

# 16

# Vector Calculus

Parametric surfaces, which are studied in Section 16.6, are frequently used by programmers in creating the sophisticated software used in the development of computer animated films like the *Shrek* series. The software employs parametric and other types of surfaces to create 3D models of the characters and objects in a scene. Color, texture, and lighting is then rendered to bring the scene to life.

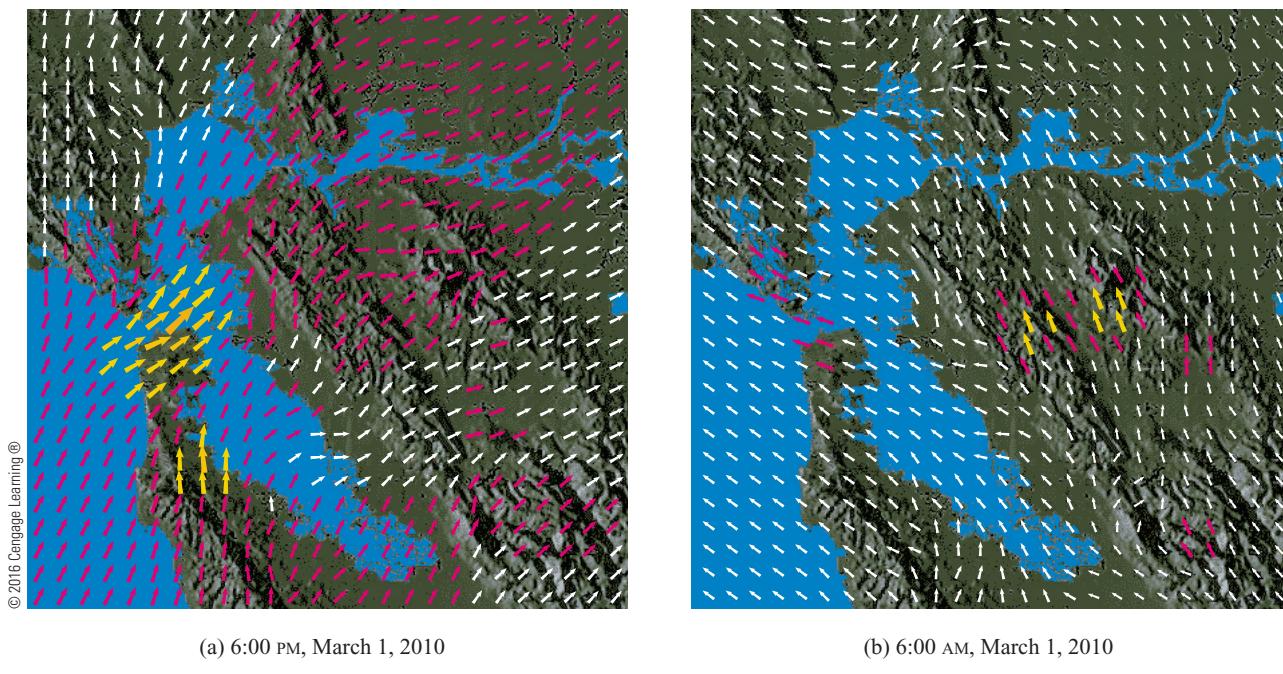


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**IN THIS CHAPTER WE STUDY** the calculus of vector fields. (These are functions that assign vectors to points in space.) In particular we define line integrals (which can be used to find the work done by a force field in moving an object along a curve). Then we define surface integrals (which can be used to find the rate of fluid flow across a surface). The connections between these new types of integrals and the single, double, and triple integrals that we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

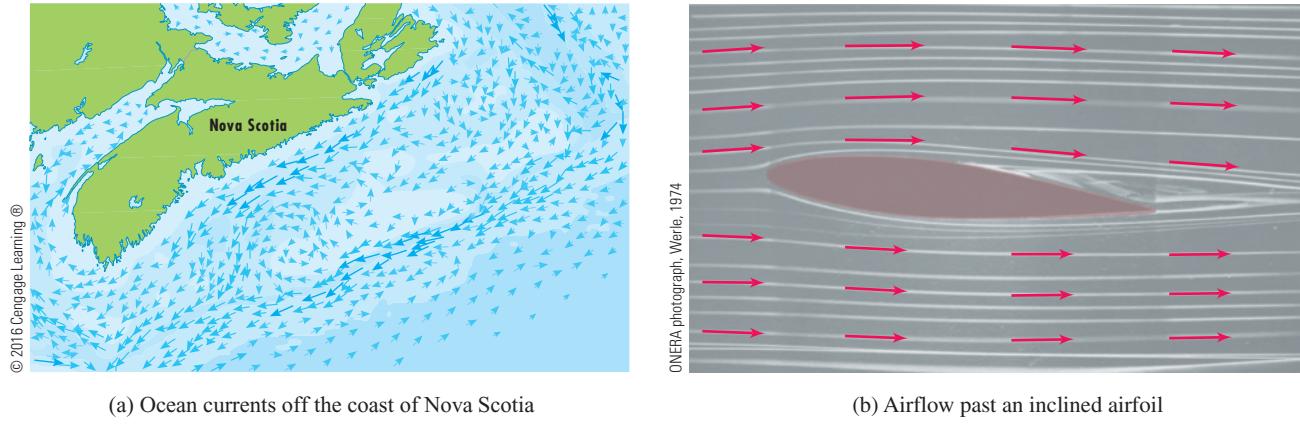
## 16.1 Vector Fields

The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area. We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern 12 hours earlier. Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field*.



**FIGURE 1** Velocity vector fields showing San Francisco Bay wind patterns

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.

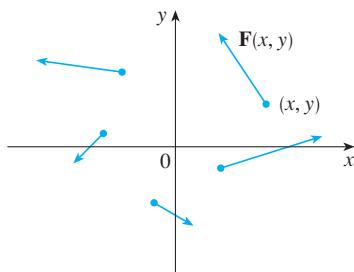


**FIGURE 2**  
Velocity vector fields

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field that we will look at in Example 4.

In general, a vector field is a function whose domain is a set of points in  $\mathbb{R}^2$  (or  $\mathbb{R}^3$ ) and whose range is a set of vectors in  $V_2$  (or  $V_3$ ).

**1 Definition** Let  $D$  be a set in  $\mathbb{R}^2$  (a plane region). A **vector field on  $\mathbb{R}^2$**  is a function  $\mathbf{F}$  that assigns to each point  $(x, y)$  in  $D$  a two-dimensional vector  $\mathbf{F}(x, y)$ .



**FIGURE 3**  
Vector field on  $\mathbb{R}^2$

The best way to picture a vector field is to draw the arrow representing the vector  $\mathbf{F}(x, y)$  starting at the point  $(x, y)$ . Of course, it's impossible to do this for all points  $(x, y)$ , but we can gain a reasonable impression of  $\mathbf{F}$  by doing it for a few representative points in  $D$  as in Figure 3. Since  $\mathbf{F}(x, y)$  is a two-dimensional vector, we can write it in terms of its **component functions**  $P$  and  $Q$  as follows:

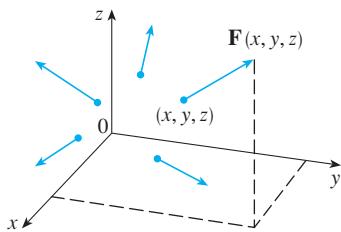
$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

or, for short,

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$$

Notice that  $P$  and  $Q$  are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

**2 Definition** Let  $E$  be a subset of  $\mathbb{R}^3$ . A **vector field on  $\mathbb{R}^3$**  is a function  $\mathbf{F}$  that assigns to each point  $(x, y, z)$  in  $E$  a three-dimensional vector  $\mathbf{F}(x, y, z)$ .



**FIGURE 4**  
Vector field on  $\mathbb{R}^3$

A vector field  $\mathbf{F}$  on  $\mathbb{R}^3$  is pictured in Figure 4. We can express it in terms of its component functions  $P$ ,  $Q$ , and  $R$  as

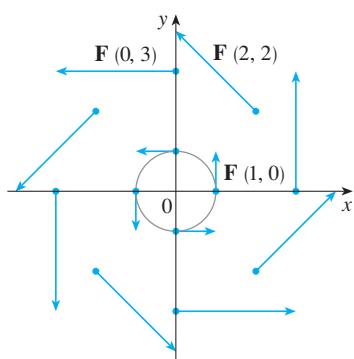
$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

As with the vector functions in Section 13.1, we can define continuity of vector fields and show that  $\mathbf{F}$  is continuous if and only if its component functions  $P$ ,  $Q$ , and  $R$  are continuous.

We sometimes identify a point  $(x, y, z)$  with its position vector  $\mathbf{x} = \langle x, y, z \rangle$  and write  $\mathbf{F}(\mathbf{x})$  instead of  $\mathbf{F}(x, y, z)$ . Then  $\mathbf{F}$  becomes a function that assigns a vector  $\mathbf{F}(\mathbf{x})$  to a vector  $\mathbf{x}$ .

**EXAMPLE 1** A vector field on  $\mathbb{R}^2$  is defined by  $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$ . Describe  $\mathbf{F}$  by sketching some of the vectors  $\mathbf{F}(x, y)$  as in Figure 3.

**SOLUTION** Since  $\mathbf{F}(1, 0) = \mathbf{j}$ , we draw the vector  $\mathbf{j} = \langle 0, 1 \rangle$  starting at the point  $(1, 0)$  in Figure 5. Since  $\mathbf{F}(0, 1) = -\mathbf{i}$ , we draw the vector  $\langle -1, 0 \rangle$  with starting point  $(0, 1)$ . Continuing in this way, we calculate several other representative values of  $\mathbf{F}(x, y)$  in the table and draw the corresponding vectors to represent the vector field in Figure 5.



**FIGURE 5**  
 $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$

$(x, y)$	$\mathbf{F}(x, y)$	$(x, y)$	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle 2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$

It appears from Figure 5 that each arrow is tangent to a circle with center the origin. To confirm this, we take the dot product of the position vector  $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$  with the vector  $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$ :

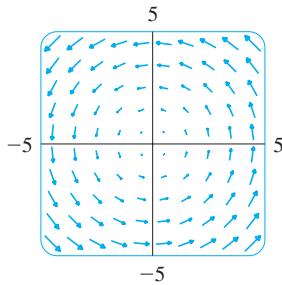
$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\mathbf{i} + y\mathbf{j}) \cdot (-y\mathbf{i} + x\mathbf{j}) = -xy + yx = 0$$

This shows that  $\mathbf{F}(x, y)$  is perpendicular to the position vector  $\langle x, y \rangle$  and is therefore tangent to a circle with center the origin and radius  $|\mathbf{x}| = \sqrt{x^2 + y^2}$ . Notice also that

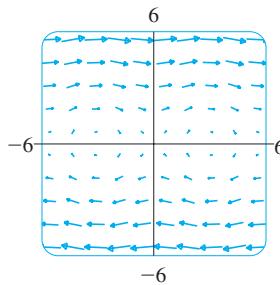
$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

so the magnitude of the vector  $\mathbf{F}(x, y)$  is equal to the radius of the circle. ■

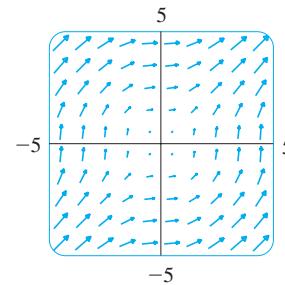
Some computer algebra systems are capable of plotting vector fields in two or three dimensions. They give a better impression of the vector field than is possible by hand because the computer can plot a large number of representative vectors. Figure 6 shows a computer plot of the vector field in Example 1; Figures 7 and 8 show two other vector fields. Notice that the computer scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.



**FIGURE 6**



**FIGURE 7**

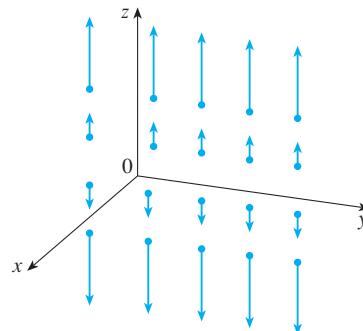


**FIGURE 8**

$$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$$

**EXAMPLE 2** Sketch the vector field on  $\mathbb{R}^3$  given by  $\mathbf{F}(x, y, z) = z\mathbf{k}$ .

**SOLUTION** The sketch is shown in Figure 9. Notice that all vectors are vertical and point upward above the  $xy$ -plane or downward below it. The magnitude increases with the distance from the  $xy$ -plane.

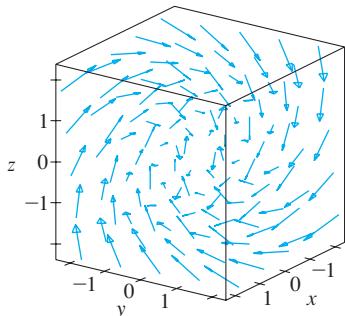


**FIGURE 9**

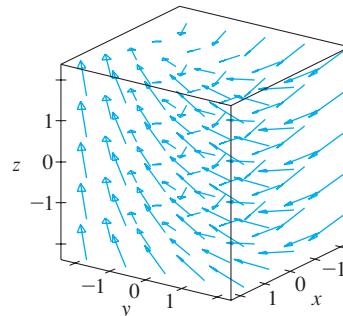
$$\mathbf{F}(x, y, z) = z\mathbf{k}$$

We were able to draw the vector field in Example 2 by hand because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible

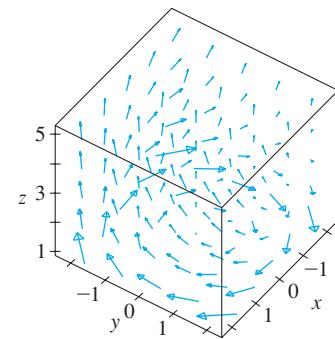
to sketch by hand and so we need to resort to computer software. Examples are shown in Figures 10, 11, and 12. Notice that the vector fields in Figures 10 and 11 have similar formulas, but all the vectors in Figure 11 point in the general direction of the negative  $y$ -axis because their  $y$ -components are all  $-2$ . If the vector field in Figure 12 represents a velocity field, then a particle would be swept upward and would spiral around the  $z$ -axis in the clockwise direction as viewed from above.



**FIGURE 10**  
 $\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$

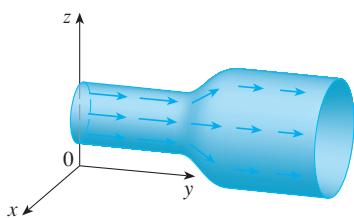


**FIGURE 11**  
 $\mathbf{F}(x, y, z) = y \mathbf{i} - 2 \mathbf{j} + x \mathbf{k}$



**FIGURE 12**  
 $\mathbf{F}(x, y, z) = \frac{y}{z} \mathbf{i} - \frac{x}{z} \mathbf{j} + \frac{z}{4} \mathbf{k}$

**TEC** In Visual 16.1 you can rotate the vector fields in Figures 10–12 as well as additional fields.



**FIGURE 13**  
Velocity field in fluid flow

**EXAMPLE 3** Imagine a fluid flowing steadily along a pipe and let  $\mathbf{V}(x, y, z)$  be the velocity vector at a point  $(x, y, z)$ . Then  $\mathbf{V}$  assigns a vector to each point  $(x, y, z)$  in a certain domain  $E$  (the interior of the pipe) and so  $\mathbf{V}$  is a vector field on  $\mathbb{R}^3$  called a **velocity field**. A possible velocity field is illustrated in Figure 13. The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics. For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel. We have seen other examples of velocity fields in Figures 1 and 2. ■

**EXAMPLE 4** Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses  $m$  and  $M$  is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

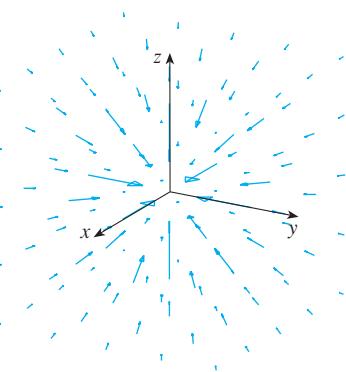
where  $r$  is the distance between the objects and  $G$  is the gravitational constant. (This is an example of an inverse square law.) Let's assume that the object with mass  $M$  is located at the origin in  $\mathbb{R}^3$ . (For instance,  $M$  could be the mass of the earth and the origin would be at its center.) Let the position vector of the object with mass  $m$  be  $\mathbf{x} = \langle x, y, z \rangle$ . Then  $r = |\mathbf{x}|$ , so  $r^2 = |\mathbf{x}|^2$ . The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at  $\mathbf{x} = \langle x, y, z \rangle$  is

$$\boxed{3} \quad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

[Physicists often use the notation  $\mathbf{r}$  instead of  $\mathbf{x}$  for the position vector, so you may see Formula 3 written in the form  $\mathbf{F} = -(mMG/r^3)\mathbf{r}$ .] The function given by Equation 3 is



**FIGURE 14**  
Gravitational force field

an example of a vector field, called the **gravitational field**, because it associates a vector [the force  $\mathbf{F}(\mathbf{x})$ ] with every point  $\mathbf{x}$  in space.

Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that  $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$ :

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mM Gy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

The gravitational field  $\mathbf{F}$  is pictured in Figure 14. ■

**EXAMPLE 5** Suppose an electric charge  $Q$  is located at the origin. According to Coulomb's Law, the electric force  $\mathbf{F}(\mathbf{x})$  exerted by this charge on a charge  $q$  located at a point  $(x, y, z)$  with position vector  $\mathbf{x} = \langle x, y, z \rangle$  is

4

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$$

where  $\varepsilon$  is a constant (that depends on the units used). For like charges, we have  $qQ > 0$  and the force is repulsive; for unlike charges, we have  $qQ < 0$  and the force is attractive. Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force  $\mathbf{F}$ , physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

Then  $\mathbf{E}$  is a vector field on  $\mathbb{R}^3$  called the **electric field** of  $Q$ . ■

### ■ Gradient Fields

If  $f$  is a scalar function of two variables, recall from Section 14.6 that its gradient  $\nabla f$  (or grad  $f$ ) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

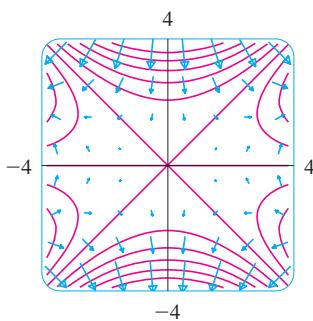
Therefore  $\nabla f$  is really a vector field on  $\mathbb{R}^2$  and is called a **gradient vector field**. Likewise, if  $f$  is a scalar function of three variables, its gradient is a vector field on  $\mathbb{R}^3$  given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

**EXAMPLE 6** Find the gradient vector field of  $f(x, y) = x^2y - y^3$ . Plot the gradient vector field together with a contour map of  $f$ . How are they related?

**SOLUTION** The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 2xy\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$



**FIGURE 15**

Figure 15 shows a contour map of  $f$  with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from

Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of  $f$  and closely spaced level curves indicate a steep graph.

A vector field  $\mathbf{F}$  is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function  $f$  such that  $\mathbf{F} = \nabla f$ . In this situation  $f$  is called a **potential function** for  $\mathbf{F}$ .

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field  $\mathbf{F}$  in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-mM Gy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

## 16.1 EXERCISES

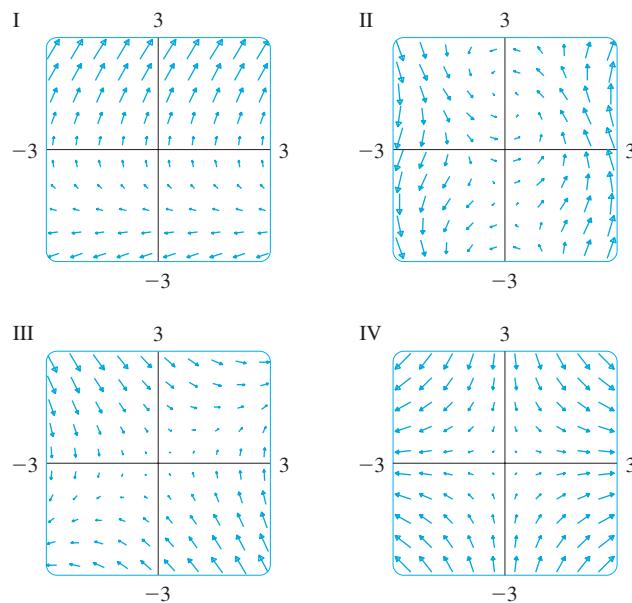
- 1–10** Sketch the vector field  $\mathbf{F}$  by drawing a diagram like Figure 5 or Figure 9.

1.  $\mathbf{F}(x, y) = 0.3 \mathbf{i} - 0.4 \mathbf{j}$
2.  $\mathbf{F}(x, y) = \frac{1}{2}x \mathbf{i} + y \mathbf{j}$
3.  $\mathbf{F}(x, y) = -\frac{1}{2} \mathbf{i} + (y - x) \mathbf{j}$
4.  $\mathbf{F}(x, y) = y \mathbf{i} + (x + y) \mathbf{j}$
5.  $\mathbf{F}(x, y) = \frac{y \mathbf{i} + x \mathbf{j}}{\sqrt{x^2 + y^2}}$
6.  $\mathbf{F}(x, y) = \frac{y \mathbf{i} - x \mathbf{j}}{\sqrt{x^2 + y^2}}$
7.  $\mathbf{F}(x, y, z) = \mathbf{i}$
8.  $\mathbf{F}(x, y, z) = z \mathbf{i}$
9.  $\mathbf{F}(x, y, z) = -y \mathbf{i}$
10.  $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$

- 11–14** Match the vector fields  $\mathbf{F}$  with the plots labeled I–IV. Give reasons for your choices.

11.  $\mathbf{F}(x, y) = \langle x, -y \rangle$
12.  $\mathbf{F}(x, y) = \langle y, x - y \rangle$

- 13.  $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$**   
**14.  $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$**



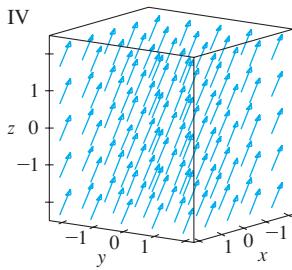
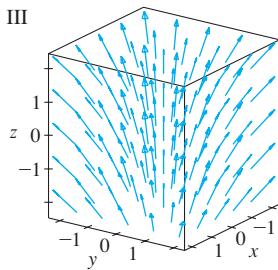
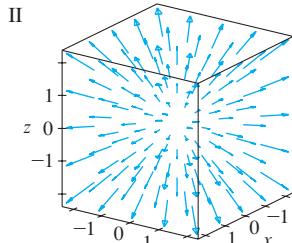
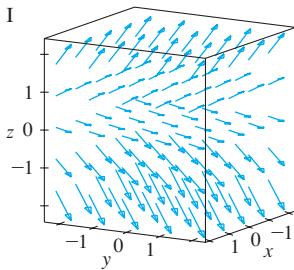
**15–18** Match the vector fields  $\mathbf{F}$  on  $\mathbb{R}^3$  with the plots labeled I–IV. Give reasons for your choices.

**15.**  $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

**16.**  $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$

**17.**  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 3\mathbf{k}$

**18.**  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



- CAS** **19.** If you have a CAS that plots vector fields (the command is `fieldplot` in Maple and `PlotVectorField` or `VectorPlot` in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}$$

Explain the appearance by finding the set of points  $(x, y)$  such that  $\mathbf{F}(x, y) = \mathbf{0}$ .

- CAS** **20.** Let  $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$ , where  $\mathbf{x} = \langle x, y \rangle$  and  $r = |\mathbf{x}|$ . Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where  $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ .

**21–24** Find the gradient vector field of  $f$ .

**21.**  $f(x, y) = y \sin(xy)$

**22.**  $f(s, t) = \sqrt{2s + 3t}$

**23.**  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

**24.**  $f(x, y, z) = x^2ye^{y/z}$

**25–26** Find the gradient vector field  $\nabla f$  of  $f$  and sketch it.

**25.**  $f(x, y) = \frac{1}{2}(x - y)^2$

**26.**  $f(x, y) = \frac{1}{2}(x^2 - y^2)$

- CAS** **27–28** Plot the gradient vector field of  $f$  together with a contour map of  $f$ . Explain how they are related to each other.

**27.**  $f(x, y) = \ln(1 + x^2 + 2y^2)$

**28.**  $f(x, y) = \cos x - 2 \sin y$

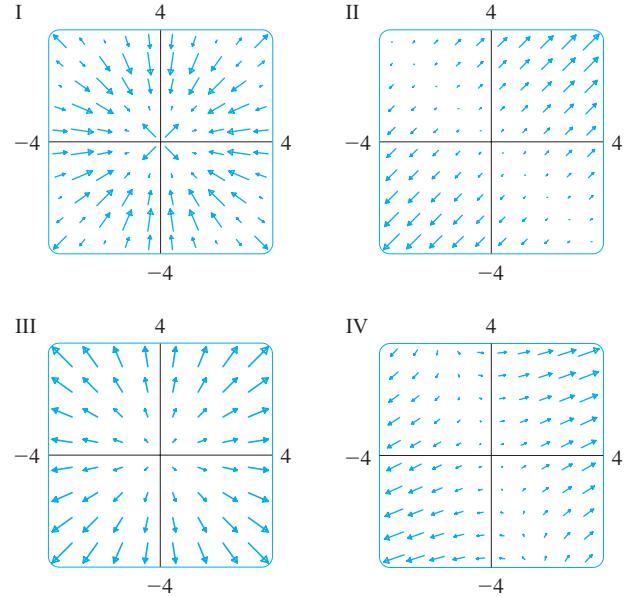
**29–32** Match the functions  $f$  with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

**29.**  $f(x, y) = x^2 + y^2$

**30.**  $f(x, y) = x(x + y)$

**31.**  $f(x, y) = (x + y)^2$

**32.**  $f(x, y) = \sin \sqrt{x^2 + y^2}$



- 33.** A particle moves in a velocity field  $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$ . If it is at position  $(2, 1)$  at time  $t = 3$ , estimate its location at time  $t = 3.01$ .

- 34.** At time  $t = 1$ , a particle is located at position  $(1, 3)$ . If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time  $t = 1.05$ .

- 35.** The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

- (a) Use a sketch of the vector field  $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$  to draw some flow lines. From your sketches, can you guess the equations of the flow lines?  
(b) If parametric equations of a flow line are  $x = x(t)$ ,  $y = y(t)$ , explain why these functions satisfy the differential equations  $dx/dt = x$  and  $dy/dt = -y$ . Then solve the differential equations to find an equation of the flow line that passes through the point  $(1, 1)$ .

- 36.** (a) Sketch the vector field  $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$  and then sketch some flow lines. What shape do these flow lines appear to have?  
(b) If parametric equations of the flow lines are  $x = x(t)$ ,  $y = y(t)$ , what differential equations do these functions satisfy? Deduce that  $dy/dx = x$ .  
(c) If a particle starts at the origin in the velocity field given by  $\mathbf{F}$ , find an equation of the path it follows.

## 16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval  $[a, b]$ , we integrate over a curve  $C$ . Such integrals are called *line integrals*, although “curve integrals” would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve  $C$  given by the parametric equations

**1**

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

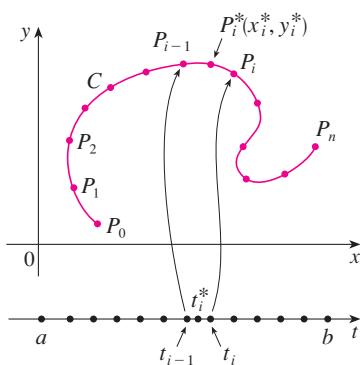


FIGURE 1

or, equivalently, by the vector equation  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ , and we assume that  $C$  is a smooth curve. [This means that  $\mathbf{r}'$  is continuous and  $\mathbf{r}'(t) \neq \mathbf{0}$ . See Section 13.3.] If we divide the parameter interval  $[a, b]$  into  $n$  subintervals  $[t_{i-1}, t_i]$  of equal width and we let  $x_i = x(t_i)$  and  $y_i = y(t_i)$ , then the corresponding points  $P_i(x_i, y_i)$  divide  $C$  into  $n$  subarcs with lengths  $\Delta s_1, \Delta s_2, \dots, \Delta s_n$ . (See Figure 1.) We choose any point  $P_i^*(x_i^*, y_i^*)$  in the  $i$ th subarc. (This corresponds to a point  $t_i^*$  in  $[t_{i-1}, t_i]$ .) Now if  $f$  is any function of two variables whose domain includes the curve  $C$ , we evaluate  $f$  at the point  $(x_i^*, y_i^*)$ , multiply by the length  $\Delta s_i$  of the subarc, and form the sum

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

**2**

**Definition** If  $f$  is defined on a smooth curve  $C$  given by Equations 1, then the **line integral of  $f$  along  $C$**  is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

In Section 10.2 we found that the length of  $C$  is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

A similar type of argument can be used to show that if  $f$  is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

**3**

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as  $t$  increases from  $a$  to  $b$ .

The arc length function  $s$  is discussed in Section 13.3.

If  $s(t)$  is the length of  $C$  between  $\mathbf{r}(a)$  and  $\mathbf{r}(t)$ , then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter  $t$ : Use the parametric equations to express  $x$  and  $y$  in terms of  $t$  and write  $ds$  as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

In the special case where  $C$  is the line segment that joins  $(a, 0)$  to  $(b, 0)$ , using  $x$  as the parameter, we can write the parametric equations of  $C$  as follows:  $x = x$ ,  $y = 0$ ,  $a \leq x \leq b$ . Formula 3 then becomes

$$\int_C f(x, y) ds = \int_a^b f(x, 0) dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a positive function as an area. In fact, if  $f(x, y) \geq 0$ ,  $\int_C f(x, y) ds$  represents the area of one side of the “fence” or “curtain” in Figure 2, whose base is  $C$  and whose height above the point  $(x, y)$  is  $f(x, y)$ .

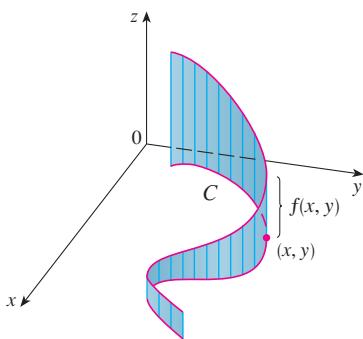


FIGURE 2

**EXAMPLE 1** Evaluate  $\int_C (2 + x^2y) ds$ , where  $C$  is the upper half of the unit circle  $x^2 + y^2 = 1$ ,  $(y \geq 0)$ .

**SOLUTION** In order to use Formula 3, we first need parametric equations to represent  $C$ . Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t$$

and the upper half of the circle is described by the parameter interval  $0 \leq t \leq \pi$ . (See Figure 3.) Therefore Formula 3 gives

$$\begin{aligned} \int_C (2 + x^2y) ds &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) dt = \left[ 2t - \frac{\cos^3 t}{3} \right]_0^\pi \\ &= 2\pi + \frac{2}{3} \end{aligned}$$

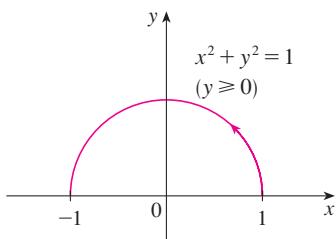


FIGURE 3

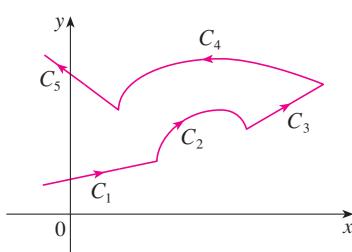


FIGURE 4

A piecewise-smooth curve

Suppose now that  $C$  is a **piecewise-smooth curve**; that is,  $C$  is a union of a finite number of smooth curves  $C_1, C_2, \dots, C_n$ , where, as illustrated in Figure 4, the initial point of  $C_{i+1}$  is the terminal point of  $C_i$ . Then we define the integral of  $f$  along  $C$  as the sum of the integrals of  $f$  along each of the smooth pieces of  $C$ :

$$\int_C f(x, y) ds = \int_{C_1} f(x, y) ds + \int_{C_2} f(x, y) ds + \cdots + \int_{C_n} f(x, y) ds$$

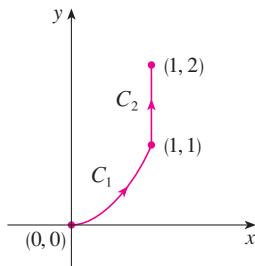
**EXAMPLE 2** Evaluate  $\int_C 2x \, ds$ , where  $C$  consists of the arc  $C_1$  of the parabola  $y = x^2$  from  $(0, 0)$  to  $(1, 1)$  followed by the vertical line segment  $C_2$  from  $(1, 1)$  to  $(1, 2)$ .

