Sequences, Series, and Probability

9

Poker has become a popular card game in recent years. You can use the probability theory developed in this chapter to calculate the likelihood of getting different poker hands.

Jeff Greenberg/PhotoEdit, Inc.

SELECTED APPLICATIONS

Sequences, series, and probability have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Federal Debt, Exercise 111, page 651
- Falling Object, Exercises 87 and 88, page 661
- Multiplier Effect, Exercises 113–116, page 671
- Data Analysis: Tax Returns, Exercise 61, page 682
- Child Support, Exercise 80, page 690
- Poker Hand Exercise 57, page 699
- Lottery, Exercise 65, page 700
- Defective Units, Exercise 47, page 711
- Population Growth, Exercise 139, page 718

9.1 Sequences and Series

What you should learn

- Use sequence notation to write the terms of sequences.
- Use factorial notation.
- Use summation notation to write sums.
- Find the sums of infinite series.
- Use sequences and series to model and solve real-life problems.

Why you should learn it

Sequences and series can be used to model real-life problems. For instance, in Exercise 109 on page 651, sequences are used to model the number of Best Buy stores from 1998 through 2003.

Scott Olson/Getty Images

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers.

$$f(1) = a_1,\ f(2) = a_2,\ f(3) = a_3,\ f(4) = a_4, \ldots,\ f(n) = a_n, \ldots$$

Rather than using function notation, however, sequences are usually written using subscript notation, as indicated in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. If the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots$.

Example 1 Writing the Terms of a Sequence

Write the first four terms of the sequences given by

a. $a_n = 3n - 2$ **b.** $a_n = 3 + (-1)^n$.

Solution

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

CHECKPOINT Now try Exercise 1.

Exploration

Write out the first five terms of the sequence whose nth term is

$$a_n = \frac{(-1)^{n+1}}{2n-1}.$$

Are they the same as the first five terms of the sequence in Example 2? If not, how do they differ?

Example 2 A Sequence Whose Terms Alternate in Sign

Write the first five terms of the sequence given by $a_n = \dfrac{(-1)^n}{2n-1}$.

Solution

The first five terms of the sequence are as follows.

$$a_1 = \frac{(-1)^1}{2(1)-1} = \frac{-1}{2-1} = -1 \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2)-1} = \frac{1}{4-1} = \frac{1}{3} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3)-1} = \frac{-1}{6-1} = -\frac{1}{5} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4)-1} = \frac{1}{8-1} = \frac{1}{7} \qquad \text{4th term}$$

$$a_5 = \frac{(-1)^5}{2(5)-1} = \frac{-1}{10-1} = -\frac{1}{9} \qquad \text{5th term}$$

CHECKPOINT Now try Exercise 17.

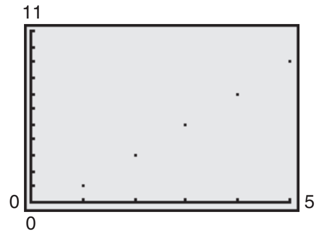
Simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given*. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

Technology

To graph a sequence using a graphing utility, set the mode to *sequence* and *dot* and enter the sequence. The graph of the sequence in Example 3(a) is shown below. You can use the *trace* feature or *value* feature to identify the terms.



Example 3 Finding the nth Term of a Sequence

Write an expression for the apparent nth term (a_n) of each sequence.

a. 1, 3, 5, 7, . . . **b.** 2, −5, 10, −17, . . .

Solution

a. *n:* 1 2 3 4 . . . n

Terms: 1 3 5 7 . . . a_n

Apparent pattern: Each term is 1 less than twice n, which implies that

$$a_n = 2n - 1.$$

b. *n:* 1 2 3 4 . . . n

Terms: 2 −5 10 −17 . . . a_n

Apparent pattern: The terms have alternating signs with those in the even positions being negative. Each term is 1 more than the square of n, which implies that

$$a_n = (-1)^{n+1}(n^2+1)$$

CHECKPOINT Now try Exercise 37.

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms. A well-known example is the Fibonacci sequence shown in Example 4.

Example 4 The Fibonacci Sequence: A Recursive Sequence

The Fibonacci sequence is defined recursively, as follows.

$$a_0 = 1,\ a_1 = 1,\ a_k = a_{k-2} + a_{k-1}, \text{ where } k \geq 2$$

Write the first six terms of this sequence.

Solution

$a_0 = 1$ — 0th term is given.

$a_1 = 1$ — 1st term is given.

$a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2$ — Use recursion formula.

$a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3$ — Use recursion formula.

$a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 2 + 3 = 5$ — Use recursion formula.

$a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 3 + 5 = 8$ — Use recursion formula.

CHECKPOINT Now try Exercise 51.

STUDY TIP

The subscripts of a sequence make up the domain of the sequence and they serve to identify the location of a term within the sequence. For example, a_4 is the fourth term of the sequence, and a_n is the nth term of the sequence. Any variable can be used as a subscript. The most commonly used variable subscripts in sequence and series notation are i, j, k, and n.

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n - 1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Here are some values of $n!$ for the first several nonnegative integers. Notice that $0!$ is 1 by definition.

$$0! = 1$$

$$1! = 1$$

$$2! = 1 \cdot 2 = 2$$

$$3! = 1 \cdot 2 \cdot 3 = 6$$

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

The value of n does not have to be very large before the value of $n!$ becomes extremely large. For instance, $10! = 3{,}628{,}800$.

Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n)$$

whereas $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot 2n$.

Example 5 Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by

$$a_n = \frac{2^n}{n!}.$$

Begin with $n = 0$. Then graph the terms on a set of coordinate axes.

Solution

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \qquad \text{0th term}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \qquad \text{1st term}$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \qquad \text{2nd term}$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \qquad \text{3rd term}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3} \qquad \text{4th term}$$

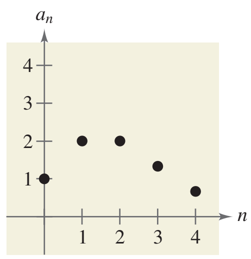


FIGURE 9.1

Figure 9.1 shows the first five terms of the sequence.

CHECKPOINT Now try Exercise 59.

When working with fractions involving factorials, you will often find that the fractions can be reduced to simplify the computations.

Example 6 Evaluating Factorial Expressions

Evaluate each factorial expression.

a. $\dfrac{8!}{2! \cdot 6!}$ **b.** $\dfrac{2! \cdot 6!}{3! \cdot 5!}$ **c.** $\dfrac{n!}{(n-1)!}$

Solution

a. $$\frac{8!}{2! \cdot 6!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}} = \frac{7 \cdot 8}{2} = 28$$

b. $$\frac{2! \cdot 6!}{3! \cdot 5!} = \frac{\cancel{1 \cdot 2} \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6}{\cancel{1 \cdot 2} \cdot 3 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}} = \frac{6}{3} = 2$$

c. $$\frac{n!}{(n-1)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n-1)} \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n-1)}} = n$$

CHECKPOINT Now try Exercise 69.

STUDY TIP

Note in Example 6(a) that you can simplify the computation as follows.

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2! \cdot \cancel{6!}}$$

$$= \frac{8 \cdot 7}{2 \cdot 1} = 28$$

Summation Notation

There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

Technology

Most graphing utilities are able to sum the first n terms of a sequence. Check your user's guide for a *sum sequence* feature or a *series* feature.

Example 7 Summation Notation for Sums

Find each sum.

a. $\sum_{i=1}^{5} 3i$ **b.** $\sum_{k=3}^{6} (1 + k^2)$ **c.** $\sum_{i=0}^{8} \frac{1}{i!}$

Solution

a. $$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$$
$$= 3(1 + 2 + 3 + 4 + 5)$$
$$= 3(15)$$
$$= 45$$

b. $$\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$$
$$= 10 + 17 + 26 + 37$$
$$= 90$$

c. $$\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$$
$$\approx 2.71828$$

For this summation, note that the sum is very close to the irrational number $e \approx 2.718281828$. It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

CHECKPOINT Now try Exercise 73.

STUDY TIP

Summation notation is an instruction to add the terms of a sequence. From the definition at the right, the upper limit of summation tells you where to end the sum. Summation notation helps you generate the appropriate terms of the sequence prior to finding the actual sum, which may be unclear.

In Example 7, note that the lower limit of a summation does not have to be 1. Also note that the index of summation does not have to be the letter i. For instance, in part (b), the letter k is the index of summation.

STUDY TIP

Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum*. For example, the following two sums have the same terms.

$$\sum_{i=1}^{3} 3(2^i) = 3(2^1 + 2^2 + 2^3)$$

$$\sum_{i=0}^{2} 3(2^{i+1}) = 3(2^1 + 2^2 + 2^3)$$

Properties of Sums

1. $\sum_{i=1}^{n} c = cn,$ c is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i,$ c is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

For proofs of these properties, see Proofs in Mathematics on page 722.

Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a **series.**

Definition of Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$.

1. The sum of the first n terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

Example 8 Finding the Sum of a Series

For the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$, find (a) the third partial sum and (b) the sum.

Solution

a. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 = 0.333.$$

b. The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots$$

$$= 0.33333\ldots = \frac{1}{3}.$$

CHECKPOINT Now try Exercise 99.

Application

Sequences have many applications in business and science. One such application is illustrated in Example 9.

Example 9 Population of the United States

For the years 1980 to 2003, the resident population of the United States can be approximated by the model

$$a_n = 226.9 + 2.05n + 0.035n^2, \qquad n = 0, 1, \ldots, 23$$

where a_n is the population (in millions) and n represents the year, with $n = 0$ corresponding to 1980. Find the last five terms of this finite sequence, which represent the U.S. population for the years 1999 to 2003. (Source: U.S. Census Bureau)

Solution

The last five terms of this finite sequence are as follows.

$$a_{19} = 226.9 + 2.05(19) + 0.035(19)^2 \approx 278.5 \qquad \text{1999 population}$$

$$a_{20} = 226.9 + 2.05(20) + 0.035(20)^2 = 281.9 \qquad \text{2000 population}$$

$$a_{21} = 226.9 + 2.05(21) + 0.035(21)^2 \approx 285.4 \qquad \text{2001 population}$$

$$a_{22} = 226.9 + 2.05(22) + 0.035(22)^2 \approx 288.9 \qquad \text{2002 population}$$

$$a_{23} = 226.9 + 2.05(23) + 0.035(23)^2 \approx 292.6 \qquad \text{2003 population}$$

CHECKPOINT Now try Exercise 111.

Exploration

A $3 \times 3 \times 3$ cube is created using 27 unit cubes (a unit cube has a length, width, and height of 1 unit) and only the faces of each cube that are visible are painted blue (see Figure 9.2). Complete the table below to determine how many unit cubes of the $3 \times 3 \times 3$ cube have 0 blue faces, 1 blue face, 2 blue faces, and 3 blue faces. Do the same for a $4 \times 4 \times 4$ cube, a $5 \times 5 \times 5$ cube, and a $6 \times 6 \times 6$ cube and add your results to the table below. What type of pattern do you observe in the table? Write a formula you could use to determine the column values for an $n \times n \times n$ cube.

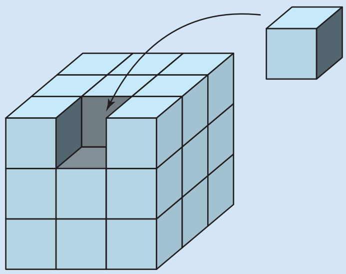


FIGURE 9.2

Number of blue cube faces	0	1	2	3
$3 \times 3 \times 3$				

9.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

VOCABULARY CHECK: Fill in the blanks.

1. An ________ ________ is a function whose domain is the set of positive integers.
2. The function values $a_1, a_2, a_3, a_4, \ldots$ are called the ________ of a sequence.
3. A sequence is a ________ sequence if the domain of the function consists of the first n positive integers.
4. If you are given one or more of the first few terms of a sequence, and all other terms of the sequence are defined using previous terms, then the sequence is said to be defined ________.
5. If n is a positive integer, n ________ is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.
6. The notation used to represent the sum of the terms of a finite sequence is ________ ________ or sigma notation.
7. For the sum $\sum_{i=1}^{n} a_i$, i is called the ________ of summation, n is the ________ limit of summation, and 1 is the ________ limit of summation.
8. The sum of the terms of a finite or infinite sequence is called a ________.
9. The ________ ________ ________ of a sequence is the sum of the first n terms of the sequence.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–22, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 3n + 1$
2. $a_n = 5n - 3$
3. $a_n = 2^n$
4. $a_n = \left(\frac{1}{2}\right)^n$
5. $a_n = (-2)^n$
6. $a_n = \left(-\frac{1}{2}\right)^n$
7. $a_n = \dfrac{n+2}{n}$
8. $a_n = \dfrac{n}{n+2}$
9. $a_n = \dfrac{6n}{3n^2 - 1}$
10. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$
11. $a_n = \dfrac{1 + (-1)^n}{n}$
12. $a_n = 1 + (-1)^n$
13. $a_n = 2 - \dfrac{1}{3^n}$
14. $a_n = \dfrac{2^n}{3^n}$
15. $a_n = \dfrac{1}{n^{3/2}}$
16. $a_n = \dfrac{10}{n^{2/3}}$
17. $a_n = \dfrac{(-1)^n}{n^2}$
18. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$
19. $a_n = \frac{2}{3}$
20. $a_n = 0.3$
21. $a_n = n(n-1)(n-2)$
22. $a_n = n(n^2 - 6)$

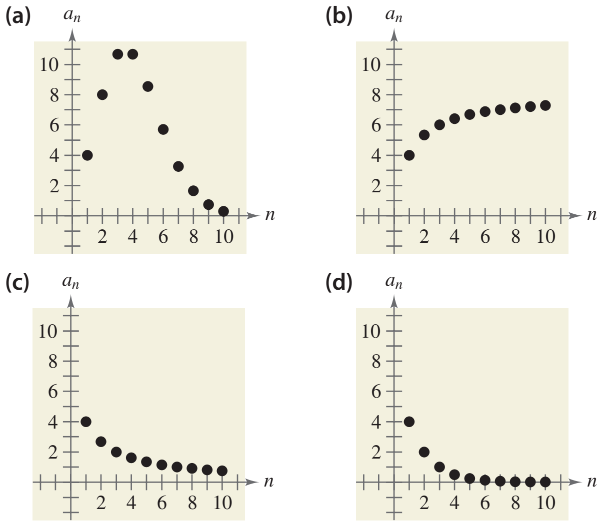
In Exercises 23–26, find the indicated term of the sequence.

23. $a_n = (-1)^n(3n - 2)$
    $a_{25} = \square$
24. $a_n = (-1)^{n-1}[n(n-1)]$
    $a_{16} = \square$
25. $a_n = \dfrac{4n}{2n^2 - 3}$
    $a_{11} = \square$
26. $a_n = \dfrac{4n^2 - n + 3}{n(n-1)(n+2)}$
    $a_{13} = \square$

In Exercises 27–32, use a graphing utility to graph the first 10 terms of the sequence. (Assume that n begins with 1.)

27. $a_n = \dfrac{3}{4}n$
28. $a_n = 2 - \dfrac{4}{n}$
29. $a_n = 16(-0.5)^{n-1}$
30. $a_n = 8(0.75)^{n-1}$
31. $a_n = \dfrac{2n}{n+1}$
32. $a_n = \dfrac{n^2}{n^2 + 2}$

In Exercises 33–36, match the sequence with the graph of its first 10 terms. [The graphs are labeled (a), (b), (c), and (d).]



33. $a_n = \dfrac{8}{n+1}$
34. $a_n = \dfrac{8n}{n+1}$
35. $a_n = 4(0.5)^{n-1}$
36. $a_n = \dfrac{4^n}{n!}$

In Exercises 37–50, write an expression for the apparent nth term of the sequence. (Assume that n begins with 1.)

37. $1, 4, 7, 10, 13, \ldots$

38. $3, 7, 11, 15, 19, \ldots$

39. $0, 3, 8, 15, 24, \ldots$

40. $2, -4, 6, -8, 10, \ldots$

41. $\frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \frac{-6}{7}, \ldots$

42. $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \ldots$

43. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

44. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \ldots$

45. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

46. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

47. $1, -1, 1, -1, 1, \ldots$

48. $1, 2, \frac{2^2}{2}, \frac{2^3}{6}, \frac{2^4}{24}, \frac{2^5}{120}, \ldots$

49. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

50. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

In Exercises 51–54, write the first five terms of the sequence defined recursively.

51. $a_1 = 28, \quad a_{k+1} = a_k - 4$

52. $a_1 = 15, \quad a_{k+1} = a_k + 3$

53. $a_1 = 3, \quad a_{k+1} = 2(a_k - 1)$

54. $a_1 = 32, \quad a_{k+1} = \frac{1}{2}a_k$

In Exercises 55–58, write the first five terms of the sequence defined recursively. Use the pattern to write the nth term of the sequence as a function of n. (Assume that n begins with 1.)

55. $a_1 = 6, \quad a_{k+1} = a_k + 2$

56. $a_1 = 25, \quad a_{k+1} = a_k - 5$

57. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

58. $a_1 = 14, \quad a_{k+1} = (-2)a_k$

In Exercises 59–64, write the first five terms of the sequence. (Assume that n begins with 0.)

59. $a_n = \dfrac{3^n}{n!}$

60. $a_n = \dfrac{n!}{n}$

61. $a_n = \dfrac{1}{(n+1)!}$

62. $a_n = \dfrac{n^2}{(n+1)!}$

63. $a_n = \dfrac{(-1)^{2n}}{(2n)!}$

64. $a_n = \dfrac{(-1)^{2n+1}}{(2n+1)!}$

In Exercises 65–72, simplify the factorial expression.

65. $\dfrac{4!}{6!}$

66. $\dfrac{5!}{8!}$

67. $\dfrac{10!}{8!}$

68. $\dfrac{25!}{23!}$

69. $\dfrac{(n+1)!}{n!}$

70. $\dfrac{(n+2)!}{n!}$

71. $\dfrac{(2n-1)!}{(2n+1)!}$

72. $\dfrac{(3n+1)!}{(3n)!}$

In Exercises 73–84, find the sum.

73. $\displaystyle\sum_{i=1}^{5} (2i + 1)$

74. $\displaystyle\sum_{i=1}^{6} (3i - 1)$

75. $\displaystyle\sum_{k=1}^{4} 10$

76. $\displaystyle\sum_{k=1}^{5} 5$

77. $\displaystyle\sum_{i=0}^{4} i^2$

78. $\displaystyle\sum_{i=0}^{5} 2i^2$

79. $\displaystyle\sum_{k=0}^{3} \frac{1}{k^2 + 1}$

80. $\displaystyle\sum_{j=3}^{5} \frac{1}{j^2 - 3}$

81. $\displaystyle\sum_{k=2}^{5} (k+1)^2(k-3)$

82. $\displaystyle\sum_{i=1}^{4} [(i-1)^2 + (i+1)^3]$

83. $\displaystyle\sum_{i=1}^{4} 2^i$

84. $\displaystyle\sum_{j=0}^{4} (-2)^j$

In Exercises 85–88, use a calculator to find the sum.

85. $\displaystyle\sum_{j=1}^{6} (24 - 3j)$

86. $\displaystyle\sum_{j=1}^{10} \frac{3}{j+1}$

87. $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k+1}$

88. $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k!}$

In Exercises 89–98, use sigma notation to write the sum.

89. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

90. $\dfrac{5}{1+1} + \dfrac{5}{1+2} + \dfrac{5}{1+3} + \cdots + \dfrac{5}{1+15}$

91. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right]$

92. $\left[1 - \left(\frac{1}{6}\right)^2\right] + \left[1 - \left(\frac{2}{6}\right)^2\right] + \cdots + \left[1 - \left(\frac{6}{6}\right)^2\right]$

93. $3 - 9 + 27 - 81 + 243 - 729$

94. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$

95. $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots - \dfrac{1}{20^2}$

96. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{10 \cdot 12}$

97. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64}$

98. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}$

In Exercises 99–102, find the indicated partial sum of the series.

99. $\displaystyle\sum_{i=1}^{\infty} 5\left(\tfrac{1}{2}\right)^i$

Fourth partial sum

100. $\displaystyle\sum_{i=1}^{\infty} 2\left(\tfrac{1}{3}\right)^i$

Fifth partial sum

101. $\displaystyle\sum_{n=1}^{\infty} 4\left(-\tfrac{1}{2}\right)^n$

Third partial sum

102. $\displaystyle\sum_{n=1}^{\infty} 8\left(-\tfrac{1}{4}\right)^n$

Fourth partial sum

In Exercises 103–106, find the sum of the infinite series.

103. $\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i$

104. $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$

105. $\sum_{k=1}^{\infty} 7\left(\frac{1}{10}\right)^k$

106. $\sum_{i=1}^{\infty} 2\left(\frac{1}{10}\right)^i$

107. ***Compound Interest*** A deposit of \$5000 is made in an account that earns 8% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 5000\left(1 + \frac{0.08}{4}\right)^n, \qquad n = 1, 2, 3, \ldots.$$

(a) Write the first eight terms of this sequence.

(b) Find the balance in this account after 10 years by finding the 40th term of the sequence.

108. ***Compound Interest*** A deposit of \$100 is made each month in an account that earns 12% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 100(101)[(1.01)^n - 1], \quad n = 1, 2, 3, \ldots.$$

(a) Write the first six terms of this sequence.

(b) Find the balance in this account after 5 years by finding the 60th term of the sequence.

(c) Find the balance in this account after 20 years by finding the 240th term of the sequence.

Model It

109. ***Data Analysis: Number of Stores*** The table shows the numbers a_n of Best Buy stores for the years 1998 to 2003. (Source: Best Buy Company, Inc.)

Year	Number of stores, a_n
1998	311
1999	357
2000	419
2001	481
2002	548
2003	608

Model It (continued)

(a) Use the *regression* feature of a graphing utility to find a linear sequence that models the data. Let n represent the year, with $n = 8$ corresponding to 1998.

(b) Use the *regression* feature of a graphing utility to find a quadratic sequence that models the data.

(c) Evaluate the sequences from parts (a) and (b) for $n = 8, 9, \ldots, 13$. Compare these values with those shown in the table. Which model is a better fit for the data? Explain.

(d) Which model do you think would better predict the number of Best Buy stores in the future? Use the model you chose to predict the number of Best Buy stores in 2008.

110. ***Medicine*** The numbers a_n (in thousands) of AIDS cases reported from 1995 to 2003 can be approximated by the model

$$a_n = 0.0457n^3 - 0.352n^2 - 9.05n + 121.4,$$
$$n = 5, 6, \ldots, 13$$

where n is the year, with $n = 5$ corresponding to 1995. (Source: U.S. Centers for Disease Control and Prevention)

(a) Find the terms of this finite sequence. Use the *statistical plotting* feature of a graphing utility to construct a bar graph that represents the sequence.

(b) What does the graph in part (a) say about reported cases of AIDS?

111. ***Federal Debt*** From 1990 to 2003, the federal debt of the United States rose from just over \$3 trillion to almost \$7 trillion. The federal debt a_n (in billions of dollars) from 1990 to 2003 is approximated by the model

$$a_n = 2.7698n^3 - 61.372n^2 + 600.00n + 3102.9,$$
$$n = 0, 1, \ldots, 13$$

where n is the year, with $n = 0$ corresponding to 1990. (Source: U.S. Office of Management and Budget)

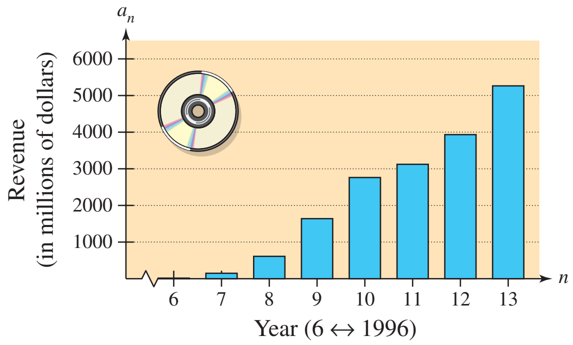
(a) Find the terms of this finite sequence. Use the *statistical plotting* feature of a graphing utility to construct a bar graph that represents the sequence.

(b) What does the pattern in the bar graph in part (a) say about the future of the federal debt?

