

# Binscatter Methods

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## References

- ▶ CCFF (2024): “Nonlinear Binscatter Methods”, working paper.
- ▶ CCFF (2024): “Binscatter Regression”, *Stata Journal*.
- ▶ CCFF (2024): “On Binscatter”, *American Economic Review*, forthcoming.

`https://nppackages.github.io/binsreg/`

# Outline

## 1. Introduction

## 2. Overview

## 3. Theoretical Contributions

## 4. Final Remarks

# Introduction

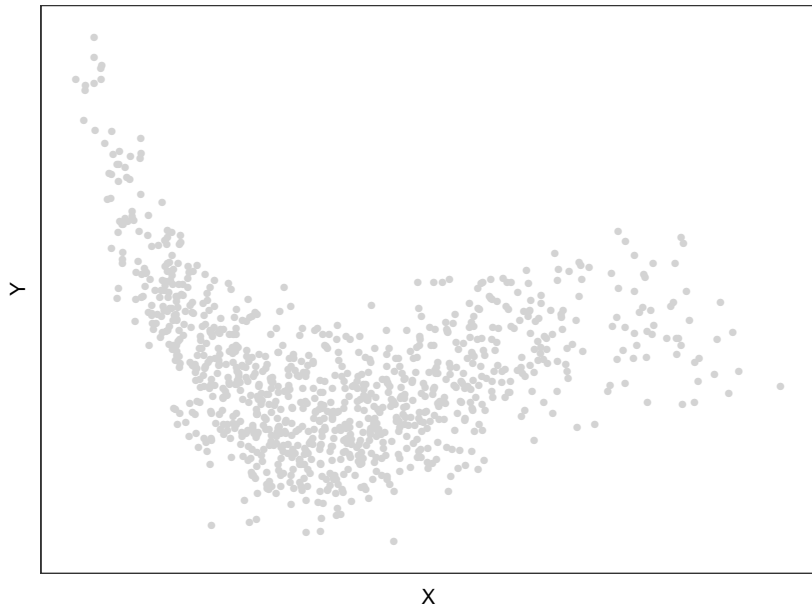
**Binscatter** is widely used in economics and other disciplines.

- ▶ Popularized by Chetty, Friedman, Rockoff, Saez, many others.
- ▶ Previous incarnations:
  - ▶ *Regressogram* (Tukey, 1961).
  - ▶ *Subclassification* (Cochran, 1968).
  - ▶ *Portfolio Sorting* (Fama, 1976).
  - ▶ *Regression Trees* (Friedman, 1977).
  - ▶ you tell me...
- ▶ Today: foundational, thorough study of Binscatter.
  - ▶ *Methodology*: guidance on valid and invalid current practices, and more.
  - ▶ *Theory*: novel strong approximation approach, and more.
  - ▶ *Practice*: new **Python**, **R** and **Stata** software (Binsreg package).

<https://nppackages.github.io/binsreg/>

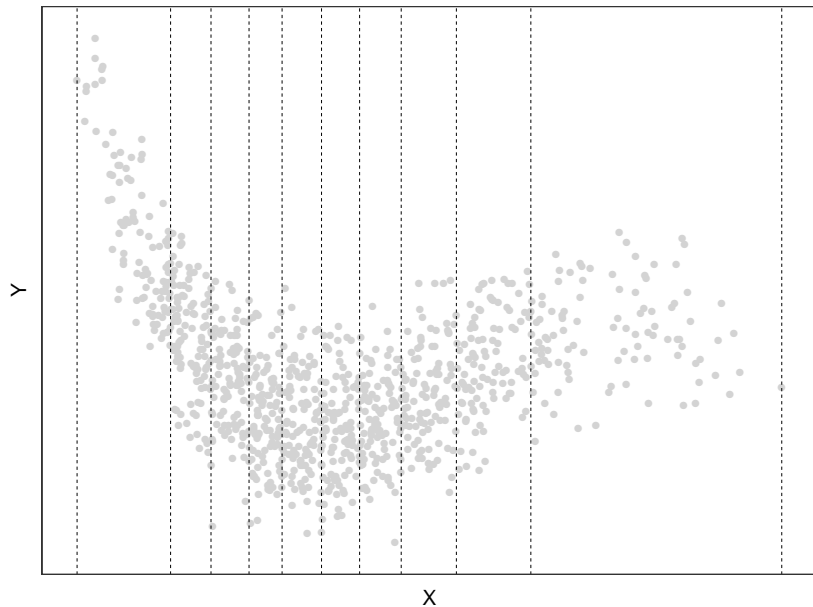
## What is a binned scatter plot?

**Step 1:** Start with a familiar scatter plot



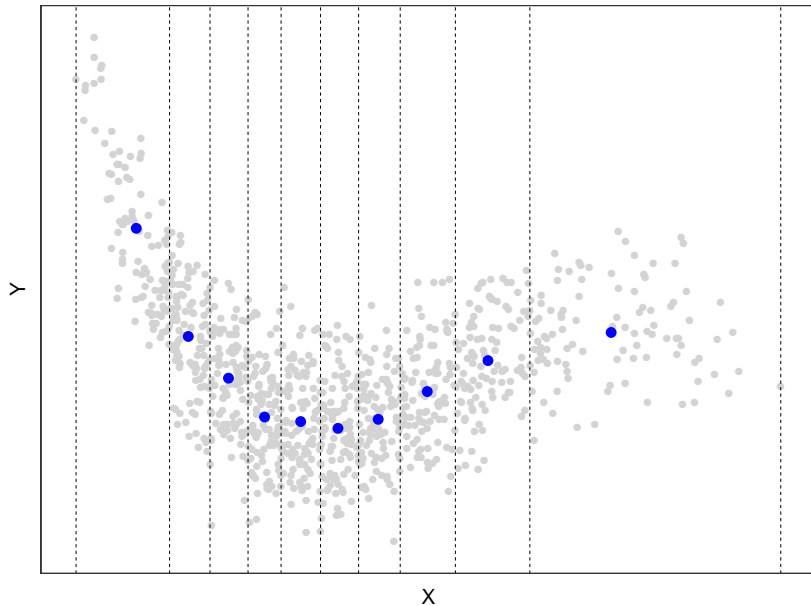
## What is a binned scatter plot?

**Step 2:** Partition the support of  $X$  into bins



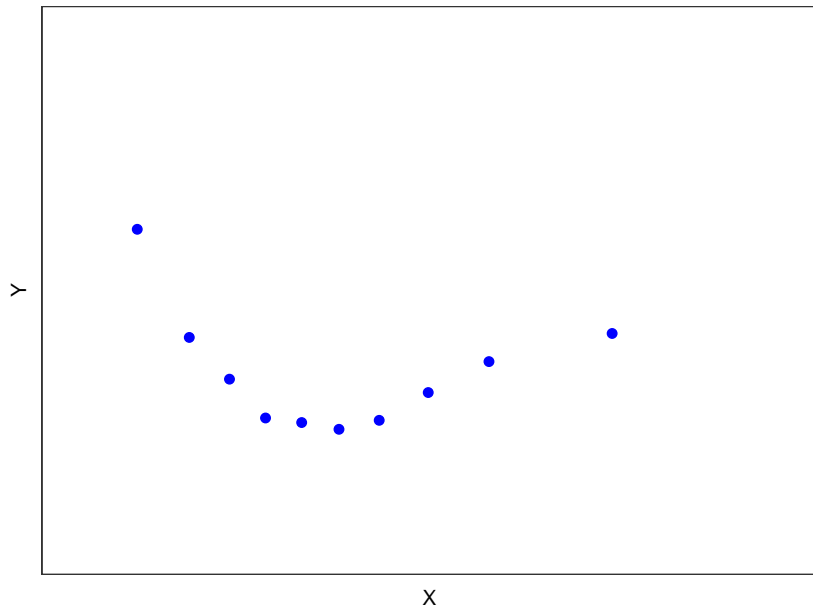
## What is a binned scatter plot?

