Robust Inference for Pairwise Estimation

Matias D. Cattaneo, Princeton University

Michael Jansson, UC Berkeley

Kenichi Nagasawa, University of Warwick

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Talk based on:

- "Robust Inference for Mean-Square Differentiable Pairwise Difference Estimation".

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- "Pairwise Estimation: Distribution Theory and Bootstrap Inference".

Outline

Introduction

Pairwise Difference Estimation is used in economics and other disciplines.

- Identification idea:
  - Localize pair of observations with similar observable characteristics to remove unobserved confounders/heterogeneity.
- Background references:
  - Honoré & Powell (2005): "Pairwise Difference Estimators for Nonlinear Models".
  - Aradillas-Lopez, Honoré & Powell (2007): "Pairwise Difference Estimation with Nonparametric Control Variables".
  - Many, many more...

Contribution: Robust Distribution Theory and Inference Methods.

- New distributional approximation for possibly "more strict" localization.
- Two asymptotic regimes depending on smoothness of objective function.
  - *Smooth*: "Small Bandwidth" Gaussian law.
  - *Non-Smooth*: Mixture of "Small Bandwidth" Gaussian and Chernoff-type laws.

Setup: Pairwise Difference Estimation

DGP: $\mathbf{z}_i = (y_i, \mathbf{x}_i, \mathbf{w}_i)^\top$, $i = 1, \ldots, n$, random sample + regularity conditions.

Parameter of Interest:

$$\boldsymbol{\theta}_0 = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_0(\boldsymbol{\theta}), \qquad M_0(\boldsymbol{\theta}) = \lim_{n\to\infty} \mathbb{E}\Big[m(\mathbf{z}_1, \mathbf{z}_2; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_1-\mathbf{w}_2}{h}\big)\Big].$$

- Loss function $m(\cdot)$ is example-specific.
- Kernel function K re-weights for localization.
- Bandwidth $h \to 0$ (as $n \to \infty$) key tuning parameter for localization.

Estimator:

$$\widehat{\boldsymbol{\theta}}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n(\boldsymbol{\theta}), \qquad \widehat{M}_n(\boldsymbol{\theta}) = \binom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i-\mathbf{w}_j}{h}\big).$$

- Inference sensitive to choice of h (lack of robustness).
- $m(\cdot)$ can be non-convex, non-differentiable, discontinuous, etc.

Goal: robust inference with respect to choice of h and features of $m(\cdot)$.

Example 1: Partially Linear Model

Model:

$$y_i = \mathbf{x}_i^\top \boldsymbol{\theta}_0 + g(\mathbf{w}_i) + \varepsilon_i, \qquad \mathbb{E}[\varepsilon_i | \mathbf{a}_i] = 0, \qquad \mathbf{a}_i = (\mathbf{x}_i, \mathbf{w}_i)^\top.$$

Identification: $\mathbf{w}_i = \mathbf{w}_j \implies \mathbb{E}[y_i - y_j | \mathbf{a}_i, \mathbf{a}_j, \mathbf{w}_i - \mathbf{w}_j = 0] = (\mathbf{x}_i - \mathbf{x}_j)^\top \boldsymbol{\theta}_0$, and

$$\boldsymbol{\theta}_0 = \underset{\boldsymbol{\theta} \in \Theta}{\arg\min}\, \mathbb{E}\big[m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \big| \mathbf{w}_i - \mathbf{w}_j = 0\big]$$

with

$$m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) = \big[(y_i - y_j) - (\mathbf{x}_i - \mathbf{x}_j)^\top \boldsymbol{\theta}\big]^2.$$

Estimator: localize $\mathbf{w}_i \approx \mathbf{w}_j$, and thus

$$\widehat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta} \in \Theta}{\arg\min} \binom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \tfrac{1}{h} K\big(\tfrac{\mathbf{w}_i - \mathbf{w}_j}{h}\big).$$

- $m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})$ is differentiable.
- $\widehat{\boldsymbol{\theta}}_n$ closed-form.