112. ***Revenue*** The revenues a_n (in millions of dollars) for Amazon.com for the years 1996 through 2003 are shown in the figure. The revenues can be approximated by the model

$$a_n = 46.609n^2 - 119.84n - 1125.8, \quad n = 6, 7, \ldots, 13$$

where n is the year, with $n = 6$ corresponding to 1996. Use this model to approximate the total revenue from 1996 through 2003. Compare this sum with the result of adding the revenues shown in the figure. (Source: Amazon.com)



Synthesis

True or False? **In Exercises 113 and 114, determine whether the statement is true or false. Justify your answer.**

113. $\sum_{i=1}^{4}(i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$ **114.** $\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

Fibonacci Sequence **In Exercises 115 and 116, use the Fibonacci sequence. (See Example 4.)**

115. Write the first 12 terms of the Fibonacci sequence a_n and the first 10 terms of the sequence given by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

116. Using the definition for b_n in Exercise 115, show that b_n can be defined recursively by

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

Arithmetic Mean **In Exercises 117–120, use the following definition of the arithmetic mean $\bar{x}$ of a set of n measurements $x_1, x_2, x_3, \ldots, x_n$.**

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

117. Find the arithmetic mean of the six checking account balances \$327.15, \$785.69, \$433.04, \$265.38, \$604.12, and \$590.30. Use the statistical capabilities of a graphing utility to verify your result.

118. Find the arithmetic mean of the following prices per gallon for regular unleaded gasoline at five gasoline stations in a city: \$1.899, \$1.959, \$1.919, \$1.939, and \$1.999. Use the statistical capabilities of a graphing utility to verify your result.

119. ***Proof*** Prove that $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$.

120. ***Proof*** Prove that $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$.

In Exercises 121–124, find the first five terms of the sequence.

121. $a_n = \dfrac{x^n}{n!}$ **122.** $a_n = \dfrac{(-1)^n x^{2n+1}}{2n+1}$

123. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$ **124.** $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

Skills Review

In Exercises 125–128, determine whether the function has an inverse function. If it does, find its inverse function.

125. $f(x) = 4x - 3$ **126.** $g(x) = \dfrac{3}{x}$

127. $h(x) = \sqrt{5x + 1}$ **128.** $f(x) = (x - 1)^2$

In Exercises 129–132, find (a) $A - B$, (b) $4B - 3A$, (c) AB, and (d) BA.

129. $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix}$

130. $A = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix}$

131. $A = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix}$

132. $A = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

In Exercises 133–136, find the determinant of the matrix.

133. $A = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$ **134.** $A = \begin{bmatrix} -2 & 8 \\ 12 & 15 \end{bmatrix}$

135. $A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 7 & 3 \\ 4 & 9 & -1 \end{bmatrix}$

136. $A = \begin{bmatrix} 16 & 11 & 10 & 2 \\ 9 & 8 & 3 & 7 \\ -2 & -1 & 12 & 3 \\ -4 & 6 & 2 & 1 \end{bmatrix}$

9.2 Arithmetic Sequences and Partial Sums

What you should learn

- Recognize, write, and find the nth terms of arithmetic sequences.
- Find nth partial sums of arithmetic sequences.
- Use arithmetic sequences to model and solve real-life problems.

Why you should learn it

Arithmetic sequences have practical real-life applications. For instance, in Exercise 83 on page 660, an arithmetic sequence is used to model the seating capacity of an auditorium.

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** if the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic if there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

The number d is the **common difference** of the arithmetic sequence.

Example 1 **Examples of Arithmetic Sequences**

a. The sequence whose nth term is $4n + 3$ is arithmetic. For this sequence, the common difference between consecutive terms is 4.

$$\underbrace{7, 11}_{11 - 7 = 4}, 15, 19, \ldots, 4n + 3, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

$$\underbrace{2, -3}_{-3 - 2 = -5}, -8, -13, \ldots, 7 - 5n, \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}_{\frac{5}{4} - 1 = \frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n + 3}{4}, \ldots \qquad \text{Begin with } n = 1.$$

CHECKPOINT Now try Exercise 1.

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

In Example 1, notice that each of the arithmetic sequences has an nth term that is of the form $dn + c$, where the common difference of the sequence is d. An arithmetic sequence may be thought of as a linear function whose domain is the set of natural numbers.

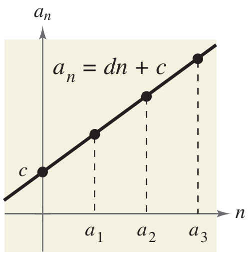


FIGURE 9.3

The nth Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = dn + c \qquad \text{Linear form}$$

where d is the common difference between consecutive terms of the sequence and $c = a_1 - d$. A graphical representation of this definition is shown in Figure 9.3. Substituting $a_1 - d$ for c in $a_n = dn + c$ yields an alternative *recursion* form for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n - 1)\,d \qquad \text{Alternative form}$$

STUDY TIP

The alternative recursion form of the nth term of an arithmetic sequence can be derived from the pattern below.

$a_1 = a_1$ 1st term

$a_2 = a_1 + d$ 2nd term

$a_3 = a_1 + 2d$ 3rd term

$a_4 = a_1 + 3d$ 4th term

$a_5 = a_1 + 4d$ 5th term

1 less

$\vdots$

$a_n = a_1 + (n - 1)\,d$ nth term

1 less

Example 2 Finding the nth Term of an Arithmetic Sequence

Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

Solution

Because the sequence is arithmetic, you know that the formula for the nth term is of the form $a_n = dn + c$. Moreover, because the common difference is $d = 3$, the formula must have the form

$$a_n = 3n + c. \qquad \text{Substitute 3 for } d.$$

Because $a_1 = 2$, it follows that

$$\begin{aligned} c &= a_1 - d \\ &= 2 - 3 && \text{Substitute 2 for } a_1 \text{ and 3 for } d. \\ &= -1. \end{aligned}$$

So, the formula for the nth term is

$$a_n = 3n - 1.$$

The sequence therefore has the following form.

$$2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots$$

CHECKPOINT Now try Exercise 21.

Another way to find a formula for the nth term of the sequence in Example 2 is to begin by writing the terms of the sequence.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\ldots$
2	2 + 3	5 + 3	8 + 3	11 + 3	14 + 3	17 + 3	$\ldots$
2	5	8	11	14	17	20	$\ldots$

From these terms, you can reason that the nth term is of the form

$$a_n = dn + c = 3n - 1.$$

STUDY TIP

You can find a_1 in Example 3 by using the alternative recursion form of the nth term of an arithmetic sequence, as follows.

$$a_n = a_1 + (n - 1)d$$
$$a_4 = a_1 + (4 - 1)d$$
$$20 = a_1 + (4 - 1)5$$
$$20 = a_1 + 15$$
$$5 = a_1$$

Example 3 Writing the Terms of an Arithmetic Sequence

The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first 11 terms of this sequence.

Solution

You know that $a_4 = 20$ and $a_{13} = 65$. So, you must add the common difference d nine times to the fourth term to obtain the 13th term. Therefore, the fourth and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d. \qquad a_4 \text{ and } a_{13} \text{ are nine terms apart.}$$

Using $a_4 = 20$ and $a_{13} = 65$, you can conclude that $d = 5$, which implies that the sequence is as follows.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11} . . .
5	10	15	20	25	30	35	40	45	50	55 . . .

CHECKPOINT Now try Exercise 37.

If you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the *recursion formula*

$$a_{n+1} = a_n + d. \qquad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term. For instance, if you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

Example 4 Using a Recursion Formula

Find the ninth term of the arithmetic sequence that begins with 2 and 9.

Solution

For this sequence, the common difference is $d = 9 - 2 = 7$. There are two ways to find the ninth term. One way is simply to write out the first nine terms (by repeatedly adding 7).

2, 9, 16, 23, 30, 37, 44, 51, 58

Another way to find the ninth term is to first find a formula for the nth term. Because the first term is 2, it follows that

$$c = a_1 - d = 2 - 7 = -5.$$

Therefore, a formula for the nth term is

$$a_n = 7n - 5$$

which implies that the ninth term is

$$a_9 = 7(9) - 5 = 58.$$

CHECKPOINT Now try Exercise 45.

The Sum of a Finite Arithmetic Sequence

There is a simple formula for the *sum* of a finite arithmetic sequence.

STUDY TIP

Note that this formula works only for *arithmetic* sequences.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

For a proof of the sum of a finite arithmetic sequence, see Proofs in Mathematics on page 723.

Example 5 Finding the Sum of a Finite Arithmetic Sequence

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

Solution

To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Formula for the sum of an arithmetic sequence}$$

$$= \frac{10}{2}(1 + 19) \qquad \text{Substitute 10 for } n, \text{ 1 for } a_1, \text{ and 19 for } a_n.$$

$$= 5(20) = 100. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 63.

Example 6 Finding the Sum of a Finite Arithmetic Sequence

Find the sum of the integers (a) from 1 to 100 and (b) from 1 to N.

Solution

a. The integers from 1 to 100 form an arithmetic sequence that has 100 terms. So, you can use the formula for the sum of an arithmetic sequence, as follows.

$$S_n = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 99 + 100$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Formula for sum of an arithmetic sequence}$$

$$= \frac{100}{2}(1 + 100) \qquad \text{Substitute 100 for } n, \text{ 1 for } a_1, \text{ 100 for } a_n.$$

$$= 50(101) = 5050 \qquad \text{Simplify.}$$

b. $S_n = 1 + 2 + 3 + 4 + \cdots + N$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Formula for sum of an arithmetic sequence}$$

$$= \frac{N}{2}(1 + N) \qquad \text{Substitute } N \text{ for } n, \text{ 1 for } a_1, \text{ and } N \text{ for } a_n.$$

CHECKPOINT Now try Exercise 65.

The Granger Collection

Historical Note

A teacher of Carl Friedrich Gauss (1777–1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcrcrcrcl} S_n & = & 1 & + & 2 & + & 3 & + \cdots + & 100 \\ S_n & = & 100 & + & 99 & + & 98 & + \cdots + & 1 \\ \hline 2S_n & = & 101 & + & 101 & + & 101 & + \cdots + & 101 \end{array}$$

$$S_n = \frac{100 \times 101}{2} = 5050$$

The sum of the first n terms of an infinite sequence is the *nth partial sum*. The nth partial sum can be found by using the formula for the sum of a finite arithmetic sequence.

Example 7 Finding a Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence

$$5, 16, 27, 38, 49, \ldots .$$

Solution

For this arithmetic sequence, $a_1 = 5$ and $d = 16 - 5 = 11$. So,

$$c = a_1 - d = 5 - 11 = -6$$

and the nth term is $a_n = 11n - 6$. Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$S_{150} = \frac{n}{2}(a_1 + a_{150}) \qquad n\text{th partial sum formula}$$

$$= \frac{150}{2}(5 + 1644) \qquad \text{Substitute 150 for } n, \text{ 5 for } a_1, \text{ and 1644 for } a_{150}.$$

$$= 75(1649) \qquad \text{Simplify.}$$

$$= 123{,}675. \qquad n\text{th partial sum}$$

CHECKPOINT Now try Exercise 69.

Applications

Example 8 Prize Money

In a golf tournament, the 16 golfers with the lowest scores win cash prizes. First place receives a cash prize of \$1000, second place receives \$950, third place receives \$900, and so on. What is the total amount of prize money?

Solution

The cash prizes awarded form an arithmetic sequence in which the common difference is $d = -50$. Because

$$c = a_1 - d = 1000 - (-50) = 1050$$

you can determine that the formula for the nth term of the sequence is $a_n = -50n + 1050$. So, the 16th term of the sequence is $a_{16} = -50(16) + 1050 = 250$, and the total amount of prize money is

$$S_{16} = 1000 + 950 + 900 + \cdots + 250$$

$$S_{16} = \frac{n}{2}(a_1 + a_{16}) \qquad n\text{th partial sum formula}$$

$$= \frac{16}{2}(1000 + 250) \qquad \text{Substitute 16 for } n, \text{ 1000 for } a_1, \text{ and 250 for } a_{16}.$$

$$= 8(1250) = \$10{,}000. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 89.

Example 9 Total Sales

A small business sells \$10,000 worth of skin care products during its first year. The owner of the business has set a goal of increasing annual sales by \$7500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation.

Solution

The annual sales form an arithmetic sequence in which $a_1 = 10{,}000$ and $d = 7500$. So,

$$c = a_1 - d$$
$$= 10{,}000 - 7500$$
$$= 2500$$

and the nth term of the sequence is

$$a_n = 7500n + 2500.$$

This implies that the 10th term of the sequence is

$$a_{10} = 7500(10) + 2500$$
$$= 77{,}500. \qquad \text{See Figure 9.4.}$$

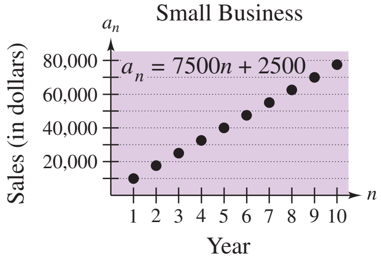


FIGURE 9.4

The sum of the first 10 terms of the sequence is

$$S_{10} = \frac{n}{2}(a_1 + a_{10}) \qquad n\text{th partial sum formula}$$
$$= \frac{10}{2}(10{,}000 + 77{,}500) \qquad \text{Substitute 10 for } n\text{, 10,000 for } a_1\text{, and 77,500 for } a_{10}.$$
$$= 5(87{,}500) \qquad \text{Simplify.}$$
$$= 437{,}500. \qquad \text{Simplify.}$$

So, the total sales for the first 10 years will be \$437,500.

CHECKPOINT Now try Exercise 91.

WRITING ABOUT MATHEMATICS

Numerical Relationships Decide whether it is possible to fill in the blanks in each of the sequences such that the resulting sequence is arithmetic. If so, find a recursion formula for the sequence.

a. -7, ___, ___, ___, ___, ___, 11

b. 17, ___, ___, ___, ___, ___, ___, ___, ___, 71

c. 2, 6, ___, ___, 162

d. 4, 7.5, ___, ___, ___, ___, ___, ___, ___, 39

e. 8, 12, ___, ___, ___, 60.75

9.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A sequence is called an ________ sequence if the differences between two consecutive terms are the same. This difference is called the ________ difference.
2. The nth term of an arithmetic sequence has the form ________.
3. The formula $S_n = \frac{n}{2}(a_1 + a_n)$ can be used to find the sum of the first n terms of an arithmetic sequence, called the ________ of a ________ ________ ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, determine whether the sequence is arithmetic. If so, find the common difference.

1. $10, 8, 6, 4, 2, \ldots$
2. $4, 7, 10, 13, 16, \ldots$
3. $1, 2, 4, 8, 16, \ldots$
4. $80, 40, 20, 10, 5, \ldots$
5. $\frac{9}{4}, 2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, \ldots$
6. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \ldots$
7. $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{6}, \ldots$
8. $5.3, 5.7, 6.1, 6.5, 6.9, \ldots$
9. $\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, \ldots$
10. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

In Exercises 11–18, write the first five terms of the sequence. Determine whether the sequence is arithmetic. If so, find the common difference. (Assume that *n* begins with 1.)

11. $a_n = 5 + 3n$
12. $a_n = 100 - 3n$
13. $a_n = 3 - 4(n - 2)$
14. $a_n = 1 + (n - 1)4$
15. $a_n = (-1)^n$
16. $a_n = 2^{n-1}$
17. $a_n = \dfrac{(-1)^n 3}{n}$
18. $a_n = (2^n)n$

In Exercises 19–30, find a formula for a_n for the arithmetic sequence.

19. $a_1 = 1, d = 3$
20. $a_1 = 15, d = 4$
21. $a_1 = 100, d = -8$
22. $a_1 = 0, d = -\frac{2}{3}$
23. $a_1 = x, d = 2x$
24. $a_1 = -y, d = 5y$
25. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$
26. $10, 5, 0, -5, -10, \ldots$
27. $a_1 = 5, a_4 = 15$
28. $a_1 = -4, a_5 = 16$
29. $a_3 = 94, a_6 = 85$
30. $a_5 = 190, a_{10} = 115$

In Exercises 31–38, write the first five terms of the arithmetic sequence.

31. $a_1 = 5, d = 6$
32. $a_1 = 5, d = -\frac{3}{4}$
33. $a_1 = -2.6, d = -0.4$
34. $a_1 = 16.5, d = 0.25$
35. $a_1 = 2, a_{12} = 46$
36. $a_4 = 16, a_{10} = 46$
37. $a_8 = 26, a_{12} = 42$
38. $a_3 = 19, a_{15} = -1.7$

In Exercises 39–44, write the first five terms of the arithmetic sequence. Find the common difference and write the *n*th term of the sequence as a function of *n*.

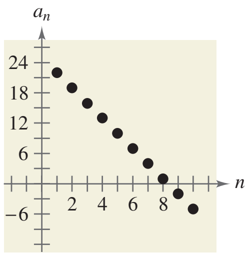
39. $a_1 = 15, \quad a_{k+1} = a_k + 4$
40. $a_1 = 6, \quad a_{k+1} = a_k + 5$
41. $a_1 = 200, \quad a_{k+1} = a_k - 10$
42. $a_1 = 72, \quad a_{k+1} = a_k - 6$
43. $a_1 = \frac{5}{8}, \quad a_{k+1} = a_k - \frac{1}{8}$
44. $a_1 = 0.375, \quad a_{k+1} = a_k + 0.25$

In Exercises 45–48, the first two terms of the arithmetic sequence are given. Find the missing term.

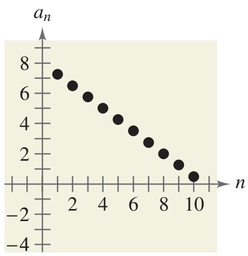
45. $a_1 = 5, a_2 = 11, a_{10} =$ ____
46. $a_1 = 3, a_2 = 13, a_9 =$ ____
47. $a_1 = 4.2, a_2 = 6.6, a_7 =$ ____
48. $a_1 = -0.7, a_2 = -13.8, a_8 =$ ____

In Exercises 49–52, match the arithmetic sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

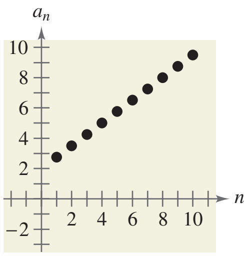
(a)



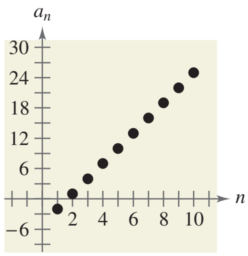
(b)



(c)



(d)



49. $a_n = -\frac{3}{4}n + 8$ **50.** $a_n = 3n - 5$

51. $a_n = 2 + \frac{3}{4}n$ **52.** $a_n = 25 - 3n$

In Exercises 53–56, use a graphing utility to graph the first 10 terms of the sequence. (Assume that *n* begins with 1.)

53. $a_n = 15 - \frac{3}{2}n$ **54.** $a_n = -5 + 2n$

55. $a_n = 0.2n + 3$ **56.** $a_n = -0.3n + 8$

In Exercises 57–64, find the indicated *n*th partial sum of the arithmetic sequence.

57. 8, 20, 32, 44, . . . , $n = 10$

58. 2, 8, 14, 20, . . . , $n = 25$

59. 4.2, 3.7, 3.2, 2.7, . . . , $n = 12$

60. 0.5, 0.9, 1.3, 1.7, . . . , $n = 10$

61. 40, 37, 34, 31, . . . , $n = 10$

62. 75, 70, 65, 60, . . . , $n = 25$

63. $a_1 = 100$, $a_{25} = 220$, $n = 25$

64. $a_1 = 15$, $a_{100} = 307$, $n = 100$

65. Find the sum of the first 100 positive odd integers.

66. Find the sum of the integers from -10 to 50.

In Exercises 67–74, find the partial sum.

67. $\sum_{n=1}^{50} n$ **68.** $\sum_{n=1}^{100} 2n$

69. $\sum_{n=10}^{100} 6n$ **70.** $\sum_{n=51}^{100} 7n$

71. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n$ **72.** $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n$

73. $\sum_{n=1}^{400} (2n - 1)$ **74.** $\sum_{n=1}^{250} (1000 - n)$

In Exercises 75–80, use a graphing utility to find the partial sum.

75. $\sum_{n=1}^{20} (2n + 5)$ **76.** $\sum_{n=0}^{50} (1000 - 5n)$

77. $\sum_{n=1}^{100} \frac{n + 4}{2}$ **78.** $\sum_{n=0}^{100} \frac{8 - 3n}{16}$

79. $\sum_{i=1}^{60} \left(250 - \frac{8}{3}i\right)$ **80.** $\sum_{j=1}^{200} (4.5 + 0.025j)$

Job Offer **In Exercises 81 and 82, consider a job offer with the given starting salary and the given annual raise.**

(a) Determine the salary during the sixth year of employment.

(b) Determine the total compensation from the company through six full years of employment.

	Starting Salary	*Annual Raise*
81.	\$32,500	\$1500
82.	\$36,800	\$1750

83. ***Seating Capacity*** Determine the seating capacity of an auditorium with 30 rows of seats if there are 20 seats in the first row, 24 seats in the second row, 28 seats in the third row, and so on.

84. ***Seating Capacity*** Determine the seating capacity of an auditorium with 36 rows of seats if there are 15 seats in the first row, 18 seats in the second row, 21 seats in the third row, and so on.

85. ***Brick Pattern*** A brick patio has the approximate shape of a trapezoid (see figure). The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are in the patio?

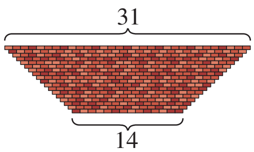


FIGURE FOR 85

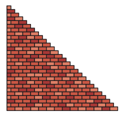


FIGURE FOR 86

86. ***Brick Pattern*** A triangular brick wall is made by cutting some bricks in half to use in the first column of every other row. The wall has 28 rows. The top row is one-half brick wide and the bottom row is 14 bricks wide. How many bricks are used in the finished wall?

87. ***Falling Object*** An object with negligible air resistance is dropped from a plane. During the first second of fall, the object falls 4.9 meters; during the second second, it falls 14.7 meters; during the third second, it falls 24.5 meters; during the fourth second, it falls 34.3 meters. If this arithmetic pattern continues, how many meters will the object fall in 10 seconds?

88. ***Falling Object*** An object with negligible air resistance is dropped from the top of the Sears Tower in Chicago at a height of 1454 feet. During the first second of fall, the object falls 16 feet; during the second second, it falls 48 feet; during the third second, it falls 80 feet; during the fourth second, it falls 112 feet. If this arithmetic pattern continues, how many feet will the object fall in 7 seconds?

89. ***Prize Money*** A county fair is holding a baked goods competition in which the top eight bakers receive cash prizes. First places receives a cash prize of $200, second place receives $175, third place receives $150, and so on.

(a) Write a sequence a_n that represents the cash prize awarded in terms of the place n in which the baked good places.

(b) Find the total amount of prize money awarded at the competition.

90. ***Prize Money*** A city bowling league is holding a tournament in which the top 12 bowlers with the highest three-game totals are awarded cash prizes. First place will win $1200, second place $1100, third place $1000, and so on.

(a) Write a sequence a_n that represents the cash prize awarded in terms of the place n in which the bowler finishes.

(b) Find the total amount of prize money awarded at the tournament.

91. ***Total Profit*** A small snowplowing company makes a profit of $8000 during its first year. The owner of the company sets a goal of increasing profit by $1500 each year for 5 years. Assuming that this goal is met, find the total profit during the first 6 years of this business. What kinds of economic factors could prevent the company from meeting its profit goal? Are there any other factors that could prevent the company from meeting its goal? Explain.

92. ***Total Sales*** An entrepreneur sells $15,000 worth of sports memorabilia during one year and sets a goal of increasing annual sales by $5000 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years of this business. What kinds of economic factors could prevent the business from meeting its goals?

93. ***Borrowing Money*** You borrowed $2000 from a friend to purchase a new laptop computer and have agreed to pay back the loan with monthly payments of $200 plus 1% interest on the unpaid balance.

(a) Find the first six monthly payments you will make, and the unpaid balance after each month.

(b) Find the total amount of interest paid over the term of the loan.

94. ***Borrowing Money*** You borrowed $5000 from your parents to purchase a used car. The arrangements of the loan are such that you will make payments of $250 per month plus 1% interest on the unpaid balance.

(a) Find the first year's monthly payments you will make, and the unpaid balance after each month.

(b) Find the total amount of interest paid over the term of the loan.