**Step 3:** Find the average Y in each bin



## What is a binned scatter plot?

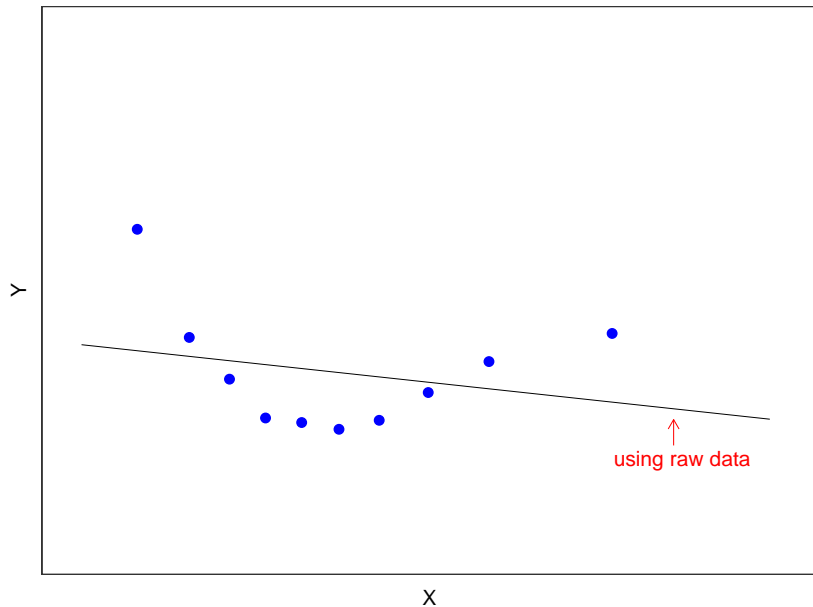
**Step 4:** Plot only bin means



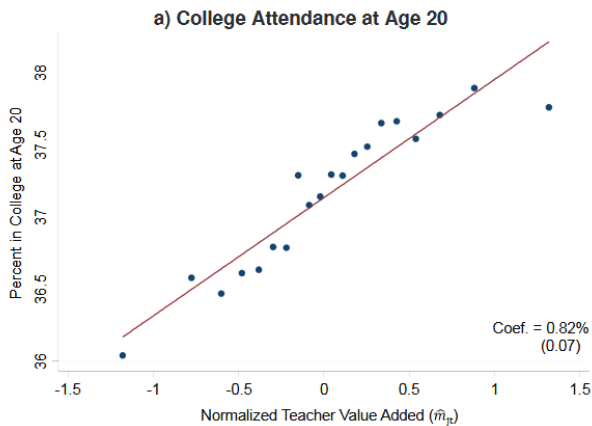


## What is a binned scatter plot?

**Step 5:** Add a polynomial fit to raw data



## Typical Example: Chetty, Friedman and Rockoff (2014, AER)



**Note:**  $n = 4,170,905$  with # of bins  $J = 20$

# Outline

1. Introduction

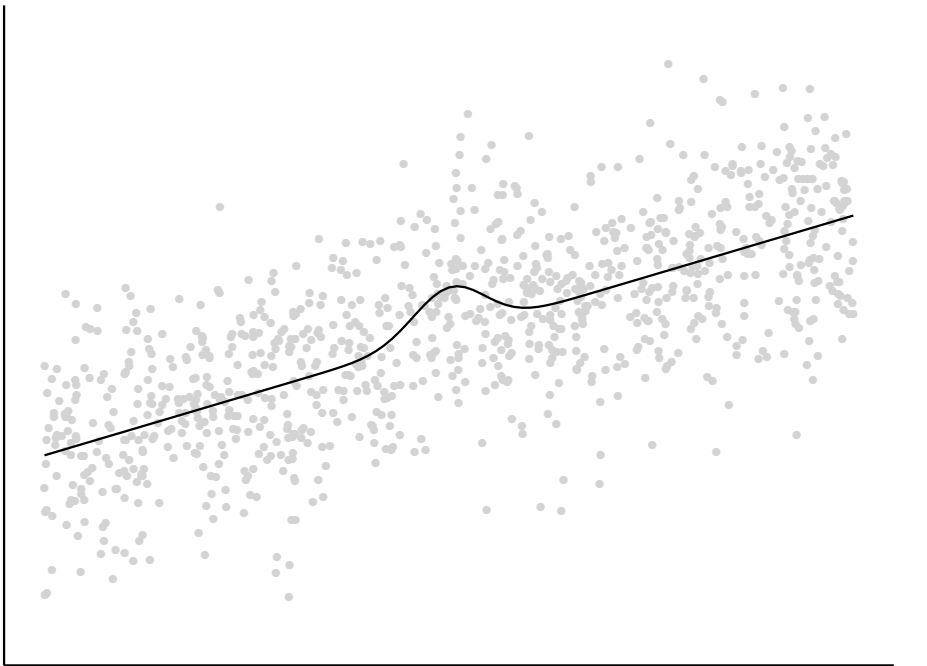
2. Overview

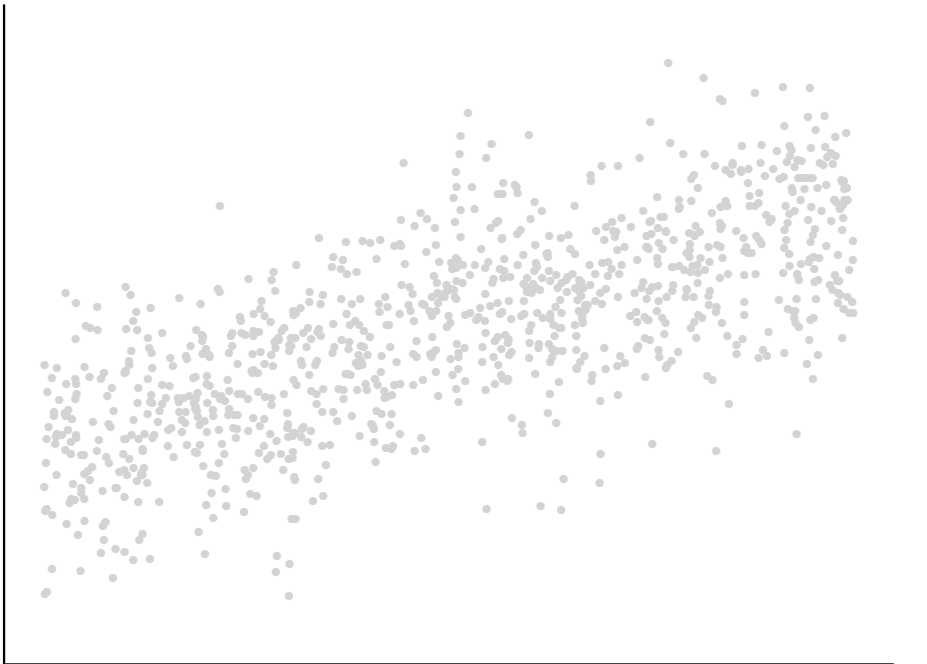
3. Theoretical Contributions

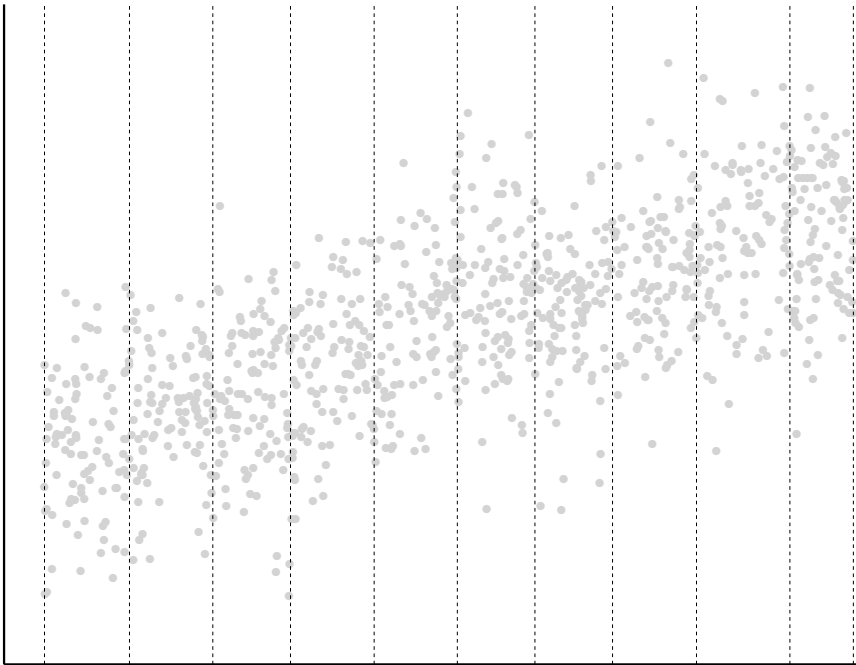
4. Final Remarks

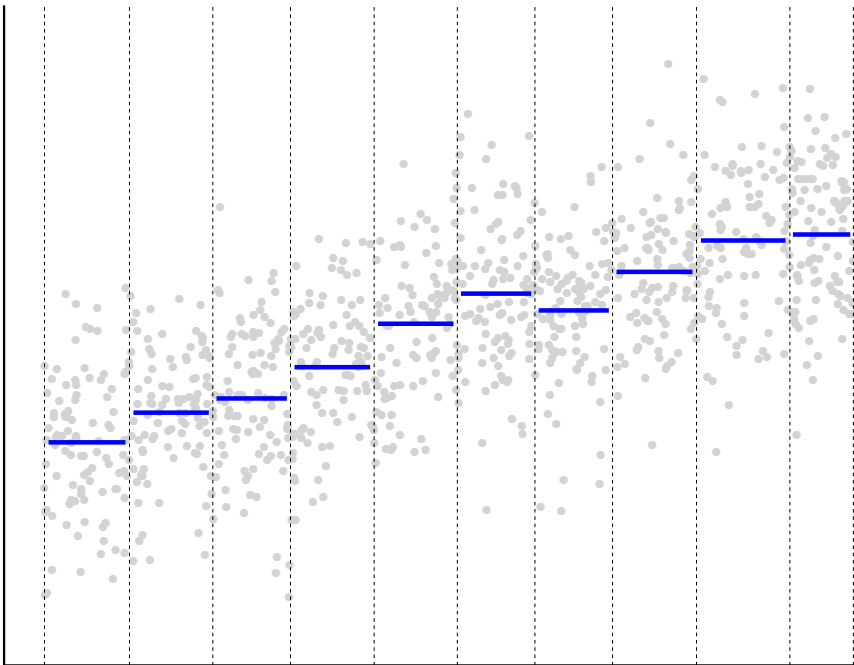
## Overview: Contributions

1. Set up formal, general framework for studying **Binscatter**.
  - ▶ *Respects practice*: quantile-spaced binning, covariate adjustment.
  - ▶ *Extensions*: higher-order polynomial, smoothness-restricted approximations.
  - ▶ *Generalizations*: semi-linear QMLE (quantiles, logistic, etc.).
2. IMSE-Optimal choice of binning structure.
3. Valid point estimators, confidence intervals, and confidence bands.
4. Valid hypothesis testing of parametric specification and shape restrictions.
5. Novel theoretical results specifically developed for binscatter.
6. **Python**, **R**, and **Stata** software resolving valid and invalid current practices.

