain moment, car A is 0.3 km from the intersection and traveling at 90 km/h while car B is 0.4 km from the intersection and traveling at 80 km/h . How fast is the distance between the cars changing at that moment?

43. One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s . If the area of the

triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

44. If a sound with frequency f_s is produced by a source traveling along a line with speed v_s and an observer is traveling with speed v_o along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s} \right) f_s$$

where c is the speed of sound, about 332 m/s. (This is the **Doppler effect**.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s². A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s², and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

- 45–48 Assume that all the given functions are differentiable.

45. If $z = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, (a) find $\partial z / \partial r$ and $\partial z / \partial \theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2$$

46. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = e^{-2s} \left[\left(\frac{\partial u}{\partial s} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right]$$

47. If $z = f(x - y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

48. If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that

$$\left(\frac{\partial z}{\partial x} \right)^2 - \left(\frac{\partial z}{\partial y} \right)^2 = \frac{\partial z}{\partial s} \frac{\partial z}{\partial t}$$

- 49–54 Assume that all the given functions have continuous second-order partial derivatives.

49. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

[Hint: Let $u = x + at$, $v = x - at$.]

50. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s} \left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right]$$

51. If $z = f(x, y)$, where $x = r^2 + s^2$ and $y = 2rs$, find $\partial^2 z / \partial r \partial s$. (Compare with Example 7.)

52. If $z = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, find
(a) $\partial z / \partial r$, (b) $\partial z / \partial \theta$, and (c) $\partial^2 z / \partial r \partial \theta$.

53. If $z = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r}$$

54. Suppose $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$.

- (a) Show that

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 \\ &\quad + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

- (b) Find a similar formula for $\partial^2 z / \partial s \partial t$.

55. A function f is called **homogeneous of degree n** if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t , where n is a positive integer and f has continuous second-order partial derivatives.

- (a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

- (b) Show that if f is homogeneous of degree n , then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)$$

[Hint: Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t .]

56. If f is homogeneous of degree n , show that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n - 1)f(x, y)$$

57. If f is homogeneous of degree n , show that

$$f_x(tx, ty) = t^{n-1} f_x(x, y)$$

58. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x , y , and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x , F_y , and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$$

14.6

DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

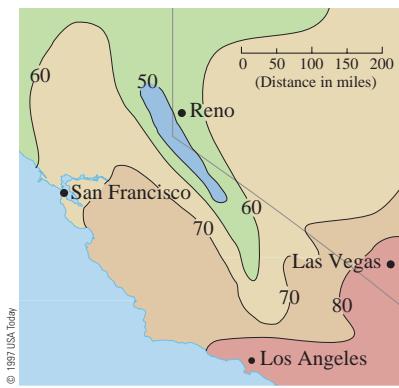


FIGURE 1

The weather map in Figure 1 shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3:00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature. The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

DIRECTIONAL DERIVATIVES

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

I

$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x - and y -directions, that is, in the directions of the unit vectors \mathbf{i} and \mathbf{j} .

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.) To do this we consider the surface S with equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S . The vertical plane that passes through P in the direction of \mathbf{u} intersects S in a curve C . (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of \mathbf{u} .

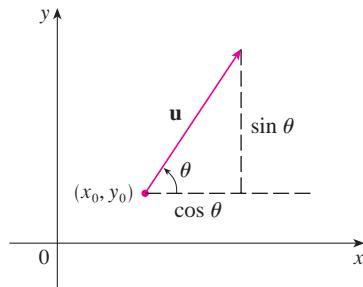


FIGURE 2

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos \theta, \sin \theta \rangle$

TEC Visual 14.6A animates Figure 3 by rotating \mathbf{u} and therefore T .

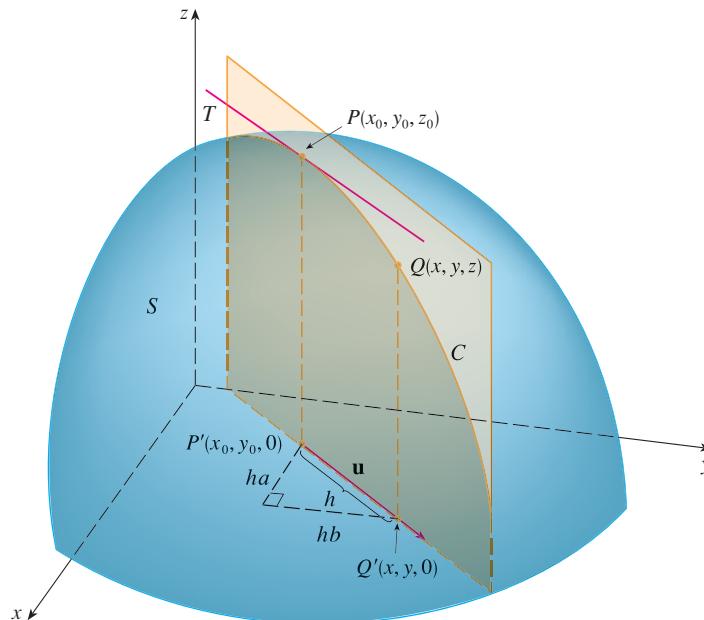


FIGURE 3

If $\overrightarrow{Q(x, y, z)}$ is another point on C and P' , Q' are the projections of P , Q on the xy -plane, then the vector $\overrightarrow{P'Q'}$ is parallel to \mathbf{u} and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h . Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \rightarrow 0$, we obtain the rate of change of z (with respect to distance) in the direction of \mathbf{u} , which is called the directional derivative of f in the direction of \mathbf{u} .

2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations (1), we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

EXAMPLE 1 Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

SOLUTION The unit vector directed toward the southeast is $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, but we won't need to use this expression. We start by drawing a line through Reno toward the southeast. (See Figure 4.)

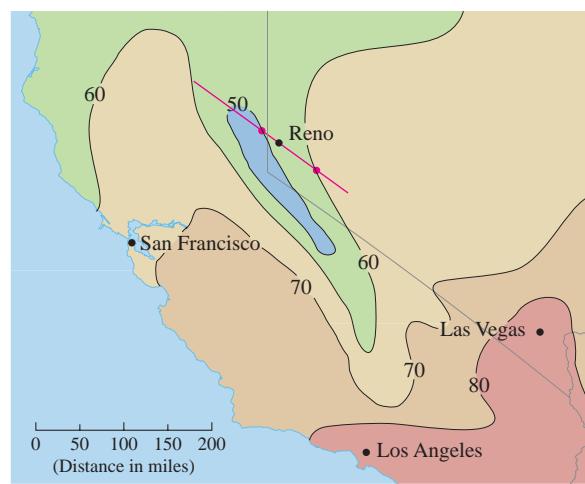


FIGURE 4

We approximate the directional derivative $D_{\mathbf{u}}T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and

$T = 60$. The temperature at the point southeast of Reno is $T = 60^\circ\text{F}$ and the temperature at the point northwest of Reno is $T = 50^\circ\text{F}$. The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}} T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13^\circ\text{F/mi}$$

■

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 THEOREM If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\begin{aligned} \mathbf{4} \quad g'(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}} f(x_0, y_0) \end{aligned}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y) a + f_y(x, y) b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\mathbf{5} \quad g'(0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b$$

■

If the unit vector \mathbf{u} makes an angle θ with the positive x -axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and the formula in Theorem 3 becomes

$$\mathbf{6} \quad D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and \mathbf{u} is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

The directional derivative $D_{\mathbf{u}}f(1, 2)$ in Example 2 represents the rate of change of z in the direction of \mathbf{u} . This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of \mathbf{u} shown in Figure 5.

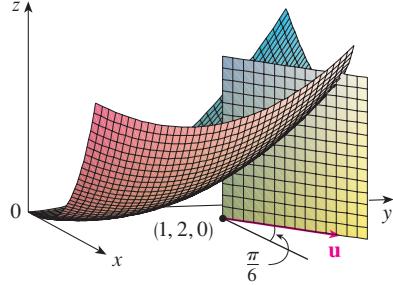


FIGURE 5

SOLUTION Formula 6 gives

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} \\ &= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \frac{1}{2} \\ &= \frac{1}{2}[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y] \end{aligned}$$

Therefore

$$D_{\mathbf{u}}f(1, 2) = \frac{1}{2}[3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13 - 3\sqrt{3}}{2} \quad \blacksquare$$

THE GRADIENT VECTOR

Notice from Theorem 3 that the directional derivative can be written as the dot product of two vectors:

7

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y) a + f_y(x, y) b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f , which is read “del f ”).

8

DEFINITION If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and

$$\nabla f(0, 1) = \langle 2, 0 \rangle \quad \blacksquare$$

With this notation for the gradient vector, we can rewrite the expression (7) for the directional derivative as

9

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of \mathbf{u} as the scalar projection of the gradient vector onto \mathbf{u} .

The gradient vector $\nabla f(2, -1)$ in Example 4 is shown in Figure 6 with initial point $(2, -1)$. Also shown is the vector \mathbf{v} that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f .

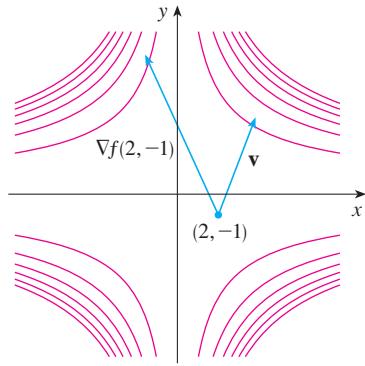


FIGURE 6

V EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

Note that \mathbf{v} is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of \mathbf{v} is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore, by Equation 9, we have

$$\begin{aligned} D_{\mathbf{u}}f(2, -1) &= \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j} \right) \\ &= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}} \end{aligned}$$

■

FUNCTIONS OF THREE VARIABLES

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector \mathbf{u} .

