Adaptive Decision Tree Methods

Matias D. Cattaneo

Princeton University

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Talk based on:

- Cattaneo, Klusowski & Tian (2022, CKT): "On the Pointwise Behavior of Recursive Partitioning and Its Implications for Heterogeneous Causal Effect Estimation", `arXiv:2211.10805`.

- Cattaneo, Chandak & Klusowski (2022, CCK): "Convergence Rates of Oblique Regression Trees for Flexible Function Libraries", `arXiv:2210.14429`.

Outline

1. Introduction and Overview

2. Pointwise Inconsistency of Axis-Aligned Decision Trees

3. Mean-Square Optimality of Oblique Decision Trees

4. Takeaways

Introduction

Adaptive Decision Trees are widely used in academia and industry.

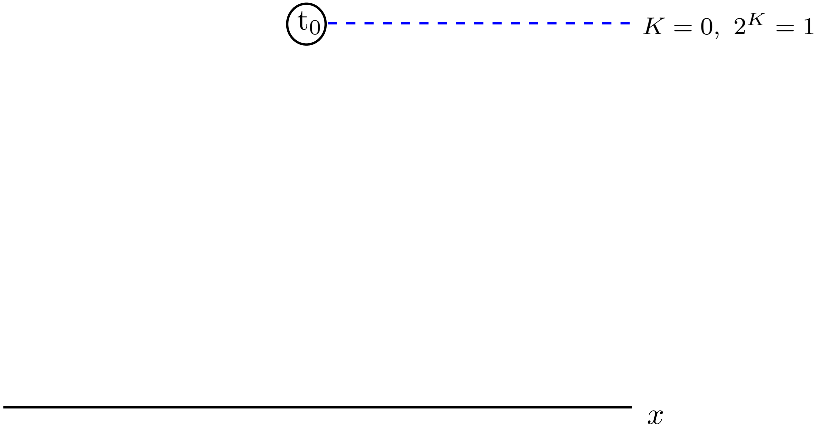
- CART: Breiman, Friedman, Olshen & Stone (1984).
- Adaptivity: incorporate data features in their construction.
- Popularity: prime example of "modern" machine learning toolkit.
- Preferred for interpretability or pointwise learning:

$$y_i = \mu(\mathbf{x}_i) + \varepsilon_i, \qquad \mathbb{E}[\varepsilon_i \mid \mathbf{x}_i] = 0, \qquad \mathbb{E}\big[\varepsilon_i^2 \mid \mathbf{x}_i\big] = \sigma^2(\mathbf{x}_i),$$

  where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ covariates supported on $\mathcal{X}$.

- Today: two foundational results for Adaptive Decision Trees.
  - Axis-aligned: pointwise inconsistent $\Longrightarrow$ uniformly inconsistent.
  - Oblique: mean square consistent $\Longleftrightarrow$ Single-hidden layer NN performance.

Adaptive Axis-Aligned Decision Tree (CART)



for each K : $$\min_{j=1,2,\cdots,p} \min_{\beta_1,\beta_2,\tau} \sum_{\mathbf{x}_i \in \mathbf{t}} \Big(y_i - \beta_1 \mathbb{1}(x_{ij} \leq \tau) - \beta_2 \mathbb{1}(x_{ij} > \tau)\Big)^2$$

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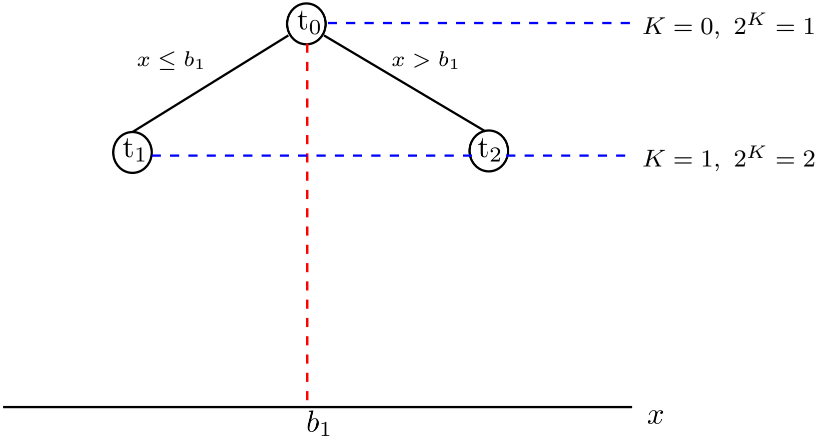