Example 2: Partially Linear Logistic Model

Model:

$$y_i = \mathbb{1}(\mathbf{x}_i^\top \boldsymbol{\theta}_0 + g(\mathbf{w}_i) + \varepsilon_i \geq 0), \qquad \varepsilon_i \perp\!\!\!\perp \mathbf{x}_i, \mathbf{w}_i, \qquad \varepsilon_i \sim \text{Logistic with cdf } \Lambda(\cdot).$$

Identification: $\mathbb{P}[y_i = 1 | \mathbf{a}_i, \mathbf{a}_j, y_i + y_j = 1, \mathbf{w}_i - \mathbf{w}_j = \mathbf{0}] = \Lambda\big((\mathbf{x}_i - \mathbf{x}_j)^\top \boldsymbol{\theta}_0\big)$, and

$$\boldsymbol{\theta}_0 = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \mathbb{E}\big[m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \big| \mathbf{w}_i - \mathbf{w}_j = \mathbf{0}\big]$$

with

$$m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) = -\mathbb{1}(y_1 \neq y_2)\big[y_1 \ln \Lambda(\mathbf{x}_2'\boldsymbol{\theta} - \mathbf{x}_1'\boldsymbol{\theta}) + y_2 \ln \Lambda(\mathbf{x}_1'\boldsymbol{\theta} - \mathbf{x}_2'\boldsymbol{\theta})\big].$$

Estimator: localize $\mathbf{w}_i \approx \mathbf{w}_j$, and thus

$$\widehat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min} \binom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \frac{1}{h} K\Big(\frac{\mathbf{w}_i - \mathbf{w}_j}{h}\Big).$$

- $m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})$ is differentiable.
- $\widehat{\boldsymbol{\theta}}_n$ not closed-form.

Example 3: Partially Linear Tobit Model

Model:

$$y_i = \max\{\mathbf{x}_i^\top \boldsymbol{\theta}_0 + g(\mathbf{w}_i) + \varepsilon_i, 0\}.$$

Identification: Honoré (1992), and

$$\boldsymbol{\theta}_0 = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \mathbb{E}\big[m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \big| \mathbf{w}_i - \mathbf{w}_j = \mathbf{0}\big]$$

with

$$m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) = \begin{cases} |y_1| - ((\mathbf{x}_1 - \mathbf{x}_2)'\boldsymbol{\theta} + y_2)\,\mathrm{sgn}(y_1) & \text{if } (\mathbf{x}_1 - \mathbf{x}_2)'\boldsymbol{\theta} \leq -y_2 \\ |y_1 - y_2 - (\mathbf{x}_1 - \mathbf{x}_2)'\boldsymbol{\theta}| & \text{if } -y_2 < (\mathbf{x}_1 - \mathbf{x}_2)'\boldsymbol{\theta} < y_1 \\ |y_2| + ((\mathbf{x}_1 - \mathbf{x}_2)'\boldsymbol{\theta} - y_1)\,\mathrm{sgn}(y_2) & \text{if } y_1 \leq (\mathbf{x}_1 - \mathbf{x}_2)'\boldsymbol{\theta} \end{cases}.$$

Estimator: localize $\mathbf{w}_i \approx \mathbf{w}_j$, and thus

$$\widehat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \binom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \frac{1}{h} K\Big(\frac{\mathbf{w}_i - \mathbf{w}_j}{h}\Big).$$

- $m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})$ is stochastically differentiable.
- $\widehat{\boldsymbol{\theta}}_n$ not closed-form.

Example 4: Monotone Index Model (not today)

Model:

$$y_i = G\big(\phi(\mathbf{x}_i'\boldsymbol{\theta}_0, \varepsilon_i), \mathbf{w}_i\big), \qquad \mathbf{x}_i = (x_{i1}, \mathbf{x}_{i2})^\top, \qquad \boldsymbol{\theta}_0 \mapsto (1, \boldsymbol{\theta}_0).$$

with G weakly increasing in first argument, and ϕ strictly increasing.

Identification: Jochmans (2013), and

$$\boldsymbol{\theta}_0 = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \mathbb{E}\big[m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})\big|\mathbf{w}_i - \mathbf{w}_j = \mathbf{0}\big]$$

with

$$m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) = \mathbb{1}(y_i > y_j)\mathbb{1}((\mathbf{x}_i - \mathbf{x}_j)^\top(1, \boldsymbol{\theta}) > 0) + \mathbb{1}(y_i < y_j)\mathbb{1}((\mathbf{x}_i - \mathbf{x}_j)^\top(1, \boldsymbol{\theta}) < 0).$$

Estimator: localize $\mathbf{w}_i \approx \mathbf{w}_j$, and thus

$$\widehat{\boldsymbol{\theta}}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \binom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \tfrac{1}{h} K\big(\tfrac{\mathbf{w}_i - \mathbf{w}_j}{h}\big).$$

- $m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})$ is discontinuous.
- $\widehat{\boldsymbol{\theta}}_n$ not closed-form.