**SOLUTION** The curve  $C$  is shown in Figure 5.  $C_1$  is the graph of a function of  $x$ , so we can choose  $x$  as the parameter and the equations for  $C_1$  become

$$x = x \quad y = x^2 \quad 0 \leq x \leq 1$$

Therefore

$$\begin{aligned} \int_{C_1} 2x \, ds &= \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 2x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{4} \cdot \frac{2}{3} (1 + 4x^2)^{3/2} \Big|_0^1 = \frac{5\sqrt{5} - 1}{6} \end{aligned}$$



**FIGURE 5**  
 $C = C_1 \cup C_2$

On  $C_2$  we choose  $y$  as the parameter, so the equations of  $C_2$  are

$$x = 1 \quad y = y \quad 1 \leq y \leq 2$$

$$\text{and} \quad \int_{C_2} 2x \, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} dy = \int_1^2 2 \, dy = 2$$

$$\text{Thus} \quad \int_C 2x \, ds = \int_{C_1} 2x \, ds + \int_{C_2} 2x \, ds = \frac{5\sqrt{5} - 1}{6} + 2 \quad \blacksquare$$

Any physical interpretation of a line integral  $\int_C f(x, y) \, ds$  depends on the physical interpretation of the function  $f$ . Suppose that  $\rho(x, y)$  represents the linear density at a point  $(x, y)$  of a thin wire shaped like a curve  $C$ . Then the mass of the part of the wire from  $P_{i-1}$  to  $P_i$  in Figure 1 is approximately  $\rho(x_i^*, y_i^*) \Delta s_i$  and so the total mass of the wire is approximately  $\sum \rho(x_i^*, y_i^*) \Delta s_i$ . By taking more and more points on the curve, we obtain the **mass**  $m$  of the wire as the limiting value of these approximations:

$$m = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(x_i^*, y_i^*) \Delta s_i = \int_C \rho(x, y) \, ds$$

[For example, if  $f(x, y) = 2 + x^2 y$  represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function  $\rho$  is located at the point  $(\bar{x}, \bar{y})$ , where

$$\boxed{4} \quad \bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds$$

Other physical interpretations of line integrals will be discussed later in this chapter.

**EXAMPLE 3** A wire takes the shape of the semicircle  $x^2 + y^2 = 1$ ,  $y \geq 0$ , and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line  $y = 1$ .

**SOLUTION** As in Example 1 we use the parametrization  $x = \cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq \pi$ , and find that  $ds = dt$ . The linear density is

$$\rho(x, y) = k(1 - y)$$

where  $k$  is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y) ds = \int_0^\pi k(1 - \sin t) dt = k[t + \cos t]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\begin{aligned}\bar{y} &= \frac{1}{m} \int_C y \rho(x, y) ds = \frac{1}{k(\pi - 2)} \int_C y k(1 - y) ds \\ &= \frac{1}{\pi - 2} \int_0^\pi (\sin t - \sin^2 t) dt = \frac{1}{\pi - 2} \left[ -\cos t - \frac{1}{2}t + \frac{1}{4} \sin 2t \right]_0^\pi \\ &= \frac{4 - \pi}{2(\pi - 2)}\end{aligned}$$

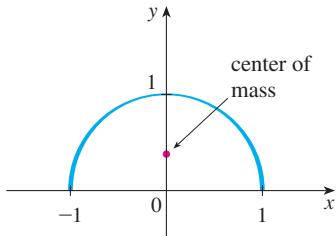


FIGURE 6

By symmetry we see that  $\bar{x} = 0$ , so the center of mass is

$$\left(0, \frac{4 - \pi}{2(\pi - 2)}\right) \approx (0, 0.38)$$

See Figure 6. ■

Two other line integrals are obtained by replacing  $\Delta s_i$  by either  $\Delta x_i = x_i - x_{i-1}$  or  $\Delta y_i = y_i - y_{i-1}$  in Definition 2. They are called the **line integrals of  $f$  along  $C$  with respect to  $x$  and  $y$** :

$$5 \quad \int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i$$

$$6 \quad \int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i$$

When we want to distinguish the original line integral  $\int_C f(x, y) ds$  from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to  $x$  and  $y$  can also be evaluated by expressing everything in terms of  $t$ :  $x = x(t)$ ,  $y = y(t)$ ,  $dx = x'(t) dt$ ,  $dy = y'(t) dt$ .

$$7 \quad \int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

It frequently happens that line integrals with respect to  $x$  and  $y$  occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a

vector representation of the line segment that starts at  $\mathbf{r}_0$  and ends at  $\mathbf{r}_1$  is given by

8

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

(See Equation 12.5.4.)

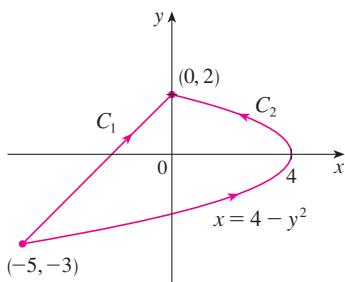


FIGURE 7

**EXAMPLE 4** Evaluate  $\int_C y^2 dx + x dy$ , where (a)  $C = C_1$  is the line segment from  $(-5, -3)$  to  $(0, 2)$  and (b)  $C = C_2$  is the arc of the parabola  $x = 4 - y^2$  from  $(-5, -3)$  to  $(0, 2)$ . (See Figure 7.)

**SOLUTION**

(a) A parametric representation for the line segment is

$$x = 5t - 5 \quad y = 5t - 3 \quad 0 \leq t \leq 1$$

(Use Equation 8 with  $\mathbf{r}_0 = \langle -5, -3 \rangle$  and  $\mathbf{r}_1 = \langle 0, 2 \rangle$ .) Then  $dx = 5 dt$ ,  $dy = 5 dt$ , and Formulas 7 give

$$\begin{aligned} \int_{C_1} y^2 dx + x dy &= \int_0^1 (5t - 3)^2(5 dt) + (5t - 5)(5 dt) \\ &= 5 \int_0^1 (25t^2 - 25t + 4) dt \\ &= 5 \left[ \frac{25t^3}{3} - \frac{25t^2}{2} + 4t \right]_0^1 = -\frac{5}{6} \end{aligned}$$

(b) Since the parabola is given as a function of  $y$ , let's take  $y$  as the parameter and write  $C_2$  as

$$x = 4 - y^2 \quad y = y \quad -3 \leq y \leq 2$$

Then  $dx = -2y dy$  and by Formulas 7 we have

$$\begin{aligned} \int_{C_2} y^2 dx + x dy &= \int_{-3}^2 y^2(-2y) dy + (4 - y^2) dy \\ &= \int_{-3}^2 (-2y^3 - y^2 + 4) dy \\ &= \left[ -\frac{y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^2 = 40\frac{5}{6} \end{aligned}$$

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 16.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If  $-C_1$  denotes the line segment from  $(0, 2)$  to  $(-5, -3)$ , you can verify, using the parametrization

$$x = -5t \quad y = 2 - 5t \quad 0 \leq t \leq 1$$

that

$$\int_{-C_1} y^2 dx + x dy = \frac{5}{6}$$

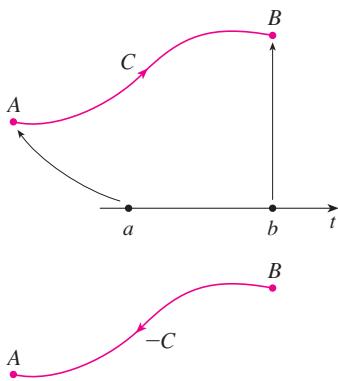


FIGURE 8

In general, a given parametrization  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$ , determines an **orientation** of a curve  $C$ , with the positive direction corresponding to increasing values of the parameter  $t$ . (See Figure 8, where the initial point  $A$  corresponds to the parameter value  $a$  and the terminal point  $B$  corresponds to  $t = b$ .)

If  $-C$  denotes the curve consisting of the same points as  $C$  but with the opposite orientation (from initial point  $B$  to terminal point  $A$  in Figure 8), then we have

$$\int_{-C} f(x, y) dx = - \int_C f(x, y) dx \quad \int_{-C} f(x, y) dy = - \int_C f(x, y) dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y) ds = \int_C f(x, y) ds$$

This is because  $\Delta s_i$  is always positive, whereas  $\Delta x_i$  and  $\Delta y_i$  change sign when we reverse the orientation of  $C$ .

### ■ Line Integrals in Space

We now suppose that  $C$  is a smooth space curve given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or by a vector equation  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . If  $f$  is a function of three variables that is continuous on some region containing  $C$ , then we define the **line integral of  $f$  along  $C$**  (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$9 \quad \int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$$

For the special case  $f(x, y, z) = 1$ , we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)| dt = L$$

where  $L$  is the length of the curve  $C$  (see Formula 13.3.3).

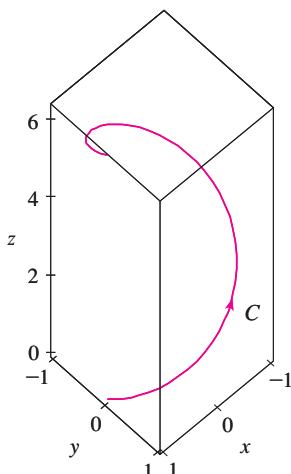
Line integrals along  $C$  with respect to  $x$ ,  $y$ , and  $z$  can also be defined. For example,

$$\begin{aligned}\int_C f(x, y, z) dz &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta z_i \\ &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt\end{aligned}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{10} \quad \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

by expressing everything  $(x, y, z, dx, dy, dz)$  in terms of the parameter  $t$ .



**FIGURE 9**

**EXAMPLE 5** Evaluate  $\int_C y \sin z ds$ , where  $C$  is the circular helix given by the equations  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ ,  $0 \leq t \leq 2\pi$ . (See Figure 9.)

**SOLUTION** Formula 9 gives

$$\begin{aligned}\int_C y \sin z ds &= \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) dt \\ &= \frac{\sqrt{2}}{2} \left[ t - \frac{1}{2} \sin 2t \right]_0^{2\pi} = \sqrt{2} \pi\end{aligned}$$

■

**EXAMPLE 6** Evaluate  $\int_C y dx + z dy + x dz$ , where  $C$  consists of the line segment  $C_1$  from  $(2, 0, 0)$  to  $(3, 4, 5)$ , followed by the vertical line segment  $C_2$  from  $(3, 4, 5)$  to  $(3, 4, 0)$ .

**SOLUTION** The curve  $C$  is shown in Figure 10. Using Equation 8, we write  $C_1$  as

$$\mathbf{r}(t) = (1-t)\langle 2, 0, 0 \rangle + t\langle 3, 4, 5 \rangle = \langle 2+t, 4t, 5t \rangle$$

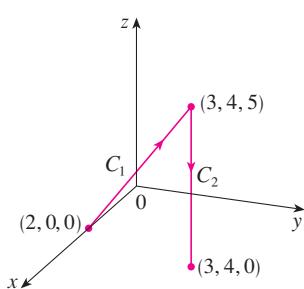
or, in parametric form, as

$$x = 2 + t \quad y = 4t \quad z = 5t \quad 0 \leq t \leq 1$$

Thus

$$\int_{C_1} y dx + z dy + x dz = \int_0^1 (4t) dt + (5t)4 dt + (2+t)5 dt$$

$$= \int_0^1 (10 + 29t) dt = 10t + 29 \frac{t^2}{2} \Big|_0^1 = 24.5$$



**FIGURE 10**

Likewise,  $C_2$  can be written in the form

$$\mathbf{r}(t) = (1-t)\langle 3, 4, 5 \rangle + t\langle 3, 4, 0 \rangle = \langle 3, 4, 5 - 5t \rangle$$

$$\text{or} \quad x = 3 \quad y = 4 \quad z = 5 - 5t \quad 0 \leq t \leq 1$$

Then  $dx = 0 = dy$ , so

$$\int_{C_2} y \, dx + z \, dy + x \, dz = \int_0^1 3(-5) \, dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y \, dx + z \, dy + x \, dz = 24.5 - 15 = 9.5 \quad \blacksquare$$

### ■ Line Integrals of Vector Fields

Recall from Section 6.4 that the work done by a variable force  $f(x)$  in moving a particle from  $a$  to  $b$  along the  $x$ -axis is  $W = \int_a^b f(x) \, dx$ . Then in Section 12.3 we found that the work done by a constant force  $\mathbf{F}$  in moving an object from a point  $P$  to another point  $Q$  in space is  $W = \mathbf{F} \cdot \mathbf{D}$ , where  $\mathbf{D} = \vec{PQ}$  is the displacement vector.

Now suppose that  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$  is a continuous force field on  $\mathbb{R}^3$ , such as the gravitational field of Example 16.1.4 or the electric force field of Example 16.1.5. (A force field on  $\mathbb{R}^2$  could be regarded as a special case where  $R = 0$  and  $P$  and  $Q$  depend only on  $x$  and  $y$ .) We wish to compute the work done by this force in moving a particle along a smooth curve  $C$ .

We divide  $C$  into subarcs  $P_{i-1}P_i$  with lengths  $\Delta s_i$  by dividing the parameter interval  $[a, b]$  into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.) Choose a point  $P_i^*(x_i^*, y_i^*, z_i^*)$  on the  $i$ th subarc corresponding to the parameter value  $t_i^*$ . If  $\Delta s_i$  is small, then as the particle moves from  $P_{i-1}$  to  $P_i$  along the curve, it proceeds approximately in the direction of  $\mathbf{T}(t_i^*)$ , the unit tangent vector at  $P_i^*$ . Thus the work done by the force  $\mathbf{F}$  in moving the particle from  $P_{i-1}$  to  $P_i$  is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

and the total work done in moving the particle along  $C$  is approximately

$$(11) \quad \sum_{i=1}^n [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

where  $\mathbf{T}(x, y, z)$  is the unit tangent vector at the point  $(x, y, z)$  on  $C$ . Intuitively, we see that these approximations ought to become better as  $n$  becomes larger. Therefore we define the **work**  $W$  done by the force field  $\mathbf{F}$  as the limit of the Riemann sums in (11), namely,

$$(12) \quad W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force*.

If the curve  $C$  is given by the vector equation  $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$ , then  $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ , so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[ \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

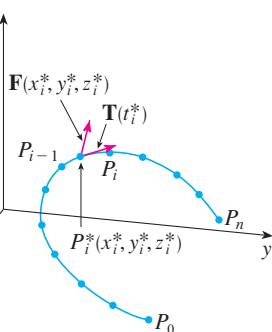


FIGURE 11

This integral is often abbreviated as  $\int_C \mathbf{F} \cdot d\mathbf{r}$  and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

**13 Definition** Let  $\mathbf{F}$  be a continuous vector field defined on a smooth curve  $C$  given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then the **line integral of  $\mathbf{F}$  along  $C$**  is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

When using Definition 13, bear in mind that  $\mathbf{F}(\mathbf{r}(t))$  is just an abbreviation for the vector field  $\mathbf{F}(x(t), y(t), z(t))$ , so we evaluate  $\mathbf{F}(\mathbf{r}(t))$  simply by putting  $x = x(t)$ ,  $y = y(t)$ , and  $z = z(t)$  in the expression for  $\mathbf{F}(x, y, z)$ . Notice also that we can formally write  $d\mathbf{r} = \mathbf{r}'(t) dt$ .

Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

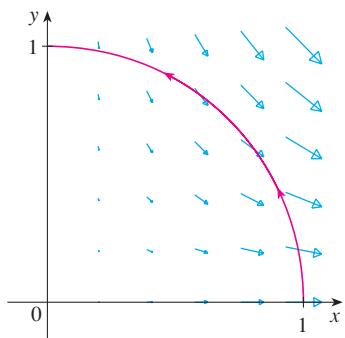


FIGURE 12

Figure 13 shows the twisted cubic  $C$  in Example 8 and some typical vectors acting at three points on  $C$ .

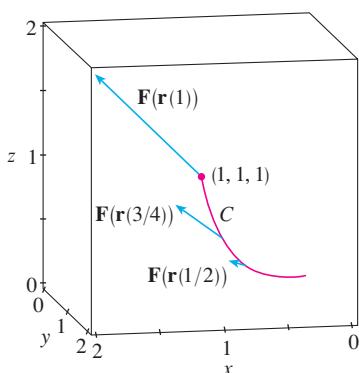


FIGURE 13

**EXAMPLE 7** Find the work done by the force field  $\mathbf{F}(x, y) = x^2 \mathbf{i} - xy \mathbf{j}$  in moving a particle along the quarter-circle  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$ ,  $0 \leq t \leq \pi/2$ .

**SOLUTION** Since  $x = \cos t$  and  $y = \sin t$ , we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \mathbf{i} - \cos t \sin t \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Therefore the work done is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{\pi/2} (-2 \cos^2 t \sin t) dt \\ &= 2 \left[ \frac{\cos^3 t}{3} \right]_0^{\pi/2} = -\frac{2}{3} \end{aligned}$$

■

**NOTE** Even though  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$  and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector  $\mathbf{T}$  is replaced by its negative when  $C$  is replaced by  $-C$ .

**EXAMPLE 8** Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$  and  $C$  is the twisted cubic given by

$$x = t \quad y = t^2 \quad z = t^3 \quad 0 \leq t \leq 1$$

**SOLUTION** We have

$$\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3 \mathbf{i} + t^5 \mathbf{j} + t^4 \mathbf{k}$$

Thus

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (t^3 + 5t^6) dt = \left[ \frac{t^4}{4} + \frac{5t^7}{7} \right]_0^1 = \frac{27}{28}\end{aligned}$$

■

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field  $\mathbf{F}$  on  $\mathbb{R}^3$  is given in component form by the equation  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ . We use Definition 13 to compute its line integral along  $C$ :

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b (P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}) \cdot (x'(t) \mathbf{i} + y'(t) \mathbf{j} + z'(t) \mathbf{k}) dt \\ &= \int_a^b [P(x(t), y(t), z(t)) x'(t) + Q(x(t), y(t), z(t)) y'(t) + R(x(t), y(t), z(t)) z'(t)] dt\end{aligned}$$

But this last integral is precisely the line integral in (10). Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz \quad \text{where } \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

For example, the integral  $\int_C y dx + z dy + x dz$  in Example 6 could be expressed as  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where

$$\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$$

## 16.2 EXERCISES

**1–16** Evaluate the line integral, where  $C$  is the given curve.

1.  $\int_C y ds$ ,  $C: x = t^2, y = 2t, 0 \leq t \leq 3$

2.  $\int_C (x/y) ds$ ,  $C: x = t^3, y = t^4, 1 \leq t \leq 2$

3.  $\int_C xy^4 ds$ ,  $C$  is the right half of the circle  $x^2 + y^2 = 16$

4.  $\int_C xe^y ds$ ,  $C$  is the line segment from  $(2, 0)$  to  $(5, 4)$

5.  $\int_C (x^2 y + \sin x) dy$ ,  
 $C$  is the arc of the parabola  $y = x^2$  from  $(0, 0)$  to  $(\pi, \pi^2)$

6.  $\int_C e^x dx$ ,  
 $C$  is the arc of the curve  $x = y^3$  from  $(-1, -1)$  to  $(1, 1)$

7.  $\int_C (x + 2y) dx + x^2 dy$ ,  $C$  consists of line segments from  $(0, 0)$  to  $(2, 1)$  and from  $(2, 1)$  to  $(3, 0)$

8.  $\int_C x^2 dx + y^2 dy$ ,  $C$  consists of the arc of the circle  $x^2 + y^2 = 4$  from  $(2, 0)$  to  $(0, 2)$  followed by the line segment from  $(0, 2)$  to  $(4, 3)$

9.  $\int_C x^2 y ds$ ,  
 $C: x = \cos t, y = \sin t, z = t, 0 \leq t \leq \pi/2$

10.  $\int_C y^2 z ds$ ,  
 $C$  is the line segment from  $(3, 1, 2)$  to  $(1, 2, 5)$

11.  $\int_C x e^{yz} ds$ ,  
 $C$  is the line segment from  $(0, 0, 0)$  to  $(1, 2, 3)$

12.  $\int_C (x^2 + y^2 + z^2) ds$ ,  
 $C: x = t, y = \cos 2t, z = \sin 2t, 0 \leq t \leq 2\pi$

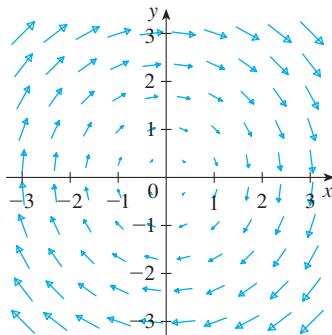
13.  $\int_C x y e^{yz} dy$ ,  $C: x = t, y = t^2, z = t^3, 0 \leq t \leq 1$

14.  $\int_C y dx + z dy + x dz$ ,  
 $C: x = \sqrt{t}, y = t, z = t^2, 1 \leq t \leq 4$

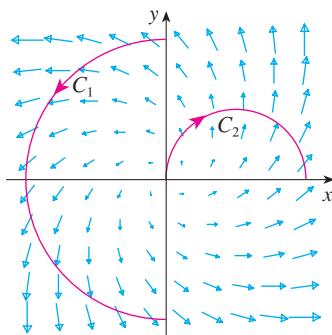
15.  $\int_C z^2 dx + x^2 dy + y^2 dz$ ,  
 $C$  is the line segment from  $(1, 0, 0)$  to  $(4, 1, 2)$

16.  $\int_C (y + z) dx + (x + z) dy + (x + y) dz$ ,  
 $C$  consists of line segments from  $(0, 0, 0)$  to  $(1, 0, 1)$  and from  $(1, 0, 1)$  to  $(0, 1, 2)$

17. Let  $\mathbf{F}$  be the vector field shown in the figure.
- If  $C_1$  is the vertical line segment from  $(-3, -3)$  to  $(-3, 3)$ , determine whether  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$  is positive, negative, or zero.
  - If  $C_2$  is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether  $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  is positive, negative, or zero.



18. The figure shows a vector field  $\mathbf{F}$  and two curves  $C_1$  and  $C_2$ . Are the line integrals of  $\mathbf{F}$  over  $C_1$  and  $C_2$  positive, negative, or zero? Explain.



- 19–22 Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is given by the vector function  $\mathbf{r}(t)$ .

19.  $\mathbf{F}(x, y) = xy^2 \mathbf{i} - x^2 \mathbf{j}$ ,  
 $\mathbf{r}(t) = t^3 \mathbf{i} + t^2 \mathbf{j}, \quad 0 \leq t \leq 1$

20.  $\mathbf{F}(x, y, z) = (x + y^2) \mathbf{i} + xz \mathbf{j} + (y + z) \mathbf{k}$ ,  
 $\mathbf{r}(t) = t^2 \mathbf{i} + t^3 \mathbf{j} - 2t \mathbf{k}, \quad 0 \leq t \leq 2$

21.  $\mathbf{F}(x, y, z) = \sin x \mathbf{i} + \cos y \mathbf{j} + xz \mathbf{k}$ ,  
 $\mathbf{r}(t) = t^3 \mathbf{i} - t^2 \mathbf{j} + t \mathbf{k}, \quad 0 \leq t \leq 1$

22.  $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + xy \mathbf{k}$ ,  
 $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}, \quad 0 \leq t \leq \pi$

- 23–26 Use a calculator to evaluate the line integral correct to four decimal places.

23.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = \sqrt{x+y} \mathbf{i} + (y/x) \mathbf{j}$  and  
 $\mathbf{r}(t) = \sin^2 t \mathbf{i} + \sin t \cos t \mathbf{j}, \quad \pi/6 \leq t \leq \pi/3$

24.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = yze^x \mathbf{i} + zx e^y \mathbf{j} + xy e^z \mathbf{k}$  and  
 $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \tan t \mathbf{k}, \quad 0 \leq t \leq \pi/4$
25.  $\int_C xy \arctan z \, ds$ , where  $C$  has parametric equations  
 $x = t^2, y = t^3, z = \sqrt{t}, \quad 1 \leq t \leq 2$
26.  $\int_C z \ln(x+y) \, ds$ , where  $C$  has parametric equations  
 $x = 1 + 3t, y = 2 + t^2, z = t^4, \quad -1 \leq t \leq 1$

**CAS** 27–28 Use a graph of the vector field  $\mathbf{F}$  and the curve  $C$  to guess whether the line integral of  $\mathbf{F}$  over  $C$  is positive, negative, or zero. Then evaluate the line integral.

27.  $\mathbf{F}(x, y) = (x - y) \mathbf{i} + xy \mathbf{j}$ ,  
 $C$  is the arc of the circle  $x^2 + y^2 = 4$  traversed counterclockwise from  $(2, 0)$  to  $(0, -2)$

28.  $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j}$ ,  
 $C$  is the parabola  $y = 1 + x^2$  from  $(-1, 2)$  to  $(1, 2)$

29. (a) Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  
 $\mathbf{F}(x, y) = e^{x-1} \mathbf{i} + xy \mathbf{j}$  and  $C$  is given by  
 $\mathbf{r}(t) = t^2 \mathbf{i} + t^3 \mathbf{j}, \quad 0 \leq t \leq 1$ .

- (b) Illustrate part (a) by using a graphing calculator or computer to graph  $C$  and the vectors from the vector field corresponding to  $t = 0, 1/\sqrt{2}$ , and 1 (as in Figure 13).

30. (a) Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  
 $\mathbf{F}(x, y, z) = x \mathbf{i} - z \mathbf{j} + y \mathbf{k}$  and  $C$  is given by  
 $\mathbf{r}(t) = 2t \mathbf{i} + 3t \mathbf{j} - t^2 \mathbf{k}, \quad -1 \leq t \leq 1$ .

- (b) Illustrate part (a) by using a computer to graph  $C$  and the vectors from the vector field corresponding to  $t = \pm 1$  and  $\pm \frac{1}{2}$  (as in Figure 13).

- CAS** 31. Find the exact value of  $\int_C x^3 y^2 z \, ds$ , where  $C$  is the curve with parametric equations  $x = e^{-t} \cos 4t, y = e^{-t} \sin 4t, z = e^{-t}, \quad 0 \leq t \leq 2\pi$ .

32. (a) Find the work done by the force field  
 $\mathbf{F}(x, y) = x^2 \mathbf{i} + xy \mathbf{j}$  on a particle that moves once around the circle  $x^2 + y^2 = 4$  oriented in the counterclockwise direction.

- CAS** (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

33. A thin wire is bent into the shape of a semicircle  $x^2 + y^2 = 4, x \geq 0$ . If the linear density is a constant  $k$ , find the mass and center of mass of the wire.

34. A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius  $a$ . If the density function is  $\rho(x, y) = kxy$ , find the mass and center of mass of the wire.

35. (a) Write the formulas similar to Equations 4 for the center of mass  $(\bar{x}, \bar{y}, \bar{z})$  of a thin wire in the shape of a space curve  $C$  if the wire has density function  $\rho(x, y, z)$ .

- (b) Find the center of mass of a wire in the shape of the helix  $x = 2 \sin t, y = 2 \cos t, z = 3t, 0 \leq t \leq 2\pi$ , if the density is a constant  $k$ .
36. Find the mass and center of mass of a wire in the shape of the helix  $x = t, y = \cos t, z = \sin t, 0 \leq t \leq 2\pi$ , if the density at any point is equal to the square of the distance from the origin.
37. If a wire with linear density  $\rho(x, y)$  lies along a plane curve  $C$ , its **moments of inertia** about the  $x$ - and  $y$ -axes are defined as

$$I_x = \int_C y^2 \rho(x, y) ds \quad I_y = \int_C x^2 \rho(x, y) ds$$

Find the moments of inertia for the wire in Example 3.

38. If a wire with linear density  $\rho(x, y, z)$  lies along a space curve  $C$ , its **moments of inertia** about the  $x$ -,  $y$ -, and  $z$ -axes are defined as

$$I_x = \int_C (y^2 + z^2) \rho(x, y, z) ds$$

$$I_y = \int_C (x^2 + z^2) \rho(x, y, z) ds$$

$$I_z = \int_C (x^2 + y^2) \rho(x, y, z) ds$$

Find the moments of inertia for the wire in Exercise 35.

39. Find the work done by the force field

$$\mathbf{F}(x, y) = x \mathbf{i} + (y + 2) \mathbf{j}$$

in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j} \quad 0 \leq t \leq 2\pi$$

40. Find the work done by the force field  $\mathbf{F}(x, y) = x^2 \mathbf{i} + ye^x \mathbf{j}$  on a particle that moves along the parabola  $x = y^2 + 1$  from  $(1, 0)$  to  $(2, 1)$ .

41. Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from  $(0, 0, 1)$  to  $(2, 1, 0)$ .

42. The force exerted by an electric charge at the origin on a charged particle at a point  $(x, y, z)$  with position vector  $\mathbf{r} = \langle x, y, z \rangle$  is  $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$  where  $K$  is a constant. (See Example 16.1.5.) Find the work done as the particle moves along a straight line from  $(2, 0, 0)$  to  $(2, 1, 5)$ .

43. The position of an object with mass  $m$  at time  $t$  is  $\mathbf{r}(t) = at^2 \mathbf{i} + bt^3 \mathbf{j}, 0 \leq t \leq 1$ .

- (a) What is the force acting on the object at time  $t$ ?  
 (b) What is the work done by the force during the time interval  $0 \leq t \leq 1$ ?

44. An object with mass  $m$  moves with position function  $\mathbf{r}(t) = a \sin t \mathbf{i} + b \cos t \mathbf{j} + ct \mathbf{k}, 0 \leq t \leq \pi/2$ . Find the work done on the object during this time period.

45. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

46. Suppose there is a hole in the can of paint in Exercise 45 and 9 lb of paint leaks steadily out of the can during the man's ascent. How much work is done?

47. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle  $x^2 + y^2 = 1$ .

- (b) Is this also true for a force field  $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$ , where  $k$  is a constant and  $\mathbf{x} = \langle x, y \rangle$ ?

48. The base of a circular fence with radius 10 m is given by  $x = 10 \cos t, y = 10 \sin t$ . The height of the fence at position  $(x, y)$  is given by the function  $h(x, y) = 4 + 0.01(x^2 - y^2)$ , so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m<sup>2</sup>. Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

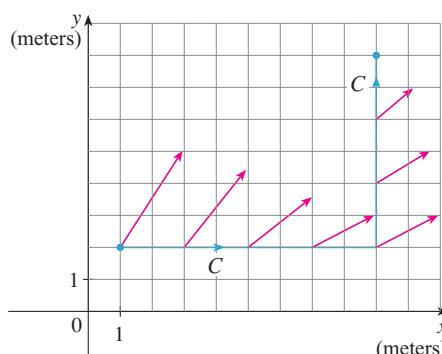
49. If  $C$  is a smooth curve given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , and  $\mathbf{v}$  is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

50. If  $C$  is a smooth curve given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} [|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2]$$

51. An object moves along the curve  $C$  shown in the figure from  $(1, 2)$  to  $(9, 8)$ . The lengths of the vectors in the force field  $\mathbf{F}$  are measured in newtons by the scales on the axes. Estimate the work done by  $\mathbf{F}$  on the object.



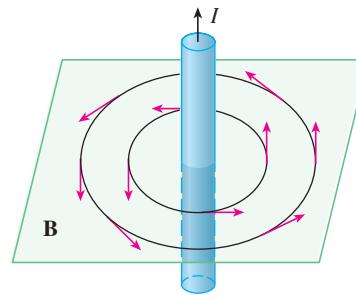
52. Experiments show that a steady current  $I$  in a long wire produces a magnetic field  $\mathbf{B}$  that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). Ampère's Law relates the electric

current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where  $I$  is the net current that passes through any surface bounded by a closed curve  $C$ , and  $\mu_0$  is a constant called the permeability of free space. By taking  $C$  to be a circle with radius  $r$ , show that the magnitude  $B = |\mathbf{B}|$  of the magnetic field at a distance  $r$  from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



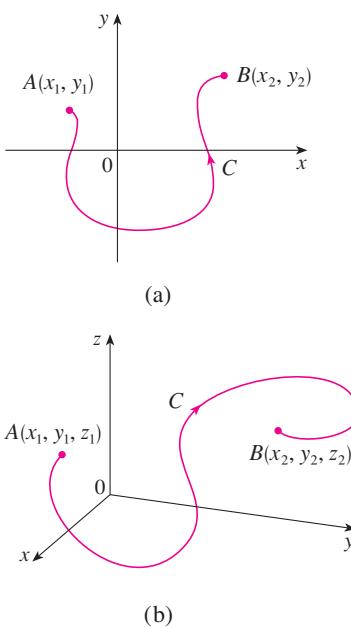
## 16.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.3 that Part 2 of the Fundamental Theorem of Calculus can be written as

$$1 \quad \int_a^b F'(x) dx = F(b) - F(a)$$

where  $F'$  is continuous on  $[a, b]$ . We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

If we think of the gradient vector  $\nabla f$  of a function  $f$  of two or three variables as a sort of derivative of  $f$ , then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.



**2 Theorem** Let  $C$  be a smooth curve given by the vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Let  $f$  be a differentiable function of two or three variables whose gradient vector  $\nabla f$  is continuous on  $C$ . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

**NOTE** Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function  $f$ ) simply by knowing the value of  $f$  at the endpoints of  $C$ . In fact, Theorem 2 says that the line integral of  $\nabla f$  is the net change in  $f$ . If  $f$  is a function of two variables and  $C$  is a plane curve with initial point  $A(x_1, y_1)$  and terminal point  $B(x_2, y_2)$ , as in Figure 1(a), then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If  $f$  is a function of three variables and  $C$  is a space curve joining the point  $A(x_1, y_1, z_1)$  to the point  $B(x_2, y_2, z_2)$ , as in Figure 1(b), then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Let's prove Theorem 2 for this case.

**PROOF OF THEOREM 2** Using Definition 16.2.13, we have

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \quad (\text{by the Chain Rule}) \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

Although we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing  $C$  into a finite number of smooth curves and adding the resulting integrals.

**EXAMPLE 1** Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

in moving a particle with mass  $m$  from the point  $(3, 4, 12)$  to the point  $(2, 2, 0)$  along a piecewise-smooth curve  $C$ . (See Example 16.1.4.)