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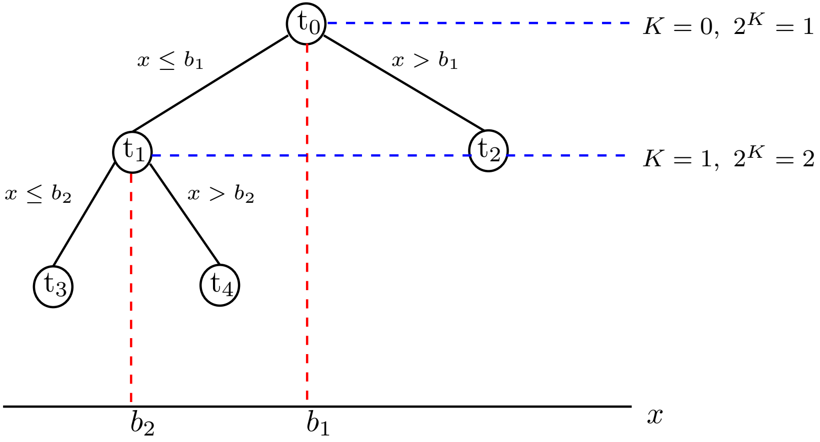

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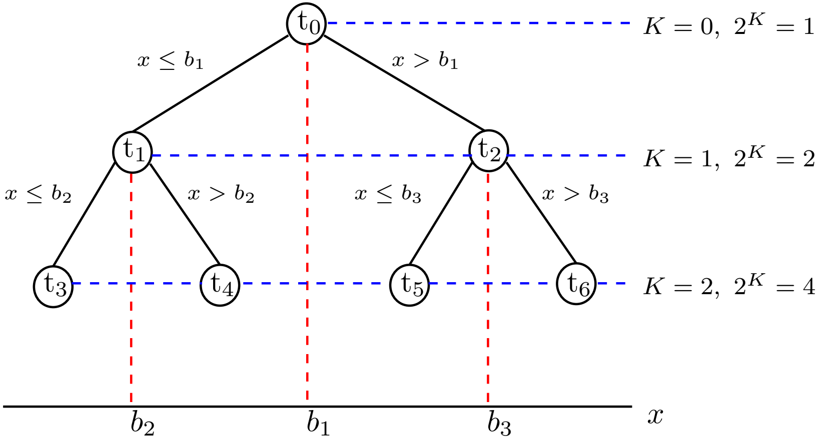

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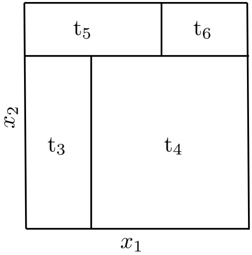
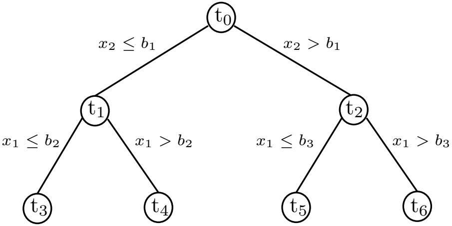




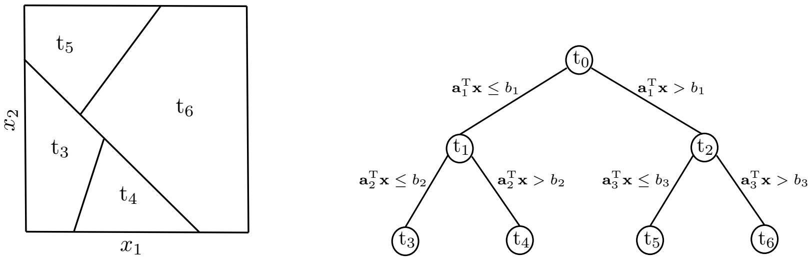
$$\widehat{\mu}(T_K)(\mathbf{x}) = \overline{y}_{\mathbf{t}} = \frac{1}{n(\mathbf{t})} \sum_{\mathbf{x}_i \in \mathbf{t}} y_i, \qquad n(\mathbf{t}) = \sum_{\mathbf{x}_i \in \mathbf{t}} \mathbb{1}(\mathbf{x}_i \in \mathbf{t}).$$

