

Additional Topics in Trigonometry

6

- 6.1 Law of Sines
- 6.2 Law of Cosines
- 6.3 Vectors in the Plane
- 6.4 Vectors and Dot Products
- 6.5 Trigonometric Form of a Complex Number

The work done by a force, such as pushing and pulling objects, can be calculated using vector operations.

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SELECTED APPLICATIONS

Triangles and vectors have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Bridge Design,
Exercise 39, page 437
- Paper Manufacturing,
Exercise 45, page 445
- Revenue,
Exercise 65, page 468
- Glide Path,
Exercise 41, page 437
- Cable Tension,
Exercises 79 and 80, page 458
- Work,
Exercise 73, page 469
- Surveying,
Exercise 31, page 444
- Navigation,
Exercise 84, page 459

6.1 Law of Sines

What you should learn

- Use the Law of Sines to solve oblique triangles (AAS or ASA).
- Use the Law of Sines to solve oblique triangles (SSA).
- Find the areas of oblique triangles.
- Use the Law of Sines to model and solve real-life problems.

Why you should learn it

You can use the Law of Sines to solve real-life problems involving oblique triangles. For instance, in Exercise 44 on page 438, you can use the Law of Sines to determine the length of the shadow of the Leaning Tower of Pisa.



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Introduction

In Chapter 4, you studied techniques for solving right triangles. In this section and the next, you will solve **oblique triangles**—triangles that have no right angles. As standard notation, the angles of a triangle are labeled A , B , and C , and their opposite sides are labeled a , b , and c , as shown in Figure 6.1.

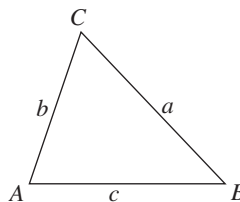


FIGURE 6.1

To solve an oblique triangle, you need to know the measure of at least one side and any two other parts of the triangle—either two sides, two angles, or one angle and one side. This breaks down into the following four cases.

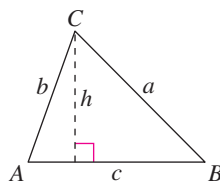
1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

The first two cases can be solved using the **Law of Sines**, whereas the last two cases require the Law of Cosines (see Section 6.2).

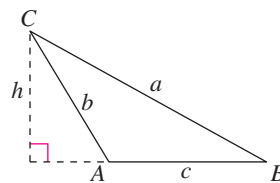
Law of Sines

If ABC is a triangle with sides a , b , and c , then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



A is acute.



A is obtuse.

The Law of Sines can also be written in the reciprocal form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

For a proof of the Law of Sines, see Proofs in Mathematics on page 489.

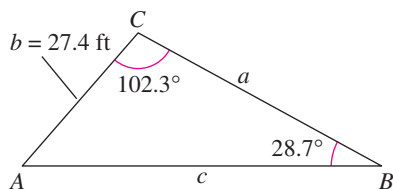


FIGURE 6.2

STUDY TIP

When solving triangles, a careful sketch is useful as a quick test for the feasibility of an answer. Remember that the longest side lies opposite the largest angle, and the shortest side lies opposite the smallest angle.

Example 1 Given Two Angles and One Side—AAS

For the triangle in Figure 6.2, $C = 102.3^\circ$, $B = 28.7^\circ$, and $b = 27.4$ feet. Find the remaining angle and sides.

Solution

The third angle of the triangle is

$$\begin{aligned} A &= 180^\circ - B - C \\ &= 180^\circ - 28.7^\circ - 102.3^\circ \\ &= 49.0^\circ. \end{aligned}$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Using $b = 27.4$ produces

$$a = \frac{b}{\sin B}(\sin A) = \frac{27.4}{\sin 28.7^\circ}(\sin 49.0^\circ) \approx 43.06 \text{ feet}$$

and

$$c = \frac{b}{\sin B}(\sin C) = \frac{27.4}{\sin 28.7^\circ}(\sin 102.3^\circ) \approx 55.75 \text{ feet.}$$

CHECKPOINT Now try Exercise 1.

Example 2 Given Two Angles and One Side—ASA



A pole tilts *toward* the sun at an 8° angle from the vertical, and it casts a 22-foot shadow. The angle of elevation from the tip of the shadow to the top of the pole is 43° . How tall is the pole?

Solution

From Figure 6.3, note that $A = 43^\circ$ and $B = 90^\circ + 8^\circ = 98^\circ$. So, the third angle is

$$\begin{aligned} C &= 180^\circ - A - B \\ &= 180^\circ - 43^\circ - 98^\circ \\ &= 39^\circ. \end{aligned}$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Because $c = 22$ feet, the length of the pole is

$$a = \frac{c}{\sin C}(\sin A) = \frac{22}{\sin 39^\circ}(\sin 43^\circ) \approx 23.84 \text{ feet.}$$

CHECKPOINT Now try Exercise 35.

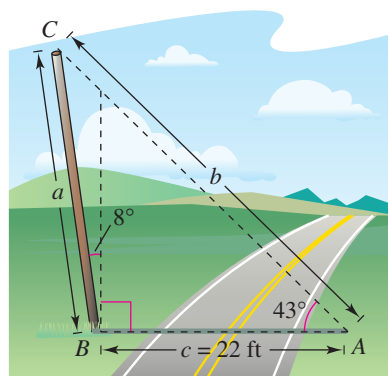


FIGURE 6.3

For practice, try reworking Example 2 for a pole that tilts *away from* the sun under the same conditions.

The Ambiguous Case (SSA)

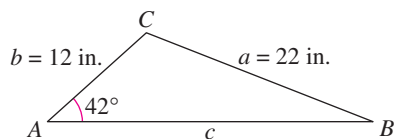
In Examples 1 and 2 you saw that two angles and one side determine a unique triangle. However, if two sides and one opposite angle are given, three possible situations can occur: (1) no such triangle exists, (2) one such triangle exists, or (3) two distinct triangles may satisfy the conditions.

The Ambiguous Case (SSA)

Consider a triangle in which you are given a , b , and A . ($h = b \sin A$)

| | A is acute. | A is acute. | A is acute. | A is acute. | A is obtuse. | A is obtuse. |
|---------------------|---------------|---------------|---------------|---------------|----------------|----------------|
| Sketch | | | | | | |
| Necessary condition | $a < h$ | $a = h$ | $a > b$ | $h < a < b$ | $a \leq b$ | $a > b$ |
| Triangles possible | None | One | One | Two | None | One |

Example 3 Single-Solution Case—SSA



One solution: $a > b$

FIGURE 6.4

For the triangle in Figure 6.4, $a = 22$ inches, $b = 12$ inches, and $A = 42^\circ$. Find the remaining side and angles.

Solution

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \text{Reciprocal form}$$

$$\sin B = b \left(\frac{\sin A}{a} \right) \quad \text{Multiply each side by } b.$$

$$\sin B = 12 \left(\frac{\sin 42^\circ}{22} \right) \quad \text{Substitute for } A, a, \text{ and } b.$$

$$B \approx 21.41^\circ. \quad B \text{ is acute.}$$

Now, you can determine that

$$C \approx 180^\circ - 42^\circ - 21.41^\circ = 116.59^\circ.$$

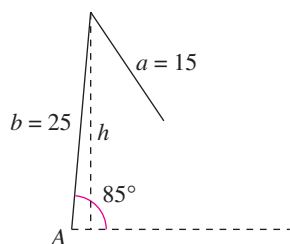
Then, the remaining side is

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a}{\sin A} (\sin C) = \frac{22}{\sin 42^\circ} (\sin 116.59^\circ) \approx 29.40 \text{ inches.}$$



CHECKPOINT Now try Exercise 19.



No solution: $a < h$
FIGURE 6.5

Example 4 No-Solution Case—SSA

Show that there is no triangle for which $a = 15$, $b = 25$, and $A = 85^\circ$.

Solution

Begin by making the sketch shown in Figure 6.5. From this figure it appears that no triangle is formed. You can verify this using the Law of Sines.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \text{Reciprocal form}$$

$$\sin B = b \left(\frac{\sin A}{a} \right) \quad \text{Multiply each side by } b.$$

$$\sin B = 25 \left(\frac{\sin 85^\circ}{15} \right) \approx 1.660 > 1$$

This contradicts the fact that $|\sin B| \leq 1$. So, no triangle can be formed having sides $a = 15$ and $b = 25$ and an angle of $A = 85^\circ$.

CHECKPOINT Now try Exercise 21.

Example 5 Two-Solution Case—SSA

Find two triangles for which $a = 12$ meters, $b = 31$ meters, and $A = 20.5^\circ$.

Solution

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \text{Reciprocal form}$$

$$\sin B = b \left(\frac{\sin A}{a} \right) = 31 \left(\frac{\sin 20.5^\circ}{12} \right) \approx 0.9047.$$

There are two angles $B_1 \approx 64.8^\circ$ and $B_2 \approx 180^\circ - 64.8^\circ = 115.2^\circ$ between 0° and 180° whose sine is 0.9047. For $B_1 \approx 64.8^\circ$, you obtain

$$C \approx 180^\circ - 20.5^\circ - 64.8^\circ = 94.7^\circ$$

$$c = \frac{a}{\sin A} (\sin C) = \frac{12}{\sin 20.5^\circ} (\sin 94.7^\circ) \approx 34.15 \text{ meters.}$$

For $B_2 \approx 115.2^\circ$, you obtain

$$C \approx 180^\circ - 20.5^\circ - 115.2^\circ = 44.3^\circ$$

$$c = \frac{a}{\sin A} (\sin C) = \frac{12}{\sin 20.5^\circ} (\sin 44.3^\circ) \approx 23.93 \text{ meters.}$$

The resulting triangles are shown in Figure 6.6.

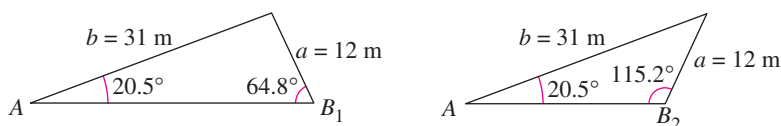


FIGURE 6.6

CHECKPOINT Now try Exercise 23.

STUDY TIP

To see how to obtain the height of the obtuse triangle in Figure 6.7, notice the use of the reference angle $180^\circ - A$ and the difference formula for sine, as follows.

$$\begin{aligned} h &= b \sin(180^\circ - A) \\ &= b(\sin 180^\circ \cos A \\ &\quad - \cos 180^\circ \sin A) \\ &= b[0 \cdot \cos A - (-1) \cdot \sin A] \\ &= b \sin A \end{aligned}$$

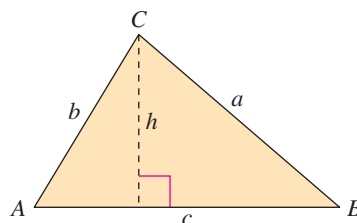
Area of an Oblique Triangle

The procedure used to prove the Law of Sines leads to a simple formula for the area of an oblique triangle. Referring to Figure 6.7, note that each triangle has a height of $h = b \sin A$. Consequently, the area of each triangle is

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(c)(b \sin A) = \frac{1}{2}bc \sin A.$$

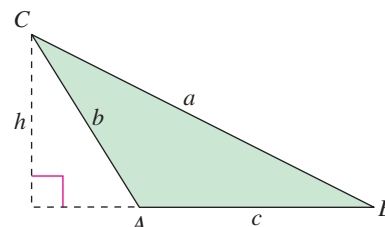
By similar arguments, you can develop the formulas

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$



A is acute

FIGURE 6.7



A is obtuse

Area of an Oblique Triangle

The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$

Note that if angle A is 90° , the formula gives the area for a right triangle:

$$\text{Area} = \frac{1}{2}bc(\sin 90^\circ) = \frac{1}{2}bc = \frac{1}{2}(\text{base})(\text{height}). \quad \sin 90^\circ = 1$$

Similar results are obtained for angles C and B equal to 90° .

Example 6 Finding the Area of a Triangular Lot



Find the area of a triangular lot having two sides of lengths 90 meters and 52 meters and an included angle of 102° .

Solution

Consider $a = 90$ meters, $b = 52$ meters, and angle $C = 102^\circ$, as shown in Figure 6.8. Then, the area of the triangle is

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(90)(52)(\sin 102^\circ) \approx 2289 \text{ square meters.}$$

CHECKPOINT Now try Exercise 29.

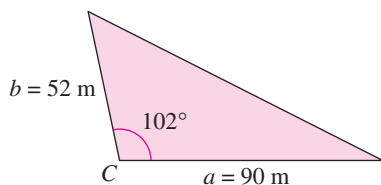


FIGURE 6.8

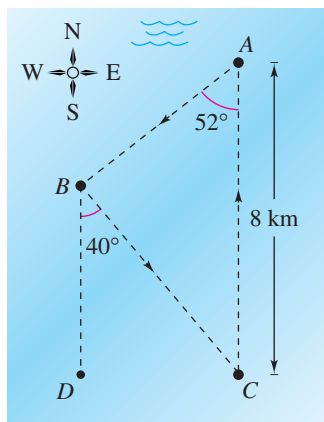


FIGURE 6.9

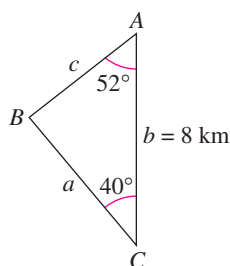


FIGURE 6.10

Application

Example 7 An Application of the Law of Sines



The course for a boat race starts at point A in Figure 6.9 and proceeds in the direction $S\ 52^\circ\ W$ to point B , then in the direction $S\ 40^\circ\ E$ to point C , and finally back to A . Point C lies 8 kilometers directly south of point A . Approximate the total distance of the race course.

Solution

Because lines BD and AC are parallel, it follows that $\angle BCA \cong \angle DBC$. Consequently, triangle ABC has the measures shown in Figure 6.10. For angle B , you have $B = 180^\circ - 52^\circ - 40^\circ = 88^\circ$. Using the Law of Sines

$$\frac{a}{\sin 52^\circ} = \frac{b}{\sin 88^\circ} = \frac{c}{\sin 40^\circ}$$

you can let $b = 8$ and obtain

$$a = \frac{8}{\sin 88^\circ}(\sin 52^\circ) \approx 6.308$$

and

$$c = \frac{8}{\sin 88^\circ}(\sin 40^\circ) \approx 5.145.$$

The total length of the course is approximately

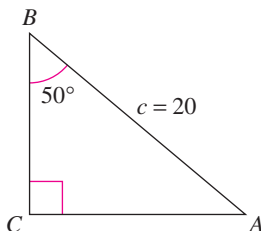
$$\begin{aligned} \text{Length} &\approx 8 + 6.308 + 5.145 \\ &= 19.453 \text{ kilometers.} \end{aligned}$$

CHECKPOINT Now try Exercise 39.

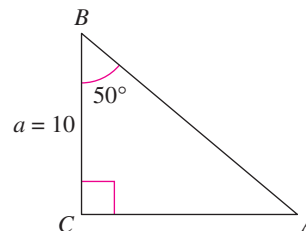
WRITING ABOUT MATHEMATICS

Using the Law of Sines In this section, you have been using the Law of Sines to solve *oblique* triangles. Can the Law of Sines also be used to solve a right triangle? If so, write a short paragraph explaining how to use the Law of Sines to solve each triangle. Is there an easier way to solve these triangles?

a. (AAS)



b. (ASA)



6.1

Exercises

The HM mathSpace® CD-ROM and Eduspace® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

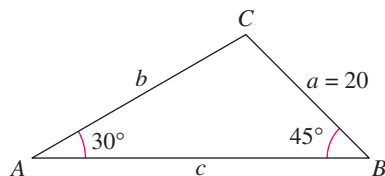
VOCABULARY CHECK: Fill in the blanks.

1. An _____ triangle is a triangle that has no right angle.
2. For triangle ABC , the Law of Sines is given by $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
3. The area of an oblique triangle is given by $\frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$.

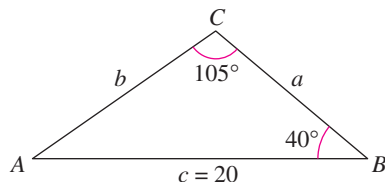
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–18, use the Law of Sines to solve the triangle. Round your answers to two decimal places.

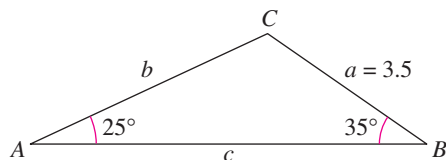
1.



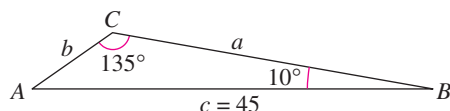
2.



3.



4.



5. $A = 36^\circ$, $a = 8$, $b = 5$
6. $A = 60^\circ$, $a = 9$, $c = 10$
7. $A = 102.4^\circ$, $C = 16.7^\circ$, $a = 21.6$
8. $A = 24.3^\circ$, $C = 54.6^\circ$, $c = 2.68$
9. $A = 83^\circ 20'$, $C = 54.6^\circ$, $c = 18.1$
10. $A = 5^\circ 40'$, $B = 8^\circ 15'$, $b = 4.8$
11. $B = 15^\circ 30'$, $a = 4.5$, $b = 6.8$
12. $B = 2^\circ 45'$, $b = 6.2$, $c = 5.8$
13. $C = 145^\circ$, $b = 4$, $c = 14$

14. $A = 100^\circ$, $a = 125$, $c = 10$
15. $A = 110^\circ 15'$, $a = 48$, $b = 16$
16. $C = 85^\circ 20'$, $a = 35$, $c = 50$
17. $A = 55^\circ$, $B = 42^\circ$, $c = \frac{3}{4}$
18. $B = 28^\circ$, $C = 104^\circ$, $a = 3\frac{5}{8}$

In Exercises 19–24, use the Law of Sines to solve (if possible) the triangle. If two solutions exist, find both. Round your answers to two decimal places.

19. $A = 110^\circ$, $a = 125$, $b = 100$
20. $A = 110^\circ$, $a = 125$, $b = 200$
21. $A = 76^\circ$, $a = 18$, $b = 20$
22. $A = 76^\circ$, $a = 34$, $b = 21$
23. $A = 58^\circ$, $a = 11.4$, $b = 12.8$
24. $A = 58^\circ$, $a = 4.5$, $b = 12.8$

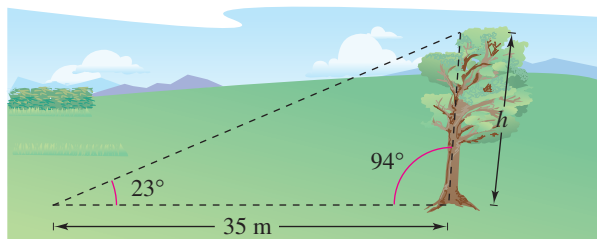
In Exercises 25–28, find values for b such that the triangle has (a) one solution, (b) two solutions, and (c) no solution.

25. $A = 36^\circ$, $a = 5$
26. $A = 60^\circ$, $a = 10$
27. $A = 10^\circ$, $a = 10.8$
28. $A = 88^\circ$, $a = 315.6$

In Exercises 29–34, find the area of the triangle having the indicated angle and sides.

29. $C = 120^\circ$, $a = 4$, $b = 6$
30. $B = 130^\circ$, $a = 62$, $c = 20$
31. $A = 43^\circ 45'$, $b = 57$, $c = 85$
32. $A = 5^\circ 15'$, $b = 4.5$, $c = 22$
33. $B = 72^\circ 30'$, $a = 105$, $c = 64$
34. $C = 84^\circ 30'$, $a = 16$, $b = 20$

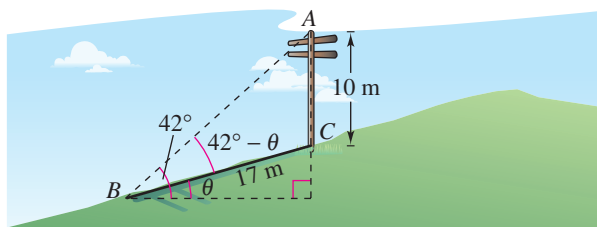
- 35. Height** Because of prevailing winds, a tree grew so that it was leaning 4° from the vertical. At a point 35 meters from the tree, the angle of elevation to the top of the tree is 23° (see figure). Find the height h of the tree.



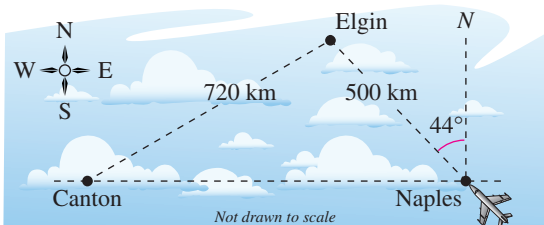
- 36. Height** A flagpole at a right angle to the horizontal is located on a slope that makes an angle of 12° with the horizontal. The flagpole's shadow is 16 meters long and points directly up the slope. The angle of elevation from the tip of the shadow to the sun is 20° .

- Draw a triangle that represents the problem. Show the known quantities on the triangle and use a variable to indicate the height of the flagpole.
- Write an equation involving the unknown quantity.
- Find the height of the flagpole.

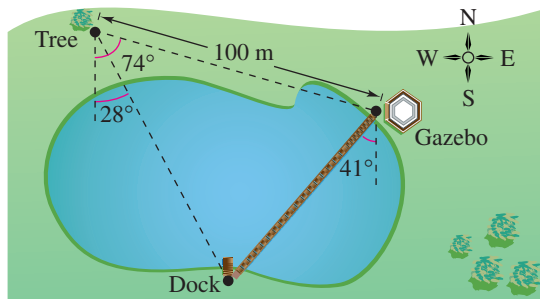
- 37. Angle of Elevation** A 10-meter telephone pole casts a 17-meter shadow directly down a slope when the angle of elevation of the sun is 42° (see figure). Find θ , the angle of elevation of the ground.



- 38. Flight Path** A plane flies 500 kilometers with a bearing of 316° from Naples to Elgin (see figure). The plane then flies 720 kilometers from Elgin to Canton. Find the bearing of the flight from Elgin to Canton.



- 39. Bridge Design** A bridge is to be built across a small lake from a gazebo to a dock (see figure). The bearing from the gazebo to the dock is $S 41^\circ W$. From a tree 100 meters from the gazebo, the bearings to the gazebo and the dock are $S 74^\circ E$ and $S 28^\circ E$, respectively. Find the distance from the gazebo to the dock.



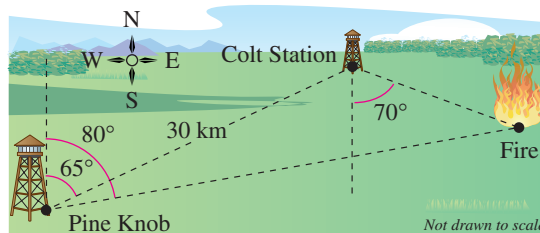
- 40. Railroad Track Design** The circular arc of a railroad curve has a chord of length 3000 feet and a central angle of 40° .

- Draw a diagram that visually represents the problem. Show the known quantities on the diagram and use the variables r and s to represent the radius of the arc and the length of the arc, respectively.
- Find the radius r of the circular arc.
- Find the length s of the circular arc.

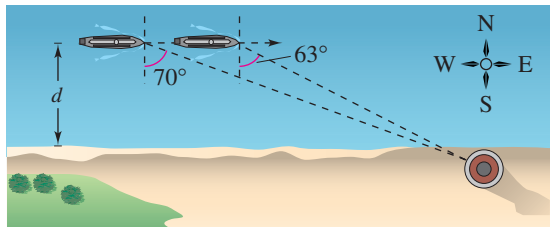
- 41. Glide Path** A pilot has just started on the glide path for landing at an airport with a runway of length 9000 feet. The angles of depression from the plane to the ends of the runway are 17.5° and 18.8° .

- Draw a diagram that visually represents the problem.
- Find the air distance the plane must travel until touching down on the near end of the runway.
- Find the ground distance the plane must travel until touching down.
- Find the altitude of the plane when the pilot begins the descent.

- 42. Locating a Fire** The bearing from the Pine Knob fire tower to the Colt Station fire tower is $N 65^\circ E$, and the two towers are 30 kilometers apart. A fire spotted by rangers in each tower has a bearing of $N 80^\circ E$ from Pine Knob and $S 70^\circ E$ from Colt Station (see figure). Find the distance of the fire from each tower.

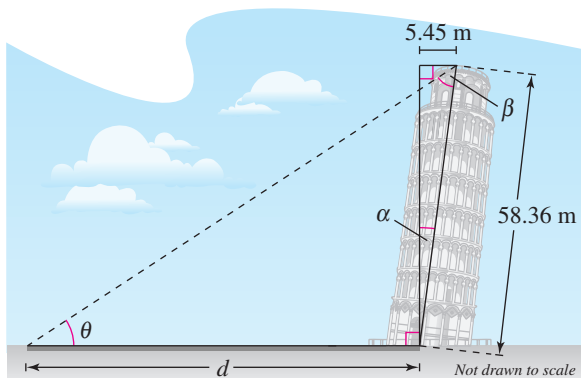


- 43. Distance** A boat is sailing due east parallel to the shoreline at a speed of 10 miles per hour. At a given time, the bearing to the lighthouse is S 70° E, and 15 minutes later the bearing is S 63° E (see figure). The lighthouse is located at the shoreline. What is the distance from the boat to the shoreline?