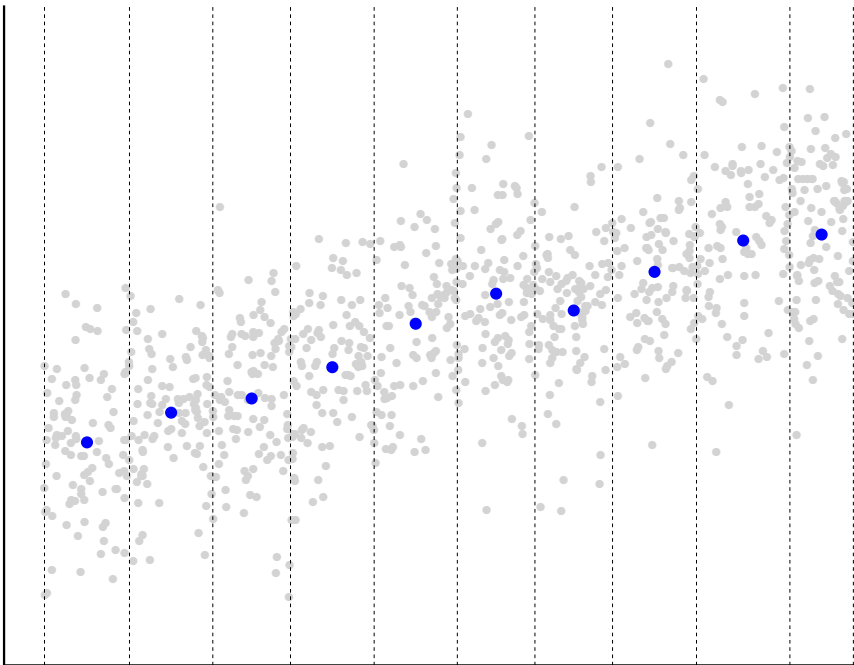


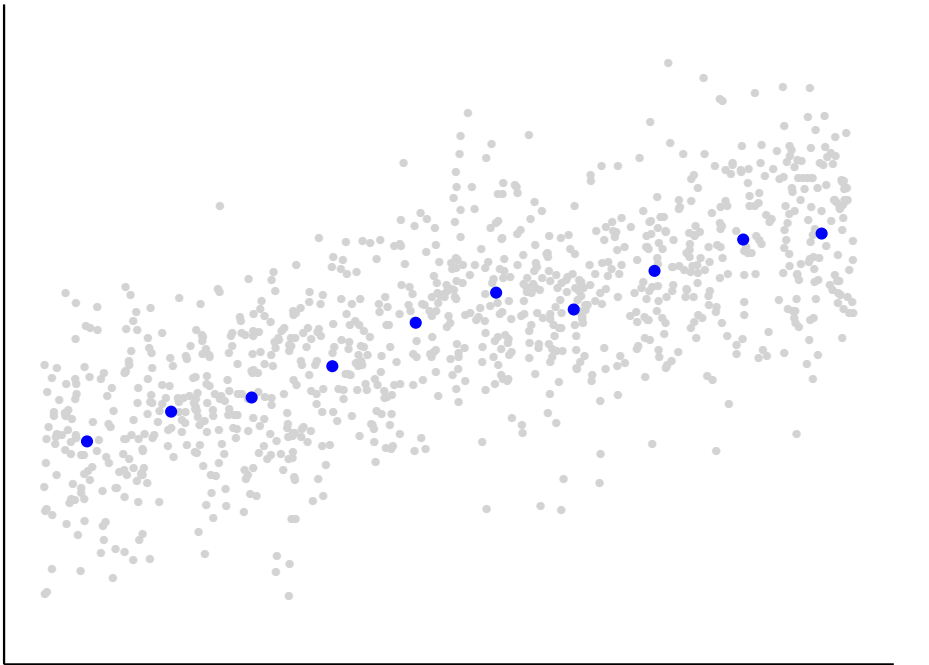


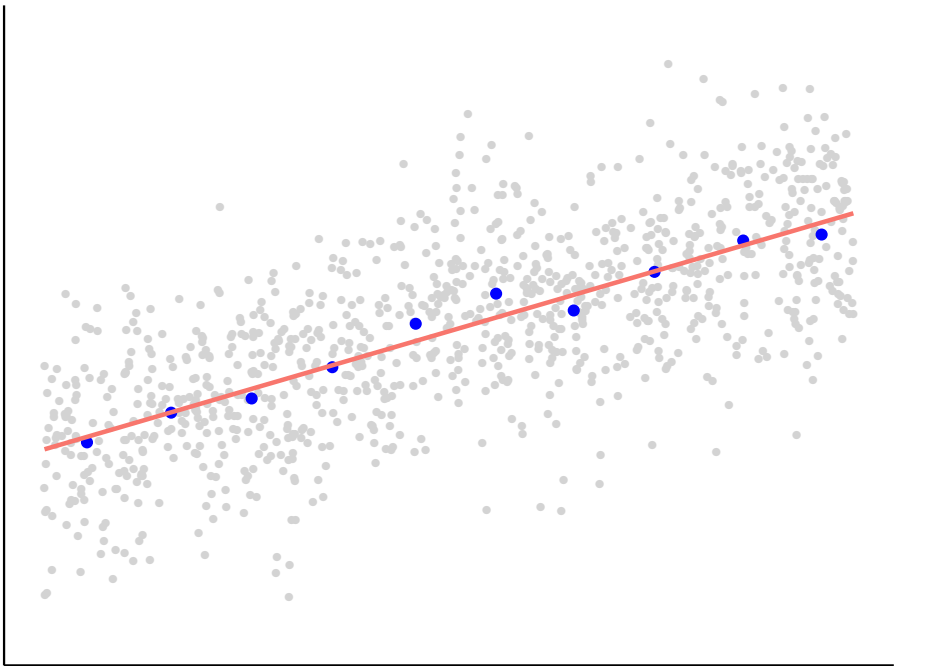


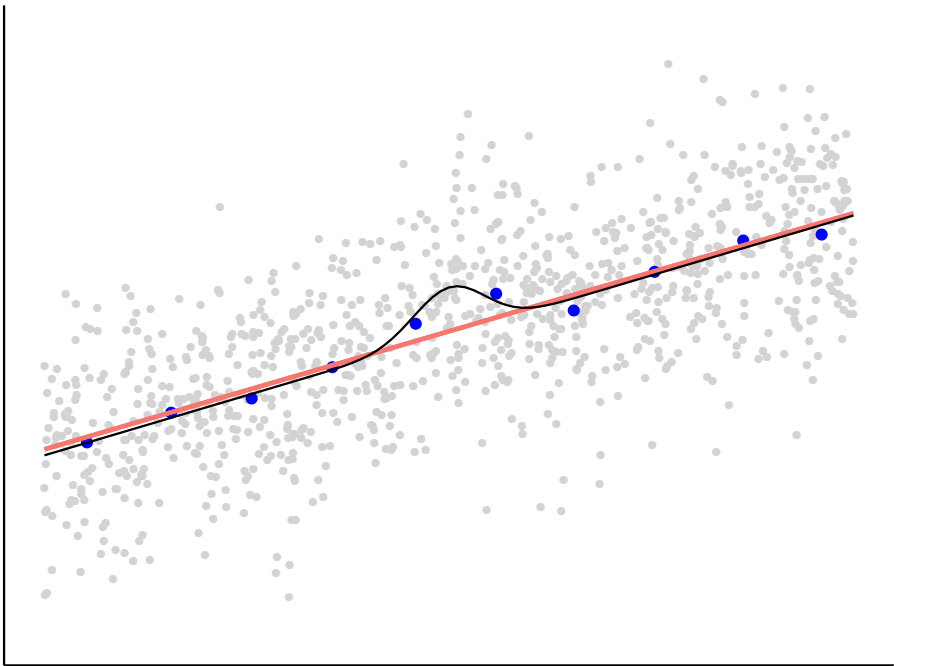


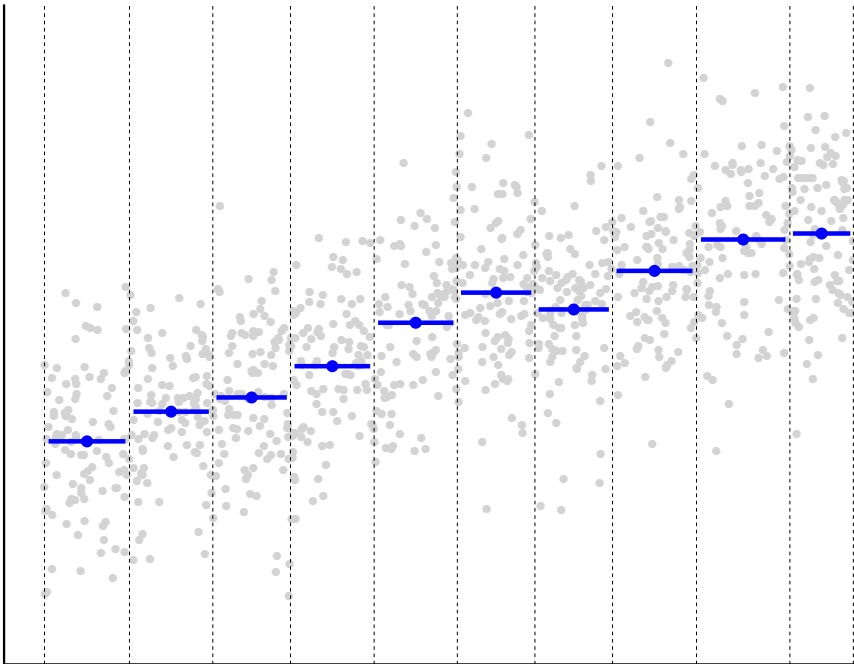












## Framework: Canonical Binscatter

$$y_i = \mu(x_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | x_i] = 0$$