**SOLUTION** From Section 16.1 we know that  $\mathbf{F}$  is a conservative vector field and, in fact,  $\mathbf{F} = \nabla f$ , where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2, 2, 0) - f(3, 4, 12) \\ &= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG \left( \frac{1}{2\sqrt{2}} - \frac{1}{13} \right)\end{aligned}$$

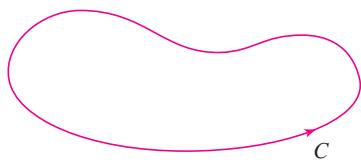
### ■ Independence of Path

Suppose  $C_1$  and  $C_2$  are two piecewise-smooth curves (which are called **paths**) that have the same initial point  $A$  and terminal point  $B$ . We know from Example 16.2.4 that, in general,  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ . But one implication of Theorem 2 is that

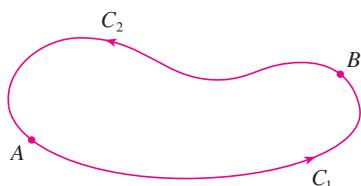
$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever  $\nabla f$  is continuous. In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

In general, if  $\mathbf{F}$  is a continuous vector field with domain  $D$ , we say that the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is **independent of path** if  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  for any two paths  $C_1$  and  $C_2$  in  $D$  that have the same initial points and the same terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path*.



**FIGURE 2**  
A closed curve



**FIGURE 3**

A curve is called **closed** if its terminal point coincides with its initial point, that is,  $\mathbf{r}(b) = \mathbf{r}(a)$ . (See Figure 2.) If  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  and  $C$  is any closed path in  $D$ , we can choose any two points  $A$  and  $B$  on  $C$  and regard  $C$  as being composed of the path  $C_1$  from  $A$  to  $B$  followed by the path  $C_2$  from  $B$  to  $A$ . (See Figure 3.) Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since  $C_1$  and  $-C_2$  have the same initial and terminal points.

Conversely, if it is true that  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  whenever  $C$  is a closed path in  $D$ , then we demonstrate independence of path as follows. Take any two paths  $C_1$  and  $C_2$  from  $A$  to  $B$  in  $D$  and define  $C$  to be the curve consisting of  $C_1$  followed by  $-C_2$ . Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ . Thus we have proved the following theorem.

**3 Theorem**  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  if and only if  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed path  $C$  in  $D$ .

Since we know that the line integral of any conservative vector field  $\mathbf{F}$  is independent of path, it follows that  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 16.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that  $D$  is **open**, which means that for every point  $P$  in  $D$  there is a disk with center  $P$  that lies entirely in  $D$ . (So  $D$  doesn't contain any of its boundary points.) In addition, we assume that  $D$  is **connected**: this means that any two points in  $D$  can be joined by a path that lies in  $D$ .

**4 Theorem** Suppose  $\mathbf{F}$  is a vector field that is continuous on an open connected region  $D$ . If  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$ , then  $\mathbf{F}$  is a conservative vector field on  $D$ ; that is, there exists a function  $f$  such that  $\nabla f = \mathbf{F}$ .

**PROOF** Let  $A(a, b)$  be a fixed point in  $D$ . We construct the desired potential function  $f$  by defining

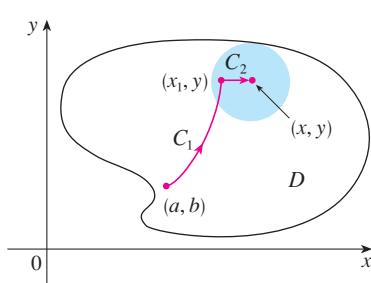
$$f(x, y) = \int_{(a, b)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point  $(x, y)$  in  $D$ . Since  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path, it does not matter which path  $C$  from  $(a, b)$  to  $(x, y)$  is used to evaluate  $f(x, y)$ . Since  $D$  is open, there exists a disk contained in  $D$  with center  $(x, y)$ . Choose any point  $(x_1, y)$  in the disk with  $x_1 < x$  and let  $C$  consist of any path  $C_1$  from  $(a, b)$  to  $(x_1, y)$  followed by the horizontal line segment  $C_2$  from  $(x_1, y)$  to  $(x, y)$ . (See Figure 4.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a, b)}^{(x_1, y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the first of these integrals does not depend on  $x$ , so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$



**FIGURE 4**

If we write  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ , then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P dx + Q dy$$

On  $C_2$ ,  $y$  is constant, so  $dy = 0$ . Using  $t$  as the parameter, where  $x_1 \leq t \leq x$ , we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P dx + Q dy = \frac{\partial}{\partial x} \int_{x_1}^x P(t, y) dt = P(x, y)$$

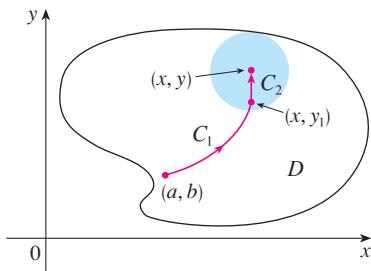
by Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). A similar argument, using a vertical line segment (see Figure 5), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P dx + Q dy = \frac{\partial}{\partial y} \int_{y_1}^y Q(x, t) dt = Q(x, y)$$

Thus

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \nabla f$$

which says that  $\mathbf{F}$  is conservative. ■



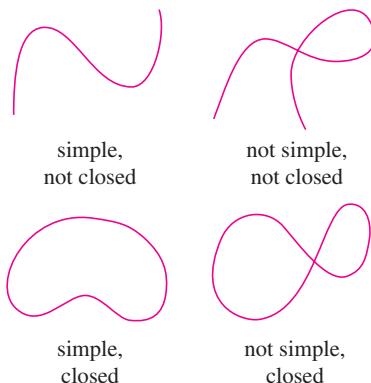
**FIGURE 5**

The question remains: how is it possible to determine whether or not a vector field  $\mathbf{F}$  is conservative? Suppose it is known that  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$  is conservative, where  $P$  and  $Q$  have continuous first-order partial derivatives. Then there is a function  $f$  such that  $\mathbf{F} = \nabla f$ , that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

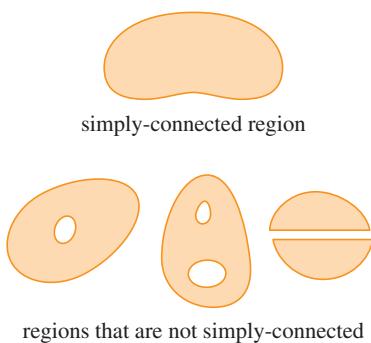
$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$



**FIGURE 6**  
Types of curves

**5 Theorem** If  $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$  is a conservative vector field, where  $P$  and  $Q$  have continuous first-order partial derivatives on a domain  $D$ , then throughout  $D$  we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$



**FIGURE 7**

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6;  $\mathbf{r}(a) = \mathbf{r}(b)$  for a simple closed curve, but  $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$  when  $a < t_1 < t_2 < b$ .]

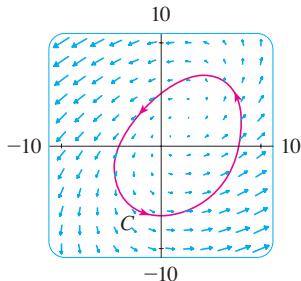
In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region  $D$  such that every simple closed curve in  $D$  encloses only points that are in  $D$ . Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on  $\mathbb{R}^2$  is conservative. The proof will be sketched in the next section as a consequence of Green's Theorem.

**6 Theorem** Let  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$  be a vector field on an open simply-connected region  $D$ . Suppose that  $P$  and  $Q$  have continuous first-order partial derivatives and

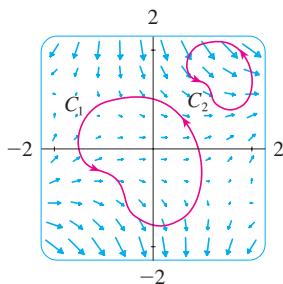
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then  $\mathbf{F}$  is conservative.



**FIGURE 8**

Figures 8 and 9 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 8 that start on the closed curve  $C$  all appear to point in roughly the same direction as  $C$ . So it looks as if  $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$  and therefore  $\mathbf{F}$  is not conservative. The calculation in Example 2 confirms this impression. Some of the vectors near the curves  $C_1$  and  $C_2$  in Figure 9 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 3 shows that  $\mathbf{F}$  is indeed conservative.



**FIGURE 9**

**EXAMPLE 2** Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y) \mathbf{i} + (x - 2) \mathbf{j}$$

is conservative.

**SOLUTION** Let  $P(x, y) = x - y$  and  $Q(x, y) = x - 2$ . Then

$$\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1$$

Since  $\partial P / \partial y \neq \partial Q / \partial x$ ,  $\mathbf{F}$  is not conservative by Theorem 5. ■

**EXAMPLE 3** Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$$

is conservative.

**SOLUTION** Let  $P(x, y) = 3 + 2xy$  and  $Q(x, y) = x^2 - 3y^2$ . Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of  $\mathbf{F}$  is the entire plane ( $D = \mathbb{R}^2$ ), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that  $\mathbf{F}$  is conservative. ■

In Example 3, Theorem 6 told us that  $\mathbf{F}$  is conservative, but it did not tell us how to find the (potential) function  $f$  such that  $\mathbf{F} = \nabla f$ . The proof of Theorem 4 gives us a clue as to how to find  $f$ . We use “partial integration” as in the following example.

**EXAMPLE 4**

- (a) If  $\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$ , find a function  $f$  such that  $\mathbf{F} = \nabla f$ .
- (b) Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is the curve given by

$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} \quad 0 \leq t \leq \pi$$

**SOLUTION**

- (a) From Example 3 we know that  $\mathbf{F}$  is conservative and so there exists a function  $f$  with  $\nabla f = \mathbf{F}$ , that is,

**7**

$$f_x(x, y) = 3 + 2xy$$

**8**

$$f_y(x, y) = x^2 - 3y^2$$

Integrating (7) with respect to  $x$ , we obtain

$$\boxed{9} \quad f(x, y) = 3x + x^2y + g(y)$$

Notice that the constant of integration is a constant with respect to  $x$ , that is, a function of  $y$ , which we have called  $g(y)$ . Next we differentiate both sides of (9) with respect to  $y$ :

$$\boxed{10} \quad f_y(x, y) = x^2 + g'(y)$$

Comparing (8) and (10), we see that

$$g'(y) = -3y^2$$

Integrating with respect to  $y$ , we have

$$g(y) = -y^3 + K$$

where  $K$  is a constant. Putting this in (9), we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

- (b) To use Theorem 2 all we have to know are the initial and terminal points of  $C$ , namely,  $\mathbf{r}(0) = (0, 1)$  and  $\mathbf{r}(\pi) = (0, -e^\pi)$ . In the expression for  $f(x, y)$  in part (a), any value of the constant  $K$  will do, so let's choose  $K = 0$ . Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 16.2. ■

A criterion for determining whether or not a vector field  $\mathbf{F}$  on  $\mathbb{R}^3$  is conservative is given in Section 16.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on  $\mathbb{R}^2$ .

**EXAMPLE 5** If  $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + (2xy + e^{3z}) \mathbf{j} + 3ye^{3z} \mathbf{k}$ , find a function  $f$  such that  $\nabla f = \mathbf{F}$ .

**SOLUTION** If there is such a function  $f$ , then

$$\boxed{11} \quad f_x(x, y, z) = y^2$$

$$\boxed{12} \quad f_y(x, y, z) = 2xy + e^{3z}$$

$$\boxed{13} \quad f_z(x, y, z) = 3ye^{3z}$$

Integrating (11) with respect to  $x$ , we get

$$\boxed{14} \quad f(x, y, z) = xy^2 + g(y, z)$$

where  $g(y, z)$  is a constant with respect to  $x$ . Then differentiating (14) with respect to  $y$ , we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with (12) gives

$$g_y(y, z) = e^{3z}$$

Thus  $g(y, z) = ye^{3z} + h(z)$  and we rewrite (14) as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to  $z$  and comparing with (13), we obtain  $h'(z) = 0$  and therefore  $h(z) = K$ , a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that  $\nabla f = \mathbf{F}$ . ■

### ■ Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field  $\mathbf{F}$  that moves an object along a path  $C$  given by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , where  $\mathbf{r}(a) = A$  is the initial point and  $\mathbf{r}(b) = B$  is the terminal point of  $C$ . According to Newton's Second Law of Motion (see Section 13.4), the force  $\mathbf{F}(\mathbf{r}(t))$  at a point on  $C$  is related to the acceleration  $\mathbf{a}(t) = \mathbf{r}''(t)$  by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t) dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] dt \quad (\text{Theorem 13.2.3, Formula 4}) \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2 dt = \frac{m}{2} [|\mathbf{r}'(t)|^2]_a^b \quad (\text{Fundamental Theorem of Calculus}) \\ &= \frac{m}{2} (|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) \end{aligned}$$

Therefore

$$15 \quad W = \frac{1}{2}m|\mathbf{v}(b)|^2 - \frac{1}{2}m|\mathbf{v}(a)|^2$$

where  $\mathbf{v} = \mathbf{r}'$  is the velocity.

The quantity  $\frac{1}{2}m|\mathbf{v}(t)|^2$ , that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$16 \quad W = K(B) - K(A)$$

which says that the work done by the force field along  $C$  is equal to the change in kinetic energy at the endpoints of  $C$ .

Now let's further assume that  $\mathbf{F}$  is a conservative force field; that is, we can write  $\mathbf{F} = \nabla f$ . In physics, the **potential energy** of an object at the point  $(x, y, z)$  is defined as  $P(x, y, z) = -f(x, y, z)$ , so we have  $\mathbf{F} = -\nabla P$ . Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

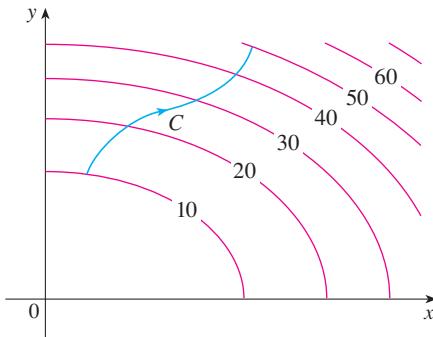
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point  $A$  to another point  $B$  under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

### 16.3 EXERCISES

1. The figure shows a curve  $C$  and a contour map of a function  $f$  whose gradient is continuous. Find  $\int_C \nabla f \cdot d\mathbf{r}$ .



2. A table of values of a function  $f$  with continuous gradient is given. Find  $\int_C \nabla f \cdot d\mathbf{r}$ , where  $C$  has parametric equations

$$x = t^2 + 1 \quad y = t^3 + t \quad 0 \leq t \leq 1$$

$x \backslash y$	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

- 3–10 Determine whether or not  $\mathbf{F}$  is a conservative vector field. If it is, find a function  $f$  such that  $\mathbf{F} = \nabla f$ .

3.  $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$

4.  $\mathbf{F}(x, y) = (y^2 - 2x)\mathbf{i} + 2xy\mathbf{j}$

5.  $\mathbf{F}(x, y) = y^2 e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$

6.  $\mathbf{F}(x, y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$

7.  $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y)\mathbf{j}$

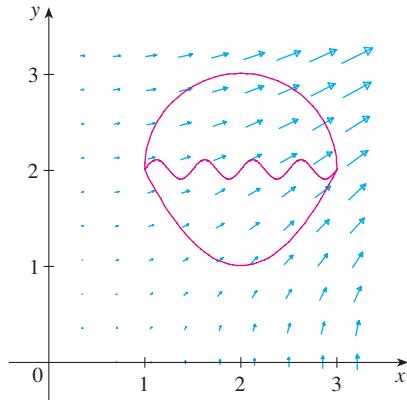
8.  $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}, \quad y > 0$

9.  $\mathbf{F}(x, y) = (y^2 \cos x + \cos y)\mathbf{i} + (2y \sin x - x \sin y)\mathbf{j}$

10.  $\mathbf{F}(x, y) = (\ln y + y/x)\mathbf{i} + (\ln x + x/y)\mathbf{j}$

11. The figure shows the vector field  $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$  and three curves that start at  $(1, 2)$  and end at  $(3, 2)$ .

- (a) Explain why  $\int_C \mathbf{F} \cdot d\mathbf{r}$  has the same value for all three curves.  
 (b) What is this common value?



- 12–18 (a) Find a function  $f$  such that  $\mathbf{F} = \nabla f$  and (b) use part (a) to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  along the given curve  $C$ .

12.  $\mathbf{F}(x, y) = (3 + 2xy^2)\mathbf{i} + 2x^2y\mathbf{j}$ ,  
 $C$  is the arc of the hyperbola  $y = 1/x$  from  $(1, 1)$  to  $(4, \frac{1}{4})$

13.  $\mathbf{F}(x, y) = x^2y^3\mathbf{i} + x^3y^2\mathbf{j}$ ,  
 $C: \mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle, \quad 0 \leq t \leq 1$

14.  $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$ ,  
 $C: \mathbf{r}(t) = \cos t\mathbf{i} + 2 \sin t\mathbf{j}, \quad 0 \leq t \leq \pi/2$

15.  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ ,  
 $C$  is the line segment from  $(1, 0, -2)$  to  $(4, 6, 3)$

16.  $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ ,  
 $C: x = \sqrt{t}, y = t + 1, z = t^2, 0 \leq t \leq 1$

17.  $\mathbf{F}(x, y, z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + xye^{xz}\mathbf{k}$ ,  
 $C: \mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + (t^2 - 2t)\mathbf{k}, 0 \leq t \leq 2$

18.  $\mathbf{F}(x, y, z) = \sin y \mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z \mathbf{k}$ ,  
 $C: \mathbf{r}(t) = \sin t \mathbf{i} + t \mathbf{j} + 2t \mathbf{k}, 0 \leq t \leq \pi/2$

**19–20** Show that the line integral is independent of path and evaluate the integral.

19.  $\int_C 2xe^{-y} dx + (2y - x^2e^{-y}) dy$ ,  
 $C$  is any path from  $(1, 0)$  to  $(2, 1)$

20.  $\int_C \sin y dx + (x \cos y - \sin y) dy$ ,  
 $C$  is any path from  $(2, 0)$  to  $(1, \pi)$

21. Suppose you're asked to determine the curve that requires the least work for a force field  $\mathbf{F}$  to move a particle from one point to another point. You decide to check first whether  $\mathbf{F}$  is conservative, and indeed it turns out that it is. How would you reply to the request?

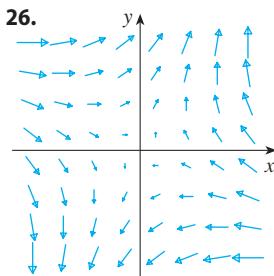
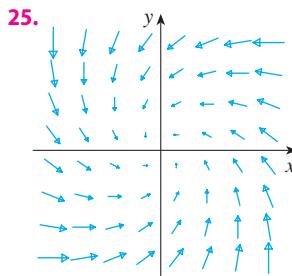
22. Suppose an experiment determines that the amount of work required for a force field  $\mathbf{F}$  to move a particle from the point  $(1, 2)$  to the point  $(5, -3)$  along a curve  $C_1$  is 1.2 J and the work done by  $\mathbf{F}$  in moving the particle along another curve  $C_2$  between the same two points is 1.4 J. What can you say about  $\mathbf{F}$ ? Why?

**23–24** Find the work done by the force field  $\mathbf{F}$  in moving an object from  $P$  to  $Q$ .

23.  $\mathbf{F}(x, y) = x^3 \mathbf{i} + y^3 \mathbf{j}; P(1, 0), Q(2, 2)$

24.  $\mathbf{F}(x, y) = (2x + y) \mathbf{i} + x \mathbf{j}; P(1, 1), Q(4, 3)$

**25–26** Is the vector field shown in the figure conservative? Explain.



**CAS** 27. If  $\mathbf{F}(x, y) = \sin y \mathbf{i} + (1 + x \cos y) \mathbf{j}$ , use a plot to guess whether  $\mathbf{F}$  is conservative. Then determine whether your guess is correct.

28. Let  $\mathbf{F} = \nabla f$ , where  $f(x, y) = \sin(x - 2y)$ . Find curves  $C_1$  and  $C_2$  that are not closed and satisfy the equation.

(a)  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$       (b)  $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

29. Show that if the vector field  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$  is conservative and  $P, Q, R$  have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

30. Use Exercise 29 to show that the line integral  $\int_C y dx + x dy + xyz dz$  is not independent of path.

**31–34** Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

31.  $\{(x, y) \mid 0 < y < 3\}$

32.  $\{(x, y) \mid 1 < |x| < 2\}$

33.  $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$

34.  $\{(x, y) \mid (x, y) \neq (2, 3)\}$

35. Let  $\mathbf{F}(x, y) = \frac{-y \mathbf{i} + x \mathbf{j}}{x^2 + y^2}$ .

(a) Show that  $\partial P/\partial y = \partial Q/\partial x$ .

(b) Show that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is not independent of path.

[Hint: Compute  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$  and  $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ , where  $C_1$  and  $C_2$  are the upper and lower halves of the circle  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(-1, 0)$ .] Does this contradict Theorem 6?

36. (a) Suppose that  $\mathbf{F}$  is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant  $c$ , where  $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ .

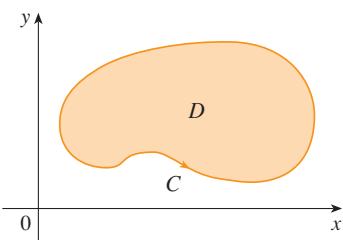
Find the work done by  $\mathbf{F}$  in moving an object from a point  $P_1$  along a path to a point  $P_2$  in terms of the distances  $d_1$  and  $d_2$  from these points to the origin.

(b) An example of an inverse square field is the gravitational field  $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$  discussed in Example 16.1.4. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of  $1.52 \times 10^8$  km from the sun) to perihelion (at a minimum distance of  $1.47 \times 10^8$  km). (Use the values

$m = 5.97 \times 10^{24}$  kg,  $M = 1.99 \times 10^{30}$  kg, and  $G = 6.67 \times 10^{-11}$  N·m<sup>2</sup>/kg<sup>2</sup>.)

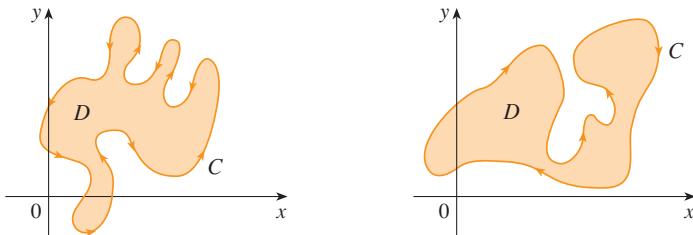
(c) Another example of an inverse square field is the electric force field  $\mathbf{F} = \epsilon qQ\mathbf{r}/|\mathbf{r}|^3$  discussed in Example 16.1.5. Suppose that an electron with a charge of  $-1.6 \times 10^{-19}$  C is located at the origin. A positive unit charge is positioned a distance  $10^{-12}$  m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value  $\epsilon = 8.985 \times 10^9$ .)

## 16.4 Green's Theorem



**FIGURE 1**

Green's Theorem gives the relationship between a line integral around a simple closed curve  $C$  and a double integral over the plane region  $D$  bounded by  $C$ . (See Figure 1. We assume that  $D$  consists of all points inside  $C$  as well as all points on  $C$ .) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve  $C$  refers to a single *counterclockwise* traversal of  $C$ . Thus if  $C$  is given by the vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , then the region  $D$  is always on the left as the point  $\mathbf{r}(t)$  traverses  $C$ . (See Figure 2.)



**FIGURE 2**

(a) Positive orientation

(b) Negative orientation

**Green's Theorem** Let  $C$  be a positively oriented, piecewise-smooth, simple closed curve in the plane and let  $D$  be the region bounded by  $C$ . If  $P$  and  $Q$  have continuous partial derivatives on an open region that contains  $D$ , then

$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

**NOTE** The notation

$$\oint_C P dx + Q dy \quad \text{or} \quad \oint_C P dx + Q dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve  $C$ . Another notation for the positively oriented boundary curve of  $D$  is  $\partial D$ , so the equation in Green's Theorem can be written as

$$\boxed{1} \quad \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P dx + Q dy$$

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x) dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives ( $F'$ ,  $\partial Q/\partial x$ , and  $\partial P/\partial y$ ) on the left side of the equation. And in both cases the right side involves the values of the original functions ( $F$ ,  $Q$ , and  $P$ ) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval  $[a, b]$  whose boundary consists of just two points,  $a$  and  $b$ .)

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both type I and type II (see Section 15.2). Let's call such regions **simple regions**.

### George Green

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

**PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH  $D$  IS A SIMPLE REGION** Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \quad \int_C P \, dx = - \iint_D \frac{\partial P}{\partial y} \, dA$$

and

$$\boxed{3} \quad \int_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA$$

We prove Equation 2 by expressing  $D$  as a type I region:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where  $g_1$  and  $g_2$  are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \quad \iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} (x, y) \, dy \, dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))] \, dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up  $C$  as the union of the four curves  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  shown in Figure 3. On  $C_1$  we take  $x$  as the parameter and write the parametric equations as  $x = x$ ,  $y = g_1(x)$ ,  $a \leq x \leq b$ . Thus

$$\int_{C_1} P(x, y) \, dx = \int_a^b P(x, g_1(x)) \, dx$$

Observe that  $C_3$  goes from right to left but  $-C_3$  goes from left to right, so we can write the parametric equations of  $-C_3$  as  $x = x$ ,  $y = g_2(x)$ ,  $a \leq x \leq b$ . Therefore

$$\int_{C_3} P(x, y) \, dx = - \int_{-C_3} P(x, y) \, dx = - \int_a^b P(x, g_2(x)) \, dx$$

On  $C_2$  or  $C_4$  (either of which might reduce to just a single point),  $x$  is constant, so  $dx = 0$  and

$$\int_{C_2} P(x, y) \, dx = 0 = \int_{C_4} P(x, y) \, dx$$

Hence

$$\begin{aligned} \int_C P(x, y) \, dx &= \int_{C_1} P(x, y) \, dx + \int_{C_2} P(x, y) \, dx + \int_{C_3} P(x, y) \, dx + \int_{C_4} P(x, y) \, dx \\ &= \int_a^b P(x, g_1(x)) \, dx - \int_a^b P(x, g_2(x)) \, dx \end{aligned}$$

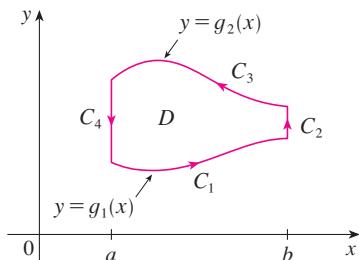


FIGURE 3

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y) dx = - \iint_D \frac{\partial P}{\partial y} dA$$

Equation 3 can be proved in much the same way by expressing  $D$  as a type II region (see Exercise 30). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

**EXAMPLE 1** Evaluate  $\int_C x^4 dx + xy dy$ , where  $C$  is the triangular curve consisting of the line segments from  $(0, 0)$  to  $(1, 0)$ , from  $(1, 0)$  to  $(0, 1)$ , and from  $(0, 1)$  to  $(0, 0)$ .

**SOLUTION** Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region  $D$  enclosed by  $C$  is simple and  $C$  has positive orientation (see Figure 4). If we let  $P(x, y) = x^4$  and  $Q(x, y) = xy$ , then we have

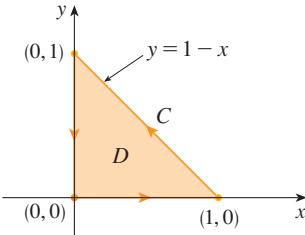


FIGURE 4

$$\begin{aligned} \int_C x^4 dx + xy dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (y - 0) dy dx \\ &= \int_0^1 \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=1-x} dx = \frac{1}{2} \int_0^1 (1-x)^2 dx \\ &= -\frac{1}{6}(1-x)^3 \Big|_0^1 = \frac{1}{6} \end{aligned}$$

**EXAMPLE 2** Evaluate  $\oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy$ , where  $C$  is the circle  $x^2 + y^2 = 9$ .

**SOLUTION** The region  $D$  bounded by  $C$  is the disk  $x^2 + y^2 \leq 9$ , so let's change to polar coordinates after applying Green's Theorem:

$$\begin{aligned} \oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy &= \iint_D \left[ \frac{\partial}{\partial x} (7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y} (3y - e^{\sin x}) \right] dA \\ &= \int_0^{2\pi} \int_0^3 (7 - 3) r dr d\theta = 4 \int_0^{2\pi} d\theta \int_0^3 r dr = 36\pi \end{aligned}$$

Instead of using polar coordinates, we could simply use the fact that  $D$  is a disk of radius 3 and write

$$\iint_D 4 dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that  $P(x, y) = Q(x, y) = 0$  on the curve  $C$ , then Green's Theorem gives

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P dx + Q dy = 0$$

no matter what values  $P$  and  $Q$  assume in the region  $D$ .

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of  $D$  is  $\iint_D 1 dA$ , we wish to choose  $P$  and  $Q$  so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0$$

$$P(x, y) = -y$$

$$P(x, y) = -\frac{1}{2}y$$

$$Q(x, y) = x$$

$$Q(x, y) = 0$$

$$Q(x, y) = \frac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of  $D$ :

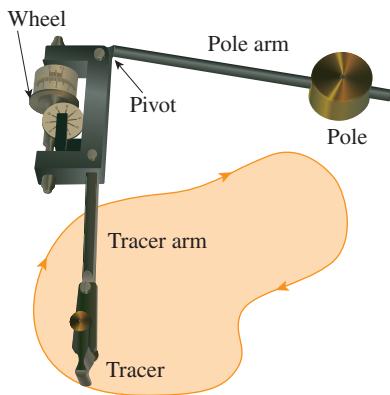
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$$A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx$$

**EXAMPLE 3** Find the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

**SOLUTION** The ellipse has parametric equations  $x = a \cos t$  and  $y = b \sin t$ , where  $0 \leq t \leq 2\pi$ . Using the third formula in Equation 5, we have

$$\begin{aligned} A &= \frac{1}{2} \int_C x \, dy - y \, dx \\ &= \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) \, dt - (b \sin t)(-a \sin t) \, dt \\ &= \frac{ab}{2} \int_0^{2\pi} dt = \pi ab \end{aligned}$$



**FIGURE 5**  
A Keuffel and Esser polar planimeter

Formula 5 can be used to explain how planimeters work. A **planimeter** is a mechanical instrument used for measuring the area of a region by tracing its boundary curve. These devices are useful in all the sciences: in biology for measuring the area of leaves or wings, in medicine for measuring the size of cross-sections of organs or tumors, in forestry for estimating the size of forested regions from photographs.

Figure 5 shows the operation of a polar planimeter: the pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region. The explanation as a consequence of Formula 5 can be found in the following articles:

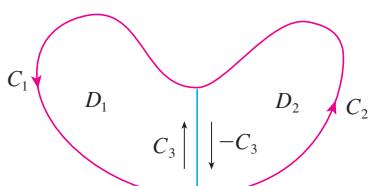
- R. W. Gatterman, “The planimeter as an example of Green’s Theorem” *Amer. Math. Monthly*, Vol. 88 (1981), pp. 701–4.
- Tanya Leise, “As the planimeter wheel turns” *College Math. Journal*, Vol. 38 (2007), pp. 24–31.

### ■ Extended Versions of Green’s Theorem

Although we have proved Green’s Theorem only for the case where  $D$  is simple, we can now extend it to the case where  $D$  is a finite union of simple regions. For example, if  $D$  is the region shown in Figure 6, then we can write  $D = D_1 \cup D_2$ , where  $D_1$  and  $D_2$  are both simple. The boundary of  $D_1$  is  $C_1 \cup C_3$  and the boundary of  $D_2$  is  $C_2 \cup (-C_3)$  so, applying Green’s Theorem to  $D_1$  and  $D_2$  separately, we get

$$\int_{C_1 \cup C_3} P \, dx + Q \, dy = \iint_{D_1} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P \, dx + Q \, dy = \iint_{D_2} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



**FIGURE 6**

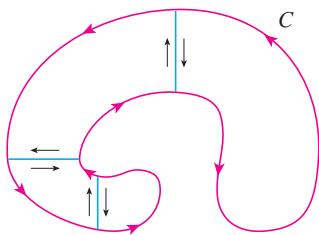


FIGURE 7

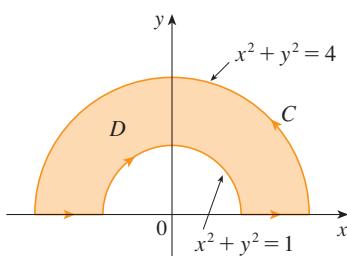


FIGURE 8

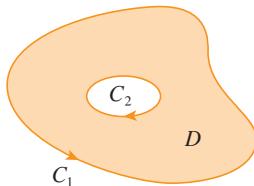


FIGURE 9

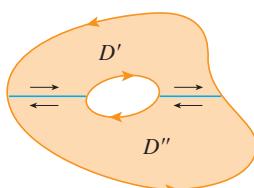


FIGURE 10

If we add these two equations, the line integrals along  $C_3$  and  $-C_3$  cancel, so we get

$$\int_{C_1 \cup C_2} P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for  $D = D_1 \cup D_2$ , since its boundary is  $C = C_1 \cup C_2$ .

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

**EXAMPLE 4** Evaluate  $\oint_C y^2 dx + 3xy dy$ , where  $C$  is the boundary of the semiannular region  $D$  in the upper half-plane between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

**SOLUTION** Notice that although  $D$  is not simple, the  $y$ -axis divides it into two simple regions (see Figure 8). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

Therefore Green's Theorem gives

$$\begin{aligned} \oint_C y^2 dx + 3xy dy &= \iint_D \left[ \frac{\partial}{\partial x} (3xy) - \frac{\partial}{\partial y} (y^2) \right] dA \\ &= \iint_D y dA = \int_0^\pi \int_1^2 (r \sin \theta) r dr d\theta \\ &= \int_0^\pi \sin \theta d\theta \int_1^2 r^2 dr = \left[ -\cos \theta \right]_0^\pi \left[ \frac{1}{3} r^3 \right]_1^2 = \frac{14}{3} \quad \blacksquare \end{aligned}$$

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary  $C$  of the region  $D$  in Figure 9 consists of two simple closed curves  $C_1$  and  $C_2$ . We assume that these boundary curves are oriented so that the region  $D$  is always on the left as the curve  $C$  is traversed. Thus the positive direction is counterclockwise for the outer curve  $C_1$  but clockwise for the inner curve  $C_2$ . If we divide  $D$  into two regions  $D'$  and  $D''$  by means of the lines shown in Figure 10 and then apply Green's Theorem to each of  $D'$  and  $D''$ , we get

$$\begin{aligned} \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \int_{\partial D'} P dx + Q dy + \int_{\partial D''} P dx + Q dy \end{aligned}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy = \int_C P dx + Q dy$$

which is Green's Theorem for the region  $D$ .