Model It

95. ***Data Analysis: Personal Income*** The table shows the per capita personal income a_n in the United States from 1993 to 2003. (Source: U.S. Bureau of Economic Analysis)

Year	Per capita personal income, a_n
1993	$21,356
1994	$22,176
1995	$23,078
1996	$24,176
1997	$25,334
1998	$26,880
1999	$27,933
2000	$29,848
2001	$30,534
2002	$30,913
2003	$31,633

(a) Find an arithmetic sequence that models the data. Let n represent the year, with $n = 3$ corresponding to 1993.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. How does this model compare with the arithmetic sequence you found in part (a)?

(c) Use a graphing utility to graph the terms of the finite sequence you found in part (a).

(d) Use the sequence from part (a) to estimate the per capita personal income in 2004 and 2005.

(e) Use your school's library, the Internet, or some other reference source to find the actual per capita personal income in 2004 and 2005, and compare these values with the estimates from part (d).

96. ***Data Analysis: Revenue*** The table shows the annual revenue a_n (in millions of dollars) for Nextel Communications, Inc. from 1997 to 2003. (Source: Nextel Communications, Inc.)

Year	Revenue, a_n
1997	739
1998	1847
1999	3326
2000	5714
2001	7689
2002	8721
2003	10,820

(a) Construct a bar graph showing the annual revenue from 1997 to 2003.

(b) Use the *linear regression* feature of a graphing utility to find an arithmetic sequence that approximates the annual revenue from 1997 to 2003.

(c) Use summation notation to represent the *total* revenue from 1997 to 2003. Find the total revenue.

(d) Use the sequence from part (b) to estimate the annual revenue in 2008.

Synthesis

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.**

97. Given an arithmetic sequence for which only the first two terms are known, it is possible to find the nth term.

98. If the only known information about a finite arithmetic sequence is its first term and its last term, then it is possible to find the sum of the sequence.

99. ***Writing*** In your own words, explain what makes a sequence arithmetic.

100. ***Writing*** Explain how to use the first two terms of an arithmetic sequence to find the nth term.

101. ***Exploration***

(a) Graph the first 10 terms of the arithmetic sequence $a_n = 2 + 3n$.

(b) Graph the equation of the line $y = 3x + 2$.

(c) Discuss any differences between the graph of

$a_n = 2 + 3n$

and the graph of

$y = 3x + 2$.

(d) Compare the slope of the line in part (b) with the common difference of the sequence in part (a). What can you conclude about the slope of a line and the common difference of an arithmetic sequence?

102. ***Pattern Recognition***

(a) Compute the following sums of positive odd integers.

$1 + 3 = \square$

$1 + 3 + 5 = \square$

$1 + 3 + 5 + 7 = \square$

$1 + 3 + 5 + 7 + 9 = \square$

$1 + 3 + 5 + 7 + 9 + 11 = \square$

(b) Use the sums in part (a) to make a conjecture about the sums of positive odd integers. Check your conjecture for the sum

$1 + 3 + 5 + 7 + 9 + 11 + 13 = \square$.

(c) Verify your conjecture algebraically.

103. ***Think About It*** The sum of the first 20 terms of an arithmetic sequence with a common difference of 3 is 650. Find the first term.

104. ***Think About It*** The sum of the first n terms of an arithmetic sequence with first term a_1 and common difference d is S_n. Determine the sum if each term is increased by 5. Explain.

Skills Review

In Exercises 105–108, find the slope and y-intercept (if possible) of the equation of the line. Sketch the line.

105. $2x - 4y = 3$

106. $9x + y = -8$

107. $x - 7 = 0$

108. $y + 11 = 0$

In Exercises 109 and 110, use Gauss-Jordan elimination to solve the system of equations.

109. $\begin{cases} 2x - y + 7z = -10 \\ 3x + 2y - 4z = 17 \\ 6x - 5y + z = -20 \end{cases}$

110. $\begin{cases} -x + 4y + 10z = 4 \\ 5x - 3y + z = 31 \\ 8x + 2y - 3z = -5 \end{cases}$

111. **Make a Decision** To work an extended application analyzing the median sales price of existing one-family homes in the United States from 1987 to 2003, visit this text's website at *college.hmco.com*. *(Data Source: National Association of Realtors)*

9.3 Geometric Sequences and Series

What you should learn

- Recognize, write, and find the *n*th terms of geometric sequences.
- Find *n*th partial sums of geometric sequences.
- Find the sum of an infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

Why you should learn it

Geometric sequences can be used to model and solve real-life problems. For instance, in Exercise 99 on page 670, you will use a geometric sequence to model the population of China.

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Geometric Sequences

In Section 9.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is **geometric** if the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n \ldots$ is geometric if there is a number r such that

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \quad r \neq 0$$

and so the number r is the **common ratio** of the sequence.

Example 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio of consecutive terms is 2.

$$\underbrace{2, 4}_{\frac{4}{2} = 2}, 8, 16, \ldots, 2^n, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio of consecutive terms is 3.

$$\underbrace{12, 36}_{\frac{36}{12} = 3}, 108, 324, \ldots, 4(3^n), \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio of consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9}}_{\frac{1/9}{-1/3} = -\frac{1}{3}}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots \qquad \text{Begin with } n = 1.$$

CHECKPOINT Now try Exercise 1.

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is $\frac{a_3}{a_2} = \frac{9}{4}$.

In Example 1, notice that each of the geometric sequences has an nth term that is of the form ar^n, where the common ratio of the sequence is r. A geometric sequence may be thought of as an exponential function whose domain is the set of natural numbers.

The nth Term of a Geometric Sequence

The nth term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{ccccccc} a_1, & a_2, & a_3, & a_4, & a_5, \ldots\ldots, & a_n, \ldots\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1, & a_1r, & a_1r^2, & a_1r^3, & a_1r^4, \ldots, & a_1r^{n-1}, \ldots \end{array}$$

If you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = ra_n$.

Example 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$. Then graph the terms on a set of coordinate axes.

Solution

Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$ 1st term

$a_2 = 3(2^1) = 6$ 2nd term

$a_3 = 3(2^2) = 12$ 3rd term

$a_4 = 3(2^3) = 24$ 4th term

$a_5 = 3(2^4) = 48$ 5th term

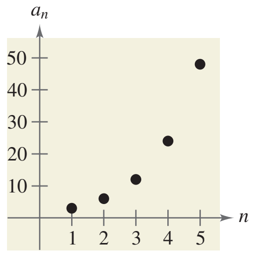


FIGURE 9.5

Figure 9.5 shows the first five terms of this geometric sequence.

CHECKPOINT Now try Exercise 11.

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Solution

$a_{15} = a_1 r^{n-1}$ Formula for geometric sequence

$= 20(1.05)^{15-1}$ Substitute 20 for a_1, 1.05 for r, and 15 for n.

≈ 39.599 Use a calculator.

CHECKPOINT Now try Exercise 27.

Example 4 Finding a Term of a Geometric Sequence

Find the 12th term of the geometric sequence

$5, 15, 45, \ldots$

Solution

The common ratio of this sequence is

$$r = \frac{15}{5} = 3.$$

Because the first term is $a_1 = 5$, you can determine the 12th term $(n = 12)$ to be

$$a_n = a_1 r^{n-1} \qquad \text{Formula for geometric sequence}$$

$$a_{12} = 5(3)^{12-1} \qquad \text{Substitute 5 for } a_1\text{, 3 for } r\text{, and 12 for } n.$$

$$= 5(177{,}147) \qquad \text{Use a calculator.}$$

$$= 885{,}735. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 35.

If you know any two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

STUDY TIP

Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5,

$$a_{10} = a_1 r^9$$

$$= a_1 \cdot r \cdot r \cdot r \cdot r^6$$

$$= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6$$

$$= a_4 r^6.$$

Example 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is $125/64$. Find the 14th term. (Assume that the terms of the sequence are positive.)

Solution

The 10th term is related to the fourth term by the equation

$$a_{10} = a_4 r^6. \qquad \text{Multiply 4th term by } r^{10-4}.$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6 \qquad \text{Substitute } \tfrac{125}{64} \text{ for } a_{10} \text{ and 125 for } a_4.$$

$$\frac{1}{64} = r^6 \qquad \text{Divide each side by 125.}$$

$$\frac{1}{2} = r \qquad \text{Take the sixth root of each side.}$$

You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10} r^4 \qquad \text{Multiply the 10th term by } r^{14-10}.$$

$$= \frac{125}{64}\left(\frac{1}{2}\right)^4 \qquad \text{Substitute } \tfrac{125}{64} \text{ for } a_{10} \text{ and } \tfrac{1}{2} \text{ for } r.$$

$$= \frac{125}{1024} \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 41.

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ .\ .\ .\ ,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

For a proof of the sum of a finite geometric sequence, see Proofs in Mathematics on page 723.

Example 6 Finding the Sum of a Finite Geometric Sequence

Find the sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$.

Solution

By writing out a few terms, you have

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + \cdot\cdot\cdot + 4(0.3)^{11}.$$

Now, because $a_1 = 4$, $r = 0.3$, and $n = 12$, you can apply the formula for the sum of a finite geometric sequence to obtain

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) \qquad \text{Formula for the sum of a sequence}$$

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \qquad \text{Substitute 4 for } a_1,\ 0.3 \text{ for } r, \text{ and 12 for } n.$$

$$\approx 5.714. \qquad \text{Use a calculator.}$$

CHECKPOINT Now try Exercise 57.

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form

$$\sum_{i=1}^{n} a_1 r^{i-1}. \qquad \text{Exponent for } r \text{ is } i-1.$$

If the sum is not of this form, you must adjust the formula. For instance, if the sum in Example 6 were $\sum_{i=1}^{12} 4(0.3)^i$, then you would evaluate the sum as follows.

$$\sum_{i=1}^{12} 4(0.3)^i = 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdot\cdot\cdot + 4(0.3)^{12}$$

$$= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdot\cdot\cdot + [4(0.3)](0.3)^{11}$$

$$= 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \approx 1.714. \qquad a_1 = 4(0.3),\ r = 0.3,\ n = 12$$

Exploration

Use a graphing utility to graph

$$y = \left(\frac{1 - r^x}{1 - r}\right)$$

for $r = \frac{1}{2}, \frac{2}{3}$, and $\frac{4}{5}$. What happens as $x \to \infty$?

Use a graphing utility to graph

$$y = \left(\frac{1 - r^x}{1 - r}\right)$$

for $r = 1.5$, 2, and 3. What happens as $x \to \infty$?

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric *series*. Specifically, if the common ratio r has the property that $|r| < 1$, it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \longrightarrow a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Note that if $|r| \geq 1$, the series does not have a sum.

Example 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\displaystyle\sum_{n=1}^{\infty} 4(0.6)^{n-1}$

b. $3 + 0.3 + 0.03 + 0.003 + \cdots$

Solution

a.
$$\begin{aligned}\sum_{n=1}^{\infty} 4(0.6)^{n-1} &= 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^{n-1} + \cdots \\ &= \frac{4}{1 - 0.6} && \frac{a_1}{1 - r} \\ &= 10\end{aligned}$$

b.
$$\begin{aligned}3 + 0.3 + 0.03 + 0.003 + \cdots &= 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots \\ &= \frac{3}{1 - 0.1} && \frac{a_1}{1 - r} \\ &= \frac{10}{3} \\ &\approx 3.33\end{aligned}$$

CHECKPOINT Now try Exercise 79.

Application

STUDY TIP

Recall from Section 3.1 that the formula for compound interest is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

So, in Example 8, \$50 is the principal P, 0.06 is the interest rate r, 12 is the number of compoundings per year n, and 2 is the time t in years. If you substitute these values into the formula, you obtain

$$A = 50\left(1 + \frac{0.06}{12}\right)^{12(2)}$$

$$= 50\left(1 + \frac{0.06}{12}\right)^{24}.$$

Example 8 Increasing Annuity

A deposit of \$50 is made on the first day of each month in a savings account that pays 6% compounded monthly. What is the balance at the end of 2 years? (This type of savings plan is called an **increasing annuity.**)

Solution

The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24}$$

$$= 50(1.005)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23}$$

$$= 50(1.005)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_{1} = 50\left(1 + \frac{0.06}{12}\right)^{1}$$

$$= 50(1.005).$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.005)$ and $r = 1.005$, you have

$$S_{24} = 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right]$$ Substitute $50(1.005)$ for A_1, 1.005 for r, and 24 for n.

$$= \$1277.96.$$ Simplify.

CHECKPOINT Now try Exercise 107.

WRITING ABOUT MATHEMATICS

An Experiment You will need a piece of string or yarn, a pair of scissors, and a tape measure. Measure out any length of string at least 5 feet long. Double over the string and cut it in half. Take one of the resulting halves, double it over, and cut it in half. Continue this process until you are no longer able to cut a length of string in half. How many cuts were you able to make? Construct a sequence of the resulting string lengths after each cut, starting with the original length of the string. Find a formula for the *n*th term of this sequence. How many cuts could you theoretically make? Discuss why you were not able to make that many cuts.

9.3 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A sequence is called a ________ sequence if the ratios between consecutive terms are the same. This ratio is called the ________ ratio.

2. The nth term of a geometric sequence has the form ________.

3. The formula for the sum of a finite geometric sequence is given by ________.

4. The sum of the terms of an infinite geometric sequence is called a ________ ________.

5. The formula for the sum of an infinite geometric series is given by ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, determine whether the sequence is geometric. If so, find the common ratio.

1. $5, 15, 45, 135, \ldots$

2. $3, 12, 48, 192, \ldots$

3. $3, 12, 21, 30, \ldots$

4. $36, 27, 18, 9, \ldots$

5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

6. $5, 1, 0.2, 0.04, \ldots$

7. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

8. $9, -6, 4, -\frac{8}{3}, \ldots$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

10. $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

In Exercises 11–20, write the first five terms of the geometric sequence.

11. $a_1 = 2, r = 3$

12. $a_1 = 6, r = 2$

13. $a_1 = 1, r = \frac{1}{2}$

14. $a_1 = 1, r = \frac{1}{3}$

15. $a_1 = 5, r = -\frac{1}{10}$

16. $a_1 = 6, r = -\frac{1}{4}$

17. $a_1 = 1, r = e$

18. $a_1 = 3, r = \sqrt{5}$

19. $a_1 = 2, r = \dfrac{x}{4}$

20. $a_1 = 5, r = 2x$

In Exercises 21–26, write the first five terms of the geometric sequence. Determine the common ratio and write the *n*th term of the sequence as a function of *n*.

21. $a_1 = 64, \quad a_{k+1} = \frac{1}{2}a_k$

22. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

23. $a_1 = 7, \quad a_{k+1} = 2a_k$

24. $a_1 = 5, \quad a_{k+1} = -2a_k$

25. $a_1 = 6, \quad a_{k+1} = -\frac{3}{2}a_k$

26. $a_1 = 48, \quad a_{k+1} = -\frac{1}{2}a_k$

In Exercises 27–34, write an expression for the *n*th term of the geometric sequence. Then find the indicated term.

27. $a_1 = 4, r = \frac{1}{2}, n = 10$

28. $a_1 = 5, r = \frac{3}{2}, n = 8$

29. $a_1 = 6, r = -\frac{1}{3}, n = 12$

30. $a_1 = 64, r = -\frac{1}{4}, n = 10$

31. $a_1 = 100, r = e^x, n = 9$

32. $a_1 = 1, r = \sqrt{3}, n = 8$

33. $a_1 = 500, r = 1.02, n = 40$

34. $a_1 = 1000, r = 1.005, n = 60$

In Exercises 35–42, find the indicated *n*th term of the geometric sequence.

35. 9th term: $7, 21, 63, \ldots$

36. 7th term: $3, 36, 432, \ldots$

37. 10th term: $5, 30, 180, \ldots$

38. 22nd term: $4, 8, 16, \ldots$

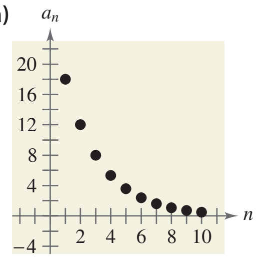
39. 3rd term: $a_1 = 16, a_4 = \frac{27}{4}$

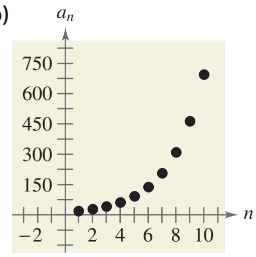
40. 1st term: $a_2 = 3, a_5 = \frac{3}{64}$

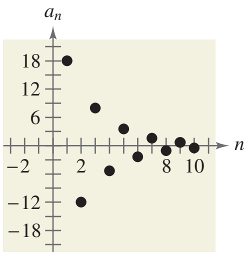
41. 6th term: $a_4 = -18, a_7 = \frac{2}{3}$

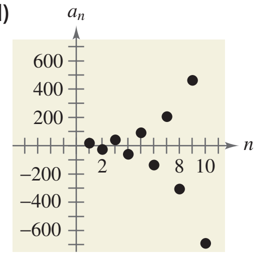
42. 7th term: $a_3 = \frac{16}{3}, a_5 = \frac{64}{27}$

In Exercises 43–46, match the geometric sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)


(b)


(c)


(d)


43. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$

44. $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$

45. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$

46. $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$

In Exercises 47–52, use a graphing utility to graph the first 10 terms of the sequence.

47. $a_n = 12(-0.75)^{n-1}$ **48.** $a_n = 10(1.5)^{n-1}$

49. $a_n = 12(-0.4)^{n-1}$ **50.** $a_n = 20(-1.25)^{n-1}$

51. $a_n = 2(1.3)^{n-1}$ **52.** $a_n = 10(1.2)^{n-1}$

In Exercises 53–72, find the sum of the finite geometric sequence.

53. $\sum_{n=1}^{9} 2^{n-1}$ **54.** $\sum_{n=1}^{10} \left(\frac{5}{2}\right)^{n-1}$

55. $\sum_{n=1}^{9} (-2)^{n-1}$ **56.** $\sum_{n=1}^{8} 5\left(-\frac{3}{2}\right)^{n-1}$

57. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$ **58.** $\sum_{i=1}^{10} 2\left(\frac{1}{4}\right)^{i-1}$

59. $\sum_{i=1}^{6} 32\left(\frac{1}{4}\right)^{i-1}$ **60.** $\sum_{i=1}^{12} 16\left(\frac{1}{2}\right)^{i-1}$

61. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$ **62.** $\sum_{n=0}^{40} 5\left(\frac{3}{5}\right)^{n}$

63. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^{n}$ **64.** $\sum_{n=0}^{20} 10\left(\frac{1}{5}\right)^{n}$

65. $\sum_{n=0}^{5} 300(1.06)^{n}$ **66.** $\sum_{n=0}^{6} 500(1.04)^{n}$

67. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^{n}$ **68.** $\sum_{n=0}^{50} 10\left(\frac{2}{3}\right)^{n-1}$

69. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$ **70.** $\sum_{i=0}^{25} 8\left(-\frac{1}{2}\right)^{i}$

71. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1}$ **72.** $\sum_{i=1}^{100} 15\left(\frac{2}{3}\right)^{i-1}$

In Exercises 73–78, use summation notation to write the sum.

73. $5 + 15 + 45 + \cdots + 3645$

74. $7 + 14 + 28 + \cdots + 896$

75. $2 - \dfrac{1}{2} + \dfrac{1}{8} - \cdots + \dfrac{1}{2048}$

76. $15 - 3 + \dfrac{3}{5} - \cdots - \dfrac{3}{625}$

77. $0.1 + 0.4 + 1.6 + \cdots + 102.4$

78. $32 + 24 + 18 + \cdots + 10.125$

In Exercises 79–92, find the sum of the infinite geometric series.

79. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}$ **80.** $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^{n}$

81. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n}$ **82.** $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^{n}$

83. $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^{n}$ **84.** $\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^{n}$

85. $\sum_{n=0}^{\infty} (0.4)^{n}$ **86.** $\sum_{n=0}^{\infty} 4(0.2)^{n}$

87. $\sum_{n=0}^{\infty} -3(0.9)^{n}$ **88.** $\sum_{n=0}^{\infty} -10(0.2)^{n}$

89. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$ **90.** $9 + 6 + 4 + \frac{8}{3} + \cdots$

91. $\frac{1}{9} - \frac{1}{3} + 1 - 3 + \cdots$ **92.** $-\frac{125}{36} + \frac{25}{6} - 5 + 6 - \cdots$

In Exercises 93–96, find the rational number representation of the repeating decimal.

93. $0.\overline{36}$ **94.** $0.\overline{297}$

95. $0.3\overline{18}$ **96.** $1.\overline{38}$

Graphical Reasoning **In Exercises 97 and 98, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.**

97. $f(x) = 6\left[\dfrac{1-(0.5)^x}{1-(0.5)}\right], \quad \sum_{n=0}^{\infty} 6\left(\dfrac{1}{2}\right)^n$

98. $f(x) = 2\left[\dfrac{1-(0.8)^x}{1-(0.8)}\right], \quad \sum_{n=0}^{\infty} 2\left(\dfrac{4}{5}\right)^n$

Model It

99. ***Data Analysis: Population*** The table shows the population a_n of China (in millions) from 1998 through 2004. (Source: U.S. Census Bureau)

Year	Population, a_n
1998	1250.4
1999	1260.1
2000	1268.9
2001	1276.9
2002	1284.3
2003	1291.5
2004	1298.8

(a) Use the *exponential regression* feature of a graphing utility to find a geometric sequence that models the data. Let n represent the year, with $n = 8$ corresponding to 1998.

(b) Use the sequence from part (a) to describe the rate at which the population of China is growing.

Model It (continued)

(c) Use the sequence from part (a) to predict the population of China in 2010. The U.S. Census Bureau predicts the population of China will be 1374.6 million in 2010. How does this value compare with your prediction?

(d) Use the sequence from part (a) to determine when the population of China will reach 1.32 billion.

100. ***Compound Interest*** A principal of \$1000 is invested at 6% interest. Find the amount after 10 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

101. ***Compound Interest*** A principal of \$2500 is invested at 2% interest. Find the amount after 20 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

102. ***Depreciation*** A tool and die company buys a machine for \$135,000 and it depreciates at a rate of 30% per year. (In other words, at the end of each year the depreciated value is 70% of what it was at the beginning of the year.) Find the depreciated value of the machine after 5 full years.

103. ***Annuities*** A deposit of \$100 is made at the beginning of each month in an account that pays 6%, compounded monthly. The balance A in the account at the end of 5 years is

$$A = 100\left(1 + \frac{0.06}{12}\right)^1 + \cdots + 100\left(1 + \frac{0.06}{12}\right)^{60}.$$

Find A.

104. ***Annuities*** A deposit of \$50 is made at the beginning of each month in an account that pays 8%, compounded monthly. The balance A in the account at the end of 5 years is

$$A = 50\left(1 + \frac{0.08}{12}\right)^1 + \cdots + 50\left(1 + \frac{0.08}{12}\right)^{60}.$$

Find A.

105. ***Annuities*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded monthly. The balance A after t years is

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \cdots + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

106. ***Annuities*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded continuously. The balance A after t years is $A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{12tr/12}$. Show that the balance is

$$A = \frac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}.$$

Annuities **In Exercises 107–110, consider making monthly deposits of P dollars in a savings account earning an annual interest rate r. Use the results of Exercises 105 and 106 to find the balance A after t years if the interest is compounded (a) monthly and (b) continuously.**

107. $P = \$50,\ r = 7\%,\ t = 20$ years

108. $P = \$75,\ r = 3\%,\ t = 25$ years

109. $P = \$100,\ r = 10\%,\ t = 40$ years

110. $P = \$20,\ r = 6\%,\ t = 50$ years

111. ***Annuities*** Consider an initial deposit of P dollars in an account earning an annual interest rate r, compounded monthly. At the end of each month, a withdrawal of W dollars will occur and the account will be depleted in t years. The amount of the initial deposit required is

$$P = W\left(1 + \frac{r}{12}\right)^{-1} + W\left(1 + \frac{r}{12}\right)^{-2} + \cdots + W\left(1 + \frac{r}{12}\right)^{-12t}.$$

Show that the initial deposit is

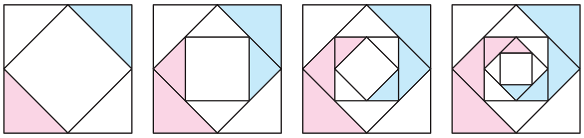
$$P = W\left(\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right].$$

112. ***Annuities*** Determine the amount required in a retirement account for an individual who retires at age 65 and wants an income of \$2000 from the account each month for 20 years. Use the result of Exercise 111 and assume that the account earns 9% compounded monthly.