[10] DEFINITION The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

[II]

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$. This is reasonable because the vector equation of the line through \mathbf{x}_0 in the direction of the vector \mathbf{u} is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 12.5.1) and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that

$$\boxed{12} \quad D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f , is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

$$\boxed{13} \quad \nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

$$\boxed{14} \quad D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

V EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle \end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}} \mathbf{i} + \frac{2}{\sqrt{6}} \mathbf{j} - \frac{1}{\sqrt{6}} \mathbf{k}$$

Therefore Equation 14 gives

$$\begin{aligned} D_{\mathbf{u}}f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}} \mathbf{i} + \frac{2}{\sqrt{6}} \mathbf{j} - \frac{1}{\sqrt{6}} \mathbf{k} \right) \\ &= 3 \left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$
■

MAXIMIZING THE DIRECTIONAL DERIVATIVE

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

TEC Visual 14.6B provides visual confirmation of Theorem 15.

15 THEOREM Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when \mathbf{u} has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 we have

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

where θ is the angle between ∇f and \mathbf{u} . The maximum value of $\cos \theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when \mathbf{u} has the same direction as ∇f . ■

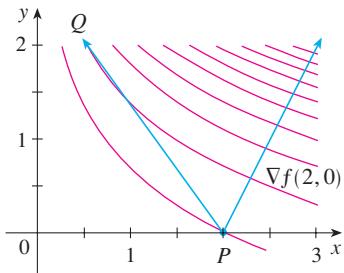


FIGURE 7

At $(2, 0)$ the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. Notice from Figure 7 that this vector appears to be perpendicular to the level curve through $(2, 0)$. Figure 8 shows the graph of f and the gradient vector.

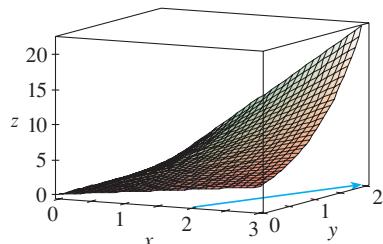


FIGURE 8

EXAMPLE 6

- If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
- In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

- We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathbf{u}}f(2, 0) &= \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle \\ &= 1\left(-\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = 1 \end{aligned}$$

- According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x} \mathbf{i} + \frac{\partial T}{\partial y} \mathbf{j} + \frac{\partial T}{\partial z} \mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2} (-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k}) \end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \frac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \frac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \frac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4^\circ\text{C}/\text{m}$. ■

TANGENT PLANES TO LEVEL SURFACES

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S . Let C be any curve that lies on the surface S and passes through the point P . Recall from Section 13.1 that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Let t_0 be the parameter value corresponding to P ; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S , any point $(x(t), y(t), z(t))$ must satisfy the equation of S , that is,

$$16 \quad F(x(t), y(t), z(t)) = k$$

If x , y , and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$17 \quad \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$18 \quad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector at P , $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P .* (See Figure 9.) If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$** as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 12.5.7), we can write the equation of this tangent plane as

$$19 \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

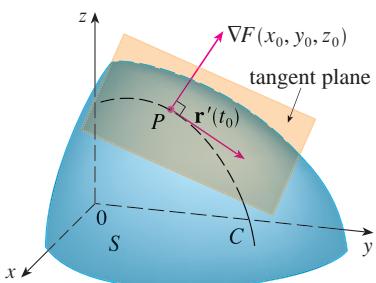


FIGURE 9

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 12.5.3, its symmetric equations are

$$\boxed{20} \quad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F . Then

$$\begin{aligned} F_x(x_0, y_0, z_0) &= f_x(x_0, y_0) \\ F_y(x_0, y_0, z_0) &= f_y(x_0, y_0) \\ F_z(x_0, y_0, z_0) &= -1 \end{aligned}$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to Equation 14.4.2. Thus our new, more general, definition of a tangent plane is consistent with the definition that was given for the special case of Section 14.4.

V EXAMPLE 8 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Figure 10 shows the ellipsoid, tangent plane, and normal line in Example 8.

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \quad F_y(-2, 1, -3) = 2 \quad F_z(-2, 1, -3) = -\frac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

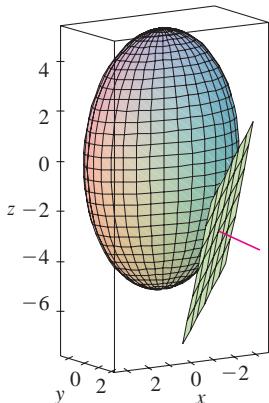
$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

FIGURE 10



SIGNIFICANCE OF THE GRADIENT VECTOR

We now summarize the ways in which the gradient vector is significant. We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain. On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f . On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P . (Refer to Figure 9.) These two properties are quite compatible intuitively because as we move away from P on the level surface S , the value of f does not change at all. So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain. Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f . Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P . Again this is intuitively plausible because the values of f remain constant as we move along the curve. (See Figure 11.)

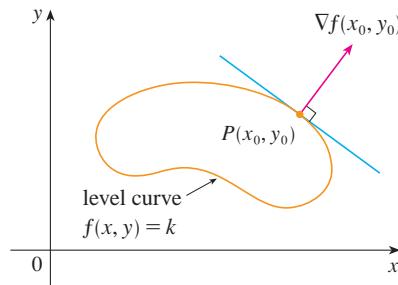


FIGURE 11

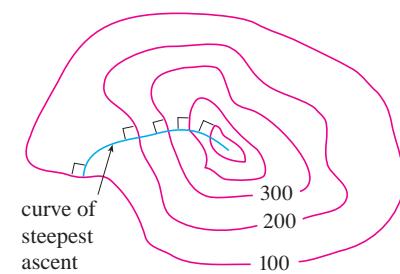


FIGURE 12

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y) , then a curve of steepest ascent can be drawn as in Figure 12 by making it perpendicular to all of the contour lines. This phenomenon can also be noticed in Figure 12 in Section 14.1, where Lonesome Creek follows a curve of steepest descent.

Computer algebra systems have commands that plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b) . Figure 13 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f . As expected, the gradient vectors point “uphill” and are perpendicular to the level curves.

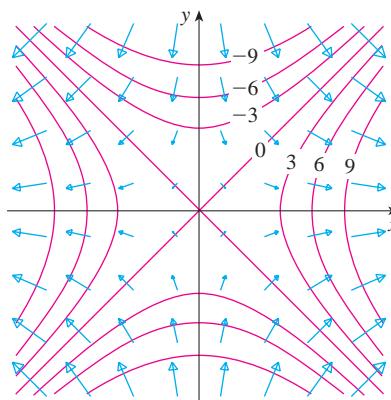
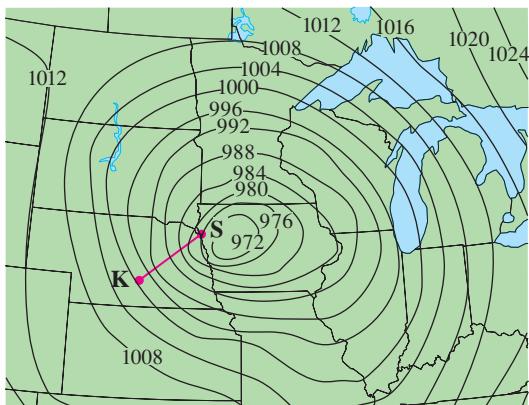


FIGURE 13

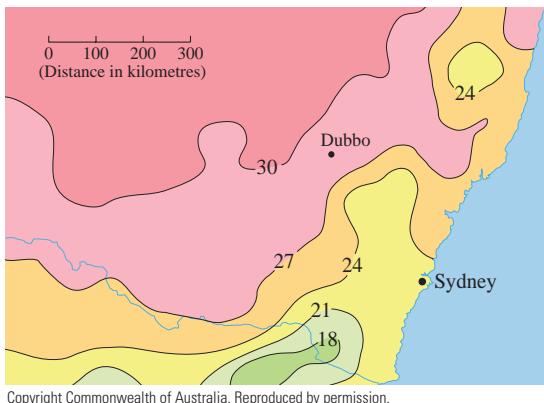
14.6 EXERCISES

- 1.** Level curves for barometric pressure (in millibars) are shown for 6:00 AM on November 10, 1998. A deep low with pressure 972 mb is moving over northeast Iowa. The distance along the red line from K (Kearney, Nebraska) to S (Sioux City, Iowa) is 300 km. Estimate the value of the directional derivative of the pressure function at Kearney in the direction of Sioux City. What are the units of the directional derivative?



From Meteorology Today, 8E by C. Donald Ahrens (2007 Thomson Brooks/Cole).

- 2.** The contour map shows the average maximum temperature for November 2004 (in °C). Estimate the value of the directional derivative of this temperature function at Dubbo, New South Wales, in the direction of Sydney. What are the units?



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- 3.** A table of values for the wind-chill index $W = f(T, v)$ is given in Exercise 3 on page 888. Use the table to estimate the value of $D_u f(-20, 30)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

- 4–6** Find the directional derivative of f at the given point in the direction indicated by the angle θ .

4. $f(x, y) = x^2y^3 - y^4$, $(2, 1)$, $\theta = \pi/4$
5. $f(x, y) = ye^{-x}$, $(0, 4)$, $\theta = 2\pi/3$
6. $f(x, y) = x \sin(xy)$, $(2, 0)$, $\theta = \pi/3$

7–10

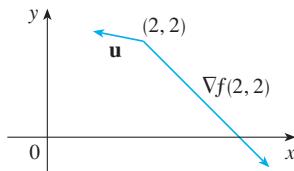
- (a) Find the gradient of f .
- (b) Evaluate the gradient at the point P .
- (c) Find the rate of change of f at P in the direction of the vector \mathbf{u} .

7. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}(\sqrt{3}\mathbf{i} - \mathbf{j})$
8. $f(x, y) = y^2/x$, $P(1, 2)$, $\mathbf{u} = \frac{1}{3}(2\mathbf{i} + \sqrt{5}\mathbf{j})$
9. $f(x, y, z) = xe^{2yz}$, $P(3, 0, 2)$, $\mathbf{u} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$
10. $f(x, y, z) = \sqrt{x + yz}$, $P(1, 3, 1)$, $\mathbf{u} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$

- 11–17** Find the directional derivative of the function at the given point in the direction of the vector \mathbf{v} .

11. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$
12. $f(x, y) = \ln(x^2 + y^2)$, $(2, 1)$, $\mathbf{v} = \langle -1, 2 \rangle$
13. $g(p, q) = p^4 - p^2q^3$, $(2, 1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$
14. $g(r, s) = \tan^{-1}(rs)$, $(1, 2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$
15. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$
16. $f(x, y, z) = \sqrt{xyz}$, $(3, 2, 6)$, $\mathbf{v} = \langle -1, -2, 2 \rangle$
17. $g(x, y, z) = (x + 2y + 3z)^{3/2}$, $(1, 1, 2)$, $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$

- 18.** Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.

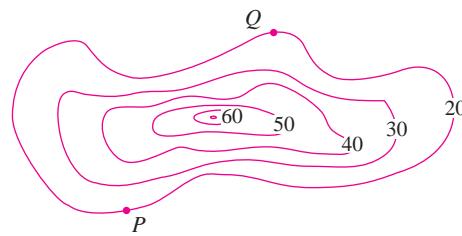
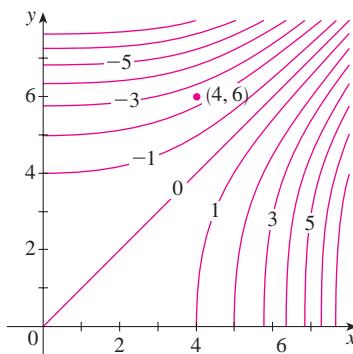


- 19.** Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

- 20.** Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

- 21–26** Find the maximum rate of change of f at the given point and the direction in which it occurs.

21. $f(x, y) = y^2/x$, $(2, 4)$
22. $f(p, q) = qe^{-p} + pe^{-q}$, $(0, 0)$
23. $f(x, y) = \sin(xy)$, $(1, 0)$
24. $f(x, y, z) = (x + y)/z$, $(1, 1, -1)$
25. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(3, 6, -2)$
26. $f(x, y, z) = \tan(x + 2y + 3z)$, $(-5, 1, 1)$

- 27.** (a) Show that a differentiable function f decreases most rapidly at \mathbf{x} in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$.
 (b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.
- 28.** Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.
- 29.** Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.
- 30.** Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x , y , and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.
- 31.** The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120° .
 (a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
 (b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.
- 32.** The temperature at a point (x, y, z) is given by
- $$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$
- where T is measured in $^\circ\text{C}$ and x, y, z in meters.
 (a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.
 (b) In which direction does the temperature increase fastest at P ?
 (c) Find the maximum rate of increase at P .
- 33.** Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.
 (a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.
 (b) In which direction does V change most rapidly at P ?
 (c) What is the maximum rate of change at P ?
- 34.** Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x , y , and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x -axis points east and the positive y -axis points north.
 (a) If you walk due south, will you start to ascend or descend? At what rate?
 (b) If you walk northwest, will you start to ascend or descend? At what rate?
 (c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?
- 35.** Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector \overrightarrow{AB} is 3 and the directional derivative at A in the direction of \overrightarrow{AC} is 26. Find the directional derivative of f at A in the direction of the vector \overrightarrow{AD} .
- 36.** For the given contour map draw the curves of steepest ascent starting at P and at Q .
- 
- 37.** Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.
 (a) $\nabla(au + bv) = a\nabla u + b\nabla v$ (b) $\nabla(uv) = u\nabla v + v\nabla u$
 (c) $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\nabla u$
- 38.** Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.
- 
- 39–44** Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.
- 39.** $2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10$, $(3, 3, 5)$
- 40.** $y = x^2 - z^2$, $(4, 7, 3)$
- 41.** $x^2 - 2y^2 + z^2 + yz = 2$, $(2, 1, -1)$
- 42.** $x - z = 4 \arctan(yz)$, $(1 + \pi, 1, 1)$
- 43.** $z + 1 = xe^y \cos z$, $(1, 0, 0)$
- 44.** $yz = \ln(x + z)$, $(0, 0, 1)$

45–46 Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

45. $xy + yz + zx = 3, \quad (1, 1, 1)$

46. $xyz = 6, \quad (1, 2, 3)$

47. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

48. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

49. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

50. Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at (x_0, y_0, z_0) and express it in a form similar to the one in Exercise 49.

51. Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

52. At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

53. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

54. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

55. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

56. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

57. Show that the sum of the x -, y -, and z -intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

58. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

59. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

60. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

61. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

62. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.

(b) Graph f near the origin and comment on how the graph confirms part (a).

63. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors \mathbf{u} and \mathbf{v} . Is it possible to find ∇f at this point? If so, how would you do it?

64. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[Hint: Use Definition 14.4.7 directly.]

14.7

MAXIMUM AND MINIMUM VALUES

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values. In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 6 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.

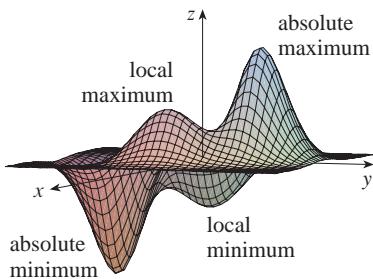


FIGURE 1

Look at the hills and valleys in the graph of f shown in Figure 1. There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the *absolute maximum*. Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the *absolute minimum*.

1 ■ DEFINITION A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) . [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f , then f has an **absolute maximum** (or **absolute minimum**) at (a, b) .

■ Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

2 ■ THEOREM If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

PROOF Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b) , then g has a local maximum (or minimum) at a , so $g'(a) = 0$ by Fermat's Theorem (see Theorem 4.1.4). But $g'(a) = f_x(a, b)$ (see Equation 14.3.1) and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. ■

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 14.4.2), we get $z = z_0$. Thus the geometric interpretation of Theorem 2 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b) , then (a, b) is a critical point of f . However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \quad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y . Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f . This can be confirmed geometrically from the graph of f , which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2. ■

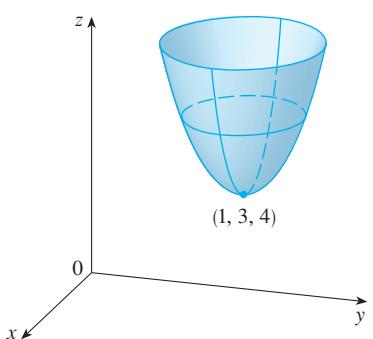


FIGURE 2

$$z = x^2 + y^2 - 2x - 6y + 14$$

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

SOLUTION Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x -axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y -axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f , so f has no extreme value. ■

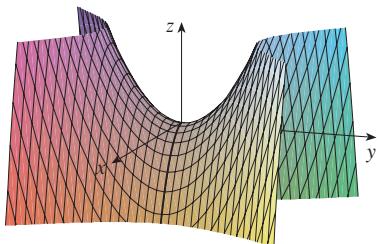


FIGURE 3

$$z = y^2 - x^2$$

Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point. Figure 3 shows how this is possible. The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$, which has a horizontal tangent plane ($z = 0$) at the origin. You can see that $f(0, 0) = 0$ is a maximum in the direction of the x -axis but a minimum in the direction of the y -axis. Near the origin the graph has the shape of a saddle and so $(0, 0)$ is called a *saddle point* of f .

We need to be able to determine whether or not a function has an extreme value at a critical point. The following test, which is proved at the end of this section, is analogous to the Second Derivative Test for functions of one variable.

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b) .

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b) , or (a, b) could be a saddle point of f .

NOTE 3 To remember the formula for D , it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

V EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

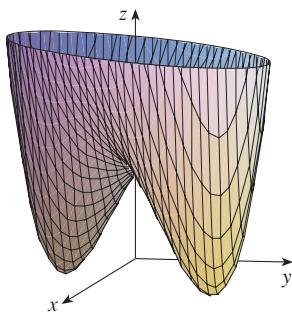


FIGURE 4
 $z = x^4 + y^4 - 4xy + 1$

■ A contour map of the function f in Example 3 is shown in Figure 5. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found in Example 3.

FIGURE 5

TEC In Module 14.7 you can use contour maps to estimate the locations of critical points.

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f .

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \quad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

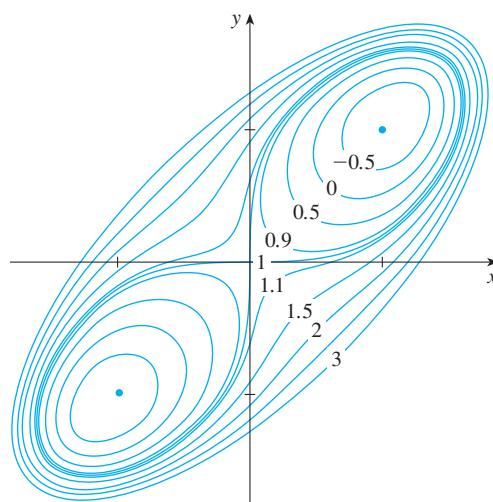
$$\boxed{4} \quad 2x(10y - 5 - 2x^2) = 0$$

$$\boxed{5} \quad 5x^2 - 4y - 8y^3 = 0$$

From Equation 4 we see that either

$$x = 0 \quad \text{or} \quad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.