**Binscatter:**

$$\hat{\mu}(x) = \hat{\mathbf{b}}(x)' \hat{\boldsymbol{\beta}}, \quad \hat{\boldsymbol{\beta}} = \arg \min_{\boldsymbol{\beta}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}(x_i)' \boldsymbol{\beta})^2$$

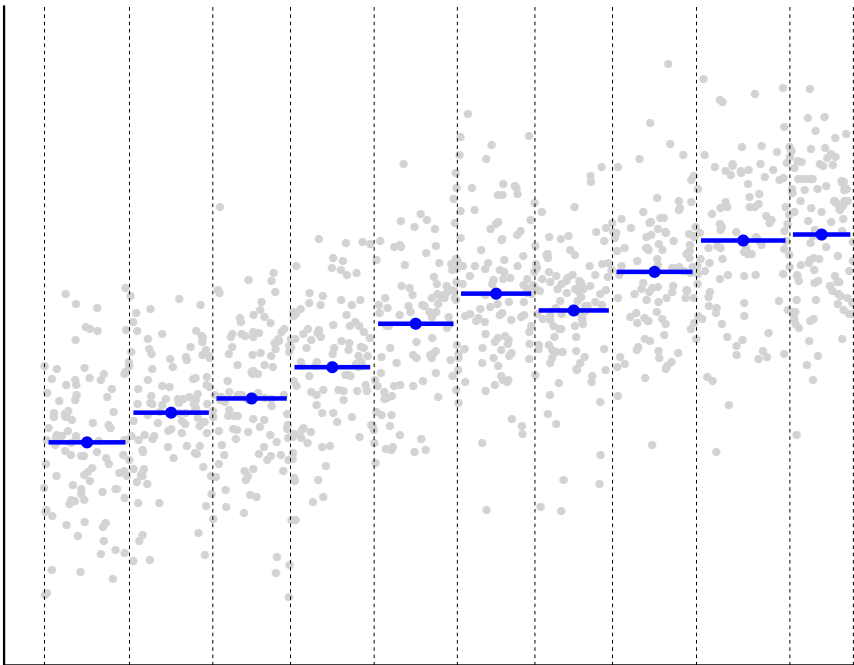
► Partitioning/Binning:

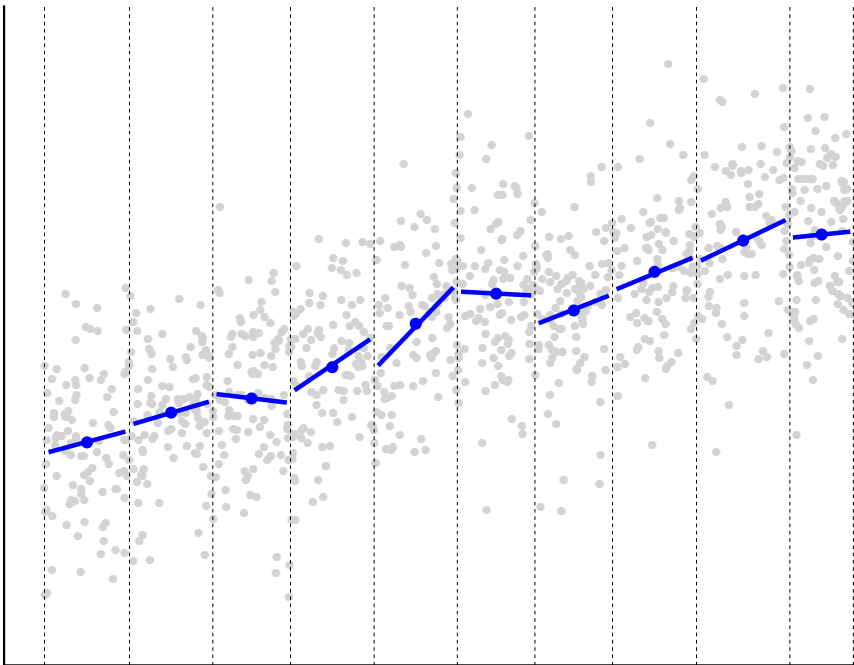
$$\hat{\Delta} = \{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}, \quad \hat{\mathcal{B}}_j = \begin{cases} [x_{(1)}, x_{(\lfloor n/J \rfloor)}] & \text{if } j = 1 \\ [x_{(\lfloor n(j-1)/J \rfloor)}, x_{(\lfloor nj/J \rfloor)}] & \text{if } j = 2, \dots, J-1 \\ [x_{(\lfloor n(J-1)/J \rfloor)}, x_{(n)}] & \text{if } j = J \end{cases}$$

► Within-Bin Constant Approximation:

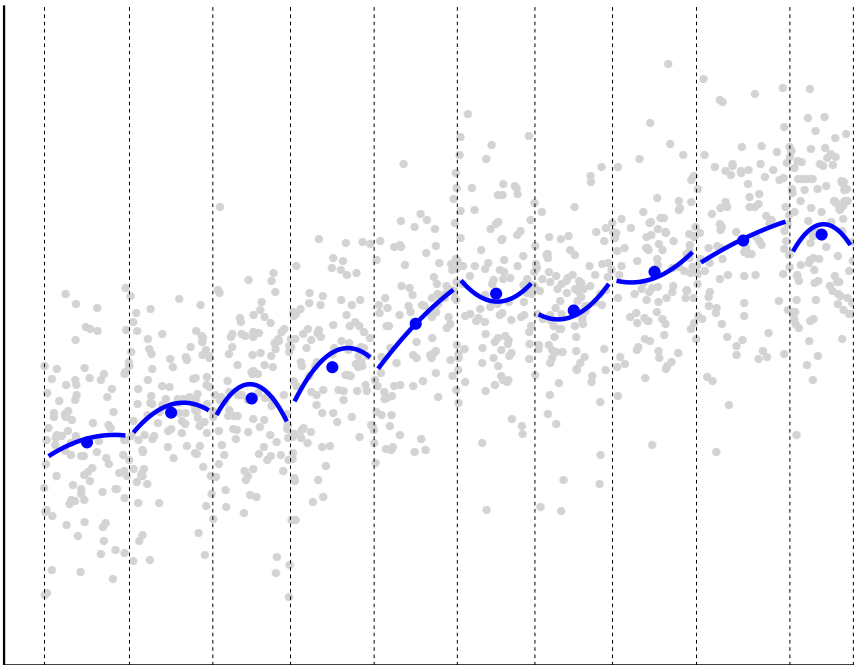
$$\hat{\mathbf{b}}(x) = [\mathbb{1}_{\hat{\mathcal{B}}_1}(x) \quad \mathbb{1}_{\hat{\mathcal{B}}_2}(x) \quad \cdots \quad \mathbb{1}_{\hat{\mathcal{B}}_J}(x)]'$$

► Dimension:  $J$ .









## Framework: Within-Bin Polynomial Approximation

$$y_i = \mu(x_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | x_i] = 0$$

**Binscatter:**

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \hat{\boldsymbol{\beta}} = \arg \min_{\boldsymbol{\beta}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}(x_i)' \boldsymbol{\beta})^2$$

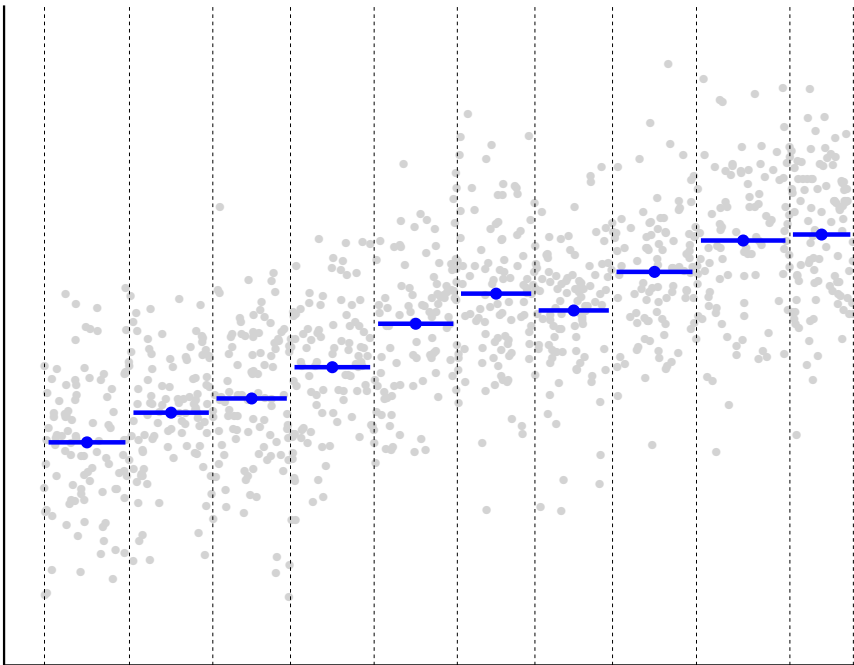
► Partitioning/Binning:  $\hat{\Delta} = \{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$ .

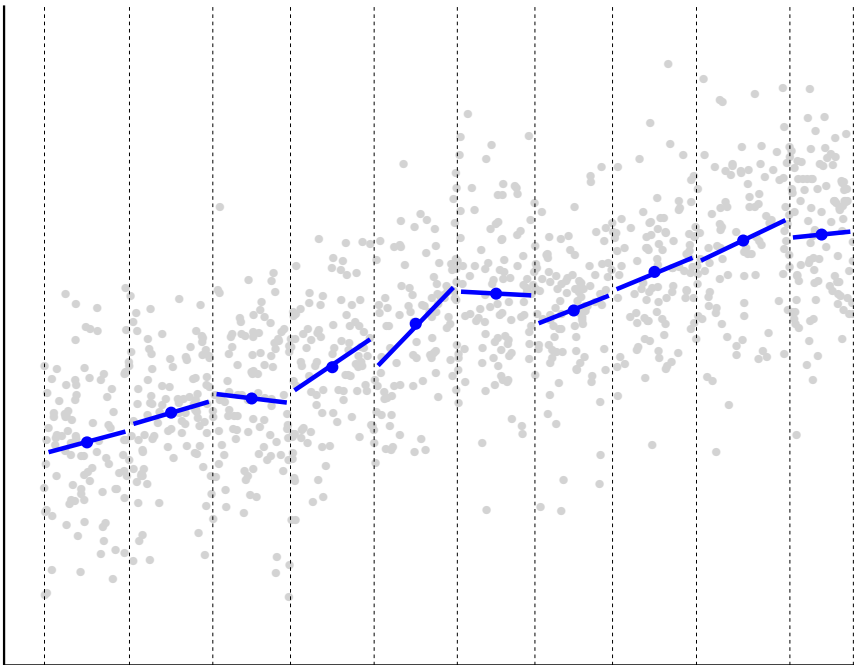
► Within-Bin Polynomial Approximation:

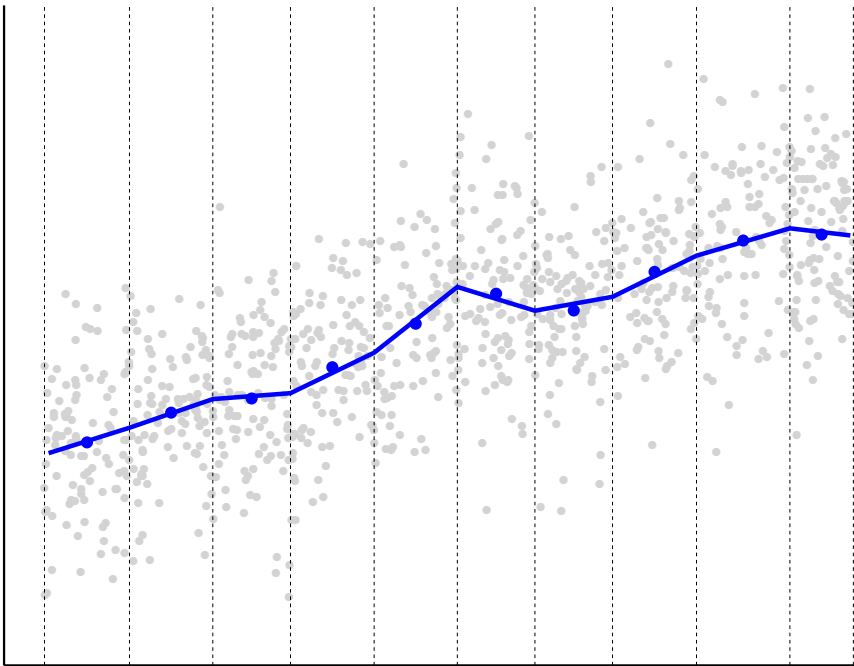
$$\hat{\mathbf{b}}(x) = \begin{bmatrix} \mathbb{1}_{\hat{\mathcal{B}}_1}(x) & \mathbb{1}_{\hat{\mathcal{B}}_2}(x) & \cdots & \mathbb{1}_{\hat{\mathcal{B}}_J}(x) \end{bmatrix}' \otimes \begin{bmatrix} 1 & x & \cdots & x^p \end{bmatrix}'$$

► Dimension:  $(p+1) \cdot J$ .

► Restrictions:  $0 \leq v \leq p$ .







## Framework: Across-Bins Smoothness Restriction

$$y_i = \mu(x_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | x_i] = 0$$

**Binscatter:**

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \hat{\boldsymbol{\beta}} = \arg \min_{\boldsymbol{\beta}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta})^2$$

► Partitioning/Binning:  $\hat{\Delta} = \{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$ .

► Across-Bins Smoothness Restriction:

$$\hat{\mathbf{b}}_s(x) = \hat{\mathbf{T}}_s \hat{\mathbf{b}}(x), \quad \hat{\mathbf{b}}(x) = [\mathbb{1}_{\hat{\mathcal{B}}_1}(x) \quad \cdots \quad \mathbb{1}_{\hat{\mathcal{B}}_J}(x)]' \otimes [1 \quad \cdots \quad x^p]'$$

► Dimension  $\hat{\mathbf{T}}_s$ :  $[(p+1)J - (J-1)s] \times (p+1)J$ .

► Restrictions:  $0 \leq s, v \leq p$ .

## Framework: Covariate Adjustment

$$y_i = \mu(x_i) + \mathbf{w}_i' \boldsymbol{\gamma} + \epsilon_i, \quad \mathbb{E}[\epsilon_i | x_i, \mathbf{w}_i] = 0$$

### Covariate-Adjusted Binscatter:

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} - \mathbf{w}_i' \boldsymbol{\gamma})^2$$

- Partitioning/Binning:  $\{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$  — Binscatter Basis:  $\hat{\mathbf{b}}_s(x)$ .
- Dimension:  $[(p+1)J - (J-1)s] + d$  — Restrictions:  $0 \leq s, v \leq p$ .

## Framework: Covariate Adjustment

$$y_i = \mu(x_i) + \mathbf{w}_i' \boldsymbol{\gamma} + \epsilon_i, \quad \mathbb{E}[\epsilon_i | x_i, \mathbf{w}_i] = 0$$

### Covariate-Adjusted Binscatter:

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} - \mathbf{w}_i' \boldsymbol{\gamma})^2$$

- Partitioning/Binning:  $\{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$  — Binscatter Basis:  $\hat{\mathbf{b}}_s(x)$ .
- Dimension:  $[(p+1)J - (J-1)s] + d$  — Restrictions:  $0 \leq s, v \leq p$ .

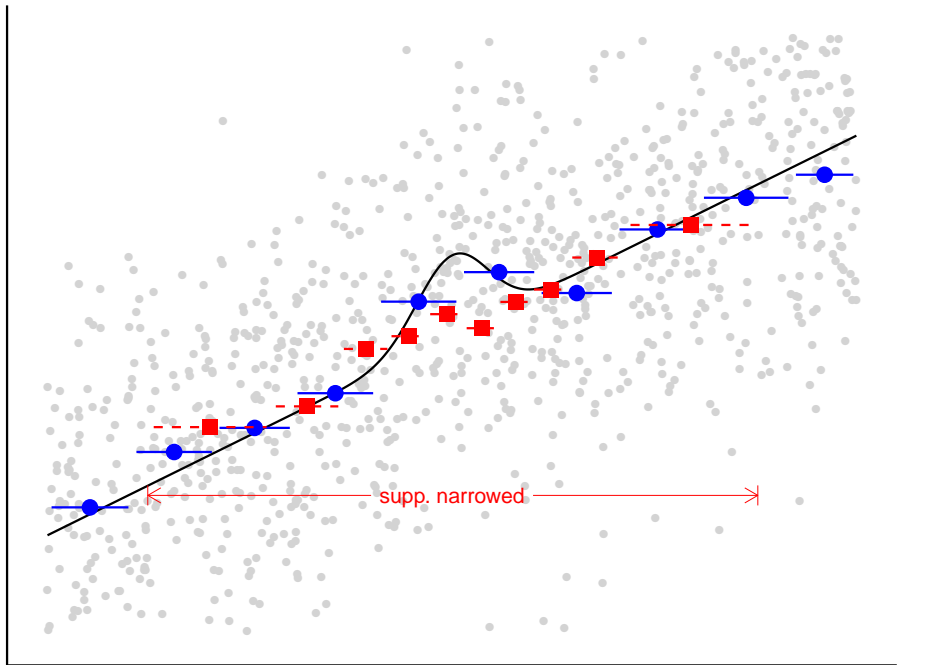
### Residualized Binscatter (a No, No!):

$$\tilde{\mu}(x) = \hat{\mathbf{b}}(x)' \tilde{\boldsymbol{\beta}}, \quad \tilde{\boldsymbol{\beta}} = \arg \min_{\boldsymbol{\beta}} \sum_{i=1}^n (\tilde{y}_i - \hat{\mathbf{b}}(\tilde{x}_i)' \boldsymbol{\beta})^2$$

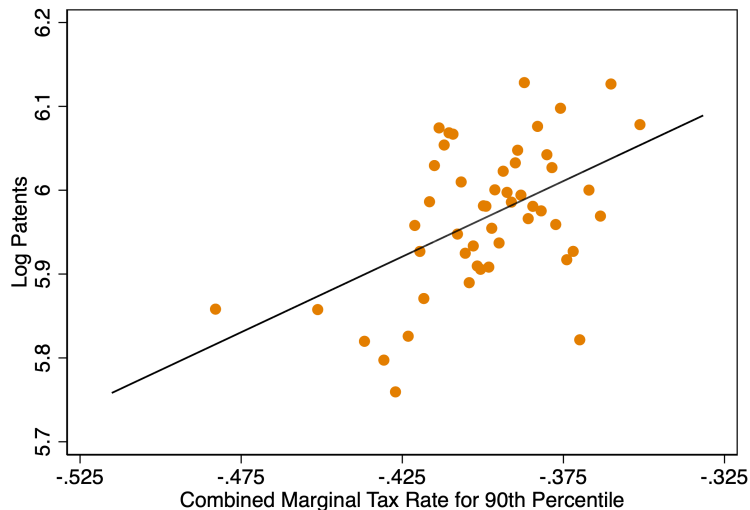
where

$$\tilde{y}_i = y_i - (1, \mathbf{w}_i)' \hat{\boldsymbol{\delta}}_{y.w} \quad \text{and} \quad \tilde{x}_i = x_i - (1, \mathbf{w}_i)' \hat{\boldsymbol{\delta}}_{x.w}$$