Multiplier Effect **In Exercises 113–116, use the following information. A tax rebate has been given to property owners by the state government with the anticipation that each property owner spends approximately p% of the rebate, and in turn each recipient of this amount spends p% of what they receive, and so on. Economists refer to this exchange of money and its circulation within the economy as the "multiplier effect." The multiplier effect operates on the idea that the expenditures of one individual become the income of another individual. For the given tax rebate, find the total amount put back into the state's economy, if this effect continues without end.**

	Tax rebate	p%
113.	\$400	75%
114.	\$250	80%
115.	\$600	72.5%
116.	\$450	77.5%

117. ***Geometry*** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the resulting triangles are shaded (see figure). If this process is repeated five more times, determine the total area of the shaded region.



118. ***Sales*** The annual sales a_n (in millions of dollars) for Urban Outfitters for 1994 through 2003 can be approximated by the model

$$a_n = 54.6e^{0.172n}, \qquad n = 4, 5, \ldots, 13$$

where n represents the year, with $n = 4$ corresponding to 1994. Use this model and the formula for the sum of a finite geometric sequence to approximate the total sales earned during this 10-year period. (Source: Urban Outfitters Inc.)

119. ***Salary*** An investment firm has a job opening with a salary of \$30,000 for the first year. Suppose that during the next 39 years, there is a 5% raise each year. Find the total compensation over the 40-year period.

120. ***Distance*** A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet.

(a) Find the total vertical distance traveled by the ball.

(b) The ball takes the following times (in seconds) for each fall.

$$\begin{aligned}
s_1 &= -16t^2 + 16, & s_1 &= 0 \text{ if } t = 1\\
s_2 &= -16t^2 + 16(0.81), & s_2 &= 0 \text{ if } t = 0.9\\
s_3 &= -16t^2 + 16(0.81)^2, & s_3 &= 0 \text{ if } t = (0.9)^2\\
s_4 &= -16t^2 + 16(0.81)^3, & s_4 &= 0 \text{ if } t = (0.9)^3\\
&\vdots & &\vdots\\
s_n &= -16t^2 + 16(0.81)^{n-1}, & s_n &= 0 \text{ if } t = (0.9)^{n-1}
\end{aligned}$$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n.$$

Find this total time.

Synthesis

True or False? **In Exercises 121 and 122, determine whether the statement is true or false. Justify your answer.**

121. A sequence is geometric if the ratios of consecutive differences of consecutive terms are the same.

122. You can find the nth term of a geometric sequence by multiplying its common ratio by the first term of the sequence raised to the $(n - 1)$th power.

123. ***Writing*** Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.

124. Find two different geometric series with sums of 4.

Skills Review

In Exercises 125–128, evaluate the function for $f(x) = 3x + 1$ and $g(x) = x^2 - 1$.

125. $g(x + 1)$

126. $f(x + 1)$

127. $f(g(x + 1))$

128. $g(f(x + 1))$

In Exercises 129–132, completely factor the expression.

129. $9x^3 - 64x$

130. $x^2 + 4x - 63$

131. $6x^2 - 13x - 5$

132. $16x^2 - 4x^4$

In Exercises 133–138, perform the indicated operation(s) and simplify.

133. $\dfrac{3}{x + 3} \cdot \dfrac{x(x + 3)}{x - 3}$

134. $\dfrac{x - 2}{x + 7} \cdot \dfrac{2x(x + 7)}{6x(x - 2)}$

135. $\dfrac{x}{3} \div \dfrac{3x}{6x + 3}$

136. $\dfrac{x - 5}{x - 3} \div \dfrac{10 - 2x}{2(3 - x)}$

137. $5 + \dfrac{7}{x + 2} + \dfrac{2}{x - 2}$

138. $8 - \dfrac{x - 1}{x + 4} - \dfrac{4}{x - 1} - \dfrac{x + 4}{(x - 1)(x + 4)}$

139. **Make a Decision** To work an extended application analyzing the amounts spent on research and development in the United States from 1980 to 2003, visit this text's website at *college.hmco.com*. *(Data Source: U.S. Census Bureau)*

9.4 Mathematical Induction

What you should learn

- Use mathematical induction to prove statements involving a positive integer n.
- Recognize patterns and write the nth term of a sequence.
- Find the sums of powers of integers.
- Find finite differences of sequences.

Why you should learn it

Finite differences can be used to determine what type of model can be used to represent a sequence. For instance, in Exercise 61 on page 682, you will use finite differences to find a model that represents the number of individual income tax returns filed in the United States from 1998 to 2003.

Mario Tama/Getty Images

Introduction

In this section, you will study a form of mathematical proof called **mathematical induction.** It is important that you see clearly the logical need for it, so take a closer look at the problem discussed in Example 5 in Section 9.2.

$$S_1 = 1 = 1^2$$

$$S_2 = 1 + 3 = 2^2$$

$$S_3 = 1 + 3 + 5 = 3^2$$

$$S_4 = 1 + 3 + 5 + 7 = 4^2$$

$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2.$$

Although this particular formula *is* valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of n is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of n and then at some point the pattern fails. One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \ldots$$

are prime. For $n = 0, 1, 2, 3,$ and 4, the conjecture is true.

$$F_0 = 3$$

$$F_1 = 5$$

$$F_2 = 17$$

$$F_3 = 257$$

$$F_4 = 65{,}537$$

The size of the next Fermat number ($F_5 = 4{,}294{,}967{,}297$) is so great that it was difficult for Fermat to determine whether it was prime or not. However, another well-known mathematician, Leonhard Euler (1707–1783), later found the factorization

$$\begin{aligned} F_5 &= 4{,}294{,}967{,}297 \\ &= 641(6{,}700{,}417) \end{aligned}$$

which proved that F_5 is not prime and therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of n, you cannot simply decide that it is valid for all values of n without going through a *legitimate proof*. Mathematical induction is one method of proof.

STUDY TIP

It is important to recognize that in order to prove a statement by induction, both parts of the Principle of Mathematical Induction are necessary.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. for every positive integer k, the truth of P_k implies the truth of P_{k+1}

then the statement P_n must be true for all positive integers n.

To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k. To determine P_{k+1}, substitute the quantity $k + 1$ for k in the statement P_k.

Example 1 A Preliminary Example

Find the statement P_{k+1} for each given statement P_k.

a. $P_k: \; S_k = \dfrac{k^2(k+1)^2}{4}$

b. $P_k: \; S_k = 1 + 5 + 9 + \cdot \cdot \cdot + [4(k-1) - 3] + (4k - 3)$

c. $P_k: \; k + 3 < 5k^2$

d. $P_k: \; 3^k \geq 2k + 1$

Solution

a. $P_{k+1}: \; S_{k+1} = \dfrac{(k+1)^2(k+1+1)^2}{4}$ Replace k by $k + 1$.

$= \dfrac{(k+1)^2(k+2)^2}{4}$ Simplify.

b. $P_{k+1}: \; S_{k+1} = 1 + 5 + 9 + \cdot \cdot \cdot + \{4[(k+1) - 1] - 3\} + [4(k+1) - 3]$

$= 1 + 5 + 9 + \cdot \cdot \cdot + (4k - 3) + (4k + 1)$

c. $P_{k+1}: \; (k+1) + 3 < 5(k+1)^2$

$k + 4 < 5(k^2 + 2k + 1)$

d. $P_{k+1}: \; 3^{k+1} \geq 2(k+1) + 1$

$3^{k+1} \geq 2k + 3$

CHECKPOINT Now try Exercise 1.

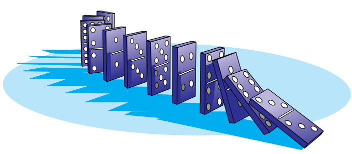


FIGURE 9.6

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes shown in Figure 9.6. If the line actually contains infinitely many dominoes, it is clear that you could not knock the entire line down by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, the chain reaction proceeds as follows: P_1 implies P_2, P_2 implies P_3, P_3 implies P_4, and so on.

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as

$$S_{k+1} = S_k + a_{k+1}$$

where a_{k+1} is the $(k + 1)$th term of the original sum.

Example 2 Using Mathematical Induction

Use mathematical induction to prove the following formula.

$$\begin{aligned} S_n &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) \\ &= n^2 \end{aligned}$$

Solution

Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

1. When $n = 1$, the formula is valid, because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to *assume* that the formula is valid for some integer k. The second step is to use this assumption to prove that the formula is valid for the *next* integer, $k + 1$.

2. Assuming that the formula

$$\begin{aligned} S_k &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) \\ &= k^2 \end{aligned}$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Replace } S_k \text{ by } k^2. \\ &= (k + 1)^2 \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all positive integer values of n.

CHECKPOINT Now try Exercise 5.

It occasionally happens that a statement involving natural numbers is not true for the first $k - 1$ positive integers but is true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1. This variation is called the *extended principle of mathematical induction*. To see the validity of this, note from Figure 9.6 that all but the first $k - 1$ dominoes can be knocked down by knocking over the kth domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1}. In Exercises 17–22 of this section, you are asked to apply this extension of mathematical induction.

Example 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Solution

1. When $n = 1$, the formula is valid, because

$$S_1 = 1^2 = \frac{1(2)(3)}{6}.$$

2. Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 \qquad a_k = k^2$$
$$= \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}.$$

To do this, write the following.

$$S_{k+1} = S_k + a_{k+1}$$
$$= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2 \qquad \text{Substitute for } S_k.$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{By assumption}$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \qquad \text{Combine fractions.}$$
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \qquad \text{Factor.}$$
$$= \frac{(k+1)(2k^2 + 7k + 6)}{6} \qquad \text{Simplify.}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6} \qquad S_k \text{ implies } S_{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

CHECKPOINT Now try Exercise 11.

STUDY TIP

Remember that when adding rational expressions, you must first find the least common denominator (LCD). In Example 3, the LCD is 6.

When proving a formula using mathematical induction, the only statement that you *need* to verify is P_1. As a check, however, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying P_2 and P_3.

Example 4 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all positive integers n.

Solution

1. For $n = 1$ and $n = 2$, the statement is true because

$$1 < 2^1 \quad \text{and} \quad 2 < 2^2.$$

2. Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. For $n = k$, you have

$$2^{k+1} = 2(2^k) > 2(k) = 2k. \qquad \text{By assumption}$$

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$$2^{k+1} > 2k > k + 1 \quad \text{or} \quad k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all integers $n \geq 1$.

CHECKPOINT Now try Exercise 17.

STUDY TIP

To check a result that you have proved by mathematical induction, it helps to list the statement for several values of n. For instance, in Example 4, you could list

$$1 < 2^1 = 2, \quad 2 < 2^2 = 4,$$
$$2 < 2^3 = 8, \quad 4 < 2^4 = 16,$$
$$5 < 2^5 = 32, \quad 6 < 2^6 = 64,$$

From this list, your intuition confirms that the statement $n < 2^n$ is reasonable.

Example 5 Proving Factors by Mathematical Induction

Prove that 3 is a factor of $4^n - 1$ for all positive integers n.

Solution

1. For $n = 1$, the statement is true because

$$4^1 - 1 = 3.$$

So, 3 is a factor.

2. Assuming that 3 is a factor of $4^k - 1$, you must show that 3 is a factor of $4^{k+1} - 1$. To do this, write the following.

$$\begin{aligned} 4^{k+1} - 1 &= 4^{k+1} - 4^k + 4^k - 1 && \text{Subtract and add } 4^k. \\ &= 4^k(4 - 1) + (4^k - 1) && \text{Regroup terms.} \\ &= 4^k \cdot 3 + (4^k - 1) && \text{Simplify.} \end{aligned}$$

Because 3 is a factor of $4^k \cdot 3$ and 3 is also a factor of $4^k - 1$, it follows that 3 is a factor of $4^{k+1} - 1$. Combining the results of parts (1) and (2), you can conclude by mathematical induction that 3 is a factor of $4^n - 1$ for all positive integers n.

CHECKPOINT Now try Exercise 29.

Pattern Recognition

Although choosing a formula on the basis of a few observations does *not* guarantee the validity of the formula, pattern recognition *is* important. Once you have a pattern or formula that you think works, you can try using mathematical induction to prove your formula.

Finding a Formula for the *n*th Term of a Sequence

To find a formula for the nth term of a sequence, consider these guidelines.

1. Calculate the first several terms of the sequence. It is often a good idea to write the terms in both simplified and factored forms.
2. Try to find a recognizable pattern for the terms and write a formula for the nth term of the sequence. This is your *hypothesis* or *conjecture*. You might try computing one or two more terms in the sequence to test your hypothesis.
3. Use mathematical induction to prove your hypothesis.

Example 6 Finding a Formula for a Finite Sum

Find a formula for the finite sum and prove its validity.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)}$$

Solution

Begin by writing out the first few sums.

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3} = \frac{2}{2+1}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{9}{12} = \frac{3}{4} = \frac{3}{3+1}$$

$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{48}{60} = \frac{4}{5} = \frac{4}{4+1}$$

From this sequence, it appears that the formula for the kth sum is

$$S_k = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

To prove the validity of this hypothesis, use mathematical induction. Note that you have already verified the formula for $n = 1$, so you can begin by assuming that the formula is valid for $n = k$ and trying to show that it is valid for $n = k + 1$.

$$S_{k+1} = \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{By assumption}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

So, by mathematical induction, you can conclude that the hypothesis is valid.

CHECKPOINT Now try Exercise 35.

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are as follows.

Sums of Powers of Integers

1. $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$

2. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

4. $1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

5. $1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$

Example 7 Finding a Sum of Powers of Integers

Find each sum.

a. $\displaystyle\sum_{i=1}^{7} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$ **b.** $\displaystyle\sum_{i=1}^{4} (6i - 4i^2)$

Solution

a. Using the formula for the sum of the cubes of the first n positive integers, you obtain

$$\sum_{i=1}^{7} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$$

$$= \frac{7^2(7+1)^2}{4} = \frac{49(64)}{4} = 784. \qquad \text{Formula 3}$$

b. $$\sum_{i=1}^{4} (6i - 4i^2) = \sum_{i=1}^{4} 6i - \sum_{i=1}^{4} 4i^2$$

$$= 6\sum_{i=1}^{4} i - 4\sum_{i=1}^{4} i^2$$

$$= 6\left[\frac{4(4+1)}{2}\right] - 4\left[\frac{4(4+1)(8+1)}{6}\right] \qquad \text{Formula 1 and 2}$$

$$= 6(10) - 4(30)$$

$$= 60 - 120 = -60$$

CHECKPOINT Now try Exercise 47.

> **STUDY TIP**
>
> For a linear model, the *first* differences should be the same nonzero number. For a quadratic model, the *second* differences are the same nonzero number.

Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences. The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as follows.

n:	1		2		3		4		5		6
a_n:	3		5		8		12		17		23
First differences:		2		3		4		5		6	
Second differences:			1		1		1		1		

For this sequence, the second differences are all the same. When this happens, the sequence has a perfect *quadratic* model. If the first differences are all the same, the sequence has a *linear* model. That is, it is arithmetic.

Example 8 Finding a Quadratic Model

Find the quadratic model for the sequence

3, 5, 8, 12, 17, 23, ...

Solution

You know from the second differences shown above that the model is quadratic and has the form

$$a_n = an^2 + bn + c.$$

By substituting 1, 2, and 3 for n, you can obtain a system of three linear equations in three variables.

$$a_1 = a(1)^2 + b(1) + c = 3 \qquad \text{Substitute 1 for } n.$$

$$a_2 = a(2)^2 + b(2) + c = 5 \qquad \text{Substitute 2 for } n.$$

$$a_3 = a(3)^2 + b(3) + c = 8 \qquad \text{Substitute 3 for } n.$$

You now have a system of three equations in a, b, and c.

$$\begin{cases} a + b + c = 3 & \text{Equation 1} \\ 4a + 2b + c = 5 & \text{Equation 2} \\ 9a + 3b + c = 8 & \text{Equation 3} \end{cases}$$

Using the techniques discussed in Chapter 7, you can find the solution to be $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 2$. So, the quadratic model is

$$a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 2.$$

Try checking the values of a_1, a_2, and a_3.

CHECKPOINT Now try Exercise 57.

9.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The first step in proving a formula by ________ ________ is to show that the formula is true when $n = 1$.
2. The ________ differences of a sequence are found by subtracting consecutive terms.
3. A sequence is an ________ sequence if the first differences are all the same nonzero number.
4. If the ________ differences of a sequence are all the same nonzero number, then the sequence has a perfect quadratic model.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, find P_{k+1} for the given P_k.

1. $P_k = \dfrac{5}{k(k+1)}$
2. $P_k = \dfrac{1}{2(k+2)}$
3. $P_k = \dfrac{k^2(k+1)^2}{4}$
4. $P_k = \dfrac{k}{3}(2k+1)$

In Exercises 5–16, use mathematical induction to prove the formula for every positive integer n.

5. $2 + 4 + 6 + 8 + \cdots + 2n = n(n+1)$
6. $3 + 7 + 11 + 15 + \cdots + (4n-1) = n(2n+1)$
7. $2 + 7 + 12 + 17 + \cdots + (5n-3) = \dfrac{n}{2}(5n-1)$
8. $1 + 4 + 7 + 10 + \cdots + (3n-2) = \dfrac{n}{2}(3n-1)$
9. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$
10. $2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}) = 3^n - 1$
11. $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$
12. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
13. $\displaystyle\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$
14. $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
15. $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$
16. $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

In Exercises 17–22, prove the inequality for the indicated integer values of n.

17. $n! > 2^n, \quad n \ge 4$
18. $\left(\frac{4}{3}\right)^n > n, \quad n \ge 7$
19. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \ge 2$
20. $\left(\dfrac{x}{y}\right)^{n+1} < \left(\dfrac{x}{y}\right)^n, \quad n \ge 1$ and $0 < x < y$
21. $(1 + a)^n \ge na, \quad n \ge 1$ and $a > 0$
22. $2n^2 > (n+1)^2, \quad n \ge 3$

In Exercises 23–34, use mathematical induction to prove the property for all positive integers n.

23. $(ab)^n = a^n b^n$
24. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
25. If $x_1 \ne 0, x_2 \ne 0, \ldots, x_n \ne 0$, then
$(x_1 x_2 x_3 \cdots x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdots x_n^{-1}$.
26. If $x_1 > 0, x_2 > 0, \ldots, x_n > 0$, then
$\ln(x_1 x_2 \cdots x_n) = \ln x_1 + \ln x_2 + \cdots + \ln x_n$.
27. Generalized Distributive Law:
$x(y_1 + y_2 + \cdots + y_n) = xy_1 + xy_2 + \cdots + xy_n$
28. $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \ge 1$.
29. A factor of $(n^3 + 3n^2 + 2n)$ is 3.
30. A factor of $(n^3 - n + 3)$ is 3.
31. A factor of $(n^4 - n + 4)$ is 2.
32. A factor of $(2^{2n+1} + 1)$ is 3.
33. A factor of $(2^{4n-2} + 1)$ is 5.
34. A factor of $(2^{2n-1} + 3^{2n-1})$ is 5.

In Exercises 35–40, find a formula for the sum of the first n terms of the sequence.

35. $1, 5, 9, 13, \ldots$
36. $25, 22, 19, 16, \ldots$
37. $1, \frac{9}{10}, \frac{81}{100}, \frac{729}{1000}, \ldots$
38. $3, -\frac{9}{2}, \frac{27}{4}, -\frac{81}{8}, \ldots$
39. $\dfrac{1}{4}, \dfrac{1}{12}, \dfrac{1}{24}, \dfrac{1}{40}, \ldots, \dfrac{1}{2n(n+1)}, \ldots$
40. $\dfrac{1}{2 \cdot 3}, \dfrac{1}{3 \cdot 4}, \dfrac{1}{4 \cdot 5}, \dfrac{1}{5 \cdot 6}, \ldots, \dfrac{1}{(n+1)(n+2)}, \ldots$

In Exercises 41–50, find the sum using the formulas for the sums of powers of integers.

41. $\sum_{n=1}^{15} n$

42. $\sum_{n=1}^{30} n$

43. $\sum_{n=1}^{6} n^2$

44. $\sum_{n=1}^{10} n^3$

45. $\sum_{n=1}^{5} n^4$

46. $\sum_{n=1}^{8} n^5$

47. $\sum_{n=1}^{6} (n^2 - n)$

48. $\sum_{n=1}^{20} (n^3 - n)$

49. $\sum_{i=1}^{6} (6i - 8i^3)$

50. $\sum_{j=1}^{10} \left(3 - \frac{1}{2}j + \frac{1}{2}j^2\right)$

In Exercises 51–56, write the first six terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. State whether the sequence has a linear model, a quadratic model, or neither.

51. $a_1 = 0$
$a_n = a_{n-1} + 3$

52. $a_1 = 2$
$a_n = a_{n-1} + 2$

53. $a_1 = 3$
$a_n = a_{n-1} - n$

54. $a_2 = -3$
$a_n = -2a_{n-1}$

55. $a_0 = 2$
$a_n = (a_{n-1})^2$

56. $a_0 = 0$
$a_n = a_{n-1} + n$

In Exercises 57–60, find a quadratic model for the sequence with the indicated terms.

57. $a_0 = 3,\ a_1 = 3,\ a_4 = 15$

58. $a_0 = 7,\ a_1 = 6,\ a_3 = 10$

59. $a_0 = -3,\ a_2 = 1,\ a_4 = 9$

60. $a_0 = 3,\ a_2 = 0,\ a_6 = 36$

Model It

61. ***Data Analysis: Tax Returns*** The table shows the number a_n (in millions) of individual tax returns filed in the United States from 1998 to 2003. (Source: Internal Revenue Service)

Year	Number of returns, a_n
1998	120.3
1999	122.5
2000	124.9
2001	127.1
2002	129.4
2003	130.3

Model It (continued)

(a) Find the first differences of the data shown in the table.

(b) Use your results from part (a) to determine whether a linear model can be used to approximate the data. If so, find a model algebraically. Let n represent the year, with $n = 8$ corresponding to 1998.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Compare this model with the one from part (b).

(d) Use the models found in parts (b) and (c) to estimate the number of individual tax returns filed in 2008. How do these values compare?

Synthesis

62. ***Writing*** In your own words, explain what is meant by a proof by mathematical induction.

True or False? **In Exercises 63–66, determine whether the statement is true or false. Justify your answer.**

63. If the statement P_1 is true but the true statement P_6 does *not* imply that the statement P_7 is true, then P_n is not necessarily true for all positive integers n.

64. If the statement P_k is true and P_k implies P_{k+1}, then P_1 is also true.

65. If the second differences of a sequence are all zero, then the sequence is arithmetic.

66. A sequence with n terms has $n - 1$ second differences.

Skills Review

In Exercises 67–70, find the product.

67. $(2x^2 - 1)^2$

68. $(2x - y)^2$

69. $(5 - 4x)^3$

70. $(2x - 4y)^3$

In Exercises 71–74, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical and horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

71. $f(x) = \dfrac{x}{x + 3}$

72. $g(x) = \dfrac{x^2}{x^2 - 4}$

73. $h(t) = \dfrac{t - 7}{t}$

74. $f(x) = \dfrac{5 + x}{1 - x}$

9.5 The Binomial Theorem

What you should learn

- Use the Binomial Theorem to calculate binomial coefficients.
- Use Pascal's Triangle to calculate binomial coefficients.
- Use binomial coefficients to write binomial expansions.

Why you should learn it

You can use binomial coefficients to model and solve real-life problems. For instance, in Exercise 80 on page 690, you will use binomial coefficients to write the expansion of a model that represents the amounts of child support collected in the U. S.

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Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that gives a quick method of raising a binomial to a power. To begin, look at the expansion of $(x + y)^n$ for several values of n.

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.

2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.

3. The sum of the powers of each term is n. For instance, in the expansion of $(x + y)^5$, the sum of the powers of each term is 5.

$$(x + y)^5 = x^5 + 5\overbrace{x^4y^1}^{4+1=5} + 10\overbrace{x^3y^2}^{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-1}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

For a proof of the Binomial Theorem, see Proofs in Mathematics on page 724.

Technology

Most graphing calculators are programmed to evaluate ${}_nC_r$. Consult the user's guide for your calculator and then evaluate ${}_8C_5$. You should get an answer of 56.

Example 1 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$ **b.** $\binom{10}{3}$ **c.** ${}_7C_0$ **d.** $\binom{8}{8}$

Solution

a. $${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

b. $$\binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$

d. $$\binom{8}{8} = \frac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$$

CHECKPOINT Now try Exercise 1.