In the second case ($10y - 5 - 2x^2 = 0$), we get

$$6 \quad x^2 = 5y - 2.5$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$7 \quad 4y^3 - 21y + 12.5 = 0$$

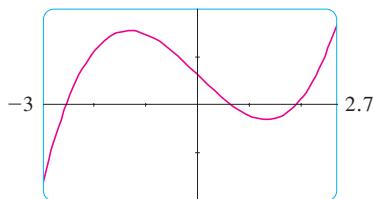
Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \quad y \approx 0.6468 \quad y \approx 1.8984$$

FIGURE 6



(Alternatively, we could have used Newton's method or a rootfinder to locate these roots.) From Equation 6, the corresponding x -values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
(0, 0)	0.00	-10.00	80.00	local maximum
(± 2.64 , 1.90)	8.50	-55.93	2488.72	local maximum
(± 0.86 , 0.65)	-1.48	-5.87	-187.64	saddle point

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: The dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

TEC Visual 14.7 shows several families of surfaces. The surface in Figures 7 and 8 is a member of one of these families.

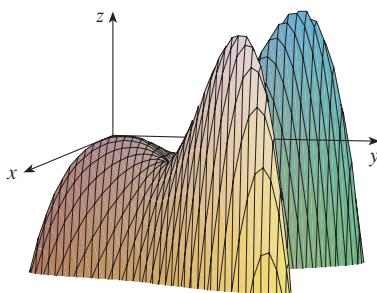


FIGURE 7

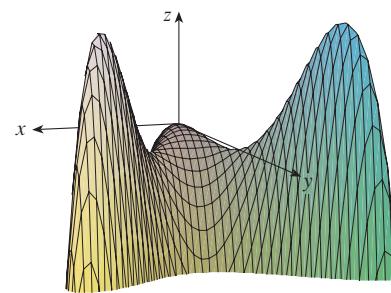


FIGURE 8

- The five critical points of the function f in Example 4 are shown in red in the contour map of f in Figure 9.

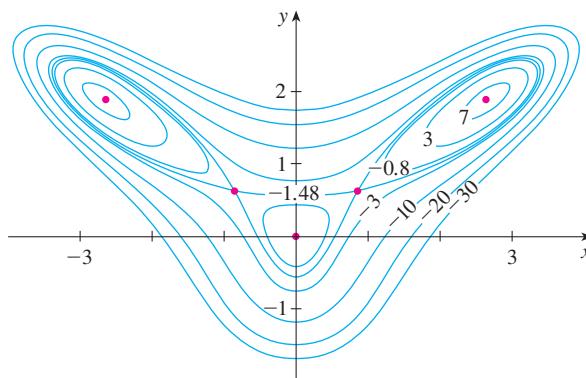


FIGURE 9

EXAMPLE 5 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x - 1)^2 + y^2 + (z + 2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x - 1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x - 1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test $(\frac{11}{6}, \frac{5}{3})$ has a local minimum. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$. ■

EXAMPLE 6 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION Let the length, width, and height of the box (in meters) be x , y , and z , as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

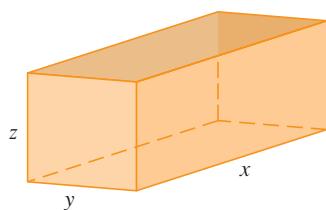


FIGURE 10

Solving this equation for z , we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy \frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \quad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \quad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V , or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V , so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3 . ■

ABSOLUTE MAXIMUM AND MINIMUM VALUES

For a function f of one variable the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. According to the Closed Interval Method in Section 4.1, we found these by evaluating f not only at the critical numbers but also at the endpoints a and b .

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in \mathbb{R}^2 is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D .] For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)

A **bounded set** in \mathbb{R}^2 is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 EXTREME VALUE THEOREM FOR FUNCTIONS OF TWO VARIABLES If f is continuous on a closed, bounded set D in \mathbb{R}^2 , then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

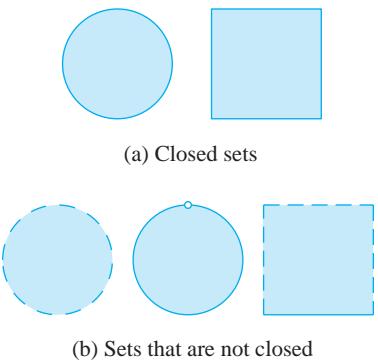


FIGURE 11

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1) , then (x_1, y_1) is either a critical point of f or a boundary point of D . Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D , so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \quad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D , which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \quad 0 \leq x \leq 3$$

This is an increasing function of x , so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \quad 0 \leq y \leq 2$$

This is a decreasing function of y , so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3$$

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f . ■

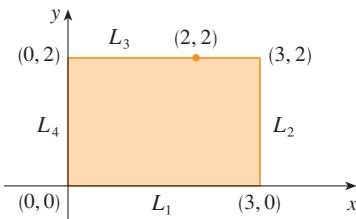


FIGURE 12

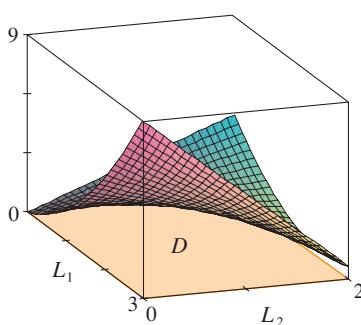


FIGURE 13

$$f(x, y) = x^2 - 2xy + 2y$$

We close this section by giving a proof of the first part of the Second Derivatives Test. Part (b) has a similar proof.

PROOF OF THEOREM 3, PART (A) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}}f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f &= D_{\mathbf{u}}(D_{\mathbf{u}}f) = \frac{\partial}{\partial x}(D_{\mathbf{u}}f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}}f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \end{aligned} \quad (\text{by Clairaut's Theorem})$$

If we complete the square in this expression, we obtain

$$10 \quad D_{\mathbf{u}}^2 f = f_{xx} \left(h + \frac{f_{xy}}{f_{xx}} k \right)^2 + \frac{k^2}{f_{xx}} (f_{xx}f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B . Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B . This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of \mathbf{u} , then C is concave upward on an interval of length 2δ . This is true in the direction of every vector \mathbf{u} , so if we restrict (x, y) to lie in B , the graph of f lies above its horizontal tangent plane at P . Thus $f(x, y) \geq f(a, b)$ whenever (x, y) is in B . This shows that $f(a, b)$ is a local minimum. ■

14.7 EXERCISES

1. Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f ?

- (a) $f_{xx}(1, 1) = 4, f_{xy}(1, 1) = 1, f_{yy}(1, 1) = 2$
 (b) $f_{xx}(1, 1) = 4, f_{xy}(1, 1) = 3, f_{yy}(1, 1) = 2$

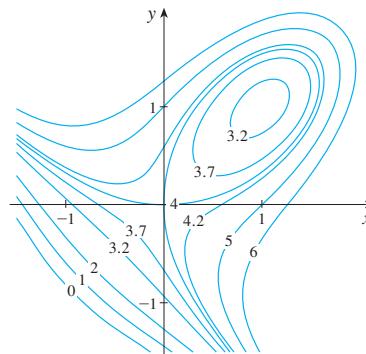
2. Suppose $(0, 2)$ is a critical point of a function g with continuous second derivatives. In each case, what can you say about g ?

- (a) $g_{xx}(0, 2) = -1, g_{xy}(0, 2) = 6, g_{yy}(0, 2) = 1$
 (b) $g_{xx}(0, 2) = -1, g_{xy}(0, 2) = 2, g_{yy}(0, 2) = -8$
 (c) $g_{xx}(0, 2) = 4, g_{xy}(0, 2) = 6, g_{yy}(0, 2) = 9$

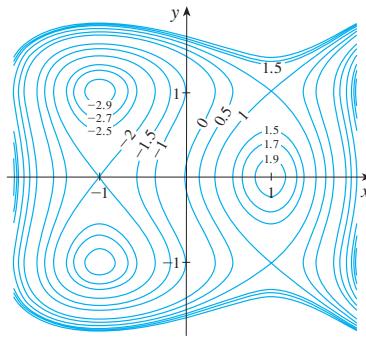
- 3–4 Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your

reasoning. Then use the Second Derivatives Test to confirm your predictions.

3. $f(x, y) = 4 + x^3 + y^3 - 3xy$



4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



5–18 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$

6. $f(x, y) = x^3y + 12x^2 - 8y$

7. $f(x, y) = x^4 + y^4 - 4xy + 2$

8. $f(x, y) = e^{4y-x^2-y^2}$

9. $f(x, y) = (1 + xy)(x + y)$

10. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

11. $f(x, y) = x^3 - 12xy + 8y^3$

12. $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

13. $f(x, y) = e^x \cos y$

14. $f(x, y) = y \cos x$

15. $f(x, y) = (x^2 + y^2)e^{y^2-x^2}$

16. $f(x, y) = e^y(y^2 - x^2)$

17. $f(x, y) = y^2 - 2y \cos x, \quad 1 \leq x \leq 7$

18. $f(x, y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi$

19. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

20. Show that $f(x, y) = x^2ye^{-x^2-y^2}$ has maximum values at $(\pm 1, 1/\sqrt{2})$ and minimum values at $(\pm 1, -1/\sqrt{2})$. Show also that f has infinitely many other critical points and $D = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

21–24 Use a graph and/or level curves to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

21. $f(x, y) = x^2 + y^2 + x^{-2}y^{-2}$

22. $f(x, y) = xye^{-x^2-y^2}$

23. $f(x, y) = \sin x + \sin y + \sin(x + y), \quad 0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi$

24. $f(x, y) = \sin x + \sin y + \cos(x + y), \quad 0 \leq x \leq \pi/4, \quad 0 \leq y \leq \pi/4$

25–28 Use a graphing device as in Example 4 (or Newton's method or a rootfinder) to find the critical points of f correct to three decimal places. Then classify the critical points and find the highest or lowest points on the graph.

25. $f(x, y) = x^4 - 5x^2 + y^2 + 3x + 2$

26. $f(x, y) = 5 - 10xy - 4x^2 + 3y - y^4$

27. $f(x, y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4$

28. $f(x, y) = e^x + y^4 - x^3 + 4 \cos y$

29–36 Find the absolute maximum and minimum values of f on the set D .