Model It

- 44. Shadow Length** The Leaning Tower of Pisa in Italy is characterized by its tilt. The tower leans because it was built on a layer of unstable soil—clay, sand, and water. The tower is approximately 58.36 meters tall from its foundation (see figure). The top of the tower leans about 5.45 meters off center.



- Find the angle of lean α of the tower.
- Write β as a function of d and θ , where θ is the angle of elevation to the sun.
- Use the Law of Sines to write an equation for the length d of the shadow cast by the tower.
- Use a graphing utility to complete the table.

| θ | 10° | 20° | 30° | 40° | 50° | 60° |
|----------|------------|------------|------------|------------|------------|------------|
| d | | | | | | |

Synthesis

True or False? In Exercises 45 and 46, determine whether the statement is true or false. Justify your answer.

- If a triangle contains an obtuse angle, then it must be oblique.
- Two angles and one side of a triangle do not necessarily determine a unique triangle.

- 47. Graphical and Numerical Analysis** In the figure, α and β are positive angles.

- Write α as a function of β .
- Use a graphing utility to graph the function. Determine its domain and range.
- Use the result of part (a) to write c as a function of β .
- Use a graphing utility to graph the function in part (c). Determine its domain and range.
- Complete the table. What can you infer?

| β | 0.4 | 0.8 | 1.2 | 1.6 | 2.0 | 2.4 | 2.8 |
|----------|-----|-----|-----|-----|-----|-----|-----|
| α | | | | | | | |
| c | | | | | | | |

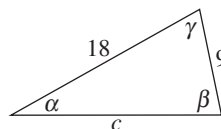


FIGURE FOR 47

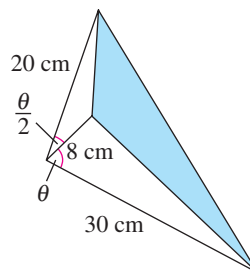


FIGURE FOR 48

- 48. Graphical Analysis**

- Write the area A of the shaded region in the figure as a function of θ .
- Use a graphing utility to graph the area function.
- Determine the domain of the area function. Explain how the area of the region and the domain of the function would change if the eight-centimeter line segment were decreased in length.

Skills Review

In Exercises 49–52, use the fundamental trigonometric identities to simplify the expression.

49. $\sin x \cot x$

50. $\tan x \cos x \sec x$

51. $1 - \sin^2\left(\frac{\pi}{2} - x\right)$

52. $1 + \cot^2\left(\frac{\pi}{2} - x\right)$

6.2 Law of Cosines

What you should learn

- Use the Law of Cosines to solve oblique triangles (SSS or SAS).
- Use the Law of Cosines to model and solve real-life problems.
- Use Heron's Area Formula to find the area of a triangle.

Why you should learn it

You can use the Law of Cosines to solve real-life problems involving oblique triangles. For instance, in Exercise 31 on page 444, you can use the Law of Cosines to approximate the length of a marsh.



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Introduction

Two cases remain in the list of conditions needed to solve an oblique triangle—SSS and SAS. If you are given three sides (SSS), or two sides and their included angle (SAS), none of the ratios in the Law of Sines would be complete. In such cases, you can use the **Law of Cosines**.

Law of Cosines

Standard Form

Alternative Form

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

For a proof of the Law of Cosines, see Proofs in Mathematics on page 490.

Example 1 Three Sides of a Triangle—SSS

Find the three angles of the triangle in Figure 6.11.

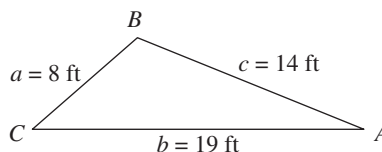


FIGURE 6.11

Solution

It is a good idea first to find the angle opposite the longest side—side b in this case. Using the alternative form of the Law of Cosines, you find that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{8^2 + 14^2 - 19^2}{2(8)(14)} \approx -0.45089.$$

Because $\cos B$ is negative, you know that B is an *obtuse* angle given by $B \approx 116.80^\circ$. At this point, it is simpler to use the Law of Sines to determine A .

$$\sin A = a \left(\frac{\sin B}{b} \right) \approx 8 \left(\frac{\sin 116.80^\circ}{19} \right) \approx 0.37583$$

Because B is obtuse, A must be acute, because a triangle can have, at most, one obtuse angle. So, $A \approx 22.08^\circ$ and $C \approx 180^\circ - 22.08^\circ - 116.80^\circ = 41.12^\circ$.



CHECKPOINT

Now try Exercise 1.

Exploration

What familiar formula do you obtain when you use the third form of the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

and you let $C = 90^\circ$? What is the relationship between the Law of Cosines and this formula?

Do you see why it was wise to find the largest angle *first* in Example 1? Knowing the cosine of an angle, you can determine whether the angle is acute or obtuse. That is,

$$\cos \theta > 0 \quad \text{for} \quad 0^\circ < \theta < 90^\circ \quad \text{Acute}$$

$$\cos \theta < 0 \quad \text{for} \quad 90^\circ < \theta < 180^\circ. \quad \text{Obtuse}$$

So, in Example 1, once you found that angle B was obtuse, you knew that angles A and C were both acute. If the largest angle is acute, the remaining two angles are acute also.

Example 2 Two Sides and the Included Angle—SAS

Find the remaining angles and side of the triangle in Figure 6.12.

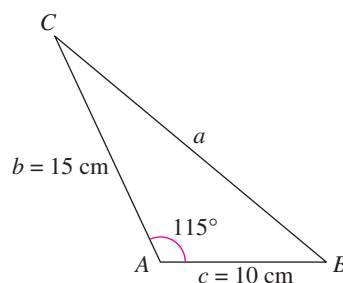


FIGURE 6.12

Solution

Use the Law of Cosines to find the unknown side a in the figure.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 15^2 + 10^2 - 2(15)(10) \cos 115^\circ$$

$$a^2 \approx 451.79$$

$$a \approx 21.26$$

Because $a \approx 21.26$ centimeters, you now know the ratio $\sin A/a$ and you can use the reciprocal form of the Law of Sines to solve for B .

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin A}{a} \\ \sin B &= b \left(\frac{\sin A}{a} \right) \\ &= 15 \left(\frac{\sin 115^\circ}{21.26} \right) \\ &\approx 0.63945 \end{aligned}$$

So, $B = \arcsin 0.63945 \approx 39.75^\circ$ and $C \approx 180^\circ - 115^\circ - 39.75^\circ = 25.25^\circ$.

CHECKPOINT Now try Exercise 3.

STUDY TIP

When solving an oblique triangle given three sides, you use the alternative form of the Law of Cosines to solve for an angle. When solving an oblique triangle given two sides and their included angle, you use the standard form of the Law of Cosines to solve for an unknown.

Applications

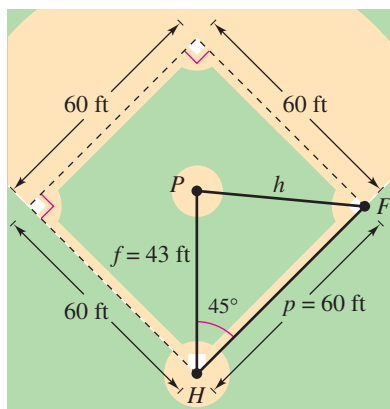


FIGURE 6.13

Example 3 An Application of the Law of Cosines



The pitcher's mound on a women's softball field is 43 feet from home plate and the distance between the bases is 60 feet, as shown in Figure 6.13. (The pitcher's mound is not halfway between home plate and second base.) How far is the pitcher's mound from first base?

Solution

In triangle HPF , $H = 45^\circ$ (line HP bisects the right angle at H), $f = 43$, and $p = 60$. Using the Law of Cosines for this SAS case, you have

$$\begin{aligned} h^2 &= f^2 + p^2 - 2fp \cos H \\ &= 43^2 + 60^2 - 2(43)(60) \cos 45^\circ \approx 1800.3 \end{aligned}$$

So, the approximate distance from the pitcher's mound to first base is

$$h \approx \sqrt{1800.3} \approx 42.43 \text{ feet.}$$



CHECKPOINT

Now try Exercise 31.

Example 4 An Application of the Law of Cosines



A ship travels 60 miles due east, then adjusts its course northward, as shown in Figure 6.14. After traveling 80 miles in that direction, the ship is 139 miles from its point of departure. Describe the bearing from point B to point C .

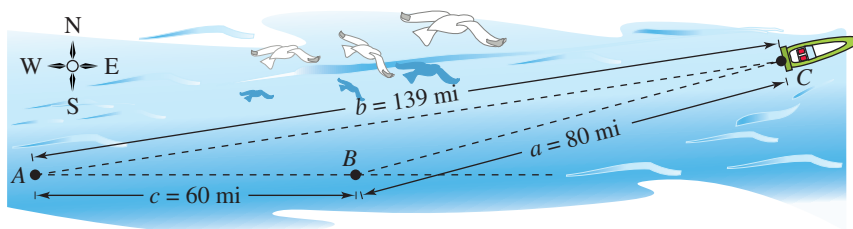


FIGURE 6.14

Solution

You have $a = 80$, $b = 139$, and $c = 60$; so, using the alternative form of the Law of Cosines, you have

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{80^2 + 60^2 - 139^2}{2(80)(60)} \\ &\approx -0.97094. \end{aligned}$$

So, $B \approx \arccos(-0.97094) \approx 166.15^\circ$, and thus the bearing measured from due north from point B to point C is $166.15^\circ - 90^\circ = 76.15^\circ$, or N 76.15° E.



CHECKPOINT

Now try Exercise 37.

Historical Note

Heron of Alexandria (c. 100 B.C.) was a Greek geometer and inventor. His works describe how to find the areas of triangles, quadrilaterals, regular polygons having 3 to 12 sides, and circles as well as the surface areas and volumes of three-dimensional objects.

Heron's Area Formula

The Law of Cosines can be used to establish the following formula for the area of a triangle. This formula is called **Heron's Area Formula** after the Greek mathematician Heron (c. 100 B.C.).

Heron's Area Formula

Given any triangle with sides of lengths a , b , and c , the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a + b + c)/2$.

For a proof of Heron's Area Formula, see Proofs in Mathematics on page 491.

Example 5 Using Heron's Area Formula

Find the area of a triangle having sides of lengths $a = 43$ meters, $b = 53$ meters, and $c = 72$ meters.

Solution

Because $s = (a + b + c)/2 = 168/2 = 84$, Heron's Area Formula yields

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{84(41)(31)(12)} \approx 1131.89 \text{ square meters.}\end{aligned}$$



CHECKPOINT

Now try Exercise 47.

You have now studied three different formulas for the area of a triangle.

Standard Formula $\text{Area} = \frac{1}{2}bh$

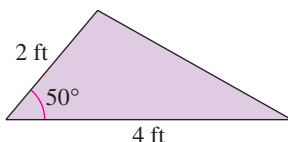
Oblique Triangle $\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$

Heron's Area Formula $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

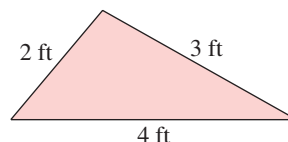
WRITING ABOUT MATHEMATICS

The Area of a Triangle Use the most appropriate formula to find the area of each triangle below. Show your work and give your reasons for choosing each formula.

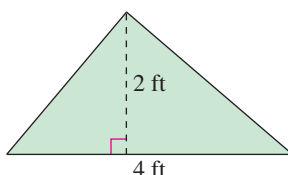
a.



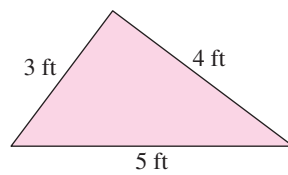
b.



c.



d.



6.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

- If you are given three sides of a triangle, you would use the Law of _____ to find the three angles of the triangle.
- The standard form of the Law of Cosines for $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ is _____.
- The Law of Cosines can be used to establish a formula for finding the area of a triangle called _____ Formula.

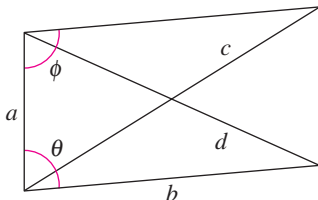
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–16, use the Law of Cosines to solve the triangle. Round your answers to two decimal places.

-
-
-
-

- $a = 11$, $b = 14$, $c = 20$
- $a = 55$, $b = 25$, $c = 72$
- $a = 75.4$, $b = 52$, $c = 52$
- $a = 1.42$, $b = 0.75$, $c = 1.25$
- $A = 135^\circ$, $b = 4$, $c = 9$
- $A = 55^\circ$, $b = 3$, $c = 10$
- $B = 10^\circ 35'$, $a = 40$, $c = 30$
- $B = 75^\circ 20'$, $a = 6.2$, $c = 9.5$
- $B = 125^\circ 40'$, $a = 32$, $c = 32$
- $C = 15^\circ 15'$, $a = 6.25$, $b = 2.15$
- $C = 43^\circ$, $a = \frac{4}{9}$, $b = \frac{7}{9}$
- $C = 103^\circ$, $a = \frac{3}{8}$, $b = \frac{3}{4}$

In Exercises 17–22, complete the table by solving the parallelogram shown in the figure. (The lengths of the diagonals are given by c and d .)



| | a | b | c | d | θ | ϕ |
|-----|-----|-----|-----|-----|------------|-------------|
| 17. | 5 | 8 | | | 45° | |
| 18. | 25 | 35 | | | | 120° |
| 19. | 10 | 14 | 20 | | | |
| 20. | 40 | 60 | | 80 | | |
| 21. | 15 | | 25 | 20 | | |
| 22. | | 25 | 50 | 35 | | |

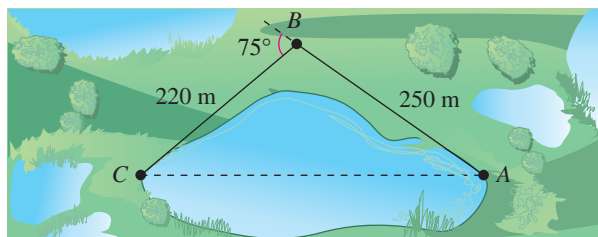
In Exercises 23–28, use Heron's Area Formula to find the area of the triangle.

- $a = 5$, $b = 7$, $c = 10$
- $a = 12$, $b = 15$, $c = 9$
- $a = 2.5$, $b = 10.2$, $c = 9$
- $a = 75.4$, $b = 52$, $c = 52$
- $a = 12.32$, $b = 8.46$, $c = 15.05$
- $a = 3.05$, $b = 0.75$, $c = 2.45$

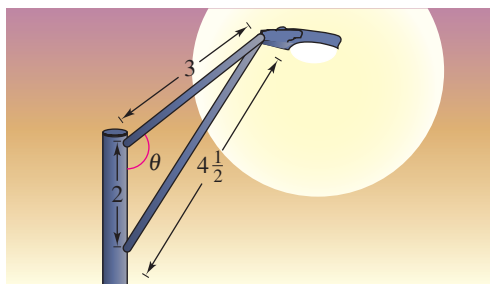
29. Navigation A boat race runs along a triangular course marked by buoys A , B , and C . The race starts with the boats headed west for 3700 meters. The other two sides of the course lie to the north of the first side, and their lengths are 1700 meters and 3000 meters. Draw a figure that gives a visual representation of the problem, and find the bearings for the last two legs of the race.

30. Navigation A plane flies 810 miles from Franklin to Centerville with a bearing of 75° . Then it flies 648 miles from Centerville to Rosemount with a bearing of 32° . Draw a figure that visually represents the problem, and find the straight-line distance and bearing from Franklin to Rosemount.

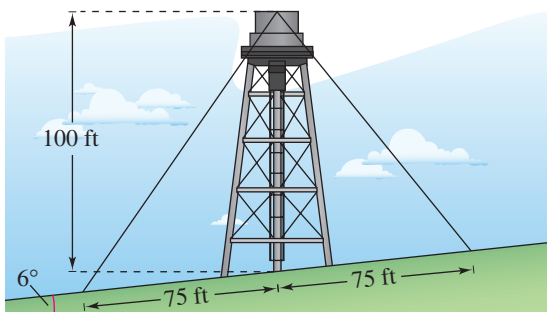
- 31. Surveying** To approximate the length of a marsh, a surveyor walks 250 meters from point A to point B , then turns 75° and walks 220 meters to point C (see figure). Approximate the length AC of the marsh.



- 32. Surveying** A triangular parcel of land has 115 meters of frontage, and the other boundaries have lengths of 76 meters and 92 meters. What angles does the frontage make with the two other boundaries?
- 33. Surveying** A triangular parcel of ground has sides of lengths 725 feet, 650 feet, and 575 feet. Find the measure of the largest angle.
- 34. Streetlight Design** Determine the angle θ in the design of the streetlight shown in the figure.



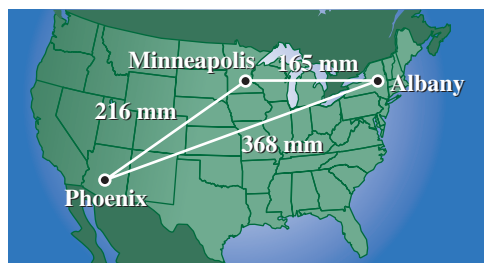
- 35. Distance** Two ships leave a port at 9 A.M. One travels at a bearing of $N 53^\circ W$ at 12 miles per hour, and the other travels at a bearing of $S 67^\circ W$ at 16 miles per hour. Approximate how far apart they are at noon that day.
- 36. Length** A 100-foot vertical tower is to be erected on the side of a hill that makes a 6° angle with the horizontal (see figure). Find the length of each of the two guy wires that will be anchored 75 feet uphill and downhill from the base of the tower.



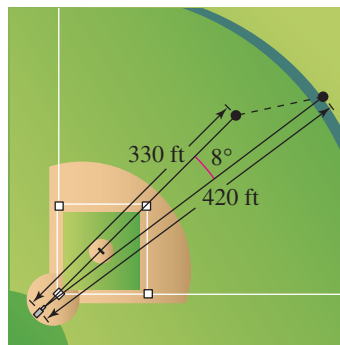
- 37. Navigation** On a map, Orlando is 178 millimeters due south of Niagara Falls, Denver is 273 millimeters from Orlando, and Denver is 235 millimeters from Niagara Falls (see figure).



- (a) Find the bearing of Denver from Orlando.
- (b) Find the bearing of Denver from Niagara Falls.
- 38. Navigation** On a map, Minneapolis is 165 millimeters due west of Albany, Phoenix is 216 millimeters from Minneapolis, and Phoenix is 368 millimeters from Albany (see figure).



- (a) Find the bearing of Minneapolis from Phoenix.
- (b) Find the bearing of Albany from Phoenix.
- 39. Baseball** On a baseball diamond with 90-foot sides, the pitcher's mound is 60.5 feet from home plate. How far is it from the pitcher's mound to third base?
- 40. Baseball** The baseball player in center field is playing approximately 330 feet from the television camera that is behind home plate. A batter hits a fly ball that goes to the wall 420 feet from the camera (see figure). The camera turns 8° to follow the play. Approximately how far does the center fielder have to run to make the catch?



- 41. Aircraft Tracking** To determine the distance between two aircraft, a tracking station continuously determines the distance to each aircraft and the angle A between them (see figure). Determine the distance a between the planes when $A = 42^\circ$, $b = 35$ miles, and $c = 20$ miles.

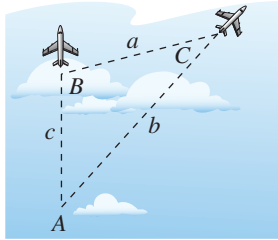
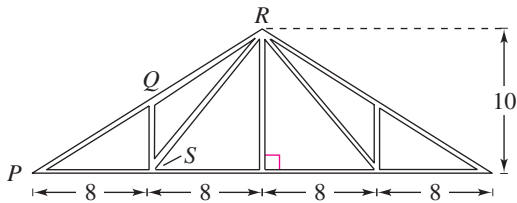


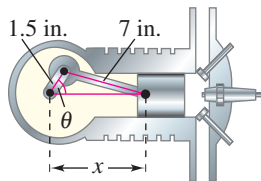
FIGURE FOR 41

- 42. Aircraft Tracking** Use the figure for Exercise 41 to determine the distance a between the planes when $A = 11^\circ$, $b = 20$ miles, and $c = 20$ miles.
- 43. Trusses** Q is the midpoint of the line segment \overline{PR} in the truss rafter shown in the figure. What are the lengths of the line segments PQ , QS , and RS ?



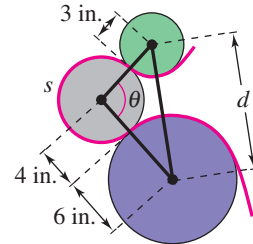
Model It

- 44. Engine Design** An engine has a seven-inch connecting rod fastened to a crank (see figure).



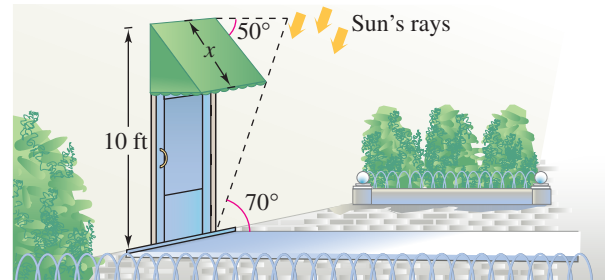
- Use the Law of Cosines to write an equation giving the relationship between x and θ .
- Write x as a function of θ . (Select the sign that yields positive values of x .)
- Use a graphing utility to graph the function in part (b).
- Use the graph in part (c) to determine the maximum distance the piston moves in one cycle.

- 45. Paper Manufacturing** In a process with continuous paper, the paper passes across three rollers of radii 3 inches, 4 inches, and 6 inches (see figure). The centers of the three-inch and six-inch rollers are d inches apart, and the length of the arc in contact with the paper on the four-inch roller is s inches. Complete the table.

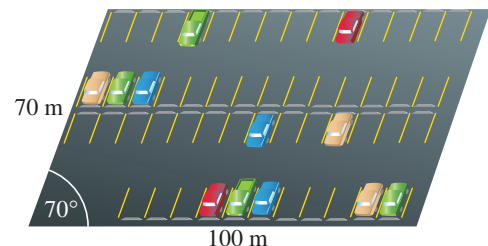


| | | | | | | | |
|--------------------|---|----|----|----|----|----|----|
| d (inches) | 9 | 10 | 12 | 13 | 14 | 15 | 16 |
| θ (degrees) | | | | | | | |
| s (inches) | | | | | | | |

- 46. Awning Design** A retractable awning above a patio door lowers at an angle of 50° from the exterior wall at a height of 10 feet above the ground (see figure). No direct sunlight is to enter the door when the angle of elevation of the sun is greater than 70° . What is the length x of the awning?



- 47. Geometry** The lengths of the sides of a triangular parcel of land are approximately 200 feet, 500 feet, and 600 feet. Approximate the area of the parcel.
- 48. Geometry** A parking lot has the shape of a parallelogram (see figure). The lengths of two adjacent sides are 70 meters and 100 meters. The angle between the two sides is 70° . What is the area of the parking lot?



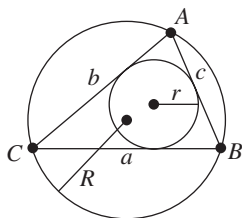
- 49. Geometry** You want to buy a triangular lot measuring 510 yards by 840 yards by 1120 yards. The price of the land is \$2000 per acre. How much does the land cost? (Hint: 1 acre = 4840 square yards)
- 50. Geometry** You want to buy a triangular lot measuring 1350 feet by 1860 feet by 2490 feet. The price of the land is \$2200 per acre. How much does the land cost? (Hint: 1 acre = 43,560 square feet)

Synthesis

True or False? In Exercises 51–53, determine whether the statement is true or false. Justify your answer.

- 51.** In Heron's Area Formula, s is the average of the lengths of the three sides of the triangle.
- 52.** In addition to SSS and SAS, the Law of Cosines can be used to solve triangles with SSA conditions.
- 53.** A triangle with side lengths of 10 centimeters, 16 centimeters, and 5 centimeters can be solved using the Law of Cosines.
- 54. Circumscribed and Inscribed Circles** Let R and r be the radii of the circumscribed and inscribed circles of a triangle ABC , respectively (see figure), and let

$$s = \frac{a + b + c}{2}.$$



- (a) Prove that $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- (b) Prove that $r = \frac{\sqrt{(s-a)(s-b)(s-c)}}{s}$.

Circumscribed and Inscribed Circles In Exercises 55 and 56, use the results of Exercise 54.

- 55.** Given a triangle with $a = 25$, $b = 55$, and $c = 72$ find the areas of (a) the triangle, (b) the circumscribed circle, and (c) the inscribed circle.
- 56.** Find the length of the largest circular running track that can be built on a triangular piece of property with sides of lengths 200 feet, 250 feet, and 325 feet.

57. Proof Use the Law of Cosines to prove that

$$\frac{1}{2}bc(1 + \cos A) = \frac{a + b + c}{2} \cdot \frac{-a + b + c}{2}.$$

58. Proof Use the Law of Cosines to prove that

$$\frac{1}{2}bc(1 - \cos A) = \frac{a - b + c}{2} \cdot \frac{a + b - c}{2}.$$

Skills Review

In Exercises 59–64, evaluate the expression without using a calculator.

- 59.** $\arcsin(-1)$
- 60.** $\arccos 0$
- 61.** $\arctan \sqrt{3}$
- 62.** $\arctan(-\sqrt{3})$
- 63.** $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
- 64.** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

f In Exercises 65–68, write an algebraic expression that is equivalent to the expression.