**EXAMPLE 5** If  $\mathbf{F}(x, y) = (-y \mathbf{i} + x \mathbf{j})/(x^2 + y^2)$ , show that  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$  for every positively oriented simple closed path that encloses the origin.

**SOLUTION** Since  $C$  is an arbitrary closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle  $C'$

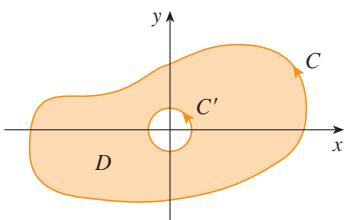


FIGURE 11

with center the origin and radius  $a$ , where  $a$  is chosen to be small enough that  $C'$  lies inside  $C$ . (See Figure 11.) Let  $D$  be the region bounded by  $C$  and  $C'$ . Then its positively oriented boundary is  $C \cup (-C')$  and so the general version of Green's Theorem gives

$$\begin{aligned} \int_C P \, dx + Q \, dy + \int_{-C'} P \, dx + Q \, dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D \left[ \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dA = 0 \end{aligned}$$

Therefore

$$\int_C P \, dx + Q \, dy = \int_{C'} P \, dx + Q \, dy$$

that is,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by  $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$ ,  $0 \leq t \leq 2\pi$ . Thus

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} \, dt = \int_0^{2\pi} dt = 2\pi \quad \blacksquare \end{aligned}$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

**SKETCH OF PROOF OF THEOREM 16.3.6** We're assuming that  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$  is a vector field on an open simply-connected region  $D$ , that  $P$  and  $Q$  have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

If  $C$  is any simple closed path in  $D$  and  $R$  is the region that  $C$  encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R 0 \, dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of  $\mathbf{F}$  around these simple curves are all 0 and, adding these integrals, we see that  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed curve  $C$ . Therefore  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  by Theorem 16.3.3. It follows that  $\mathbf{F}$  is a conservative vector field. ■

## 16.4 EXERCISES

- 1–4** Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1.  $\oint_C y^2 \, dx + x^2 y \, dy$ ,

$C$  is the rectangle with vertices  $(0, 0)$ ,  $(5, 0)$ ,  $(5, 4)$ , and  $(0, 4)$

2.  $\oint_C y \, dx - x \, dy$ ,

$C$  is the circle with center the origin and radius 4

3.  $\oint_C xy \, dx + x^2 y^3 \, dy$ ,

$C$  is the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 2)$

4.  $\oint_C x^2 y^2 dx + xy dy$ ,  $C$  consists of the arc of the parabola  $y = x^2$  from  $(0, 0)$  to  $(1, 1)$  and the line segments from  $(1, 1)$  to  $(0, 1)$  and from  $(0, 1)$  to  $(0, 0)$

**5–10** Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5.  $\int_C ye^x dx + 2e^x dy$ ,  
 $C$  is the rectangle with vertices  $(0, 0)$ ,  $(3, 0)$ ,  $(3, 4)$ , and  $(0, 4)$
6.  $\int_C (x^2 + y^2) dx + (x^2 - y^2) dy$ ,  
 $C$  is the triangle with vertices  $(0, 0)$ ,  $(2, 1)$ , and  $(0, 1)$
7.  $\int_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy$ ,  
 $C$  is the boundary of the region enclosed by the parabolas  $y = x^2$  and  $x = y^2$
8.  $\int_C y^4 dx + 2xy^3 dy$ ,  $C$  is the ellipse  $x^2 + 2y^2 = 2$
9.  $\int_C y^3 dx - x^3 dy$ ,  $C$  is the circle  $x^2 + y^2 = 4$
10.  $\int_C (1 - y^3) dx + (x^3 + e^{y^2}) dy$ ,  $C$  is the boundary of the region between the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 = 9$

**11–14** Use Green's Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ . (Check the orientation of the curve before applying the theorem.)

11.  $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$ ,  
 $C$  is the triangle from  $(0, 0)$  to  $(0, 4)$  to  $(2, 0)$  to  $(0, 0)$
12.  $\mathbf{F}(x, y) = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$ ,  
 $C$  consists of the arc of the curve  $y = \cos x$  from  $(-\pi/2, 0)$  to  $(\pi/2, 0)$  and the line segment from  $(\pi/2, 0)$  to  $(-\pi/2, 0)$
13.  $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y \rangle$ ,  
 $C$  is the circle  $(x - 3)^2 + (y + 4)^2 = 4$  oriented clockwise
14.  $\mathbf{F}(x, y) = \langle \sqrt{x^2 + 1}, \tan^{-1} x \rangle$ ,  $C$  is the triangle from  $(0, 0)$  to  $(1, 1)$  to  $(0, 1)$  to  $(0, 0)$

**CAS 15–16** Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15.  $P(x, y) = x^3 y^4$ ,  $Q(x, y) = x^5 y^4$ ,  
 $C$  consists of the line segment from  $(-\pi/2, 0)$  to  $(\pi/2, 0)$  followed by the arc of the curve  $y = \cos x$  from  $(\pi/2, 0)$  to  $(-\pi/2, 0)$

16.  $P(x, y) = 2x - x^3 y^5$ ,  $Q(x, y) = x^3 y^8$ ,  
 $C$  is the ellipse  $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force  $\mathbf{F}(x, y) = x(x + y)\mathbf{i} + xy^2\mathbf{j}$  in moving a particle from the origin along the  $x$ -axis to  $(1, 0)$ , then along the line segment to  $(0, 1)$ , and then back to the origin along the  $y$ -axis.

18. A particle starts at the origin, moves along the  $x$ -axis to  $(5, 0)$ , then along the quarter-circle  $x^2 + y^2 = 25$ ,  $x \geq 0$ ,  $y \geq 0$  to the point  $(0, 5)$ , and then down the  $y$ -axis back to the origin. Use Green's Theorem to find

the work done on this particle by the force field  $\mathbf{F}(x, y) = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle$ .

19. Use one of the formulas in (5) to find the area under one arch of the cycloid  $x = t - \sin t$ ,  $y = 1 - \cos t$ .
20. If a circle  $C$  with radius 1 rolls along the outside of the circle  $x^2 + y^2 = 16$ , a fixed point  $P$  on  $C$  traces out a curve called an *epicycloid*, with parametric equations  $x = 5 \cos t - \cos 5t$ ,  $y = 5 \sin t - \sin 5t$ . Graph the epicycloid and use (5) to find the area it encloses.
21. (a) If  $C$  is the line segment connecting the point  $(x_1, y_1)$  to the point  $(x_2, y_2)$ , show that
- $$\int_C x dy - y dx = x_1 y_2 - x_2 y_1$$
- (b) If the vertices of a polygon, in counterclockwise order, are  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , show that the area of the polygon is
- $$A = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \dots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$$
- (c) Find the area of the pentagon with vertices  $(0, 0), (2, 1), (1, 3), (0, 2)$ , and  $(-1, 1)$ .
22. Let  $D$  be a region bounded by a simple closed path  $C$  in the  $xy$ -plane. Use Green's Theorem to prove that the coordinates of the centroid  $(\bar{x}, \bar{y})$  of  $D$  are
- $$\bar{x} = \frac{1}{2A} \int_C x^2 dy \quad \bar{y} = -\frac{1}{2A} \int_C y^2 dx$$
- where  $A$  is the area of  $D$ .
23. Use Exercise 22 to find the centroid of a quarter-circular region of radius  $a$ .
24. Use Exercise 22 to find the centroid of the triangle with vertices  $(0, 0), (a, 0)$ , and  $(a, b)$ , where  $a > 0$  and  $b > 0$ .
25. A plane lamina with constant density  $\rho(x, y) = \rho$  occupies a region in the  $xy$ -plane bounded by a simple closed path  $C$ . Show that its moments of inertia about the axes are
- $$I_x = -\frac{\rho}{3} \int_C y^3 dx \quad I_y = \frac{\rho}{3} \int_C x^3 dy$$
26. Use Exercise 25 to find the moment of inertia of a circular disk of radius  $a$  with constant density  $\rho$  about a diameter. (Compare with Example 15.4.4.)
27. Use the method of Example 5 to calculate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = \frac{2xy\mathbf{i} + (y^2 - x^2)\mathbf{j}}{(x^2 + y^2)^2}$
- and  $C$  is any positively oriented simple closed curve that encloses the origin.
28. Calculate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$  and  $C$  is the positively oriented boundary curve of a region  $D$  that has area 6.
29. If  $\mathbf{F}$  is the vector field of Example 5, show that  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every simple closed path that does not pass through or enclose the origin.

30. Complete the proof of the special case of Green's Theorem by proving Equation 3.
31. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where  $f(x, y) = 1$ :

$$\iint_R dx dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Here  $R$  is the region in the  $xy$ -plane that corresponds to the region  $S$  in the  $uv$ -plane under the transformation given by  $x = g(u, v)$ ,  $y = h(u, v)$ .

[Hint: Note that the left side is  $A(R)$  and apply the first part of Equation 5. Convert the line integral over  $\partial R$  to a line integral over  $\partial S$  and apply Green's Theorem in the  $uv$ -plane.]

## 16.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

### Curl

If  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and the partial derivatives of  $P$ ,  $Q$ , and  $R$  all exist, then the curl of  $\mathbf{F}$  is the vector field on  $\mathbb{R}^3$  defined by

$$1 \quad \text{curl } \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator  $\nabla$  ("del") as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of  $f$ :

$$\nabla f = \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

If we think of  $\nabla$  as a vector with components  $\partial/\partial x$ ,  $\partial/\partial y$ , and  $\partial/\partial z$ , we can also consider the formal cross product of  $\nabla$  with the vector field  $\mathbf{F}$  as follows:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= \text{curl } \mathbf{F} \end{aligned}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

2

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

**EXAMPLE 1** If  $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$ , find  $\operatorname{curl} \mathbf{F}$ .

**SOLUTION** Using Equation 2, we have

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} \\ &= \left[ \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz) \right] \mathbf{i} - \left[ \frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz) \right] \mathbf{j} \\ &\quad + \left[ \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \right] \mathbf{k} \\ &= (-2y - xy) \mathbf{i} - (0 - x) \mathbf{j} + (yz - 0) \mathbf{k} \\ &= -y(2 + x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k}\end{aligned}$$

■

**CAS** Most computer algebra systems have commands that compute the curl and divergence of vector fields. If you have access to a CAS, use these commands to check the answers to the examples and exercises in this section.

Recall that the gradient of a function  $f$  of three variables is a vector field on  $\mathbb{R}^3$  and so we can compute its curl. The following theorem says that the curl of a gradient vector field is  $\mathbf{0}$ .

**3 Theorem** If  $f$  is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

**PROOF** We have

Notice the similarity to what we know from Section 12.4:  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$  for every three-dimensional vector  $\mathbf{a}$ .

$$\begin{aligned}\operatorname{curl}(\nabla f) &= \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} \\ &= 0 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} = \mathbf{0}\end{aligned}$$

by Clairaut's Theorem. ■

Since a conservative vector field is one for which  $\mathbf{F} = \nabla f$ , Theorem 3 can be rephrased as follows:

Compare this with Exercise 16.3.29.

If  $\mathbf{F}$  is conservative, then  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ .

This gives us a way of verifying that a vector field is not conservative.

**EXAMPLE 2** Show that the vector field  $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$  is not conservative.

**SOLUTION** In Example 1 we showed that

$$\operatorname{curl} \mathbf{F} = -y(2+x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k}$$

This shows that  $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$  and so, by the remarks preceding this example,  $\mathbf{F}$  is not conservative. ■

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if  $\mathbf{F}$  is defined everywhere. (More generally it is true if the domain is simply-connected, that is, “has no hole.”) Theorem 4 is the three-dimensional version of Theorem 16.3.6. Its proof requires Stokes’ Theorem and is sketched at the end of Section 16.8.

**4 Theorem** If  $\mathbf{F}$  is a vector field defined on all of  $\mathbb{R}^3$  whose component functions have continuous partial derivatives and  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ , then  $\mathbf{F}$  is a conservative vector field.

### EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$$

is a conservative vector field.

(b) Find a function  $f$  such that  $\mathbf{F} = \nabla f$ .

**SOLUTION**

(a) We compute the curl of  $\mathbf{F}$ :

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} \\ &= (6xyz^2 - 6xyz^2) \mathbf{i} - (3y^2 z^2 - 3y^2 z^2) \mathbf{j} + (2yz^3 - 2yz^3) \mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Since  $\operatorname{curl} \mathbf{F} = \mathbf{0}$  and the domain of  $\mathbf{F}$  is  $\mathbb{R}^3$ ,  $\mathbf{F}$  is a conservative vector field by Theorem 4.

(b) The technique for finding  $f$  was given in Section 16.3. We have

$$5 \quad f_x(x, y, z) = y^2 z^3$$

$$6 \quad f_y(x, y, z) = 2xyz^3$$

$$7 \quad f_z(x, y, z) = 3xy^2 z^2$$

Integrating (5) with respect to  $x$ , we obtain

$$8 \quad f(x, y, z) = xy^2 z^3 + g(y, z)$$

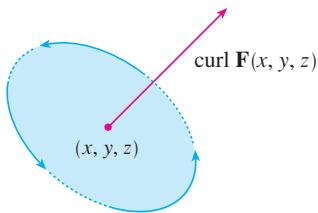
Differentiating (8) with respect to  $y$ , we get  $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$ , so comparison with (6) gives  $g_y(y, z) = 0$ . Thus  $g(y, z) = h(z)$  and

$$f_z(x, y, z) = 3xy^2z^2 + h'(z)$$

Then (7) gives  $h'(z) = 0$ . Therefore

$$f(x, y, z) = xy^2z^3 + K$$

■



**FIGURE 1**

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 37. Another occurs when  $\mathbf{F}$  represents the velocity field in fluid flow (see Example 16.1.3). Particles near  $(x, y, z)$  in the fluid tend to rotate about the axis that points in the direction of  $\text{curl } \mathbf{F}(x, y, z)$ , and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If  $\text{curl } \mathbf{F} = \mathbf{0}$  at a point  $P$ , then the fluid is free from rotations at  $P$  and  $\mathbf{F}$  is called **irrotational** at  $P$ . In other words, there is no whirlpool or eddy at  $P$ . If  $\text{curl } \mathbf{F} \neq \mathbf{0}$ , then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. We give a more detailed explanation in Section 16.8 as a consequence of Stokes' Theorem.

### ■ Divergence

If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and  $\partial P/\partial x$ ,  $\partial Q/\partial y$ , and  $\partial R/\partial z$  exist, then the **divergence of  $\mathbf{F}$**  is the function of three variables defined by

9

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that  $\text{curl } \mathbf{F}$  is a vector field but  $\text{div } \mathbf{F}$  is a scalar field. In terms of the gradient operator  $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$ , the divergence of  $\mathbf{F}$  can be written symbolically as the dot product of  $\nabla$  and  $\mathbf{F}$ :

10

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

**EXAMPLE 4** If  $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ , find  $\text{div } \mathbf{F}$ .

**SOLUTION** By the definition of divergence (Equation 9 or 10) we have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

■

If  $\mathbf{F}$  is a vector field on  $\mathbb{R}^3$ , then  $\text{curl } \mathbf{F}$  is also a vector field on  $\mathbb{R}^3$ . As such, we can compute its divergence. The next theorem shows that the result is 0.

11

**Theorem** If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and  $P, Q$ , and  $R$  have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

**PROOF** Using the definitions of divergence and curl, we have

Note the analogy with the scalar triple product:  $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ .

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F})$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\ &= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\ &= 0 \end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. ■

**EXAMPLE 5** Show that the vector field  $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$  can't be written as the curl of another vector field, that is,  $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$ .

**SOLUTION** In Example 4 we showed that

$$\operatorname{div} \mathbf{F} = z + xz$$

and therefore  $\operatorname{div} \mathbf{F} \neq 0$ . If it were true that  $\mathbf{F} = \operatorname{curl} \mathbf{G}$ , then Theorem 11 would give

$$\operatorname{div} \mathbf{F} = \operatorname{div} \operatorname{curl} \mathbf{G} = 0$$

which contradicts  $\operatorname{div} \mathbf{F} \neq 0$ . Therefore  $\mathbf{F}$  is not the curl of another vector field. ■

The reason for this interpretation of  $\operatorname{div} \mathbf{F}$  will be explained at the end of Section 16.9 as a consequence of the Divergence Theorem.

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If  $\mathbf{F}(x, y, z)$  is the velocity of a fluid (or gas), then  $\operatorname{div} \mathbf{F}(x, y, z)$  represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point  $(x, y, z)$  per unit volume. In other words,  $\operatorname{div} \mathbf{F}(x, y, z)$  measures the tendency of the fluid to diverge from the point  $(x, y, z)$ . If  $\operatorname{div} \mathbf{F} = 0$ , then  $\mathbf{F}$  is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field  $\nabla f$ . If  $f$  is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as  $\nabla^2 f$ . The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator  $\nabla^2$  to a vector field

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P \mathbf{i} + \nabla^2 Q \mathbf{j} + \nabla^2 R \mathbf{k}$$

### ■ Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work. We suppose that the plane region  $D$ , its boundary curve  $C$ , and the functions  $P$  and  $Q$  satisfy the hypotheses of Green's Theorem. Then we consider the vector field  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ . Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy$$

and, regarding  $\mathbf{F}$  as a vector field on  $\mathbb{R}^3$  with third component 0, we have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Therefore

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

12

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$$

Equation 12 expresses the line integral of the tangential component of  $\mathbf{F}$  along  $C$  as the double integral of the vertical component of  $\text{curl } \mathbf{F}$  over the region  $D$  enclosed by  $C$ . We now derive a similar formula involving the *normal* component of  $\mathbf{F}$ .

If  $C$  is given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad a \leq t \leq b$$

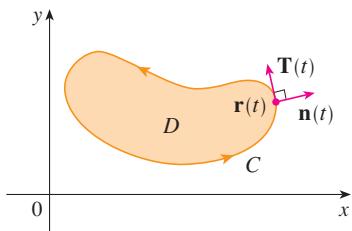
then the unit tangent vector (see Section 13.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

You can verify that the outward unit normal vector to  $C$  is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

(See Figure 2.) Then, from Equation 16.2.3, we have



$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{n} ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t) |\mathbf{r}'(t)| dt \\ &= \int_a^b \left[ \frac{P(x(t), y(t)) y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t)) x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| dt \\ &= \int_a^b P(x(t), y(t)) y'(t) dt - Q(x(t), y(t)) x'(t) dt \\ &= \int_C P dy - Q dx = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of  $\mathbf{F}$ . So we have a second vector form of Green's Theorem.

13

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

This version says that the line integral of the normal component of  $\mathbf{F}$  along  $C$  is equal to the double integral of the divergence of  $\mathbf{F}$  over the region  $D$  enclosed by  $C$ .

## 16.5 EXERCISES

- 1–8** Find (a) the curl and (b) the divergence of the vector field.

1.  $\mathbf{F}(x, y, z) = xy^2z^2 \mathbf{i} + x^2yz^2 \mathbf{j} + x^2y^2z \mathbf{k}$

2.  $\mathbf{F}(x, y, z) = x^3yz^2 \mathbf{j} + y^4z^3 \mathbf{k}$

3.  $\mathbf{F}(x, y, z) = xye^z \mathbf{i} + yze^x \mathbf{k}$

4.  $\mathbf{F}(x, y, z) = \sin yz \mathbf{i} + \sin zx \mathbf{j} + \sin xy \mathbf{k}$

5.  $\mathbf{F}(x, y, z) = \frac{\sqrt{x}}{1+z} \mathbf{i} + \frac{\sqrt{y}}{1+x} \mathbf{j} + \frac{\sqrt{z}}{1+y} \mathbf{k}$

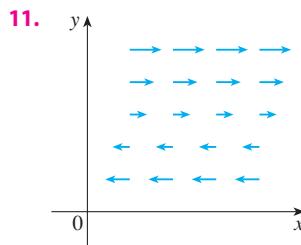
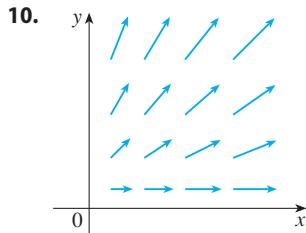
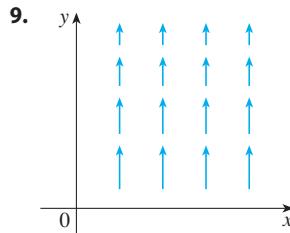
6.  $\mathbf{F}(x, y, z) = \ln(2y + 3z) \mathbf{i} + \ln(x + 3z) \mathbf{j} + \ln(x + 2y) \mathbf{k}$

7.  $\mathbf{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$

8.  $\mathbf{F}(x, y, z) = \langle \arctan(xy), \arctan(yz), \arctan(zx) \rangle$

- 9–11** The vector field  $\mathbf{F}$  is shown in the  $xy$ -plane and looks the same in all other horizontal planes. (In other words,  $\mathbf{F}$  is independent of  $z$  and its  $z$ -component is 0.)

- (a) Is  $\operatorname{div} \mathbf{F}$  positive, negative, or zero? Explain.  
 (b) Determine whether  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ . If not, in which direction does  $\operatorname{curl} \mathbf{F}$  point?



- 12.** Let  $f$  be a scalar field and  $\mathbf{F}$  a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

- |  |  |
|--|--|
| (a) $\operatorname{curl} f$  | (b) $\operatorname{grad} f$  |
| (c) $\operatorname{div} \mathbf{F}$                                  | (d) $\operatorname{curl}(\operatorname{grad} f)$                     |
| (e) $\operatorname{grad} \mathbf{F}$                                 | (f) $\operatorname{grad}(\operatorname{div} \mathbf{F})$             |
| (g) $\operatorname{div}(\operatorname{grad} f)$                      | (h) $\operatorname{grad}(\operatorname{div} f)$                      |
| (i) $\operatorname{curl}(\operatorname{curl} \mathbf{F})$            | (j) $\operatorname{div}(\operatorname{div} \mathbf{F})$              |
| (k) $(\operatorname{grad} f) \times (\operatorname{div} \mathbf{F})$ | (l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ |

- 13–18** Determine whether or not the vector field is conservative. If it is conservative, find a function  $f$  such that  $\mathbf{F} = \nabla f$ .

13.  $\mathbf{F}(x, y, z) = y^2z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2z^2 \mathbf{k}$

14.  $\mathbf{F}(x, y, z) = xyz^4 \mathbf{i} + x^2z^4 \mathbf{j} + 4x^2yz^3 \mathbf{k}$

15.  $\mathbf{F}(x, y, z) = z \cos y \mathbf{i} + xz \sin y \mathbf{j} + x \cos y \mathbf{k}$

16.  $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z \mathbf{j} + y \cos z \mathbf{k}$

17.  $\mathbf{F}(x, y, z) = e^{yz} \mathbf{i} + xze^{yz} \mathbf{j} + xye^{yz} \mathbf{k}$

18.  $\mathbf{F}(x, y, z) = e^x \sin yz \mathbf{i} + ze^x \cos yz \mathbf{j} + ye^x \cos yz \mathbf{k}$

19. Is there a vector field  $\mathbf{G}$  on  $\mathbb{R}^3$  such that  $\operatorname{curl} \mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$ ? Explain.

20. Is there a vector field  $\mathbf{G}$  on  $\mathbb{R}^3$  such that  $\operatorname{curl} \mathbf{G} = \langle x, y, z \rangle$ ? Explain.

21. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x) \mathbf{i} + g(y) \mathbf{j} + h(z) \mathbf{k}$$

where  $f, g, h$  are differentiable functions, is irrotational.

22. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z) \mathbf{i} + g(x, z) \mathbf{j} + h(x, y) \mathbf{k}$$

is incompressible.

**23–29** Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If  $f$  is a scalar field and  $\mathbf{F}, \mathbf{G}$  are vector fields, then  $f\mathbf{F}$ ,  $\mathbf{F} \cdot \mathbf{G}$ , and  $\mathbf{F} \times \mathbf{G}$  are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$$

$$(\mathbf{F} \cdot \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)$$

$$(\mathbf{F} \times \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)$$

**23.**  $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$

**24.**  $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$

**25.**  $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f$

**26.**  $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F}$

**27.**  $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$

**28.**  $\operatorname{div}(\nabla f \times \nabla g) = 0$

**29.**  $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2 \mathbf{F}$

**30–32** Let  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $r = |\mathbf{r}|$ .

**30.** Verify each identity.

(a)  $\nabla \cdot \mathbf{r} = 3$

(b)  $\nabla \cdot (r\mathbf{r}) = 4r$

(c)  $\nabla^2 r^3 = 12r$

**31.** Verify each identity.

(a)  $\nabla r = \mathbf{r}/r$

(b)  $\nabla \times \mathbf{r} = \mathbf{0}$

(c)  $\nabla(1/r) = -\mathbf{r}/r^3$

(d)  $\nabla \ln r = \mathbf{r}/r^2$

**32.** If  $\mathbf{F} = \mathbf{r}/r^p$ , find  $\operatorname{div} \mathbf{F}$ . Is there a value of  $p$  for which  $\operatorname{div} \mathbf{F} = 0$ ?

**33.** Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f \nabla^2 g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

where  $D$  and  $C$  satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of  $f$  and  $g$  exist and are continuous. (The quantity  $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}} g$  occurs in the line integral. This is the directional derivative in the direction of the normal vector  $\mathbf{n}$  and is called the **normal derivative** of  $g$ .)

**34.** Use Green's first identity (Exercise 33) to prove **Green's second identity**:

$$\iint_D (f \nabla^2 g - g \nabla^2 f) \, dA = \oint_C (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds$$

where  $D$  and  $C$  satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of  $f$  and  $g$  exist and are continuous.

**35.** Recall from Section 14.3 that a function  $g$  is called *harmonic* on  $D$  if it satisfies Laplace's equation, that is,  $\nabla^2 g = 0$  on  $D$ . Use Green's first identity (with the same hypotheses as in

Exercise 33) to show that if  $g$  is harmonic on  $D$ , then  $\oint_C D_{\mathbf{n}} g \, ds = 0$ . Here  $D_{\mathbf{n}} g$  is the normal derivative of  $g$  defined in Exercise 33.

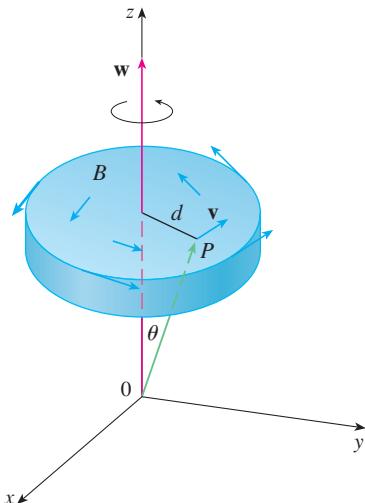
**36.** Use Green's first identity to show that if  $f$  is harmonic on  $D$ , and if  $f(x, y) = 0$  on the boundary curve  $C$ , then  $\iint_D |\nabla f|^2 \, dA = 0$ . (Assume the same hypotheses as in Exercise 33.)

**37.** This exercise demonstrates a connection between the curl vector and rotations. Let  $B$  be a rigid body rotating about the  $z$ -axis. The rotation can be described by the vector  $\mathbf{w} = \omega \mathbf{k}$ , where  $\omega$  is the angular speed of  $B$ , that is, the tangential speed of any point  $P$  in  $B$  divided by the distance  $d$  from the axis of rotation. Let  $\mathbf{r} = \langle x, y, z \rangle$  be the position vector of  $P$ .

(a) By considering the angle  $\theta$  in the figure, show that the velocity field of  $B$  is given by  $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ .

(b) Show that  $\mathbf{v} = -\omega y \mathbf{i} + \omega x \mathbf{j}$ .

(c) Show that  $\operatorname{curl} \mathbf{v} = 2\mathbf{w}$ .



**38.** Maxwell's equations relating the electric field  $\mathbf{E}$  and magnetic field  $\mathbf{H}$  as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div} \mathbf{E} = 0$$

$$\operatorname{div} \mathbf{H} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \quad \operatorname{curl} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

where  $c$  is the speed of light. Use these equations to prove the following:

$$(a) \nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$(b) \nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

$$(c) \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad [\text{Hint: Use Exercise 29.}]$$

$$(d) \nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

39. We have seen that all vector fields of the form  $\mathbf{F} = \nabla g$  satisfy the equation  $\operatorname{curl} \mathbf{F} = \mathbf{0}$  and that all vector fields of the form  $\mathbf{F} = \operatorname{curl} \mathbf{G}$  satisfy the equation  $\operatorname{div} \mathbf{F} = 0$  (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the

form  $f = \operatorname{div} \mathbf{G}$  must satisfy? Show that the answer to this question is “No” by proving that every continuous function  $f$  on  $\mathbb{R}^3$  is the divergence of some vector field.

[Hint: Let  $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$ , where  $g(x, y, z) = \int_0^x f(t, y, z) dt$ .]

## 16.6 Parametric Surfaces and Their Areas

So far we have considered special types of surfaces: cylinders, quadric surfaces, graphs of functions of two variables, and level surfaces of functions of three variables. Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas. Then we take the general surface area formula and see how it applies to special surfaces.

### Parametric Surfaces

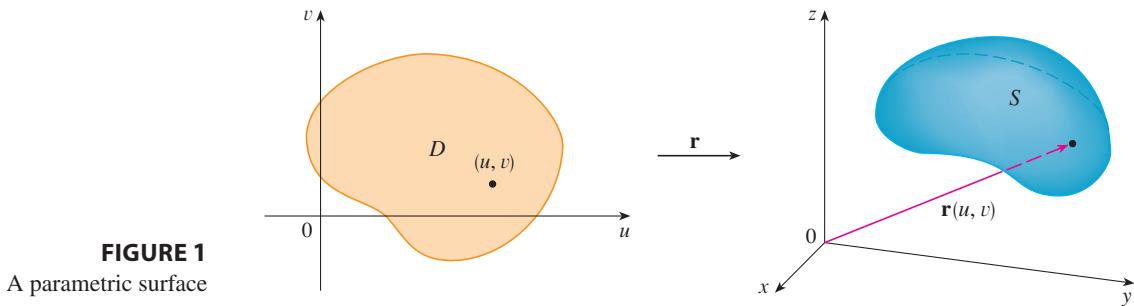
In much the same way that we describe a space curve by a vector function  $\mathbf{r}(t)$  of a single parameter  $t$ , we can describe a surface by a vector function  $\mathbf{r}(u, v)$  of two parameters  $u$  and  $v$ . We suppose that

$$1 \quad \mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

is a vector-valued function defined on a region  $D$  in the  $uv$ -plane. So  $x$ ,  $y$ , and  $z$ , the component functions of  $\mathbf{r}$ , are functions of the two variables  $u$  and  $v$  with domain  $D$ . The set of all points  $(x, y, z)$  in  $\mathbb{R}^3$  such that

$$2 \quad x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

and  $(u, v)$  varies throughout  $D$ , is called a **parametric surface**  $S$  and Equations 2 are called **parametric equations** of  $S$ . Each choice of  $u$  and  $v$  gives a point on  $S$ ; by making all choices, we get all of  $S$ . In other words, the surface  $S$  is traced out by the tip of the position vector  $\mathbf{r}(u, v)$  as  $(u, v)$  moves throughout the region  $D$ . (See Figure 1.)



**FIGURE 1**  
A parametric surface

**EXAMPLE 1** Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \mathbf{i} + v \mathbf{j} + 2 \sin u \mathbf{k}$$

**SOLUTION** The parametric equations for this surface are

$$x = 2 \cos u \quad y = v \quad z = 2 \sin u$$

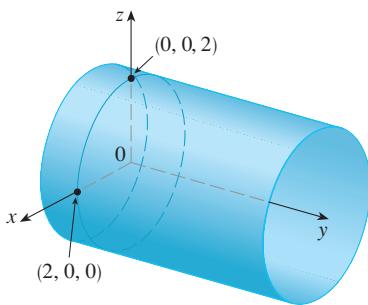


FIGURE 2

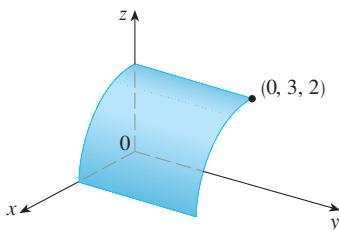


FIGURE 3

**TEC** Visual 16.6 shows animated versions of Figures 4 and 5, with moving grid curves, for several parametric surfaces.

So for any point  $(x, y, z)$  on the surface, we have

$$x^2 + z^2 = 4 \cos^2 u + 4 \sin^2 u = 4$$

This means that vertical cross-sections parallel to the  $xz$ -plane (that is, with  $y$  constant) are all circles with radius 2. Since  $y = v$  and no restriction is placed on  $v$ , the surface is a circular cylinder with radius 2 whose axis is the  $y$ -axis (see Figure 2). ■

In Example 1 we placed no restrictions on the parameters  $u$  and  $v$  and so we obtained the entire cylinder. If, for instance, we restrict  $u$  and  $v$  by writing the parameter domain as

$$0 \leq u \leq \pi/2 \quad 0 \leq v \leq 3$$

then  $x \geq 0$ ,  $z \geq 0$ ,  $0 \leq y \leq 3$ , and we get the quarter-cylinder with length 3 illustrated in Figure 3.