29. $f(x, y) = 1 + 4x - 5y, \quad D$ is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$

30. $f(x, y) = 3 + xy - x - 2y, \quad D$ is the closed triangular region with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$

31. $f(x, y) = x^2 + y^2 + x^2y + 4, \quad D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$

32. $f(x, y) = 4x + 6y - x^2 - y^2, \quad D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 5\}$

33. $f(x, y) = x^4 + y^4 - 4xy + 2, \quad D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$

34. $f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

35. $f(x, y) = 2x^3 + y^4, \quad D = \{(x, y) \mid x^2 + y^2 \leq 1\}$

36. $f(x, y) = x^3 - 3x - y^3 + 12y, \quad D$ is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$.

37. For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions exist. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has local maxima at both of them. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

38. If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be

an absolute maximum. But this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

39. Find the shortest distance from the point $(2, 1, -1)$ to the plane $x + y - z = 1$.
40. Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.
41. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.
42. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.
43. Find three positive numbers whose sum is 100 and whose product is a maximum.
44. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.
45. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r .
46. Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.
47. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.
48. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2 .
49. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c .
50. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.
51. A cardboard box without a lid is to have a volume of $32,000 \text{ cm}^3$. Find the dimensions that minimize the amount of cardboard used.
52. A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of $10 \text{ units/m}^2 \text{ per day}$, the north and south walls at a rate of $8 \text{ units/m}^2 \text{ per day}$, the floor at a rate of $1 \text{ unit/m}^2 \text{ per day}$, and the roof at a rate of $5 \text{ units/m}^2 \text{ per day}$. Each wall must be at least 30 m long, the height must be at least 4 m, and the volume must be exactly 4000 m^3 .
 - (a) Find and sketch the domain of the heat loss as a function of the lengths of the sides.

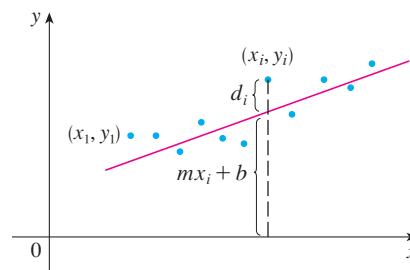
- (b) Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)
- (c) Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed?

53. If the length of the diagonal of a rectangular box must be L , what is the largest possible volume?
54. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p , q , and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

55. Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b . The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible. (See the figure.)



Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The **method of least squares** determines m and b so as to minimize $\sum_{i=1}^n d_i^2$, the sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^n x_i + bn = \sum_{i=1}^n y_i$$

$$m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

Thus the line is found by solving these two equations in the two unknowns m and b . (See Section 1.2 for a further discussion and applications of the method of least squares.)

56. Find an equation of the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first octant.

**APPLIED
PROJECT****DESIGNING A DUMPSTER**

For this project we locate a trash dumpster in order to study its shape and construction. We then attempt to determine the dimensions of a container of similar design that minimize construction cost.

1. First locate a trash dumpster in your area. Carefully study and describe all details of its construction, and determine its volume. Include a sketch of the container.
2. While maintaining the general shape and method of construction, determine the dimensions such a container of the same volume should have in order to minimize the cost of construction. Use the following assumptions in your analysis:
 - The sides, back, and front are to be made from 12-gauge (0.1046 inch thick) steel sheets, which cost \$0.70 per square foot (including any required cuts or bends).
 - The base is to be made from a 10-gauge (0.1345 inch thick) steel sheet, which costs \$0.90 per square foot.
 - Lids cost approximately \$50.00 each, regardless of dimensions.
 - Welding costs approximately \$0.18 per foot for material and labor combined.
 Give justification of any further assumptions or simplifications made of the details of construction.
3. Describe how any of your assumptions or simplifications may affect the final result.
4. If you were hired as a consultant on this investigation, what would your conclusions be? Would you recommend altering the design of the dumpster? If so, describe the savings that would result.

**DISCOVERY
PROJECT****QUADRATIC APPROXIMATIONS AND CRITICAL POINTS**

The Taylor polynomial approximation to functions of one variable that we discussed in Chapter 11 can be extended to functions of two or more variables. Here we investigate quadratic approximations to functions of two variables and use them to give insight into the Second Derivatives Test for classifying critical points.

In Section 14.4 we discussed the linearization of a function f of two variables at a point (a, b) :

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that the graph of L is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ and the corresponding linear approximation is $f(x, y) \approx L(x, y)$. The linearization L is also called the **first-degree Taylor polynomial** of f at (a, b) .

1. If f has continuous second-order partial derivatives at (a, b) , then the **second-degree Taylor polynomial** of f at (a, b) is

$$\begin{aligned} Q(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{1}{2}f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2 \end{aligned}$$

and the approximation $f(x, y) \approx Q(x, y)$ is called the **quadratic approximation** to f at (a, b) . Verify that Q has the same first- and second-order partial derivatives as f at (a, b) .

2. (a) Find the first- and second-degree Taylor polynomials L and Q of $f(x, y) = e^{-x^2-y^2}$ at $(0, 0)$.
graph (b) Graph f , L , and Q . Comment on how well L and Q approximate f .
3. (a) Find the first- and second-degree Taylor polynomials L and Q for $f(x, y) = xe^y$ at $(1, 0)$.
graph (b) Compare the values of L , Q , and f at $(0.9, 0.1)$.
graph (c) Graph f , L , and Q . Comment on how well L and Q approximate f .
4. In this problem we analyze the behavior of the polynomial $f(x, y) = ax^2 + bxy + cy^2$ (without using the Second Derivatives Test) by identifying the graph as a paraboloid.
(a) By completing the square, show that if $a \neq 0$, then
- $$f(x, y) = ax^2 + bxy + cy^2 = a \left[\left(x + \frac{b}{2a} y \right)^2 + \left(\frac{4ac - b^2}{4a^2} \right) y^2 \right]$$
- (b) Let $D = 4ac - b^2$. Show that if $D > 0$ and $a > 0$, then f has a local minimum at $(0, 0)$.
(c) Show that if $D > 0$ and $a < 0$, then f has a local maximum at $(0, 0)$.
(d) Show that if $D < 0$, then $(0, 0)$ is a saddle point.
5. (a) Suppose f is any function with continuous second-order partial derivatives such that $f(0, 0) = 0$ and $(0, 0)$ is a critical point of f . Write an expression for the second-degree Taylor polynomial, Q , of f at $(0, 0)$.
(b) What can you conclude about Q from Problem 4?
(c) In view of the quadratic approximation $f(x, y) \approx Q(x, y)$, what does part (b) suggest about f ?

14.8 LAGRANGE MULTIPLIERS

In Example 6 in Section 14.7 we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, which expressed the side condition that the surface area was 12 m^2 . In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f . These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ .

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

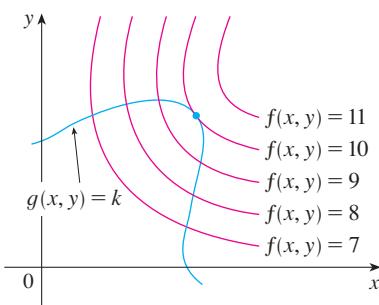


FIGURE 1

TEC Visual 14.8 animates Figure 1 for both level curves and level surfaces.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P . If t_0 is the parameter value corresponding to the point P , then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C . Since f has an extreme value at (x_0, y_0, z_0) , it follows that h has an extreme value at t_0 , so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C . But we already know from Section 14.6 that the gradient vector of g , $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 14.6.18.) This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

■ Lagrange multipliers are named after the French-Italian mathematician Joseph-Louis Lagrange (1736–1813). See page 283 for a biographical sketch of Lagrange.

1

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure based on Equation 1 is as follows.

METHOD OF LAGRANGE MULTIPLIERS To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z , and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$\text{and } g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f ; the smallest is the minimum value of f .

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of its components, then the equations in step (a) become

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z , and λ , but it is not necessary to find explicit values for λ .

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y , and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = k$$

Our first illustration of Lagrange's method is to reconsider the problem given in Example 6 in Section 14.7.

V EXAMPLE 1 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 6 in Section 14.7, we let x , y , and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x , y , z , and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x \quad V_y = \lambda g_y \quad V_z = \lambda g_z \quad 2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \quad yz = \lambda(2z + y)$$

$$\boxed{3} \quad xz = \lambda(2z + x)$$

$$\boxed{4} \quad xy = \lambda(2x + 2y)$$

$$\boxed{5} \quad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x , (3) by y , and (4) by z , then the left sides of these equations will be identical. Doing this, we have

$$\boxed{6} \quad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \quad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \quad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (2), (3), and (4) and this would contradict (5). Therefore, from (6) and (7), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x , y , and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7. ■

- Another method for solving the system of equations (2–5) is to solve each of Equations 2, 3, and 4 for λ and then to equate the resulting expressions.

In geometric terms, Example 2 asks for the highest and lowest points on the curve C in Figure 2 that lies on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.

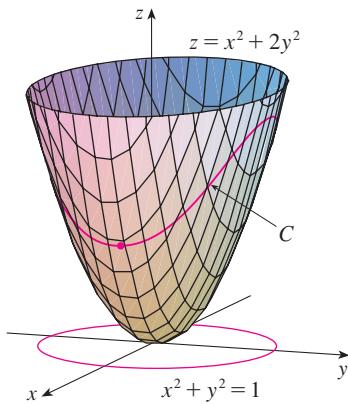


FIGURE 2

The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.