Today: Differentiable Pairwise Difference Estimation

DGP: $\mathbf{z}_i = (y_i, \mathbf{x}_i, \mathbf{w}_i)^\top$, $i = 1, \dots, n$, random sample + regularity conditions.

Parameter of Interest:

$$\boldsymbol{\theta}_0 = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, M_0(\boldsymbol{\theta}), \qquad M_0(\boldsymbol{\theta}) = \lim_{n\to\infty} \mathbb{E}\Big[m(\mathbf{z}_1, \mathbf{z}_2; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_1-\mathbf{w}_2}{h}\big)\Big].$$

- Loss function $m(\cdot)$ is example-specific.
- Kernel function K re-weights for localization.
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Estimator:

$$\widehat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \widehat{M}_n(\boldsymbol{\theta}), \qquad \widehat{M}_n(\boldsymbol{\theta}) = \binom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i-\mathbf{w}_j}{h}\big).$$

- Inference sensitive to choice of h (lack of robustness).
- $\boldsymbol{\theta} \mapsto m(\mathbf{z}_1, \mathbf{z}_2; \boldsymbol{\theta})$ stochastically differentiable $\implies$ Gaussian limiting distribution.

Goal: robust inference with respect to choice of h and features of $m(\cdot)$.

Outline

Today: Differentiable Pairwise Difference Estimation

DGP: $\mathbf{z}_i = (y_i, \mathbf{x}_i, \mathbf{w}_i)^\top$, $i = 1, \dots, n$, random sample + regularity conditions.

Parameter of Interest:

$$\boldsymbol{\theta}_0 = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_0(\boldsymbol{\theta}), \qquad M_0(\boldsymbol{\theta}) = \lim_{n\to\infty} M_n(\boldsymbol{\theta}).$$

Fixed-h Estimands:

$$\boldsymbol{\theta}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_n(\boldsymbol{\theta}), \qquad M_n(\boldsymbol{\theta}) = \mathbb{E}\Big[m(\mathbf{z}_1, \mathbf{z}_2; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_1-\mathbf{w}_2}{h}\big)\Big].$$

Estimator:

$$\widehat{\boldsymbol{\theta}}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n(\boldsymbol{\theta}), \qquad \widehat{M}_n(\boldsymbol{\theta}) = \tbinom{n}{2}^{-1}\sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i-\mathbf{w}_j}{h}\big).$$

- Inference sensitive to choice of h (lack of robustness).
- $\boldsymbol{\theta} \mapsto m(\mathbf{z}_1, \mathbf{z}_2; \boldsymbol{\theta})$ stochastically differentiable $\implies$ Gaussian limiting distribution.

Distribution Theory: Basics + Bias

$$\boldsymbol{\theta}_0 = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_0(\boldsymbol{\theta}), \qquad M_0(\boldsymbol{\theta}) = \lim_{n\to\infty} M_n(\boldsymbol{\theta}).$$

$$\boldsymbol{\theta}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_n(\boldsymbol{\theta}), \qquad M_n(\boldsymbol{\theta}) = \mathbb{E}\Big[m(\mathbf{z}_1,\mathbf{z}_2;\boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_1-\mathbf{w}_2}{h}\big)\Big].$$

$$\widehat{\boldsymbol{\theta}}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n(\boldsymbol{\theta}), \qquad \widehat{M}_n(\boldsymbol{\theta}) = \binom{n}{2}^{-1}\sum_{i<j} m(\mathbf{z}_i,\mathbf{z}_j;\boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i-\mathbf{w}_j}{h}\big).$$

Basic Decomposition. Stochastic contribution and "Localization" bias:

$$r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) = \underbrace{r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n)}_{\text{“Variance”}} + \underbrace{r_n(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0)}_{\text{“Bias”}}$$

- For bias, not difficult to show that:

$$r_n(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0) = \operatorname*{arg\,min}_{\mathbf{s}\in\Theta_n} M_n(\boldsymbol{\theta}_0 + \mathbf{s}/r_n) - M_n(\boldsymbol{\theta}_0) = O(r_n h^P)$$

  - P denotes the order of the kernel K.
  - Thus, we need: $r_n h^P \to 0$.