If a parametric surface  $S$  is given by a vector function  $\mathbf{r}(u, v)$ , then there are two useful families of curves that lie on  $S$ , one family with  $u$  constant and the other with  $v$  constant. These families correspond to vertical and horizontal lines in the  $uv$ -plane. If we keep  $u$  constant by putting  $u = u_0$ , then  $\mathbf{r}(u_0, v)$  becomes a vector function of the single parameter  $v$  and defines a curve  $C_1$  lying on  $S$ . (See Figure 4.)

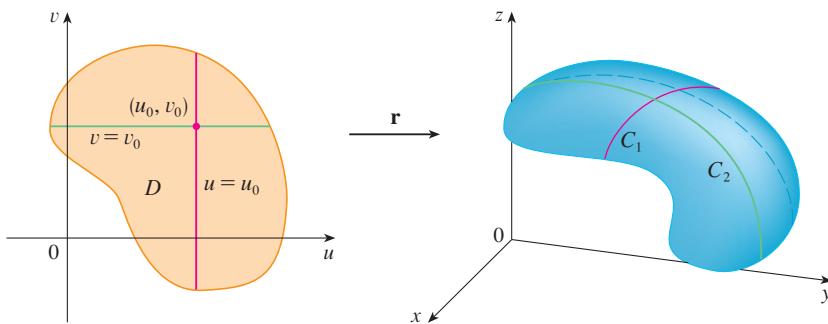


FIGURE 4

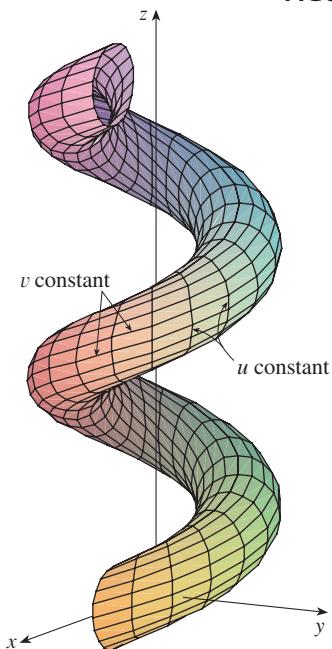


FIGURE 5

Similarly, if we keep  $v$  constant by putting  $v = v_0$ , we get a curve  $C_2$  given by  $\mathbf{r}(u, v_0)$  that lies on  $S$ . We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting  $u$  be constant are horizontal lines whereas the grid curves with  $v$  constant are circles.) In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves, as we see in the following example.

**EXAMPLE 2** Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

Which grid curves have  $u$  constant? Which have  $v$  constant?

**SOLUTION** We graph the portion of the surface with parameter domain  $0 \leq u \leq 4\pi$ ,  $0 \leq v \leq 2\pi$  in Figure 5. It has the appearance of a spiral tube. To identify the grid curves, we write the corresponding parametric equations:

$$x = (2 + \sin v) \cos u \quad y = (2 + \sin v) \sin u \quad z = u + \cos v$$

If  $v$  is constant, then  $\sin v$  and  $\cos v$  are constant, so the parametric equations resemble those of the helix in Example 13.1.4. Thus the grid curves with  $v$  constant are the spiral curves in Figure 5. We deduce that the grid curves with  $u$  constant must be the curves

that look like circles in the figure. Further evidence for this assertion is that if  $u$  is kept constant,  $u = u_0$ , then the equation  $z = u_0 + \cos v$  shows that the  $z$ -values vary from  $u_0 - 1$  to  $u_0 + 1$ . ■

In Examples 1 and 2 we were given a vector equation and asked to graph the corresponding parametric surface. In the following examples, however, we are given the more challenging problem of finding a vector function to represent a given surface. In the rest of this chapter we will often need to do exactly that.

**EXAMPLE 3** Find a vector function that represents the plane that passes through the point  $P_0$  with position vector  $\mathbf{r}_0$  and that contains two nonparallel vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

**SOLUTION** If  $P$  is any point in the plane, we can get from  $P_0$  to  $P$  by moving a certain distance in the direction of  $\mathbf{a}$  and another distance in the direction of  $\mathbf{b}$ . So there are scalars  $u$  and  $v$  such that  $\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$ . (Figure 6 illustrates how this works, by means of the Parallelogram Law, for the case where  $u$  and  $v$  are positive. See also Exercise 12.2.46.) If  $\mathbf{r}$  is the position vector of  $P$ , then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

So the vector equation of the plane can be written as

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

where  $u$  and  $v$  are real numbers.

If we write  $\mathbf{r} = \langle x, y, z \rangle$ ,  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ ,  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ , and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then we can write the parametric equations of the plane through the point  $(x_0, y_0, z_0)$  as follows:

$$x = x_0 + ua_1 + vb_1 \quad y = y_0 + ua_2 + vb_2 \quad z = z_0 + ua_3 + vb_3$$

**EXAMPLE 4** Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

**SOLUTION** The sphere has a simple representation  $\rho = a$  in spherical coordinates, so let's choose the angles  $\phi$  and  $\theta$  in spherical coordinates as the parameters (see Section 15.8). Then, putting  $\rho = a$  in the equations for conversion from spherical to rectangular coordinates (Equations 15.8.1), we obtain

$$x = a \sin \phi \cos \theta \quad y = a \sin \phi \sin \theta \quad z = a \cos \phi$$

as the parametric equations of the sphere. The corresponding vector equation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$$

We have  $0 \leq \phi \leq \pi$  and  $0 \leq \theta \leq 2\pi$ , so the parameter domain is the rectangle  $D = [0, \pi] \times [0, 2\pi]$ . The grid curves with  $\phi$  constant are the circles of constant latitude (including the equator). The grid curves with  $\theta$  constant are the meridians (semi-circles), which connect the north and south poles (see Figure 7). ■

**NOTE** We saw in Example 4 that the grid curves for a sphere are curves of constant latitude or constant longitude. For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude. Describing a point on a parametric surface (like the one in Figure 5) by giving specific values of  $u$  and  $v$  is like giving the latitude and longitude of a point.

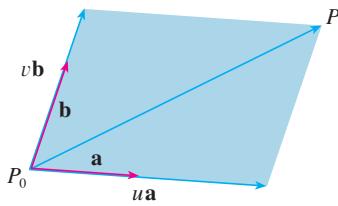


FIGURE 6

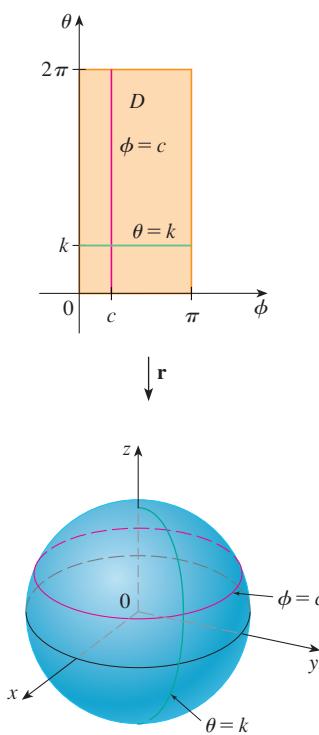


FIGURE 7

One of the uses of parametric surfaces is in computer graphics. Figure 8 shows the result of trying to graph the sphere  $x^2 + y^2 + z^2 = 1$  by solving the equation for  $z$  and graphing the top and bottom hemispheres separately. Part of the sphere appears to be missing because of the rectangular grid system used by the computer. The much better picture in Figure 9 was produced by a computer using the parametric equations found in Example 4.

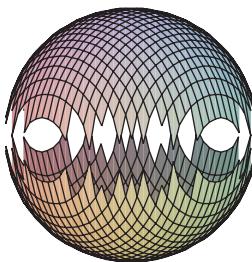


FIGURE 8

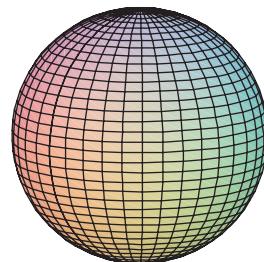


FIGURE 9

**EXAMPLE 5** Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \quad 0 \leq z \leq 1$$

**SOLUTION** The cylinder has a simple representation  $r = 2$  in cylindrical coordinates, so we choose as parameters  $\theta$  and  $z$  in cylindrical coordinates. Then the parametric equations of the cylinder are

$$x = 2 \cos \theta \quad y = 2 \sin \theta \quad z = z$$

where  $0 \leq \theta \leq 2\pi$  and  $0 \leq z \leq 1$ . ■

**EXAMPLE 6** Find a vector function that represents the elliptic paraboloid  $z = x^2 + 2y^2$ .

**SOLUTION** If we regard  $x$  and  $y$  as parameters, then the parametric equations are simply

$$x = x \quad y = y \quad z = x^2 + 2y^2$$

and the vector equation is

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (x^2 + 2y^2)\mathbf{k}$$

**TEC** In Module 16.6 you can investigate several families of parametric surfaces.

In general, a surface given as the graph of a function of  $x$  and  $y$ , that is, with an equation of the form  $z = f(x, y)$ , can always be regarded as a parametric surface by taking  $x$  and  $y$  as parameters and writing the parametric equations as

$$x = x \quad y = y \quad z = f(x, y)$$

Parametric representations (also called parametrizations) of surfaces are not unique. The next example shows two ways to parametrize a cone.

**EXAMPLE 7** Find a parametric representation for the surface  $z = 2\sqrt{x^2 + y^2}$ , that is, the top half of the cone  $z^2 = 4x^2 + 4y^2$ .

**SOLUTION 1** One possible representation is obtained by choosing  $x$  and  $y$  as parameters:

$$x = x \quad y = y \quad z = 2\sqrt{x^2 + y^2}$$

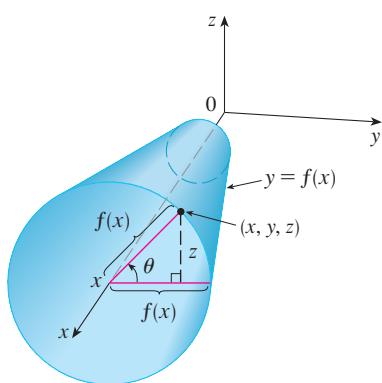
So the vector equation is

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + 2\sqrt{x^2 + y^2}\mathbf{k}$$

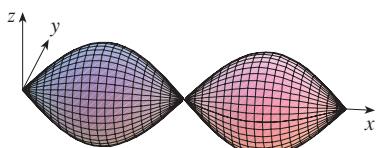
**SOLUTION 2** Another representation results from choosing as parameters the polar coordinates  $r$  and  $\theta$ . A point  $(x, y, z)$  on the cone satisfies  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and

For some purposes the parametric representations in Solutions 1 and 2 are equally good, but Solution 2 might be preferable in certain situations. If we are interested only in the part of the cone that lies below the plane  $z = 1$ , for instance, all we have to do in Solution 2 is change the parameter domain to

$$0 \leq r \leq \frac{1}{2} \quad 0 \leq \theta \leq 2\pi$$



**FIGURE 10**



**FIGURE 11**

$z = 2\sqrt{x^2 + y^2} = 2r$ . So a vector equation for the cone is

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 2r \mathbf{k}$$

where  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$ . ■

### Surfaces of Revolution

Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface  $S$  obtained by rotating the curve  $y = f(x)$ ,  $a \leq x \leq b$ , about the  $x$ -axis, where  $f(x) \geq 0$ . Let  $\theta$  be the angle of rotation as shown in Figure 10. If  $(x, y, z)$  is a point on  $S$ , then

**3**

$$x = x \quad y = f(x) \cos \theta \quad z = f(x) \sin \theta$$

Therefore we take  $x$  and  $\theta$  as parameters and regard Equations 3 as parametric equations of  $S$ . The parameter domain is given by  $a \leq x \leq b$ ,  $0 \leq \theta \leq 2\pi$ .

**EXAMPLE 8** Find parametric equations for the surface generated by rotating the curve  $y = \sin x$ ,  $0 \leq x \leq 2\pi$ , about the  $x$ -axis. Use these equations to graph the surface of revolution.

**SOLUTION** From Equations 3, the parametric equations are

$$x = x \quad y = \sin x \cos \theta \quad z = \sin x \sin \theta$$

and the parameter domain is  $0 \leq x \leq 2\pi$ ,  $0 \leq \theta \leq 2\pi$ . Using a computer to plot these equations and then rotating the image, we obtain the graph in Figure 11. ■

We can adapt Equations 3 to represent a surface obtained through revolution about the  $y$ - or  $z$ -axis (see Exercise 30).

### Tangent Planes

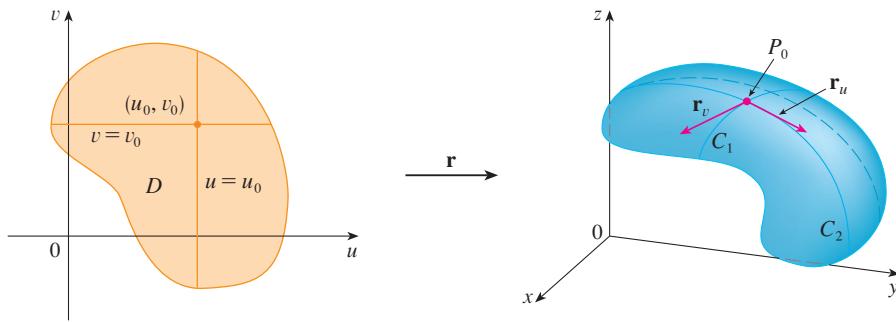
We now find the tangent plane to a parametric surface  $S$  traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

at a point  $P_0$  with position vector  $\mathbf{r}(u_0, v_0)$ . If we keep  $u$  constant by putting  $u = u_0$ , then  $\mathbf{r}(u_0, v)$  becomes a vector function of the single parameter  $v$  and defines a grid curve  $C_1$  lying on  $S$ . (See Figure 12.) The tangent vector to  $C_1$  at  $P_0$  is obtained by taking the partial derivative of  $\mathbf{r}$  with respect to  $v$ :

**4**

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0) \mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0) \mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0) \mathbf{k}$$



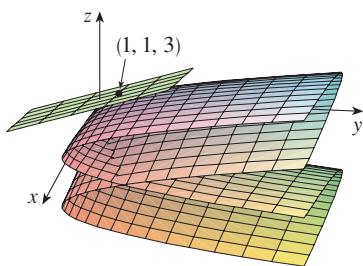
**FIGURE 12**

Similarly, if we keep  $v$  constant by putting  $v = v_0$ , we get a grid curve  $C_2$  given by  $\mathbf{r}(u, v_0)$  that lies on  $S$ , and its tangent vector at  $P_0$  is

$$\boxed{5} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0) \mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0) \mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0) \mathbf{k}$$

If  $\mathbf{r}_u \times \mathbf{r}_v$  is not  $\mathbf{0}$ , then the surface  $S$  is called **smooth** (it has no “corners”). For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ , and the vector  $\mathbf{r}_u \times \mathbf{r}_v$  is a normal vector to the tangent plane.

Figure 13 shows the self-intersecting surface in Example 9 and its tangent plane at  $(1, 1, 3)$ .



**FIGURE 13**

**EXAMPLE 9** Find the tangent plane to the surface with parametric equations  $x = u^2$ ,  $y = v^2$ ,  $z = u + 2v$  at the point  $(1, 1, 3)$ .

**SOLUTION** We first compute the tangent vectors:

$$\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} = 2u \mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k} = 2v \mathbf{j} + 2 \mathbf{k}$$

Thus a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v \mathbf{i} - 4u \mathbf{j} + 4uv \mathbf{k}$$

Notice that the point  $(1, 1, 3)$  corresponds to the parameter values  $u = 1$  and  $v = 1$ , so the normal vector there is

$$-2 \mathbf{i} - 4 \mathbf{j} + 4 \mathbf{k}$$

Therefore an equation of the tangent plane at  $(1, 1, 3)$  is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

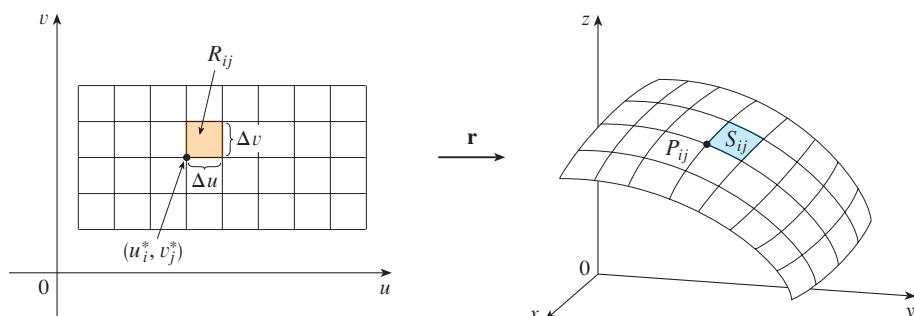
or

$$x + 2y - 2z + 3 = 0$$



## ■ Surface Area

Now we define the surface area of a general parametric surface given by Equation 1. For simplicity we start by considering a surface whose parameter domain  $D$  is a rectangle, and we divide it into subrectangles  $R_{ij}$ . Let's choose  $(u_i^*, v_j^*)$  to be the lower left corner of  $R_{ij}$ . (See Figure 14.)



**FIGURE 14**

The image of the subrectangle  $R_{ij}$  is the patch  $S_{ij}$ .

The part  $S_{ij}$  of the surface  $S$  that corresponds to  $R_{ij}$  is called a *patch* and has the point  $P_{ij}$  with position vector  $\mathbf{r}(u_i^*, v_j^*)$  as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \quad \text{and} \quad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at  $P_{ij}$  as given by Equations 5 and 4.

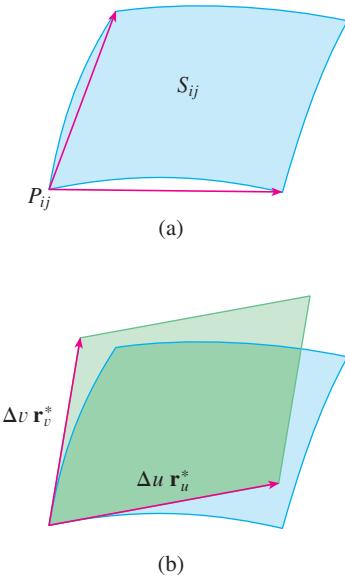
Figure 15(a) shows how the two edges of the patch that meet at  $P_{ij}$  can be approximated by vectors. These vectors, in turn, can be approximated by the vectors  $\Delta u \mathbf{r}_u^*$  and  $\Delta v \mathbf{r}_v^*$  because partial derivatives can be approximated by difference quotients. So we approximate  $S_{ij}$  by the parallelogram determined by the vectors  $\Delta u \mathbf{r}_u^*$  and  $\Delta v \mathbf{r}_v^*$ . This parallelogram is shown in Figure 15(b) and lies in the tangent plane to  $S$  at  $P_{ij}$ . The area of this parallelogram is

$$|(\Delta u \mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

and so an approximation to the area of  $S$  is

$$\sum_{i=1}^m \sum_{j=1}^n |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral  $\iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$ . This motivates the following definition.



**FIGURE 15**  
Approximating a patch by a parallelogram

**6 Definition** If a smooth parametric surface  $S$  is given by the equation

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D$$

and  $S$  is covered just once as  $(u, v)$  ranges throughout the parameter domain  $D$ , then the **surface area** of  $S$  is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

$$\text{where } \mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad \mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$$

**EXAMPLE 10** Find the surface area of a sphere of radius  $a$ .

**SOLUTION** In Example 4 we found the parametric representation

$$x = a \sin \phi \cos \theta \quad y = a \sin \phi \sin \theta \quad z = a \cos \phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k} \end{aligned}$$

Thus

$$\begin{aligned} |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi} \\ &= \sqrt{a^4 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi} = a^2 \sqrt{\sin^2 \phi} = a^2 \sin \phi \end{aligned}$$

since  $\sin \phi \geq 0$  for  $0 \leq \phi \leq \pi$ . Therefore, by Definition 6, the area of the sphere is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA = \int_0^{2\pi} \int_0^\pi a^2 \sin \phi \, d\phi \, d\theta \\ &= a^2 \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \, d\phi = a^2(2\pi)2 = 4\pi a^2 \end{aligned}$$

■

### ■ Surface Area of the Graph of a Function

For the special case of a surface  $S$  with equation  $z = f(x, y)$ , where  $(x, y)$  lies in  $D$  and  $f$  has continuous partial derivatives, we take  $x$  and  $y$  as parameters. The parametric equations are

$$x = x \quad y = y \quad z = f(x, y)$$

so  $\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right) \mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right) \mathbf{k}$

and

$$7 \quad \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}$$

Thus we have

$$8 \quad |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area formula in Definition 6 becomes

$$9 \quad A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

Notice the similarity between the surface area formula in Equation 9 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

from Section 8.1.

**EXAMPLE 11** Find the area of the part of the paraboloid  $z = x^2 + y^2$  that lies under the plane  $z = 9$ .

**SOLUTION** The plane intersects the paraboloid in the circle  $x^2 + y^2 = 9$ ,  $z = 9$ . Therefore the given surface lies above the disk  $D$  with center the origin and radius 3. (See

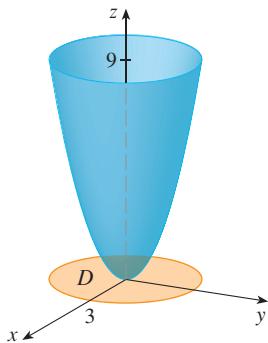
**FIGURE 16**

Figure 16.) Using Formula 9, we have

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2} dA \\ &= \iint_D \sqrt{1 + 4(x^2 + y^2)} dA \end{aligned}$$

Converting to polar coordinates, we obtain

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^3 r \sqrt{1 + 4r^2} dr \\ &= 2\pi \left[ \frac{1}{8} r^2 (1 + 4r^2)^{3/2} \right]_0^3 = \frac{\pi}{6} (37\sqrt{37} - 1) \quad \blacksquare \end{aligned}$$

The question remains whether our definition of surface area (6) is consistent with the surface area formula from single-variable calculus (8.2.4).

We consider the surface  $S$  obtained by rotating the curve  $y = f(x)$ ,  $a \leq x \leq b$ , about the  $x$ -axis, where  $f(x) \geq 0$  and  $f'$  is continuous. From Equations 3 we know that parametric equations of  $S$  are

$$x = x \quad y = f(x) \cos \theta \quad z = f(x) \sin \theta \quad a \leq x \leq b \quad 0 \leq \theta \leq 2\pi$$

To compute the surface area of  $S$  we need the tangent vectors

$$\mathbf{r}_x = \mathbf{i} + f'(x) \cos \theta \mathbf{j} + f'(x) \sin \theta \mathbf{k}$$

$$\mathbf{r}_\theta = -f(x) \sin \theta \mathbf{j} + f(x) \cos \theta \mathbf{k}$$

Thus

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) \cos \theta & f'(x) \sin \theta \\ 0 & -f(x) \sin \theta & f(x) \cos \theta \end{vmatrix} \\ &= f(x) f'(x) \mathbf{i} - f(x) \cos \theta \mathbf{j} - f(x) \sin \theta \mathbf{k} \end{aligned}$$

and so

$$\begin{aligned} |\mathbf{r}_x \times \mathbf{r}_\theta| &= \sqrt{[f(x)]^2 [f'(x)]^2 + [f(x)]^2 \cos^2 \theta + [f(x)]^2 \sin^2 \theta} \\ &= \sqrt{[f(x)]^2 [1 + [f'(x)]^2]} = f(x) \sqrt{1 + [f'(x)]^2} \end{aligned}$$

because  $f(x) \geq 0$ . Therefore the area of  $S$  is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta| dA \\ &= \int_0^{2\pi} \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx d\theta \\ &= 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx \end{aligned}$$

This is precisely the formula that was used to define the area of a surface of revolution in single-variable calculus (8.2.4).

## 16.6 EXERCISES

- 1–2** Determine whether the points  $P$  and  $Q$  lie on the given surface.

**1.**  $\mathbf{r}(u, v) = \langle u + v, u - 2v, 3 + u - v \rangle$   
 $P(4, -5, 1), Q(0, 4, 6)$

**2.**  $\mathbf{r}(u, v) = \langle 1 + u - v, u + v^2, u^2 - v^2 \rangle$   
 $P(1, 2, 1), Q(2, 3, 3)$

- 3–6** Identify the surface with the given vector equation.

**3.**  $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (3 - v)\mathbf{j} + (1 + 4u + 5v)\mathbf{k}$

**4.**  $\mathbf{r}(u, v) = u^2\mathbf{i} + u \cos v\mathbf{j} + u \sin v\mathbf{k}$

**5.**  $\mathbf{r}(s, t) = \langle s \cos t, s \sin t, s \rangle$

**6.**  $\mathbf{r}(s, t) = \langle 3 \cos t, s, \sin t \rangle, -1 \leq s \leq 1$

- 7–12** Use a computer to graph the parametric surface. Get a printout and indicate on it which grid curves have  $u$  constant and which have  $v$  constant.

**7.**  $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle, -1 \leq u \leq 1, -1 \leq v \leq 1$

**8.**  $\mathbf{r}(u, v) = \langle u, v^3, -v \rangle, -2 \leq u \leq 2, -2 \leq v \leq 2$

**9.**  $\mathbf{r}(u, v) = \langle u^3, u \sin v, u \cos v \rangle, -1 \leq u \leq 1, 0 \leq v \leq 2\pi$

**10.**  $\mathbf{r}(u, v) = \langle u, \sin(u + v), \sin v \rangle, -\pi \leq u \leq \pi, -\pi \leq v \leq \pi$

**11.**  $x = \sin v, y = \cos u \sin 4v, z = \sin 2u \sin 4v, 0 \leq u \leq 2\pi, -\pi/2 \leq v \leq \pi/2$

**12.**  $x = \cos u, y = \sin u \sin v, z = \cos v, 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$

- 13–18** Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have  $u$  constant and which have  $v$  constant.

**13.**  $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

**14.**  $\mathbf{r}(u, v) = uv^2\mathbf{i} + u^2v\mathbf{j} + (u^2 - v^2)\mathbf{k}$

**15.**  $\mathbf{r}(u, v) = (u^3 - u)\mathbf{i} + v^2\mathbf{j} + u^2\mathbf{k}$

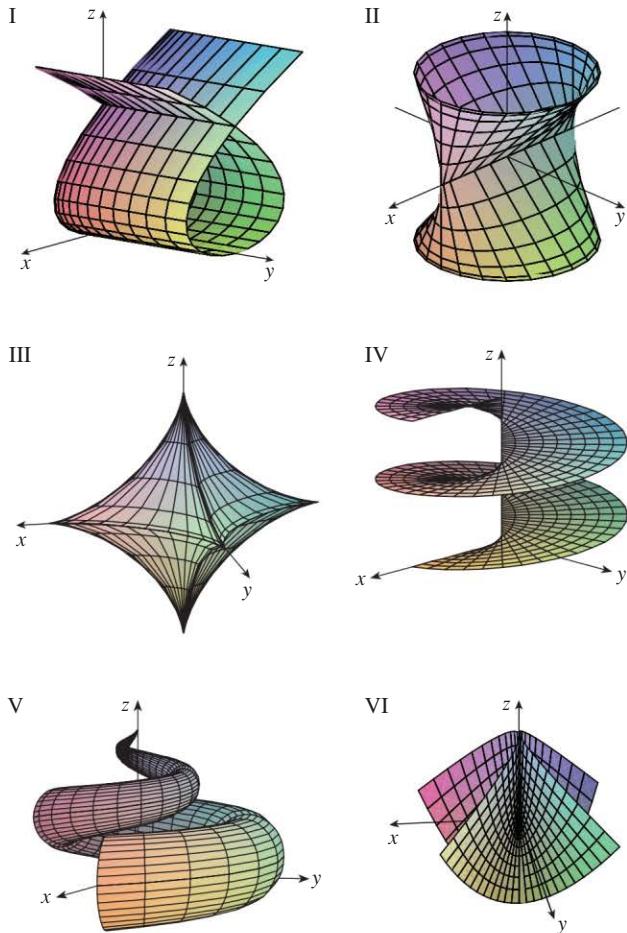
**16.**  $x = (1 - u)(3 + \cos v) \cos 4\pi u,$

$y = (1 - u)(3 + \cos v) \sin 4\pi u,$

$z = 3u + (1 - u) \sin v$

**17.**  $x = \cos^3 u \cos^3 v, y = \sin^3 u \cos^3 v, z = \sin^3 v$

**18.**  $x = \sin u, y = \cos u \sin v, z = \sin v$



- 19–26** Find a parametric representation for the surface.

**19.** The plane through the origin that contains the vectors  $\mathbf{i} - \mathbf{j}$  and  $\mathbf{j} - \mathbf{k}$

**20.** The plane that passes through the point  $(0, -1, 5)$  and contains the vectors  $\langle 2, 1, 4 \rangle$  and  $\langle -3, 2, 5 \rangle$

**21.** The part of the hyperboloid  $4x^2 - 4y^2 - z^2 = 4$  that lies in front of the  $yz$ -plane

**22.** The part of the ellipsoid  $x^2 + 2y^2 + 3z^2 = 1$  that lies to the left of the  $xz$ -plane

**23.** The part of the sphere  $x^2 + y^2 + z^2 = 4$  that lies above the cone  $z = \sqrt{x^2 + y^2}$

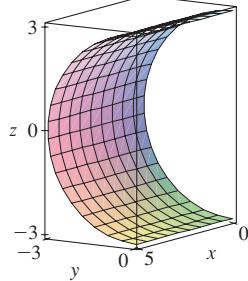
**24.** The part of the cylinder  $x^2 + z^2 = 9$  that lies above the  $xy$ -plane and between the planes  $y = -4$  and  $y = 4$

**25.** The part of the sphere  $x^2 + y^2 + z^2 = 36$  that lies between the planes  $z = 0$  and  $z = 3\sqrt{3}$

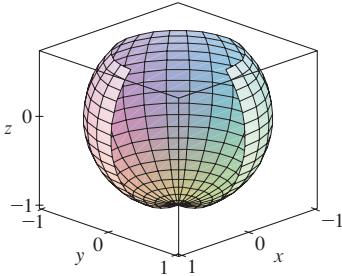
- 26.** The part of the plane  $z = x + 3$  that lies inside the cylinder  $x^2 + y^2 = 1$

**27–28** Use a graphing device to produce a graph that looks like the given one.

**27.**



**28.**



**29.** Find parametric equations for the surface obtained by rotating the curve  $y = 1/(1 + x^2)$ ,  $-2 \leq x \leq 2$ , about the  $x$ -axis and use them to graph the surface.

**30.** Find parametric equations for the surface obtained by rotating the curve  $x = 1/y$ ,  $y \geq 1$ , about the  $y$ -axis and use them to graph the surface.

**31.** (a) What happens to the spiral tube in Example 2 (see Figure 5) if we replace  $\cos u$  by  $\sin u$  and  $\sin u$  by  $\cos u$ ?  
(b) What happens if we replace  $\cos u$  by  $\cos 2u$  and  $\sin u$  by  $\sin 2u$ ?

**32.** The surface with parametric equations

$$x = 2 \cos \theta + r \cos(\theta/2)$$

$$y = 2 \sin \theta + r \sin(\theta/2)$$

$$z = r \sin(\theta/2)$$

where  $-\frac{1}{2} \leq r \leq \frac{1}{2}$  and  $0 \leq \theta \leq 2\pi$ , is called a **Möbius strip**. Graph this surface with several viewpoints. What is unusual about it?

**33–36** Find an equation of the tangent plane to the given parametric surface at the specified point.

**33.**  $x = u + v$ ,  $y = 3u^2$ ,  $z = u - v$ ;  $(2, 3, 0)$

**34.**  $x = u^2 + 1$ ,  $y = v^3 + 1$ ,  $z = u + v$ ;  $(5, 2, 3)$

**35.**  $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$ ;  $u = 1$ ,  $v = \pi/3$

**36.**  $\mathbf{r}(u, v) = \sin u \mathbf{i} + \cos u \sin v \mathbf{j} + \sin v \mathbf{k}$ ;  
 $u = \pi/6$ ,  $v = \pi/6$

**CAS** **37–38** Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane.

**37.**  $\mathbf{r}(u, v) = u^2 \mathbf{i} + 2u \sin v \mathbf{j} + u \cos v \mathbf{k}$ ;  $u = 1$ ,  $v = 0$

**38.**  $\mathbf{r}(u, v) = (1 - u^2 - v^2) \mathbf{i} - v \mathbf{j} - u \mathbf{k}$ ;  $(-1, -1, -1)$

**39–50** Find the area of the surface.

**39.** The part of the plane  $3x + 2y + z = 6$  that lies in the first octant

**40.** The part of the plane with vector equation  $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v \rangle$  that is given by  $0 \leq u \leq 2$ ,  $-1 \leq v \leq 1$

**41.** The part of the plane  $x + 2y + 3z = 1$  that lies inside the cylinder  $x^2 + y^2 = 3$

**42.** The part of the cone  $z = \sqrt{x^2 + y^2}$  that lies between the plane  $y = x$  and the cylinder  $y = x^2$

**43.** The surface  $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$

**44.** The part of the surface  $z = 4 - 2x^2 + y$  that lies above the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 1)$

**45.** The part of the surface  $z = xy$  that lies within the cylinder  $x^2 + y^2 = 1$

**46.** The part of the surface  $x = z^2 + y$  that lies between the planes  $y = 0$ ,  $y = 2$ ,  $z = 0$ , and  $z = 2$

**47.** The part of the paraboloid  $y = x^2 + z^2$  that lies within the cylinder  $x^2 + z^2 = 16$

**48.** The helicoid (or spiral ramp) with vector equation  $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$ ,  $0 \leq u \leq 1$ ,  $0 \leq v \leq \pi$

**49.** The surface with parametric equations  $x = u^2$ ,  $y = uv$ ,  $z = \frac{1}{2}v^2$ ,  $0 \leq u \leq 1$ ,  $0 \leq v \leq 2$

**50.** The part of the sphere  $x^2 + y^2 + z^2 = b^2$  that lies inside the cylinder  $x^2 + y^2 = a^2$ , where  $0 < a < b$

**51.** If the equation of a surface  $S$  is  $z = f(x, y)$ , where  $x^2 + y^2 \leq R^2$ , and you know that  $|f_x| \leq 1$  and  $|f_y| \leq 1$ , what can you say about  $A(S)$ ?