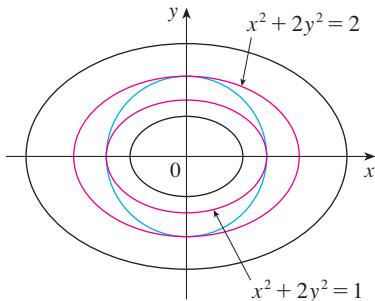


FIGURE 3

V EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = 1$$

or as

9

$$2x = 2x\lambda$$

10

$$4y = 2y\lambda$$

11

$$x^2 + y^2 = 1$$

From (9) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (11) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (10), so then (11) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \quad f(0, -1) = 2 \quad f(1, 0) = 1 \quad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. Checking with Figure 2, we see that these values look reasonable. ■

EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

SOLUTION According to the procedure in (14.7.9), we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \quad f(\pm 1, 0) = 1 \quad f(0, \pm 1) = 2$$

Therefore the maximum value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. ■

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

$$\boxed{12} \quad 2(x - 3) = 2x\lambda$$

$$\boxed{13} \quad 2(y - 1) = 2y\lambda$$

$$\boxed{14} \quad 2(z + 1) = 2z\lambda$$

$$\boxed{15} \quad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x , y , and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \quad \text{or} \quad x(1 - \lambda) = 3 \quad \text{or} \quad x = \frac{3}{1 - \lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1 - \lambda} \quad z = -\frac{1}{1 - \lambda}$$

Therefore, from (15), we have

$$\frac{3^2}{(1 - \lambda)^2} + \frac{1^2}{(1 - \lambda)^2} + \frac{(-1)^2}{(1 - \lambda)^2} = 4$$

which gives $(1 - \lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z) :

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}} \right) \quad \text{and} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11})$ and the farthest is $(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11})$. ■

- Figure 4 shows the sphere and the nearest point P in Example 4. Can you see how to find the coordinates of P without using calculus?

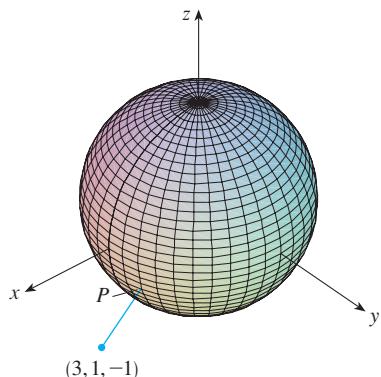


FIGURE 4

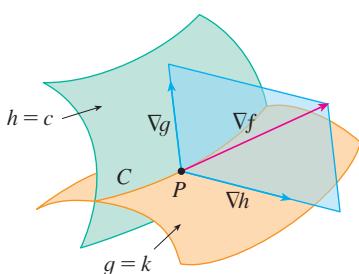


FIGURE 5

TWO CONSTRAINTS

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P . But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C . This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ

(called Lagrange multipliers) such that

16

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x , y , z , λ , and μ . These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

- The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse (Figure 6). Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

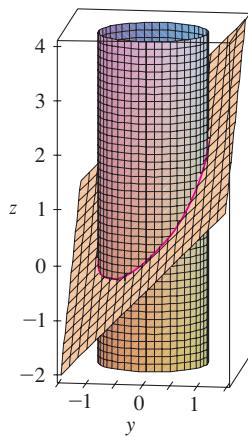


FIGURE 6

V EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

17

$$1 = \lambda + 2x\mu$$

18

$$2 = -\lambda + 2y\mu$$

19

$$3 = \lambda$$

20

$$x - y + z = 1$$

21

$$x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from (19)] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

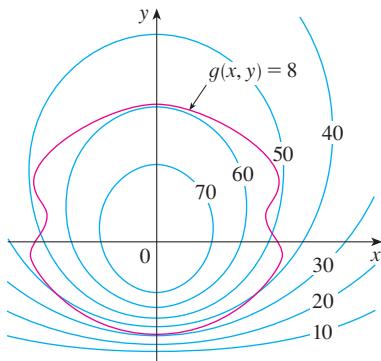
and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (20), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$. ■

14.8 EXERCISES

- 1.** Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



- 2.** (a) Use a graphing calculator or computer to graph the circle $x^2 + y^2 = 1$. On the same screen, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?
 (b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

3–17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. $f(x, y) = x^2 + y^2; \quad xy = 1$

4. $f(x, y) = 4x + 6y; \quad x^2 + y^2 = 13$

5. $f(x, y) = x^2y; \quad x^2 + 2y^2 = 6$

6. $f(x, y) = e^{xy}; \quad x^3 + y^3 = 16$

7. $f(x, y, z) = 2x + 6y + 10z; \quad x^2 + y^2 + z^2 = 35$

8. $f(x, y, z) = 8x - 4z; \quad x^2 + 10y^2 + z^2 = 5$

9. $f(x, y, z) = xyz; \quad x^2 + 2y^2 + 3z^2 = 6$

10. $f(x, y, z) = x^2y^2z^2; \quad x^2 + y^2 + z^2 = 1$

11. $f(x, y, z) = x^2 + y^2 + z^2; \quad x^4 + y^4 + z^4 = 1$

12. $f(x, y, z) = x^4 + y^4 + z^4; \quad x^2 + y^2 + z^2 = 1$

13. $f(x, y, z, t) = x + y + z + t; \quad x^2 + y^2 + z^2 + t^2 = 1$

14. $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n;$

$$x_1^2 + x_2^2 + \dots + x_n^2 = 1$$

15. $f(x, y, z) = x + 2y; \quad x + y + z = 1, \quad y^2 + z^2 = 4$

16. $f(x, y, z) = 3x - y - 3z; \quad x + y - z = 0, \quad x^2 + 2z^2 = 1$

17. $f(x, y, z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$

18–19 Find the extreme values of f on the region described by the inequality.

18. $f(x, y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16$

19. $f(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leq 1$

- 20.** Consider the problem of maximizing the function

$$f(x, y) = 2x + 3y \text{ subject to the constraint } \sqrt{x} + \sqrt{y} = 5.$$

- (a) Try using Lagrange multipliers to solve the problem.
- (b) Does $f(25, 0)$ give a larger value than the one in part (a)?
- (c) Solve the problem by graphing the constraint equation and several level curves of f .
- (d) Explain why the method of Lagrange multipliers fails to solve the problem.
- (e) What is the significance of $f(9, 4)$?

- 21.** Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).

- (a) Try using Lagrange multipliers to solve the problem.
- (b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ .
- (c) Explain why Lagrange multipliers fail to find the minimum value in this case.

- CAS 22.** (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$ by graphical methods.

- (b) Solve the problem in part (a) with the aid of Lagrange multipliers. Use your CAS to solve the equations numerically. Compare your answers with those in part (a).

- 23.** The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Sections 14.1 and 14.3 we discussed how the Cobb-Douglas model $P = bL^\alpha K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n , and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \quad \text{and} \quad K = \frac{(1 - \alpha)p}{n}$$

- 24.** Referring to Exercise 23, we now suppose that the production is fixed at $bL^\alpha K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?
- 25.** Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.
- 26.** Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.

Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

27–39 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 14.7.

- | | |
|------------------------|------------------------|
| 27. Exercise 39 | 28. Exercise 40 |
| 29. Exercise 41 | 30. Exercise 42 |
| 31. Exercise 43 | 32. Exercise 44 |
| 33. Exercise 45 | 34. Exercise 46 |
| 35. Exercise 47 | 36. Exercise 48 |
| 37. Exercise 49 | 38. Exercise 50 |
| 39. Exercise 53 | |

- 40.** Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm .
- 41.** The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.
- 42.** The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.
- (a) Graph the cone, the plane, and the ellipse.
(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

CAS **43–44** Find the maximum and minimum values of f subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

- 43.** $f(x, y, z) = ye^{x-z}; \quad 9x^2 + 4y^2 + 36z^2 = 36, \quad xy + yz = 1$
- 44.** $f(x, y, z) = x + y + z; \quad x^2 - y^2 = z, \quad x^2 + z^2 = 4$

- 45.** (a) Find the maximum value of

$$f(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that x_1, x_2, \dots, x_n are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

- (b) Deduce from part (a) that if x_1, x_2, \dots, x_n are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

- 46.** (a) Maximize $\sum_{i=1}^n x_i y_i$ subject to the constraints $\sum_{i=1}^n x_i^2 = 1$ and $\sum_{i=1}^n y_i^2 = 1$.
(b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$

to show that

$$\sum a_i b_i \leq \sqrt{\sum a_j^2} \sqrt{\sum b_j^2}$$

for any numbers $a_1, \dots, a_n, b_1, \dots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

APPLIED PROJECT

ROCKET SCIENCE

Many rockets, such as the Pegasus XL currently used to launch satellites and the Saturn V that first put men on the moon, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed in such a way as to minimize the total mass of the rocket while enabling it to reach a desired velocity.



For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta V = -c \ln\left(1 - \frac{(1-S)M_r}{P + M_r}\right)$$

where M_r is the mass of the rocket engine including initial fuel, P is the mass of the payload, S is a *structural factor* determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload), and c is the (constant) speed of exhaust relative to the rocket.

Now consider a rocket with three stages and a payload of mass A . Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the i th stage, we can initially consider the rocket engine to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

- 1.** Show that the velocity attained after all three stages have been jettisoned is given by

$$v_f = c \left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right) \right]$$

- 2.** We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the desired velocity v_f from Problem 1 is attained. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions. To simplify, we define variables N_i so that the constraint equation may be expressed as $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. Since M is now difficult to express in terms of the N_i 's, we wish to use a simpler function that will be minimized at the same place as M . Show that

$$\begin{aligned} \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} &= \frac{(1-S)N_1}{1 - SN_1} \\ \frac{M_2 + M_3 + A}{M_3 + A} &= \frac{(1-S)N_2}{1 - SN_2} \\ \frac{M_3 + A}{A} &= \frac{(1-S)N_3}{1 - SN_3} \end{aligned}$$

and conclude that

$$\frac{M + A}{A} = \frac{(1-S)^3 N_1 N_2 N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}$$

- 3.** Verify that $\ln((M + A)/A)$ is minimized at the same location as M ; use Lagrange multipliers and the results of Problem 2 to find expressions for the values of N_i where the minimum occurs subject to the constraint $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. [Hint: Use properties of logarithms to help simplify the expressions.]
- 4.** Find an expression for the minimum value of M as a function of v_f .
- 5.** If we want to put a three-stage rocket into orbit 100 miles above the earth's surface, a final velocity of approximately 17,500 mi/h is required. Suppose that each stage is built with a structural factor $S = 0.2$ and an exhaust speed of $c = 6000$ mi/h.
- (a) Find the minimum total mass M of the rocket engines as a function of A .
 - (b) Find the mass of each individual stage as a function of A . (They are not equally sized!)
- 6.** The same rocket would require a final velocity of approximately 24,700 mi/h in order to escape earth's gravity. Find the mass of each individual stage that would minimize the total mass of the rocket engines and allow the rocket to propel a 500-pound probe into deep space.