Distribution Theory: First-Order Representation

$$\boldsymbol{\theta}_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, M_n(\boldsymbol{\theta}), \quad M_n(\boldsymbol{\theta}) = \mathbb{E}\Big[m_{12,n}(\boldsymbol{\theta})\Big], \qquad m_{ij,n}(\boldsymbol{\theta}) = m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i - \mathbf{w}_j}{h}\big).$$

$$\widehat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \widehat{M}_n(\boldsymbol{\theta}), \quad \widehat{M}_n(\boldsymbol{\theta}) = \tbinom{n}{2}^{-1}\sum_{i<j} m_{ij,n}(\boldsymbol{\theta}).$$

Quadratic Approximation. For all $\|\boldsymbol{\theta} - \boldsymbol{\theta}_n\|$ small:

$$\widehat{M}_n(\boldsymbol{\theta}) - \widehat{M}_n(\boldsymbol{\theta}_n) \approx (\boldsymbol{\theta} - \boldsymbol{\theta}_n)^\top \widehat{\mathbf{U}}_n + \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_n)^\top \widehat{\mathbf{H}}_n (\boldsymbol{\theta} - \boldsymbol{\theta}_n)$$

with

$$\widehat{\mathbf{U}}_n = \tbinom{n}{2}^{-1}\sum_{i<j} \frac{\partial}{\partial\boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta})\Big|_{\boldsymbol{\theta}_n}, \qquad \widehat{\mathbf{H}}_n = \tbinom{n}{2}^{-1}\sum_{i<j} \frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta})\Big|_{\boldsymbol{\theta}_n}.$$

First-Order Representation. For $\|\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n\| = o_{\mathbb{P}}(1)$:

$$r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) = \underset{\mathbf{s}\in\Theta_n}{\arg\min}\, \widehat{M}_n(\boldsymbol{\theta}_n + \mathbf{s}/r_n) - \widehat{M}_n(\boldsymbol{\theta}_n) = r_n \widehat{\mathbf{H}}_n^{-1}\widehat{\mathbf{U}}_n$$

Distribution Theory: Hessian

$$r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) = r_n \widehat{\mathbf{H}}_n^{-1} \widehat{\mathbf{U}}_n$$

Hessian Approximation. Using the Hoeffding decomposition:

$$\widehat{\mathbf{H}}_n = \binom{n}{2}^{-1} \sum_{i<j} \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta}_n) = \mathbf{H}_n + \widehat{\mathcal{H}}_{1,n} + \widehat{\mathcal{H}}_{2,n}$$

where

$$\mathbf{H}_n = \mathbb{E}\Big[\tfrac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta}_n)\Big] \to \mathbf{H}_0,$$

$$\widehat{\mathcal{H}}_{1,n} = \frac{1}{n} \sum_{i=1}^{n} 2\Big(\mathbb{E}\big[\tfrac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta}_n) \big| \mathbf{z}_i\big] - \mathbf{H}_n\Big) = O_{\mathbb{P}}(\tfrac{1}{\sqrt{n}}),$$

$$\begin{aligned}\widehat{\mathcal{H}}_{2,n} &= \binom{n}{2}^{-1} \sum_{i<j} \Big(\tfrac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta}_n) - \mathbb{E}[\tfrac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta}_n)|\mathbf{z}_i] - \mathbb{E}[\tfrac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} m_{ij,n}(\boldsymbol{\theta}_n)|\mathbf{z}_j] + \mathbf{H}_n\Big) \\ &= O_{\mathbb{P}}(\tfrac{1}{\sqrt{n^2 h^d}}).\end{aligned}$$

Distribution Theory: Gaussian Approximation I

$$r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) = r_n \widehat{\mathbf{H}}_n^{-1} \widehat{\mathbf{U}}_n$$