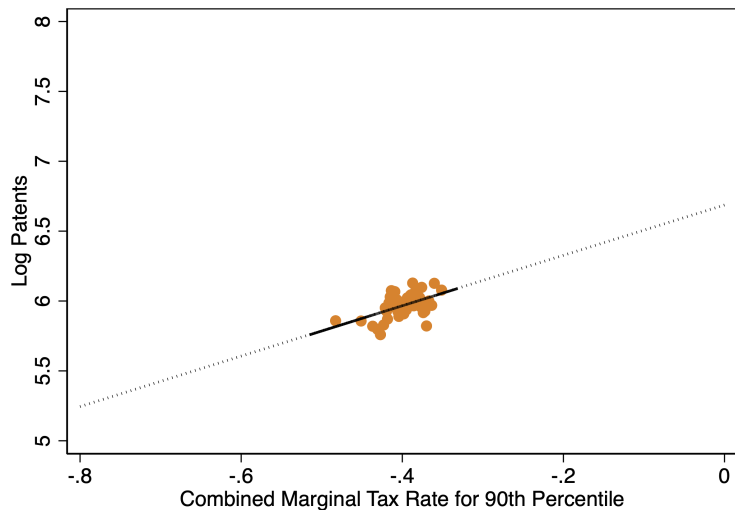


Example: Akcigit, Grigsby, Nicholas, and Stantcheva (2022, QJE)



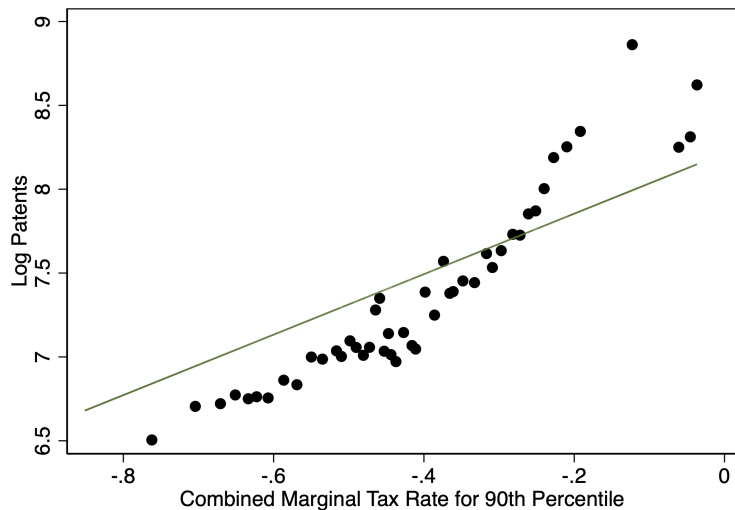
**Method:** Residualized binscatter (`binscatter`) – original.

Example: Akcigit, Grigsby, Nicholas, and Stantcheva (2022, QJE)



**Method:** Residualized binscatter (`binscatter`) – original + true data scale.

Example: Akcigit, Grigsby, Nicholas, and Stantcheva (2022, QJE)



**Method:** Semi-linear binscatter (binsreg).

## Framework: Uncertainty Quantification

$$y_i = \mu(x_i) + \mathbf{w}_i' \boldsymbol{\gamma} + \epsilon_i, \quad \mathbb{E}[\epsilon_i | x_i, \mathbf{w}_i] = 0$$

### Covariate-Adjusted Binscatter:

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} - \mathbf{w}_i' \boldsymbol{\gamma})^2$$

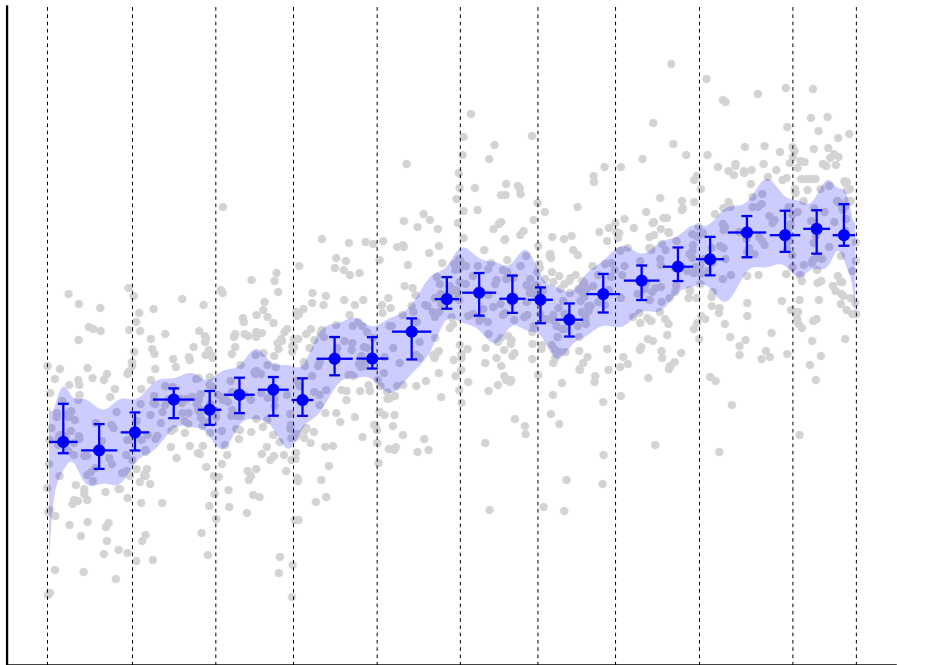
- Partitioning/Binning:  $\{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$  — Binscatter Basis:  $\hat{\mathbf{b}}_s(x)$ .
- Dimension:  $[(p+1)J - (J-1)s] + d$  — Restrictions:  $0 \leq s, v \leq p$ .

### Confidence Intervals vs. Confidence Bands:

$$\hat{I}_p(x) = \left[ \hat{\mu}^{(v)}(x) \pm \mathbf{c} \cdot \sqrt{\hat{\Omega}(x)/n} \right]$$

$$\text{CI} \implies \mathbf{c} = \Phi^{-1}(1 - \alpha/2)$$

$$\text{CB} \implies \mathbf{c} = \inf \left\{ c \in \mathbb{R}_+ : \mathbb{P}^* \left[ \sup_{x \in \mathcal{X}} |\hat{Z}_p(x)| \leq c \right] \geq 1 - \alpha \right\}$$



## Framework: Specification and Shape Testing

$$y_i = \mu(x_i) + \mathbf{w}_i' \boldsymbol{\gamma} + \epsilon_i, \quad \mathbb{E}[\epsilon_i | x_i, \mathbf{w}_i] = 0$$