**52–53** Find the area of the surface correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

**52.** The part of the surface  $z = \cos(x^2 + y^2)$  that lies inside the cylinder  $x^2 + y^2 = 1$

**53.** The part of the surface  $z = \ln(x^2 + y^2 + 2)$  that lies above the disk  $x^2 + y^2 \leq 1$

**CAS** **54.** Find, to four decimal places, the area of the part of the surface  $z = (1 + x^2)/(1 + y^2)$  that lies above the square  $|x| + |y| \leq 1$ . Illustrate by graphing this part of the surface.

**55.** (a) Use the Midpoint Rule for double integrals (see Section 15.1) with six squares to estimate the area of the surface  $z = 1/(1 + x^2 + y^2)$ ,  $0 \leq x \leq 6$ ,  $0 \leq y \leq 4$ .

**CAS** (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

- CAS** 56. Find the area of the surface with vector equation

$\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$ ,  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ . State your answer correct to four decimal places.

- CAS** 57. Find the exact area of the surface  $z = 1 + 2x + 3y + 4y^2$ ,  $1 \leq x \leq 4$ ,  $0 \leq y \leq 1$ .

58. (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations  $x = au \cos v$ ,  $y = bu \sin v$ ,  $z = u^2$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$ .  
 (b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.  
 (c) Use the parametric equations in part (a) with  $a = 2$  and  $b = 3$  to graph the surface.  
**CAS** (d) For the case  $a = 2$ ,  $b = 3$ , use a computer algebra system to find the surface area correct to four decimal places.

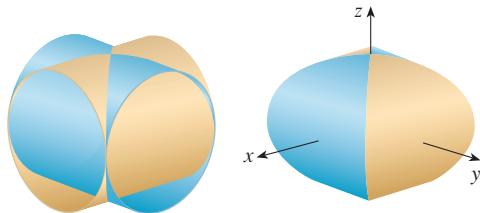
59. (a) Show that the parametric equations  $x = a \sin u \cos v$ ,  $y = b \sin u \sin v$ ,  $z = c \cos u$ ,  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ , represent an ellipsoid.  
**CAS** (b) Use the parametric equations in part (a) to graph the ellipsoid for the case  $a = 1$ ,  $b = 2$ ,  $c = 3$ .

- (c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).

60. (a) Show that the parametric equations  $x = a \cosh u \cos v$ ,  $y = b \cosh u \sin v$ ,  $z = c \sinh u$ , represent a hyperboloid of one sheet.  
**CAS** (b) Use the parametric equations in part (a) to graph the hyperboloid for the case  $a = 1$ ,  $b = 2$ ,  $c = 3$ .  
 (c) Set up, but do not evaluate, a double integral for the surface area of the part of the hyperboloid in part (b) that lies between the planes  $z = -3$  and  $z = 3$ .

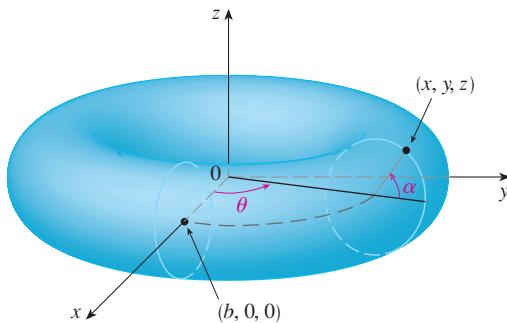
61. Find the area of the part of the sphere  $x^2 + y^2 + z^2 = 4z$  that lies inside the paraboloid  $z = x^2 + y^2$ .

62. The figure shows the surface created when the cylinder  $y^2 + z^2 = 1$  intersects the cylinder  $x^2 + z^2 = 1$ . Find the area of this surface.



63. Find the area of the part of the sphere  $x^2 + y^2 + z^2 = a^2$  that lies inside the cylinder  $x^2 + y^2 = ax$ .

64. (a) Find a parametric representation for the torus obtained by rotating about the  $z$ -axis the circle in the  $xz$ -plane with center  $(b, 0, 0)$  and radius  $a < b$ . [Hint: Take as parameters the angles  $\theta$  and  $\alpha$  shown in the figure.]  
 (b) Use the parametric equations found in part (a) to graph the torus for several values of  $a$  and  $b$ .  
 (c) Use the parametric representation from part (a) to find the surface area of the torus.



## 16.7 Surface Integrals

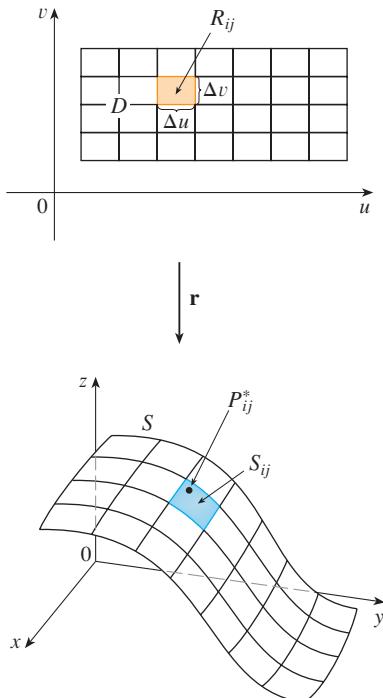
The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose  $f$  is a function of three variables whose domain includes a surface  $S$ . We will define the surface integral of  $f$  over  $S$  in such a way that, in the case where  $f(x, y, z) = 1$ , the value of the surface integral is equal to the surface area of  $S$ . We start with parametric surfaces and then deal with the special case where  $S$  is the graph of a function of two variables.

### Parametric Surfaces

Suppose that a surface  $S$  has a vector equation

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D$$

We first assume that the parameter domain  $D$  is a rectangle and we divide it into subrect-

**FIGURE 1**

We assume that the surface is covered only once as  $(u, v)$  ranges throughout  $D$ . The value of the surface integral does not depend on the parametrization that is used.

angles  $R_{ij}$  with dimensions  $\Delta u$  and  $\Delta v$ . Then the surface  $S$  is divided into corresponding patches  $S_{ij}$  as in Figure 1. We evaluate  $f$  at a point  $P_{ij}^*$  in each patch, multiply by the area  $\Delta S_{ij}$  of the patch, and form the Riemann sum

$$\sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of  $f$  over the surface  $S$**  as

**1**

$$\iint_S f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

Notice the analogy with the definition of a line integral (16.2.2) and also the analogy with the definition of a double integral (15.1.5).

To evaluate the surface integral in Equation 1 we approximate the patch area  $\Delta S_{ij}$  by the area of an approximating parallelogram in the tangent plane. In our discussion of surface area in Section 16.6 we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

where  $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \mathbf{i} + \frac{\partial \mathbf{r}}{\partial u} \mathbf{j} + \frac{\partial \mathbf{r}}{\partial u} \mathbf{k}$        $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} \mathbf{i} + \frac{\partial \mathbf{r}}{\partial v} \mathbf{j} + \frac{\partial \mathbf{r}}{\partial v} \mathbf{k}$

are the tangent vectors at a corner of  $S_{ij}$ . If the components are continuous and  $\mathbf{r}_u$  and  $\mathbf{r}_v$  are nonzero and nonparallel in the interior of  $D$ , it can be shown from Definition 1, even when  $D$  is not a rectangle, that

**2**

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$$

Observe also that

$$\iint_S 1 dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain  $D$ . When using this formula, remember that  $f(\mathbf{r}(u, v))$  is evaluated by writing  $x = x(u, v)$ ,  $y = y(u, v)$ , and  $z = z(u, v)$  in the formula for  $f(x, y, z)$ .

**EXAMPLE 1** Compute the surface integral  $\iint_S x^2 dS$ , where  $S$  is the unit sphere  $x^2 + y^2 + z^2 = 1$ .

**SOLUTION** As in Example 16.6.4, we use the parametric representation

$$x = \sin \phi \cos \theta \quad y = \sin \phi \sin \theta \quad z = \cos \phi \quad 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

that is,  $\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$

As in Example 16.6.10, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin \phi$$

Therefore, by Formula 2,

$$\begin{aligned} \iint_S x^2 dS &= \iint_D (\sin \phi \cos \theta)^2 |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA \\ &= \int_0^{2\pi} \int_0^\pi \sin^2 \phi \cos^2 \theta \sin \phi d\phi d\theta = \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\pi \sin^3 \phi d\phi \end{aligned}$$

Here we use the identities

$$\begin{aligned} \cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta) \\ \sin^2 \phi &= 1 - \cos^2 \phi \end{aligned}$$

Instead, we could use Formulas 64 and 67 in the Table of Integrals.

$$= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) d\theta \int_0^\pi (\sin \phi - \sin \phi \cos^2 \phi) d\phi$$

■

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface  $S$  and the density (mass per unit area) at the point  $(x, y, z)$  is  $\rho(x, y, z)$ , then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z) dS$$

and the **center of mass** is  $(\bar{x}, \bar{y}, \bar{z})$ , where

$$\bar{x} = \frac{1}{m} \iint_S x \rho(x, y, z) dS \quad \bar{y} = \frac{1}{m} \iint_S y \rho(x, y, z) dS \quad \bar{z} = \frac{1}{m} \iint_S z \rho(x, y, z) dS$$

Moments of inertia can also be defined as before (see Exercise 41).

### Graphs of Functions

Any surface  $S$  with equation  $z = g(x, y)$  can be regarded as a parametric surface with parametric equations

$$x = x \quad y = y \quad z = g(x, y)$$

and so we have  $\mathbf{r}_x = \mathbf{i} + \left( \frac{\partial g}{\partial x} \right) \mathbf{k}$   $\mathbf{r}_y = \mathbf{j} + \left( \frac{\partial g}{\partial y} \right) \mathbf{k}$

Thus

$$\boxed{3} \quad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}$$

and

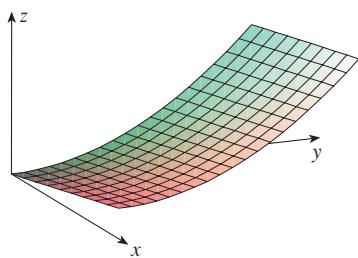
$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

$$\boxed{4} \quad \iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

Similar formulas apply when it is more convenient to project  $S$  onto the  $yz$ -plane or  $xz$ -plane. For instance, if  $S$  is a surface with equation  $y = h(x, z)$  and  $D$  is its projection onto the  $xz$ -plane, then

$$\iint_S f(x, y, z) dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1} dA$$



**FIGURE 2**

**EXAMPLE 2** Evaluate  $\iint_S y dS$ , where  $S$  is the surface  $z = x + y^2$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2$ . (See Figure 2.)

**SOLUTION** Since

$$\frac{\partial z}{\partial x} = 1 \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

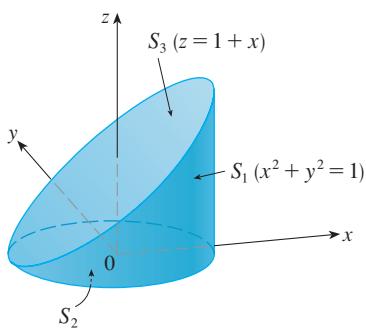
Formula 4 gives

$$\begin{aligned} \iint_S y dS &= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \int_0^1 \int_0^2 y \sqrt{1 + 1 + 4y^2} dy dx \\ &= \int_0^1 dx \sqrt{2} \int_0^2 y \sqrt{1 + 2y^2} dy \\ &= \sqrt{2} \left(\frac{1}{4}\right)_0^2 (1 + 2y^2)^{3/2} = \frac{13\sqrt{2}}{3} \end{aligned}$$

■

If  $S$  is a piecewise-smooth surface, that is, a finite union of smooth surfaces  $S_1, S_2, \dots, S_n$  that intersect only along their boundaries, then the surface integral of  $f$  over  $S$  is defined by

$$\iint_S f(x, y, z) dS = \iint_{S_1} f(x, y, z) dS + \dots + \iint_{S_n} f(x, y, z) dS$$



**FIGURE 3**

**EXAMPLE 3** Evaluate  $\iint_S z dS$ , where  $S$  is the surface whose sides  $S_1$  are given by the cylinder  $x^2 + y^2 = 1$ , whose bottom  $S_2$  is the disk  $x^2 + y^2 \leq 1$  in the plane  $z = 0$ , and whose top  $S_3$  is the part of the plane  $z = 1 + x$  that lies above  $S_2$ .

**SOLUTION** The surface  $S$  is shown in Figure 3. (We have changed the usual position of the axes to get a better look at  $S$ .) For  $S_1$  we use  $\theta$  and  $z$  as parameters (see Example 16.6.5) and write its parametric equations as

$$x = \cos \theta \quad y = \sin \theta \quad z = z$$

where

$$0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq z \leq 1 + x = 1 + \cos \theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus the surface integral over  $S_1$  is

$$\begin{aligned} \iint_{S_1} z \, dS &= \iint_D z |\mathbf{r}_\theta \times \mathbf{r}_z| \, dA \\ &= \int_0^{2\pi} \int_0^{1+\cos\theta} z \, dz \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 + \cos\theta)^2 \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [1 + 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)] \, d\theta \\ &= \frac{1}{2} \left[ \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

Since  $S_2$  lies in the plane  $z = 0$ , we have

$$\iint_{S_2} z \, dS = \iint_{S_2} 0 \, dS = 0$$

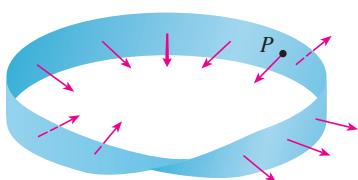
The top surface  $S_3$  lies above the unit disk  $D$  and is part of the plane  $z = 1 + x$ . So, taking  $g(x, y) = 1 + x$  in Formula 4 and converting to polar coordinates, we have

$$\begin{aligned} \iint_{S_3} z \, dS &= \iint_D (1 + x) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \\ &= \int_0^{2\pi} \int_0^1 (1 + r\cos\theta) \sqrt{1 + 1 + 0} \, r \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 (r + r^2\cos\theta) \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left( \frac{1}{2} + \frac{1}{3}\cos\theta \right) \, d\theta \\ &= \sqrt{2} \left[ \frac{\theta}{2} + \frac{\sin\theta}{3} \right]_0^{2\pi} = \sqrt{2}\pi \end{aligned}$$

Therefore

$$\begin{aligned} \iint_S z \, dS &= \iint_{S_1} z \, dS + \iint_{S_2} z \, dS + \iint_{S_3} z \, dS \\ &= \frac{3\pi}{2} + 0 + \sqrt{2}\pi = (\frac{3}{2} + \sqrt{2})\pi \end{aligned}$$





**FIGURE 4**  
A Möbius strip

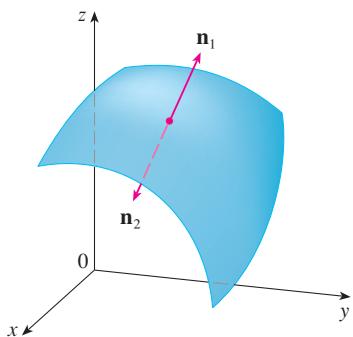
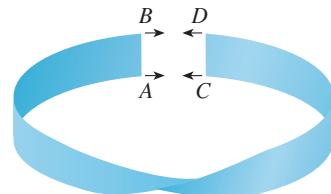
### Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5. If an ant were to crawl along the Möbius strip starting at a point  $P$ , it would end up on the “other side” of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point  $P$  without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore a Möbius strip really has only one side. You can graph the Möbius strip using the parametric equations in Exercise 16.6.32.

**TEC** Visual 16.7 shows a Möbius strip with a normal vector that can be moved along the surface.



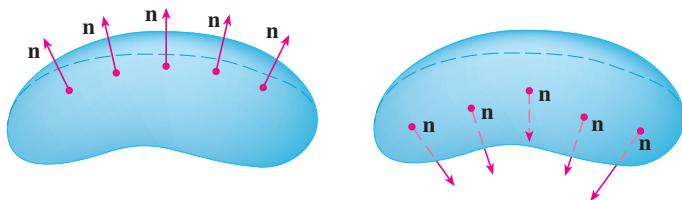
**FIGURE 5**  
Constructing a Möbius strip



**FIGURE 6**

From now on we consider only orientable (two-sided) surfaces. We start with a surface  $S$  that has a tangent plane at every point  $(x, y, z)$  on  $S$  (except at any boundary point). There are two unit normal vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2 = -\mathbf{n}_1$  at  $(x, y, z)$ . (See Figure 6.)

If it is possible to choose a unit normal vector  $\mathbf{n}$  at every such point  $(x, y, z)$  so that  $\mathbf{n}$  varies continuously over  $S$ , then  $S$  is called an **oriented surface** and the given choice of  $\mathbf{n}$  provides  $S$  with an **orientation**. There are two possible orientations for any orientable surface (see Figure 7).



**FIGURE 7**  
The two orientations  
of an orientable surface

For a surface  $z = g(x, y)$  given as the graph of  $g$ , we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} \quad (5)$$

Since the  $\mathbf{k}$ -component is positive, this gives the *upward* orientation of the surface.

If  $S$  is a smooth orientable surface given in parametric form by a vector function  $\mathbf{r}(u, v)$ , then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad (6)$$

and the opposite orientation is given by  $-\mathbf{n}$ . For instance, in Example 16.6.4 we found

the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$$

for the sphere  $x^2 + y^2 + z^2 = a^2$ . Then in Example 16.6.10 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k}$$

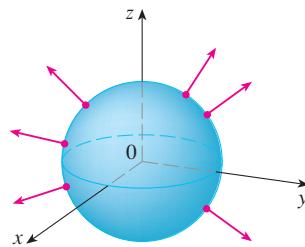
and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$$

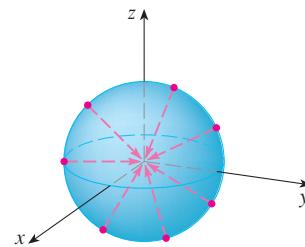
So the orientation induced by  $\mathbf{r}(\phi, \theta)$  is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k} = \frac{1}{a} \mathbf{r}(\phi, \theta)$$

Observe that  $\mathbf{n}$  points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because  $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$ .



**FIGURE 8**  
Positive orientation



**FIGURE 9**  
Negative orientation

For a **closed surface**, that is, a surface that is the boundary of a solid region  $E$ , the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from  $E$ , and inward-pointing normals give the negative orientation (see Figures 8 and 9).

### ■ Surface Integrals of Vector Fields

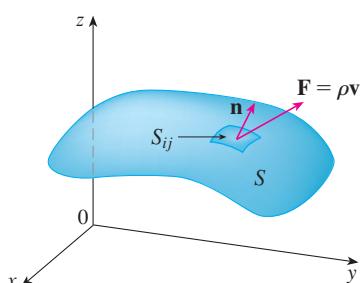
Suppose that  $S$  is an oriented surface with unit normal vector  $\mathbf{n}$ , and imagine a fluid with density  $\rho(x, y, z)$  and velocity field  $\mathbf{v}(x, y, z)$  flowing through  $S$ . (Think of  $S$  as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is  $\rho \mathbf{v}$ . If we divide  $S$  into small patches  $S_{ij}$ , as in Figure 10 (compare with Figure 1), then  $S_{ij}$  is nearly planar and so we can approximate the mass of fluid per unit time crossing  $S_{ij}$  in the direction of the normal  $\mathbf{n}$  by the quantity

$$(\rho \mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where  $\rho$ ,  $\mathbf{v}$ , and  $\mathbf{n}$  are evaluated at some point on  $S_{ij}$ . (Recall that the component of the vector  $\rho \mathbf{v}$  in the direction of the unit vector  $\mathbf{n}$  is  $\rho \mathbf{v} \cdot \mathbf{n}$ .) By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function  $\rho \mathbf{v} \cdot \mathbf{n}$  over  $S$ :

$$\boxed{7} \quad \iint_S \rho \mathbf{v} \cdot \mathbf{n} dS = \iint_S \rho(x, y, z) \mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z) dS$$

and this is interpreted physically as the rate of flow through  $S$ .



**FIGURE 10**

If we write  $\mathbf{F} = \rho \mathbf{v}$ , then  $\mathbf{F}$  is also a vector field on  $\mathbb{R}^3$  and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$

A surface integral of this form occurs frequently in physics, even when  $\mathbf{F}$  is not  $\rho \mathbf{v}$ , and is called the *surface integral* (or *flux integral*) of  $\mathbf{F}$  over  $S$ .

**8 Definition** If  $\mathbf{F}$  is a continuous vector field defined on an oriented surface  $S$  with unit normal vector  $\mathbf{n}$ , then the **surface integral of  $\mathbf{F}$  over  $S$**  is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

This integral is also called the **flux** of  $\mathbf{F}$  across  $S$ .

In words, Definition 8 says that the surface integral of a vector field over  $S$  is equal to the surface integral of its normal component over  $S$  (as previously defined).

If  $S$  is given by a vector function  $\mathbf{r}(u, v)$ , then  $\mathbf{n}$  is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} dS \\ &= \iint_D \left[ \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| dA \end{aligned}$$

where  $D$  is the parameter domain. Thus we have

Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 16.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

Figure 11 shows the vector field  $\mathbf{F}$  in Example 4 at points on the unit sphere.

**9**

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

**EXAMPLE 4** Find the flux of the vector field  $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$  across the unit sphere  $x^2 + y^2 + z^2 = 1$ .

**SOLUTION** As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k} \quad 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

Then

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos \phi \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \sin \phi \cos \theta \mathbf{k}$$

and, from Example 16.6.10,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2 \phi \cos \theta \mathbf{i} + \sin^2 \phi \sin \theta \mathbf{j} + \sin \phi \cos \phi \mathbf{k}$$

Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos \phi \sin^2 \phi \cos \theta + \sin^3 \phi \sin^2 \theta + \sin^2 \phi \cos \phi \cos \theta$$

FIGURE 11

and, by Formula 9, the flux is

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) dA \\
 &= \int_0^{2\pi} \int_0^\pi (2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta) d\phi d\theta \\
 &= 2 \int_0^\pi \sin^2 \phi \cos \phi d\phi \int_0^{2\pi} \cos \theta d\theta + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta \\
 &= 0 + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta \quad \left( \text{since } \int_0^{2\pi} \cos \theta d\theta = 0 \right) \\
 &= \frac{4\pi}{3}
 \end{aligned}$$

by the same calculation as in Example 1. ■

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer,  $4\pi/3$ , represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface  $S$  given by a graph  $z = g(x, y)$ , we can think of  $x$  and  $y$  as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left( -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right)$$

Thus Formula 9 becomes

10

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

This formula assumes the upward orientation of  $S$ ; for a downward orientation we multiply by  $-1$ . Similar formulas can be worked out if  $S$  is given by  $y = h(x, z)$  or  $x = k(y, z)$ . (See Exercises 37 and 38.)

**EXAMPLE 5** Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$  and  $S$  is the boundary of the solid region  $E$  enclosed by the paraboloid  $z = 1 - x^2 - y^2$  and the plane  $z = 0$ .

**SOLUTION**  $S$  consists of a parabolic top surface  $S_1$  and a circular bottom surface  $S_2$ . (See Figure 12.) Since  $S$  is a closed surface, we use the convention of positive (outward) orientation. This means that  $S_1$  is oriented upward and we can use Equation 10 with  $D$  being the projection of  $S_1$  onto the  $xy$ -plane, namely, the disk  $x^2 + y^2 \leq 1$ . Since

$$P(x, y, z) = y \quad Q(x, y, z) = x \quad R(x, y, z) = z = 1 - x^2 - y^2$$

on  $S_1$  and

$$\frac{\partial g}{\partial x} = -2x \quad \frac{\partial g}{\partial y} = -2y$$

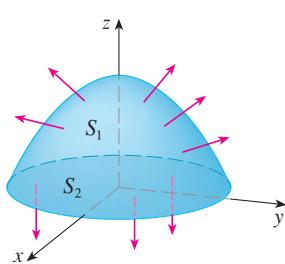


FIGURE 12

we have

$$\begin{aligned}
 \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA \\
 &= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2] dA \\
 &= \iint_D (1 + 4xy - x^2 - y^2) dA \\
 &= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos \theta \sin \theta - r^2) r dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos \theta \sin \theta) dr d\theta \\
 &= \int_0^{2\pi} \left( \frac{1}{4} + \cos \theta \sin \theta \right) d\theta = \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
 \end{aligned}$$

The disk  $S_2$  is oriented downward, so its unit normal vector is  $\mathbf{n} = -\mathbf{k}$  and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k}) dS = \iint_D (-z) dA = \iint_D 0 dA = 0$$

since  $z = 0$  on  $S_2$ . Finally, we compute, by definition,  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  as the sum of the surface integrals of  $\mathbf{F}$  over the pieces  $S_1$  and  $S_2$ :

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2} \quad \blacksquare$$

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if  $\mathbf{E}$  is an electric field (see Example 16.1.5), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of  $\mathbf{E}$  through the surface  $S$ . One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface  $S$  is

$$Q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} \quad (11)$$

where  $\epsilon_0$  is a constant (called the permittivity of free space) that depends on the units used. (In the SI system,  $\epsilon_0 \approx 8.8542 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$ .) Therefore, if the vector field  $\mathbf{F}$  in Example 4 represents an electric field, we can conclude that the charge enclosed by  $S$  is  $Q = \frac{4}{3}\pi\epsilon_0$ .

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point  $(x, y, z)$  in a body is  $u(x, y, z)$ . Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K \nabla u$$

where  $K$  is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface  $S$  in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

**EXAMPLE 6** The temperature  $u$  in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere  $S$  of radius  $a$  with center at the center of the ball.

**SOLUTION** Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where  $C$  is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K \nabla u = -KC(2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k})$$

where  $K$  is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere  $x^2 + y^2 + z^2 = a^2$  at the point  $(x, y, z)$  is

$$\mathbf{n} = \frac{1}{a}(x \mathbf{i} + y \mathbf{j} + z \mathbf{k})$$

and so

$$\mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

But on  $S$  we have  $x^2 + y^2 + z^2 = a^2$ , so  $\mathbf{F} \cdot \mathbf{n} = -2aKC$ . Therefore the rate of heat flow across  $S$  is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} dS = -2aKC \iint_S dS \\ &= -2aKC A(S) = -2aKC(4\pi a^2) = -8KC\pi a^3 \end{aligned}$$

■

## 16.7 EXERCISES

- Let  $S$  be the surface of the box enclosed by the planes  $x = \pm 1$ ,  $y = \pm 1$ ,  $z = \pm 1$ . Approximate  $\iint_S \cos(x + 2y + 3z) dS$  by using a Riemann sum as in Definition 1, taking the patches  $S_{ij}$  to be the squares that are the faces of the box  $S$  and the points  $P_{ij}^*$  to be the centers of the squares.
- A surface  $S$  consists of the cylinder  $x^2 + y^2 = 1$ ,  $-1 \leq z \leq 1$ , together with its top and bottom disks. Suppose you know that  $f$  is a continuous function with
 
$$f(\pm 1, 0, 0) = 2 \quad f(0, \pm 1, 0) = 3 \quad f(0, 0, \pm 1) = 4$$

Estimate the value of  $\iint_S f(x, y, z) dS$  by using a Riemann sum, taking the patches  $S_{ij}$  to be four quarter-cylinders and the top and bottom disks.

- Let  $H$  be the hemisphere  $x^2 + y^2 + z^2 = 50$ ,  $z \geq 0$ , and suppose  $f$  is a continuous function with  $f(3, 4, 5) = 7$ ,  $f(3, -4, 5) = 8$ ,  $f(-3, 4, 5) = 9$ , and  $f(-3, -4, 5) = 12$ . By dividing  $H$  into four patches, estimate the value of  $\iint_H f(x, y, z) dS$ .
- Suppose that  $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$ , where  $g$  is a function of one variable such that  $g(2) = -5$ . Evaluate  $\iint_S f(x, y, z) dS$ , where  $S$  is the sphere  $x^2 + y^2 + z^2 = 4$ .

**5–20** Evaluate the surface integral.

- $\iint_S (x + y + z) dS$ ,  
 $S$  is the parallelogram with parametric equations  $x = u + v$ ,  $y = u - v$ ,  $z = 1 + 2u + v$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 1$

6.  $\iint_S xyz \, dS$ ,

$S$  is the cone with parametric equations  $x = u \cos v$ ,  $y = u \sin v$ ,  $z = u$ ,  $0 \leq u \leq 1$ ,  $0 \leq v \leq \pi/2$

7.  $\iint_S y \, dS$ ,  $S$  is the helicoid with vector equation  $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$ ,  $0 \leq u \leq 1$ ,  $0 \leq v \leq \pi$

8.  $\iint_S (x^2 + y^2) \, dS$ ,

$S$  is the surface with vector equation  $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$ ,  $u^2 + v^2 \leq 1$

9.  $\iint_S x^2yz \, dS$ ,

$S$  is the part of the plane  $z = 1 + 2x + 3y$  that lies above the rectangle  $[0, 3] \times [0, 2]$

10.  $\iint_S xz \, dS$ ,

$S$  is the part of the plane  $2x + 2y + z = 4$  that lies in the first octant

11.  $\iint_S x \, dS$ ,

$S$  is the triangular region with vertices  $(1, 0, 0)$ ,  $(0, -2, 0)$ , and  $(0, 0, 4)$

12.  $\iint_S y \, dS$ ,

$S$  is the surface  $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$

13.  $\iint_S z^2 \, dS$ ,

$S$  is the part of the paraboloid  $x = y^2 + z^2$  given by  $0 \leq x \leq 1$

14.  $\iint_S y^2z^2 \, dS$ ,

$S$  is the part of the cone  $y = \sqrt{x^2 + z^2}$  given by  $0 \leq y \leq 5$

15.  $\iint_S x \, dS$ ,

$S$  is the surface  $y = x^2 + 4z$ ,  $0 \leq x \leq 1$ ,  $0 \leq z \leq 1$

16.  $\iint_S y^2 \, dS$ ,

$S$  is the part of the sphere  $x^2 + y^2 + z^2 = 1$  that lies above the cone  $z = \sqrt{x^2 + y^2}$

17.  $\iint_S (x^2z + y^2z) \, dS$ ,

$S$  is the hemisphere  $x^2 + y^2 + z^2 = 4$ ,  $z \geq 0$

18.  $\iint_S (x + y + z) \, dS$ ,

$S$  is the part of the half-cylinder  $x^2 + z^2 = 1$ ,  $z \geq 0$ , that lies between the planes  $y = 0$  and  $y = 2$

19.  $\iint_S xz \, dS$ ,

$S$  is the boundary of the region enclosed by the cylinder  $y^2 + z^2 = 9$  and the planes  $x = 0$  and  $x + y = 5$

20.  $\iint_S (x^2 + y^2 + z^2) \, dS$ ,

$S$  is the part of the cylinder  $x^2 + y^2 = 9$  between the planes  $z = 0$  and  $z = 2$ , together with its top and bottom disks

**21–32** Evaluate the surface integral  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  for the given vector field  $\mathbf{F}$  and the oriented surface  $S$ . In other words, find the flux of  $\mathbf{F}$  across  $S$ . For closed surfaces, use the positive (outward) orientation.

21.  $\mathbf{F}(x, y, z) = ze^{xy} \mathbf{i} - 3ze^{xy} \mathbf{j} + xy \mathbf{k}$ ,

$S$  is the parallelogram of Exercise 5 with upward orientation

22.  $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$ ,

$S$  is the helicoid of Exercise 7 with upward orientation

23.  $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$ ,  $S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$  that lies above the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , and has upward orientation

24.  $\mathbf{F}(x, y, z) = -x \mathbf{i} - y \mathbf{j} + z^3 \mathbf{k}$ ,  $S$  is the part of the cone  $z = \sqrt{x^2 + y^2}$  between the planes  $z = 1$  and  $z = 3$  with downward orientation

25.  $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z^2 \mathbf{k}$ ,  $S$  is the sphere with radius 1 and center the origin

26.  $\mathbf{F}(x, y, z) = y \mathbf{i} - x \mathbf{j} + 2z \mathbf{k}$ ,  $S$  is the hemisphere  $x^2 + y^2 + z^2 = 4$ ,  $z \geq 0$ , oriented downward

27.  $\mathbf{F}(x, y, z) = y \mathbf{j} - z \mathbf{k}$ ,  
 $S$  consists of the paraboloid  $y = x^2 + z^2$ ,  $0 \leq y \leq 1$ , and the disk  $x^2 + z^2 \leq 1$ ,  $y = 1$

28.  $\mathbf{F}(x, y, z) = yz \mathbf{i} + zx \mathbf{j} + xy \mathbf{k}$ ,

$S$  is the surface  $z = x \sin y$ ,  $0 \leq x \leq 2$ ,  $0 \leq y \leq \pi$ , with upward orientation

29.  $\mathbf{F}(x, y, z) = x \mathbf{i} + 2y \mathbf{j} + 3z \mathbf{k}$ ,

$S$  is the cube with vertices  $(\pm 1, \pm 1, \pm 1)$

30.  $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + 5 \mathbf{k}$ ,  $S$  is the boundary of the region enclosed by the cylinder  $x^2 + z^2 = 1$  and the planes  $y = 0$  and  $x + y = 2$

31.  $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$ ,  $S$  is the boundary of the solid half-cylinder  $0 \leq z \leq \sqrt{1 - y^2}$ ,  $0 \leq x \leq 2$

32.  $\mathbf{F}(x, y, z) = y \mathbf{i} + (z - y) \mathbf{j} + x \mathbf{k}$ ,

$S$  is the surface of the tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$

**CAS** 33. Evaluate  $\iint_S (x^2 + y^2 + z^2) \, dS$  correct to four decimal places, where  $S$  is the surface  $z = xe^y$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ .

**CAS** 34. Find the exact value of  $\iint_S xyz \, dS$ , where  $S$  is the surface  $z = x^2y^2$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2$ .

**CAS** 35. Find the value of  $\iint_S x^2y^2z^2 \, dS$  correct to four decimal places, where  $S$  is the part of the paraboloid  $z = 3 - 2x^2 - y^2$  that lies above the  $xy$ -plane.

**CAS** 36. Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz) \mathbf{i} + x^2y \mathbf{j} + z^2e^{x/5} \mathbf{k}$$

across the part of the cylinder  $4y^2 + z^2 = 4$  that lies above the  $xy$ -plane and between the planes  $x = -2$  and  $x = 2$  with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

37. Find a formula for  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  similar to Formula 10 for the case where  $S$  is given by  $y = h(x, z)$  and  $\mathbf{n}$  is the unit normal that points toward the left.

- 38.** Find a formula for  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  similar to Formula 10 for the case where  $S$  is given by  $x = k(y, z)$  and  $\mathbf{n}$  is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).
- 39.** Find the center of mass of the hemisphere  $x^2 + y^2 + z^2 = a^2$ ,  $z \geq 0$ , if it has constant density.
- 40.** Find the mass of a thin funnel in the shape of a cone  $z = \sqrt{x^2 + y^2}$ ,  $1 \leq z \leq 4$ , if its density function is  $\rho(x, y, z) = 10 - z$ .
- 41.** (a) Give an integral expression for the moment of inertia  $I_z$  about the  $z$ -axis of a thin sheet in the shape of a surface  $S$  if the density function is  $\rho$ .  
 (b) Find the moment of inertia about the  $z$ -axis of the funnel in Exercise 40.
- 42.** Let  $S$  be the part of the sphere  $x^2 + y^2 + z^2 = 25$  that lies above the plane  $z = 4$ . If  $S$  has constant density  $k$ , find (a) the center of mass and (b) the moment of inertia about the  $z$ -axis.
- 43.** A fluid has density  $870 \text{ kg/m}^3$  and flows with velocity  $\mathbf{v} = z \mathbf{i} + y^2 \mathbf{j} + x^2 \mathbf{k}$ , where  $x$ ,  $y$ , and  $z$  are measured in meters and the components of  $\mathbf{v}$  in meters per second. Find the rate of flow outward through the cylinder  $x^2 + y^2 = 4$ ,  $0 \leq z \leq 1$ .
- 44.** Seawater has density  $1025 \text{ kg/m}^3$  and flows in a velocity field  $\mathbf{v} = y \mathbf{i} + x \mathbf{j}$ , where  $x$ ,  $y$ , and  $z$  are measured in meters and the components of  $\mathbf{v}$  in meters per second. Find the rate of flow outward through the hemisphere  $x^2 + y^2 + z^2 = 9$ ,  $z \geq 0$ .
- 45.** Use Gauss's Law to find the charge contained in the solid hemisphere  $x^2 + y^2 + z^2 \leq a^2$ ,  $z \geq 0$ , if the electric field is  $\mathbf{E}(x, y, z) = x \mathbf{i} + y \mathbf{j} + 2z \mathbf{k}$
- 46.** Use Gauss's Law to find the charge enclosed by the cube with vertices  $(\pm 1, \pm 1, \pm 1)$  if the electric field is  $\mathbf{E}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$
- 47.** The temperature at the point  $(x, y, z)$  in a substance with conductivity  $K = 6.5$  is  $u(x, y, z) = 2y^2 + 2z^2$ . Find the rate of heat flow inward across the cylindrical surface  $y^2 + z^2 = 6$ ,  $0 \leq x \leq 4$ .
- 48.** The temperature at a point in a ball with conductivity  $K$  is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere  $S$  of radius  $a$  with center at the center of the ball.
- 49.** Let  $\mathbf{F}$  be an inverse square field, that is,  $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$  for some constant  $c$ , where  $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ . Show that the flux of  $\mathbf{F}$  across a sphere  $S$  with center the origin is independent of the radius of  $S$ .

## 16.8 Stokes' Theorem

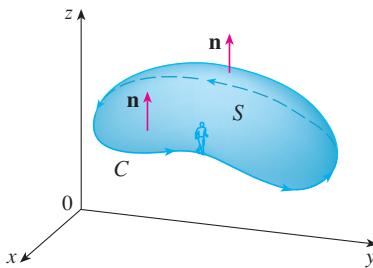


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region  $D$  to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface  $S$  to a line integral around the boundary curve of  $S$  (which is a space curve). Figure 1 shows an oriented surface with unit normal vector  $\mathbf{n}$ . The orientation of  $S$  induces the **positive orientation of the boundary curve  $C$**  shown in the figure. This means that if you walk in the positive direction around  $C$  with your head pointing in the direction of  $\mathbf{n}$ , then the surface will always be on your left.

**Stokes' Theorem** Let  $S$  be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve  $C$  with positive orientation. Let  $\mathbf{F}$  be a vector field whose components have continuous partial derivatives on an open region in  $\mathbb{R}^3$  that contains  $S$ . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds \quad \text{and} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS$$

**George Stokes**

Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students was able to do so.

Stokes' Theorem says that the line integral around the boundary curve of  $S$  of the tangential component of  $\mathbf{F}$  is equal to the surface integral over  $S$  of the normal component of the curl of  $\mathbf{F}$ .

The positively oriented boundary curve of the oriented surface  $S$  is often written as  $\partial S$ , so Stokes' Theorem can be expressed as

$$\boxed{1} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

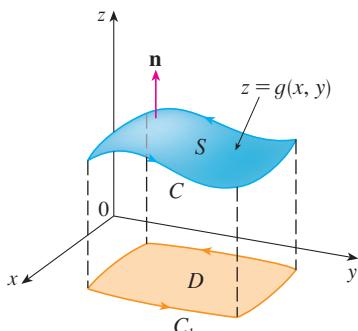
There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that  $\operatorname{curl} \mathbf{F}$  is a sort of derivative of  $\mathbf{F}$ ) and the right side involves the values of  $\mathbf{F}$  only on the *boundary* of  $S$ .

In fact, in the special case where the surface  $S$  is flat and lies in the  $xy$ -plane with upward orientation, the unit normal is  $\mathbf{k}$ , the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when  $S$  is a graph and  $\mathbf{F}$ ,  $S$ , and  $C$  are well behaved.



**FIGURE 2**

**PROOF OF A SPECIAL CASE OF STOKES' THEOREM** We assume that the equation of  $S$  is  $z = g(x, y)$ ,  $(x, y) \in D$ , where  $g$  has continuous second-order partial derivatives and  $D$  is a simple plane region whose boundary curve  $C_1$  corresponds to  $C$ . If the orientation of  $S$  is upward, then the positive orientation of  $C$  corresponds to the positive orientation of  $C_1$ . (See Figure 2.) We are also given that  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ , where the partial derivatives of  $P$ ,  $Q$ , and  $R$  are continuous.

Since  $S$  is a graph of a function, we can apply Formula 16.7.10 with  $\mathbf{F}$  replaced by  $\operatorname{curl} \mathbf{F}$ . The result is

$$\boxed{2} \quad \begin{aligned} & \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_D \left[ -\left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA \end{aligned}$$

where the partial derivatives of  $P$ ,  $Q$ , and  $R$  are evaluated at  $(x, y, g(x, y))$ . If

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

is a parametric representation of  $C_1$ , then a parametric representation of  $C$  is

$$x = x(t) \quad y = y(t) \quad z = g(x(t), y(t)) \quad a \leq t \leq b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt \\
 &= \int_a^b \left[ P \frac{dx}{dt} + Q \frac{dy}{dt} + R \left( \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right) \right] dt \\
 &= \int_a^b \left[ \left( P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left( Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt \\
 &= \int_{C_1} \left( P + R \frac{\partial z}{\partial x} \right) dx + \left( Q + R \frac{\partial z}{\partial y} \right) dy \\
 &= \iint_D \left[ \frac{\partial}{\partial x} \left( Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left( P + R \frac{\partial z}{\partial x} \right) \right] dA
 \end{aligned}$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that  $P$ ,  $Q$ , and  $R$  are functions of  $x$ ,  $y$ , and  $z$  and that  $z$  is itself a function of  $x$  and  $y$ , we get

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left[ \left( \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial x \partial y} \right) \right. \\
 &\quad \left. - \left( \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial y \partial x} \right) \right] dA
 \end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

**EXAMPLE 1** Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = -y^2 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$  and  $C$  is the curve of intersection of the plane  $y + z = 2$  and the cylinder  $x^2 + y^2 = 1$ . (Orient  $C$  to be counterclockwise when viewed from above.)

**SOLUTION** The curve  $C$  (an ellipse) is shown in Figure 3. Although  $\int_C \mathbf{F} \cdot d\mathbf{r}$  could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y) \mathbf{k}$$

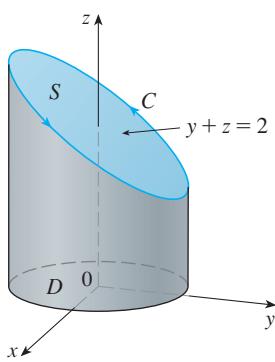
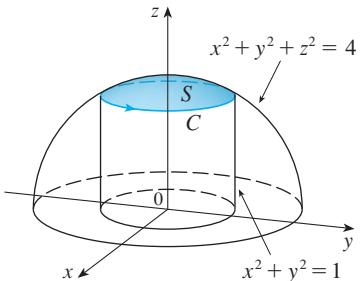


FIGURE 3

Although there are many surfaces with boundary  $C$ , the most convenient choice is the elliptical region  $S$  in the plane  $y + z = 2$  that is bounded by  $C$ . If we orient  $S$  upward, then  $C$  has the induced positive orientation. The projection  $D$  of  $S$  onto the  $xy$ -plane is

the disk  $x^2 + y^2 \leq 1$  and so using Equation 16.7.10 with  $z = g(x, y) = 2 - y$ , we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y) dA \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{r^2}{2} + 2 \frac{r^3}{3} \sin \theta \right]_0^1 d\theta = \int_0^{2\pi} \left( \frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta \\ &= \frac{1}{2}(2\pi) + 0 = \pi\end{aligned}$$



**FIGURE 4**

**EXAMPLE 2** Use Stokes' Theorem to compute the integral  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = xz \mathbf{i} + yz \mathbf{j} + xy \mathbf{k}$  and  $S$  is the part of the sphere  $x^2 + y^2 + z^2 = 4$  that lies inside the cylinder  $x^2 + y^2 = 1$  and above the  $xy$ -plane. (See Figure 4.)

**SOLUTION** To find the boundary curve  $C$  we solve the equations  $x^2 + y^2 + z^2 = 4$  and  $x^2 + y^2 = 1$ . Subtracting, we get  $z^2 = 3$  and so  $z = \sqrt{3}$  (since  $z > 0$ ). Thus  $C$  is the circle given by the equations  $x^2 + y^2 = 1$ ,  $z = \sqrt{3}$ . A vector equation of  $C$  is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sqrt{3} \mathbf{k} \quad 0 \leq t \leq 2\pi$$

so

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3} \cos t \mathbf{i} + \sqrt{3} \sin t \mathbf{j} + \cos t \sin t \mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t) dt \\ &= \sqrt{3} \int_0^{2\pi} 0 dt = 0\end{aligned}$$

Note that in Example 2 we computed a surface integral simply by knowing the values of  $\mathbf{F}$  on the boundary curve  $C$ . This means that if we have another oriented surface with the same boundary curve  $C$ , then we get exactly the same value for the surface integral!

In general, if  $S_1$  and  $S_2$  are oriented surfaces with the same oriented boundary curve  $C$  and both satisfy the hypotheses of Stokes' Theorem, then

$$3 \quad \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

We now use Stokes' Theorem to throw some light on the meaning of the curl vector. Suppose that  $C$  is an oriented closed curve and  $\mathbf{v}$  represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T} ds$$

and recall that  $\mathbf{v} \cdot \mathbf{T}$  is the component of  $\mathbf{v}$  in the direction of the unit tangent vector  $\mathbf{T}$ . This means that the closer the direction of  $\mathbf{v}$  is to the direction of  $\mathbf{T}$ , the larger the value of  $\mathbf{v} \cdot \mathbf{T}$ . Thus  $\int_C \mathbf{v} \cdot d\mathbf{r}$  is a measure of the tendency of the fluid to move around  $C$  and is called the **circulation** of  $\mathbf{v}$  around  $C$ . (See Figure 5.)

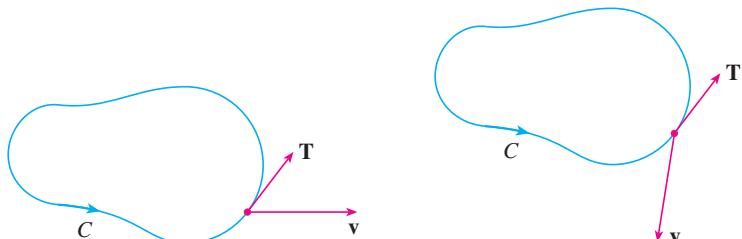


FIGURE 5

(a)  $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$ , positive circulation(b)  $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$ , negative circulation

Now let  $P_0(x_0, y_0, z_0)$  be a point in the fluid and let  $S_a$  be a small disk with radius  $a$  and center  $P_0$ . Then  $(\text{curl } \mathbf{F})(P) \approx (\text{curl } \mathbf{F})(P_0)$  for all points  $P$  on  $S_a$  because  $\text{curl } \mathbf{F}$  is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle  $C_a$ :

$$\begin{aligned}\int_{C_a} \mathbf{v} \cdot d\mathbf{r} &= \iint_{S_a} \text{curl } \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl } \mathbf{v} \cdot \mathbf{n} dS \\ &\approx \iint_{S_a} \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) dS = \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \pi a^2\end{aligned}$$

Imagine a tiny paddle wheel placed in the fluid at a point  $P$ , as in Figure 6; the paddle wheel rotates fastest when its axis is parallel to  $\text{curl } \mathbf{v}$ .

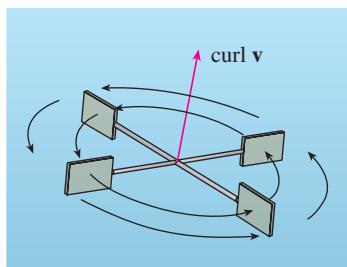


FIGURE 6

This approximation becomes better as  $a \rightarrow 0$  and we have

$$4 \quad \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \rightarrow 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that  $\text{curl } \mathbf{v} \cdot \mathbf{n}$  is a measure of the rotating effect of the fluid about the axis  $\mathbf{n}$ . The curling effect is greatest about the axis parallel to  $\text{curl } \mathbf{v}$ .

Finally, we mention that Stokes' Theorem can be used to prove Theorem 16.5.4 (which states that if  $\text{curl } \mathbf{F} = \mathbf{0}$  on all of  $\mathbb{R}^3$ , then  $\mathbf{F}$  is conservative). From our previous work (Theorems 16.3.3 and 16.3.4), we know that  $\mathbf{F}$  is conservative if  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed path  $C$ . Given  $C$ , suppose we can find an orientable surface  $S$  whose boundary is  $C$ . (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

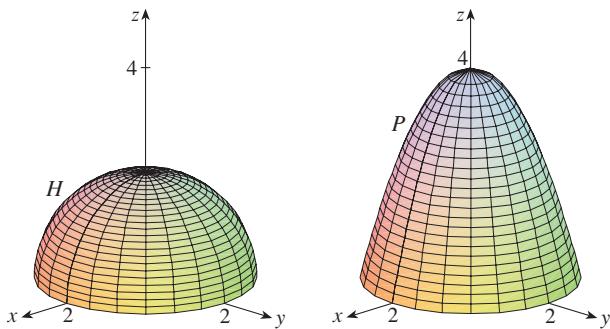
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed curve  $C$ .

## 16.8 EXERCISES

1. A hemisphere  $H$  and a portion  $P$  of a paraboloid are shown. Suppose  $\mathbf{F}$  is a vector field on  $\mathbb{R}^3$  whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



- 2–6 Use Stokes' Theorem to evaluate  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ .

2.  $\mathbf{F}(x, y, z) = x^2 \sin z \mathbf{i} + y^2 \mathbf{j} + xy \mathbf{k}$ ,  
 $S$  is the part of the paraboloid  $z = 1 - x^2 - y^2$  that lies above the  $xy$ -plane, oriented upward
3.  $\mathbf{F}(x, y, z) = ze^y \mathbf{i} + x \cos y \mathbf{j} + xz \sin y \mathbf{k}$ ,  
 $S$  is the hemisphere  $x^2 + y^2 + z^2 = 16$ ,  $y \geq 0$ , oriented in the direction of the positive  $y$ -axis
4.  $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2) \mathbf{i} + x^2y \mathbf{j} + x^2z^2 \mathbf{k}$ ,  
 $S$  is the cone  $x = \sqrt{y^2 + z^2}$ ,  $0 \leq x \leq 2$ , oriented in the direction of the positive  $x$ -axis
5.  $\mathbf{F}(x, y, z) = xyz \mathbf{i} + xy \mathbf{j} + x^2yz \mathbf{k}$ ,  
 $S$  consists of the top and the four sides (but not the bottom) of the cube with vertices  $(\pm 1, \pm 1, \pm 1)$ , oriented outward
6.  $\mathbf{F}(x, y, z) = e^{xy} \mathbf{i} + e^{xz} \mathbf{j} + x^2z \mathbf{k}$ ,  
 $S$  is the half of the ellipsoid  $4x^2 + y^2 + 4z^2 = 4$  that lies to the right of the  $xz$ -plane, oriented in the direction of the positive  $y$ -axis

- 7–10 Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ . In each case  $C$  is oriented counterclockwise as viewed from above.

7.  $\mathbf{F}(x, y, z) = (x + y^2) \mathbf{i} + (y + z^2) \mathbf{j} + (z + x^2) \mathbf{k}$ ,  
 $C$  is the triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$
8.  $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz) \mathbf{j} + (xy - \sqrt{z}) \mathbf{k}$ ,  
 $C$  is the boundary of the part of the plane  $3x + 2y + z = 1$  in the first octant
9.  $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$ ,  $C$  is the boundary of the part of the paraboloid  $z = 1 - x^2 - y^2$  in the first octant

10.  $\mathbf{F}(x, y, z) = 2y \mathbf{i} + xz \mathbf{j} + (x + y) \mathbf{k}$ ,  $C$  is the curve of intersection of the plane  $z = y + 2$  and the cylinder  $x^2 + y^2 = 1$

11. (a) Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where

$$\mathbf{F}(x, y, z) = x^2z \mathbf{i} + xy^2 \mathbf{j} + z^2 \mathbf{k}$$

and  $C$  is the curve of intersection of the plane  $x + y + z = 1$  and the cylinder  $x^2 + y^2 = 9$ , oriented counterclockwise as viewed from above.

- (b) Graph both the plane and the cylinder with domains chosen so that you can see the curve  $C$  and the surface that you used in part (a).
- (c) Find parametric equations for  $C$  and use them to graph  $C$ .

12. (a) Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = x^2y \mathbf{i} + \frac{1}{3}x^3 \mathbf{j} + xy \mathbf{k}$  and  $C$  is the curve of intersection of the hyperbolic paraboloid  $z = y^2 - x^2$  and the cylinder  $x^2 + y^2 = 1$ , oriented counterclockwise as viewed from above.
- (b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve  $C$  and the surface that you used in part (a).
- (c) Find parametric equations for  $C$  and use them to graph  $C$ .

- 13–15 Verify that Stokes' Theorem is true for the given vector field  $\mathbf{F}$  and surface  $S$ .

13.  $\mathbf{F}(x, y, z) = -y \mathbf{i} + x \mathbf{j} - 2 \mathbf{k}$ ,  
 $S$  is the cone  $z^2 = x^2 + y^2$ ,  $0 \leq z \leq 4$ , oriented downward
14.  $\mathbf{F}(x, y, z) = -2yz \mathbf{i} + y \mathbf{j} + 3x \mathbf{k}$ ,  
 $S$  is the part of the paraboloid  $z = 5 - x^2 - y^2$  that lies above the plane  $z = 1$ , oriented upward
15.  $\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$ ,  
 $S$  is the hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $y \geq 0$ , oriented in the direction of the positive  $y$ -axis

16. Let  $C$  be a simple closed smooth curve that lies in the plane  $x + y + z = 1$ . Show that the line integral

$$\int_C z \, dx - 2x \, dy + 3y \, dz$$

depends only on the area of the region enclosed by  $C$  and not on the shape of  $C$  or its location in the plane.

17. A particle moves along line segments from the origin to the points  $(1, 0, 0)$ ,  $(1, 2, 1)$ ,  $(0, 2, 1)$ , and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2 \mathbf{i} + 2xy \mathbf{j} + 4y^2 \mathbf{k}$$

Find the work done.

**18.** Evaluate

$$\int_C (y + \sin x) dx + (z^2 + \cos y) dy + x^3 dz$$

where  $C$  is the curve  $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$ ,  $0 \leq t \leq 2\pi$ .  
 [Hint: Observe that  $C$  lies on the surface  $z = 2xy$ .]

**19.** If  $S$  is a sphere and  $\mathbf{F}$  satisfies the hypotheses of Stokes'

Theorem, show that  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ .

**20.** Suppose  $S$  and  $C$  satisfy the hypotheses of Stokes' Theorem and  $f, g$  have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

$$(a) \int_C (f \nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$$

$$(b) \int_C (f \nabla f) \cdot d\mathbf{r} = 0$$

$$(c) \int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$$

## WRITING PROJECT

## THREE MEN AND TWO THEOREMS

The photograph shows a stained-glass window at Cambridge University in honor of George Green.



Courtesy of the Masters and Fellows of Gonville and Caius College, Cambridge University, England

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 1097 and 1135.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.

The dictionary edited by Gillispie [2] is a good source for both biographical and scientific information. The book by Hutchinson [5] gives an account of Stokes' life and the book by Thompson [8] is a biography of Lord Kelvin. The articles by Grattan-Guinness [3] and Gray [4] and the book by Cannell [1] give background on the extraordinary life and works of Green. Additional historical and mathematical information is found in the books by Katz [6] and Kline [7].

1. D. M. Cannell, *George Green, Mathematician and Physicist 1793–1841: The Background to His Life and Work* (Philadelphia: Society for Industrial and Applied Mathematics, 2001).
2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.
3. I. Grattan-Guinness, "Why Did George Green Write his Essay of 1828 on Electricity and Magnetism?" *Amer. Math. Monthly*, Vol. 102 (1995), pp. 387–96.
4. J. Gray, "There Was a Jolly Miller." *The New Scientist*, Vol. 139 (1993), pp. 24–27.
5. G. E. Hutchinson, *The Enchanted Voyage and Other Studies* (Westport, CT: Greenwood Press, 1978).
6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 678–80.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 683–85.
8. Sylvanus P. Thompson, *The Life of Lord Kelvin* (New York: Chelsea, 1976).

## 16.9 The Divergence Theorem

In Section 16.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

where  $C$  is the positively oriented boundary curve of the plane region  $D$ . If we were seeking to extend this theorem to vector fields on  $\mathbb{R}^3$ , we might make the guess that

$$1 \quad \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z) \, dV$$

where  $S$  is the boundary surface of the solid region  $E$ . It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function ( $\operatorname{div} \mathbf{F}$  in this case) over a region to the integral of the original function  $\mathbf{F}$  over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 15.6. We state and prove the Divergence Theorem for regions  $E$  that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of  $E$  is a closed surface, and we use the convention, introduced in Section 16.7, that the positive orientation is outward; that is, the unit normal vector  $\mathbf{n}$  is directed outward from  $E$ .

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

**The Divergence Theorem** Let  $E$  be a simple solid region and let  $S$  be the boundary surface of  $E$ , given with positive (outward) orientation. Let  $\mathbf{F}$  be a vector field whose component functions have continuous partial derivatives on an open region that contains  $E$ . Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of  $\mathbf{F}$  across the boundary surface of  $E$  is equal to the triple integral of the divergence of  $\mathbf{F}$  over  $E$ .

**PROOF** Let  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ . Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so 
$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E \frac{\partial P}{\partial x} \, dV + \iiint_E \frac{\partial Q}{\partial y} \, dV + \iiint_E \frac{\partial R}{\partial z} \, dV$$

If  $\mathbf{n}$  is the unit outward normal of  $S$ , then the surface integral on the left side of the Divergence Theorem is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S (P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}) \cdot \mathbf{n} \, dS \\ &= \iint_S P \mathbf{i} \cdot \mathbf{n} \, dS + \iint_S Q \mathbf{j} \cdot \mathbf{n} \, dS + \iint_S R \mathbf{k} \cdot \mathbf{n} \, dS \end{aligned}$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three

equations:

$$\boxed{2} \quad \iint_S P \mathbf{i} \cdot \mathbf{n} dS = \iiint_E \frac{\partial P}{\partial x} dV$$

$$\boxed{3} \quad \iint_S Q \mathbf{j} \cdot \mathbf{n} dS = \iiint_E \frac{\partial Q}{\partial y} dV$$

$$\boxed{4} \quad \iint_S R \mathbf{k} \cdot \mathbf{n} dS = \iiint_E \frac{\partial R}{\partial z} dV$$

To prove Equation 4 we use the fact that  $E$  is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where  $D$  is the projection of  $E$  onto the  $xy$ -plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z} dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} \frac{\partial R}{\partial z}(x, y, z) dz \right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

$$\boxed{5} \quad \iiint_E \frac{\partial R}{\partial z} dV = \iint_D [R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))] dA$$

The boundary surface  $S$  consists of three pieces: the bottom surface  $S_1$ , the top surface  $S_2$ , and possibly a vertical surface  $S_3$ , which lies above the boundary curve of  $D$ . (See Figure 1. It might happen that  $S_3$  doesn't appear, as in the case of a sphere.) Notice that on  $S_3$  we have  $\mathbf{k} \cdot \mathbf{n} = 0$ , because  $\mathbf{k}$  is vertical and  $\mathbf{n}$  is horizontal, and so

$$\iint_{S_3} R \mathbf{k} \cdot \mathbf{n} dS = \iint_{S_3} 0 dS = 0$$

Thus, regardless of whether there is a vertical surface, we can write

$$\boxed{6} \quad \iint_S R \mathbf{k} \cdot \mathbf{n} dS = \iint_{S_1} R \mathbf{k} \cdot \mathbf{n} dS + \iint_{S_2} R \mathbf{k} \cdot \mathbf{n} dS$$

The equation of  $S_2$  is  $z = u_2(x, y)$ ,  $(x, y) \in D$ , and the outward normal  $\mathbf{n}$  points upward, so from Equation 16.7.10 (with  $\mathbf{F}$  replaced by  $R \mathbf{k}$ ) we have

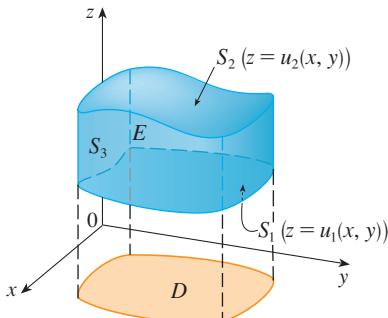
$$\iint_{S_2} R \mathbf{k} \cdot \mathbf{n} dS = \iint_D R(x, y, u_2(x, y)) dA$$

On  $S_1$  we have  $z = u_1(x, y)$ , but here the outward normal  $\mathbf{n}$  points downward, so we multiply by  $-1$ :

$$\iint_{S_1} R \mathbf{k} \cdot \mathbf{n} dS = - \iint_D R(x, y, u_1(x, y)) dA$$

Therefore Equation 6 gives

$$\iint_S R \mathbf{k} \cdot \mathbf{n} dS = \iint_D [R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))] dA$$



**FIGURE 1**

Comparison with Equation 5 shows that

$$\iint_S R \mathbf{k} \cdot \mathbf{n} dS = \iiint_E \frac{\partial R}{\partial z} dV$$

Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

Equations 2 and 3 are proved in a similar manner using the expressions for  $E$  as a type 2 or type 3 region, respectively. ■

**EXAMPLE 1** Find the flux of the vector field  $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$  over the unit sphere  $x^2 + y^2 + z^2 = 1$ .

**SOLUTION** First we compute the divergence of  $\mathbf{F}$ :

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere  $S$  is the boundary of the unit ball  $B$  given by  $x^2 + y^2 + z^2 \leq 1$ . Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \operatorname{div} \mathbf{F} dV = \iiint_B 1 dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$

**EXAMPLE 2** Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where

$$\mathbf{F}(x, y, z) = xy \mathbf{i} + (y^2 + e^{xz^2}) \mathbf{j} + \sin(xy) \mathbf{k}$$

and  $S$  is the surface of the region  $E$  bounded by the parabolic cylinder  $z = 1 - x^2$  and the planes  $z = 0$ ,  $y = 0$ , and  $y + z = 2$ . (See Figure 2.)

**SOLUTION** It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of  $S$ .) Furthermore, the divergence of  $\mathbf{F}$  is much less complicated than  $\mathbf{F}$  itself:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express  $E$  as a type 3 region:

$$E = \{(x, y, z) \mid -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2, 0 \leq y \leq 2 - z\}$$

Then we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 3y dV \\ &= 3 \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} y dy dz dx = 3 \int_{-1}^1 \int_0^{1-x^2} \frac{(2-z)^2}{2} dz dx \\ &= \frac{3}{2} \int_{-1}^1 \left[ -\frac{(2-z)^3}{3} \right]_0^{1-x^2} dx = -\frac{1}{2} \int_{-1}^1 [(x^2 + 1)^3 - 8] dx \\ &= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7) dx = \frac{184}{35} \end{aligned}$$

The solution in Example 1 should be compared with the solution in Example 16.7.4.

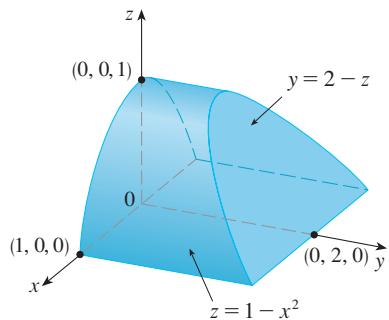


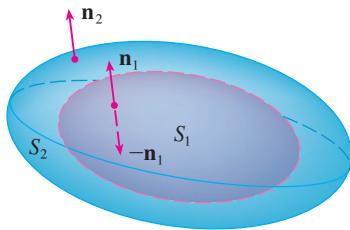
FIGURE 2

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 16.4 to extend Green's Theorem.)

For example, let's consider the region  $E$  that lies between the closed surfaces  $S_1$  and  $S_2$ , where  $S_1$  lies inside  $S_2$ . Let  $\mathbf{n}_1$  and  $\mathbf{n}_2$  be outward normals of  $S_1$  and  $S_2$ . Then the boundary surface of  $E$  is  $S = S_1 \cup S_2$  and its normal  $\mathbf{n}$  is given by  $\mathbf{n} = -\mathbf{n}_1$  on  $S_1$  and  $\mathbf{n} = \mathbf{n}_2$  on  $S_2$ . (See Figure 3.) Applying the Divergence Theorem to  $S$ , we get

$$\begin{aligned} 7 \quad \iiint_E \operatorname{div} \mathbf{F} dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS \\ &= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS \\ &= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \end{aligned}$$

**FIGURE 3**



**EXAMPLE 3** In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\epsilon Q}{|\mathbf{x}|^3} \mathbf{x}$$

where the electric charge  $Q$  is located at the origin and  $\mathbf{x} = \langle x, y, z \rangle$  is a position vector. Use the Divergence Theorem to show that the electric flux of  $\mathbf{E}$  through any closed surface  $S_2$  that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\epsilon Q$$

**SOLUTION** The difficulty is that we don't have an explicit equation for  $S_2$  because it is *any* closed surface enclosing the origin. The simplest such surface would be a sphere, so we let  $S_1$  be a small sphere with radius  $a$  and center the origin. You can verify that  $\operatorname{div} \mathbf{E} = 0$ . (See Exercise 23.) Therefore Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \operatorname{div} \mathbf{E} dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} dS$$

The point of this calculation is that we can compute the surface integral over  $S_1$  because  $S_1$  is a sphere. The normal vector at  $\mathbf{x}$  is  $\mathbf{x}/|\mathbf{x}|$ . Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\epsilon Q}{|\mathbf{x}|^3} \mathbf{x} \cdot \left( \frac{\mathbf{x}}{|\mathbf{x}|} \right) = \frac{\epsilon Q}{|\mathbf{x}|^4} \mathbf{x} \cdot \mathbf{x} = \frac{\epsilon Q}{|\mathbf{x}|^2} = \frac{\epsilon Q}{a^2}$$

since the equation of  $S_1$  is  $|\mathbf{x}| = a$ . Thus we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} dS = \frac{\epsilon Q}{a^2} \iint_{S_1} dS = \frac{\epsilon Q}{a^2} A(S_1) = \frac{\epsilon Q}{a^2} 4\pi a^2 = 4\pi\epsilon Q$$

This shows that the electric flux of  $\mathbf{E}$  is  $4\pi\epsilon Q$  through *any* closed surface  $S_2$  that contains the origin. [This is a special case of Gauss's Law (Equation 16.7.11) for a single charge. The relationship between  $\epsilon$  and  $\epsilon_0$  is  $\epsilon = 1/(4\pi\epsilon_0)$ .]