- 65.** $\sec(\arcsin 2x)$
- 66.** $\tan(\arccos 3x)$
- 67.** $\cot[\arctan(x - 2)]$
- 68.** $\cos\left(\arcsin \frac{x - 1}{2}\right)$

f In Exercises 69–72, use trigonometric substitution to write the algebraic equation as a trigonometric function of θ , where $-\pi/2 < \theta < \pi/2$. Then find $\sec \theta$ and $\csc \theta$.

- 69.** $5 = \sqrt{25 - x^2}$, $x = 5 \sin \theta$
- 70.** $-\sqrt{2} = \sqrt{4 - x^2}$, $x = 2 \cos \theta$
- 71.** $-\sqrt{3} = \sqrt{x^2 - 9}$, $x = 3 \sec \theta$
- 72.** $12 = \sqrt{36 + x^2}$, $x = 6 \tan \theta$

In Exercises 73 and 74, write the sum or difference as a product.

- 73.** $\cos \frac{5\pi}{6} - \cos \frac{\pi}{3}$
- 74.** $\sin\left(x - \frac{\pi}{2}\right) - \sin\left(x + \frac{\pi}{2}\right)$

6.3 Vectors in the Plane

What you should learn

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent them graphically.
- Write vectors as linear combinations of unit vectors.
- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.

Why you should learn it

You can use vectors to model and solve real-life problems involving magnitude and direction. For instance, in Exercise 84 on page 459, you can use vectors to determine the true direction of a commercial jet.



Bill Bachman/Photo Researchers, Inc.

Introduction

Quantities such as force and velocity involve both *magnitude* and *direction* and cannot be completely characterized by a single real number. To represent such a quantity, you can use a **directed line segment**, as shown in Figure 6.15. The directed line segment \overrightarrow{PQ} has **initial point** P and **terminal point** Q . Its **magnitude** (or length) is denoted by $\|\overrightarrow{PQ}\|$ and can be found using the Distance Formula.

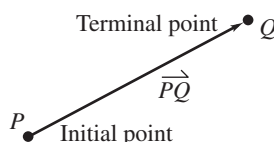


FIGURE 6.15

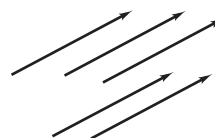


FIGURE 6.16

Two directed line segments that have the same magnitude and direction are equivalent. For example, the directed line segments in Figure 6.16 are all equivalent. The set of all directed line segments that are equivalent to the directed line segment \overrightarrow{PQ} is a **vector \mathbf{v} in the plane**, written $\mathbf{v} = \overrightarrow{PQ}$. Vectors are denoted by lowercase, boldface letters such as \mathbf{u} , \mathbf{v} , and \mathbf{w} .

Example 1 Vector Representation by Directed Line Segments

Let \mathbf{u} be represented by the directed line segment from $P = (0, 0)$ to $Q = (3, 2)$, and let \mathbf{v} be represented by the directed line segment from $R = (1, 2)$ to $S = (4, 4)$, as shown in Figure 6.17. Show that $\mathbf{u} = \mathbf{v}$.

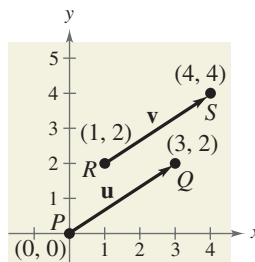


FIGURE 6.17

Solution

From the Distance Formula, it follows that \overrightarrow{PQ} and \overrightarrow{RS} have the *same magnitude*.

$$\|\overrightarrow{PQ}\| = \sqrt{(3 - 0)^2 + (2 - 0)^2} = \sqrt{13}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4 - 1)^2 + (4 - 2)^2} = \sqrt{13}$$

Moreover, both line segments have the *same direction* because they are both directed toward the upper right on lines having a slope of $\frac{2}{3}$. So, \overrightarrow{PQ} and \overrightarrow{RS} have the same magnitude and direction, and it follows that $\mathbf{u} = \mathbf{v}$.



CHECKPOINT

Now try Exercise 1.

Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This representative of the vector \mathbf{v} is in **standard position**.

A vector whose initial point is the origin $(0, 0)$ can be uniquely represented by the coordinates of its terminal point (v_1, v_2) . This is the **component form of a vector \mathbf{v}** , written as

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are the *components* of \mathbf{v} . If both the initial point and the terminal point lie at the origin, \mathbf{v} is the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Technology

You can graph vectors with a graphing utility by graphing directed line segments. Consult the user's guide for your graphing utility for specific instructions.

Component Form of a Vector

The component form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of \mathbf{v} is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, \mathbf{v} is a **unit vector**. Moreover, $\|\mathbf{v}\| = 0$ if and only if \mathbf{v} is the zero vector $\mathbf{0}$.

Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are *equal* if and only if $u_1 = v_1$ and $u_2 = v_2$. For instance, in Example 1, the vector \mathbf{u} from $P = (0, 0)$ to $Q = (3, 2)$ is

$$\mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$$

and the vector \mathbf{v} from $R = (1, 2)$ to $S = (4, 4)$ is

$$\mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle.$$

Example 2 Finding the Component Form of a Vector

Find the component form and magnitude of the vector \mathbf{v} that has initial point $(4, -7)$ and terminal point $(-1, 5)$.

Solution

Let $P = (4, -7) = (p_1, p_2)$ and let $Q = (-1, 5) = (q_1, q_2)$, as shown in Figure 6.18. Then, the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

$$v_1 = q_1 - p_1 = -1 - 4 = -5$$

$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

So, $\mathbf{v} = \langle -5, 12 \rangle$ and the magnitude of \mathbf{v} is

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(-5)^2 + 12^2} \\ &= \sqrt{169} = 13. \end{aligned}$$

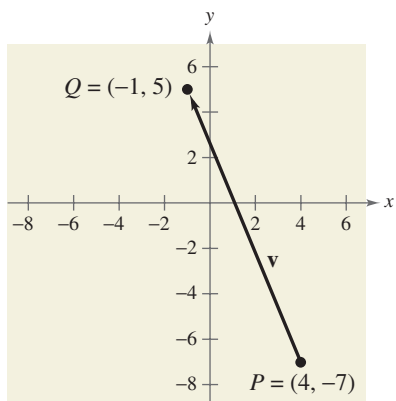


FIGURE 6.18



CHECKPOINT

Now try Exercise 9.

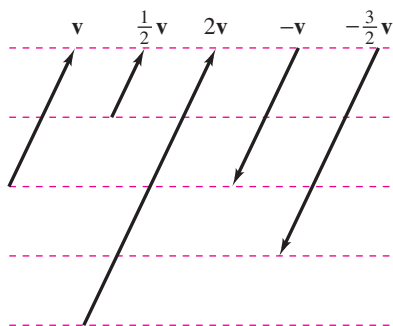


FIGURE 6.19

Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition**. In operations with vectors, numbers are usually referred to as **scalars**. In this text, scalars will always be real numbers. Geometrically, the product of a vector \mathbf{v} and a scalar k is the vector that is $|k|$ times as long as \mathbf{v} . If k is positive, $k\mathbf{v}$ has the same direction as \mathbf{v} , and if k is negative, $k\mathbf{v}$ has the direction opposite that of \mathbf{v} , as shown in Figure 6.19.

To add two vectors geometrically, position them (without changing their lengths or directions) so that the initial point of one coincides with the terminal point of the other. The sum $\mathbf{u} + \mathbf{v}$ is formed by joining the initial point of the second vector \mathbf{v} with the terminal point of the first vector \mathbf{u} , as shown in Figure 6.20. This technique is called the **parallelogram law** for vector addition because the vector $\mathbf{u} + \mathbf{v}$, often called the **resultant** of vector addition, is the diagonal of a parallelogram having \mathbf{u} and \mathbf{v} as its adjacent sides.

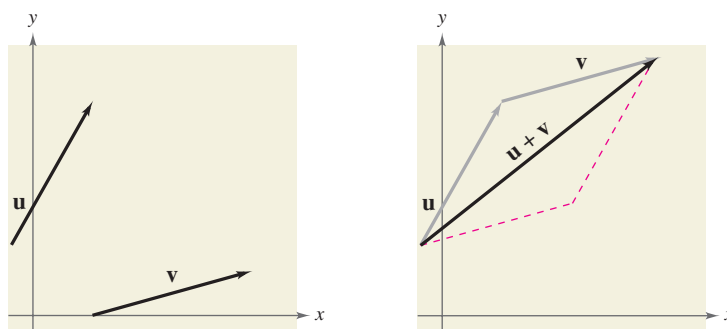


FIGURE 6.20

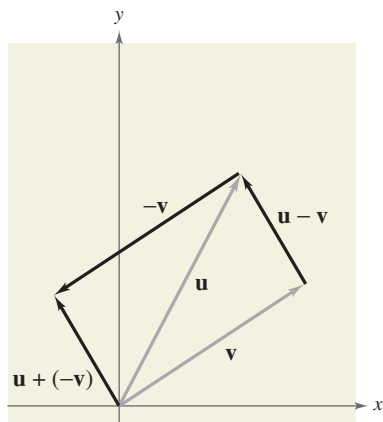
Definitions of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the **sum** of \mathbf{u} and \mathbf{v} is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \quad \text{Sum}$$

and the **scalar multiple** of k times \mathbf{u} is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \quad \text{Scalar multiple}$$



$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

FIGURE 6.21

The **negative** of $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\begin{aligned} -\mathbf{v} &= (-1)\mathbf{v} \\ &= \langle -v_1, -v_2 \rangle \end{aligned} \quad \text{Negative}$$

and the **difference** of \mathbf{u} and \mathbf{v} is

$$\begin{aligned} \mathbf{u} - \mathbf{v} &= \mathbf{u} + (-\mathbf{v}) \\ &= \langle u_1 - v_1, u_2 - v_2 \rangle. \end{aligned} \quad \begin{array}{l} \text{Add } (-\mathbf{v}). \text{ See Figure 8.21.} \\ \text{Difference} \end{array}$$

To represent $\mathbf{u} - \mathbf{v}$ geometrically, you can use directed line segments with the *same* initial point. The difference $\mathbf{u} - \mathbf{v}$ is the vector from the terminal point of \mathbf{v} to the terminal point of \mathbf{u} , which is equal to $\mathbf{u} + (-\mathbf{v})$, as shown in Figure 6.21.

The component definitions of vector addition and scalar multiplication are illustrated in Example 3. In this example, notice that each of the vector operations can be interpreted geometrically.

Example 3 Vector Operations

Let $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$, and find each of the following vectors.

- a. $2\mathbf{v}$ b. $\mathbf{w} - \mathbf{v}$ c. $\mathbf{v} + 2\mathbf{w}$

Solution

- a. Because $\mathbf{v} = \langle -2, 5 \rangle$, you have

$$\begin{aligned} 2\mathbf{v} &= 2\langle -2, 5 \rangle \\ &= \langle 2(-2), 2(5) \rangle \\ &= \langle -4, 10 \rangle. \end{aligned}$$

A sketch of $2\mathbf{v}$ is shown in Figure 6.22.

- b. The difference of \mathbf{w} and \mathbf{v} is

$$\begin{aligned} \mathbf{w} - \mathbf{v} &= \langle 3 - (-2), 4 - 5 \rangle \\ &= \langle 5, -1 \rangle. \end{aligned}$$

A sketch of $\mathbf{w} - \mathbf{v}$ is shown in Figure 6.23. Note that the figure shows the vector difference $\mathbf{w} - \mathbf{v}$ as the sum $\mathbf{w} + (-\mathbf{v})$.

- c. The sum of \mathbf{v} and $2\mathbf{w}$ is

$$\begin{aligned} \mathbf{v} + 2\mathbf{w} &= \langle -2, 5 \rangle + 2\langle 3, 4 \rangle \\ &= \langle -2, 5 \rangle + \langle 2(3), 2(4) \rangle \\ &= \langle -2, 5 \rangle + \langle 6, 8 \rangle \\ &= \langle -2 + 6, 5 + 8 \rangle \\ &= \langle 4, 13 \rangle. \end{aligned}$$

A sketch of $\mathbf{v} + 2\mathbf{w}$ is shown in Figure 6.24.

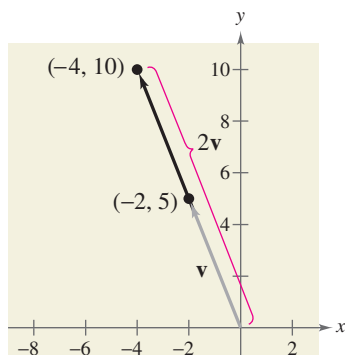


FIGURE 6.22

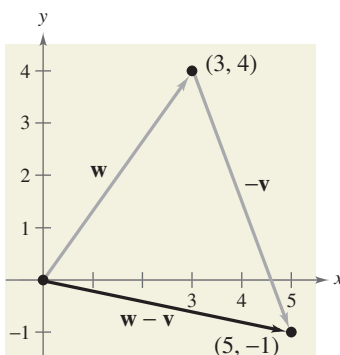


FIGURE 6.23

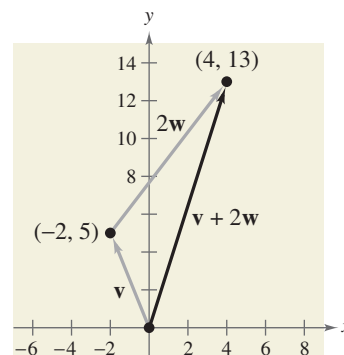


FIGURE 6.24



CHECKPOINT

Now try Exercise 21.

Vector addition and scalar multiplication share many of the properties of ordinary arithmetic.

Properties of Vector Addition and Scalar Multiplication

Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors and let c and d be scalars. Then the following properties are true.

- | | |
|---|--|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ |
| 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ |
| 5. $c(d\mathbf{u}) = (cd)\mathbf{u}$ | 6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | 8. $1(\mathbf{u}) = \mathbf{u}, 0(\mathbf{u}) = \mathbf{0}$ |
| 9. $\ c\mathbf{v}\ = c \ \mathbf{v}\ $ | |

Property 9 can be stated as follows: the magnitude of the vector $c\mathbf{v}$ is the absolute value of c times the magnitude of \mathbf{v} .

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector \mathbf{v} . To do this, you can divide \mathbf{v} by its magnitude to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|} \right) \mathbf{v}. \quad \text{Unit vector in direction of } \mathbf{v}$$

Note that \mathbf{u} is a scalar multiple of \mathbf{v} . The vector \mathbf{u} has a magnitude of 1 and the same direction as \mathbf{v} . The vector \mathbf{u} is called a **unit vector in the direction of \mathbf{v}** .

Example 4 Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5 \rangle$ and verify that the result has a magnitude of 1.

Solution

The unit vector in the direction of \mathbf{v} is

$$\begin{aligned} \frac{\mathbf{v}}{\|\mathbf{v}\|} &= \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + (5)^2}} \\ &= \frac{1}{\sqrt{29}} \langle -2, 5 \rangle \\ &= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle. \end{aligned}$$

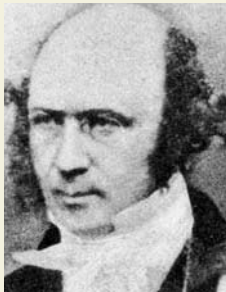
This vector has a magnitude of 1 because

$$\sqrt{\left(\frac{-2}{\sqrt{29}} \right)^2 + \left(\frac{5}{\sqrt{29}} \right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$



CHECKPOINT Now try Exercise 31.

The Granger Collection



Historical Note

William Rowan Hamilton (1805–1865), an Irish mathematician, did some of the earliest work with vectors. Hamilton spent many years developing a system of vector-like quantities called quaternions. Although Hamilton was convinced of the benefits of quaternions, the operations he defined did not produce good models for physical phenomena. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.

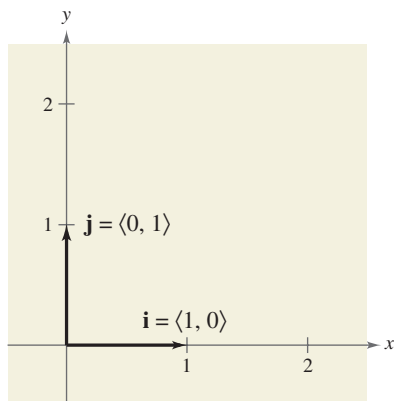


FIGURE 6.25

The unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are called the **standard unit vectors** and are denoted by

$$\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle$$

as shown in Figure 6.25. (Note that the lowercase letter **i** is written in boldface to distinguish it from the imaginary number $i = \sqrt{-1}$.) These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle$, as follows.

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} \end{aligned}$$

The scalars v_1 and v_2 are called the **horizontal** and **vertical components of \mathbf{v}** , respectively. The vector sum

$$v_1 \mathbf{i} + v_2 \mathbf{j}$$

is called a **linear combination** of the vectors \mathbf{i} and \mathbf{j} . Any vector in the plane can be written as a linear combination of the standard unit vectors \mathbf{i} and \mathbf{j} .

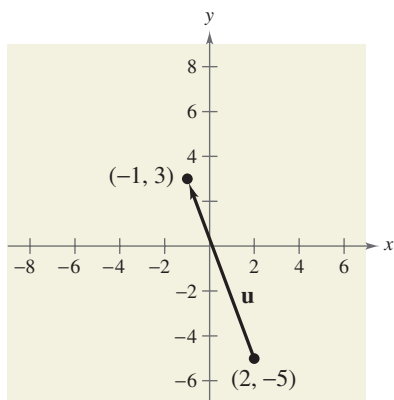


FIGURE 6.26

Example 5 Writing a Linear Combination of Unit Vectors

Let \mathbf{u} be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$. Write \mathbf{u} as a linear combination of the standard unit vectors \mathbf{i} and \mathbf{j} .

Solution

Begin by writing the component form of the vector \mathbf{u} .

$$\begin{aligned} \mathbf{u} &= \langle -1 - 2, 3 - (-5) \rangle \\ &= \langle -3, 8 \rangle \\ &= -3\mathbf{i} + 8\mathbf{j} \end{aligned}$$

This result is shown graphically in Figure 6.26.

 **CHECKPOINT** Now try Exercise 43.

Example 6 Vector Operations

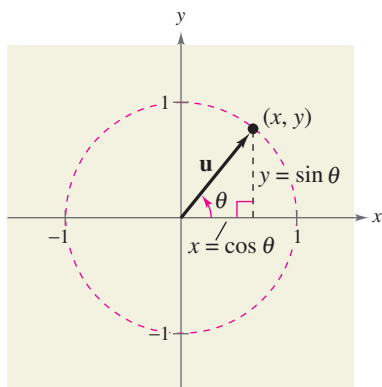
Let $\mathbf{u} = -3\mathbf{i} + 8\mathbf{j}$ and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Find $2\mathbf{u} - 3\mathbf{v}$.

Solution

You could solve this problem by converting \mathbf{u} and \mathbf{v} to component form. This, however, is not necessary. It is just as easy to perform the operations in unit vector form.

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} &= 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) \\ &= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} \\ &= -12\mathbf{i} + 19\mathbf{j} \end{aligned}$$

 **CHECKPOINT** Now try Exercise 49.

FIGURE 6.27 $\|u\| = 1$

Direction Angles

If u is a *unit vector* such that θ is the angle (measured counterclockwise) from the positive x -axis to u , the terminal point of u lies on the unit circle and you have

$$u = \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$$

as shown in Figure 6.27. The angle θ is the **direction angle** of the vector u .

Suppose that u is a unit vector with direction angle θ . If $v = a\mathbf{i} + b\mathbf{j}$ is any vector that makes an angle θ with the positive x -axis, it has the same direction as u and you can write

$$\begin{aligned} v &= \|v\| \langle \cos \theta, \sin \theta \rangle \\ &= \|v\| (\cos \theta)\mathbf{i} + \|v\| (\sin \theta)\mathbf{j}. \end{aligned}$$

Because $v = a\mathbf{i} + b\mathbf{j} = \|v\| (\cos \theta)\mathbf{i} + \|v\| (\sin \theta)\mathbf{j}$, it follows that the direction angle θ for v is determined from

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} && \text{Quotient identity} \\ &= \frac{\|v\| \sin \theta}{\|v\| \cos \theta} && \text{Multiply numerator and denominator by } \|v\|. \\ &= \frac{b}{a}. && \text{Simplify.} \end{aligned}$$

Example 7 Finding Direction Angles of Vectors

Find the direction angle of each vector.

a. $u = 3\mathbf{i} + 3\mathbf{j}$

b. $v = 3\mathbf{i} - 4\mathbf{j}$

Solution

a. The direction angle is

$$\tan \theta = \frac{b}{a} = \frac{3}{3} = 1.$$

So, $\theta = 45^\circ$, as shown in Figure 6.28.

b. The direction angle is

$$\tan \theta = \frac{b}{a} = \frac{-4}{3}.$$

Moreover, because $v = 3\mathbf{i} - 4\mathbf{j}$ lies in Quadrant IV, θ lies in Quadrant IV and its reference angle is

$$\theta = \left| \arctan\left(-\frac{4}{3}\right) \right| \approx |-53.13^\circ| = 53.13^\circ.$$

So, it follows that $\theta \approx 360^\circ - 53.13^\circ = 306.87^\circ$, as shown in Figure 6.29.

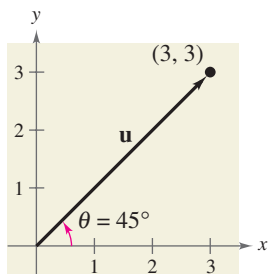


FIGURE 6.28

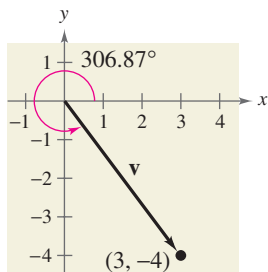


FIGURE 6.29



CHECKPOINT Now try Exercise 55.

Applications of Vectors

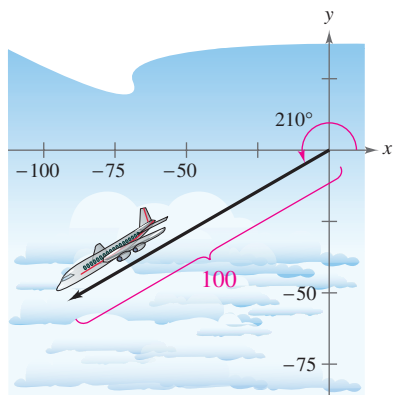


FIGURE 6.30

Example 8 Finding the Component Form of a Vector

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 100 miles per hour at an angle 30° below the horizontal, as shown in Figure 6.30.

Solution

The velocity vector \mathbf{v} has a magnitude of 100 and a direction angle of $\theta = 210^\circ$.

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\| (\cos \theta)\mathbf{i} + \|\mathbf{v}\| (\sin \theta)\mathbf{j} \\ &= 100(\cos 210^\circ)\mathbf{i} + 100(\sin 210^\circ)\mathbf{j} \\ &= 100\left(-\frac{\sqrt{3}}{2}\right)\mathbf{i} + 100\left(-\frac{1}{2}\right)\mathbf{j} \\ &= -50\sqrt{3}\mathbf{i} - 50\mathbf{j} \\ &= \langle -50\sqrt{3}, -50 \rangle\end{aligned}$$

You can check that \mathbf{v} has a magnitude of 100, as follows.

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-50\sqrt{3})^2 + (-50)^2} \\ &= \sqrt{7500 + 2500} \\ &= \sqrt{10,000} = 100\end{aligned}$$

**CHECKPOINT**

Now try Exercise 77.

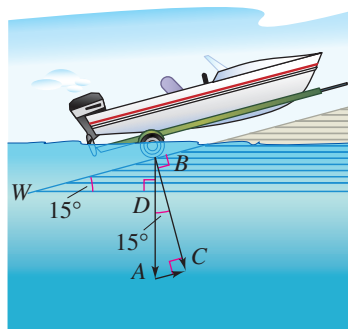


FIGURE 6.31

Example 9 Using Vectors to Determine Weight

A force of 600 pounds is required to pull a boat and trailer up a ramp inclined at 15° from the horizontal. Find the combined weight of the boat and trailer.

Solution

Based on Figure 6.31, you can make the following observations.

$$\begin{aligned}\|\vec{BA}\| &= \text{force of gravity} = \text{combined weight of boat and trailer} \\ \|\vec{BC}\| &= \text{force against ramp} \\ \|\vec{AC}\| &= \text{force required to move boat up ramp} = 600 \text{ pounds}\end{aligned}$$

By construction, triangles BWD and ABC are similar. So, angle ABC is 15° , and so in triangle ABC you have

$$\begin{aligned}\sin 15^\circ &= \frac{\|\vec{AC}\|}{\|\vec{BA}\|} = \frac{600}{\|\vec{BA}\|} \\ \|\vec{BA}\| &= \frac{600}{\sin 15^\circ} \approx 2318.\end{aligned}$$

Consequently, the combined weight is approximately 2318 pounds. (In Figure 6.31, note that \vec{AC} is parallel to the ramp.)

**CHECKPOINT**

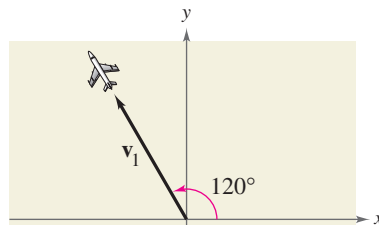
Now try Exercise 81.

STUDY TIP

Recall from Section 4.8 that in air navigation, bearings are measured in degrees clockwise from north.