Gaussian Approximation. Using the Hoeffding decomposition:

$$\widehat{\mathbf{U}}_n = \binom{n}{2}^{-1} \sum_{i<j} \frac{\partial}{\partial \boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) = \mathbf{U}_n + \mathcal{U}_{1,n} + \mathcal{U}_{2,n}$$

where

$$\mathbf{U}_n = \mathbb{E}\Big[\tfrac{\partial}{\partial \boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) \Big] = \mathbf{0},$$

$$\widehat{\mathcal{U}}_{1,n} = \frac{r_n}{n} \sum_{i=1}^{n} \frac{2}{h^d} \mathbb{E}\big[\tfrac{\partial}{\partial \boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) \big| \mathbf{z}_i \big] = O_{\mathbb{P}}(\tfrac{r_n}{\sqrt{n}}),$$

$$\begin{aligned}\widehat{\mathcal{U}}_{2,n} &= r_n \binom{n}{2}^{-1} \sum_{i<j} \Big(\tfrac{\partial}{\partial \boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) - \mathbb{E}[\tfrac{\partial}{\partial \boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) | \mathbf{z}_i] - \mathbb{E}[\tfrac{\partial}{\partial \boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) | \mathbf{z}_j] \Big) \\ &= O_{\mathbb{P}}(\tfrac{r_n}{\sqrt{n^2 h^d}}).\end{aligned}$$

Distribution Theory: Gaussian Approximation II

$$r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) = r_n \widehat{\mathbf{H}}_n^{-1} \widehat{\mathbf{U}}_n, \qquad \mathbb{V}[\widehat{\mathbf{U}}_n] = \mathbb{V}[\widehat{\mathcal{U}}_{1,n}] + \mathbb{V}[\widehat{\mathcal{U}}_{2,n}] \approx \frac{1}{n}\boldsymbol{\Sigma}_0 + \frac{1}{n}\frac{2}{nh^d}\boldsymbol{\Delta}_0$$

Gaussian Approximation. For $nh^d \to \kappa \in (0,\infty] \implies r_n = \sqrt{n}$,

$$\sqrt{n}\,\widehat{\mathbf{U}}_n = \sqrt{n}\mathcal{U}_{1,n} + \sqrt{n}\mathcal{U}_{2,n} \rightsquigarrow \mathsf{N}\Big(\mathbf{0}, \boldsymbol{\Sigma}_0 + \frac{2}{\kappa}\boldsymbol{\Delta}_0\Big)$$

because

$$\sqrt{n}\,\widehat{\mathcal{U}}_{1,n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{2}{h^d}\mathbb{E}\big[\tfrac{\partial}{\partial\boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n)\big|\mathbf{z}_i\big] \rightsquigarrow \mathsf{N}(\mathbf{0}, \boldsymbol{\Sigma}_0),$$

$$\begin{aligned}\sqrt{n}\,\widehat{\mathcal{U}}_{2,n} &= \sqrt{n}\tbinom{n}{2}^{-1}\sum_{i<j}\Big(\tfrac{\partial}{\partial\boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n) - \mathbb{E}[\tfrac{\partial}{\partial\boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n)|\mathbf{z}_i] - \mathbb{E}[\tfrac{\partial}{\partial\boldsymbol{\theta}} m_{ij,n}(\boldsymbol{\theta}_n)|\mathbf{z}_j]\Big)\\ &\rightsquigarrow \mathsf{N}\Big(\mathbf{0}, \frac{2}{\kappa}\boldsymbol{\Delta}_0\Big)\end{aligned}$$

- Distribution theory for $(\widehat{\mathcal{U}}_{1,n}, \widehat{\mathcal{U}}_{2,n})$ follows from martingale CLT.
- We need: $n^2h^d \to \infty$.