**APPLIED
PROJECT****HYDRO-TURBINE OPTIMIZATION**

The Katahdin Paper Company in Millinocket, Maine, operates a hydroelectric generating station on the Penobscot River. Water is piped from a dam to the power station. The rate at which the water flows through the pipe varies, depending on external conditions.

The power station has three different hydroelectric turbines, each with a known (and unique) power function that gives the amount of electric power generated as a function of the water flow arriving at the turbine. The incoming water can be apportioned in different volumes to each turbine, so the goal is to determine how to distribute water among the turbines to give the maximum total energy production for any rate of flow.

Using experimental evidence and *Bernoulli's equation*, the following quadratic models were determined for the power output of each turbine, along with the allowable flows of operation:

$$KW_1 = (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$KW_2 = (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$KW_3 = (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}Q_3^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$250 \leq Q_1 \leq 1110, \quad 250 \leq Q_2 \leq 1110, \quad 250 \leq Q_3 \leq 1225$$

where

Q_i = flow through turbine i in cubic feet per second

KW_i = power generated by turbine i in kilowatts

Q_T = total flow through the station in cubic feet per second

1. If all three turbines are being used, we wish to determine the flow Q_i to each turbine that will give the maximum total energy production. Our limitations are that the flows must sum to the total incoming flow and the given domain restrictions must be observed. Consequently, use Lagrange multipliers to find the values for the individual flows (as functions of Q_T) that maximize the total energy production $KW_1 + KW_2 + KW_3$ subject to the constraints $Q_1 + Q_2 + Q_3 = Q_T$ and the domain restrictions on each Q_i .
2. For which values of Q_T is your result valid?
3. For an incoming flow of 2500 ft³/s, determine the distribution to the turbines and verify (by trying some nearby distributions) that your result is indeed a maximum.
4. Until now we have assumed that all three turbines are operating; is it possible in some situations that more power could be produced by using only one turbine? Make a graph of the three power functions and use it to help decide if an incoming flow of 1000 ft³/s should be distributed to all three turbines or routed to just one. (If you determine that only one turbine should be used, which one would it be?) What if the flow is only 600 ft³/s?
5. Perhaps for some flow levels it would be advantageous to use two turbines. If the incoming flow is 1500 ft³/s, which two turbines would you recommend using? Use Lagrange multipliers to determine how the flow should be distributed between the two turbines to maximize the energy produced. For this flow, is using two turbines more efficient than using all three?
6. If the incoming flow is 3400 ft³/s, what would you recommend to the company?

14 REVIEW

CONCEPT CHECK

1. (a) What is a function of two variables?
 (b) Describe three methods for visualizing a function of two variables.
2. What is a function of three variables? How can you visualize such a function?
3. What does

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$
 mean? How can you show that such a limit does not exist?
4. (a) What does it mean to say that f is continuous at (a, b) ?
 (b) If f is continuous on \mathbb{R}^2 , what can you say about its graph?
5. (a) Write expressions for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ as limits.
 (b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically?
 How do you interpret them as rates of change?
 (c) If $f(x, y)$ is given by a formula, how do you calculate f_x and f_y ?
6. What does Clairaut's Theorem say?
7. How do you find a tangent plane to each of the following types of surfaces?
 (a) A graph of a function of two variables, $z = f(x, y)$
 (b) A level surface of a function of three variables,
 $F(x, y, z) = k$
8. Define the linearization of f at (a, b) . What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?
9. (a) What does it mean to say that f is differentiable at (a, b) ?
 (b) How do you usually verify that f is differentiable?
10. If $z = f(x, y)$, what are the differentials dx , dy , and dz ?
11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $f_y(a, b) = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b}$
2. There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z / \partial x$ and $\partial z / \partial y$?
13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$. How do you interpret it as a rate? How do you interpret it geometrically?
 (b) If f is differentiable, write an expression for $D_{\mathbf{u}}f(x_0, y_0)$ in terms of f_x and f_y .
14. (a) Define the gradient vector ∇f for a function f of two or three variables.
 (b) Express $D_{\mathbf{u}}f$ in terms of ∇f .
 (c) Explain the geometric significance of the gradient.
15. What do the following statements mean?
 (a) f has a local maximum at (a, b) .
 (b) f has an absolute maximum at (a, b) .
 (c) f has a local minimum at (a, b) .
 (d) f has an absolute minimum at (a, b) .
 (e) f has a saddle point at (a, b) .
16. (a) If f has a local maximum at (a, b) , what can you say about its partial derivatives at (a, b) ?
 (b) What is a critical point of f ?
17. State the Second Derivatives Test.
18. (a) What is a closed set in \mathbb{R}^2 ? What is a bounded set?
 (b) State the Extreme Value Theorem for functions of two variables.
 (c) How do you find the values that the Extreme Value Theorem guarantees?
19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

3. $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$
4. $D_k f(x, y, z) = f_z(x, y, z)$
5. If $f(x, y) \rightarrow L$ as $(x, y) \rightarrow (a, b)$ along every straight line through (a, b) , then $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$.
6. If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b) .

7. If f has a local minimum at (a, b) and f is differentiable at (a, b) , then $\nabla f(a, b) = \mathbf{0}$.

8. If f is a function, then

$$\lim_{(x, y) \rightarrow (2, 5)} f(x, y) = f(2, 5)$$

9. If $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$.

EXERCISES

- 1–2** Find and sketch the domain of the function.

1. $f(x, y) = \ln(x + y + 1)$

2. $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$

- 3–4** Sketch the graph of the function.

3. $f(x, y) = 1 - y^2$

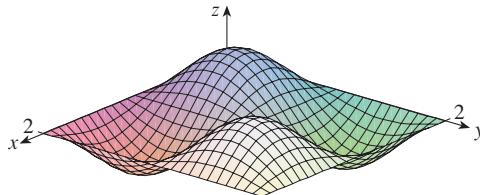
4. $f(x, y) = x^2 + (y - 2)^2$

- 5–6** Sketch several level curves of the function.

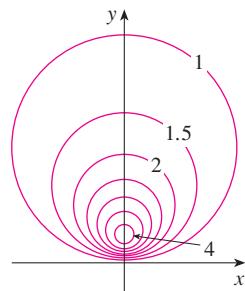
5. $f(x, y) = \sqrt{4x^2 + y^2}$

6. $f(x, y) = e^x + y$

7. Make a rough sketch of a contour map for the function whose graph is shown.



8. A contour map of a function f is shown. Use it to make a rough sketch of the graph of f .



10. If $(2, 1)$ is a critical point of f and

$$f_{xx}(2, 1)f_{yy}(2, 1) < [f_{xy}(2, 1)]^2$$

then f has a saddle point at $(2, 1)$.

11. If $f(x, y) = \sin x + \sin y$, then $-\sqrt{2} \leq D_u f(x, y) \leq \sqrt{2}$.

12. If $f(x, y)$ has two local maxima, then f must have a local minimum.

- 9–10** Evaluate the limit or show that it does not exist.

9. $\lim_{(x, y) \rightarrow (1, 1)} \frac{2xy}{x^2 + 2y^2}$

10. $\lim_{(x, y) \rightarrow (0, 0)} \frac{2xy}{x^2 + 2y^2}$

11. A metal plate is situated in the xy -plane and occupies the rectangle $0 \leq x \leq 10$, $0 \leq y \leq 8$, where x and y are measured in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures at equally spaced points were measured and recorded in the table.

(a) Estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$. What are the units?

(b) Estimate the value of $D_u T(6, 4)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Interpret your result.

(c) Estimate the value of $T_{xy}(6, 4)$.

$x \backslash y$	0	2	4	6	8
0	30	38	45	51	55
2	52	56	60	62	61
4	78	74	72	68	66
6	98	87	80	75	71
8	96	90	86	80	75
10	92	92	91	87	78

12. Find a linear approximation to the temperature function $T(x, y)$ in Exercise 11 near the point $(6, 4)$. Then use it to estimate the temperature at the point $(5, 3.8)$.

- 13–17** Find the first partial derivatives.

13. $f(x, y) = \sqrt{2x + y^2}$

14. $u = e^{-r} \sin 2\theta$

15. $g(u, v) = u \tan^{-1} v$

16. $w = \frac{x}{y - z}$

17. $T(p, q, r) = p \ln(q + e^r)$

- 18.** The speed of sound traveling through ocean water is a function of temperature, salinity, and pressure. It has been modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), S is the salinity (the concentration of salts in parts per thousand, which means the number of grams of dissolved solids per 1000 g of water), and D is the depth below the ocean surface (in meters). Compute $\partial C/\partial T$, $\partial C/\partial S$, and $\partial C/\partial D$ when $T = 10^\circ\text{C}$, $S = 35$ parts per thousand, and $D = 100$ m. Explain the physical significance of these partial derivatives.

- 19–22** Find all second partial derivatives of f .