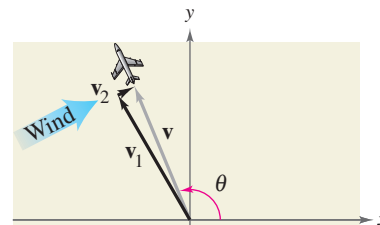
Example 10 Using Vectors to Find Speed and Direction

An airplane is traveling at a speed of 500 miles per hour with a bearing of 330° at a fixed altitude with a negligible wind velocity as shown in Figure 6.32(a). When the airplane reaches a certain point, it encounters a wind with a velocity of 70 miles per hour in the direction $N 45^\circ E$, as shown in Figure 6.32(b). What are the resultant speed and direction of the airplane?



(a)

FIGURE 6.32



(b)

Solution

Using Figure 6.32, the velocity of the airplane (alone) is

$$\begin{aligned}\mathbf{v}_1 &= 500\langle \cos 120^\circ, \sin 120^\circ \rangle \\ &= \langle -250, 250\sqrt{3} \rangle\end{aligned}$$

and the velocity of the wind is

$$\begin{aligned}\mathbf{v}_2 &= 70\langle \cos 45^\circ, \sin 45^\circ \rangle \\ &= \langle 35\sqrt{2}, 35\sqrt{2} \rangle.\end{aligned}$$

So, the velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2 \\ &= \langle -250 + 35\sqrt{2}, 250\sqrt{3} + 35\sqrt{2} \rangle \\ &\approx \langle -200.5, 482.5 \rangle\end{aligned}$$

and the resultant speed of the airplane is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-200.5)^2 + (482.5)^2} \\ &\approx 522.5 \text{ miles per hour.}\end{aligned}$$

Finally, if θ is the direction angle of the flight path, you have

$$\begin{aligned}\tan \theta &= \frac{482.5}{-200.5} \\ &\approx -2.4065\end{aligned}$$

which implies that

$$\theta \approx 180^\circ + \arctan(-2.4065) \approx 180^\circ - 67.4^\circ = 112.6^\circ.$$

So, the true direction of the airplane is 337.4° .

**CHECKPOINT**

Now try Exercise 83.

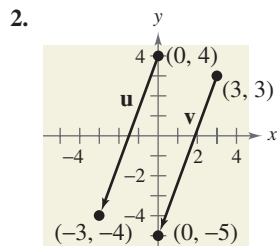
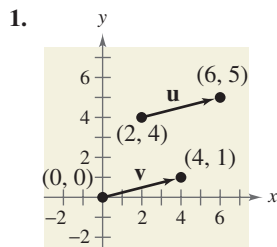
6.3 Exercises

VOCABULARY CHECK: Fill in the blanks.

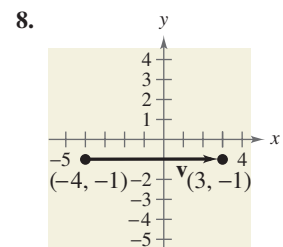
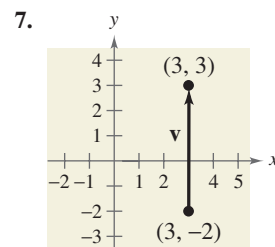
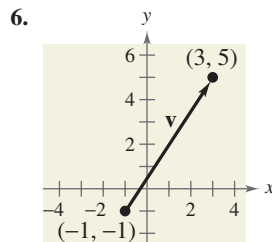
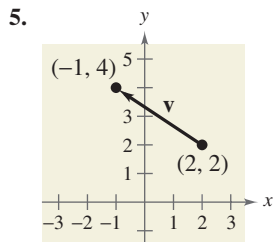
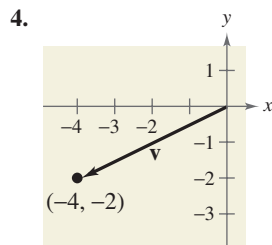
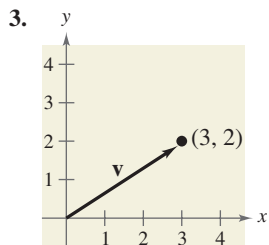
1. A _____ can be used to represent a quantity that involves both magnitude and direction.
2. The directed line segment \overrightarrow{PQ} has _____ point P and _____ point Q .
3. The _____ of the directed line segment \overrightarrow{PQ} is denoted by $\|\overrightarrow{PQ}\|$.
4. The set of all directed line segments that are equivalent to a given directed line segment \overrightarrow{PQ} is a _____ \mathbf{v} in the plane.
5. The directed line segment whose initial point is the origin is said to be in _____.
6. A vector that has a magnitude of 1 is called a _____.
7. The two basic vector operations are scalar _____ and vector _____.
8. The vector $\mathbf{u} + \mathbf{v}$ is called the _____ of vector addition.
9. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ is called a _____ of the vectors \mathbf{i} and \mathbf{j} , and the scalars v_1 and v_2 are called the _____ and _____ components of \mathbf{v} , respectively.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1 and 2, show that $\mathbf{u} = \mathbf{v}$.



In Exercises 3–14, find the component form and the magnitude of the vector \mathbf{v} .

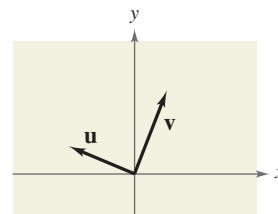


Initial Point

Terminal Point

- | | |
|----------------|------------|
| 9. $(-1, 5)$ | $(15, 12)$ |
| 10. $(1, 11)$ | $(9, 3)$ |
| 11. $(-3, -5)$ | $(5, 1)$ |
| 12. $(-3, 11)$ | $(9, 40)$ |
| 13. $(1, 3)$ | $(-8, -9)$ |
| 14. $(-2, 7)$ | $(5, -17)$ |

In Exercises 15–20, use the figure to sketch a graph of the specified vector. To print an enlarged copy of the graph, go to the website, www.mathgraphs.com.



- | | |
|--------------------------------|--|
| 15. $-\mathbf{v}$ | 16. $5\mathbf{v}$ |
| 17. $\mathbf{u} + \mathbf{v}$ | 18. $\mathbf{u} - \mathbf{v}$ |
| 19. $\mathbf{u} + 2\mathbf{v}$ | 20. $\mathbf{v} - \frac{1}{2}\mathbf{u}$ |

In Exercises 21–28, find (a) $\mathbf{u} + \mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $2\mathbf{u} - 3\mathbf{v}$. Then sketch the resultant vector.

21. $\mathbf{u} = \langle 2, 1 \rangle$, $\mathbf{v} = \langle 1, 3 \rangle$
 22. $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle 4, 0 \rangle$
 23. $\mathbf{u} = \langle -5, 3 \rangle$, $\mathbf{v} = \langle 0, 0 \rangle$
 24. $\mathbf{u} = \langle 0, 0 \rangle$, $\mathbf{v} = \langle 2, 1 \rangle$
 25. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$
 26. $\mathbf{u} = -2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$
 27. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = \mathbf{j}$
 28. $\mathbf{u} = 3\mathbf{j}$, $\mathbf{v} = 2\mathbf{i}$

In Exercises 29–38, find a unit vector in the direction of the given vector.

29. $\mathbf{u} = \langle 3, 0 \rangle$ 30. $\mathbf{u} = \langle 0, -2 \rangle$
 31. $\mathbf{v} = \langle -2, 2 \rangle$ 32. $\mathbf{v} = \langle 5, -12 \rangle$
 33. $\mathbf{v} = 6\mathbf{i} - 2\mathbf{j}$ 34. $\mathbf{v} = \mathbf{i} + \mathbf{j}$
 35. $\mathbf{w} = 4\mathbf{j}$ 36. $\mathbf{w} = -6\mathbf{i}$
 37. $\mathbf{w} = \mathbf{i} - 2\mathbf{j}$ 38. $\mathbf{w} = 7\mathbf{j} - 3\mathbf{i}$

In Exercises 39–42, find the vector \mathbf{v} with the given magnitude and the same direction as \mathbf{u} .

- | Magnitude | Direction |
|---------------------------|---------------------------------------|
| 39. $\ \mathbf{v}\ = 5$ | $\mathbf{u} = \langle 3, 3 \rangle$ |
| 40. $\ \mathbf{v}\ = 6$ | $\mathbf{u} = \langle -3, 3 \rangle$ |
| 41. $\ \mathbf{v}\ = 9$ | $\mathbf{u} = \langle 2, 5 \rangle$ |
| 42. $\ \mathbf{v}\ = 10$ | $\mathbf{u} = \langle -10, 0 \rangle$ |

In Exercises 43–46, the initial and terminal points of a vector are given. Write a linear combination of the standard unit vectors \mathbf{i} and \mathbf{j} .

- | Initial Point | Terminal Point |
|----------------|----------------|
| 43. $(-3, 1)$ | $(4, 5)$ |
| 44. $(0, -2)$ | $(3, 6)$ |
| 45. $(-1, -5)$ | $(2, 3)$ |
| 46. $(-6, 4)$ | $(0, 1)$ |

In Exercises 47–52, find the component form of \mathbf{v} and sketch the specified vector operations geometrically, where $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$.

47. $\mathbf{v} = \frac{3}{2}\mathbf{u}$
 48. $\mathbf{v} = \frac{3}{4}\mathbf{w}$
 49. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$
 50. $\mathbf{v} = -\mathbf{u} + \mathbf{w}$
 51. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$
 52. $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

In Exercises 53–56, find the magnitude and direction angle of the vector \mathbf{v} .

53. $\mathbf{v} = 3(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$
 54. $\mathbf{v} = 8(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j})$
 55. $\mathbf{v} = 6\mathbf{i} - 6\mathbf{j}$
 56. $\mathbf{v} = -5\mathbf{i} + 4\mathbf{j}$

In Exercises 57–64, find the component form of \mathbf{v} given its magnitude and the angle it makes with the positive x -axis. Sketch \mathbf{v} .

- | Magnitude | Angle |
|------------------------------------|---|
| 57. $\ \mathbf{v}\ = 3$ | $\theta = 0^\circ$ |
| 58. $\ \mathbf{v}\ = 1$ | $\theta = 45^\circ$ |
| 59. $\ \mathbf{v}\ = \frac{7}{2}$ | $\theta = 150^\circ$ |
| 60. $\ \mathbf{v}\ = \frac{5}{2}$ | $\theta = 45^\circ$ |
| 61. $\ \mathbf{v}\ = 3\sqrt{2}$ | $\theta = 150^\circ$ |
| 62. $\ \mathbf{v}\ = 4\sqrt{3}$ | $\theta = 90^\circ$ |
| 63. $\ \mathbf{v}\ = 2$ | \mathbf{v} in the direction $\mathbf{i} + 3\mathbf{j}$ |
| 64. $\ \mathbf{v}\ = 3$ | \mathbf{v} in the direction $3\mathbf{i} + 4\mathbf{j}$ |

In Exercises 65–68, find the component form of the sum of \mathbf{u} and \mathbf{v} with direction angles θ_u and θ_v .

- | Magnitude | Angle |
|---------------------------|------------------------|
| 65. $\ \mathbf{u}\ = 5$ | $\theta_u = 0^\circ$ |
| $\ \mathbf{v}\ = 5$ | $\theta_v = 90^\circ$ |
| 66. $\ \mathbf{u}\ = 4$ | $\theta_u = 60^\circ$ |
| $\ \mathbf{v}\ = 4$ | $\theta_v = 90^\circ$ |
| 67. $\ \mathbf{u}\ = 20$ | $\theta_u = 45^\circ$ |
| $\ \mathbf{v}\ = 50$ | $\theta_v = 180^\circ$ |
| 68. $\ \mathbf{u}\ = 50$ | $\theta_u = 30^\circ$ |
| $\ \mathbf{v}\ = 30$ | $\theta_v = 110^\circ$ |

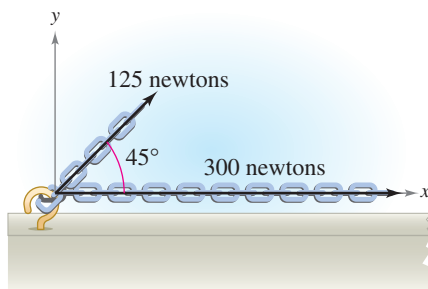
In Exercises 69 and 70, use the Law of Cosines to find the angle α between the vectors. (Assume $0^\circ \leq \alpha \leq 180^\circ$.)

69. $\mathbf{v} = \mathbf{i} + \mathbf{j}$, $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j}$
 70. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$

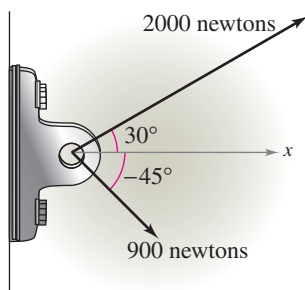
Resultant Force In Exercises 71 and 72, find the angle between the forces given the magnitude of their resultant. (Hint: Write force 1 as a vector in the direction of the positive x -axis and force 2 as a vector at an angle θ with the positive x -axis.)

- | Force 1 | Force 2 | Resultant Force |
|-----------------|-------------|-----------------|
| 71. 45 pounds | 60 pounds | 90 pounds |
| 72. 3000 pounds | 1000 pounds | 3750 pounds |

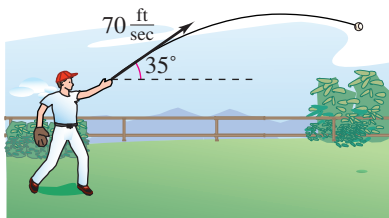
- 73. Resultant Force** Forces with magnitudes of 125 newtons and 300 newtons act on a hook (see figure). The angle between the two forces is 45° . Find the direction and magnitude of the resultant of these forces.



- 74. Resultant Force** Forces with magnitudes of 2000 newtons and 900 newtons act on a machine part at angles of 30° and -45° , respectively, with the x -axis (see figure). Find the direction and magnitude of the resultant of these forces.

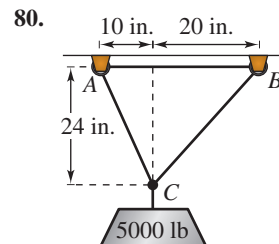
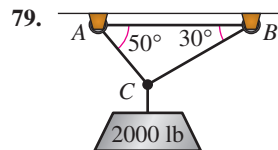


- 75. Resultant Force** Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30° , 45° , and 120° , respectively, with the positive x -axis. Find the direction and magnitude of the resultant of these forces.
- 76. Resultant Force** Three forces with magnitudes of 70 pounds, 40 pounds, and 60 pounds act on an object at angles of -30° , 445° , and 135° , respectively, with the positive x -axis. Find the direction and magnitude of the resultant of these forces.
- 77. Velocity** A ball is thrown with an initial velocity of 70 feet per second, at an angle of 35° with the horizontal (see figure). Find the vertical and horizontal components of the velocity.

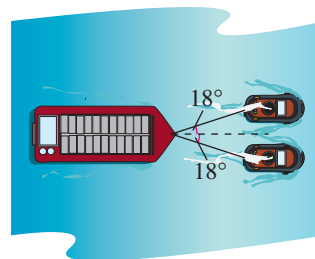


- 78. Velocity** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° with the horizontal. Find the vertical and horizontal components of the velocity.

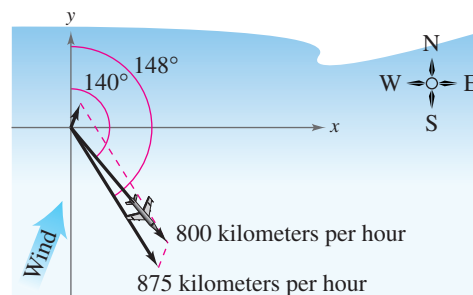
Cable Tension In Exercises 79 and 80, use the figure to determine the tension in each cable supporting the load.



- 81. Tow Line Tension** A loaded barge is being towed by two tugboats, and the magnitude of the resultant is 6000 pounds directed along the axis of the barge (see figure). Find the tension in the tow lines if they each make an 18° angle with the axis of the barge.



- 82. Rope Tension** To carry a 100-pound cylindrical weight, two people lift on the ends of short ropes that are tied to an eyelet on the top center of the cylinder. Each rope makes a 20° angle with the vertical. Draw a figure that gives a visual representation of the problem, and find the tension in the ropes.
- 83. Navigation** An airplane is flying in the direction of 148° , with an airspeed of 875 kilometers per hour. Because of the wind, its groundspeed and direction are 800 kilometers per hour and 140° , respectively (see figure). Find the direction and speed of the wind.



Model It

84. Navigation A commercial jet is flying from Miami to Seattle. The jet's velocity with respect to the air is 580 miles per hour, and its bearing is 332° . The wind, at the altitude of the plane, is blowing from the southwest with a velocity of 60 miles per hour.

- Draw a figure that gives a visual representation of the problem.
- Write the velocity of the wind as a vector in component form.
- Write the velocity of the jet relative to the air in component form.
- What is the speed of the jet with respect to the ground?
- What is the true direction of the jet?

85. Work A heavy implement is pulled 30 feet across a floor, using a force of 100 pounds. The force is exerted at an angle of 50° above the horizontal (see figure). Find the work done. (Use the formula for work, $W = FD$, where F is the component of the force in the direction of motion and D is the distance.)

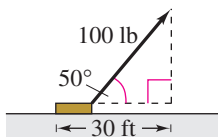


FIGURE FOR 85

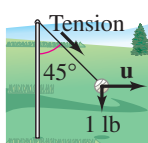


FIGURE FOR 86

86. Rope Tension A tetherball weighing 1 pound is pulled outward from the pole by a horizontal force \mathbf{u} until the rope makes a 45° angle with the pole (see figure). Determine the resulting tension in the rope and the magnitude of \mathbf{u} .

Synthesis



True or False? In Exercises 87 and 88, decide whether the statement is true or false. Justify your answer.

- If \mathbf{u} and \mathbf{v} have the same magnitude and direction, then $\mathbf{u} = \mathbf{v}$.
- If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.
- Think About It** Consider two forces of equal magnitude acting on a point.
 - If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.

- If the resultant of the forces is $\mathbf{0}$, make a conjecture about the angle between the forces.
- Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

90. Graphical Reasoning Consider two forces

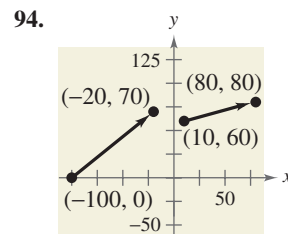
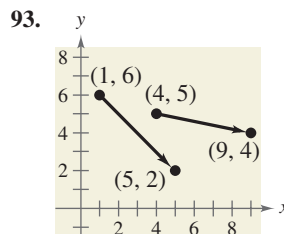
$$\mathbf{F}_1 = \langle 10, 0 \rangle \text{ and } \mathbf{F}_2 = 5\langle \cos \theta, \sin \theta \rangle.$$

- Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$ as a function of θ .
 -  Use a graphing utility to graph the function in part (a) for $0 \leq \theta < 2\pi$.
 -  Use the graph in part (b) to determine the range of the function. What is its maximum, and for what value of θ does it occur? What is its minimum, and for what value of θ does it occur?
 - Explain why the magnitude of the resultant is never 0.
- 91. Proof** Prove that $(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$ is a unit vector for any value of θ .




92. Technology Write a program for your graphing utility that graphs two vectors and their difference given the vectors in component form.

In Exercises 93 and 94, use the program in Exercise 92 to find the difference of the vectors shown in the figure.



Skills Review

 In Exercises 95–98, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ , where $0 < \theta < \pi/2$.

95. $\sqrt{x^2 - 64}$, $x = 8 \sec \theta$

96. $\sqrt{64 - x^2}$, $x = 8 \sin \theta$

97. $\sqrt{x^2 + 36}$, $x = 6 \tan \theta$

98. $\sqrt{(x^2 - 25)^3}$, $x = 5 \sec \theta$

In Exercises 99–102, solve the equation.

99. $\cos x(\cos x + 1) = 0$

100. $\sin x(2 \sin x + \sqrt{2}) = 0$

101. $3 \sec x \sin x - 2\sqrt{3} \sin x = 0$

102. $\cos x \csc x + \cos x \sqrt{2} = 0$

6.4 Vectors and Dot Products

What you should learn

- Find the dot product of two vectors and use the Properties of the Dot Product.
- Find the angle between two vectors and determine whether two vectors are orthogonal.
- Write a vector as the sum of two vector components.
- Use vectors to find the work done by a force.

Why you should learn it

You can use the dot product of two vectors to solve real-life problems involving two vector quantities. For instance, in Exercise 68 on page 468, you can use the dot product to find the force necessary to keep a sport utility vehicle from rolling down a hill.



Edward Ewert

The Dot Product of Two Vectors

So far you have studied two vector operations—vector addition and multiplication by a scalar—each of which yields another vector. In this section, you will study a third vector operation, the **dot product**. This product yields a scalar, rather than a vector.

Definition of the Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

Properties of the Dot Product

Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

For proofs of the properties of the dot product, see Proofs in Mathematics on page 492.

Example 1 Finding Dot Products

Find each dot product.

- a. $\langle 4, 5 \rangle \cdot \langle 2, 3 \rangle$ b. $\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle$ c. $\langle 0, 3 \rangle \cdot \langle 4, -2 \rangle$

Solution

$$\begin{aligned} \text{a. } \langle 4, 5 \rangle \cdot \langle 2, 3 \rangle &= 4(2) + 5(3) \\ &= 8 + 15 \\ &= 23 \end{aligned}$$

$$\text{b. } \langle 2, -1 \rangle \cdot \langle 1, 2 \rangle = 2(1) + (-1)(2) = 2 - 2 = 0$$

$$\text{c. } \langle 0, 3 \rangle \cdot \langle 4, -2 \rangle = 0(4) + 3(-2) = 0 - 6 = -6$$

CHECKPOINT Now try Exercise 1.

In Example 1, be sure you see that the dot product of two vectors is a scalar (a real number), not a vector. Moreover, notice that the dot product can be positive, zero, or negative.

Example 2 Using Properties of Dot Products

Let $\mathbf{u} = \langle -1, 3 \rangle$, $\mathbf{v} = \langle 2, -4 \rangle$, and $\mathbf{w} = \langle 1, -2 \rangle$. Find each dot product.

- a. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ b. $\mathbf{u} \cdot 2\mathbf{v}$

Solution

Begin by finding the dot product of \mathbf{u} and \mathbf{v} .

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \langle -1, 3 \rangle \cdot \langle 2, -4 \rangle \\ &= (-1)(2) + 3(-4) \\ &= -14\end{aligned}$$

$$\begin{aligned}\text{a. } (\mathbf{u} \cdot \mathbf{v})\mathbf{w} &= -14\langle 1, -2 \rangle \\ &= \langle -14, 28 \rangle\end{aligned}$$

$$\begin{aligned}\text{b. } \mathbf{u} \cdot 2\mathbf{v} &= 2(\mathbf{u} \cdot \mathbf{v}) \\ &= 2(-14) \\ &= -28\end{aligned}$$

Notice that the product in part (a) is a vector, whereas the product in part (b) is a scalar. Can you see why?

 **CHECKPOINT** Now try Exercise 11.

Example 3 Dot Product and Magnitude

The dot product of \mathbf{u} with itself is 5. What is the magnitude of \mathbf{u} ?

Solution

Because $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$ and $\mathbf{u} \cdot \mathbf{u} = 5$, it follows that

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{\mathbf{u} \cdot \mathbf{u}} \\ &= \sqrt{5}.\end{aligned}$$

 **CHECKPOINT** Now try Exercise 19.

The Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle θ , $0 \leq \theta \leq \pi$, between their respective standard position vectors, as shown in Figure 6.33. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

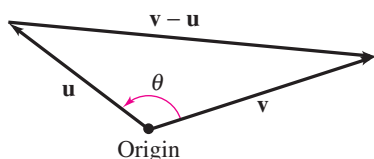


FIGURE 6.33

Angle Between Two Vectors

If θ is the angle between two nonzero vectors \mathbf{u} and \mathbf{v} , then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

For a proof of the angle between two vectors, see Proofs in Mathematics on page 492.

Example 4 Finding the Angle Between Two VectorsFind the angle between $\mathbf{u} = \langle 4, 3 \rangle$ and $\mathbf{v} = \langle 3, 5 \rangle$.**Solution**

$$\begin{aligned}
 \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \\
 &= \frac{\langle 4, 3 \rangle \cdot \langle 3, 5 \rangle}{\|\langle 4, 3 \rangle\| \|\langle 3, 5 \rangle\|} \\
 &= \frac{27}{5\sqrt{34}}
 \end{aligned}$$

This implies that the angle between the two vectors is

$$\theta = \arccos \frac{27}{5\sqrt{34}} \approx 22.2^\circ$$

as shown in Figure 6.34.

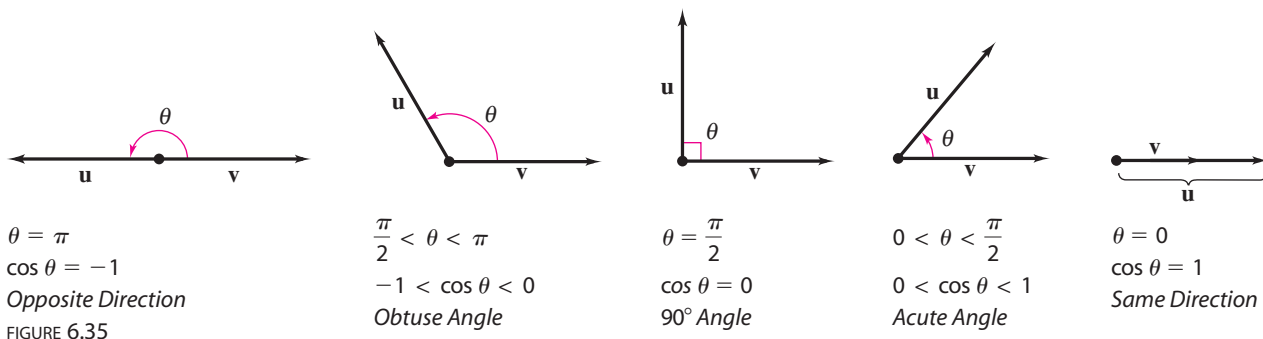
**CHECKPOINT**

Now try Exercise 29.