CKT (2022): for "honest" trees and $\mu(\mathbf{x}) = \mu$,

$$\mathbb{P}\Big(\sup_{\mathbf{x} \in \mathcal{X}} |\widehat{\mu}(T_K)(\mathbf{x}) - \mu| > C \Big) > C^2 \qquad \text{if} \quad K \gtrsim \log\log(n),$$

$$\mathbb{E}\big[\|\widehat{\mu}(T_K) - \mu\|^2\big] = \mathbb{E}\Big[\int_{\mathcal{X}} (\widehat{\mu}(T_K)(\mathbf{x}) - \mu)^2 \mathbb{P}_{\mathbf{x}}(d\mathbf{x}) \Big] \leq \frac{2^{K+1}\sigma^2}{n+1}.$$

Adaptive Oblique Decision Tree (OCART)



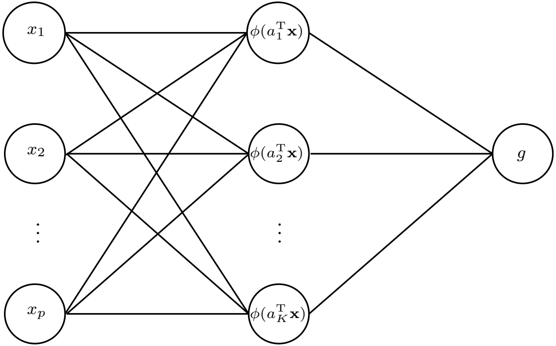
$$\widehat{\mu}(T_K)(\mathbf{x}) = \overline{y}_{\mathbf{t}} = \frac{1}{n(\mathbf{t})} \sum_{\mathbf{x}_i \in \mathbf{t}} y_i, \qquad n(\mathbf{t}) = \sum_{\mathbf{x}_i \in \mathbf{t}} \mathbb{1}(\mathbf{x}_i \in \mathbf{t}).$$

CCK (2022): for "full-sample" trees and $\mu \in$ Barron class,

$$\mathbb{E}\big[\|\widehat{\mu}(T_K) - \mu\|^2\big] \lesssim \frac{\|f\|_{\mathcal{L}_1}^2 \mathbb{E}\big[\max_{\mathbf{t}\in[T_K]} P_{\mathcal{A}_{\mathbf{t}}}^{-1}(\kappa)\big]}{\kappa K} + \frac{2^K d \log(np/d) \log^{4/\gamma}(n)}{n},$$

$$\mathbb{E}\big[\|\widehat{\mu}(T_{\mathrm{opt}}) - \mu\|^2\big] \lesssim \left(\frac{p \log^{4/\gamma+1}(n)}{n}\right)^{2/(2+q)} \approx \text{Optimal rate 1-HL NN.}$$

Single-Hidden Layer Neural Network with K Hidden Nodes



$$\text{OCART} \iff \phi(\cdot) = \text{ReLU}$$

- More generally, from the optimization community, feed-forward neural networks with Heaviside activations can be transformed into oblique decision trees with the same training error. See Bertsimas et al. (2018, 2021).

Outline

Motivation: Heterogeneous TE, Policy Decisions, Design RCTs, etc.

- $\{(y_i, \mathbf{x}_i', d_i) : i = 1, 2, \ldots, n\}$ i.i.d., and $y_i = y_i(1) \cdot d_i + y_i(0) \cdot (1 - d_i)$.

- CATE: $\quad \theta(\mathbf{x}) = \mathbb{E}[y_i(1) - y_i(0) \mid \mathbf{x}_i = \mathbf{x}]$.

- RCT: $\quad (y_i(0), y_i(1), \mathbf{x}_i^T) \perp\!\!\!\perp d_i$ and $\xi = \mathbb{P}(d_i = 1) \in (0, 1)$, so

$$\theta(\mathbf{x}_i) = \mathbb{E}[y_i \mid \mathbf{x}_i,\ d_i = 1] - \mathbb{E}[y_i \mid \mathbf{x}_i,\ d_i = 0] = \mathbb{E}\Big[y_i \frac{d_i - \xi}{\xi(1-\xi)} \mid \mathbf{x}_i\Big].$$