Distribution Theory: Smooth Case

$$r_n(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) = r_n \widehat{\mathbf{H}}_n^{-1} \widehat{\mathbf{U}}_n.$$

$$\widehat{\mathbf{H}}_n = \mathbf{H}_n + O_{\mathbb{P}}(n^{-1/2} + n^{-1} h^{d/2}) \to_{\mathbb{P}} \mathbf{H}_0.$$

$$r_n \widehat{\mathbf{U}}_n \approx_d \mathsf{N}\Big(\mathbf{0}, \frac{r_n^2}{n} \boldsymbol{\Sigma}_0 + \frac{1}{n} \frac{r_n^2}{nh^d} \boldsymbol{\Delta}_0\Big).$$

Gaussian Approximation. If $\|\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n\| = o_{\mathbb{P}}(1)$ and $n^2 h^d \to \infty$, then

$$\mathbf{V}_n^{-1/2}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) \rightsquigarrow \mathsf{N}(\mathbf{0}, \mathbf{I}_d)$$

where

$$\mathbf{V}_n = \mathbf{H}_0^{-1} \Big[\frac{1}{n} \boldsymbol{\Sigma}_0 + \binom{n}{2}^{-1} \frac{1}{h^d} \boldsymbol{\Delta}_0 \Big] \mathbf{H}_0^{-1}.$$

- If $nh^d \to \infty$, the classical result: Asymptotic linear representation.
- If $n^2 h^d \to \infty$, the small bandwidth asymptotics: Robust (to h) distribution theory.

Recap and Outstanding Issues

Gaussian Approximation. If $\|\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n\| = o_{\mathbb{P}}(1)$, and

$$n^2 h^d \to \infty \qquad \text{and} \qquad n \min\{1, nh^d\} h^{2P} \to 0,$$

then

$$\mathbf{V}_n^{-1/2}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \rightsquigarrow \mathsf{N}(\mathbf{0}, \mathbf{I}_d)$$

where

$$\mathbf{V}_n = \mathbf{H}_0^{-1}\Big[\frac{1}{n}\boldsymbol{\Sigma}_0 + \binom{n}{2}^{-1}\frac{1}{h^d}\boldsymbol{\Delta}_0\Big]\mathbf{H}_0^{-1}.$$

Outstanding Issues.

- How to handle stochastically differentiable $m_{ij}(\cdot)$?
  - Answer: empirical process theory for U-processes. (Technically tedious.)
  - Important: $\mathbf{H}_0$, $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Delta}_0$ can be function of nuisance functions. ($\approx$ quantile regression.)
  - Thus, plug-in estimates are possible but not advisable if avoidable...
- How to conduct robust inference in general?
  - Answer: careful application of the bootstrap.

Outline

Nonparametric Bootstrap

Estimator: $\mathbf{z}_i = (y_i, \mathbf{x}_i, \mathbf{w}_i)^\top$, $i = 1, \ldots, n$, random sample + regularity conditions,

$$\widehat{\boldsymbol{\theta}}_n = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n(\boldsymbol{\theta}), \qquad \widehat{M}_n(\boldsymbol{\theta}) = \tbinom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \tfrac{1}{h} K\big(\tfrac{\mathbf{w}_i - \mathbf{w}_j}{h}\big).$$

Gaussian Approximation: if $\|\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n\| = o_{\mathbb{P}}(1)$, $n^2 h^d \to \infty$, and $n \min\{1, nh^d\} h^{2P} \to 0$,

$$\mathbf{V}_n^{-1/2}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \rightsquigarrow \mathsf{N}(\mathbf{0}, \mathbf{I}_d)$$

where

$$\mathbf{V}_n = \mathbf{H}_0^{-1} \Big[\frac{1}{n} \boldsymbol{\Sigma}_0 + \binom{n}{2}^{-1} \frac{1}{h^d} \boldsymbol{\Delta}_0 \Big] \mathbf{H}_0^{-1}.$$

Bootstrap: $\mathbf{z}_i^* = (y_i^*, \mathbf{x}_i^*, \mathbf{w}_i^*)^\top$, $i = 1, \ldots, n$, random sample + regularity conditions.

$$\widehat{\boldsymbol{\theta}}_n^* = \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n^*(\boldsymbol{\theta}), \qquad \widehat{M}_n^*(\boldsymbol{\theta}) = \tbinom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i^*, \mathbf{z}_j^*; \boldsymbol{\theta}) \tfrac{1}{h} K\big(\tfrac{\mathbf{w}_i^* - \mathbf{w}_j^*}{h}\big).$$