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) above, there is a simple pattern for evaluating binomial coefficients that works because there will always be factorial terms that divide out from the expression.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{\underbrace{2 \cdot 1}_{\text{2 factors}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{\text{3 factors}}}$$

Example 2 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_7C_3$ **b.** $\binom{7}{4}$ **c.** ${}_{12}C_1$ **d.** $\binom{12}{11}$

Solution

a. $${}_7C_3 = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

b. $$\binom{7}{4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

c. $${}_{12}C_1 = \frac{12}{1} = 12$$

d. $$\binom{12}{11} = \frac{12!}{1! \cdot 11!} = \frac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \frac{12}{1} = 12$$

CHECKPOINT Now try Exercise 7.

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that

$${}_nC_r = {}_nC_{n-r}.$$

This shows the symmetric property of binomial coefficients that was identified earlier.

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 $\quad 4 + 6 = 10$
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1 $\quad 15 + 6 = 21$

The first and last numbers in each row of Pascal's Triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that numbers in this triangle are precisely the same numbers that are the coefficients of binomial expansions, as follows.

$$(x+y)^0 = 1 \qquad \text{0th row}$$
$$(x+y)^1 = 1x + 1y \qquad \text{1st row}$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$
$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$
$$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

The top row in Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion $(x + y)^0 = 1$. Similarly, the next row is called the *first row* because it corresponds to the binomial expansion $(x + y)^1 = 1(x) + 1(y)$. In general, the *nth row* in Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 3 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0,\ {}_8C_1,\ {}_8C_2,\ {}_8C_3,\ {}_8C_4,\ {}_8C_5,\ {}_8C_6,\ {}_8C_7,\ {}_8C_8$$

Solution

1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
${}_8C_0$ ${}_8C_1$ ${}_8C_2$ ${}_8C_3$ ${}_8C_4$ ${}_8C_5$ ${}_8C_6$ ${}_8C_7$ ${}_8C_8$

CHECKPOINT Now try Exercise 11.

Exploration

Complete the table and describe the result.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5		
7	1		
12	4		
6	0		
10	7		

What characteristic of Pascal's Triangle is illustrated by this table?

Historical Note
Precious Mirror "Pascal's" Triangle and forms of the Binomial Theorem were known in Eastern cultures prior to the Western "discovery" of the theorem. A Chinese text entitled *Precious Mirror* contains a triangle of binomial expansions through the eighth power.

Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next four examples.

Example 4 Expanding a Binomial

Write the expansion for the expression

$(x + 1)^3$.

Solution

The binomial coefficients from the third row of Pascal's Triangle are

1, 3, 3, 1.

So, the expansion is as follows.

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)\\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

CHECKPOINT Now try Exercise 15.

To expand binomials representing *differences* rather than sums, you alternate signs. Here are two examples.

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

$$(x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$$

Example 5 Expanding a Binomial

Write the expansion for each expression.

a. $(2x - 3)^4$ **b.** $(x - 2y)^4$

Solution

The binomial coefficients from the fourth row of Pascal's Triangle are

1, 4, 6, 4, 1.

Therefore, the expansions are as follows.

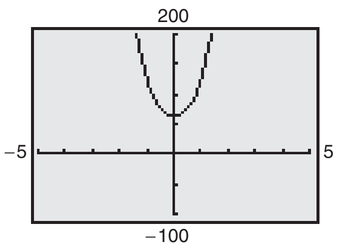
a. $$\begin{aligned}(2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

b. $$\begin{aligned}(x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

CHECKPOINT Now try Exercise 19.

Technology

You can use a graphing utility to check the expansion in Example 6. Graph the original binomial expression and the expansion in the same viewing window. The graphs should coincide as shown below.



Example 6 Expanding a Binomial

Write the expansion for $(x^2 + 4)^3$.

Solution

Use the third row of Pascal's Triangle, as follows.

$$\begin{aligned}(x^2 + 4)^3 &= (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3)\\ &= x^6 + 12x^4 + 48x^2 + 64\end{aligned}$$

CHECKPOINT Now try Exercise 29.

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$th term is ${}_nC_r\, x^{n-r} y^r$.

Example 7 Finding a Term in a Binomial Expansion

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(3a - 2b)^{11}$.

Solution

a. Remember that the formula is for the $(r + 1)$th term, so r is one less than the number of the term you are looking for. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$, and $y = 2b$, as shown.

$${}_8C_5\, a^{8-5}(2b)^5 = 56 \cdot a^3 \cdot (2b)^5 = 56(2^5)a^3b^5 = 1792a^3b^5.$$

b. In this case, $n = 11$, $r = 5$, $x = 3a$, and $y = -2b$. Substitute these values to obtain

$$\begin{aligned}{}_nC_r\, x^{n-r} y^r &= {}_{11}C_5(3a)^6(-2b)^5\\ &= (462)(729a^6)(-32b^5)\\ &= -10{,}777{,}536a^6b^5.\end{aligned}$$

So, the coefficient is $-10{,}777{,}536$.

CHECKPOINT Now try Exercise 41.

Writing About Mathematics

Error Analysis You are a math instructor and receive the following solutions from one of your students on a quiz. Find the error(s) in each solution. Discuss ways that your student could avoid the error(s) in the future.

a. Find the second term in the expansion of $(2x - 3y)^5$.

$$\cancel{5(2x)^4(3y)^2 = 720x^4y^2}$$

b. Find the fourth term in the expansion of $\left(\frac{1}{2}x + 7y\right)^6$.

$$\cancel{{}_6C_4\left(\tfrac{1}{2}x\right)^2(7y)^4 = 9003.75x^2y^4}$$

9.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The coefficients of a binomial expansion are called ________ ________.
2. To find binomial coefficients, you can use the ________ ________ or ________ ________.
3. The notation used to denote a binomial coefficient is ________ or ________.
4. When you write out the coefficients for a binomial that is raised to a power, you are ________ a ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, calculate the binomial coefficient.

1. ${}_5C_3$
2. ${}_8C_6$
3. ${}_{12}C_0$
4. ${}_{20}C_{20}$
5. ${}_{20}C_{15}$
6. ${}_{12}C_5$
7. $\binom{10}{4}$
8. $\binom{10}{6}$
9. $\binom{100}{98}$
10. $\binom{100}{2}$

In Exercises 11–14, evaluate using Pascal's Triangle.

11. $\binom{8}{5}$
12. $\binom{8}{7}$
13. ${}_7C_4$
14. ${}_6C_3$

In Exercises 15–34, use the Binomial Theorem to expand and simplify the expression.

15. $(x + 1)^4$
16. $(x + 1)^6$
17. $(a + 6)^4$
18. $(a + 5)^5$
19. $(y - 4)^3$
20. $(y - 2)^5$
21. $(x + y)^5$
22. $(c + d)^3$
23. $(r + 3s)^6$
24. $(x + 2y)^4$
25. $(3a - 4b)^5$
26. $(2x - 5y)^5$
27. $(2x + y)^3$
28. $(7a + b)^3$
29. $(x^2 + y^2)^4$
30. $(x^2 + y^2)^6$
31. $\left(\frac{1}{x} + y\right)^5$
32. $\left(\frac{1}{x} + 2y\right)^6$
33. $2(x - 3)^4 + 5(x - 3)^2$
34. $3(x + 1)^5 - 4(x + 1)^3$

In Exercises 35–38, expand the binomial by using Pascal's Triangle to determine the coefficients.

35. $(2t - s)^5$
36. $(3 - 2z)^4$
37. $(x + 2y)^5$
38. $(2v + 3)^6$

In Exercises 39–46, find the specified *n*th term in the expansion of the binomial.

39. $(x + y)^{10}, \quad n = 4$
40. $(x - y)^6, \quad n = 7$
41. $(x - 6y)^5, \quad n = 3$
42. $(x - 10z)^7, \quad n = 4$
43. $(4x + 3y)^9, \quad n = 8$
44. $(5a + 6b)^5, \quad n = 5$
45. $(10x - 3y)^{12}, \quad n = 9$
46. $(7x + 2y)^{15}, \quad n = 7$

In Exercises 47–54, find the coefficient *a* of the term in the expansion of the binomial.

	Binomial	*Term*
47.	$(x + 3)^{12}$	ax^5
48.	$(x^2 + 3)^{12}$	ax^8
49.	$(x - 2y)^{10}$	ax^8y^2
50.	$(4x - y)^{10}$	ax^2y^8
51.	$(3x - 2y)^9$	ax^4y^5
52.	$(2x - 3y)^8$	ax^6y^2
53.	$(x^2 + y)^{10}$	ax^8y^6
54.	$(z^2 - t)^{10}$	az^4t^8

In Exercises 55–58, use the Binomial Theorem to expand and simplify the expression.

55. $(\sqrt{x} + 3)^4$
56. $(2\sqrt{t} - 1)^3$
57. $(x^{2/3} - y^{1/3})^3$
58. $(u^{3/5} + 2)^5$

In Exercises 59–62, expand the expression in the difference quotient and simplify.

$$\frac{f(x + h) - f(x)}{h} \qquad \text{Difference quotient}$$

59. $f(x) = x^3$
60. $f(x) = x^4$
61. $f(x) = \sqrt{x}$
62. $f(x) = \frac{1}{x}$

In Exercises 63–68, use the Binomial Theorem to expand the complex number. Simplify your result.

63. $(1 + i)^4$

64. $(2 - i)^5$

65. $(2 - 3i)^6$

66. $\left(5 + \sqrt{-9}\right)^3$

67. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

68. $\left(5 - \sqrt{3}\,i\right)^4$

Approximation **In Exercises 69–72, use the Binomial Theorem to approximate the quantity accurate to three decimal places. For example, in Exercise 69, use the expansion**

$$(1.02)^8 = (1 + 0.02)^8 = 1 + 8(0.02) + 28(0.02)^2 + \cdots.$$

69. $(1.02)^8$

70. $(2.005)^{10}$

71. $(2.99)^{12}$

72. $(1.98)^9$

Graphical Reasoning **In Exercises 73 and 74, use a graphing utility to graph f and g in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function g in standard form.**

73. $f(x) = x^3 - 4x, \quad g(x) = f(x + 4)$

74. $f(x) = -x^4 + 4x^2 - 1, \quad g(x) = f(x - 3)$

Probability **In Exercises 75–78, consider n independent trials of an experiment in which each trial has two possible outcomes: "success" or "failure." The probability of a success on each trial is p, and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k\, p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.**

75. A fair coin is tossed seven times. To find the probability of obtaining four heads, evaluate the term

$${}_7C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

76. The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{4}$. To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^7$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

77. The probability of a sales representative making a sale with any one customer is $\frac{1}{3}$. The sales representative makes eight contacts a day. To find the probability of making four sales, evaluate the term

$${}_8C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^4$$

in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

78. To find the probability that the sales representative in Exercise 77 makes four sales if the probability of a sale with any one customer is $\frac{1}{2}$, evaluate the term

$${}_8C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

Model It

79. ***Data Analysis: Water Consumption*** The table shows the per capita consumption of bottled water $f(t)$ (in gallons) in the United States from 1990 through 2003. (Source: Economic Research Service, U.S. Department of Agriculture)

Year	Consumption, $f(t)$
1990	8.0
1991	8.0
1992	9.7
1993	10.3
1994	11.3
1995	12.1
1996	13.0
1997	13.9
1998	15.0
1999	16.4
2000	17.4
2001	18.8
2002	20.7
2003	22.0

(a) Use the *regression* feature of a graphing utility to find a cubic model for the data. Let t represent the year, with $t = 0$ corresponding to 1990.

(b) Use a graphing utility to plot the data and the model in the same viewing window.

(c) You want to adjust the model so that $t = 0$ corresponds to 2000 rather than 1990. To do this, you shift the graph of f 10 units *to the left* to obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

(d) Use a graphing utility to graph g in the same viewing window as f.

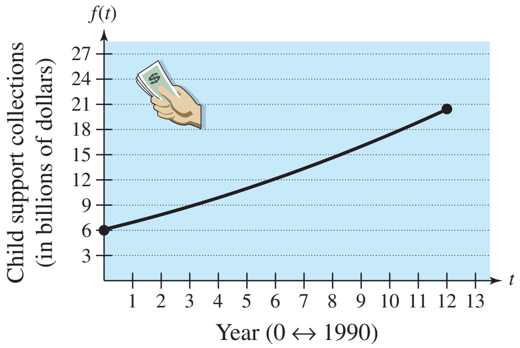
(e) Use both models to estimate the per capita consumption of bottled water in 2008. Do you obtain the same answer?

(f) Describe the overall trend in the data. What factors do you think may have contributed to the increase in the per capita consumption of bottled water?

80. ***Child Support*** The amounts $f(t)$ (in billions of dollars) of child support collected in the United States from 1990 to 2002 can be approximated by the model

$$f(t) = 0.031t^2 + 0.82t + 6.1, \quad 0 \le t \le 12$$

where t represents the year, with $t = 0$ corresponding to 1990 (see figure). (Source: U.S. Department of Health and Human Services)



(a) You want to adjust the model so that $t = 0$ corresponds to 2000 rather than 1990. To do this, you shift the graph of f 10 units *to the left* and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

(c) Use the graphs to estimate when the child support collections will exceed $30 billion.

Synthesis

True or False? **In Exercises 81–83, determine whether the statement is true or false. Justify your answer.**

81. The Binomial Theorem could be used to produce each row of Pascal's Triangle.

82. A binomial that represents a difference cannot always be accurately expanded using the Binomial Theorem.

83. The x^{10}-term and the x^{14}-term of the expansion of $(x^2 + 3)^{12}$ have identical coefficients.

84. ***Writing*** In your own words, explain how to form the rows of Pascal's Triangle.

85. Form rows 8–10 of Pascal's Triangle.

86. ***Think About It*** How many terms are in the expansion of $(x + y)^n$?

87. ***Think About It*** How do the expansions of $(x + y)^n$ and $(x - y)^n$ differ?

88. ***Graphical Reasoning*** Which two functions have identical graphs, and why? Use a graphing utility to graph the functions in the given order and in the same viewing window. Compare the graphs.

(a) $f(x) = (1 - x)^3$

(b) $g(x) = 1 - x^3$

(c) $h(x) = 1 + 3x + 3x^2 + x^3$

(d) $k(x) = 1 - 3x + 3x^2 - x^3$

(e) $p(x) = 1 + 3x - 3x^2 + x^3$

Proof **In Exercises 89–92, prove the property for all integers r and n where $0 \le r \le n$.**

89. ${}_nC_r = {}_nC_{n-r}$

90. ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdot \cdot \cdot \pm {}_nC_n = 0$

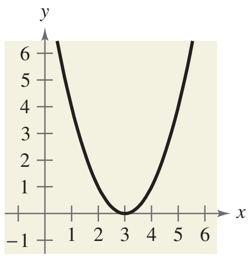
91. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

92. The sum of the numbers in the nth row of Pascal's Triangle is 2^n.

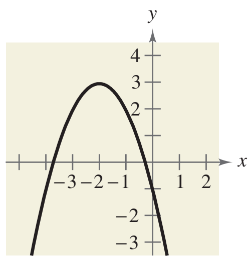
Skills Review

In Exercises 93–96, the graph of $y = g(x)$ is shown. Graph f and use the graph to write an equation for the graph of g.

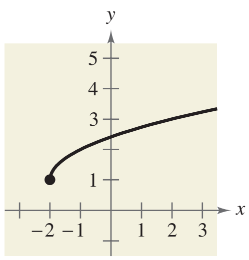
93. $f(x) = x^2$



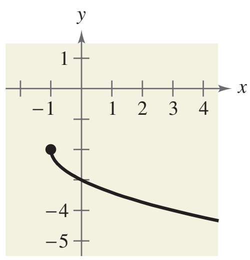
94. $f(x) = x^2$



95. $f(x) = \sqrt{x}$



96. $f(x) = \sqrt{x}$



In Exercises 97 and 98, find the inverse of the matrix.

97. $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

98. $\begin{bmatrix} 1.2 & -2.3 \\ -2 & 4 \end{bmatrix}$

9.6 Counting Principles

What you should learn

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Why you should learn it

You can use counting principles to solve counting problems that occur in real life. For instance, in Exercise 65 on page 700, you are asked to use counting principles to determine the number of possible ways of selecting the winning numbers in the Powerball lottery.

Simple Counting Problems

This section and Section 9.7 present a brief introduction to some of the basic counting principles and their application to probability. In Section 9.7, you will see that much of probability has to do with counting the number of ways an event can occur. The following two examples describe simple counting problems.

Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *replaced in the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. How many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

CHECKPOINT Now try Exercise 5.

Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. Two pieces of paper are drawn from the box *at the same time*, and the numbers on the pieces of paper are written down and totaled. How many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the different ways that a sum of 12 can be obtained *using two different numbers* from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

CHECKPOINT Now try Exercise 7.

The difference between the counting problems in Examples 1 and 2 can be described by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

The Fundamental Counting Principle

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is always the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

Solution

There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is $26 \cdot 26 = 676$.

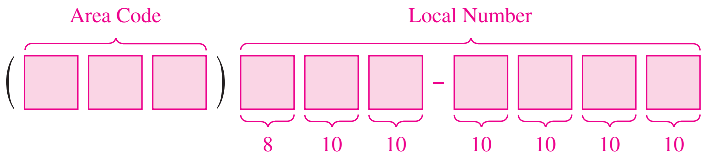
CHECKPOINT Now try Exercise 13.

Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

Solution

Because the first digit of a local telephone number cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible *within* each area code is $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$.

CHECKPOINT Now try Exercise 19.

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

How many permutations are possible for the letters A, B, C, D, E, and F?

Solution

Consider the following reasoning.

First position: Any of the *six* letters

Second position: Any of the remaining *five* letters

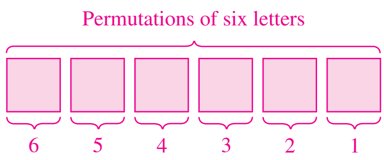
Third position: Any of the remaining *four* letters

Fourth position: Any of the remaining *three* letters

Fifth position: Any of the remaining *two* letters

Sixth position: The *one* remaining letter

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 720.$$

CHECKPOINT Now try Exercise 39.

Number of Permutations of n Elements

The number of permutations of n elements is

$$n \cdot (n - 1) \cdot \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

Vaughn Youtz/Newsmakers/Getty Images

Eleven thoroughbred racehorses hold the title of Triple Crown winner for winning the Kentucky Derby, the Preakness, and the Belmont Stakes in the same year. Forty-nine horses have won two out of the three races.

Example 6 Counting Horse Race Finishes

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

Solution

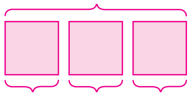
Here are the different possibilities.

Win (first position): *Eight* choices

Place (second position): *Seven* choices

Show (third position): *Six* choices

Using the Fundamental Counting Principle, multiply these three numbers together to obtain the following.



So, there are $8 \cdot 7 \cdot 6 = 336$ different orders.

CHECKPOINT Now try Exercise 43.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose and order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.**

Technology

Most graphing calculators are programmed to evaluate ${}_nP_r$. Consult the user's guide for your calculator and then evaluate ${}_8P_5$. You should get an answer of 6720.

Permutations of n Elements Taken r at a Time

The number of permutations of n elements taken r at a time is

$$\begin{aligned} {}_nP_r &= \frac{n!}{(n-r)!} \\ &= n(n-1)(n-2) \cdot \cdot \cdot (n-r+1). \end{aligned}$$

Using this formula, you can rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Remember that for permutations, order is important. So, if you are looking at the possible permutations of the letters A, B, C, and D taken three at a time, the permutations (A, B, D) and (B, A, D) are counted as different because the *order* of the elements is different.

Suppose, however, that you are asked to find the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be ${}_4P_4 = 4!$. However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdot \cdot \cdot + n_k.$$

Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

Solution

This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways the letters can be written is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{6!}{3! \cdot 2! \cdot 1!}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}$$

$$= 60.$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN	AABNAN	AABNNA
AANABN	AANANB	AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN	ABNANA	ABNNAA
ANAABN	ANAANB	ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB	ANNABA	ANNBAA
BAAANN	BAANAN	BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA	NAAABN	NAAANB
NAABAN	NAABNA	NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA	NBAAAN	NBAANA
NBANAA	NBNAAA	NNAAAB	NNAABA	NNABAA	NNBAAA

CHECKPOINT Now try Exercise 45.

Combinations

When you count the number of possible permutations of a set of elements, *order* is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not* important. Such subsets are called **combinations of n elements taken r at a time.** For instance, the combinations

$\{A, B, C\}$ and $\{B, A, C\}$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. A common example of how a combination occurs is a card game in which the player is free to reorder the cards after they have been dealt.

Example 8 Combinations of n Elements Taken r at a Time

In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

Solution

The following subsets represent the different combinations of three letters that can be chosen from the five letters.

$\{A, B, C\}$ $\{A, B, D\}$
$\{A, B, E\}$ $\{A, C, D\}$
$\{A, C, E\}$ $\{A, D, E\}$
$\{B, C, D\}$ $\{B, C, E\}$
$\{B, D, E\}$ $\{C, D, E\}$

From this list, you can conclude that there are 10 different ways that three letters can be chosen from five letters.

CHECKPOINT Now try Exercise 55.

Combinations of n Elements Taken r at a Time

The number of combinations of n elements taken r at a time is

$${}_nC_r = \frac{n!}{(n-r)!r!} \text{ which is equivalent to } {}_nC_r = \frac{{}_nP_r}{r!}.$$

Note that the formula for ${}_nC_r$ is the same one given for binomial coefficients. To see how this formula is used, solve the counting problem in Example 8. In that problem, you are asked to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$${}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

which is the same answer obtained in Example 8.

A ♥	A ♦	A ♣	A ♠
2 ♥	2 ♦	2 ♣	2 ♠
3 ♥	3 ♦	3 ♣	3 ♠
4 ♥	4 ♦	4 ♣	4 ♠
5 ♥	5 ♦	5 ♣	5 ♠
6 ♥	6 ♦	6 ♣	6 ♠
7 ♥	7 ♦	7 ♣	7 ♠
8 ♥	8 ♦	8 ♣	8 ♠
9 ♥	9 ♦	9 ♣	9 ♠
10 ♥	10 ♦	10 ♣	10 ♠
J ♥	J ♦	J ♣	J ♠
Q ♥	Q ♦	Q ♣	Q ♠
K ♥	K ♦	K ♣	K ♠

FIGURE 9.7 *Standard deck of playing cards*

Example 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52 (see Figure 9.7). How many different poker hands are possible? (After the cards are dealt, the player may reorder them, and so order is not important.)

Solution

You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$\begin{aligned}
{}_{52}C_5 &= \frac{52!}{(52-5)!5!} \\
&= \frac{52!}{47!5!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} \\
&= 2{,}598{,}960
\end{aligned}$$

CHECKPOINT Now try Exercise 63.

Example 10 Forming a Team

You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Solution

There are ${}_{10}C_5$ ways of choosing five girls. The are ${}_{15}C_7$ ways of choosing seven boys. By the Fundamental Counting Principal, there are ${}_{10}C_5 \cdot {}_{15}C_7$ ways of choosing five girls and seven boys.

$$\begin{aligned}
{}_{10}C_5 \cdot {}_{15}C_7 &= \frac{10!}{5! \cdot 5!} \cdot \frac{15!}{8! \cdot 7!} \\
&= 252 \cdot 6435 \\
&= 1{,}621{,}620
\end{aligned}$$

So, there are 1,621,620 12-member swim teams possible.

CHECKPOINT Now try Exercise 65.

When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly. To do this, ask yourself the following questions.