- *"Honest" Causal Decision Trees* (Athey and Imbens, 2019): adaptive tree T_K with sample splitting,

$$\hat{\theta}_{\mathrm{reg}}(T_K)(\mathbf{x}) = \frac{1}{\#\{\mathbf{x}_i \in \mathbf{t} : d_i = 1\}} \sum_{\mathbf{x}_i \in \mathbf{t} : d_i = 1} y_i - \frac{1}{\#\{\mathbf{x}_i \in \mathbf{t} : d_i = 0\}} \sum_{\mathbf{x}_i \in \mathbf{t} : d_i = 0} y_i$$

or

$$\hat{\theta}_{\mathrm{ipw}}(T_K)(\mathbf{x}) = \frac{1}{\#\{\mathbf{x}_i \in \mathbf{t}\}} \sum_{\mathbf{x}_i \in \mathbf{t}} y_i \frac{d_i - \xi}{\xi(1-\xi)},$$

where recall $\mathbf{t}$ denotes the unique (terminal) node containing $\mathbf{x} \in \mathcal{X}$.

Setup: Constant (Treatment Effect/Regression) Model

$$y_i = \mu(\mathbf{x}_i) + \varepsilon_i, \qquad \mathbb{E}[\varepsilon_i \mid \mathbf{x}_i] = 0, \qquad \mathbb{E}\big[\varepsilon_i^2 \mid \mathbf{x}_i\big] = \sigma^2(\mathbf{x}_i)$$

The following conditions hold.

1. $(y_i, \mathbf{x}_i')$, $i = 1, 2, \ldots, n$, is a random sample.
2. $\mu(\mathbf{x}) \equiv \mu$ is constant for all $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^p$.
3. $\mathbf{x}_i$ has a continuous distribution.
4. $\mathbf{x}_i \perp\!\!\!\perp \varepsilon_i$ for all $i = 1, 2, \ldots, n$.
5. $\mathbb{E}\big[|\varepsilon_i|^{2+\nu}\big] < \infty$ for some $\nu > 0$.

CKT (2022): axis-aligned adaptive (CART) decision trees.

1. Decision stumps ($K = 1$) split with high probability "near" the boundaries.
2. $\widehat{\mu}(T_1)(\mathbf{x})$ has at best polylog(n) convergence rate near boundaries.
3. "Honest" $\widehat{\mu}(T_K)(\mathbf{x})$ are uniformly inconsistent as soon as $K \gtrsim \log\log(n)$.
   - $n = 1$ billion implies depth $\log\log(n) \approx 3$.
   - Inconsistency occurs at countable many points on support, not just at boundaries.
4. Pruning does not solve the inconsistency.

Decision Stumps: polylog(n) Convergence Rate Near Boundaries

Recall: for each level K, adaptive (CART) decision trees solve

$$\min_{j=1,2,\cdots,p} \min_{\beta_1,\beta_2,\tau} \sum_{\mathbf{x}_i \in \mathbf{t}} \Big(y_i - \beta_1 \mathbb{1}(x_{ij} \leq \tau) - \beta_2 \mathbb{1}(x_{ij} > \tau) \Big)^2,$$

which is equivalent to maximizing the so-called *impurity gain*

$$\sum_{\mathbf{x}_l \in \mathbf{t}} (y_l - \mu)^2 - \sum_{\mathbf{x}_l \in \mathbf{t}} \big(y_l - \overline{y}_{\mathbf{t}_L} \mathbb{1}(x_{lj} \leq \tau) - \overline{y}_{\mathbf{t}_R} \mathbb{1}(x_{lj} > \tau) \big)^2$$
$$= \left(\frac{1}{\sqrt{i}} \sum_{l=1}^{i} (y_{[l]} - \mu) \right)^2 + \left(\frac{1}{\sqrt{n(\mathbf{t}) - i}} \sum_{l=i+1}^{n(\mathbf{t})} (y_{[l]} - \mu) \right)^2$$

with respect to index i and variable j, after reordering the data $\implies$ $(\hat{\imath}, \hat{\jmath})$.

- Darling-Erdös (1956) limit law (Berkes & Weber, 2006): for any non-decreasing function $1 \leq h(m) \leq m$ for which $\lim_{m\to\infty} h(m) = \infty$ and any $w \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{m/h(m) \leq i \leq m} \left| \frac{1}{\sqrt{i}} \sum_{l=1}^{i} (y_l - \mu) \right| < \lambda(h(m), w) \right) \to e^{e^{-w}}, \tag{1}$$

as $m \to \infty$, where $\lambda(\cdot, \cdot)$ is known.