Rewriting the expression for the angle between two vectors in the form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad \text{Alternative form of dot product}$$

produces an alternative way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos \theta$ will always have the same sign. Figure 6.35 shows the five possible orientations of two vectors.

**Definition of Orthogonal Vectors**The vectors \mathbf{u} and \mathbf{v} are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

The terms *orthogonal* and *perpendicular* mean essentially the same thing—meeting at right angles. Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector. In other words, the zero vector is orthogonal to every vector \mathbf{u} , because $\mathbf{0} \cdot \mathbf{u} = 0$.

Technology

The graphing utility program *Finding the Angle Between Two Vectors*, found on our website college.hmco.com, graphs two vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ in standard position and finds the measure of the angle between them. Use the program to verify the solutions for Examples 4 and 5.

Example 5 Determining Orthogonal Vectors

Are the vectors $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle 6, 4 \rangle$ orthogonal?

Solution

Begin by finding the dot product of the two vectors.

$$\mathbf{u} \cdot \mathbf{v} = \langle 2, -3 \rangle \cdot \langle 6, 4 \rangle = 2(6) + (-3)(4) = 0$$

Because the dot product is 0, the two vectors are orthogonal (see Figure 6.36).

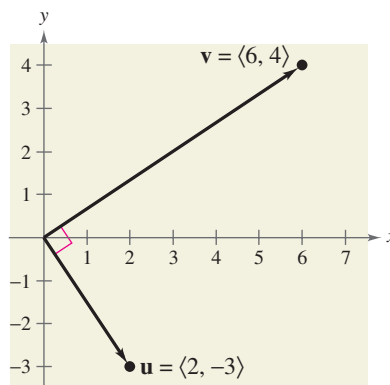


FIGURE 6.36

**CHECKPOINT**

Now try Exercise 47.

Finding Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a given vector into the sum of two **vector components**.

Consider a boat on an inclined ramp, as shown in Figure 6.37. The force \mathbf{F} due to gravity pulls the boat *down* the ramp and *against* the ramp. These two orthogonal forces, \mathbf{w}_1 and \mathbf{w}_2 , are vector components of \mathbf{F} . That is,

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2. \quad \text{Vector components of } \mathbf{F}$$

The negative of component \mathbf{w}_1 represents the force needed to keep the boat from rolling down the ramp, whereas \mathbf{w}_2 represents the force that the tires must withstand against the ramp. A procedure for finding \mathbf{w}_1 and \mathbf{w}_2 is shown on the following page.

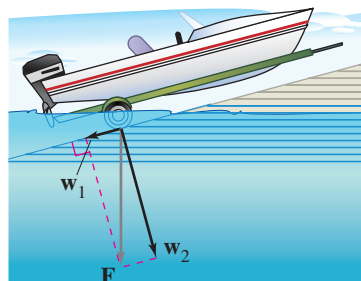


FIGURE 6.37

Definition of Vector Components

Let \mathbf{u} and \mathbf{v} be nonzero vectors such that

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where \mathbf{w}_1 and \mathbf{w}_2 are orthogonal and \mathbf{w}_1 is parallel to (or a scalar multiple of) \mathbf{v} , as shown in Figure 6.38. The vectors \mathbf{w}_1 and \mathbf{w}_2 are called **vector components** of \mathbf{u} . The vector \mathbf{w}_1 is the **projection** of \mathbf{u} onto \mathbf{v} and is denoted by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u}.$$

The vector \mathbf{w}_2 is given by $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

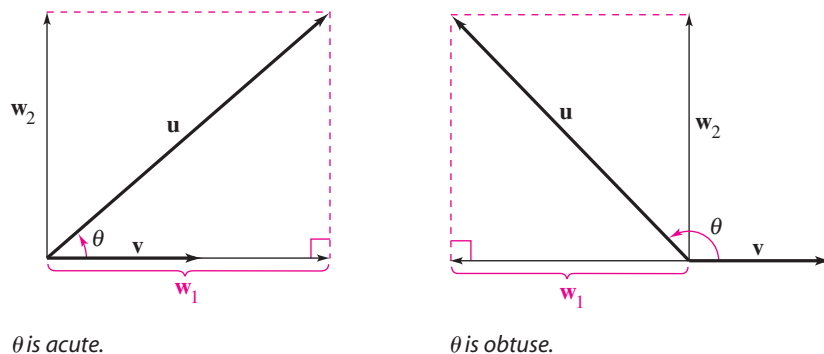


FIGURE 6.38

From the definition of vector components, you can see that it is easy to find the component \mathbf{w}_2 once you have found the projection of \mathbf{u} onto \mathbf{v} . To find the projection, you can use the dot product, as follows.

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2$$

\mathbf{w}_1 is a scalar multiple of \mathbf{v} .

$$\mathbf{u} \cdot \mathbf{v} = (c\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v}$$

Take dot product of each side with \mathbf{v} .

$$= c\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v}$$

$$= c\|\mathbf{v}\|^2 + 0$$

\mathbf{w}_2 and \mathbf{v} are orthogonal.

So,

$$c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$$

and

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = c\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

Projection of \mathbf{u} onto \mathbf{v}

Let \mathbf{u} and \mathbf{v} be nonzero vectors. The projection of \mathbf{u} onto \mathbf{v} is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

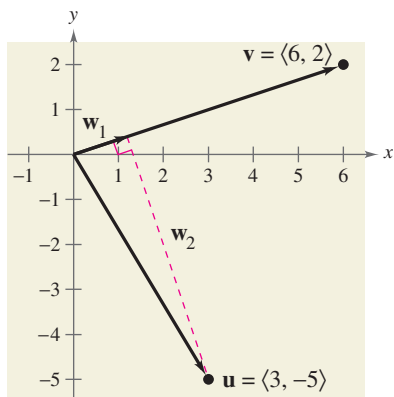


FIGURE 6.39

Example 6 Decomposing a Vector into Components

Find the projection of $\mathbf{u} = \langle 3, -5 \rangle$ onto $\mathbf{v} = \langle 6, 2 \rangle$. Then write \mathbf{u} as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution

The projection of \mathbf{u} onto \mathbf{v} is

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{8}{40} \right) \langle 6, 2 \rangle = \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle$$

as shown in Figure 6.39. The other component, \mathbf{w}_2 , is

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 3, -5 \rangle - \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle = \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle.$$

So,

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle + \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle = \langle 3, -5 \rangle.$$

 **CHECKPOINT** Now try Exercise 53.

Example 7 Finding a Force

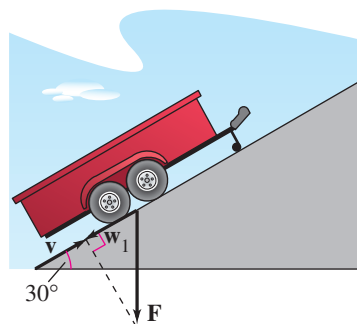


FIGURE 6.40

A 200-pound cart sits on a ramp inclined at 30° , as shown in Figure 6.40. What force is required to keep the cart from rolling down the ramp?

Solution

Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector

$$\mathbf{F} = -200\mathbf{j}. \quad \text{Force due to gravity}$$

To find the force required to keep the cart from rolling down the ramp, project \mathbf{F} onto a unit vector \mathbf{v} in the direction of the ramp, as follows.

$$\mathbf{v} = (\cos 30^\circ)\mathbf{i} + (\sin 30^\circ)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \quad \text{Unit vector along ramp}$$

Therefore, the projection of \mathbf{F} onto \mathbf{v} is

$$\begin{aligned} \mathbf{w}_1 &= \text{proj}_{\mathbf{v}}\mathbf{F} \\ &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= (\mathbf{F} \cdot \mathbf{v})\mathbf{v} \\ &= (-200)\left(\frac{1}{2}\right)\mathbf{v} \\ &= -100\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right). \end{aligned}$$

The magnitude of this force is 100, and so a force of 100 pounds is required to keep the cart from rolling down the ramp.

 **CHECKPOINT** Now try Exercise 67.

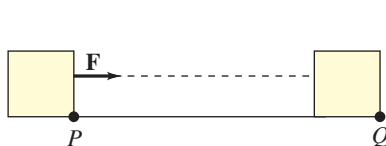
Work

The work W done by a *constant* force \mathbf{F} acting along the line of motion of an object is given by

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\| \|\overrightarrow{PQ}\|$$

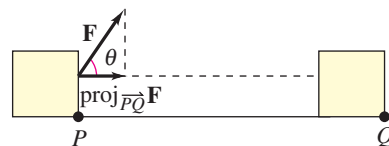
as shown in Figure 6.41. If the constant force \mathbf{F} is not directed along the line of motion, as shown in Figure 6.42, the work W done by the force is given by

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= (\cos \theta) \|\mathbf{F}\| \|\overrightarrow{PQ}\| && \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| = (\cos \theta) \|\mathbf{F}\| \\ &= \mathbf{F} \cdot \overrightarrow{PQ}. && \text{Alternative form of dot product} \end{aligned}$$



Force acts along the line of motion.

FIGURE 6.41



Force acts at angle θ with the line of motion.

FIGURE 6.42

This notion of work is summarized in the following definition.

Definition of Work

The **work** W done by a constant force \mathbf{F} as its point of application moves along the vector \overrightarrow{PQ} is given by either of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \|\overrightarrow{PQ}\|$ Projection form
2. $W = \mathbf{F} \cdot \overrightarrow{PQ}$ Dot product form

Example 8 Finding Work



To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60° , as shown in Figure 6.43. Find the work done in moving the door 12 feet to its closed position.

Solution

Using a projection, you can calculate the work as follows.

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= (\cos 60^\circ) \|\mathbf{F}\| \|\overrightarrow{PQ}\| \\ &= \frac{1}{2}(50)(12) = 300 \text{ foot-pounds} \end{aligned}$$

So, the work done is 300 foot-pounds. You can verify this result by finding the vectors \mathbf{F} and \overrightarrow{PQ} and calculating their dot product.

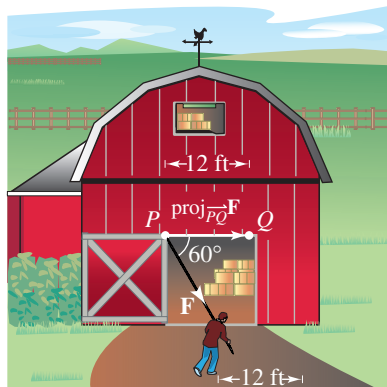


FIGURE 6.43

CHECKPOINT Now try Exercise 69.

6.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

- The _____ of two vectors yields a scalar, rather than a vector.
- If θ is the angle between two nonzero vectors \mathbf{u} and \mathbf{v} , then $\cos \theta = \underline{\hspace{2cm}}$.
- The vectors \mathbf{u} and \mathbf{v} are _____ if $\mathbf{u} \cdot \mathbf{v} = 0$.
- The projection of \mathbf{u} onto \mathbf{v} is given by $\text{proj}_{\mathbf{v}} \mathbf{u} = \underline{\hspace{2cm}}$.
- The work W done by a constant force \mathbf{F} as its point of application moves along the vector \overrightarrow{PQ} is given by $W = \underline{\hspace{2cm}}$ or $W = \underline{\hspace{2cm}}$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–8, find the dot product of \mathbf{u} and \mathbf{v} .

- | | |
|---|---|
| 1. $\mathbf{u} = \langle 6, 1 \rangle$ | 2. $\mathbf{u} = \langle 5, 12 \rangle$ |
| $\mathbf{v} = \langle -2, 3 \rangle$ | $\mathbf{v} = \langle -3, 2 \rangle$ |
| 3. $\mathbf{u} = \langle -4, 1 \rangle$ | 4. $\mathbf{u} = \langle -2, 5 \rangle$ |
| $\mathbf{v} = \langle 2, -3 \rangle$ | $\mathbf{v} = \langle -1, -2 \rangle$ |
| 5. $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j}$ | 6. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ |
| $\mathbf{v} = \mathbf{i} - \mathbf{j}$ | $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j}$ |
| 7. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ | 8. $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ |
| $\mathbf{v} = -2\mathbf{i} - 3\mathbf{j}$ | $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ |

In Exercises 9–18, use the vectors $\mathbf{u} = \langle 2, 2 \rangle$, $\mathbf{v} = \langle -3, 4 \rangle$, and $\mathbf{w} = \langle 1, -2 \rangle$ to find the indicated quantity. State whether the result is a vector or a scalar.

- | | |
|---|---|
| 9. $\mathbf{u} \cdot \mathbf{u}$ | 10. $3\mathbf{u} \cdot \mathbf{v}$ |
| 11. $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$ | 12. $(\mathbf{v} \cdot \mathbf{u})\mathbf{w}$ |
| 13. $(3\mathbf{w} \cdot \mathbf{v})\mathbf{u}$ | 14. $(\mathbf{u} \cdot 2\mathbf{v})\mathbf{w}$ |
| 15. $\ \mathbf{w}\ - 1$ | 16. $2 - \ \mathbf{u}\ $ |
| 17. $(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{w})$ | 18. $(\mathbf{v} \cdot \mathbf{u}) - (\mathbf{w} \cdot \mathbf{v})$ |

In Exercises 19–24, use the dot product to find the magnitude of \mathbf{u} .

- | | |
|--|--|
| 19. $\mathbf{u} = \langle -5, 12 \rangle$ | 20. $\mathbf{u} = \langle 2, -4 \rangle$ |
| 21. $\mathbf{u} = 20\mathbf{i} + 25\mathbf{j}$ | 22. $\mathbf{u} = 12\mathbf{i} - 16\mathbf{j}$ |
| 23. $\mathbf{u} = 6\mathbf{j}$ | 24. $\mathbf{u} = -21\mathbf{i}$ |

In Exercises 25–34, find the angle θ between the vectors.

- | | |
|--|---|
| 25. $\mathbf{u} = \langle 1, 0 \rangle$ | 26. $\mathbf{u} = \langle 3, 2 \rangle$ |
| $\mathbf{v} = \langle 0, -2 \rangle$ | $\mathbf{v} = \langle 4, 0 \rangle$ |
| 27. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ | 28. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ |
| $\mathbf{v} = -2\mathbf{j}$ | $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$ |
| 29. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ | 30. $\mathbf{u} = -6\mathbf{i} - 3\mathbf{j}$ |
| $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j}$ | $\mathbf{v} = -8\mathbf{i} + 4\mathbf{j}$ |

- | | |
|--|--|
| 31. $\mathbf{u} = 5\mathbf{i} + 5\mathbf{j}$ | 32. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ |
| $\mathbf{v} = -6\mathbf{i} + 6\mathbf{j}$ | $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ |

- | |
|--|
| 33. $\mathbf{u} = \cos\left(\frac{\pi}{3}\right)\mathbf{i} + \sin\left(\frac{\pi}{3}\right)\mathbf{j}$ |
| $\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$ |
| 34. $\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j}$ |
| $\mathbf{v} = \cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin\left(\frac{\pi}{2}\right)\mathbf{j}$ |

In Exercises 35–38, graph the vectors and find the degree measure of the angle θ between the vectors.

- | | |
|--|--|
| 35. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ | 36. $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j}$ |
| $\mathbf{v} = -7\mathbf{i} + 5\mathbf{j}$ | $\mathbf{v} = -4\mathbf{i} + 4\mathbf{j}$ |
| 37. $\mathbf{u} = 5\mathbf{i} + 5\mathbf{j}$ | 38. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ |
| $\mathbf{v} = -8\mathbf{i} + 8\mathbf{j}$ | $\mathbf{v} = 8\mathbf{i} + 3\mathbf{j}$ |

In Exercises 39–42, use vectors to find the interior angles of the triangle with the given vertices.

- | | |
|-----------------------------|------------------------------|
| 39. (1, 2), (3, 4), (2, 5) | 40. (-3, -4), (1, 7), (8, 2) |
| 41. (-3, 0), (2, 2), (0, 6) | 42. (-3, 5), (-1, 9), (7, 9) |

In Exercises 43–46, find $\mathbf{u} \cdot \mathbf{v}$, where θ is the angle between \mathbf{u} and \mathbf{v} .

- | |
|--|
| 43. $\ \mathbf{u}\ = 4, \ \mathbf{v}\ = 10, \theta = \frac{2\pi}{3}$ |
| 44. $\ \mathbf{u}\ = 100, \ \mathbf{v}\ = 250, \theta = \frac{\pi}{6}$ |
| 45. $\ \mathbf{u}\ = 9, \ \mathbf{v}\ = 36, \theta = \frac{3\pi}{4}$ |
| 46. $\ \mathbf{u}\ = 4, \ \mathbf{v}\ = 12, \theta = \frac{\pi}{3}$ |

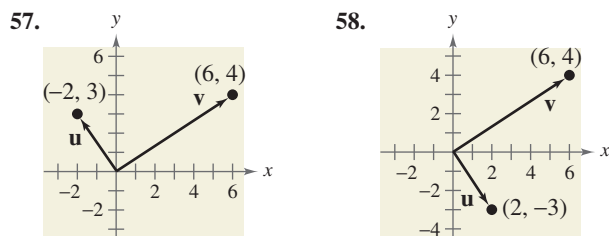
In Exercises 47–52, determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither.

47. $\mathbf{u} = \langle -12, 30 \rangle$ 48. $\mathbf{u} = \langle 3, 15 \rangle$
 $\mathbf{v} = \langle \frac{1}{2}, -\frac{5}{4} \rangle$ $\mathbf{v} = \langle -1, 5 \rangle$
49. $\mathbf{u} = \frac{1}{4}(3\mathbf{i} - \mathbf{j})$ 50. $\mathbf{u} = \mathbf{i}$
 $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$ $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j}$
51. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j}$ 52. $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$
 $\mathbf{v} = -\mathbf{i} - \mathbf{j}$ $\mathbf{v} = \langle \sin \theta, -\cos \theta \rangle$

In Exercises 53–56, find the projection of \mathbf{u} onto \mathbf{v} . Then write \mathbf{u} as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}} \mathbf{u}$.

53. $\mathbf{u} = \langle 2, 2 \rangle$ 54. $\mathbf{u} = \langle 4, 2 \rangle$
 $\mathbf{v} = \langle 6, 1 \rangle$ $\mathbf{v} = \langle 1, -2 \rangle$
55. $\mathbf{u} = \langle 0, 3 \rangle$ 56. $\mathbf{u} = \langle -3, -2 \rangle$
 $\mathbf{v} = \langle 2, 15 \rangle$ $\mathbf{v} = \langle -4, -1 \rangle$

In Exercises 57 and 58, use the graph to determine mentally the projection of \mathbf{u} onto \mathbf{v} . (The coordinates of the terminal points of the vectors in standard position are given.) Use the formula for the projection of \mathbf{u} onto \mathbf{v} to verify your result.



In Exercises 59–62, find two vectors in opposite directions that are orthogonal to the vector \mathbf{u} . (There are many correct answers.)

59. $\mathbf{u} = \langle 3, 5 \rangle$
 60. $\mathbf{u} = \langle -8, 3 \rangle$
 61. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$
 62. $\mathbf{u} = -\frac{5}{2}\mathbf{i} - 3\mathbf{j}$

Work In Exercises 63 and 64, find the work done in moving a particle from P to Q if the magnitude and direction of the force are given by \mathbf{v} .

63. $P = (0, 0)$, $Q = (4, 7)$, $\mathbf{v} = \langle 1, 4 \rangle$
 64. $P = (1, 3)$, $Q = (-3, 5)$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$

65. Revenue The vector $\mathbf{u} = \langle 1650, 3200 \rangle$ gives the numbers of units of two types of baking pans produced by a company. The vector $\mathbf{v} = \langle 15.25, 10.50 \rangle$ gives the prices (in dollars) of the two types of pans, respectively.

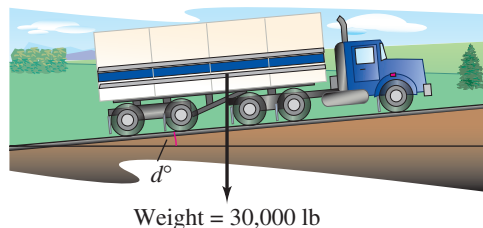
- (a) Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and interpret the result in the context of the problem.
 (b) Identify the vector operation used to increase the prices by 5%.

66. Revenue The vector $\mathbf{u} = \langle 3240, 2450 \rangle$ gives the numbers of hamburgers and hot dogs, respectively, sold at a fast-food stand in one month. The vector $\mathbf{v} = \langle 1.75, 1.25 \rangle$ gives the prices (in dollars) of the food items.

- (a) Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and interpret the result in the context of the problem.
 (b) Identify the vector operation used to increase the prices by 2.5%.

Model It

67. Braking Load A truck with a gross weight of 30,000 pounds is parked on a slope of d° (see figure). Assume that the only force to overcome is the force of gravity.



- (a) Find the force required to keep the truck from rolling down the hill in terms of the slope d .
 (b) Use a graphing utility to complete the table.

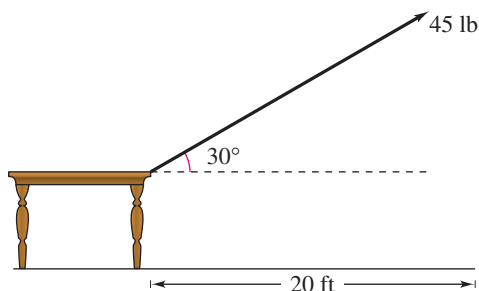
| | | | | | | |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| d | 0° | 1° | 2° | 3° | 4° | 5° |
| Force | | | | | | |

| | | | | | |
|-------|-----------|-----------|-----------|-----------|------------|
| d | 6° | 7° | 8° | 9° | 10° |
| Force | | | | | |

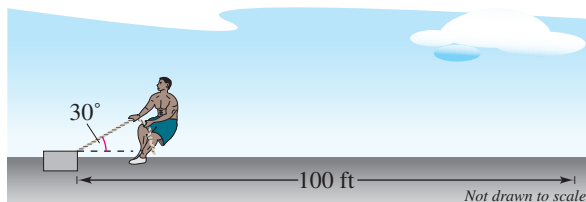
- (c) Find the force perpendicular to the hill when $d = 5^\circ$.

68. Braking Load A sport utility vehicle with a gross weight of 5400 pounds is parked on a slope of 10° . Assume that the only force to overcome is the force of gravity. Find the force required to keep the vehicle from rolling down the hill. Find the force perpendicular to the hill.

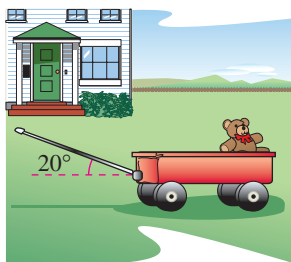
69. **Work** Determine the work done by a person lifting a 25-kilogram (245-newton) bag of sugar.
70. **Work** Determine the work done by a crane lifting a 2400-pound car 5 feet.
71. **Work** A force of 45 pounds exerted at an angle of 30° above the horizontal is required to slide a table across a floor (see figure). The table is dragged 20 feet. Determine the work done in sliding the table.



72. **Work** A tractor pulls a log 800 meters, and the tension in the cable connecting the tractor and log is approximately 1600 kilograms (15,691 newtons). The direction of the force is 35° above the horizontal. Approximate the work done in pulling the log.
73. **Work** One of the events in a local strongman contest is to pull a cement block 100 feet. One competitor pulls the block by exerting a force of 250 pounds on a rope attached to the block at an angle of 30° with the horizontal (see figure). Find the work done in pulling the block.



74. **Work** A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal (see figure). Find the work done in pulling the wagon 50 feet.



Synthesis

True or False? In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.

75. The work W done by a constant force \mathbf{F} acting along the line of motion of an object is represented by a vector.
76. A sliding door moves along the line of vector \overrightarrow{PQ} . If a force is applied to the door along a vector that is orthogonal to \overrightarrow{PQ} , then no work is done.

77. **Think About It** What is known about θ , the angle between two nonzero vectors \mathbf{u} and \mathbf{v} , under each condition?

- (a) $\mathbf{u} \cdot \mathbf{v} = 0$ (b) $\mathbf{u} \cdot \mathbf{v} > 0$ (c) $\mathbf{u} \cdot \mathbf{v} < 0$

78. **Think About It** What can be said about the vectors \mathbf{u} and \mathbf{v} under each condition?

- (a) The projection of \mathbf{u} onto \mathbf{v} equals \mathbf{u} .
 (b) The projection of \mathbf{u} onto \mathbf{v} equals $\mathbf{0}$.

79. **Proof** Use vectors to prove that the diagonals of a rhombus are perpendicular.

80. **Proof** Prove the following.

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

Skills Review

In Exercises 81–84, perform the operation and write the result in standard form.

81. $\sqrt{42} \cdot \sqrt{24}$
 82. $\sqrt{18} \cdot \sqrt{112}$
 83. $\sqrt{-3} \cdot \sqrt{-6}$
 84. $\sqrt{-12} \cdot \sqrt{-96}$

In Exercises 85–88, find all solutions of the equation in the interval $[0, 2\pi)$.