1. Is the order of the elements important? *Permutation*
2. Are the chosen elements a subset of a larger set in which order is not important? *Combination*
3. Does the problem involve two or more separate events? *Fundamental Counting Principle*

9.6 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The ________ ________ ________ states that if there are m_1 ways for one event to occur and m_2 ways for a second event to occur, there are $m_1 \cdot m_2$ ways for both events to occur.
2. An ordering of n elements is called a ________ of the elements.
3. The number of permutations of n elements taken r at a time is given by the formula ________.
4. The number of ________ ________ of n objects is given by $\dfrac{n!}{n_1!n_2!n_3! \cdot \cdot \cdot n_k!}$.
5. When selecting subsets of a larger set in which order is not important, you are finding the number of ________ of n elements taken r at a time.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

Random Selection **In Exercises 1–8, determine the number of ways a computer can randomly generate one or more such integers from 1 through 12.**

1. An odd integer
2. An even integer
3. A prime integer
4. An integer that is greater than 9
5. An integer that is divisible by 4
6. An integer that is divisible by 3
7. Two *distinct* integers whose sum is 9
8. Two *distinct* integers whose sum is 8

9. ***Entertainment Systems*** A customer can choose one of three amplifiers, one of two compact disc players, and one of five speaker models for an entertainment system. Determine the number of possible system configurations.
10. ***Job Applicants*** A college needs two additional faculty members: a chemist and a statistician. In how many ways can these positions be filled if there are five applicants for the chemistry position and three applicants for the statistics position?
11. ***Course Schedule*** A college student is preparing a course schedule for the next semester. The student may select one of two mathematics courses, one of three science courses, and one of five courses from the social sciences and humanities. How many schedules are possible?
12. ***Aircraft Boarding*** Eight people are boarding an aircraft. Two have tickets for first class and board before those in the economy class. In how many ways can the eight people board the aircraft?
13. ***True-False Exam*** In how many ways can a six-question true-false exam be answered? (Assume that no questions are omitted.)
14. ***True-False Exam*** In how many ways can a 12-question true-false exam be answered? (Assume that no questions are omitted.)
15. ***License Plate Numbers*** In the state of Pennsylvania, each standard automobile license plate number consists of three letters followed by a four-digit number. How many distinct license plate numbers can be formed in Pennsylvania?
16. ***License Plate Numbers*** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. To avoid confusion between "O" and "zero" and between "I" and "one," the letters "O" and "I" are not used. How many distinct license plate numbers can be formed in this state?
17. ***Three-Digit Numbers*** How many three-digit numbers can be formed under each condition?
    (a) The leading digit cannot be zero.
    (b) The leading digit cannot be zero and no repetition of digits is allowed.
    (c) The leading digit cannot be zero and the number must be a multiple of 5.
    (d) The number is at least 400.
18. ***Four-Digit Numbers*** How many four-digit numbers can be formed under each condition?
    (a) The leading digit cannot be zero.
    (b) The leading digit cannot be zero and no repetition of digits is allowed.
    (c) The leading digit cannot be zero and the number must be less than 5000.
    (d) The leading digit cannot be zero and the number must be even.
19. ***Combination Lock*** A combination lock will open when the right choice of three numbers (from 1 to 40, inclusive) is selected. How many different lock combinations are possible?

20. ***Combination Lock*** A combination lock will open when the right choice of three numbers (from 1 to 50, inclusive) is selected. How many different lock combinations are possible?

21. ***Concert Seats*** Four couples have reserved seats in a row for a concert. In how many different ways can they be seated if

(a) there are no seating restrictions?

(b) the two members of each couple wish to sit together?

22. ***Single File*** In how many orders can four girls and four boys walk through a doorway single file if

(a) there are no restrictions?

(b) the girls walk through before the boys?

In Exercises 23–28, evaluate $_nP_r$.

23. $_4P_4$

24. $_5P_5$

25. $_8P_3$

26. $_{20}P_2$

27. $_5P_4$

28. $_7P_4$

In Exercises 29 and 30, solve for *n*.

29. $14 \cdot {_nP_3} = {_{n+2}P_4}$

30. $_nP_5 = 18 \cdot {_{n-2}P_4}$

In Exercises 31–36, evaluate using a graphing utility.

31. $_{20}P_5$

32. $_{100}P_5$

33. $_{100}P_3$

34. $_{10}P_8$

35. $_{20}C_5$

36. $_{10}C_7$

37. ***Posing for a Photograph*** In how many ways can five children posing for a photograph line up in a row?

38. ***Riding in a Car*** In how many ways can six people sit in a six-passenger car?

39. ***Choosing Officers*** From a pool of 12 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many different ways can the offices be filled?

40. ***Assembly Line Production*** There are four processes involved in assembling a product, and these processes can be performed in any order. The management wants to test each order to determine which is the least time-consuming. How many different orders will have to be tested?

In Exercises 41–44, find the number of distinguishable permutations of the group of letters.

41. A, A, G, E, E, E, M

42. B, B, B, T, T, T, T, T

43. A, L, G, E, B, R, A

44. M, I, S, S, I, S, S, I, P, P, I

45. Write all permutations of the letters A, B, C, and D.

46. Write all permutations of the letters A, B, C, and D if the letters B and C must remain between the letters A and D.

47. ***Batting Order*** A baseball coach is creating a nine-player batting order by selecting from a team of 15 players. How many different batting orders are possible?

48. ***Athletics*** Six sprinters have qualified for the finals in the 100-meter dash at the NCAA national track meet. In how many ways can the sprinters come in first, second, and third? (Assume there are no ties.)

49. ***Jury Selection*** From a group of 40 people, a jury of 12 people is to be selected. In how many different ways can the jury be selected?

50. ***Committee Members*** As of January 2005, the U.S. Senate Committee on Indian Affairs had 14 members. Assuming party affiliation was not a factor in selection, how many different committees were possible from the 100 U.S. senators?

51. Write all possible selections of two letters that can be formed from the letters A, B, C, D, E, and F. (The order of the two letters is not important.)

52. ***Forming an Experimental Group*** In order to conduct an experiment, five students are randomly selected from a class of 20. How many different groups of five students are possible?

53. ***Lottery Choices*** In the Massachusetts Mass Cash game, a player chooses five distinct numbers from 1 to 35. In how many ways can a player select the five numbers?

54. ***Lottery Choices*** In the Louisiana Lotto game, a player chooses six distinct numbers from 1 to 40. In how many ways can a player select the six numbers?

55. ***Defective Units*** A shipment of 10 microwave ovens contains three defective units. In how many ways can a vending company purchase four of these units and receive (a) all good units, (b) two good units, and (c) at least two good units?

56. ***Interpersonal Relationships*** The complexity of interpersonal relationships increases dramatically as the size of a group increases. Determine the numbers of different two-person relationships in groups of people of sizes (a) 3, (b) 8, (c) 12, and (d) 20.

57. ***Poker Hand*** You are dealt five cards from an ordinary deck of 52 playing cards. In how many ways can you get (a) a full house and (b) a five-card combination containing two jacks and three aces? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)

58. ***Job Applicants*** A toy manufacturer interviews eight people for four openings in the research and development department of the company. Three of the eight people are women. If all eight are qualified, in how many ways can the employer fill the four positions if (a) the selection is random and (b) exactly two selections are women?

59. ***Forming a Committee*** A six-member research committee at a local college is to be formed having one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 20 students in contention for the committee. How many six-member committees are possible?

60. ***Law Enforcement*** A police department uses computer imaging to create digital photographs of alleged perpetrators from eyewitness accounts. One software package contains 195 hairlines, 99 sets of eyes and eyebrows, 89 noses, 105 mouths, and 74 chins and cheek structures.

(a) Find the possible number of different faces that the software could create.

(b) A eyewitness can clearly recall the hairline and eyes and eyebrows of a suspect. How many different faces can be produced with this information?

Geometry **In Exercises 61–64, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of the polygon.)**

61. Pentagon **62.** Hexagon

63. Octagon **64.** Decagon (10 sides)

Model It

65. ***Lottery*** Powerball is a lottery game that is operated by the Multi-State Lottery Association and is played in 27 states, Washington D.C., and the U.S. Virgin Islands. The game is played by drawing five white balls out of a drum of 53 white balls (numbered 1–53) and one red powerball out of a drum of 42 red balls (numbered 1–42). The jackpot is won by matching all five white balls in any order and the red powerball.

(a) Find the possible number of winning Powerball numbers.

(b) Find the possible number of winning Powerball numbers if the jackpot is won by matching all five white balls in order and the red power ball.

(c) Compare the results of part (a) with a state lottery in which a jackpot is won by matching six balls from a drum of 53 balls.

66. ***Permutations or Combinations?*** Decide whether each scenario should be counted using permutations or combinations. Explain your reasoning.

(a) Number of ways 10 people can line up in a row for concert tickets

(b) Number of different arrangements of three types of flowers from an array of 20 types

(c) Number of three-digit pin numbers for a debit card

(d) Number of two-scoop ice cream cones created from 31 different flavors

Synthesis

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. The number of letter pairs that can be formed in any order from any of the first 13 letters in the alphabet (A–M) is an example of a permutation.

68. The number of permutations of n elements can be determined by using the Fundamental Counting Principle.

69. What is the relationship between ${}_nC_r$ and ${}_nC_{n-r}$?

70. Without calculating the numbers, determine which of the following is greater. Explain.

(a) The number of combinations of 10 elements taken six at a time

(b) The number of permutations of 10 elements taken six at a time

Proof **In Exercises 71–74, prove the identity.**

71. ${}_nP_{n-1} = {}_nP_n$ **72.** ${}_nC_n = {}_nC_0$

73. ${}_nC_{n-1} = {}_nC_1$ **74.** ${}_nC_r = \dfrac{{}_nP_r}{r!}$

75. ***Think About It*** Can your calculator evaluate ${}_{100}P_{80}$? If not, explain why.

76. ***Writing*** Explain in words the meaning of ${}_nP_r$.

Skills Review

In Exercises 77–80, evaluate the function at each specified value of the independent variable and simplify.

77. $f(x) = 3x^2 + 8$

(a) $f(3)$ (b) $f(0)$ (c) $f(-5)$

78. $g(x) = \sqrt{x-3} + 2$

(a) $g(3)$ (b) $g(7)$ (c) $g(x+1)$

79. $f(x) = -|x-5| + 6$

(a) $f(-5)$ (b) $f(-1)$ (c) $f(11)$

80. $f(x) = \begin{cases} x^2 - 2x + 5, & x \le -4 \\ -x^2 - 2, & x > -4 \end{cases}$

(a) $f(-4)$ (b) $f(-1)$ (c) $f(-20)$

In Exercises 81–84, solve the equation. Round your answer to two decimal places, if necessary.

81. $\sqrt{x-3} = x - 6$ **82.** $\dfrac{4}{t} + \dfrac{3}{2t} = 1$

83. $\log_2(x-3) = 5$ **84.** $e^{x/3} = 16$

9.7 Probability

What you should learn

- Find the probabilities of events.
- Find the probabilities of mutually exclusive events.
- Find the probabilities of independent events.
- Find the probability of the complement of an event.

Why you should learn it

Probability applies to many games of chance. For instance, in Exercise 55, on page 712, you will calculate probabilities that relate to the game of roulette.

Hank de Lespinasse/The Image Bank

The Probability of an Event

Any happening for which the result is uncertain is called an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For instance, when a six-sided die is tossed, the sample space can be represented by the numbers 1 through 6. For this experiment, each of the outcomes is *equally likely*.

To describe sample spaces in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

Example 1 Finding a Sample Space

Find the sample space for each of the following.

a. One coin is tossed.

b. Two coins are tossed.

c. Three coins are tossed.

Solution

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space is

$$S = \{H, T\}.$$

b. Because either coin can land heads up or tails up, the possible outcomes are as follows.

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

c. Following the notation of part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases HHT, HTH, and THH, and among the cases HTT, THT, and TTH.

CHECKPOINT Now try Exercise 1.

Exploration

Toss two coins 100 times and write down the number of heads that occur on each toss (0, 1, or 2). How many times did two heads occur? How many times would you expect two heads to occur if you did the experiment 1000 times?

To calculate the probability of an event, count the number of outcomes in the event and in the sample space. The *number of outcomes* in event E is denoted by $n(E)$, and the number of outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1. That is,

$$0 \le P(E) \le 1$$

as indicated in Figure 9.8. If $P(E) = 0$, event E *cannot occur*, and E is called an **impossible event.** If $P(E) = 1$, event E *must occur*, and E is called a **certain event.**

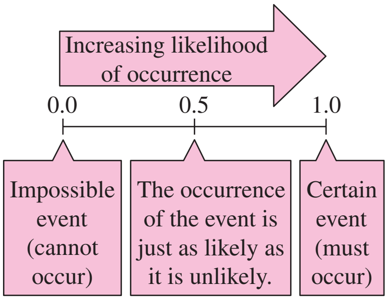


FIGURE 9.8

Example 2 Finding the Probability of an Event

a. Two coins are tossed. What is the probability that both land heads up?

b. A card is drawn from a standard deck of playing cards. What is the probability that it is an ace?

Solution

a. Following the procedure in Example 1(b), let

$$E = \{HH\}$$

and

$$S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one in each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

CHECKPOINT Now try Exercise 11.

STUDY TIP

You can write a probability as a fraction, decimal, or percent. For instance, in Example 2(a), the probability of getting two heads can be written as $\frac{1}{4}$, 0.25, or 25%.

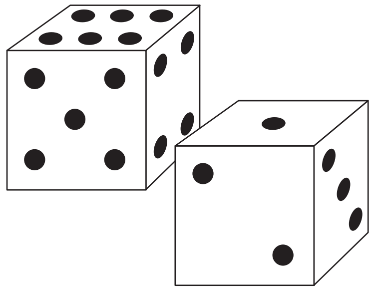


FIGURE 9.9

Example 3 Finding the Probability of an Event

Two six-sided dice are tossed. What is the probability that the total of the two dice is 7? (See Figure 9.9.)

Solution

Because there are six possible outcomes on each die, you can use the Fundamental Counting Principle to conclude that there are $6 \cdot 6$ or 36 different outcomes when two dice are tossed. To find the probability of rolling a total of 7, you must first count the number of ways in which this can occur.

First die	Second die
1	6
2	5
3	4
4	3
5	2
6	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

CHECKPOINT Now try Exercise 15.

STUDY TIP

You could have written out each sample space in Examples 2 and 3 and simply counted the outcomes in the desired events. For larger sample spaces, however, you should use the counting principles discussed in Section 9.6.

Example 4 Finding the Probability of an Event

Twelve-sided dice, as shown in Figure 9.10, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die. Prove that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes.

Solution

For an ordinary six-sided die, each of the numbers 1, 2, 3, 4, 5, and 6 occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For one of the 12-sided dice, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

CHECKPOINT Now try Exercise 17.

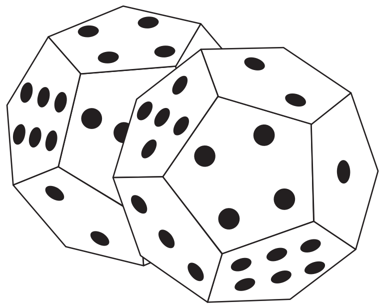


FIGURE 9.10

Example 5 The Probability of Winning a Lottery

In the Arizona state lottery, a player chooses six different numbers from 1 to 41. If these six numbers match the six numbers drawn (in any order) by the lottery commission, the player wins (or shares) the top prize. What is the probability of winning the top prize if the player buys one ticket?

Solution

To find the number of elements in the sample space, use the formula for the number of combinations of 41 elements taken six at a time.

$$\begin{aligned} n(S) &= {}_{41}C_6 \\ &= \frac{41 \cdot 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 4{,}496{,}388 \end{aligned}$$

If a person buys only one ticket, the probability of winning is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4{,}496{,}388}.$$

CHECKPOINT Now try Exercise 21.

Example 6 Random Selection

The numbers of colleges and universities in various regions of the United States in 2003 are shown in Figure 9.11. One institution is selected at random. What is the probability that the institution is in one of the three southern regions? (Source: National Center for Education Statistics)

Solution

From the figure, the total number of colleges and universities is 4163. Because there are $700 + 284 + 386 = 1370$ colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1370}{4163} \approx 0.329.$$

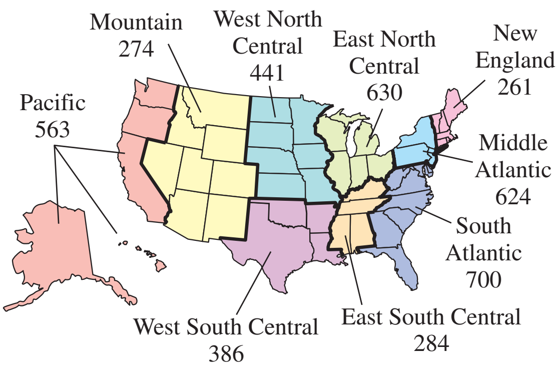


FIGURE 9.11

CHECKPOINT Now try Exercise 33.

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** if A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which is written as

$$P(A \cap B) = 0.$$

For instance, if two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 7 The Probability of a Union of Events

One card is selected from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution

Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts *and* face cards (see Figure 9.12), it follows that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} \approx 0.423.$$

CHECKPOINT Now try Exercise 45.

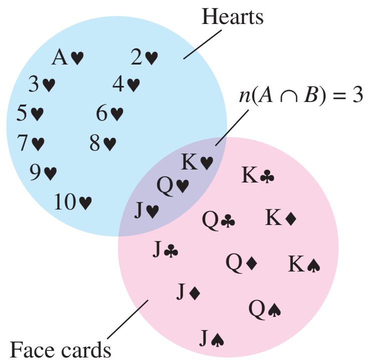


FIGURE 9.12

Example 8 Probability of Mutually Exclusive Events

The personnel department of a company has compiled data on the numbers of employees who have been with the company for various periods of time. The results are shown in the table.

Years of service	Number of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8

If an employee is chosen at random, what is the probability that the employee has (a) 4 or fewer years of service and (b) 9 or fewer years of service?

Solution

a. To begin, add the number of employees to find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Then the probability of choosing an employee who has 4 or fewer years of service is

$$P(A) = \frac{157}{529} \approx 0.297.$$

b. Let event B represent choosing an employee with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Because event A from part (a) and event B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.465. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.465.

CHECKPOINT Now try Exercise 47.

Independent Events

Two events are **independent** if the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of each.

Probability of Independent Events

If A and B are independent events, the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 9 Probability of Independent Events

A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution

The probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{64}.$$

CHECKPOINT Now try Exercise 48.

Example 10 Probability of Independent Events

In 2004, approximately 20% of the adult population of the United States got their news from the Internet every day. In a survey, 10 people were chosen at random from the adult population. What is the probability that all 10 got their news from the Internet every day? (Source: The Gallup Poll)

Solution

Let A represent choosing an adult who gets the news from the Internet every day. The probability of choosing an adult who got his or her news from the Internet every day is 0.20, the probability of choosing a second adult who got his or her news from the Internet every day is 0.20, and so on. Because these events are independent, you can conclude that the probability that all 10 people got their news from the Internet every day is

$$[P(A)]^{10} = (0.20)^{10} \approx 0.0000001.$$

CHECKPOINT Now try Exercise 49.

Exploration

You are in a class with 22 other people. What is the probability that at least two out of the 23 people will have a birthday on the same day of the year?

The complement of the probability that at least two people have the same birthday is the probability that all 23 birthdays are different. So, first find the probability that all 23 people have different birthdays and then find the complement.

Now, determine the probability that in a room with 50 people at least two people have the same birthday.

The Complement of an Event

The **complement of an event** A is the collection of all outcomes in the sample space that are *not* in A. The complement of event A is denoted by A'. Because $P(A \text{ or } A') = 1$ and because A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$. So, the probability of A' is

$$P(A') = 1 - P(A).$$

For instance, if the probability of *winning* a certain game is

$$P(A) = \frac{1}{4}$$

the probability of *losing* the game is

$$P(A') = 1 - \frac{1}{4}$$
$$= \frac{3}{4}.$$

Probability of a Complement

Let A be an event and let A' be its complement. If the probability of A is $P(A)$, the probability of the complement is

$$P(A') = 1 - P(A).$$

Example 11 Finding the Probability of a Complement

A manufacturer has determined that a machine averages one faulty unit for every 1000 it produces. What is the probability that an order of 200 units will have one or more faulty units?

Solution

To solve this problem as stated, you would need to find the probabilities of having exactly one faulty unit, exactly two faulty units, exactly three faulty units, and so on. However, using complements, you can simply find the probability that all units are perfect and then subtract this value from 1. Because the probability that any given unit is perfect is 999/1000, the probability that all 200 units are perfect is

$$P(A) = \left(\frac{999}{1000}\right)^{200}$$
$$\approx 0.819.$$

So, the probability that at least one unit is faulty is

$$P(A') = 1 - P(A)$$
$$\approx 1 - 0.819.$$
$$= 0.181$$

CHECKPOINT Now try Exercise 51.

9.7 Exercises

VOCABULARY CHECK:

In Exercises 1–7, fill in the blanks.

1. An ________ is an event whose result is uncertain, and the possible results of the event are called ________.
2. The set of all possible outcomes of an experiment is called the ________ ________.
3. To determine the ________ of an event, you can use the formula $P(E) = \frac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in the event and $n(S)$ is the number of outcomes in the sample space.
4. If $P(E) = 0$, then E is an ________ event, and if $P(E) = 1$, then E is a ________ event.
5. If two events from the same sample space have no outcomes in common, then the two events are ________ ________.
6. If the occurrence of one event has no effect on the occurrence of a second event, then the events are ________.
7. The ________ of an event A is the collection of all outcomes in the sample space that are not in A.

8. Match the probability formula with the correct probability name.

(a) Probability of the union of two events	(i) $P(A \cup B) = P(A) + P(B)$
(b) Probability of mutually exclusive events	(ii) $P(A') = 1 - P(A)$
(c) Probability of independent events	(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(d) Probability of a complement	(iv) $P(A \text{ and } B) = P(A) \cdot P(B)$

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, determine the sample space for the experiment.

1. A coin and a six-sided die are tossed.
2. A six-sided die is tossed twice and the sum of the points is recorded.
3. A taste tester has to rank three varieties of yogurt, A, B, and C, according to preference.
4. Two marbles are selected from a bag containing two red marbles, two blue marbles, and one yellow marble. The color of each marble is recorded.
5. Two county supervisors are selected from five supervisors, A, B, C, D, and E, to study a recycling plan.
6. A sales representative makes presentations about a product in three homes per day. In each home, there may be a sale (denote by S) or there may be no sale (denote by F).

Tossing a Coin **In Exercises 7–10, find the probability for the experiment of tossing a coin three times. Use the sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.**

7. The probability of getting exactly one tail
8. The probability of getting a head on the first toss
9. The probability of getting at least one head
10. The probability of getting at least two heads

Drawing a Card **In Exercises 11–14, find the probability for the experiment of selecting one card from a standard deck of 52 playing cards.**

11. The card is a face card.
12. The card is not a face card.
13. The card is a red face card.
14. The card is a 6 or lower. (Aces are low.)

Tossing a Die **In Exercises 15–20, find the probability for the experiment of tossing a six-sided die twice.**

15. The sum is 4.
16. The sum is at least 7.
17. The sum is less than 11.
18. The sum is 2, 3, or 12.
19. The sum is odd and no more than 7.
20. The sum is odd or prime.

Drawing Marbles **In Exercises 21–24, find the probability for the experiment of drawing two marbles (without replacement) from a bag containing one green, two yellow, and three red marbles.**

21. Both marbles are red.
22. Both marbles are yellow.
23. Neither marble is yellow.
24. The marbles are of different colors.

In Exercises 25–28, you are given the probability that an event will happen. Find the probability that the event *will* not happen.

25. $P(E) = 0.7$

26. $P(E) = 0.36$

27. $P(E) = \frac{1}{4}$

28. $P(E) = \frac{2}{3}$

In Exercises 29–32, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

29. $P(E') = 0.14$

30. $P(E') = 0.92$

31. $P(E') = \frac{17}{35}$

32. $P(E') = \frac{61}{100}$

33. ***Data Analysis*** A study of the effectiveness of a flu vaccine was conducted with a sample of 500 people. Some participants in the study were given no vaccine, some were given one injection, and some were given two injections. The results of the study are listed in the table.

	No vaccine	One injection	Two injections	Total
Flu	7	2	13	22
No flu	149	52	277	478
Total	156	54	290	500

A person is selected at random from the sample. Find the specified probability.

(a) The person had two injections.

(b) The person did not get the flu.

(c) The person got the flu and had one injection.

34. ***Data Analysis*** One hundred college students were interviewed to determine their political party affiliations and whether they favored a balanced-budget amendment to the Constitution. The results of the study are listed in the table, where D represents Democrat and R represents Republican.

	Favor	Not Favor	Unsure	Total
D	23	25	7	55
R	32	9	4	45
Total	55	34	11	100

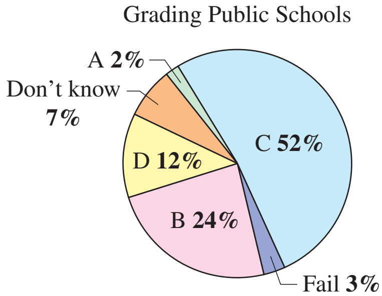
A person is selected at random from the sample. Find the probability that the described person is selected.