Decision Stumps: polylog(n) Convergence Rate Near Boundaries

Careful study of maximum over different ranges of the split index give:

Theorem

For each $\gamma \in (0, 1/2]$ and $\delta \in (0, 1)$, there exists a constant $C = C(\gamma, \delta)$ and positive integer $N = N(\gamma, \delta)$ such that, for all $n \geq N$,

$$\mathbb{P}\left(\sup_{\mathbf{x} \in \mathcal{X}} |\hat{\mu}(T_1)(\mathbf{x}) - \mu| \geq C \sigma n^{-\gamma} \sqrt{\log \log(n)} \right) \geq 1 - \delta.$$

- Decision stumps <u>cannot</u> converge at a polynomial rate, i.e., its rate is slower than any polynomial-in-n.
- With arbitrary high probability, split index $\hat{\imath}$ will concentrate near its extremes, from the beginning of any tree construction.
- The first split generates cell containing, at most, $\log^a(n)$ observations, with probability at least $(\log(n))^{-b}$, up to constant factors.
- Too few observations will be available on one of the cells after the first split for CART to deliver a polynomial-in-n consistent estimator of μ.

"Honest" (Decision/Causal) Trees: Uniform Inconsistency

Iterating nearly inconsistent decision stumps can only make things worse... Thus, employing "honesty" (i.e., sample splitting), we have:

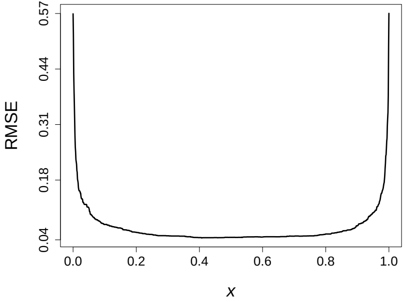
Theorem

Consider a maximal depth $K \gtrsim \log\log(n)$ tree T_K constructed with CART+ methodology. Then, there exists a positive constant Q and a positive integer N such that, for all $n \geq N$,

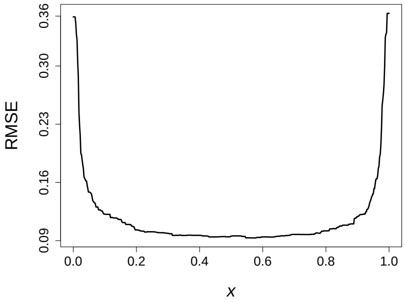
$$\mathbb{P}\left(\sup_{x \in \mathcal{X}} |\widehat{\mu}(T_K)(x) - \mu| > Q \right) > Q^2.$$

- Shallow "Honest" decision/causal trees are uniformly inconsistent.
- Inconsistency due to variance issue, not to boundary/misspecification bias.
- Inconsistency can occur at *countable* many points on the *entire* support $\mathcal{X}$.
- Pruning does not mitigate the inconsistency.
- Non-constant μ have similar problems: e.g., piecewise heterogeneity.

Simulations: Decision Stumps ($K = 1$) for Constant (Treatment) Model

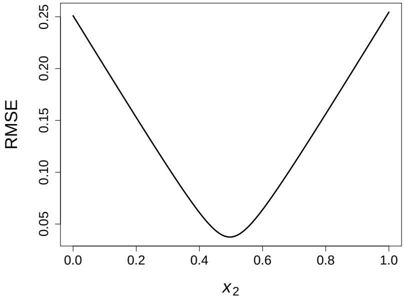


(a) Pointwise RMSE of decision stump.

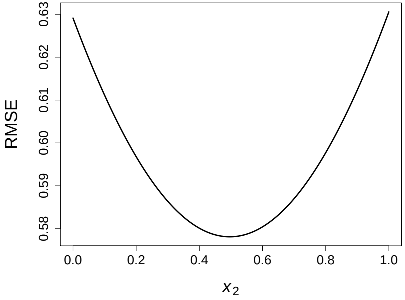


(b) Pointwise RMSE of causal decision stump.

Simulations: Decision Stumps ($K = 1$) with Pruning



(a) Pointwise RMSE for pruned tree at $\mathbf{x} = (0, x_2)^T$.



(b) Pointwise RMSE for pruned causal tree at $\mathbf{x} = (0, x_2)^T$.

Outline

Motivation

- Popular belief: decision trees compromise accuracy for being easy to use and understand, whereas neural networks are more accurate but less transparent.

- However, growing body of empirical work in optimization literature shows that certain trees are competitive with neural networks.