85. $\sin 2x - \sqrt{3} \sin x = 0$
 86. $\sin 2x + \sqrt{2} \cos x = 0$
 87. $2 \tan x = \tan 2x$
 88. $\cos 2x - 3 \sin x = 2$

In Exercises 89–92, find the exact value of the trigonometric function given that $\sin u = -\frac{12}{13}$ and $\cos v = \frac{24}{25}$. (Both u and v are in Quadrant IV.)

89. $\sin(u - v)$
 90. $\sin(u + v)$
 91. $\cos(v - u)$
 92. $\tan(u - v)$

6.5 Trigonometric Form of a Complex Number

What you should learn

- Plot complex numbers in the complex plane and find absolute values of complex numbers.
- Write the trigonometric forms of complex numbers.
- Multiply and divide complex numbers written in trigonometric form.
- Use DeMoivre's Theorem to find powers of complex numbers.
- Find n th roots of complex numbers.

Why you should learn it

You can use the trigonometric form of a complex number to perform operations with complex numbers. For instance, in Exercises 105–112 on page 480, you can use the trigonometric forms of complex numbers to help you solve polynomial equations.

The Complex Plane

Just as real numbers can be represented by points on the real number line, you can represent a complex number

$$z = a + bi$$

as the point (a, b) in a coordinate plane (the **complex plane**). The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**, as shown in Figure 6.44.

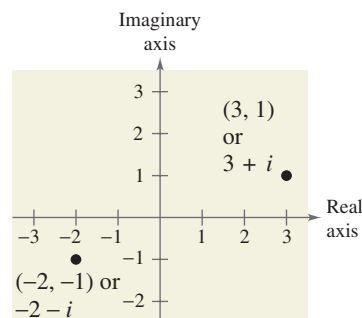


FIGURE 6.44

The **absolute value** of the complex number $a + bi$ is defined as the distance between the origin $(0, 0)$ and the point (a, b) .

Definition of the Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}.$$

If the complex number $a + bi$ is a real number (that is, if $b = 0$), then this definition agrees with that given for the absolute value of a real number

$$|a + 0i| = \sqrt{a^2 + 0^2} = |a|.$$

Example 1

Finding the Absolute Value of a Complex Number

Plot $z = -2 + 5i$ and find its absolute value.

Solution

The number is plotted in Figure 6.45. It has an absolute value of

$$\begin{aligned} |z| &= \sqrt{(-2)^2 + 5^2} \\ &= \sqrt{29}. \end{aligned}$$



CHECKPOINT

Now try Exercise 3.

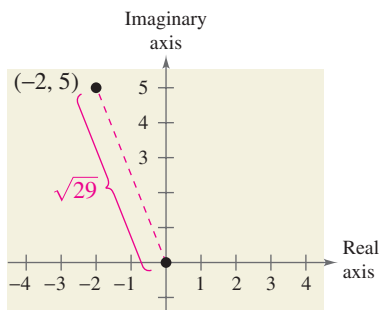


FIGURE 6.45

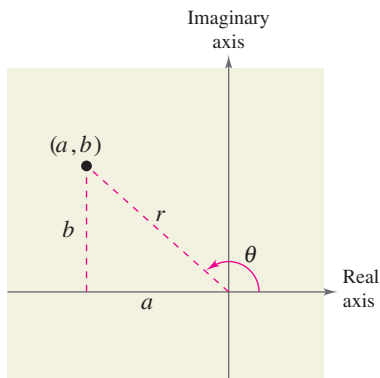


FIGURE 6.46

Trigonometric Form of a Complex Number

In Section 2.4, you learned how to add, subtract, multiply, and divide complex numbers. To work effectively with *powers* and *roots* of complex numbers, it is helpful to write complex numbers in trigonometric form. In Figure 6.46, consider the nonzero complex number $a + bi$. By letting θ be the angle from the positive real axis (measured counterclockwise) to the line segment connecting the origin and the point (a, b) , you can write

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta$$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$a + bi = (r \cos \theta) + (r \sin \theta)i$$

from which you can obtain the **trigonometric form of a complex number**.

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos \theta + i \sin \theta)$$

where $a = r \cos \theta$, $b = r \sin \theta$, $r = \sqrt{a^2 + b^2}$, and $\tan \theta = b/a$. The number r is the **modulus** of z , and θ is called an **argument** of z .

The trigonometric form of a complex number is also called the *polar form*. Because there are infinitely many choices for θ , the trigonometric form of a complex number is not unique. Normally, θ is restricted to the interval $0 \leq \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$.

Example 2 Writing a Complex Number in Trigonometric Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form.

Solution

The absolute value of z is

$$r = |-2 - 2\sqrt{3}i| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$$

and the reference angle θ' is given by

$$\tan \theta' = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Because $\tan(\pi/3) = \sqrt{3}$ and because $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, you choose θ to be $\theta = \pi + \pi/3 = 4\pi/3$. So, the trigonometric form is

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 4\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right). \end{aligned}$$

See Figure 6.47.

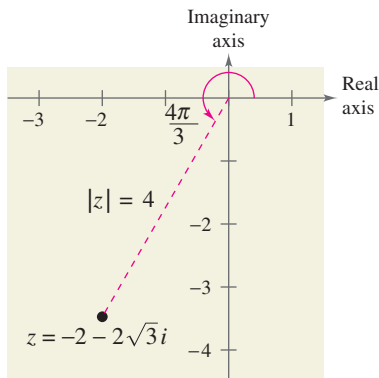


FIGURE 6.47

CHECKPOINT Now try Exercise 13.

Example 3 Writing a Complex Number in Standard Form

Write the complex number in standard form $a + bi$.

$$z = \sqrt{8} \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right]$$

Technology

A graphing utility can be used to convert a complex number in trigonometric (or polar) form to standard form. For specific keystrokes, see the user's manual for your graphing utility.

Solution

Because $\cos(-\pi/3) = \frac{1}{2}$ and $\sin(-\pi/3) = -\sqrt{3}/2$, you can write

$$\begin{aligned} z &= \sqrt{8} \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right] \\ &= 2\sqrt{2} \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \\ &= \sqrt{2} - \sqrt{6}i. \end{aligned}$$

 **CHECKPOINT** Now try Exercise 35.

Multiplication and Division of Complex Numbers

The trigonometric form adapts nicely to multiplication and division of complex numbers. Suppose you are given two complex numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

The product of z_1 and z_2 is given by

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]. \end{aligned}$$

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following rule. The second part is left for you to verify (see Exercise 117).

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \quad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \quad \text{Quotient}$$

Note that this rule says that to *multiply* two complex numbers you multiply moduli and add arguments, whereas to *divide* two complex numbers you divide moduli and subtract arguments.

Example 4 Multiplying Complex NumbersFind the product $z_1 z_2$ of the complex numbers.

$$z_1 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \quad z_2 = 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

Solution

$$\begin{aligned} z_1 z_2 &= 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \cdot 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right) \\ &= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i \sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right] \\ &= 16\left(\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right) \\ &= 16\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \\ &= 16[0 + i(1)] \\ &= 16i \end{aligned}$$

Multiply moduli and add arguments.

Technology

Some graphing utilities can multiply and divide complex numbers in trigonometric form. If you have access to such a graphing utility, use it to find $z_1 z_2$ and z_1/z_2 in Examples 4 and 5.

You can check this result by first converting the complex numbers to the standard forms $z_1 = -1 + \sqrt{3}i$ and $z_2 = 4\sqrt{3} - 4i$ and then multiplying algebraically, as in Section 2.4.

$$\begin{aligned} z_1 z_2 &= (-1 + \sqrt{3}i)(4\sqrt{3} - 4i) \\ &= -4\sqrt{3} + 4i + 12i + 4\sqrt{3} \\ &= 16i \end{aligned}$$

**CHECKPOINT**

Now try Exercise 47.

Example 5 Dividing Complex NumbersFind the quotient z_1/z_2 of the complex numbers.

$$z_1 = 24(\cos 300^\circ + i \sin 300^\circ) \quad z_2 = 8(\cos 75^\circ + i \sin 75^\circ)$$

Solution

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{24(\cos 300^\circ + i \sin 300^\circ)}{8(\cos 75^\circ + i \sin 75^\circ)} \\ &= \frac{24}{8}[\cos(300^\circ - 75^\circ) + i \sin(300^\circ - 75^\circ)] \\ &= 3(\cos 225^\circ + i \sin 225^\circ) \\ &= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i \end{aligned}$$

Divide moduli and subtract arguments.

**CHECKPOINT**

Now try Exercise 53.

Powers of Complex Numbers

The trigonometric form of a complex number is used to raise a complex number to a power. To accomplish this, consider repeated use of the multiplication rule.

$$z = r(\cos \theta + i \sin \theta)$$

$$z^2 = r(\cos \theta + i \sin \theta)r(\cos \theta + i \sin \theta) = r^2(\cos 2\theta + i \sin 2\theta)$$

$$z^3 = r^2(\cos 2\theta + i \sin 2\theta)r(\cos \theta + i \sin \theta) = r^3(\cos 3\theta + i \sin 3\theta)$$

$$z^4 = r^4(\cos 4\theta + i \sin 4\theta)$$

$$z^5 = r^5(\cos 5\theta + i \sin 5\theta)$$

$$\vdots$$

This pattern leads to DeMoivre's Theorem, which is named after the French mathematician Abraham DeMoivre (1667–1754).

The Granger Collection



Historical Note

Abraham DeMoivre (1667–1754) is remembered for his work in probability theory and DeMoivre's Theorem. His book *The Doctrine of Chances* (published in 1718) includes the theory of recurring series and the theory of partial fractions.

DeMoivre's Theorem

If $z = r(\cos \theta + i \sin \theta)$ is a complex number and n is a positive integer, then

$$\begin{aligned} z^n &= [r(\cos \theta + i \sin \theta)]^n \\ &= r^n(\cos n\theta + i \sin n\theta). \end{aligned}$$

Example 6 Finding Powers of a Complex Number

Use DeMoivre's Theorem to find $(-1 + \sqrt{3}i)^{12}$.

Solution

First convert the complex number to trigonometric form using

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \text{ and } \theta = \arctan \frac{\sqrt{3}}{-1} = \frac{2\pi}{3}.$$

So, the trigonometric form is

$$z = -1 + \sqrt{3}i = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right).$$

Then, by DeMoivre's Theorem, you have

$$\begin{aligned} (-1 + \sqrt{3}i)^{12} &= \left[2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\right]^{12} \\ &= 2^{12}\left[\cos \frac{12(2\pi)}{3} + i \sin \frac{12(2\pi)}{3}\right] \\ &= 4096(\cos 8\pi + i \sin 8\pi) \\ &= 4096(1 + 0) \\ &= 4096. \end{aligned}$$



CHECKPOINT

Now try Exercise 75.

Roots of Complex Numbers

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. So, the equation $x^6 = 1$ has six solutions, and in this particular case you can find the six solutions by factoring and using the Quadratic Formula.

$$\begin{aligned}x^6 - 1 &= (x^3 - 1)(x^3 + 1) \\&= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) = 0\end{aligned}$$

Consequently, the solutions are

$$x = \pm 1, \quad x = \frac{-1 \pm \sqrt{3}i}{2}, \quad \text{and} \quad x = \frac{1 \pm \sqrt{3}i}{2}.$$

Each of these numbers is a sixth root of 1. In general, the **n th root of a complex number** is defined as follows.

Definition of the n th Root of a Complex Number

The complex number $u = a + bi$ is an **n th root** of the complex number z if

$$z = u^n = (a + bi)^n.$$

Exploration

The n th roots of a complex number are useful for solving some polynomial equations. For instance, explain how you can use DeMoivre's Theorem to solve the polynomial equation

$$x^4 + 16 = 0.$$

[Hint: Write -16 as $16(\cos \pi + i \sin \pi)$.]

To find a formula for an n th root of a complex number, let u be an n th root of z , where

$$u = s(\cos \beta + i \sin \beta)$$

and

$$z = r(\cos \theta + i \sin \theta).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting back into the previous equation and dividing by r , you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

So, it follows that

$$\cos n\beta = \cos \theta \quad \text{and} \quad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of 2π , these last two equations have solutions if and only if the angles differ by a multiple of 2π . Consequently, there must exist an integer k such that

$$n\beta = \theta + 2\pi k$$

$$\beta = \frac{\theta + 2\pi k}{n}.$$

By substituting this value of β into the trigonometric form of u , you get the result stated on the following page.

Finding n th Roots of a Complex Number

For a positive integer n , the complex number $z = r(\cos \theta + i \sin \theta)$ has exactly n distinct n th roots given by

$$\sqrt[n]{r} \left(\cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right)$$

where $k = 0, 1, 2, \dots, n - 1$.

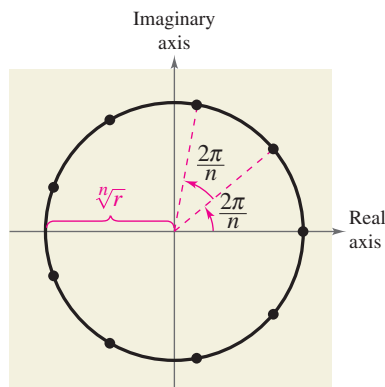


FIGURE 6.48

When k exceeds $n - 1$, the roots begin to repeat. For instance, if $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

is coterminal with θ/n , which is also obtained when $k = 0$.

The formula for the n th roots of a complex number z has a nice geometrical interpretation, as shown in Figure 6.48. Note that because the n th roots of z all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive n th roots have arguments that differ by $2\pi/n$, the n roots are equally spaced around the circle.

You have already found the sixth roots of 1 by factoring and by using the Quadratic Formula. Example 7 shows how you can solve the same problem with the formula for n th roots.

Example 7 Finding the n th Roots of a Real Number

Find all the sixth roots of 1.

Solution

First write 1 in the trigonometric form $1 = 1(\cos 0 + i \sin 0)$. Then, by the n th root formula, with $n = 6$ and $r = 1$, the roots have the form

$$\sqrt[6]{1} \left(\cos \frac{0 + 2\pi k}{6} + i \sin \frac{0 + 2\pi k}{6} \right) = \cos \frac{\pi k}{3} + i \sin \frac{\pi k}{3}.$$

So, for $k = 0, 1, 2, 3, 4$, and 5, the sixth roots are as follows. (See Figure 6.49.)

$$\cos 0 + i \sin 0 = 1$$

$$\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Increment by $\frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos \pi + i \sin \pi = -1$$

$$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

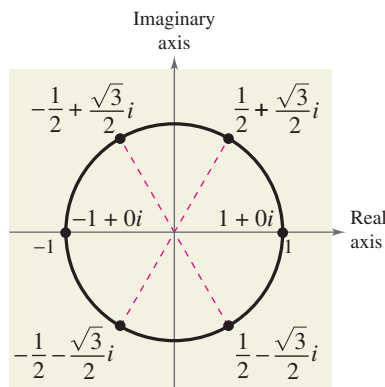


FIGURE 6.49



CHECKPOINT

Now try Exercise 97.

In Figure 8.49, notice that the roots obtained in Example 7 all have a magnitude of 1 and are equally spaced around the unit circle. Also notice that the complex roots occur in conjugate pairs, as discussed in Section 2.5. The n distinct n th roots of 1 are called the **n th roots of unity**.

Example 8 Finding the n th Roots of a Complex Number

Find the three cube roots of $z = -2 + 2i$.

Solution

Because z lies in Quadrant II, the trigonometric form of z is

$$\begin{aligned} z &= -2 + 2i \\ &= \sqrt{8} (\cos 135^\circ + i \sin 135^\circ). \quad \theta = \arctan(2/-2) = 135^\circ \end{aligned}$$

By the formula for n th roots, the cube roots have the form

$$\sqrt[3]{8} \left(\cos \frac{135^\circ + 360^\circ k}{3} + i \sin \frac{135^\circ + 360^\circ k}{3} \right).$$

Finally, for $k = 0, 1$, and 2 , you obtain the roots

$$\begin{aligned} \sqrt[3]{8} \left(\cos \frac{135^\circ + 360^\circ(0)}{3} + i \sin \frac{135^\circ + 360^\circ(0)}{3} \right) &= \sqrt{2} (\cos 45^\circ + i \sin 45^\circ) \\ &= 1 + i \end{aligned}$$

$$\begin{aligned} \sqrt[3]{8} \left(\cos \frac{135^\circ + 360^\circ(1)}{3} + i \sin \frac{135^\circ + 360^\circ(1)}{3} \right) &= \sqrt{2} (\cos 165^\circ + i \sin 165^\circ) \\ &\approx -1.3660 + 0.3660i \end{aligned}$$

$$\begin{aligned} \sqrt[3]{8} \left(\cos \frac{135^\circ + 360^\circ(2)}{3} + i \sin \frac{135^\circ + 360^\circ(2)}{3} \right) &= \sqrt{2} (\cos 285^\circ + i \sin 285^\circ) \\ &\approx 0.3660 - 1.3660i. \end{aligned}$$

See Figure 6.50.

 **CHECKPOINT** Now try Exercise 103.

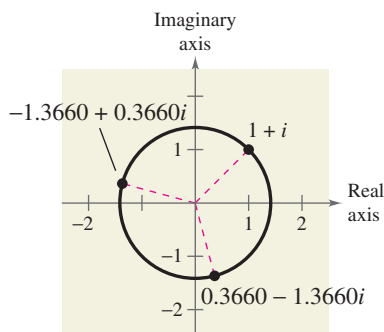


FIGURE 6.50

STUDY TIP

Note in Example 8 that the absolute value of z is

$$\begin{aligned} r &= |-2 + 2i| \\ &= \sqrt{(-2)^2 + 2^2} \\ &= \sqrt{8} \end{aligned}$$

and the angle θ is given by

$$\tan \theta = \frac{b}{a} = \frac{2}{-2} = -1.$$

WRITING ABOUT MATHEMATICS

A Famous Mathematical Formula The famous formula

$$e^{a+bi} = e^a (\cos b + i \sin b)$$

is called Euler's Formula, after the Swiss mathematician Leonhard Euler (1707–1783). Although the interpretation of this formula is beyond the scope of this text, we decided to include it because it gives rise to one of the most wonderful equations in mathematics.

$$e^{\pi i} + 1 = 0$$

This elegant equation relates the five most famous numbers in mathematics—0, 1, π , e , and i —in a single equation. Show how Euler's Formula can be used to derive this equation.

6.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

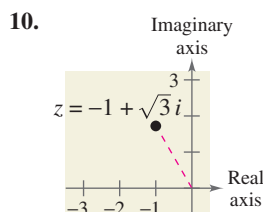
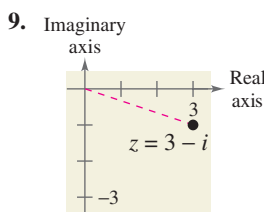
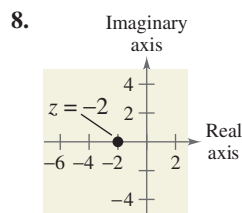
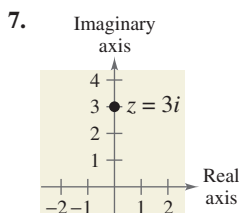
- The _____ of a complex number $a + bi$ is the distance between the origin $(0, 0)$ and the point (a, b) .
- The _____ of a complex number $z = a + bi$ is given by $z = r(\cos \theta + i \sin \theta)$, where r is the _____ of z and θ is the _____ of z .
- _____ Theorem states that if $z = r(\cos \theta + i \sin \theta)$ is a complex number and n is a positive integer, then $z^n = r^n(\cos n\theta + i \sin n\theta)$.
- The complex number $u = a + bi$ is an _____ of the complex number z if $z = u^n = (a + bi)^n$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–6, plot the complex number and find its absolute value.

- $-7i$
- -7
- $-4 + 4i$
- $5 - 12i$
- $6 - 7i$
- $-8 + 3i$

In Exercises 7–10, write the complex number in trigonometric form.



In Exercises 11–30, represent the complex number graphically, and find the trigonometric form of the number.

- $3 - 3i$
- $2 + 2i$
- $\sqrt{3} + i$
- $4 - 4\sqrt{3}i$
- $-2(1 + \sqrt{3}i)$
- $\frac{5}{2}(\sqrt{3} - i)$
- $-5i$
- $4i$
- $-7 + 4i$
- $3 - i$
- 7
- 4
- $3 + \sqrt{3}i$
- $2\sqrt{2} - i$

- $-3 - i$
- $1 + 3i$
- $5 + 2i$
- $8 + 3i$
- $-8 - 5\sqrt{3}i$
- $-9 - 2\sqrt{10}i$

In Exercises 31–40, represent the complex number graphically, and find the standard form of the number.

- $3(\cos 120^\circ + i \sin 120^\circ)$
- $5(\cos 135^\circ + i \sin 135^\circ)$
- $\frac{3}{2}(\cos 300^\circ + i \sin 300^\circ)$
- $\frac{1}{4}(\cos 225^\circ + i \sin 225^\circ)$
- $3.75\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$
- $6\left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}\right)$
- $8\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$
- $7(\cos 0 + i \sin 0)$
- $3[\cos(18^\circ 45') + i \sin(18^\circ 45')]$
- $6[\cos(230^\circ 30') + i \sin(230^\circ 30')]$



In Exercises 41–44, use a graphing utility to represent the complex number in standard form.

- $5\left(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9}\right)$
- $10\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$
- $3(\cos 165.5^\circ + i \sin 165.5^\circ)$
- $9(\cos 58^\circ + i \sin 58^\circ)$

In Exercises 45 and 46, represent the powers z , z^2 , z^3 , and z^4 graphically. Describe the pattern.

- $z = \frac{\sqrt{2}}{2}(1 + i)$
- $z = \frac{1}{2}(1 + \sqrt{3}i)$

In Exercises 47–58, perform the operation and leave the result in trigonometric form.

47. $\left[2 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right] \left[6 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \right]$
48. $\left[\frac{3}{4} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right] \left[4 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right]$
49. $\left[\frac{5}{3} (\cos 140^\circ + i \sin 140^\circ) \right] \left[\frac{2}{3} (\cos 60^\circ + i \sin 60^\circ) \right]$
50. $[0.5(\cos 100^\circ + i \sin 100^\circ)] \times [0.8(\cos 300^\circ + i \sin 300^\circ)]$
51. $[0.45(\cos 310^\circ + i \sin 310^\circ)] \times [0.60(\cos 200^\circ + i \sin 200^\circ)]$
52. $(\cos 5^\circ + i \sin 5^\circ)(\cos 20^\circ + i \sin 20^\circ)$
53. $\frac{\cos 50^\circ + i \sin 50^\circ}{\cos 20^\circ + i \sin 20^\circ}$
54. $\frac{2(\cos 120^\circ + i \sin 120^\circ)}{4(\cos 40^\circ + i \sin 40^\circ)}$
55. $\frac{\cos(5\pi/3) + i \sin(5\pi/3)}{\cos \pi + i \sin \pi}$
56. $\frac{5(\cos 4.3 + i \sin 4.3)}{4(\cos 2.1 + i \sin 2.1)}$
57. $\frac{12(\cos 52^\circ + i \sin 52^\circ)}{3(\cos 110^\circ + i \sin 110^\circ)}$
58. $\frac{6(\cos 40^\circ + i \sin 40^\circ)}{7(\cos 100^\circ + i \sin 100^\circ)}$

In Exercises 59–66, (a) write the trigonometric forms of the complex numbers, (b) perform the indicated operation using the trigonometric forms, and (c) perform the indicated operation using the standard forms, and check your result with that of part (b).

59. $(2 + 2i)(1 - i)$
60. $(\sqrt{3} + i)(1 + i)$
61. $-2i(1 + i)$
62. $4(1 - \sqrt{3}i)$
63. $\frac{3 + 4i}{1 - \sqrt{3}i}$
64. $\frac{1 + \sqrt{3}i}{6 - 3i}$
65. $\frac{5}{2 + 3i}$
66. $\frac{4i}{-4 + 2i}$

In Exercises 67–70, sketch the graph of all complex numbers z satisfying the given condition.

67. $|z| = 2$
68. $|z| = 3$
69. $\theta = \frac{\pi}{6}$
70. $\theta = \frac{5\pi}{4}$

In Exercises 71–88, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

71. $(1 + i)^5$
72. $(2 + 2i)^6$
73. $(-1 + i)^{10}$
74. $(3 - 2i)^8$
75. $2(\sqrt{3} + i)^7$
76. $4(1 - \sqrt{3}i)^3$
77. $[5(\cos 20^\circ + i \sin 20^\circ)]^3$
78. $[3(\cos 150^\circ + i \sin 150^\circ)]^4$
79. $\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^{12}$
80. $\left[2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \right]^8$
81. $[5(\cos 3.2 + i \sin 3.2)]^4$
82. $(\cos 0 + i \sin 0)^{20}$
83. $(3 - 2i)^5$
84. $(\sqrt{5} - 4i)^3$
85. $[3(\cos 15^\circ + i \sin 15^\circ)]^4$
86. $[2(\cos 10^\circ + i \sin 10^\circ)]^8$
87. $\left[2 \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right) \right]^5$
88. $\left[2 \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right) \right]^6$

In Exercises 89–104, (a) use the theorem on page 476 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.