19. $f(x, y) = 4x^3 - xy^2$

20. $z = xe^{-2y}$

21. $f(x, y, z) = x^k y^l z^m$

22. $v = r \cos(s + 2t)$

- 23.** If $z = xy + xe^{y/x}$, show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = xy + z$.

- 24.** If $z = \sin(x + \sin t)$, show that

$$\frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial x \partial t} = \frac{\partial z}{\partial t} \frac{\partial^2 z}{\partial x^2}$$

- 25–29** Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

25. $z = 3x^2 - y^2 + 2x, (1, -2, 1)$

26. $z = e^x \cos y, (0, 0, 1)$

27. $x^2 + 2y^2 - 3z^2 = 3, (2, -1, 1)$

28. $xy + yz + zx = 3, (1, 1, 1)$

29. $\sin(xyz) = x + 2y + 3z, (2, -1, 0)$

-  **30.** Use a computer to graph the surface $z = x^2 + y^4$ and its tangent plane and normal line at $(1, 1, 2)$ on the same screen. Choose the domain and viewpoint so that you get a good view of all three objects.

- 31.** Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$.

- 32.** Find du if $u = \ln(1 + se^{2t})$.

- 33.** Find the linear approximation of the function

$f(x, y, z) = x^3 \sqrt{y^2 + z^2}$ at the point $(2, 3, 4)$ and use it to estimate the number $(1.98)^3 \sqrt{(3.01)^2 + (3.97)^2}$.

- 34.** The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.

- 35.** If $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$, and $z = p \sin p$, use the Chain Rule to find du/dp .

- 36.** If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\partial v/\partial s$ and $\partial v/\partial t$ when $s = 0$ and $t = 1$.

- 37.** Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.

- 38.** Use a tree diagram to write out the Chain Rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$, and $v = v(p, q, r, s)$ are all differentiable functions.

- 39.** If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = x$$

- 40.** The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

- 41.** If $z = f(u, v)$, where $u = xy$, $v = y/x$, and f has continuous second partial derivatives, show that

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = -4uv \frac{\partial^2 z}{\partial u \partial v} + 2v \frac{\partial z}{\partial v}$$

- 42.** If $yz^4 + x^2z^3 = e^{xyz}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

- 43.** Find the gradient of the function $f(x, y, z) = z^2 e^{x\sqrt{y}}$.

- 44.** (a) When is the directional derivative of f a maximum?
 (b) When is it a minimum?
 (c) When is it 0?
 (d) When is it half of its maximum value?

- 45–46** Find the directional derivative of f at the given point in the indicated direction.

- 45.** $f(x, y) = 2\sqrt{x} - y^2, (1, 5)$,
 in the direction toward the point $(4, 1)$

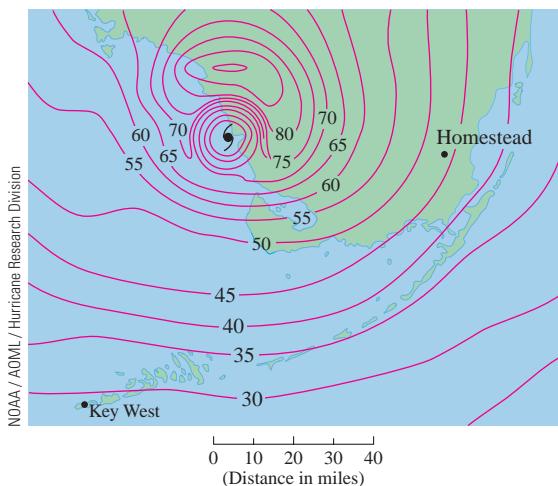
- 46.** $f(x, y, z) = x^2y + x\sqrt{1+z}, (1, 2, 3)$,
 in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

- 47.** Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$. In which direction does it occur?

- 48.** Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point $(0, 1, 2)$. What is the maximum rate of increase?

- 49.** The contour map shows wind speed in knots during Hurricane Andrew on August 24, 1992. Use it to estimate the

value of the directional derivative of the wind speed at Homestead, Florida, in the direction of the eye of the hurricane.



- 50.** Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

51–54 Find the local maximum and minimum values and saddle points of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$

52. $f(x, y) = x^3 - 6xy + 8y^3$

53. $f(x, y) = 3xy - x^2y - xy^2$

54. $f(x, y) = (x^2 + y)e^{y/2}$

55–56 Find the absolute maximum and minimum values of f on the set D .

55. $f(x, y) = 4xy^2 - x^2y^2 - xy^3$; D is the closed triangular region in the xy -plane with vertices $(0, 0)$, $(0, 6)$, and $(6, 0)$

56. $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$; D is the disk $x^2 + y^2 \leq 4$

- 57.** Use a graph and/or level curves to estimate the local maximum and minimum values and saddle points of $f(x, y) = x^3 - 3x + y^4 - 2y^2$. Then use calculus to find these values precisely.

- 58.** Use a graphing calculator or computer (or Newton's method or a computer algebra system) to find the critical points of $f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4$ correct to three decimal places. Then classify the critical points and find the highest point on the graph.

- 59–62** Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint(s).

59. $f(x, y) = x^2y$; $x^2 + y^2 = 1$

60. $f(x, y) = \frac{1}{x} + \frac{1}{y}$; $\frac{1}{x^2} + \frac{1}{y^2} = 1$

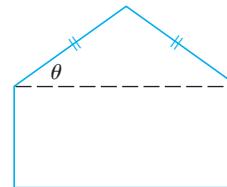
61. $f(x, y, z) = xyz$; $x^2 + y^2 + z^2 = 3$

62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$; $x + y + z = 1$, $x - y + 2z = 2$

- 63.** Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

- 64.** A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.

- 65.** A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the figure. If the pentagon has fixed perimeter P , find the lengths of the sides of the pentagon that maximize the area of the pentagon.



- 66.** A particle of mass m moves on the surface $z = f(x, y)$. Let $x = x(t)$ and $y = y(t)$ be the x - and y -coordinates of the particle at time t .

- (a) Find the velocity vector \mathbf{v} and the kinetic energy $K = \frac{1}{2}m|\mathbf{v}|^2$ of the particle.

- (b) Determine the acceleration vector \mathbf{a} .

- (c) Let $z = x^2 + y^2$ and $x(t) = t \cos t$, $y(t) = t \sin t$. Find the velocity vector, the kinetic energy, and the acceleration vector.

PROBLEMS PLUS

- A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum values of the sum of the squares of the areas of the smaller rectangles.
- Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Based on certain tests, the concentration of blood (in parts per million) at a point $P(x, y)$ on the surface of seawater is approximated by

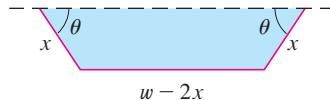
$$C(x, y) = e^{-(x^2+2y^2)/10^4}$$

where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin.

- Identify the level curves of the concentration function and sketch several members of this family together with a path that a shark will follow to the source.
- Suppose a shark is at the point (x_0, y_0) when it first detects the presence of blood in the water. Find an equation of the shark's path by setting up and solving a differential equation.

- A long piece of galvanized sheet metal with width w is to be bent into a symmetric form with three straight sides to make a rain gutter. A cross-section is shown in the figure.

 - Determine the dimensions that allow the maximum possible flow; that is, find the dimensions that give the maximum possible cross-sectional area.
 - Would it be better to bend the metal into a gutter with a semicircular cross-section?



- For what values of the number r is the function

$$f(x, y, z) = \begin{cases} \frac{(x + y + z)^r}{x^2 + y^2 + z^2} & \text{if } (x, y, z) \neq 0 \\ 0 & \text{if } (x, y, z) = 0 \end{cases}$$

continuous on \mathbb{R}^3 ?

- Suppose f is a differentiable function of one variable. Show that all tangent planes to the surface $z = xf(y/x)$ intersect in a common point.
- (a) Newton's method for approximating a root of an equation $f(x) = 0$ (see Section 4.8) can be adapted to approximating a solution of a system of equations $f(x, y) = 0$ and $g(x, y) = 0$. The surfaces $z = f(x, y)$ and $z = g(x, y)$ intersect in a curve that intersects the

xy-plane at the point (r, s) , which is the solution of the system. If an initial approximation (x_1, y_1) is close to this point, then the tangent planes to the surfaces at (x_1, y_1) intersect in a straight line that intersects the *xy*-plane in a point (x_2, y_2) , which should be closer to (r, s) . (Compare with Figure 2 in Section 4.8.) Show that

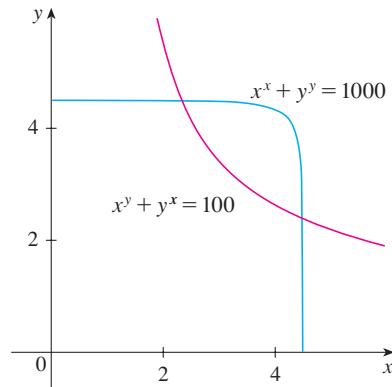
$$x_2 = x_1 - \frac{fg_y - f_y g}{f_x g_y - f_y g_x} \quad \text{and} \quad y_2 = y_1 - \frac{f_x g - f g_x}{f_x g_y - f_y g_x}$$

where f, g , and their partial derivatives are evaluated at (x_1, y_1) . If we continue this procedure, we obtain successive approximations (x_n, y_n) .

- (b) It was Thomas Simpson (1710–1761) who formulated Newton's method as we know it today and who extended it to functions of two variables as in part (a). (See the biography of Simpson on page 502.) The example that he gave to illustrate the method was to solve the system of equations

$$x^x + y^y = 1000 \quad x^y + y^x = 100$$

In other words, he found the points of intersection of the curves in the figure. Use the method of part (a) to find the coordinates of the points of intersection correct to six decimal places.



7. If the ellipse $x^2/a^2 + y^2/b^2 = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?
8. Among all planes that are tangent to the surface $xy^2z^2 = 1$, find the ones that are farthest from the origin.