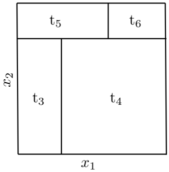
classification dataset	n	p	Number Classes	Decision Tree		2-Layer NN	
				DT depth	L2 Error	Width	L2 Error
Bank Marketing	45,211	17	2	3	89.6%	8	89.6%
Framingham Heart Study	3,658	15	2	2	83.3%	4	82.1%
Image Segmentation	210	18	7	4	88.4%	16	88.4%
Letter Recognition	20,000	16	26	8	68.2%	64	66.8%
Magic Gamma Telescope	19,020	10	2	4	86.7%	16	86.5%
Skin Segmentation	245,057	3	2	4	99.9%	16	99.9%
Thyroid Disease ANN	3,772	21	3	3	99.9%	8	97.7%

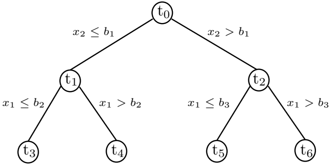
Table: Bertsimas et al., (2018)

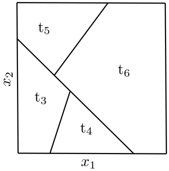
- **Question**: Is there a theoretical basis for this?

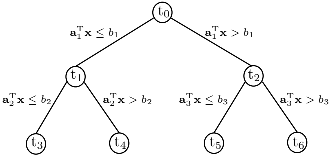
- Key advantages of binary (adaptive) decision trees:
  - Interpretability.
  - Connection to rule-based decision-making.
  - Mimics way doctor or business manager thinks.

Adaptive Axis-Aligned vs. Oblique Decision Tree (CART vs. OCART)









- Maximal decision trees with depth $K = 2$.
- OCART: splits occur along hyperplanes $\implies$ partitions are convex polytopes.

$$\widehat{\mu}(T_K)(\mathbf{x}) = \overline{y}_{\mathbf{t}} = \frac{1}{n(\mathbf{t})} \sum_{\mathbf{x}_i \in \mathbf{t}} y_i, \qquad n(\mathbf{t}) = \sum_{\mathbf{x}_i \in \mathbf{t}} \mathbb{1}(\mathbf{x}_i \in \mathbf{t}).$$

Oblique Tree Construction

- CART methodology: parent node $\mathbf{t}$ (region in $\mathbb{R}^p$) is divided into two child nodes, $\mathbf{t}_L$ and $\mathbf{t}_R$, by finding least squares decision stump

$$\psi(\mathbf{x}) = \beta_1 \mathbb{1}(\mathbf{a}'\mathbf{x} \leq b) + \beta_2 \mathbb{1}(\mathbf{a}'\mathbf{x} > b).$$

- Maximize decrease in sum-of-squares error

$$\widehat{\Delta}(b, \mathbf{a}, \mathbf{t}) = \sum_{\mathbf{x}_i \in \mathbf{t}} (y_i - \overline{y}_{\mathbf{t}})^2 - \sum_{\mathbf{x}_i \in \mathbf{t}} (y_i - \psi(\mathbf{x}_i))^2$$

  with respect to $(b, \mathbf{a})$.

- Greedy Refinement of Partition: Optimizers $(\hat{b}, \hat{\mathbf{a}})$ produce refinement of parent node $\mathbf{t}$ via child nodes

$$\mathbf{t}_L = \{\mathbf{x} \in \mathbf{t} : \hat{\mathbf{a}}'\mathbf{x} \leq \hat{b}\}, \quad \mathbf{t}_R = \{\mathbf{x} \in \mathbf{t} : \hat{\mathbf{a}}'\mathbf{x} > \hat{b}\}.$$

- Child nodes become new parent nodes at next level and can be further refined in same manner until desired depth D is reached.

Computational Challenges and Framework

- Challenging to find direction $\hat{\mathbf{a}}$ that minimizes squared error.
- Restrict search space to more tractable subset of candidate directions $\mathbf{a} \in \mathcal{A}_{\mathbf{t}}$ and allow slackness factor κ:

$$P_{\mathcal{A}_{\mathbf{t}}}(\kappa) = \mathbb{P}_{\mathcal{A}_{\mathbf{t}}}\left(\max_{(b,\mathbf{a}) \in \mathbb{R} \times \mathcal{A}_{\mathbf{t}}} \widehat{\Delta}(b, \mathbf{a}, \mathbf{t}) \geq \kappa \max_{(b,\mathbf{a}) \in \mathbb{R}^{1+p}} \widehat{\Delta}(b, \mathbf{a}, \mathbf{t}) \right)$$