89. Square roots of $5(\cos 120^\circ + i \sin 120^\circ)$
90. Square roots of $16(\cos 60^\circ + i \sin 60^\circ)$
91. Cube roots of $8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$
92. Fifth roots of $32 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$
93. Square roots of $-25i$
94. Fourth roots of $625i$
95. Cube roots of $-\frac{125}{2}(1 + \sqrt{3}i)$
96. Cube roots of $-4\sqrt{2}(1 - i)$
97. Fourth roots of 16
98. Fourth roots of i
99. Fifth roots of 1
100. Cube roots of 1000
101. Cube roots of -125
102. Fourth roots of -4
103. Fifth roots of $128(-1 + i)$
104. Sixth roots of $64i$

In Exercises 105–112, use the theorem on page 476 to find all the solutions of the equation and represent the solutions graphically.

105. $x^4 + i = 0$
 106. $x^3 + 1 = 0$
 107. $x^5 + 243 = 0$
 108. $x^3 - 27 = 0$
 109. $x^4 + 16i = 0$
 110. $x^6 + 64i = 0$
 111. $x^3 - (1 - i) = 0$
 112. $x^4 + (1 + i) = 0$

Synthesis

True or False? In Exercises 113–116, determine whether the statement is true or false. Justify your answer.

113. Although the square of the complex number bi is given by $(bi)^2 = -b^2$, the absolute value of the complex number $z = a + bi$ is defined as

$$|a + bi| = \sqrt{a^2 + b^2}.$$

114. Geometrically, the n th roots of any complex number z are all equally spaced around the unit circle centered at the origin.
 115. The product of two complex numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

is zero only when $r_1 = 0$ and/or $r_2 = 0$.

116. By DeMoivre's Theorem,

$$(4 + \sqrt{6}i)^8 = \cos(32) + i \sin(8\sqrt{6}).$$

117. Given two complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, $z_2 \neq 0$, show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

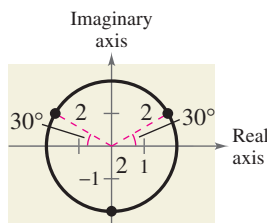
118. Show that $\bar{z} = r[\cos(-\theta) + i \sin(-\theta)]$ is the complex conjugate of $z = r(\cos \theta + i \sin \theta)$.
 119. Use the trigonometric forms of z and \bar{z} in Exercise 118 to find (a) $z\bar{z}$ and (b) z/\bar{z} , $\bar{z} \neq 0$.
 120. Show that the negative of $z = r(\cos \theta + i \sin \theta)$ is $-z = r[\cos(\theta + \pi) + i \sin(\theta + \pi)]$.
 121. Show that $-\frac{1}{2}(1 + \sqrt{3}i)$ is a sixth root of 1.
 122. Show that $2^{-1/4}(1 - i)$ is a fourth root of -2 .

Graphical Reasoning In Exercises 123 and 124, use the graph of the roots of a complex number.

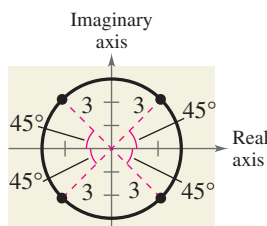
- (a) Write each of the roots in trigonometric form.
 (b) Identify the complex number whose roots are given.
 (c) Use a graphing utility to verify the results of part (b).



123.

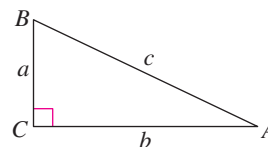


124.



Skills Review

In Exercises 125–130, solve the right triangle shown in the figure. Round your answers to two decimal places.



125. $A = 22^\circ$, $a = 8$ 126. $B = 66^\circ$, $a = 33.5$
 127. $A = 30^\circ$, $b = 112.6$ 128. $B = 6^\circ$, $b = 211.2$
 129. $A = 42^\circ 15'$, $c = 11.2$ 130. $B = 81^\circ 30'$, $c = 6.8$

Harmonic Motion In Exercises 131–134, for the simple harmonic motion described by the trigonometric function, find the maximum displacement and the least positive value of t for which $d = 0$.

131. $d = 16 \cos \frac{\pi}{4}t$ 132. $d = \frac{1}{8} \cos 12\pi t$
 133. $d = \frac{1}{16} \sin \frac{5}{4}\pi t$ 134. $d = \frac{1}{12} \sin 60\pi t$

In Exercises 135 and 136, write the product as a sum or difference.

135. $6 \sin 8\theta \cos 3\theta$ 136. $2 \cos 5\theta \sin 2\theta$

6

Chapter Summary

What did you learn?

Section 6.1

- ☐ Use the Law of Sines to solve oblique triangles (AAS, ASA, or SSA) (p. 430, 432).
- ☐ Find areas of oblique triangles (p. 434).
- ☐ Use the Law of Sines to model and solve real-life problems (p. 435).

Review Exercises

1–12

13–16

17–20

Section 6.2

- ☐ Use the Law of Cosines to solve oblique triangles (SSS or SAS) (p. 439).
- ☐ Use the Law of Cosines to model and solve real-life problems (p. 441).
- ☐ Use Heron's Area Formula to find areas of triangles (p. 442).

21–28

29–32

33–36

Section 6.3

- ☐ Represent vectors as directed line segments (p. 447).
- ☐ Write the component forms of vectors (p. 448).
- ☐ Perform basic vector operations and represent vectors graphically (p. 449).
- ☐ Write vectors as linear combinations of unit vectors (p. 451).
- ☐ Find the direction angles of vectors (p. 453).
- ☐ Use vectors to model and solve real-life problems (p. 454).

37, 38

39–44

45–56

57–62

63–68

69–72

Section 6.4

- ☐ Find the dot product of two vectors and use the properties of the dot product (p. 460).
- ☐ Find the angle between two vectors and determine whether two vectors are orthogonal (p. 461).
- ☐ Write vectors as sums of two vector components (p. 463).
- ☐ Use vectors to find the work done by a force (p. 466).

73–80

81–88

89–92

93–96

Section 6.5

- ☐ Plot complex numbers in the complex plane and find absolute values of complex numbers (p. 470).
- ☐ Write the trigonometric forms of complex numbers (p. 471).
- ☐ Multiply and divide complex numbers written in trigonometric form (p. 472).
- ☐ Use DeMoivre's Theorem to find powers of complex numbers (p. 474).
- ☐ Find n th roots of complex numbers (p. 475).

97–100

101–104

105, 106

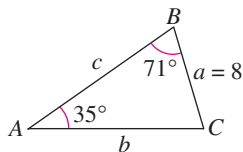
107–110

111–118

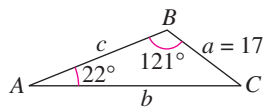
6 Review Exercises

6.1 In Exercises 1–12, use the Law of Sines to solve (if possible) the triangle. If two solutions exist, find both. Round your answers to two decimal places.

1.



2.

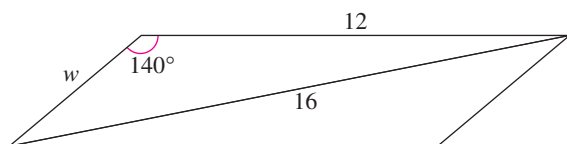


3. $B = 72^\circ$, $C = 82^\circ$, $b = 54$
4. $B = 10^\circ$, $C = 20^\circ$, $c = 33$
5. $A = 16^\circ$, $B = 98^\circ$, $c = 8.4$
6. $A = 95^\circ$, $B = 45^\circ$, $c = 104.8$
7. $A = 24^\circ$, $C = 48^\circ$, $b = 27.5$
8. $B = 64^\circ$, $C = 36^\circ$, $a = 367$
9. $B = 150^\circ$, $b = 30$, $c = 10$
10. $B = 150^\circ$, $a = 10$, $b = 3$
11. $A = 75^\circ$, $a = 51.2$, $b = 33.7$
12. $B = 25^\circ$, $a = 6.2$, $b = 4$

In Exercises 13–16, find the area of the triangle having the indicated angle and sides.

13. $A = 27^\circ$, $b = 5$, $c = 7$
14. $B = 80^\circ$, $a = 4$, $c = 8$
15. $C = 123^\circ$, $a = 16$, $b = 5$
16. $A = 11^\circ$, $b = 22$, $c = 21$

17. **Height** From a certain distance, the angle of elevation to the top of a building is 17° . At a point 50 meters closer to the building, the angle of elevation is 31° . Approximate the height of the building.
18. **Geometry** Find the length of the side w of the parallelogram.



19. **Height** A tree stands on a hillside of slope 28° from the horizontal. From a point 75 feet down the hill, the angle of elevation to the top of the tree is 45° (see figure). Find the height of the tree.

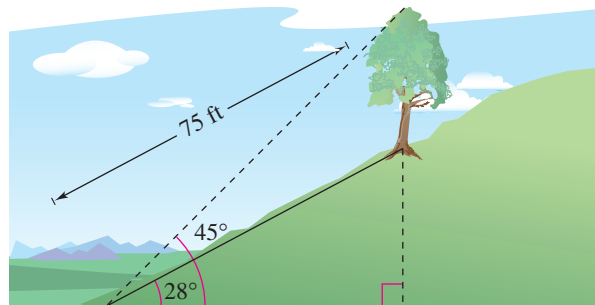


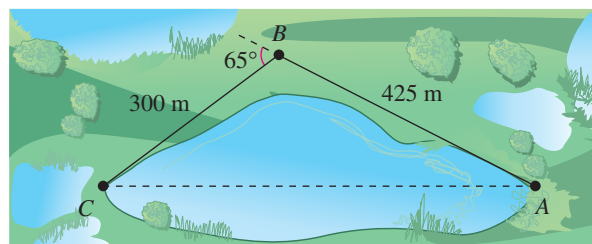
FIGURE FOR 19

20. **River Width** A surveyor finds that a tree on the opposite bank of a river, flowing due east, has a bearing of $N 22^\circ 30' E$ from a certain point and a bearing of $N 15^\circ W$ from a point 400 feet downstream. Find the width of the river.

6.2 In Exercises 21–28, use the Law of Cosines to solve the triangle. Round your answers to two decimal places.

21. $a = 5$, $b = 8$, $c = 10$
22. $a = 80$, $b = 60$, $c = 100$
23. $a = 2.5$, $b = 5.0$, $c = 4.5$
24. $a = 16.4$, $b = 8.8$, $c = 12.2$
25. $B = 110^\circ$, $a = 4$, $c = 4$
26. $B = 150^\circ$, $a = 10$, $c = 20$
27. $C = 43^\circ$, $a = 22.5$, $b = 31.4$
28. $A = 62^\circ$, $b = 11.34$, $c = 19.52$

29. **Geometry** The lengths of the diagonals of a parallelogram are 10 feet and 16 feet. Find the lengths of the sides of the parallelogram if the diagonals intersect at an angle of 28° .
30. **Geometry** The lengths of the diagonals of a parallelogram are 30 meters and 40 meters. Find the lengths of the sides of the parallelogram if the diagonals intersect at an angle of 34° .
31. **Surveying** To approximate the length of a marsh, a surveyor walks 425 meters from point A to point B . Then the surveyor turns 65° and walks 300 meters to point C (see figure). Approximate the length AC of the marsh.



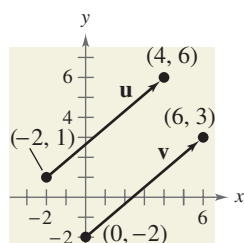
32. **Navigation** Two planes leave Raleigh-Durham Airport at approximately the same time. One is flying 425 miles per hour at a bearing of 355° , and the other is flying 530 miles per hour at a bearing of 67° . Draw a figure that gives a visual representation of the problem and determine the distance between the planes after they have flown for 2 hours.

In Exercises 33–36, use Heron's Area Formula to find the area of the triangle.

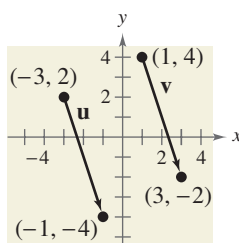
33. $a = 4$, $b = 5$, $c = 7$
 34. $a = 15$, $b = 8$, $c = 10$
 35. $a = 12.3$, $b = 15.8$, $c = 3.7$
 36. $a = 38.1$, $b = 26.7$, $c = 19.4$

6.3 In Exercises 37 and 38, show that $\mathbf{u} = \mathbf{v}$.

37.

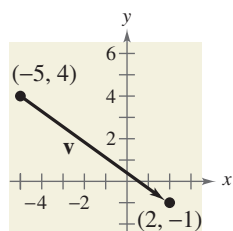


38.

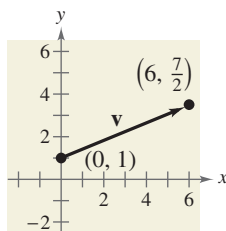


In Exercises 39–44, find the component form of the vector \mathbf{v} satisfying the conditions.

39.



40.



41. Initial point: $(0, 10)$; terminal point: $(7, 3)$
 42. Initial point: $(1, 5)$; terminal point: $(15, 9)$
 43. $\|\mathbf{v}\| = 8$, $\theta = 120^\circ$
 44. $\|\mathbf{v}\| = \frac{1}{2}$, $\theta = 225^\circ$

In Exercises 45–52, find (a) $\mathbf{u} + \mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, (c) $3\mathbf{u}$, and (d) $2\mathbf{v} + 5\mathbf{u}$.

45. $\mathbf{u} = \langle -1, -3 \rangle$, $\mathbf{v} = \langle -3, 6 \rangle$
 46. $\mathbf{u} = \langle 4, 5 \rangle$, $\mathbf{v} = \langle 0, -1 \rangle$
 47. $\mathbf{u} = \langle -5, 2 \rangle$, $\mathbf{v} = \langle 4, 4 \rangle$
 48. $\mathbf{u} = \langle 1, -8 \rangle$, $\mathbf{v} = \langle 3, -2 \rangle$
 49. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$

50. $\mathbf{u} = -7\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

51. $\mathbf{u} = 4\mathbf{i}$, $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$

52. $\mathbf{u} = -6\mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

In Exercises 53–56, find the component form of \mathbf{w} and sketch the specified vector operations geometrically, where $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j}$ and $\mathbf{v} = 1 - \mathbf{i} + 3\mathbf{j}$.

53. $\mathbf{w} = 2\mathbf{u} + \mathbf{v}$

54. $\mathbf{w} = 4\mathbf{u} - 5\mathbf{v}$

55. $\mathbf{w} = 3\mathbf{v}$

56. $\mathbf{w} = \frac{1}{2}\mathbf{v}$

In Exercises 57–60, write vector \mathbf{u} as a linear combination of the standard unit vectors \mathbf{i} and \mathbf{j} .

57. $\mathbf{u} = \langle -3, 4 \rangle$

58. $\mathbf{u} = \langle -6, -8 \rangle$

59. \mathbf{u} has initial point $(3, 4)$ and terminal point $(9, 8)$.

60. \mathbf{u} has initial point $(-2, 7)$ and terminal point $(5, -9)$.

In Exercises 61 and 62, write the vector \mathbf{v} in the form $\|\mathbf{v}\|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$.

61. $\mathbf{v} = -10\mathbf{i} + 10\mathbf{j}$

62. $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

In Exercises 63–68, find the magnitude and the direction angle of the vector \mathbf{v} .

63. $\mathbf{v} = 7(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$

64. $\mathbf{v} = 3(\cos 150^\circ \mathbf{i} + \sin 150^\circ \mathbf{j})$

65. $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$

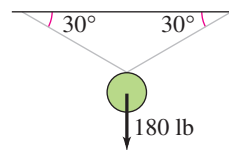
66. $\mathbf{v} = -4\mathbf{i} + 7\mathbf{j}$

67. $\mathbf{v} = -3\mathbf{i} - 3\mathbf{j}$

68. $\mathbf{v} = 8\mathbf{i} - \mathbf{j}$

69. **Resultant Force** Forces with magnitudes of 85 pounds and 50 pounds act on a single point. The angle between the forces is 15° . Describe the resultant force.

70. **Rope Tension** A 180-pound weight is supported by two ropes, as shown in the figure. Find the tension in each rope.



- 71. Navigation** An airplane has an airspeed of 430 miles per hour at a bearing of 135° . The wind velocity is 35 miles per hour in the direction of $N\ 30^\circ\ E$. Find the resultant speed and direction of the airplane.
- 72. Navigation** An airplane has an airspeed of 724 kilometers per hour at a bearing of 30° . The wind velocity is 32 kilometers per hour from the west. Find the resultant speed and direction of the airplane.

6.4 In Exercises 73–76, find the dot product of \mathbf{u} and \mathbf{v} .

- 73.** $\mathbf{u} = \langle 6, 7 \rangle$ **74.** $\mathbf{u} = \langle -7, 12 \rangle$
 $\mathbf{v} = \langle -3, 9 \rangle$ $\mathbf{v} = \langle -4, -14 \rangle$
- 75.** $\mathbf{u} = 3\mathbf{i} + 7\mathbf{j}$ **76.** $\mathbf{u} = -7\mathbf{i} + 2\mathbf{j}$
 $\mathbf{v} = 11\mathbf{i} - 5\mathbf{j}$ $\mathbf{v} = 16\mathbf{i} - 12\mathbf{j}$

In Exercises 77–80, use the vectors $\mathbf{u} = \langle -3, 4 \rangle$ and $\mathbf{v} = \langle 2, 1 \rangle$ to find the indicated quantity. State whether the result is a vector or a scalar.

- 77.** $2\mathbf{u} \cdot \mathbf{u}$
78. $\|\mathbf{v}\|^2$
79. $\mathbf{u}(\mathbf{u} \cdot \mathbf{v})$
80. $3\mathbf{u} \cdot \mathbf{v}$

In Exercises 81–84, find the angle θ between the vectors.

- 81.** $\mathbf{u} = \cos \frac{7\pi}{4} \mathbf{i} + \sin \frac{7\pi}{4} \mathbf{j}$
 $\mathbf{v} = \cos \frac{5\pi}{6} \mathbf{i} + \sin \frac{5\pi}{6} \mathbf{j}$
- 82.** $\mathbf{u} = \cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}$
 $\mathbf{v} = \cos 300^\circ \mathbf{i} + \sin 300^\circ \mathbf{j}$
- 83.** $\mathbf{u} = \langle 2\sqrt{2}, -4 \rangle$, $\mathbf{v} = \langle -\sqrt{2}, 1 \rangle$
- 84.** $\mathbf{u} = \langle 3, \sqrt{3} \rangle$, $\mathbf{v} = \langle 4, 3\sqrt{3} \rangle$

In Exercises 85–88, determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither.

- 85.** $\mathbf{u} = \langle -3, 8 \rangle$ **86.** $\mathbf{u} = \langle \frac{1}{4}, -\frac{1}{2} \rangle$
 $\mathbf{v} = \langle 8, 3 \rangle$ $\mathbf{v} = \langle -2, 4 \rangle$
- 87.** $\mathbf{u} = -\mathbf{i}$ **88.** $\mathbf{u} = -2\mathbf{i} + \mathbf{j}$
 $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j}$

In Exercises 89–92, find the projection of \mathbf{u} onto \mathbf{v} . Then write \mathbf{u} as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}} \mathbf{u}$.

- 89.** $\mathbf{u} = \langle -4, 3 \rangle$, $\mathbf{v} = \langle -8, -2 \rangle$
90. $\mathbf{u} = \langle 5, 6 \rangle$, $\mathbf{v} = \langle 10, 0 \rangle$
91. $\mathbf{u} = \langle 2, 7 \rangle$, $\mathbf{v} = \langle 1, -1 \rangle$
92. $\mathbf{u} = \langle -3, 5 \rangle$, $\mathbf{v} = \langle -5, 2 \rangle$

Work In Exercises 93 and 94, find the work done in moving a particle from P to Q if the magnitude and direction of the force are given by \mathbf{v} .

- 93.** $P = (5, 3)$, $Q = (8, 9)$, $\mathbf{v} = \langle 2, 7 \rangle$
94. $P = (-2, -9)$, $Q = (-12, 8)$, $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j}$

95. Work Determine the work done by a crane lifting an 18,000-pound truck 48 inches.

96. Work A mover exerts a horizontal force of 25 pounds on a crate as it is pushed up a ramp that is 12 feet long and inclined at an angle of 20° above the horizontal. Find the work done in pushing the crate.

6.5 In Exercises 97–100, plot the complex number and find its absolute value.

- 97.** $7i$ **98.** $-6i$
99. $5 + 3i$
100. $-10 - 4i$

In Exercises 101–104, write the complex number in trigonometric form.

- 101.** $5 - 5i$ **102.** $5 + 12i$
103. $-3\sqrt{3} + 3i$ **104.** -7

In Exercises 105 and 106, (a) write the two complex numbers in trigonometric form, and (b) use the trigonometric forms to find $z_1 z_2$ and z_1/z_2 , where $z_2 \neq 0$.

- 105.** $z_1 = 2\sqrt{3} - 2i$, $z_2 = -10i$
106. $z_1 = -3(1 + i)$, $z_2 = 2(\sqrt{3} + i)$

In Exercises 107–110, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

- 107.** $\left[5 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \right]^4$
108. $\left[2 \left(\cos \frac{4\pi}{15} + i \sin \frac{4\pi}{15} \right) \right]^5$
109. $(2 + 3i)^6$
110. $(1 - i)^8$

In Exercises 111–114, (a) use the theorem on page 476 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.

- 111.** Sixth roots of $-729i$
112. Fourth roots of $256i$
113. Cube roots of 8
114. Fifth roots of -1024

In Exercises 115–118, use the theorem on page 476 to find all solutions of the equation and represent the solutions graphically.

115. $x^4 + 81 = 0$

116. $x^5 - 32 = 0$

117. $x^3 + 8i = 0$

118. $(x^3 - 1)(x^2 + 1) = 0$

Synthesis

True or False? In Exercises 119–123, determine whether the statement is true or false. Justify your answer.

119. The Law of Sines is true if one of the angles in the triangle is a right angle.

120. When the Law of Sines is used, the solution is always unique.

121. If \mathbf{u} is a unit vector in the direction of \mathbf{v} , then $\mathbf{v} = \|\mathbf{v}\| \mathbf{u}$.

122. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

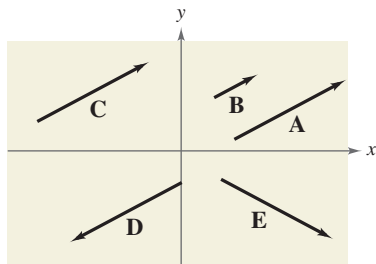
123. $x = \sqrt{3} + i$ is a solution of the equation $x^2 - 8i = 0$.

124. State the Law of Sines from memory.

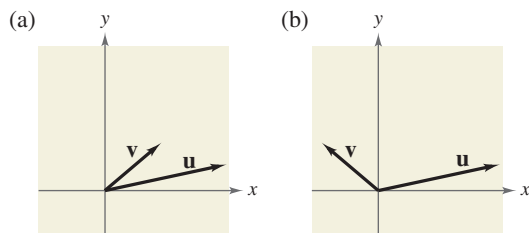
125. State the Law of Cosines from memory.

126. What characterizes a vector in the plane?

127. Which vectors in the figure appear to be equivalent?



128. The vectors \mathbf{u} and \mathbf{v} have the same magnitudes in the two figures. In which figure will the magnitude of the sum be greater? Give a reason for your answer.



129. Give a geometric description of the scalar multiple $k\mathbf{u}$ of the vector \mathbf{u} , for $k > 0$ and for $k < 0$.

130. Give a geometric description of the sum of the vectors \mathbf{u} and \mathbf{v} .

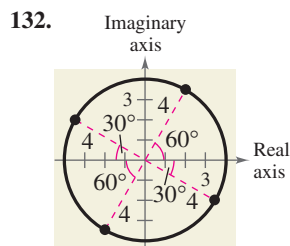
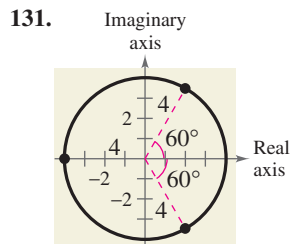
Graphical Reasoning In Exercises 131 and 132, use the graph of the roots of a complex number.

(a) Write each of the roots in trigonometric form.

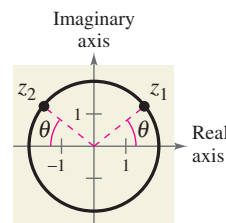
(b) Identify the complex number whose roots are given.



(c) Use a graphing utility to verify the results of part (b).



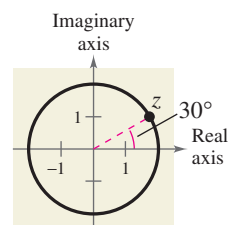
133. The figure shows z_1 and z_2 . Describe $z_1 z_2$ and z_1 / z_2 .



134. One of the fourth roots of a complex number z is shown in the figure.

(a) How many roots are not shown?

(b) Describe the other roots.



6

Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–6, use the information to solve the triangle. If two solutions exist, find both solutions. Round your answers to two decimal places.

1. $A = 24^\circ$, $B = 68^\circ$, $a = 12.2$
2. $B = 104^\circ$, $C = 33^\circ$, $a = 18.1$
3. $A = 24^\circ$, $a = 11.2$, $b = 13.4$
4. $a = 4.0$, $b = 7.3$, $c = 12.4$
5. $B = 100^\circ$, $a = 15$, $b = 23$
6. $C = 123^\circ$, $a = 41$, $b = 57$

7. A triangular parcel of land has borders of lengths 60 meters, 70 meters, and 82 meters. Find the area of the parcel of land.
8. An airplane flies 370 miles from point A to point B with a bearing of 24° . It then flies 240 miles from point B to point C with a bearing of 37° (see figure). Find the distance and bearing from point A to point C.