Distribution Theory: Bootstrap

$$\widehat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \widehat{M}_n(\boldsymbol{\theta}), \qquad \widehat{M}_n(\boldsymbol{\theta}) = \tbinom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta}) \tfrac{1}{h} K\big(\tfrac{\mathbf{w}_i - \mathbf{w}_j}{h}\big).$$

$$\widehat{\boldsymbol{\theta}}^*_n = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, \widehat{M}^*_n(\boldsymbol{\theta}), \qquad \widehat{M}^*_n(\boldsymbol{\theta}) = \tbinom{n}{2}^{-1} \sum_{i<j} m(\mathbf{z}^*_i, \mathbf{z}^*_j; \boldsymbol{\theta}) \tfrac{1}{h} K\big(\tfrac{\mathbf{w}^*_i - \mathbf{w}^*_j}{h}\big).$$

Gaussian Approximation: if $\|\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n\| = o_{\mathbb{P}}(1)$, $n^2 h^d \to \infty$, and $n \min\{1, nh^d\} h^{2P} \to 0$,

$$\mathbf{V}_n^{-1/2}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \rightsquigarrow \mathsf{N}(\mathbf{0}, \mathbf{I}_d)$$

where

$$\mathbf{V}_n = \mathbf{H}_0^{-1}\Big[\frac{1}{n}\boldsymbol{\Sigma}_0 + \binom{n}{2}^{-1}\frac{1}{h^d}\boldsymbol{\Delta}_0\Big]\mathbf{H}_0^{-1}.$$

Bootstrap Approximation: if $\|\widehat{\boldsymbol{\theta}}^*_n - \widehat{\boldsymbol{\theta}}_n\| = o_{\mathbb{P}}(1)$ and $n^2 h^d \to \infty$,

$$\mathbf{V}_n^{*\,-1/2}(\widehat{\boldsymbol{\theta}}^*_n - \widehat{\boldsymbol{\theta}}_n) \rightsquigarrow_{\mathbb{P}} \mathsf{N}(\mathbf{0}, \mathbf{I}_d)$$

where

$$\mathbf{V}^*_n = \mathbf{H}_0^{-1}\Big[\frac{1}{n}\boldsymbol{\Sigma}_0 + 3\binom{n}{2}^{-1}\frac{1}{h^d}\boldsymbol{\Delta}_0\Big]\mathbf{H}_0^{-1}.$$

Distribution Theory: Bootstrap Validity

$$\widehat{\boldsymbol{\theta}}_n(h) = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\,\widehat{M}_n(\boldsymbol{\theta}; h), \qquad \widehat{M}_n(\boldsymbol{\theta}; h) = \binom{n}{2}^{-1}\sum_{i<j} m(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\theta})\frac{1}{h}K\Big(\frac{\mathbf{w}_i - \mathbf{w}_j}{h}\Big).$$

$$\widehat{\boldsymbol{\theta}}^*_n(h) = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\,\widehat{M}^*_n(\boldsymbol{\theta}; h), \qquad \widehat{M}^*_n(\boldsymbol{\theta}; h) = \binom{n}{2}^{-1} m(\mathbf{z}^*_i, \mathbf{z}7^*_j; \boldsymbol{\theta})\frac{1}{h}K\Big(\frac{\mathbf{w}^*_i - \mathbf{w}^*_j}{h}\Big).$$

$$\mathbf{V}_n(h)^{-1/2}(\widehat{\boldsymbol{\theta}}_n(h) - \boldsymbol{\theta}_0) \rightsquigarrow \mathsf{N}(\mathbf{0}, \mathbf{I}_d), \qquad \mathbf{V}_n(h) = \mathbf{H}_0^{-1}\Big[\frac{1}{n}\boldsymbol{\Sigma}_0 + \binom{n}{2}^{-1}\frac{1}{h^d}\boldsymbol{\Delta}_0\Big]\mathbf{H}_0^{-1}.$$

$$\mathbf{V}^*_n(h)^{-1/2}(\widehat{\boldsymbol{\theta}}^*_n(h) - \widehat{\boldsymbol{\theta}}_n(h)) \rightsquigarrow_{\mathbb{P}} \mathsf{N}(\mathbf{0}, \mathbf{I}_d), \qquad \mathbf{V}^*_n(h) = \mathbf{H}_0^{-1}\Big[\frac{1}{n}\boldsymbol{\Sigma}_0 + 3\binom{n}{2}^{-1}\frac{1}{h^d}\boldsymbol{\Delta}_0\Big]\mathbf{H}_0^{-1}.$$