- Choose meaningful method for generating $\mathcal{A}_{\mathbf{t}}$ so that $P_{\mathcal{A}_{\mathbf{t}}}(\kappa) \geq \rho > 0$, a.s.
  - **Deterministic**. Direct optimization, i.e., $\mathcal{A}_{\mathbf{t}} = \mathbb{R}^p$; solve least squares problem using mixed-integer linear optimization.
  - **Purely random**. Generate candidate directions $\mathcal{A}_{\mathbf{t}}$ uniformly at random (à la random forests).
  - **Data-driven**. Use dimension-reduction techniques on separate sample, e.g., $\mathcal{A}_{\mathbf{t}}$ defined in terms of top principle components produced by PCA or LDA, or, similarly, in terms of relevant variables selected by Lasso.

Function Class Approximations: 2-Layer NN vs. Tree Expansions

- **2-Layer Neural Networks**: distributed hierarchical representations

$$\left\{ g(\mathbf{x}) = \sum_k c_k \phi(\mathbf{a}_k' \mathbf{x}),\ c_k \in \mathbb{R},\ \mathbf{a}_k \in \mathbb{R}^p \right\}$$

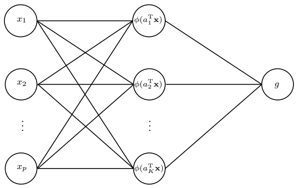
  Fixed activation function ϕ (e.g., ReLU).

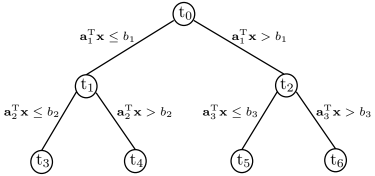
- **Decision Trees**:

$$\left\{ g(\mathbf{x}) = \sum_k c_k \mathbb{1}(\mathbf{x} \in \mathbf{t}_k) : c_k \in \mathbb{R},\ \mathbf{t}_k \text{ disjoint convex polytope} \right\}$$

  Regions $\mathbf{t}_k$ are determined by sequence of linear constraints, $\mathbf{a}'\mathbf{x} \leq b$ or $\mathbf{a}'\mathbf{x} > b$.

- Very different functional forms.





Three Key Assumptions

1. **Regularity**: Define norm of $\mu(\mathbf{x}) = \sum_k c_k \phi(\mathbf{a}_k' \mathbf{x})$ on region $\mathbf{t}$ by

$$\|\mu\|_{\mathcal{L}_1(\mathbf{t})} = \sum_k |c_k| \mathrm{V}_k(\mathbf{t}),$$

where $\mathrm{V}_k(\mathbf{t})$ is total variation of ϕ on interval $\left[\min_{\mathbf{x}\in\mathbf{t}} \mathbf{a}_k'\mathbf{x},\ \max_{\mathbf{x}\in\mathbf{t}} \mathbf{a}_k'\mathbf{x}\right]$.

   - Measures how much μ varies on region $\mathbf{t}$.
   - Example: If $\mu(\mathbf{x}) = \boldsymbol{\beta}'\mathbf{x}$, then $\|\mu\|_{\mathcal{L}_1([0,1]^p)} = \|\boldsymbol{\beta}\|_{\ell_1}$.

2. **Sparsity**: There exist $V > 0$ and $q > 2$ such that

$$\mathbb{E}\left[\sum_{\mathbf{t}\in T_K} \|\mu\|^q_{\mathcal{L}_1(\mathbf{t})}\right] \leq V^q$$

   - ℓ_q constraint on total variations of μ across all terminal nodes of tree.
   - Ensures compatibility between tree and ridge expansion.

3. **Node Size**: There exist $A = \mathrm{polylog}(n)$ and $\nu \geq 1 + 2/(q-2)$ such that

$$\left(\mathbb{E}\left[\left(\max_{\mathbf{t}\in T_K} n(\mathbf{t})\right)^{\nu}\right]\right)^{1/\nu} \leq \frac{An}{2^K}$$

   - No region contains disproportionately more observations than average $(n/2^K)$.
   - Allows for some regions to contain very few observations.