(a) A person who doesn't favor the amendment

(b) A Republican

(c) A Democrat who favors the amendment

35. ***Graphical Reasoning*** The figure shows the results of a recent survey in which 1011 adults were asked to grade U.S. public schools. (Source: Phi Delta Kappa/Gallup Poll)

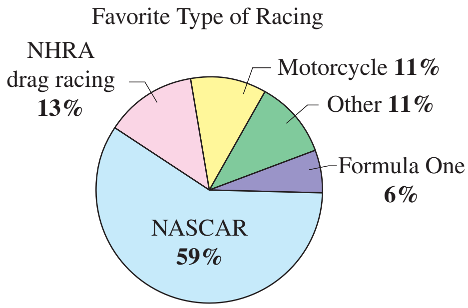


(a) Estimate the number of adults who gave U.S. public schools a B.

(b) An adult is selected at random. What is the probabilty that the adult will give the U.S. public schools an A?

(c) An adult is selected at random. What is the probabilty the adult will give the U.S. public schools a C or a D?

36. ***Graphical Reasoning*** The figure shows the results of a survey in which auto racing fans listed their favorite type of racing. (Source: ESPN Sports Poll/TNS Sports)



(a) What is the probability that an auto racing fan selected at random lists NASCAR racing as his or her favorite type of racing?

(b) What is the probability that an auto racing fan selected at random lists Formula One or motorcycle racing as his or her favorite type of racing?

(c) What is the probability that an auto racing fan selected at random does *not* list NHRA drag racing as his or her favorite type of racing?

37. ***Alumni Association*** A college sends a survey to selected members of the class of 2006. Of the 1254 people who graduated that year, 672 are women, of whom 124 went on to graduate school. Of the 582 male graduates, 198 went on to graduate school. An alumni member is selected at random. What are the probabilities that the person is (a) female, (b) male, and (c) female and did not attend graduate school?

38. ***Education*** In a high school graduating class of 202 students, 95 are on the honor roll. Of these, 71 are going on to college, and of the other 107 students, 53 are going on to college. A student is selected at random from the class. What are the probabilities that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

39. ***Winning an Election*** Taylor, Moore, and Jenkins are candidates for public office. It is estimated that Moore and Jenkins have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find the probability of each candidate winning the election.

40. ***Winning an Election*** Three people have been nominated for president of a class. From a poll, it is estimated that the first candidate has a 37% chance of winning and the second candidate has a 44% chance of winning. What is the probability that the third candidate will win?

In Exercises 41–52, the sample spaces are large and you should use the counting principles discussed in Section 9.6.

41. ***Preparing for a Test*** A class is given a list of 20 study problems, from which 10 will be part of an upcoming exam. A student knows how to solve 15 of the problems. Find the probabilities that the student will be able to answer (a) all 10 questions on the exam, (b) exactly eight questions on the exam, and (c) at least nine questions on the exam.

42. ***Payroll Mix-Up*** Five paychecks and envelopes are addressed to five different people. The paychecks are randomly inserted into the envelopes. What are the probabilities that (a) exactly one paycheck will be inserted in the correct envelope and (b) at least one paycheck will be inserted in the correct envelope?

43. ***Game Show*** On a game show, you are given five digits to arrange in the proper order to form the price of a car. If you are correct, you win the car. What is the probability of winning, given the following conditions?

(a) You guess the position of each digit.

(b) You know the first digit and guess the positions of the other digits.

44. ***Card Game*** The deck of a card game is made up of 108 cards. Twenty-five each are red, yellow, blue, and green, and eight are wild cards. Each player is randomly dealt a seven-card hand.

(a) What is the probability that a hand will contain exactly two wild cards?

(b) What is the probability that a hand will contain two wild cards, two red cards, and three blue cards?

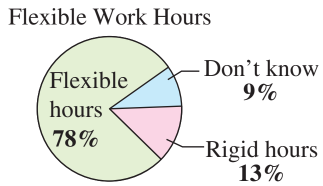
45. ***Drawing a Card*** One card is selected at random from an ordinary deck of 52 playing cards. Find the probabilities that (a) the card is an even-numbered card, (b) the card is a heart or a diamond, and (c) the card is a nine or a face card.

46. ***Poker Hand*** Five cards are drawn from an ordinary deck of 52 playing cards. What is the probability that the hand drawn is a full house? (A full house is a hand that consists of two of one kind and three of another kind.)

47. ***Defective Units*** A shipment of 12 microwave ovens contains three defective units. A vending company has ordered four of these units, and because each is identically packaged, the selection will be random. What are the probabilities that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

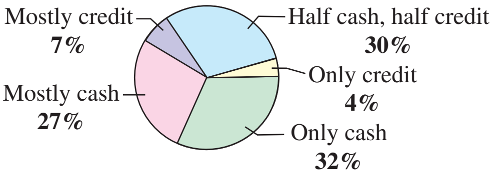
48. ***Random Number Generator*** Two integers from 1 through 40 are chosen by a random number generator. What are the probabilities that (a) the numbers are both even, (b) one number is even and one is odd, (c) both numbers are less than 30, and (d) the same number is chosen twice?

49. ***Flexible Work Hours*** In a survey, people were asked if they would prefer to work flexible hours—even if it meant slower career advancement—so they could spend more time with their families. The results of the survey are shown in the figure. Three people from the survey were chosen at random. What is the probability that all three people would prefer flexible work hours?



50. ***Consumer Awareness*** Suppose that the methods used by shoppers to pay for merchandise are as shown in the circle graph. Two shoppers are chosen at random. What is the probability that both shoppers paid for their purchases only in cash?

How Shoppers Pay for Merchandise

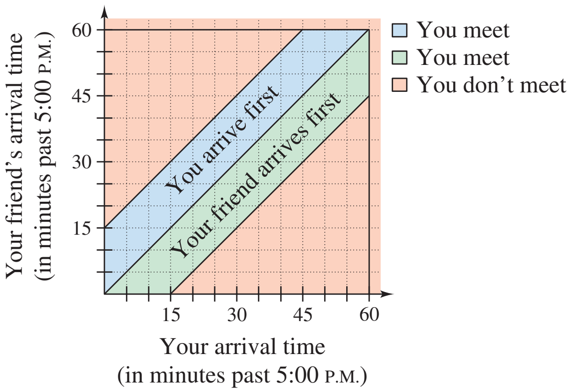


51. ***Backup System*** A space vehicle has an independent backup system for one of its communication networks. The probability that either system will function satisfactorily during a flight is 0.985. What are the probabilities that during a given flight (a) both systems function satisfactorily, (b) at least one system functions satisfactorily, and (c) both systems fail?

52. ***Backup Vehicle*** A fire company keeps two rescue vehicles. Because of the demand on the vehicles and the chance of mechanical failure, the probability that a specific vehicle is available when needed is 90%. The availability of one vehicle is *independent* of the availability of the other. Find the probabilities that (a) both vehicles are available at a given time, (b) neither vehicle is available at a given time, and (c) at least one vehicle is available at a given time.

53. ***A Boy or a Girl?*** Assume that the probability of the birth of a child of a particular sex is 50%. In a family with four children, what are the probabilities that (a) all the children are boys, (b) all the children are the same sex, and (c) there is at least one boy?

54. ***Geometry*** You and a friend agree to meet at your favorite fast-food restaurant between 5:00 and 6:00 P.M. The one who arrives first will wait 15 minutes for the other, and then will leave (see figure). What is the probability that the two of you will actually meet, assuming that your arrival times are random within the hour?



Model It

55. ***Roulette*** American roulette is a game in which a wheel turns on a spindle and is divided into 38 pockets. Thirty-six of the pockets are numbered 1–36, of which half are red and half are black. Two of the pockets are green and are numbered 0 and 00 (see figure). The dealer spins the wheel and a small ball in opposite directions. As the ball slows to a stop, it has an equal probability of landing in any of the numbered pockets.

(a) Find the probability of landing in the number 00 pocket.

(b) Find the probability of landing in a red pocket.

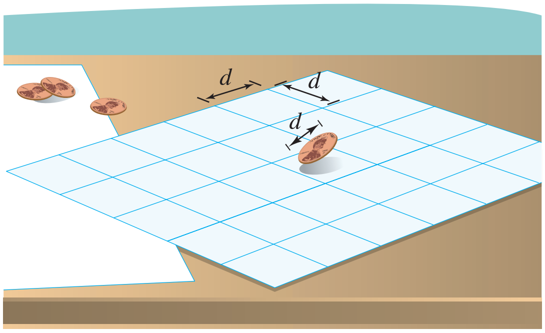
(c) Find the probability of landing in a green pocket or a black pocket.

(d) Find the probability of landing in the number 14 pocket on two consecutive spins.

(e) Find the probability of landing in a red pocket on three consecutive spins.

(f) European roulette does not contain the 00 pocket. Repeat parts (a)–(e) for European roulette. How do the probabilities for European roulette compare with the probabilities for American roulette?

56. ***Estimating π*** A coin of diameter d is dropped onto a paper that contains a grid of squares d units on a side (see figure).



(a) Find the probability that the coin covers a vertex of one of the squares on the grid.

(b) Perform the experiment 100 times and use the results to approximate π.

Synthesis

True or False? **In Exercises 57 and 58, determine whether the statement is true or false. Justify your answer.**

57. If A and B are independent events with nonzero probabilities, then A can occur when B occurs.

58. Rolling a number less than 3 on a normal six-sided die has a probability of $\frac{1}{3}$. The complement of this event is to roll a number greater than 3, and its probability is $\frac{1}{2}$.

59. ***Pattern Recognition and Exploration*** Consider a group of n people.

(a) Explain why the following pattern gives the probabilities that the n people have distinct birthdays.

$$n = 2: \quad \frac{365}{365} \cdot \frac{364}{365} = \frac{365 \cdot 364}{365^2}$$

$$n = 3: \quad \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{365 \cdot 364 \cdot 363}{365^3}$$

(b) Use the pattern in part (a) to write an expression for the probability that $n = 4$ people have distinct birthdays.

(c) Let P_n be the probability that the n people have distinct birthdays. Verify that this probability can be obtained recursively by

$$P_1 = 1 \text{ and } P_n = \frac{365 - (n - 1)}{365}P_{n-1}.$$

(d) Explain why $Q_n = 1 - P_n$ gives the probability that at least two people in a group of n people have the same birthday.

(e) Use the results of parts (c) and (d) to complete the table.

n	10	15	20	23	30	40	50
P_n							
Q_n							

(f) How many people must be in a group so that the probability of at least two of them having the same birthday is greater than $\frac{1}{2}$? Explain.

60. ***Think About It*** A weather forecast indicates that the probability of rain is 40%. What does this mean?

Skills Review

In Exercises 61–70, find all real solutions of the equation.

61. $6x^2 + 8 = 0$

62. $4x^2 + 6x - 12 = 0$

63. $x^3 - x^2 - 3x = 0$

64. $x^5 + x^3 - 2x = 0$

65. $\dfrac{12}{x} = -3$

66. $\dfrac{32}{x} = 2x$

67. $\dfrac{2}{x - 5} = 4$

68. $\dfrac{3}{2x + 3} - 4 = \dfrac{-1}{2x + 3}$

69. $\dfrac{3}{x - 2} + \dfrac{x}{x + 2} = 1$

70. $\dfrac{2}{x} - \dfrac{5}{x - 2} = \dfrac{-13}{x^2 - 2x}$

In Exercises 71–74, sketch the graph of the solution set of the system of inequalities.

71. $\begin{cases} y \ge -3 \\ x \ge -1 \\ -x - y \ge -8 \end{cases}$

72. $\begin{cases} x \le 3 \\ y \le 6 \\ 5x + 2y \ge 10 \end{cases}$

73. $\begin{cases} x^2 + y \ge -2 \\ y \ge x - 4 \end{cases}$

74. $\begin{cases} x^2 + y^2 \le 4 \\ x + y \ge -2 \end{cases}$

9 Chapter Summary

What did you learn?

Section 9.1	Review Exercises
☐ Use sequence notation to write the terms of sequences *(p. 642).*	1–8
☐ Use factorial notation *(p. 644).*	9–12
☐ Use summation notation to write sums *(p. 646).*	13–20
☐ Find the sums of infinite series *(p. 647).*	21–24
☐ Use sequences and series to model and solve real-life problems *(p. 648).*	25, 26
Section 9.2	
☐ Recognize, write, and find the *n*th terms of arithmetic sequences *(p. 653).*	27–40
☐ Find *n*th partial sums of arithmetic sequences *(p. 656).*	41–46
☐ Use arithmetic sequences to model and solve real-life problems *(p. 657).*	47, 48
Section 9.3	
☐ Recognize, write, and find the *n*th terms of geometric sequences *(p. 663).*	49–60
☐ Find *n*th partial sums of geometric sequences *(p. 666).*	61–70
☐ Find sums of infinite geometric series *(p. 667).*	71–76
☐ Use geometric sequences to model and solve real-life problems *(p. 668).*	77, 78
Section 9.4	
☐ Use mathematical induction to prove statements involving a positive integer *n (p. 673).*	79–82
☐ Recognize patterns and write the *n*th term of a sequence *(p. 677).*	83–86
☐ Find the sums of powers of integers *(p. 679).*	87–90
☐ Find finite differences of sequences *(p. 680).*	91–94
Section 9.5	
☐ Use the Binomial Theorem to calculate binomial coefficients *(p. 683).*	95–98
☐ Use Pascal's Triangle to calculate binomial coefficients *(p. 685).*	99–102
☐ Use binomial coefficients to write binomial expansions *(p. 686).*	103–108
Section 9.6	
☐ Solve simple counting problems *(p. 691).*	109, 110
☐ Use the Fundamental Counting Principle to solve counting problems *(p. 692).*	111, 112
☐ Use permutations to solve counting problems *(p. 693).*	113, 114
☐ Use combinations to solve counting problems *(p. 696).*	115, 116
Section 9.7	
☐ Find the probabilities of events *(p. 701).*	117, 118
☐ Find the probabilities of mutually exclusive events *(p. 705).*	119, 120
☐ Find the probabilities of independent events *(p. 707).*	121, 122
☐ Find the probability of the complement of an event *(p. 708).*	123, 124

9 Review Exercises

9.1 **In Exercises 1–4, write the first five terms of the sequence. (Assume that n begins with 1.)**

1. $a_n = 2 + \frac{6}{n}$

2. $a_n = \frac{(-1)^n 5n}{2n - 1}$

3. $a_n = \frac{72}{n!}$

4. $a_n = n(n - 1)$

In Exercises 5–8, write an expression for the apparent nth term of the sequence. (Assume that n begins with 1.)

5. $-2, 2, -2, 2, -2, \ldots$

6. $-1, 2, 7, 14, 23, \ldots$

7. $4, 2, \frac{4}{3}, 1, \frac{4}{5}, \ldots$

8. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

In Exercises 9–12, simplify the factorial expression.

9. $5!$

10. $3! \cdot 2!$

11. $\frac{3! \cdot 5!}{6!}$

12. $\frac{7! \cdot 6!}{6! \cdot 8!}$

In Exercises 13–18, find the sum.

13. $\sum_{i=1}^{6} 5$

14. $\sum_{k=2}^{5} 4k$

15. $\sum_{j=1}^{4} \frac{6}{j^2}$

16. $\sum_{i=1}^{8} \frac{i}{i + 1}$

17. $\sum_{k=1}^{10} 2k^3$

18. $\sum_{j=0}^{4} (j^2 + 1)$

In Exercises 19 and 20, use sigma notation to write the sum.

19. $\frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2(20)}$

20. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10}$

In Exercises 21–24, find the sum of the infinite series.

21. $\sum_{i=1}^{\infty} \frac{5}{10^i}$

22. $\sum_{i=1}^{\infty} \frac{3}{10^i}$

23. $\sum_{k=1}^{\infty} \frac{2}{100^k}$

24. $\sum_{k=2}^{\infty} \frac{9}{10^k}$

25. ***Compound Interest*** A deposit of \$10,000 is made in an account that earns 8% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 10{,}000\left(1 + \frac{0.08}{12}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Write the first 10 terms of this sequence.

(b) Find the balance in this account after 10 years by finding the 120th term of the sequence.

26. ***Education*** The enrollment a_n (in thousands) in Head Start programs in the United States from 1994 to 2002 can be approximated by the model

$$a_n = 1.07n^2 + 6.1n + 693, \quad n = 4, 5, \ldots, 12$$

where n is the year, with $n = 4$ corresponding to 1994. Find the terms of this finite sequence. Use a graphing utility to construct a bar graph that represents the sequence. (Source: U.S. Administration for Children and Families)

9.2 **In Exercises 27–30, determine whether the sequence is arithmetic. If so, find the common difference.**

27. $5, 3, 1, -1, -3, \ldots$

28. $0, 1, 3, 6, 10, \ldots$

29. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

30. $\frac{9}{9}, \frac{8}{9}, \frac{7}{9}, \frac{6}{9}, \frac{5}{9}, \ldots$

In Exercises 31–34, write the first five terms of the arithmetic sequence.

31. $a_1 = 4, d = 3$

32. $a_1 = 6, d = -2$

33. $a_1 = 25, a_{k+1} = a_k + 3$

34. $a_1 = 4.2, a_{k+1} = a_k + 0.4$

In Exercises 35–40, find a formula for a_n for the arithmetic sequence.

35. $a_1 = 7, d = 12$

36. $a_1 = 25, d = -3$

37. $a_1 = y, d = 3y$

38. $a_1 = -2x, d = x$

39. $a_2 = 93, a_6 = 65$

40. $a_7 = 8, a_{13} = 6$

In Exercises 41–44, find the partial sum.

41. $\sum_{j=1}^{10} (2j - 3)$

42. $\sum_{j=1}^{8} (20 - 3j)$

43. $\sum_{k=1}^{11} \left(\frac{2}{3}k + 4\right)$

44. $\sum_{k=1}^{25} \left(\frac{3k + 1}{4}\right)$

45. Find the sum of the first 100 positive multiples of 5.

46. Find the sum of the integers from 20 to 80 (inclusive).

47. ***Job Offer*** The starting salary for an accountant is \$34,000 with a guaranteed salary increase of \$2250 per year. Determine (a) the salary during the fifth year and (b) the total compensation through 5 full years of employment.

48. ***Baling Hay*** In the first two trips baling hay around a large field, a farmer obtains 123 bales and 112 bales, respectively. Because each round gets shorter, the farmer estimates that the same pattern will continue. Estimate the total number of bales made if the farmer takes another six trips around the field.

9.3 In Exercises 49–52, determine whether the sequence is geometric. If so, find the common ratio.

49. $5, 10, 20, 40, \ldots$ **50.** $54, -18, 6, -2, \ldots$

51. $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \ldots$ **52.** $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \ldots$

In Exercises 53–56, write the first five terms of the geometric sequence.

53. $a_1 = 4,\ r = -\frac{1}{4}$ **54.** $a_1 = 2,\ r = 2$

55. $a_1 = 9,\ a_3 = 4$ **56.** $a_1 = 2,\ a_3 = 12$

In Exercises 57–60, write an expression for the *n*th term of the geometric sequence. Then find the 20th term of the sequence.

57. $a_1 = 16, a_2 = -8$ **58.** $a_3 = 6, a_4 = 1$

59. $a_1 = 100, r = 1.05$ **60.** $a_1 = 5, r = 0.2$

In Exercises 61–66, find the sum of the finite geometric sequence.

61. $\displaystyle\sum_{i=1}^{7} 2^{i-1}$ **62.** $\displaystyle\sum_{i=1}^{5} 3^{i-1}$

63. $\displaystyle\sum_{i=1}^{4} \left(\frac{1}{2}\right)^{i}$ **64.** $\displaystyle\sum_{i=1}^{6} \left(\frac{1}{3}\right)^{i-1}$

65. $\displaystyle\sum_{i=1}^{5} (2)^{i-1}$ **66.** $\displaystyle\sum_{i=1}^{4} 6(3)^{i}$

In Exercises 67–70, use a graphing utility to find the sum of the finite geometric sequence.

67. $\displaystyle\sum_{i=1}^{10} 10\left(\frac{3}{5}\right)^{i-1}$ **68.** $\displaystyle\sum_{i=1}^{15} 20(0.2)^{i-1}$

69. $\displaystyle\sum_{i=1}^{25} 100(1.06)^{i-1}$ **70.** $\displaystyle\sum_{i=1}^{20} 8\left(\frac{6}{5}\right)^{i-1}$

In Exercises 71–76, find the sum of the infinite geometric series.

71. $\displaystyle\sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^{i-1}$ **72.** $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1}$

73. $\displaystyle\sum_{i=1}^{\infty} (0.1)^{i-1}$ **74.** $\displaystyle\sum_{i=1}^{\infty} (0.5)^{i-1}$

75. $\displaystyle\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1}$ **76.** $\displaystyle\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1}$

77. ***Depreciation*** A paper manufacturer buys a machine for \$120,000. During the next 5 years, it will depreciate at a rate of 30% per year. (That is, at the end of each year the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find the formula for the *n*th term of a geometric sequence that gives the value of the machine *t* full years after it was purchased.

(b) Find the depreciated value of the machine after 5 full years.

78. ***Annuity*** You deposit \$200 in an account at the beginning of each month for 10 years. The account pays 6% compounded monthly. What will your balance be at the end of 10 years? What would the balance be if the interest were compounded continuously?

9.4 In Exercises 79–82, use mathematical induction to prove the formula for every positive integer *n*.

79. $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$

80. $1 + \dfrac{3}{2} + 2 + \dfrac{5}{2} + \cdots + \dfrac{1}{2}(n + 1) = \dfrac{n}{4}(n + 3)$

81. $\displaystyle\sum_{i=0}^{n-1} ar^{i} = \frac{a(1 - r^{n})}{1 - r}$

82. $\displaystyle\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}[2a + (n - 1)d]$

In Exercises 83–86, find a formula for the sum of the first *n* terms of the sequence.

83. $9, 13, 17, 21, \ldots$ **84.** $68, 60, 52, 44, \ldots$

85. $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots$ **86.** $12, -1, \frac{1}{12}, -\frac{1}{144}, \ldots$

In Exercises 87–90, find the sum using the formulas for the sums of powers of integers.

87. $\displaystyle\sum_{n=1}^{30} n$ **88.** $\displaystyle\sum_{n=1}^{10} n^2$

89. $\displaystyle\sum_{n=1}^{7} (n^4 - n)$ **90.** $\displaystyle\sum_{n=1}^{6} (n^5 - n^2)$

In Exercises 91–94, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. State whether the sequence has a linear model, a quadratic model, or neither.

91. $a_1 = 5$
$a_n = a_{n-1} + 5$

92. $a_1 = -3$
$a_n = a_{n-1} - 2n$

93. $a_1 = 16$
$a_n = a_{n-1} - 1$

94. $a_0 = 0$
$a_n = n - a_{n-1}$

9.5 In Exercises 95–98, use the Binomial Theorem to calculate the binomial coefficient.

95. ${}_6C_4$

96. ${}_{10}C_7$

97. ${}_8C_5$

98. ${}_{12}C_3$

In Exercises 99–102, use Pascal's Triangle to calculate the binomial coefficient.

99. $\binom{7}{3}$

100. $\binom{9}{4}$

101. $\binom{8}{6}$

102. $\binom{5}{3}$

In Exercises 103–108, use the Binomial Theorem to expand and simplify the expression. (Remember that $i = \sqrt{-1}$.)

103. $(x + 4)^4$

104. $(x - 3)^6$

105. $(a - 3b)^5$

106. $(3x + y^2)^7$

107. $(5 + 2i)^4$

108. $(4 - 5i)^3$

9.6 **109.** ***Numbers in a Hat*** Slips of paper numbered 1 through 14 are placed in a hat. In how many ways can you draw two numbers with replacement that total 12?

110. ***Home Theater Systems*** A customer in an electronics store can choose one of six speaker systems, one of five DVD players, and one of six plasma televisions to design a home theater system. How many systems can be designed?

111. ***Telephone Numbers*** The same three-digit prefix is used for all of the telephone numbers in a small town. How many different telephone numbers are possible by changing only the last four digits?

112. ***Course Schedule*** A college student is preparing a course schedule for the next semester. The student may select one of three mathematics courses, one of four science courses, and one of six history courses. How many schedules are possible?

113. ***Bike Race*** There are 10 bicyclists entered in a race. In how many different ways could the top three places be decided?

114. ***Jury Selection*** A group of potential jurors has been narrowed down to 32 people. In how many ways can a jury of 12 people be selected?

115. ***Apparel*** You have eight different suits to choose from to take on a trip. How many combinations of three suits could you take on your trip?

116. ***Menu Choices*** A local sub shop offers five different breads, seven different meats, three different cheeses, and six different vegetables. Find the total number of combinations of sandwiches possible.