Another application of the Divergence Theorem occurs in fluid flow. Let  $\mathbf{v}(x, y, z)$  be the velocity field of a fluid with constant density  $\rho$ . Then  $\mathbf{F} = \rho\mathbf{v}$  is the rate of flow per unit area. If  $P_0(x_0, y_0, z_0)$  is a point in the fluid and  $B_a$  is a ball with center  $P_0$  and very small radius  $a$ , then  $\operatorname{div} \mathbf{F}(P) \approx \operatorname{div} \mathbf{F}(P_0)$  for all points  $P$  in  $B_a$  since  $\operatorname{div} \mathbf{F}$  is continuous. We approximate the flux over the boundary sphere  $S_a$  as follows:

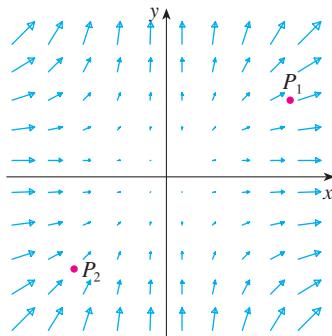
$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F} dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0) dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as  $a \rightarrow 0$  and suggests that

$$8 \quad \operatorname{div} \mathbf{F}(P_0) = \lim_{a \rightarrow 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

Equation 8 says that  $\operatorname{div} \mathbf{F}(P_0)$  is the net rate of outward flux per unit volume at  $P_0$ . (This is the reason for the name *divergence*.) If  $\operatorname{div} \mathbf{F}(P) > 0$ , the net flow is outward near  $P$  and  $P$  is called a **source**. If  $\operatorname{div} \mathbf{F}(P) < 0$ , the net flow is inward near  $P$  and  $P$  is called a **sink**.

For the vector field in Figure 4, it appears that the vectors that end near  $P_1$  are shorter than the vectors that start near  $P_1$ . Thus the net flow is outward near  $P_1$ , so  $\operatorname{div} \mathbf{F}(P_1) > 0$  and  $P_1$  is a source. Near  $P_2$ , on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so  $\operatorname{div} \mathbf{F}(P_2) < 0$  and  $P_2$  is a sink. We can use the formula for  $\mathbf{F}$  to confirm this impression. Since  $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j}$ , we have  $\operatorname{div} \mathbf{F} = 2x + 2y$ , which is positive when  $y > -x$ . So the points above the line  $y = -x$  are sources and those below are sinks.



**FIGURE 4**

The vector field  $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j}$

## 16.9 EXERCISES

- 1–4** Verify that the Divergence Theorem is true for the vector field  $\mathbf{F}$  on the region  $E$ .

1.  $\mathbf{F}(x, y, z) = 3x \mathbf{i} + xy \mathbf{j} + 2xz \mathbf{k}$ ,

$E$  is the cube bounded by the planes  $x = 0, x = 1, y = 0, y = 1, z = 0$ , and  $z = 1$

2.  $\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2yz \mathbf{j} + 4z^2 \mathbf{k}$ ,

$E$  is the solid enclosed by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 9$

3.  $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$ ,

$E$  is the solid ball  $x^2 + y^2 + z^2 \leq 16$

4.  $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$ ,

$E$  is the solid cylinder  $y^2 + z^2 \leq 9, 0 \leq x \leq 2$

- 5–15** Use the Divergence Theorem to calculate the surface integral  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ ; that is, calculate the flux of  $\mathbf{F}$  across  $S$ .

5.  $\mathbf{F}(x, y, z) = xye^z \mathbf{i} + xy^2 z^3 \mathbf{j} - ye^z \mathbf{k}$ ,

$S$  is the surface of the box bounded by the coordinate planes and the planes  $x = 3, y = 2$ , and  $z = 1$

6.  $\mathbf{F}(x, y, z) = x^2 y z \mathbf{i} + x y^2 z \mathbf{j} + x y z^2 \mathbf{k}$ ,

$S$  is the surface of the box enclosed by the planes  $x = 0, x = a, y = 0, y = b, z = 0$ , and  $z = c$ , where  $a, b$ , and  $c$  are positive numbers

7.  $\mathbf{F}(x, y, z) = 3xy^2 \mathbf{i} + xe^z \mathbf{j} + z^3 \mathbf{k}$ ,

$S$  is the surface of the solid bounded by the cylinder  $y^2 + z^2 = 1$  and the planes  $x = -1$  and  $x = 2$

8.  $\mathbf{F}(x, y, z) = (x^3 + y^3) \mathbf{i} + (y^3 + z^3) \mathbf{j} + (z^3 + x^3) \mathbf{k}$ ,

$S$  is the sphere with center the origin and radius 2

9.  $\mathbf{F}(x, y, z) = xe^y \mathbf{i} + (z - e^y) \mathbf{j} - xy \mathbf{k}$ ,

$S$  is the ellipsoid  $x^2 + 2y^2 + 3z^2 = 4$

10.  $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + zx \mathbf{k}$ ,

$S$  is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where  $a, b$ , and  $c$  are positive numbers

11.  $\mathbf{F}(x, y, z) = (2x^3 + y^3) \mathbf{i} + (y^3 + z^3) \mathbf{j} + 3y^2 z \mathbf{k}$ ,

$S$  is the surface of the solid bounded by the paraboloid  $z = 1 - x^2 - y^2$  and the  $xy$ -plane

12.  $\mathbf{F}(x, y, z) = (xy + 2xz) \mathbf{i} + (x^2 + y^2) \mathbf{j} + (xy - z^2) \mathbf{k}$ ,

$S$  is the surface of the solid bounded by the cylinder  $x^2 + y^2 = 4$  and the planes  $z = y - 2$  and  $z = 0$

13.  $\mathbf{F} = |\mathbf{r}| \mathbf{r}$ , where  $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ ,

$S$  consists of the hemisphere  $z = \sqrt{1 - x^2 - y^2}$  and the disk  $x^2 + y^2 \leq 1$  in the  $xy$ -plane

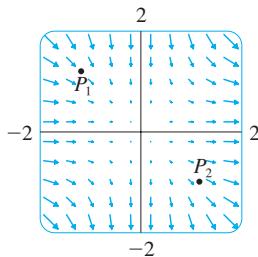
- 14.**  $\mathbf{F} = |\mathbf{r}|^2 \mathbf{r}$ , where  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ,  
 $S$  is the sphere with radius  $R$  and center the origin

- CAS 15.**  $\mathbf{F}(x, y, z) = e^y \tan z \mathbf{i} + y\sqrt{3-x^2} \mathbf{j} + x \sin y \mathbf{k}$ ,  
 $S$  is the surface of the solid that lies above the  $xy$ -plane  
and below the surface  $z = 2 - x^4 - y^4$ ,  $-1 \leq x \leq 1$ ,  
 $-1 \leq y \leq 1$

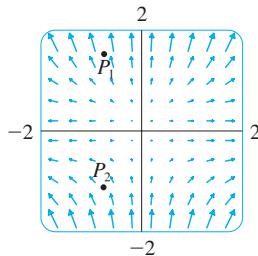
- 16.** Use a computer algebra system to plot the vector field  $\mathbf{F}(x, y, z) = \sin x \cos^2 y \mathbf{i} + \sin^3 y \cos^4 z \mathbf{j} + \sin^5 z \cos^6 x \mathbf{k}$  in the cube cut from the first octant by the planes  $x = \pi/2$ ,  $y = \pi/2$ , and  $z = \pi/2$ . Then compute the flux across the surface of the cube.
- 17.** Use the Divergence Theorem to evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = z^2 x \mathbf{i} + (\frac{1}{3}y^3 + \tan z) \mathbf{j} + (x^2 z + y^2) \mathbf{k}$  and  $S$  is the top half of the sphere  $x^2 + y^2 + z^2 = 1$ . [Hint: Note that  $S$  is not a closed surface. First compute integrals over  $S_1$  and  $S_2$ , where  $S_1$  is the disk  $x^2 + y^2 \leq 1$ , oriented downward, and  $S_2 = S \cup S_1$ .]

- 18.** Let  $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2) \mathbf{i} + z^3 \ln(x^2 + 1) \mathbf{j} + z \mathbf{k}$ . Find the flux of  $\mathbf{F}$  across the part of the paraboloid  $x^2 + y^2 + z = 2$  that lies above the plane  $z = 1$  and is oriented upward.

- 19.** A vector field  $\mathbf{F}$  is shown. Use the interpretation of divergence derived in this section to determine whether  $\operatorname{div} \mathbf{F}$  is positive or negative at  $P_1$  and at  $P_2$ .



- 20.** (a) Are the points  $P_1$  and  $P_2$  sources or sinks for the vector field  $\mathbf{F}$  shown in the figure? Give an explanation based solely on the picture.  
(b) Given that  $\mathbf{F}(x, y) = \langle x, y^2 \rangle$ , use the definition of divergence to verify your answer to part (a).



- CAS 21–22** Plot the vector field and guess where  $\operatorname{div} \mathbf{F} > 0$  and where  $\operatorname{div} \mathbf{F} < 0$ . Then calculate  $\operatorname{div} \mathbf{F}$  to check your guess.

**21.**  $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$

**22.**  $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$

- 23.** Verify that  $\operatorname{div} \mathbf{E} = 0$  for the electric field  $\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x}$ .

- 24.** Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2) dS$$

where  $S$  is the sphere  $x^2 + y^2 + z^2 = 1$ .

- 25–30** Prove each identity, assuming that  $S$  and  $E$  satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

- 25.**  $\iint_S \mathbf{a} \cdot \mathbf{n} dS = 0$ , where  $\mathbf{a}$  is a constant vector

- 26.**  $V(E) = \frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- 27.**  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$

- 28.**  $\iint_S D_{\mathbf{n}} f dS = \iiint_E \nabla^2 f dV$

- 29.**  $\iint_S (f \nabla g) \cdot \mathbf{n} dS = \iiint_E (f \nabla^2 g + \nabla f \cdot \nabla g) dV$

- 30.**  $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} dS = \iiint_E (f \nabla^2 g - g \nabla^2 f) dV$

- 31.** Suppose  $S$  and  $E$  satisfy the conditions of the Divergence Theorem and  $f$  is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f \mathbf{n} dS = \iiint_E \nabla f dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function.

[Hint: Start by applying the Divergence Theorem to  $\mathbf{F} = f\mathbf{c}$ , where  $\mathbf{c}$  is an arbitrary constant vector.]

- 32.** A solid occupies a region  $E$  with surface  $S$  and is immersed in a liquid with constant density  $\rho$ . We set up a coordinate system so that the  $xy$ -plane coincides with the surface of the liquid, and positive values of  $z$  are measured downward into the liquid. Then the pressure at depth  $z$  is  $p = \rho g z$ , where  $g$  is the acceleration due to gravity (see Section 8.3). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = - \iint_S p \mathbf{n} dS$$

where  $\mathbf{n}$  is the outer unit normal. Use the result of Exercise 31 to show that  $\mathbf{F} = -W\mathbf{k}$ , where  $W$  is the weight of the liquid displaced by the solid. (Note that  $\mathbf{F}$  is directed upward because  $z$  is directed downward.) The result is *Archimedes' Principle*: The buoyant force on an object equals the weight of the displaced liquid.

## 16.10 Summary

The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a “derivative” over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

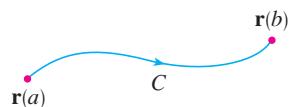
Fundamental Theorem of Calculus

$$\int_a^b F'(x) dx = F(b) - F(a)$$



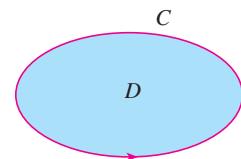
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



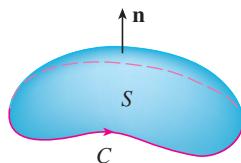
Green's Theorem

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P dx + Q dy$$



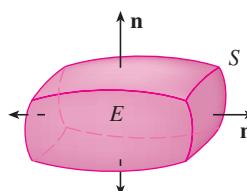
Stokes' Theorem

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



## 16 REVIEW

### CONCEPT CHECK

1. What is a vector field? Give three examples that have physical meaning.
2. (a) What is a conservative vector field?  
(b) What is a potential function?
3. (a) Write the definition of the line integral of a scalar function  $f$  along a smooth curve  $C$  with respect to arc length.  
(b) How do you evaluate such a line integral?  
(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve  $C$  if the wire has linear density function  $\rho(x, y)$ .  
(d) Write the definitions of the line integrals along  $C$  of a scalar function  $f$  with respect to  $x$ ,  $y$ , and  $z$ .  
(e) How do you evaluate these line integrals?
4. (a) Define the line integral of a vector field  $\mathbf{F}$  along a smooth curve  $C$  given by a vector function  $\mathbf{r}(t)$ .  
(b) If  $\mathbf{F}$  is a force field, what does this line integral represent?  
(c) If  $\mathbf{F} = \langle P, Q, R \rangle$ , what is the connection between the line integral of  $\mathbf{F}$  and the line integrals of the component functions  $P$ ,  $Q$ , and  $R$ ?
5. State the Fundamental Theorem for Line Integrals.
6. (a) What does it mean to say that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path?  
(b) If you know that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path, what can you say about  $\mathbf{F}$ ?
7. State Green's Theorem.
8. Write expressions for the area enclosed by a curve  $C$  in terms of line integrals around  $C$ .
9. Suppose  $\mathbf{F}$  is a vector field on  $\mathbb{R}^3$ .  
(a) Define  $\text{curl } \mathbf{F}$ .  
(b) Define  $\text{div } \mathbf{F}$ .

### TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If  $\mathbf{F}$  is a vector field, then  $\text{div } \mathbf{F}$  is a vector field.
2. If  $\mathbf{F}$  is a vector field, then  $\text{curl } \mathbf{F}$  is a vector field.
3. If  $f$  has continuous partial derivatives of all orders on  $\mathbb{R}^3$ , then  $\text{div}(\text{curl } \nabla f) = 0$ .
4. If  $f$  has continuous partial derivatives on  $\mathbb{R}^3$  and  $C$  is any circle, then  $\int_C \nabla f \cdot d\mathbf{r} = 0$ .
5. If  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$  and  $P_y = Q_x$  in an open region  $D$ , then  $\mathbf{F}$  is conservative.
6.  $\int_{-C} f(x, y) ds = -\int_C f(x, y) ds$
7. If  $\mathbf{F}$  and  $\mathbf{G}$  are vector fields and  $\text{div } \mathbf{F} = \text{div } \mathbf{G}$ , then  $\mathbf{F} = \mathbf{G}$ .

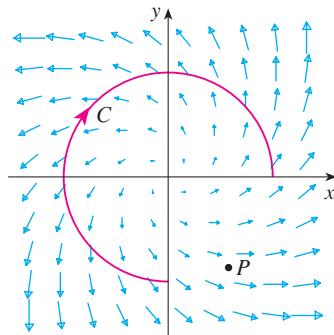
Answers to the Concept Check can be found on the back endpapers.

- (c) If  $\mathbf{F}$  is a velocity field in fluid flow, what are the physical interpretations of  $\text{curl } \mathbf{F}$  and  $\text{div } \mathbf{F}$ ?
10. If  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ , how do you determine whether  $\mathbf{F}$  is conservative? What if  $\mathbf{F}$  is a vector field on  $\mathbb{R}^3$ ?
11. (a) What is a parametric surface? What are its grid curves?  
(b) Write an expression for the area of a parametric surface.  
(c) What is the area of a surface given by an equation  
$$z = g(x, y)$$
12. (a) Write the definition of the surface integral of a scalar function  $f$  over a surface  $S$ .  
(b) How do you evaluate such an integral if  $S$  is a parametric surface given by a vector function  $\mathbf{r}(u, v)$ ?  
(c) What if  $S$  is given by an equation  $z = g(x, y)$ ?  
(d) If a thin sheet has the shape of a surface  $S$ , and the density at  $(x, y, z)$  is  $\rho(x, y, z)$ , write expressions for the mass and center of mass of the sheet.
13. (a) What is an oriented surface? Give an example of a non-orientable surface.  
(b) Define the surface integral (or flux) of a vector field  $\mathbf{F}$  over an oriented surface  $S$  with unit normal vector  $\mathbf{n}$ .  
(c) How do you evaluate such an integral if  $S$  is a parametric surface given by a vector function  $\mathbf{r}(u, v)$ ?  
(d) What if  $S$  is given by an equation  $z = g(x, y)$ ?
14. State Stokes' Theorem.
15. State the Divergence Theorem.
16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

8. The work done by a conservative force field in moving a particle around a closed path is zero.
9. If  $\mathbf{F}$  and  $\mathbf{G}$  are vector fields, then  
$$\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$$
10. If  $\mathbf{F}$  and  $\mathbf{G}$  are vector fields, then  
$$\text{curl}(\mathbf{F} \cdot \mathbf{G}) = \text{curl } \mathbf{F} \cdot \text{curl } \mathbf{G}$$
11. If  $S$  is a sphere and  $\mathbf{F}$  is a constant vector field, then  
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$$
12. There is a vector field  $\mathbf{F}$  such that  
$$\text{curl } \mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$
13. The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve  $C$  is  $A = \oint_C y dx$ .

**EXERCISES**

1. A vector field  $\mathbf{F}$ , a curve  $C$ , and a point  $P$  are shown.  
 (a) Is  $\int_C \mathbf{F} \cdot d\mathbf{r}$  positive, negative, or zero? Explain.  
 (b) Is  $\operatorname{div} \mathbf{F}(P)$  positive, negative, or zero? Explain.



**2–9** Evaluate the line integral.

2.  $\int_C x \, ds$ ,  
 $C$  is the arc of the parabola  $y = x^2$  from  $(0, 0)$  to  $(1, 1)$
3.  $\int_C yz \cos x \, ds$ ,  
 $C: x = t, y = 3 \cos t, z = 3 \sin t, 0 \leq t \leq \pi$
4.  $\int_C y \, dx + (x + y^2) \, dy$ ,  $C$  is the ellipse  $4x^2 + 9y^2 = 36$  with counterclockwise orientation
5.  $\int_C y^3 \, dx + x^2 \, dy$ ,  $C$  is the arc of the parabola  $x = 1 - y^2$  from  $(0, -1)$  to  $(0, 1)$
6.  $\int_C \sqrt{xy} \, dx + e^y \, dy + xz \, dz$ ,  
 $C$  is given by  $\mathbf{r}(t) = t^4 \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}, 0 \leq t \leq 1$
7.  $\int_C xy \, dx + y^2 \, dy + yz \, dz$ ,  
 $C$  is the line segment from  $(1, 0, -1)$ , to  $(3, 4, 2)$
8.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = xy \mathbf{i} + x^2 \mathbf{j}$  and  $C$  is given by  $\mathbf{r}(t) = \sin t \mathbf{i} + (1 + t) \mathbf{j}, 0 \leq t \leq \pi$
9.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = e^z \mathbf{i} + xz \mathbf{j} + (x + y) \mathbf{k}$  and  $C$  is given by  $\mathbf{r}(t) = t^2 \mathbf{i} + t^3 \mathbf{j} - t \mathbf{k}, 0 \leq t \leq 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z \mathbf{i} + x \mathbf{j} + y \mathbf{k}$$

in moving a particle from the point  $(3, 0, 0)$  to the point  $(0, \pi/2, 3)$  along  
 (a) a straight line  
 (b) the helix  $x = 3 \cos t, y = t, z = 3 \sin t$

- 11–12 Show that  $\mathbf{F}$  is a conservative vector field. Then find a function  $f$  such that  $\mathbf{F} = \nabla f$ .

11.  $\mathbf{F}(x, y) = (1 + xy)e^{xy} \mathbf{i} + (e^y + x^2 e^{xy}) \mathbf{j}$

12.  $\mathbf{F}(x, y, z) = \sin y \mathbf{i} + x \cos y \mathbf{j} - \sin z \mathbf{k}$

- 13–14 Show that  $\mathbf{F}$  is conservative and use this fact to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  along the given curve.

13.  $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3) \mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3) \mathbf{j}$ ,  
 $C: \mathbf{r}(t) = (t + \sin \pi t) \mathbf{i} + (2t + \cos \pi t) \mathbf{j}, 0 \leq t \leq 1$

14.  $\mathbf{F}(x, y, z) = e^y \mathbf{i} + (xe^y + e^z) \mathbf{j} + ye^z \mathbf{k}$ ,  
 $C$  is the line segment from  $(0, 2, 0)$  to  $(4, 0, 3)$

15. Verify that Green's Theorem is true for the line integral  $\int_C xy^2 \, dx - x^2y \, dy$ , where  $C$  consists of the parabola  $y = x^2$  from  $(-1, 1)$  to  $(1, 1)$  and the line segment from  $(1, 1)$  to  $(-1, 1)$ .

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3} \, dx + 2xy \, dy$$

where  $C$  is the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 3)$ .

17. Use Green's Theorem to evaluate  $\int_C x^2y \, dx - xy^2 \, dy$ , where  $C$  is the circle  $x^2 + y^2 = 4$  with counterclockwise orientation.

18. Find  $\operatorname{curl} \mathbf{F}$  and  $\operatorname{div} \mathbf{F}$  if

$$\mathbf{F}(x, y, z) = e^{-x} \sin y \mathbf{i} + e^{-y} \sin z \mathbf{j} + e^{-z} \sin x \mathbf{k}$$

19. Show that there is no vector field  $\mathbf{G}$  such that

$$\operatorname{curl} \mathbf{G} = 2x \mathbf{i} + 3yz \mathbf{j} - xz^2 \mathbf{k}$$

20. If  $\mathbf{F}$  and  $\mathbf{G}$  are vector fields whose component functions have continuous first partial derivatives, show that

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}$$

21. If  $C$  is any piecewise-smooth simple closed plane curve and  $f$  and  $g$  are differentiable functions, show that  $\int_C f(x) \, dx + g(y) \, dy = 0$ .

22. If  $f$  and  $g$  are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2g + g\nabla^2f + 2\nabla f \cdot \nabla g$$

23. If  $f$  is a harmonic function, that is,  $\nabla^2 f = 0$ , show that the line integral  $\int f_y \, dx - f_x \, dy$  is independent of path in any simple region  $D$ .

24. (a) Sketch the curve  $C$  with parametric equations

$$x = \cos t \quad y = \sin t \quad z = \sin t \quad 0 \leq t \leq 2\pi$$

- (b) Find  $\int_C 2xe^{2y} \, dx + (2x^2e^{2y} + 2y \cot z) \, dy - y^2 \csc^2 z \, dz$ .

25. Find the area of the part of the surface  $z = x^2 + 2y$  that lies above the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 2)$ .

26. (a) Find an equation of the tangent plane at the point  $(4, -2, 1)$  to the parametric surface  $S$  given by

$$\mathbf{r}(u, v) = v^2 \mathbf{i} - uv \mathbf{j} + u^2 \mathbf{k} \quad 0 \leq u \leq 3, -3 \leq v \leq 3$$

- (b) Use a computer to graph the surface  $S$  and the tangent plane found in part (a).  
(c) Set up, but do not evaluate, an integral for the surface area of  $S$ .  
(d) If

$$\mathbf{F}(x, y, z) = \frac{z^2}{1+x^2} \mathbf{i} + \frac{x^2}{1+y^2} \mathbf{j} + \frac{y^2}{1+z^2} \mathbf{k}$$

find  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  correct to four decimal places.

- 27–30 Evaluate the surface integral.

27.  $\iint_S z \, dS$ , where  $S$  is the part of the paraboloid  $z = x^2 + y^2$  that lies under the plane  $z = 4$

28.  $\iint_S (x^2 z + y^2 z) \, dS$ , where  $S$  is the part of the plane  $z = 4 + x + y$  that lies inside the cylinder  $x^2 + y^2 = 4$

29.  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = xz \mathbf{i} - 2y \mathbf{j} + 3x \mathbf{k}$  and  $S$  is the sphere  $x^2 + y^2 + z^2 = 4$  with outward orientation

30.  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k}$  and  $S$  is the part of the paraboloid  $z = x^2 + y^2$  below the plane  $z = 1$  with upward orientation

31. Verify that Stokes' Theorem is true for the vector field  $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$ , where  $S$  is the part of the paraboloid  $z = 1 - x^2 - y^2$  that lies above the  $xy$ -plane and  $S$  has upward orientation.

32. Use Stokes' Theorem to evaluate  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = x^2yz \mathbf{i} + yz^2 \mathbf{j} + z^3 e^{xy} \mathbf{k}$ ,  $S$  is the part of the sphere  $x^2 + y^2 + z^2 = 5$  that lies above the plane  $z = 1$ , and  $S$  is oriented upward.

33. Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$ , and  $C$  is the triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ , oriented counterclockwise as viewed from above.

34. Use the Divergence Theorem to calculate the surface integral  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = x^3 \mathbf{i} + y^3 \mathbf{j} + z^3 \mathbf{k}$  and  $S$  is the surface of the solid bounded by the cylinder  $x^2 + y^2 = 1$  and the planes  $z = 0$  and  $z = 2$ .

35. Verify that the Divergence Theorem is true for the vector field  $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ , where  $E$  is the unit ball  $x^2 + y^2 + z^2 \leq 1$ .

36. Compute the outward flux of

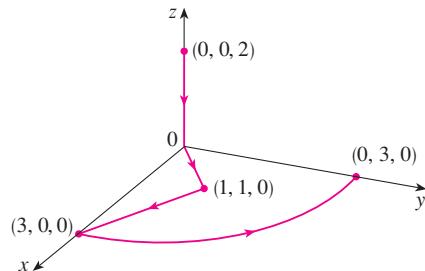
$$\mathbf{F}(x, y, z) = \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid  $4x^2 + 9y^2 + 6z^2 = 36$ .

37. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y) \mathbf{i} + (x^3z - 3x) \mathbf{j} + (x^3y + 2z) \mathbf{k}$$

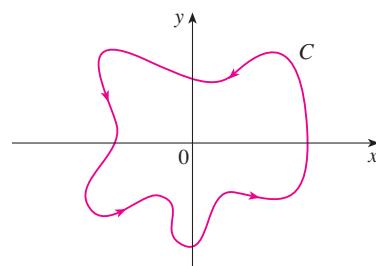
Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is the curve with initial point  $(0, 0, 2)$  and terminal point  $(0, 3, 0)$  shown in the figure.



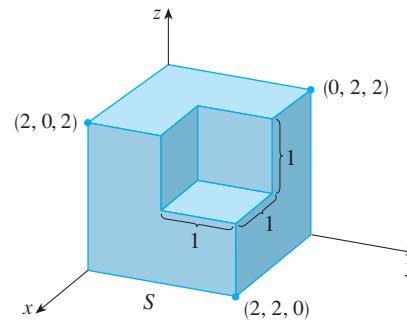
38. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y) \mathbf{i} + (2y^3 + 2x^2y + 2x) \mathbf{j}}{x^2 + y^2}$$

Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is shown in the figure.



39. Find  $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ , where  $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$  and  $S$  is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



40. If the components of  $\mathbf{F}$  have continuous second partial derivatives and  $S$  is the boundary surface of a simple solid region, show that  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ .

41. If  $\mathbf{a}$  is a constant vector,  $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ , and  $S$  is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve  $C$ , show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

## Problems Plus

1. Let  $S$  be a smooth parametric surface and let  $P$  be a point such that each line that starts at  $P$  intersects  $S$  at most once. The **solid angle**  $\Omega(S)$  subtended by  $S$  at  $P$  is the set of lines starting at  $P$  and passing through  $S$ . Let  $S(a)$  be the intersection of  $\Omega(S)$  with the surface of the sphere with center  $P$  and radius  $a$ . Then the measure of the solid angle (in *steradians*) is defined to be

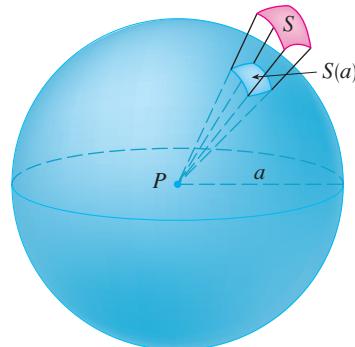
$$|\Omega(S)| = \frac{\text{area of } S(a)}{a^2}$$

Apply the Divergence Theorem to the part of  $\Omega(S)$  between  $S(a)$  and  $S$  to show that

$$|\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS$$

where  $\mathbf{r}$  is the radius vector from  $P$  to any point on  $S$ ,  $r = |\mathbf{r}|$ , and the unit normal vector  $\mathbf{n}$  is directed away from  $P$ .

This shows that the definition of the measure of a solid angle is independent of the radius  $a$  of the sphere. Thus the measure of the solid angle is equal to the area subtended on a *unit sphere*. (Note the analogy with the definition of radian measure.) The total solid angle subtended by a sphere at its center is thus  $4\pi$  steradians.



2. Find the positively oriented simple closed curve  $C$  for which the value of the line integral

$$\int_C (y^3 - y) dx - 2x^3 dy$$

is a maximum.

3. Let  $C$  be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector  $\mathbf{n} = \langle a, b, c \rangle$  and has positive orientation with respect to  $\mathbf{n}$ . Show that the plane area enclosed by  $C$  is

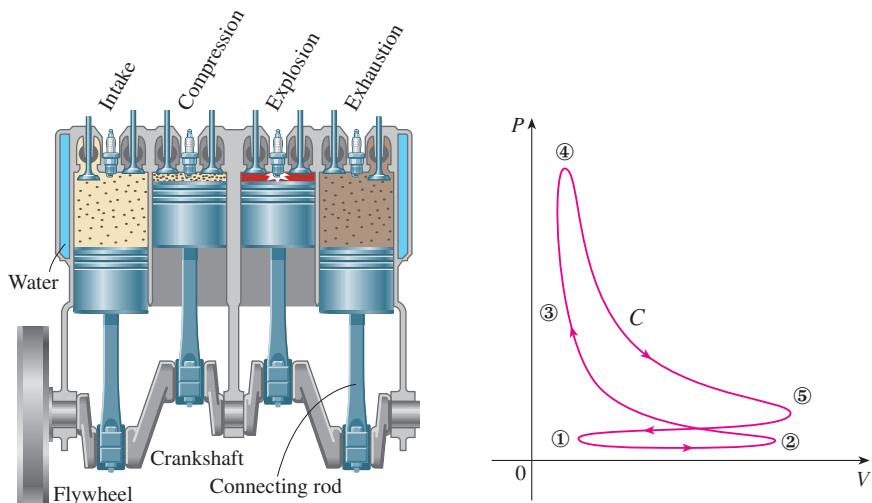
$$\frac{1}{2} \int_C (bz - cy) dx + (cx - az) dy + (ay - bx) dz$$

-  4. Investigate the shape of the surface with parametric equations  $x = \sin u$ ,  $y = \sin v$ ,  $z = \sin(u + v)$ . Start by graphing the surface from several points of view. Explain the appearance of the graphs by determining the traces in the horizontal planes  $z = 0$ ,  $z = \pm 1$ , and  $z = \pm \frac{1}{2}$ .

5. Prove the following identity:

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \operatorname{curl} \mathbf{G} + \mathbf{G} \times \operatorname{curl} \mathbf{F}$$

6. The figure depicts the sequence of events in each cylinder of a four-cylinder internal combustion engine. Each piston moves up and down and is connected by a pivoted arm to a rotating crankshaft. Let  $P(t)$  and  $V(t)$  be the pressure and volume within a cylinder at time  $t$ , where  $a \leq t \leq b$  gives the time required for a complete cycle. The graph shows how  $P$  and  $V$  vary through one cycle of a four-stroke engine.



During the intake stroke (from ① to ②) a mixture of air and gasoline at atmospheric pressure is drawn into a cylinder through the intake valve as the piston moves downward. Then the piston rapidly compresses the mix with the valves closed in the compression stroke (from ② to ③) during which the pressure rises and the volume decreases. At ③ the sparkplug ignites the fuel, raising the temperature and pressure at almost constant volume to ④. Then, with valves closed, the rapid expansion forces the piston downward during the power stroke (from ④ to ⑤). The exhaust valve opens, temperature and pressure drop, and mechanical energy stored in a rotating flywheel pushes the piston upward, forcing the waste products out of the exhaust valve in the exhaust stroke. The exhaust valve closes and the intake valve opens. We're now back at ① and the cycle starts again.

- (a) Show that the work done on the piston during one cycle of a four-stroke engine is  $W = \int_C P dV$ , where  $C$  is the curve in the  $PV$ -plane shown in the figure.  
*[Hint: Let  $x(t)$  be the distance from the piston to the top of the cylinder and note that the force on the piston is  $\mathbf{F} = AP(t) \mathbf{i}$ , where  $A$  is the area of the top of the piston. Then  $W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ , where  $C_1$  is given by  $\mathbf{r}(t) = x(t) \mathbf{i}$ ,  $a \leq t \leq b$ . An alternative approach is to work directly with Riemann sums.]*
- (b) Use Formula 16.4.5 to show that the work is the difference of the areas enclosed by the two loops of  $C$ .