Expected Training / Prediction Error

- Training error of tree: $\|y - \widehat{\mu}(T_K)\|_n^2 = \frac{1}{n}\sum_i (y_i - \widehat{\mu}(T_K)(\mathbf{x}_i))^2$
- Prediction error of tree: $\|\mu - \widehat{\mu}(T_K)\|^2 = \int(\mu(\mathbf{x}) - \widehat{\mu}(T_K)(\mathbf{x}))^2 d\mathbb{P}_{\mathbf{x}}$

Theorem

For any depth $K \geq 1$, $\mathbb{E}\big[\|y - \widehat{\mu}(T_K)\|_n^2 - \|y - \mu\|_n^2\big] \leq 4^{-(K-1)/q}\frac{AV^2}{\rho\kappa}$.

Furthermore, if $K \approx \frac{q}{2+q}\log_2(n/p)$, *then*

$$\mathbb{E}\big[\|\widehat{\mu}(T_K) - \mu\|^2\big] \leq C\Big(\frac{p}{n}\Big)^{2/(2+q)}$$

Statistical Accuracy Comparisons:

- When $q \approx 2$, convergence rate

$$\Big(\frac{p}{n}\Big)^{2/(2+q)} \approx \Big(\frac{p}{n}\Big)^{1/2}$$

  same as for least squares neural network estimators (Barron, 1994).
- q plays role of effective dimension, not ambient dimension p.
- If μ is smooth, we expect $q \leq p$, and so convergence rate always at least as fast as minimax optimal rate: $(1/n)^{2/(2+p)}$.

Discussion

- **Pruned Tree**: same guarantees hold for pruned subtree that minimizes penalized risk
$$T_{\mathrm{opt}} \in \underset{T \preceq T_{\max}}{\arg\min} \left\{ \|y - \widehat{\mu}(T)\|_n^2 + \lambda |T| \right\},$$
where $\lambda \gtrsim p/n$. Optimal subtree T_{opt} can be found efficiently.

- **Key Technical Idea**: tree output $\widehat{\mu}(T_D)$ is orthogonal projection of y onto span of orthonormal functions $\psi_{\mathbf{t}} = \psi_{\mathbf{t}}(b, \mathbf{a})$. That is,
$$\widehat{\mu}(T_D)(\mathbf{x}) = \sum_{\mathbf{t} \in T_D} \langle y, \psi_{\mathbf{t}} \rangle \, \psi_{\mathbf{t}}(\mathbf{x}),$$
where $\langle y, \psi_{\mathbf{t}} \rangle = \frac{1}{n} \sum_i (y_i - \overline{y}_{\mathbf{t}}) \psi_{\mathbf{t}}(\mathbf{x}_i)$ is empirical inner product, maximized at least squares solution $(\hat{b}, \hat{\mathbf{a}})$.

- **Connections to Linear Regression**: similar to forward-stepwise regression. At each current decision node $\mathbf{t}$, tree is grown by selecting "feature", $\psi_{\mathbf{t}}$, most correlated with residuals, $y_i - \overline{y}_{\mathbf{t}}$, and adding chosen feature along with coefficient, $\langle y, \psi_{\mathbf{t}} \rangle$, to tree output:
$$\widehat{\mu}(T_{D+1})(\mathbf{x}) = \widehat{\mu}(T_D)(\mathbf{x}) + \langle y, \psi_{\mathbf{t}} \rangle \, \psi_{\mathbf{t}}(\mathbf{x}).$$

Outline

Takeaways

Adaptive Decision Trees are a leading component of the machine learning toolkit.

- Today: two foundational results for Adaptive Decision Trees.
  - Axis-aligned: pointwise inconsistent $\implies$ uniformly inconsistent.
  - Oblique: mean square consistent $\iff$ Single-hidden layer NN performance.
- Adaptive ML methods have advantages and disadvantages.
- Statistical and algorithmic implementations must be studied together.
- Mechanical implementations of machine learning can be detrimental.
- <u>Open question</u>: do other machine learning methods have similar problems?

References

Today:

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Further Reading:

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2. C, Farrell & Feng (2020): "Large Sample Properties of Partitioning-Based Series Estimators", *Annals of Statistics* 48(3): 1718-1741.

3. C, Crump, Farrell & Feng (2023): "On Binscatter", `arXiv:1902.09608`.

4. C, Crump, Farrell & Feng (2023): "Generalized Binscatter Methods", coming soon.

5. C, Klusowski & Underwoord (2023): "Estimation and Inference using Mondrian Random Forests", coming soon.

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