Valid Bootstrap:

$$\widehat{\boldsymbol{\theta}}^*_n(3^{1/d}h) = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\,\widehat{M}^*_n(\boldsymbol{\theta}; 3^{1/d}h), \qquad \widehat{M}^*_n(\boldsymbol{\theta}; 3^{1/d}h) = \binom{n}{2}^{-1} m(\mathbf{z}^*_i, \mathbf{z}^*_j; \boldsymbol{\theta})\frac{1}{h}K\Big(\frac{\mathbf{w}^*_i - \mathbf{w}^*_j}{3^{1/d}h}\Big).$$

$$\mathbf{V}^*_n(3^{1/d}h)^{-1/2}(\widehat{\boldsymbol{\theta}}^*_n(3^{1/d}h) - \widehat{\boldsymbol{\theta}}_n(3^{1/d}h)) \rightsquigarrow_{\mathbb{P}} \mathsf{N}(\mathbf{0}, \mathbf{I}_d),$$

$$\mathbf{V}^*_n(3^{1/d}h)^{-1/2} = \mathbf{H}_0^{-1}\Big[\frac{1}{n}\boldsymbol{\Sigma}_0 + \binom{n}{2}^{-1}\frac{1}{h^d}\boldsymbol{\Delta}_0\Big]\mathbf{H}_0^{-1}.$$

Robust Bootstrap Inference

$$\begin{aligned}
\boldsymbol{\theta}_0 &= \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_0(\boldsymbol{\theta}), & M_0(\boldsymbol{\theta}) &= \lim_{n\to\infty} M_n(\boldsymbol{\theta}). \\
\boldsymbol{\theta}_n &= \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} M_n(\boldsymbol{\theta}), & M_n(\boldsymbol{\theta}) &= \mathbb{E}\Big[m(\mathbf{z}_1,\mathbf{z}_2;\boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_1-\mathbf{w}_2}{h}\big)\Big]. \\
\widehat{\boldsymbol{\theta}}_n(h) &= \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n(\boldsymbol{\theta};h), & \widehat{M}_n(\boldsymbol{\theta};h) &= \tbinom{n}{2}^{-1}\sum_{i<j} m(\mathbf{z}_i,\mathbf{z}_j;\boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i-\mathbf{w}_j}{h}\big). \\
\widehat{\boldsymbol{\theta}}_n^*(3^{1/d}h) &= \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\Theta} \widehat{M}_n^*(\boldsymbol{\theta};3^{1/d}h), & \widehat{M}_n^*(\boldsymbol{\theta};3^{1/d}h) &= \tbinom{n}{2}^{-1} m(\mathbf{z}_i^*,\mathbf{z}_j^*;\boldsymbol{\theta})\tfrac{1}{h}K\big(\tfrac{\mathbf{w}_i^*-\mathbf{w}_j^*}{3^{1/d}h}\big).
\end{aligned}$$

Robust ("Small Bandwidth Asymptotics") Confidence Intervals: $\alpha \in (0,1)$,

$$\mathrm{CI}_{1-\alpha}(h) = \Big[\widehat{\boldsymbol{\theta}}_n(h) - \mathfrak{c}^*_{1-\alpha/2}\ ,\ \widehat{\boldsymbol{\theta}}_n(h) - \mathfrak{c}^*_{\alpha/2}\Big]$$

$$\mathfrak{c}^*_\alpha = \inf\Big\{c\in\mathbb{R} : \mathbb{P}^*\big[\widehat{\boldsymbol{\theta}}_n^*(3^{1/d}h) - \widehat{\boldsymbol{\theta}}_n(3^{1/d}h) \le c\big] \ge \alpha\Big\}$$

Outline

Overview

- **Pairwise Difference Estimation** is used in economics and other disciplines.
- Rely on "localization" as determined by bandwidth h.
- Classical distributional approximations are sensitive to h.
- New distribution theory and bootstrap-based inference more robust to h.
  - Small bandwidth asymptotics.
- Results require stochastic differentiability of objective function.
- Upcoming research: discontinuous objective function.
  - Distribution theory: Mixture of "Small Bandwidth" Gaussian and Chernoff-type laws.
  - New way to conduct robust inference due to the non-Gaussian limit.