9.7 **117.** ***Apparel*** A man has five pairs of socks, of which no two pairs are the same color. He randomly selects two socks from a drawer. What is the probability that he gets a matched pair?

118. ***Bookshelf Order*** A child returns a five-volume set of books to a bookshelf. The child is not able to read, and so cannot distinguish one volume from another. What is the probability that the books are shelved in the correct order?

119. ***Students by Class*** At a particular university, the numbers of students in the four classes are broken down by percents, as shown in the table.

Class	Percent
Freshmen	31
Sophomores	26
Juniors	25
Seniors	18

A single student is picked randomly by lottery for a cash scholarship. What is the probability that the scholarship winner is

(a) a junior or senior?

(b) a freshman, sophomore, or junior?

120. ***Data Analysis*** A sample of college students, faculty, and administration were asked whether they favored a proposed increase in the annual activity fee to enhance student life on campus. The results of the study are listed in the table.

	Students	Faculty	Admin.	Total
Favor	237	37	18	292
Oppose	163	38	7	208
Total	400	75	25	500

A person is selected at random from the sample. Find each specified probability.

(a) The person is not in favor of the proposal.

(b) The person is a student.

(c) The person is a faculty member and is in favor of the proposal.

121. ***Tossing a Die*** A six-sided die is tossed three times. What is the probability of getting a 6 on each roll?

122. ***Tossing a Die*** A six-sided die is tossed six times. What is the probability that each side appears exactly once?

123. ***Drawing a Card*** You randomly select a card from a 52-card deck. What is the probability that the card is *not* a club?

124. ***Tossing a Coin*** Find the probability of obtaining at least one tail when a coin is tossed five times.

Synthesis

True or False? **In Exercises 125–129, determine whether the statement is true or false. Justify your answer.**

125. $\dfrac{(n+2)!}{n!} = (n+2)(n+1)$

126. $\displaystyle\sum_{i=1}^{5}(i^3+2i) = \sum_{i=1}^{5} i^3 + \sum_{i=1}^{5} 2i$

127. $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$

128. $\displaystyle\sum_{j=1}^{6} 2^j = \sum_{j=3}^{8} 2^{j-2}$

129. The value of ${}_nP_r$ is always greater than the value of ${}_nC_r$.

130. ***Think About It*** An infinite sequence is a function. What is the domain of the function?

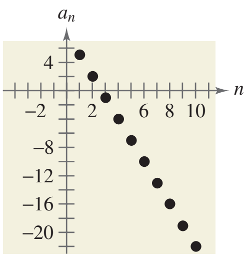
131. ***Think About It*** How do the two sequences differ?

(a) $a_n = \dfrac{(-1)^n}{n}$

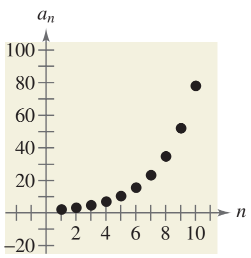
(b) $a_n = \dfrac{(-1)^{n+1}}{n}$

132. ***Graphical Reasoning*** The graphs of two sequences are shown below. Identify each sequence as arithmetic or geometric. Explain your reasoning.

(a)



(b)

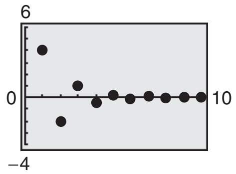


133. ***Writing*** Explain what is meant by a recursion formula.

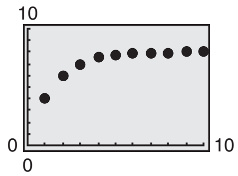
134. ***Writing*** Explain why the terms of a geometric sequence decrease when $0 < r < 1$.

Graphical Reasoning **In Exercises 135–138, match the sequence or sum of a sequence with its graph without doing any calculations. Explain your reasoning. [The graphs are labeled (a), (b), (c), and (d).]**

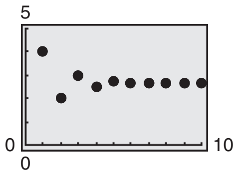
(a)



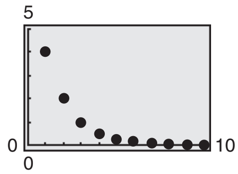
(b)



(c)



(d)



135. $a_n = 4\left(\frac{1}{2}\right)^{n-1}$

136. $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$

137. $a_n = \displaystyle\sum_{k=1}^{n} 4\left(\frac{1}{2}\right)^{k-1}$

138. $a_n = \displaystyle\sum_{k=1}^{n} 4\left(-\frac{1}{2}\right)^{k-1}$

139. ***Population Growth*** Consider an idealized population with the characteristic that each member of the population produces one offspring at the end of every time period. If each member has a life span of three time periods and the population begins with 10 newborn members, then the following table shows the population during the first five time periods.

	Time Period				
Age Bracket	**1**	**2**	**3**	**4**	**5**
0–1	10	10	20	40	70
1–2		10	10	20	40
2–3			10	10	20
Total	10	20	40	70	130

The sequence for the total population has the property that

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \quad n > 3.$$

Find the total population during the next five time periods.

140. The probability of an event must be a real number in what interval? Is the interval open or closed?

9 Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Write the first five terms of the sequence $a_n = \dfrac{(-1)^n}{3n + 2}$. (Assume that n begins with 1.)

2. Write an expression for the nth term of the sequence.

$\dfrac{3}{1!}, \dfrac{4}{2!}, \dfrac{5}{3!}, \dfrac{6}{4!}, \dfrac{7}{5!}, \ldots$

3. Find the next three terms of the series. Then find the fifth partial sum of the series.

$6 + 17 + 28 + 39 + \cdots$

4. The fifth term of an arithmetic sequence is 5.4, and the 12th term is 11.0. Find the nth term.

5. Write the first five terms of the sequence $a_n = 5(2)^{n-1}$. (Assume that n begins with 1.)

In Exercises 6–8, find the sum.

6. $\displaystyle\sum_{i=1}^{50} (2i^2 + 5)$.   **7.** $\displaystyle\sum_{n=1}^{7} (8n - 5)$ **8.** $\displaystyle\sum_{i=1}^{\infty} 4\left(\tfrac{1}{2}\right)^i$.

9. Use mathematical induction to prove the formula.

$$5 + 10 + 15 + \cdots + 5n = \frac{5n(n + 1)}{2}$$

10. Use the Binomial Theorem to expand the expression $(x + 2y)^4$.

11. Find the coefficient of the term $a^3 b^5$ in the expansion of $(2a - 3b)^8$.

In Exercises 12 and 13, evaluate each expression.

12. (a) ${}_9P_2$ (b) ${}_{70}P_3$

13. (a) ${}_{11}C_4$ (b) ${}_{66}C_4$

14. How many distinct license plates can be issued consisting of one letter followed by a three-digit number?

15. Eight people are going for a ride in a boat that seats eight people. The owner of the boat will drive, and only three of the remaining people are willing to ride in the two bow seats. How many seating arrangements are possible?

16. You attend a karaoke night and hope to hear your favorite song. The karaoke song book has 300 different songs (your favorite song is among the 300 songs). Assuming that the singers are equally likely to pick any song and no song is repeated, what is the probability that your favorite song is one of the 20 that you hear that night?

17. You are with seven of your friends at a party. Names of all of the 60 guests are placed in a hat and drawn randomly to award eight door prizes. Each guest is limited to one prize. What is the probability that you and your friends win all eight of the prizes?

18. The weather report calls for a 75% chance of snow. According to this report, what is the probability that it will *not* snow?

9 Cumulative Test for Chapters 7–9

Take this test to review the material from earlier chapters. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the system by the specified method.

1. Substitution

$$\begin{cases} y = 3 - x^2 \\ 2(y - 2) = x - 1 \end{cases}$$

2. Elimination

$$\begin{cases} x + 3y = -1 \\ 2x + 4y = 0 \end{cases}$$

3. Elimination

$$\begin{cases} -2x + 4y - z = 3 \\ x - 2y + 2z = -6 \\ x - 3y - z = 1 \end{cases}$$

4. Gauss-Jordan Elimination

$$\begin{cases} x + 3y - 2z = -7 \\ -2x + y - z = -5 \\ 4x + y + z = 3 \end{cases}$$

In Exercises 5 and 6, sketch the graph of the solution set of the system of inequalities.

5. $\begin{cases} 2x + y \geq -3 \\ x - 3y \leq 2 \end{cases}$

6. $\begin{cases} x - y > 6 \\ 5x + 2y < 10 \end{cases}$

7. Sketch the region determined by the constraints. Then find the minimum and maximum values, and where they occur, of the objective function $z = 3x + 2y$, subject to the indicated constraints.

$$\begin{aligned} x + 4y &\leq 20 \\ 2x + y &\leq 12 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

8. A custom-blend bird seed is to be mixed from seed mixtures costing \$0.75 per pound and \$1.25 per pound. How many pounds of each seed mixture are used to make 200 pounds of custom-blend bird seed costing \$0.95 per pound?

9. Find the equation of the parabola $y = ax^2 + bx + c$ passing through the points (0, 4), (3, 1), and (6, 4).

$$\begin{cases} -x + 2y - z = 9 \\ 2x - y + 2z = -9 \\ 3x + 3y - 4z = 7 \end{cases}$$

SYSTEM FOR 10 AND 11

In Exercises 10 and 11, use the system of equations at the left.

10. Write the augmented matrix corresponding to the system of equations.

11. Solve the system using the matrix found in Exercise 10 and Gauss-Jordan elimination.

In Exercises 12–15, use the following matrices to find each of the following, if possible.

$$A = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$$

12. $A + B$

13. $-2B$

14. $A - 2B$

15. AB

$$\begin{bmatrix} 8 & 0 & -5 \\ 1 & 3 & -1 \\ -2 & 6 & 4 \end{bmatrix}$$

MATRIX FOR 16

16. Find the determinant of the matrix at the left.

17. Find the inverse of the matrix (if it exists): $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$.

$$\begin{array}{ll} & \begin{array}{ccc}\text{Gym shoes} & \text{Jogging shoes} & \text{Walking shoes}\end{array} \\ \text{Age group}\begin{cases}14-17\\18-24\\25-34\end{cases} & \begin{bmatrix}0.09 & 0.09 & 0.03\\0.06 & 0.10 & 0.05\\0.12 & 0.25 & 0.12\end{bmatrix}\end{array}$$

MATRIX FOR 18

18. The percents (by age group) of the total amounts spent on three types of footwear in a recent year are shown in the matrix. The total amounts (in millions) spent by each age group on the three types of footwear were \$442.20 (14–17 age group), \$466.57(18–24 age group), and \$1088.09 (25–34 age group). How many dollars worth of gym shoes, jogging shoes, and walking shoes were sold that year? (Source: National Sporting Goods Association)

In Exercises 19 and 20, use Cramer's Rule to solve the system of equations.

19. $\begin{cases} 8x - 3y = -52 \\ 3x + 5y = 5 \end{cases}$

20. $\begin{cases} 5x + 4y + 3z = 7 \\ -3x - 8y + 7z = -9 \\ 7x - 5y - 6z = -53 \end{cases}$

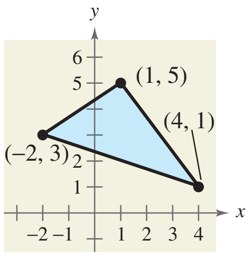


FIGURE FOR 21

21. Find the area of the triangle shown in the figure.

22. Write the first five terms of the sequence $a_n = \dfrac{(-1)^{n+1}}{2n+3}$ (assume that n begins with 1).

23. Write an expression for the nth term of the sequence.

$\dfrac{2!}{4}, \dfrac{3!}{5}, \dfrac{4!}{6}, \dfrac{5!}{7}, \dfrac{6!}{8}, \ldots$

24. Find the sum of the first 20 terms of the arithmetic sequence 8, 12, 16, 20, ...

25. The sixth term of an arithmetic sequence is 20.6, and the ninth term is 30.2.

(a) Find the 20th term.

(b) Find the nth term.

26. Write the first five terms of the sequence $a_n = 3(2)^{n-1}$ (assume that n begins with 1).

27. Find the sum: $\displaystyle\sum_{i=0}^{\infty} 1.3\left(\tfrac{1}{10}\right)^{i-1}$.

28. Use mathematical induction to prove the formula

$3 + 7 + 11 + 15 + \cdots + (4n - 1) = n(2n + 1)$.

29. Use the Binomial Theorem to expand and simplify $(z - 3)^4$.

In Exercises 30–33, evaluate the expression.

30. ${}_7P_3$ **31.** ${}_{25}P_2$ **32.** $\dbinom{8}{4}$ **33.** ${}_{10}C_3$

In Exercises 34 and 35, find the number of distinguishable permutations of the group of letters.

34. B, A, S, K, E, T, B, A, L, L **35.** A, N, T, A, R, C, T, I, C, A

36. A personnel manager at a department store has 10 applicants to fill three different sales positions. In how many ways can this be done, assuming that all the applicants are qualified for any of the three positions?

37. On a game show, the digits 3, 4, and 5 must be arranged in the proper order to form the price of an appliance. If the digits are arranged correctly, the contestant wins the appliance. What is the probability of winning if the contestant knows that the price is at least \$400?

Proofs in Mathematics

Properties of Sums *(p. 647)*

1. $\displaystyle\sum_{i=1}^{n} c = cn,$ $\quad c$ is a constant.

2. $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i,$ $\quad c$ is a constant.

3. $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Proof

Each of these properties follows directly from the properties of real numbers.

1. $\displaystyle\sum_{i=1}^{n} c = c + c + c + \cdot \cdot \cdot + c = cn$ $\qquad$ n terms

The Distributive Property is used in the proof of Property 2.

2. $$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdot \cdot \cdot + ca_n$$
$$= c(a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) \quad = c\sum_{i=1}^{n} a_i$$

The proof of Property 3 uses the Commutative and Associative Properties of Addition.

3.
$$\sum_{i=1}^{n} (a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdot \cdot \cdot + (a_n + b_n)$$
$$= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) + (b_1 + b_2 + b_3 + \cdot \cdot \cdot + b_n)$$
$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

The proof of Property 4 uses the Commutative and Associative Properties of Addition and the Distributive Property.

4.
$$\sum_{i=1}^{n} (a_i - b_i) = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdot \cdot \cdot + (a_n - b_n)$$
$$= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) + (-b_1 - b_2 - b_3 - \cdot \cdot \cdot - b_n)$$
$$= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) - (b_1 + b_2 + b_3 + \cdot \cdot \cdot + b_n)$$
$$= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Infinite Series

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity; throughout the next quarter of the interval at double the intensity; throughout the following eighth at triple the intensity and so ad infinitum; The average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdot \cdot \cdot + \frac{n}{2^n} + \cdot \cdot \cdot$$

is 2.

The Sum of a Finite Arithmetic Sequence *(p. 656)*

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Proof

Begin by generating the terms of the arithmetic sequence in two ways. In the first way, repeatedly add d to the first term to obtain

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_{n-2} + a_{n-1} + a_n \\ &= a_1 + [a_1 + d] + [a_1 + 2d] + \cdot \cdot \cdot + [a_1 + (n-1)d]. \end{aligned}$$

In the second way, repeatedly subtract d from the nth term to obtain

$$\begin{aligned} S_n &= a_n + a_{n-1} + a_{n-2} + \cdot \cdot \cdot + a_3 + a_2 + a_1 \\ &= a_n + [a_n - d] + [a_n - 2d] + \cdot \cdot \cdot + [a_n - (n-1)d]. \end{aligned}$$

If you add these two versions of S_n, the multiples of d subtract out and you obtain

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdot \cdot \cdot + (a_1 + a_n) \quad n \text{ terms}$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n).$$

The Sum of a Finite Geometric Sequence *(p. 666)*

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ .\ .\ .\ ,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \displaystyle\sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

Proof

$$S_n = a_1 + a_1r + a_1r^2 + \cdot \cdot \cdot + a_1r^{n-2} + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdot \cdot \cdot + a_1r^{n-1} + a_1r^n \qquad \text{Multiply by } r.$$

Subtracting the second equation from the first yields

$$S_n - rS_n = a_1 - a_1r^n.$$

So, $S_n(1 - r) = a_1(1 - r^n)$, and, because $r \neq 1$, you have $S_n = a_1\left(\dfrac{1-r^n}{1-r}\right)$.

The Binomial Theorem (p. 683)

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Proof

The Binomial Theorem can be proved quite nicely using mathematical induction. The steps are straightforward but look a little messy, so only an outline of the proof is presented.

1. If $n = 1$, you have $(x + y)^1 = x^1 + y^1 = {}_1C_0x + {}_1C_1y$, and the formula is valid.

2. Assuming that the formula is true for $n = k$, the coefficient of $x^{k-r}y^r$ is

$${}_kC_r = \frac{k!}{(k-r)!r!} = \frac{k(k-1)(k-2)\cdots(k-r+1)}{r!}.$$

To show that the formula is true for $n = k + 1$, look at the coefficient of $x^{k+1-r}y^r$ in the expansion of

$$(x + y)^{k+1} = (x + y)^k(x + y).$$

From the right-hand side, you can determine that the term involving $x^{k+1-r}y^r$ is the sum of two products.

$$\begin{aligned}
&({}_kC_rx^{k-r}y^r)(x) + ({}_kC_{r-1}x^{k+1-r}y^{r-1})(y)\\
&= \left[\frac{k!}{(k-r)!r!} + \frac{k!}{(k+1-r)!(r-1)!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1-r)k!}{(k+1-r)!r!} + \frac{k!r}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{k!(k+1-r+r)}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1)!}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= {}_{k+1}C_rx^{k+1-r}y^r
\end{aligned}$$

So, by mathematical induction, the Binomial Theorem is valid for all positive integers n.

P.S. Problem Solving

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. Let $x_0 = 1$ and consider the sequence x_n given by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad n = 1, 2, \ldots$$

Use a graphing utility to compute the first 10 terms of the sequence and make a conjecture about the value of x_n as n approaches infinity.

2. Consider the sequence

$$a_n = \frac{n+1}{n^2+1}.$$

(a) Use a graphing utility to graph the first 10 terms of the sequence.

(b) Use the graph from part (a) to estimate the value of a_n as n approaches infinity.

(c) Complete the table.

n	1	10	100	1000	10,000
a_n					

(d) Use the table from part (c) to determine (if possible) the value of a_n as n approaches infinity.

3. Consider the sequence

$$a_n = 3 + (-1)^n.$$

(a) Use a graphing utility to graph the first 10 terms of the sequence.

(b) Use the graph from part (a) to describe the behavior of the graph of the sequence.

(c) Complete the table.

n	1	10	101	1000	10,001
a_n					

(d) Use the table from part (c) to determine (if possible) the value of a_n as n approaches infinity.

4. The following operations are performed on each term of an arithmetic sequence. Determine if the resulting sequence is arithmetic, and if so, state the common difference.

(a) A constant C is added to each term.

(b) Each term is multiplied by a nonzero constant C.

(c) Each term is squared.

5. The following sequence of perfect squares is not arithmetic.

1, 4, 9, 16, 25, 36, 49, 64, 81, . . .

However, you can form a related sequence that is arithmetic by finding the differences of consecutive terms.

(a) Write the first eight terms of the related arithmetic sequence described above. What is the nth term of this sequence?

(b) Describe how you can find an arithmetic sequence that is related to the following sequence of perfect cubes.

1, 8, 27, 64, 125, 216, 343, 512, 729, . . .

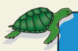
(c) Write the first seven terms of the related sequence in part (b) and find the nth term of the sequence.

(d) Describe how you can find the arithmetic sequence that is related to the following sequence of perfect fourth powers.

1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, . . .

(e) Write the first six terms of the related sequence in part (d) and find the nth term of the sequence.

6. Can the Greek hero Achilles, running at 20 feet per second, ever catch a tortoise, starting 20 feet ahead of Achilles and running at 10 feet per second? The Greek mathematician Zeno said no. When Achilles runs 20 feet, the tortoise will be 10 feet ahead. Then, when Achilles runs 10 feet, the tortoise will be 5 feet ahead. Achilles will keep cutting the distance in half but will never catch the tortoise. The table shows Zeno's reasoning. From the table you can see that both the distances and the times required to achieve them form infinite geometric series. Using the table, show that both series have finite sums. What do these sums represent?



Distance (in feet)	Time (in seconds)
20	1
10	0.5
5	0.25
2.5	0.125
1.25	0.0625
0.625	0.03125

7. Recall that a *fractal* is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. A well-known fractal is called the *Sierpinski Triangle*. In the first stage, the midpoints of the three sides are used to create the vertices of a new triangle, which is then removed, leaving three triangles. The first three stages are shown on the next page. Note that each remaining triangle is similar to the original triangle. Assume that the length of each side of the original triangle is one unit.

Write a formula that describes the side length of the triangles that will be generated in the nth stage. Write a formula for the area of the triangles that will be generated in the nth stage.

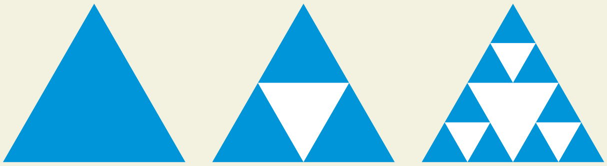


FIGURE FOR 7

8. You can define a sequence using a piecewise formula. The following is an example of a piecewise-defined sequence.

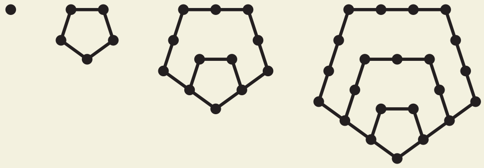
$$a_1 = 7, \quad a_n = \begin{cases} \dfrac{a_{n-1}}{2}, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1, & \text{if } a_{n-1} \text{ is odd} \end{cases}$$

(a) Write the first 10 terms of the sequence.

(b) Choose three different values for a_1 (other than $a_1 = 7$). For each value of a_1, find the first 10 terms of the sequence. What conclusions can you make about the behavior of this sequence?

9. The numbers 1, 5, 12, 22, 35, 51, . . . are called pentagonal numbers because they represent the numbers of dots used to make pentagons, as shown below. Use mathematical induction to prove that the nth pentagonal number P_n is given by

$$P_n = \frac{n(3n - 1)}{2}.$$



10. What conclusion can be drawn from the following information about the sequence of statements P_n?

(a) P_3 is true and P_k implies P_{k+1}.

(b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.

(c) P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true.

(d) P_2 is true and P_{2k} implies P_{2k+2}.

11. Let $f_1, f_2, \ldots, f_n, \ldots$ be the Fibonacci sequence.

(a) Use mathematical induction to prove that

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

(b) Find the sum of the first 20 terms of the Fibonacci sequence.

12. The odds in favor of an event occurring are the ratio of the probability that the event will occur to the probability that the event will not occur. The reciprocal of this ratio represents the odds against the event occurring.

(a) Six marbles in a bag are red. The odds against choosing a red marble are 4 to 1. How many marbles are in the bag?

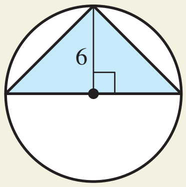
(b) A bag contains three blue marbles and seven yellow marbles. What are the odds in favor of choosing a blue marble? What are the odds against choosing a blue marble?

(c) Write a formula for converting the odds in favor of an event to the probability of the event.

(d) Write a formula for converting the probability of an event to the odds in favor of the event.

13. You are taking a test that contains only multiple choice questions (there are five choices for each question). You are on the last question and you know that the answer is not B or D, but you are not sure about answers A, C, and E. What is the probability that you will get the right answer if you take a guess?

14. A dart is thrown at the circular target shown below. The dart is equally likely to hit any point inside the target. What is the probability that it hits the region outside the triangle?



15. An event A has n possible outcomes, which have the values $x_1, x_2, \ldots, x_n$. The probabilities of the n outcomes occurring are $p_1, p_2, \ldots, p_n$. The **expected value** V of an event A is the sum of the products of the outcomes' probabilities and their values, $V = p_1x_1 + p_2x_2 + \cdots + p_nx_n$.

(a) To win California's Super Lotto Plus game, you must match five different numbers chosen from the numbers 1 to 47, plus one Mega number chosen from the numbers 1 to 27. You purchase a ticket for \$1. If the jackpot for the next drawing is \$12,000,000, what is the expected value for the ticket?

(b) You are playing a dice game in which you need to score 60 points to win. On each turn, you roll two six-sided dice. Your score for the turn is 0 if the dice do not show the same number, and the product of the numbers on the dice if they do show the same number. What is the expected value for each turn? How many turns will it take on average to score 60 points?