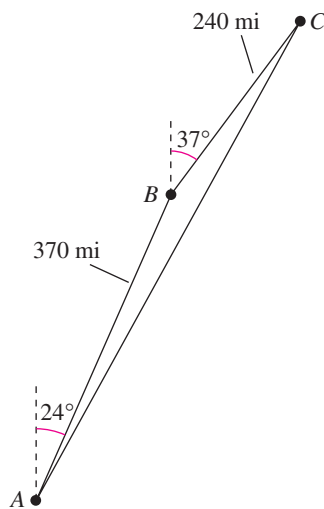


FIGURE FOR 8

In Exercises 9 and 10, find the component form of the vector \mathbf{v} satisfying the given conditions.

9. Initial point of \mathbf{v} : $(-3, 7)$; terminal point of \mathbf{v} : $(11, -16)$
10. Magnitude of \mathbf{v} : $\|\mathbf{v}\| = 12$; direction of \mathbf{v} : $\mathbf{u} = \langle 3, -5 \rangle$

In Exercises 11–13, $\mathbf{u} = \langle 3, 5 \rangle$ and $\mathbf{v} = \langle -7, 1 \rangle$. Find the resultant vector and sketch its graph.

11. $\mathbf{u} + \mathbf{v}$
12. $\mathbf{u} - \mathbf{v}$
13. $5\mathbf{u} - 3\mathbf{v}$

14. Find a unit vector in the direction of $\mathbf{u} = \langle 4, -3 \rangle$.
15. Forces with magnitudes of 250 pounds and 130 pounds act on an object at angles of 45° and -60° , respectively, with the x -axis. Find the direction and magnitude of the resultant of these forces.
16. Find the angle between the vectors $\mathbf{u} = \langle -1, 5 \rangle$ and $\mathbf{v} = \langle 3, -2 \rangle$.
17. Are the vectors $\mathbf{u} = \langle 6, 10 \rangle$ and $\mathbf{v} = \langle 2, 3 \rangle$ orthogonal?
18. Find the projection of $\mathbf{u} = \langle 6, 7 \rangle$ onto $\mathbf{v} = \langle -5, -1 \rangle$. Then write \mathbf{u} as the sum of two orthogonal vectors.
19. A 500-pound motorcycle is headed up a hill inclined at 12° . What force is required to keep the motorcycle from rolling down the hill when stopped at a red light?
20. Write the complex number $z = 5 - 5i$ in trigonometric form.
21. Write the complex number $z = 6(\cos 120^\circ + i \sin 120^\circ)$ in standard form.

In Exercises 22 and 23, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

$$22. \left[3 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) \right]^8 \quad 23. (3 - 3i)^6$$

24. Find the fourth roots of $256(1 + \sqrt{3}i)$.
25. Find all solutions of the equation $x^3 - 27i = 0$ and represent the solutions graphically.

6

Cumulative Test for Chapter 4–6

Take this test to review the material from earlier chapters. When you are finished, check your work against the answers given in the back of the book.

- Consider the angle $\theta = -120^\circ$.
 - Sketch the angle in standard position.
 - Determine a coterminal angle in the interval $[0^\circ, 360^\circ)$.
 - Convert the angle to radian measure.
 - Find the reference angle θ' .
 - Find the exact values of the six trigonometric functions of θ .
- Convert the angle $\theta = 2.35$ radians to degrees. Round the answer to one decimal place.
- Find $\cos \theta$ if $\tan \theta = -\frac{4}{3}$ and $\sin \theta < 0$.

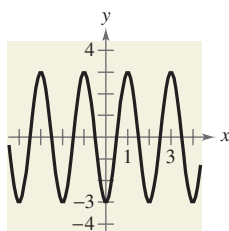


FIGURE FOR 7

In Exercises 4–6, sketch the graph of the function. (Include two full periods.)

4. $f(x) = 3 - 2 \sin \pi x$ 5. $g(x) = \frac{1}{2} \tan\left(x - \frac{\pi}{2}\right)$ 6. $h(x) = -\sec(x + \pi)$

- Find a , b , and c such that the graph of the function $h(x) = a \cos(bx + c)$ matches the graph in the figure.
- Sketch the graph of the function $f(x) = \frac{1}{2}x \sin x$ over the interval $-3\pi \leq x \leq 3\pi$.

In Exercises 9 and 10, find the exact value of the expression without using a calculator.

9. $\tan(\arctan 6.7)$ 10. $\tan(\arcsin \frac{3}{5})$

- Write an algebraic expression equivalent to $\sin(\arccos 2x)$.
- Use the fundamental identities to simplify: $\cos\left(\frac{\pi}{2} - x\right) \csc x$.
- Subtract and simplify: $\frac{\sin \theta - 1}{\cos \theta} - \frac{\cos \theta}{\sin \theta - 1}$.

In Exercises 14–16, verify the identity.

- $\cot^2 \alpha (\sec^2 \alpha - 1) = 1$
- $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$
- $\sin^2 x \cos^2 x = \frac{1}{8}(1 - \cos 4x)$

In Exercises 17 and 18, find all solutions of the equation in the interval $[0, 2\pi)$.

- $2 \cos^2 \beta - \cos \beta = 0$
- $3 \tan \theta - \cot \theta = 0$
- Use the Quadratic Formula to solve the equation in the interval $[0, 2\pi)$: $\sin^2 x + 2 \sin x + 1 = 0$.
- Given that $\sin u = \frac{12}{13}$, $\cos v = \frac{3}{5}$, and angles u and v are both in Quadrant I, find $\tan(u - v)$.
- If $\tan \theta = \frac{1}{2}$, find the exact value of $\tan(2\theta)$.

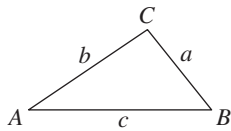


FIGURE FOR 25–28

22. If $\tan \theta = \frac{4}{3}$, find the exact value of $\sin \frac{\theta}{2}$.
23. Write the product $5 \sin \frac{3\pi}{4} \cdot \cos \frac{7\pi}{4}$ as a sum or difference.
24. Write $\cos 8x + \cos 4x$ as a product.

In Exercises 25–28, use the information to solve the triangle shown in the figure. Round your answers to two decimal places.

25. $A = 30^\circ$, $a = 9$, $b = 8$
26. $A = 30^\circ$, $b = 8$, $c = 10$
27. $A = 30^\circ$, $C = 90^\circ$, $b = 10$
28. $a = 4$, $b = 8$, $c = 9$
29. Two sides of a triangle have lengths 7 inches and 12 inches. Their included angle measures 60° . Find the area of the triangle.
30. Find the area of a triangle with sides of lengths 11 inches, 16 inches, and 17 inches.
31. Write the vector $\mathbf{u} = \langle 3, 5 \rangle$ as a linear combination of the standard unit vectors \mathbf{i} and \mathbf{j} .
32. Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j}$.
33. Find $\mathbf{u} \cdot \mathbf{v}$ for $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$.
34. Find the projection of $\mathbf{u} = \langle 8, -2 \rangle$ onto $\mathbf{v} = \langle 1, 5 \rangle$. Then write \mathbf{u} as the sum of two orthogonal vectors.
35. Write the complex number $-2 + 2i$ in trigonometric form.
36. Find the product of $[4(\cos 30^\circ + i \sin 30^\circ)][6(\cos 120^\circ + i \sin 120^\circ)]$. Write the answer in standard form.
37. Find the three cube roots of 1.
38. Find all the solutions of the equation $x^5 + 243 = 0$.
39. A ceiling fan with 21-inch blades makes 63 revolutions per minute. Find the angular speed of the fan in radians per minute. Find the linear speed of the tips of the blades in inches per minute.
40. Find the area of the sector of a circle with a radius of 8 yards and a central angle of 114° .
41. From a point 200 feet from a flagpole, the angles of elevation to the bottom and top of the flag are $16^\circ 45'$ and 18° , respectively. Approximate the height of the flag to the nearest foot.
42. To determine the angle of elevation of a star in the sky, you get the star in your line of vision with the backboard of a basketball hoop that is 5 feet higher than your eyes (see figure). Your horizontal distance from the backboard is 12 feet. What is the angle of elevation of the star?
43. Write a model for a particle in simple harmonic motion with a displacement of 4 inches and a period of 8 seconds.
44. An airplane's velocity with respect to the air is 500 kilometers per hour, with a bearing of 30° . The wind at the altitude of the plane has a velocity of 50 kilometers per hour with a bearing of N 60° E. What is the true direction of the plane, and what is its speed relative to the ground?
45. A force of 85 pounds exerted at an angle of 60° above the horizontal is required to slide an object across a floor. The object is dragged 10 feet. Determine the work done in sliding the object.

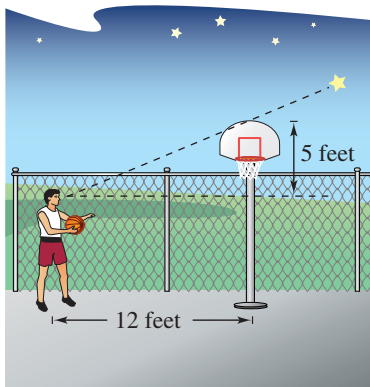


FIGURE FOR 42

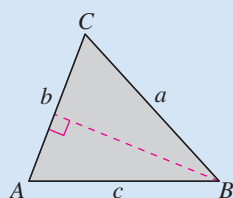
Proofs in Mathematics

Law of Tangents

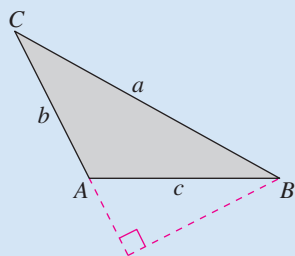
Besides the Law of Sines and the Law of Cosines, there is also a Law of Tangents, which was developed by Francois Viète (1540–1603). The Law of Tangents follows from the Law of Sines and the sum-to-product formulas for sine and is defined as follows.

$$\frac{a+b}{a-b} = \frac{\tan[(A+B)/2]}{\tan[(A-B)/2]}$$

The Law of Tangents can be used to solve a triangle when two sides and the included angle are given (SAS). Before calculators were invented, the Law of Tangents was used to solve the SAS case instead of the Law of Cosines, because computation with a table of tangent values was easier.



A is acute.

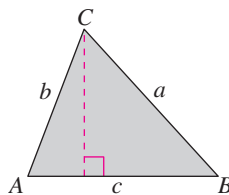


A is obtuse.

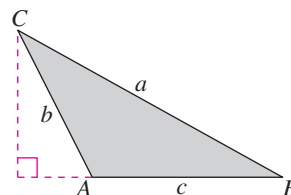
Law of Sines (p. 430)

If ABC is a triangle with sides a , b , and c , then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



A is acute.



A is obtuse.

Proof

Let h be the altitude of either triangle found in the figure above. Then you have

$$\sin A = \frac{h}{b} \quad \text{or} \quad h = b \sin A$$

$$\sin B = \frac{h}{a} \quad \text{or} \quad h = a \sin B.$$

Equating these two values of h , you have

$$a \sin B = b \sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}.$$

Note that $\sin A \neq 0$ and $\sin B \neq 0$ because no angle of a triangle can have a measure of 0° or 180° . In a similar manner, construct an altitude from vertex B to side AC (extended in the obtuse triangle), as shown at the left. Then you have

$$\sin A = \frac{h}{c} \quad \text{or} \quad h = c \sin A$$

$$\sin C = \frac{h}{a} \quad \text{or} \quad h = a \sin C.$$

Equating these two values of h , you have

$$a \sin C = c \sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{c}{\sin C}.$$

By the Transitive Property of Equality you know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

So, the Law of Sines is established.

Law of Cosines (p. 439)

Standard Form

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Alternative Form

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Proof

To prove the first formula, consider the top triangle at the left, which has three acute angles. Note that vertex B has coordinates $(c, 0)$. Furthermore, C has coordinates (x, y) , where $x = b \cos A$ and $y = b \sin A$. Because a is the distance from vertex C to vertex B , it follows that

$$a = \sqrt{(x - c)^2 + (y - 0)^2}$$

Distance Formula

$$a^2 = (x - c)^2 + (y - 0)^2$$

Square each side.

$$a^2 = (b \cos A - c)^2 + (b \sin A)^2$$

Substitute for x and y .

$$a^2 = b^2 \cos^2 A - 2bc \cos A + c^2 + b^2 \sin^2 A$$

Expand.

$$a^2 = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A$$

Factor out b^2 .

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$\sin^2 A + \cos^2 A = 1$

To prove the second formula, consider the bottom triangle at the left, which also has three acute angles. Note that vertex A has coordinates $(c, 0)$. Furthermore, C has coordinates (x, y) , where $x = a \cos B$ and $y = a \sin B$. Because b is the distance from vertex C to vertex A , it follows that

$$b = \sqrt{(x - c)^2 + (y - 0)^2}$$

Distance Formula

$$b^2 = (x - c)^2 + (y - 0)^2$$

Square each side.

$$b^2 = (a \cos B - c)^2 + (a \sin B)^2$$

Substitute for x and y .

$$b^2 = a^2 \cos^2 B - 2ac \cos B + c^2 + a^2 \sin^2 B$$

Expand.

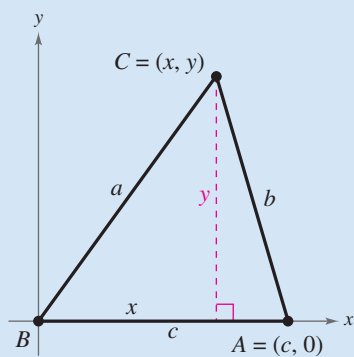
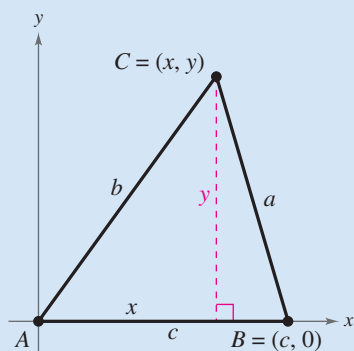
$$b^2 = a^2(\sin^2 B + \cos^2 B) + c^2 - 2ac \cos B$$

Factor out a^2 .

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

$\sin^2 B + \cos^2 B = 1$

A similar argument is used to establish the third formula.



Heron's Area Formula (p. 442)

Given any triangle with sides of lengths a , b , and c , the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{(a+b+c)}{2}$.

Proof

From Section 6.1, you know that

$$\text{Area} = \frac{1}{2}bc \sin A$$

Formula for the area of an oblique triangle

$$(\text{Area})^2 = \frac{1}{4}b^2c^2 \sin^2 A$$

Square each side.

$$\text{Area} = \sqrt{\frac{1}{4}b^2c^2 \sin^2 A}$$

Take the square root of each side.

$$= \sqrt{\frac{1}{4}b^2c^2(1 - \cos^2 A)}$$

Pythagorean Identity

$$= \sqrt{\left[\frac{1}{2}bc(1 + \cos A)\right]\left[\frac{1}{2}bc(1 - \cos A)\right]}$$

Factor.

Using the Law of Cosines, you can show that

$$\frac{1}{2}bc(1 + \cos A) = \frac{a+b+c}{2} \cdot \frac{-a+b+c}{2}$$

and

$$\frac{1}{2}bc(1 - \cos A) = \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}$$

Letting $s = (a+b+c)/2$, these two equations can be rewritten as

$$\frac{1}{2}bc(1 + \cos A) = s(s-a)$$

and

$$\frac{1}{2}bc(1 - \cos A) = (s-b)(s-c).$$

By substituting into the last formula for area, you can conclude that

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Properties of the Dot Product (p. 460)

Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Proof

Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, $\mathbf{w} = \langle w_1, w_2 \rangle$, $\mathbf{0} = \langle 0, 0 \rangle$, and let c be a scalar.

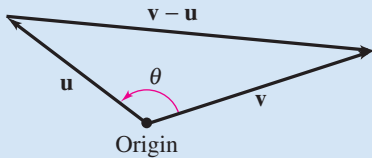
1. $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0 \cdot v_1 + 0 \cdot v_2 = 0$
3. $\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \langle v_1 + w_1, v_2 + w_2 \rangle \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 \\ &= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \end{aligned}$
4. $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 = (\sqrt{v_1^2 + v_2^2})^2 = \|\mathbf{v}\|^2$
5. $\begin{aligned} c(\mathbf{u} \cdot \mathbf{v}) &= c(\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle) \\ &= c(u_1v_1 + u_2v_2) \\ &= (cu_1)v_1 + (cu_2)v_2 \\ &= \langle cu_1, cu_2 \rangle \cdot \langle v_1, v_2 \rangle \\ &= c\mathbf{u} \cdot \mathbf{v} \end{aligned}$

Angle Between Two Vectors (p. 461)

If θ is the angle between two nonzero vectors \mathbf{u} and \mathbf{v} , then $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

Proof

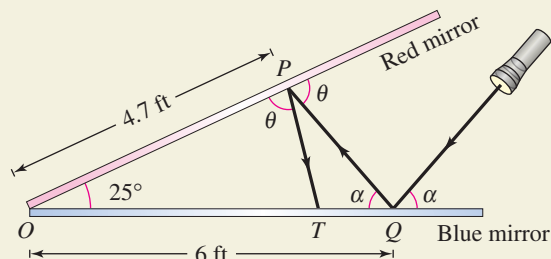
Consider the triangle determined by vectors \mathbf{u} , \mathbf{v} , and $\mathbf{v} - \mathbf{u}$, as shown in the figure. By the Law of Cosines, you can write



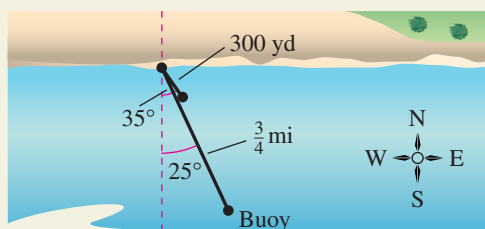
$$\begin{aligned} \|\mathbf{v} - \mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \end{aligned}$$

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. In the figure, a beam of light is directed at the blue mirror, reflected to the red mirror, and then reflected back to the blue mirror. Find the distance PT that the light travels from the red mirror back to the blue mirror.



2. A triathlete sets a course to swim $S 25^\circ E$ from a point on shore to a buoy $\frac{3}{4}$ mile away. After swimming 300 yards through a strong current, the triathlete is off course at a bearing of $S 35^\circ E$. Find the bearing and distance the triathlete needs to swim to correct her course.



3. A hiking party is lost in a national park. Two ranger stations have received an emergency SOS signal from the party. Station B is 75 miles due east of station A. The bearing from station A to the signal is $S 60^\circ E$ and the bearing from station B to the signal is $S 75^\circ W$.

- Draw a diagram that gives a visual representation of the problem.
- Find the distance from each station to the SOS signal.
- A rescue party is in the park 20 miles from station A at a bearing of $S 80^\circ E$. Find the distance and the bearing the rescue party must travel to reach the lost hiking party.

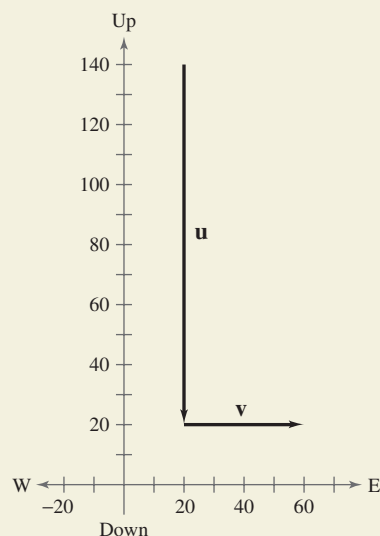
4. You are seeding a triangular courtyard. One side of the courtyard is 52 feet long and another side is 46 feet long. The angle opposite the 52-foot side is 65° .

- Draw a diagram that gives a visual representation of the problem.
- How long is the third side of the courtyard?
- One bag of grass covers an area of 50 square feet. How many bags of grass will you need to cover the courtyard?

5. For each pair of vectors, find the following.

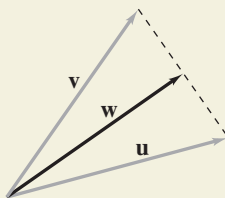
- | | | |
|---|--|---|
| (i) $\ \mathbf{u}\ $ | (ii) $\ \mathbf{v}\ $ | (iii) $\ \mathbf{u} + \mathbf{v}\ $ |
| (iv) $\left\ \frac{\mathbf{u}}{\ \mathbf{u}\ } \right\ $ | (v) $\left\ \frac{\mathbf{v}}{\ \mathbf{v}\ } \right\ $ | (vi) $\left\ \frac{\mathbf{u} + \mathbf{v}}{\ \mathbf{u} + \mathbf{v}\ } \right\ $ |
| (a) $\mathbf{u} = \langle 1, -1 \rangle$ | | (b) $\mathbf{u} = \langle 0, 1 \rangle$ |
| $\mathbf{v} = \langle -1, 2 \rangle$ | | $\mathbf{v} = \langle 3, -3 \rangle$ |
| (c) $\mathbf{u} = \langle 1, \frac{1}{2} \rangle$ | | (d) $\mathbf{u} = \langle 2, -4 \rangle$ |
| $\mathbf{v} = \langle 2, 3 \rangle$ | | $\mathbf{v} = \langle 5, 5 \rangle$ |

6. A skydiver is falling at a constant downward velocity of 120 miles per hour. In the figure, vector \mathbf{u} represents the skydiver's velocity. A steady breeze pushes the skydiver to the east at 40 miles per hour. Vector \mathbf{v} represents the wind velocity.

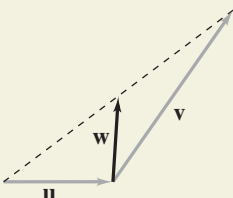


- Write the vectors \mathbf{u} and \mathbf{v} in component form.
- Let $\mathbf{s} = \mathbf{u} + \mathbf{v}$. Use the figure to sketch \mathbf{s} . To print an enlarged copy of the graph, go to the website, www.mathgraphs.com.
- Find the magnitude of \mathbf{s} . What information does the magnitude give you about the skydiver's fall?
- If there were no wind, the skydiver would fall in a path perpendicular to the ground. At what angle to the ground is the path of the skydiver when the skydiver is affected by the 40 mile per hour wind from due west?
- The skydiver is blown to the west at 30 miles per hour. Draw a new figure that gives a visual representation of the problem and find the skydiver's new velocity.

7. Write the vector \mathbf{w} in terms of \mathbf{u} and \mathbf{v} , given that the terminal point of \mathbf{w} bisects the line segment (see figure).



8. Prove that if \mathbf{u} is orthogonal to \mathbf{v} and \mathbf{w} , then \mathbf{u} is orthogonal to $c\mathbf{v} + d\mathbf{w}$ for any scalars c and d (see figure).



9. Two forces of the same magnitude \mathbf{F}_1 and \mathbf{F}_2 act at angles θ_1 and θ_2 , respectively. Use a diagram to compare the work done by \mathbf{F}_1 with the work done by \mathbf{F}_2 in moving along the vector PQ if

- (a) $\theta_1 = -\theta_2$
 (b) $\theta_1 = 60^\circ$ and $\theta_2 = 30^\circ$.

10. Four basic forces are in action during flight: weight, lift, thrust, and drag. To fly through the air, an object must overcome its own *weight*. To do this, it must create an upward force called *lift*. To generate lift, a forward motion called *thrust* is needed. The thrust must be great enough to overcome air resistance, which is called *drag*.

For a commercial jet aircraft, a quick climb is important to maximize efficiency, because the performance of an aircraft at high altitudes is enhanced. In addition, it is necessary to clear obstacles such as buildings and mountains and reduce noise in residential areas. In the diagram, the angle θ is called the climb angle. The velocity of the plane can be represented by a vector \mathbf{v} with a vertical component $\|\mathbf{v}\| \sin \theta$ (called climb speed) and a horizontal component $\|\mathbf{v}\| \cos \theta$, where $\|\mathbf{v}\|$ is the speed of the plane.

When taking off, a pilot must decide how much of the thrust to apply to each component. The more the thrust is applied to the horizontal component, the faster the airplane will gain speed. The more the thrust is applied to the vertical component, the quicker the airplane will climb.

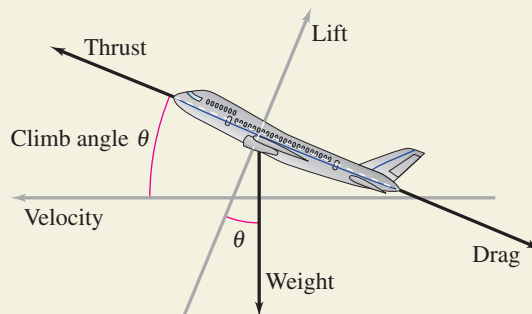


FIGURE FOR 10

- (a) Complete the table for an airplane that has a speed of $\|\mathbf{v}\| = 100$ miles per hour.

| θ | 0.5° | 1.0° | 1.5° | 2.0° | 2.5° | 3.0° |
|------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\ \mathbf{v}\ \sin \theta$ | | | | | | |
| $\ \mathbf{v}\ \cos \theta$ | | | | | | |

- (b) Does an airplane's speed equal the sum of the vertical and horizontal components of its velocity? If not, how could you find the speed of an airplane whose velocity components were known?
- (c) Use the result of part (b) to find the speed of an airplane with the given velocity components.
- (i) $\|\mathbf{v}\| \sin \theta = 5.235$ miles per hour
 $\|\mathbf{v}\| \cos \theta = 149.909$ miles per hour
- (ii) $\|\mathbf{v}\| \sin \theta = 10.463$ miles per hour
 $\|\mathbf{v}\| \cos \theta = 149.634$ miles per hour