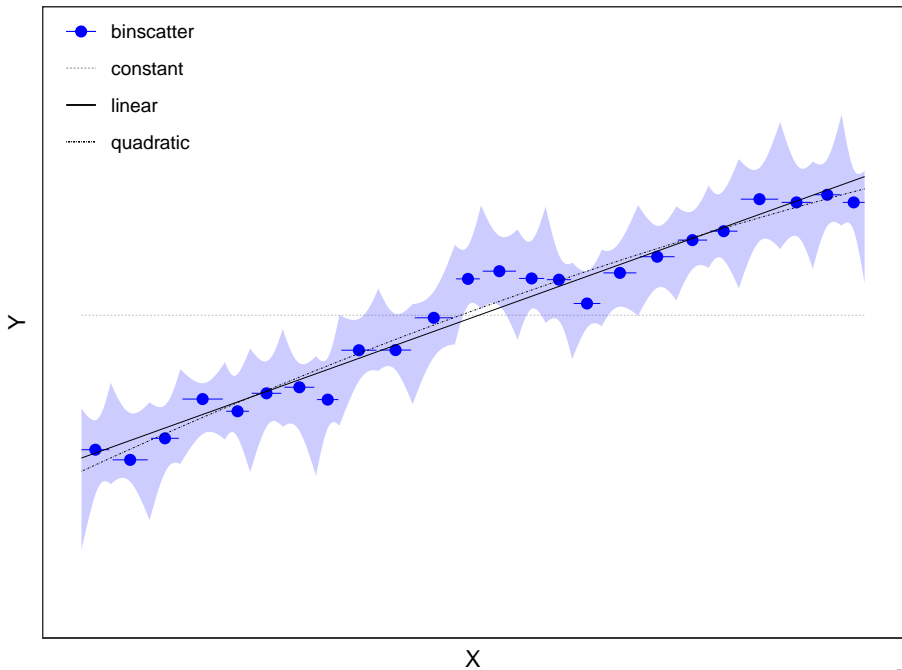
### Covariate-Adjusted Binscatter:

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} - \mathbf{w}_i' \boldsymbol{\gamma})^2$$

- ▶ Partitioning/Binning:  $\{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$  — Binscatter Basis:  $\hat{\mathbf{b}}_s(x)$ .
- ▶ Dimension:  $[(p+1)J - (J-1)s] + d$  — Restrictions:  $0 \leq s, v \leq p$ .

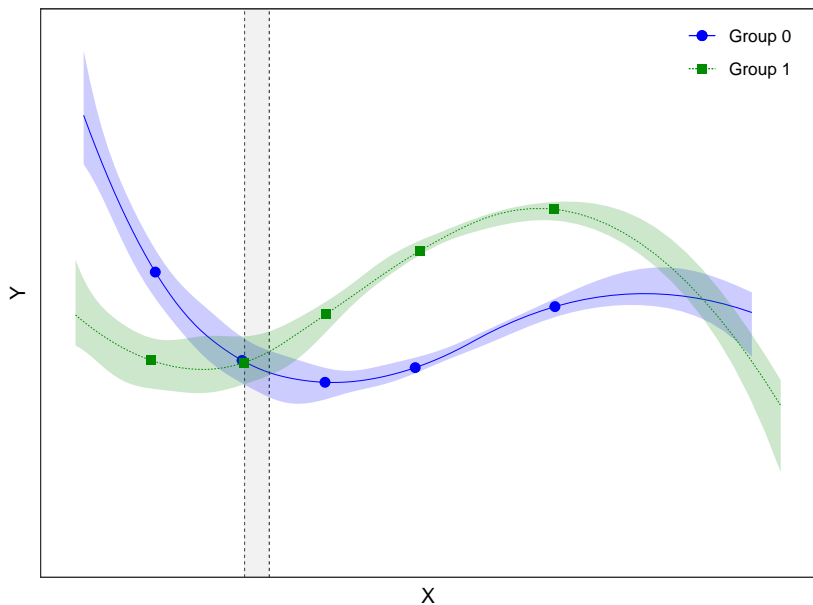
### Questions:

- ▶ Is  $\mu(x)$  constant, linear or quadratic?
- ▶ Is  $\mu(x)$  positive, increasing or convex?
- ▶ What about  $\mathbb{E}[y_i | x_i = x, \mathbf{w}_i = \mathbf{w}]$ ?
- ▶ What about more general regression-like models?





## Application: Treatment Effect Heterogeneity



## Framework: Other Parameters & QMLE

### QMLE Binscatter:

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \sum_{i=1}^n \rho(y_i - \eta(\hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} + \mathbf{w}_i' \boldsymbol{\gamma})).$$

- ▶  $\rho(u) = u^2 \implies$  Binscatter ( $\eta(u) = u$ ), GLM Binscatter ( $\eta(u) = \Lambda(u)$ ).
- ▶  $\rho(u; \tau) = (2\tau - 1)(y - u) + |y - u| \implies \tau$ -th Quantile Binscatter.
- ▶ Huber loss, MLE, etc.

### Parameters of interest:

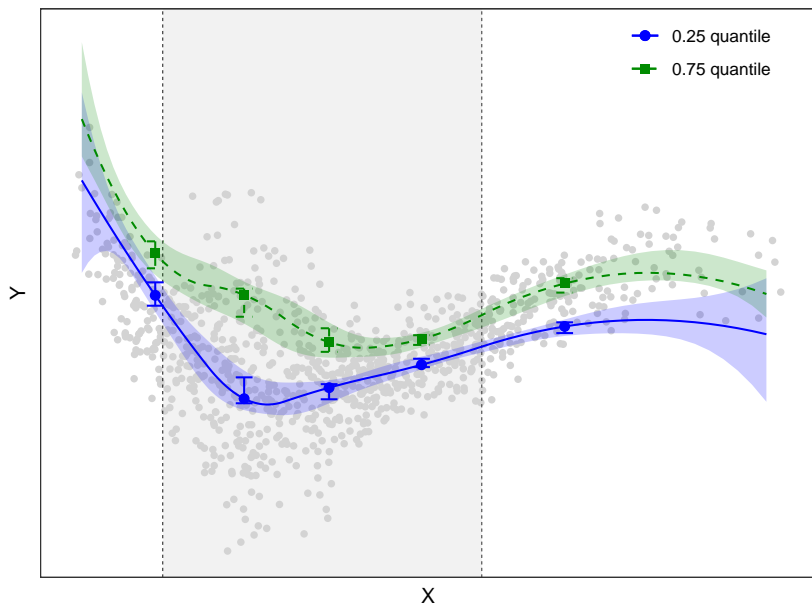
$$(\mu_0(\cdot), \boldsymbol{\gamma}_0) = \arg \min_{\mu \in \mathcal{M}, \boldsymbol{\gamma} \in \mathbb{R}^d} \mathbb{E}[\rho(y_i; \eta(\mu(x_i) + \mathbf{w}_i' \boldsymbol{\gamma}))]$$

$$\vartheta(x, \mathbf{a}_w) = \eta(\mu_0(x) + \mathbf{a}_w' \boldsymbol{\gamma}_0) \quad \text{and} \quad \vartheta_x^{(1)}(x, \mathbf{a}_w) = \left. \frac{\partial}{\partial x} \vartheta(x, \mathbf{w}) \right|_{\mathbf{w}=\mathbf{a}_w}$$

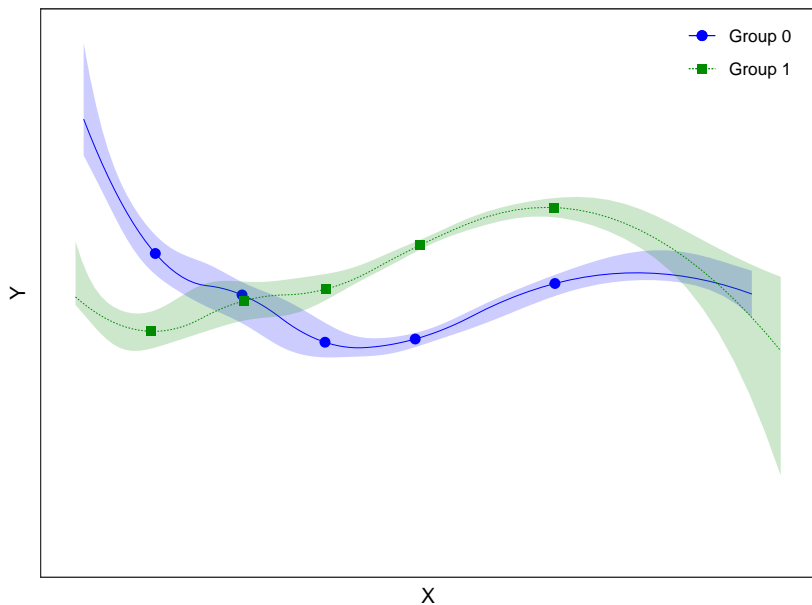
### Generalized Binscatter:

$$\hat{\vartheta}(x, \hat{\mathbf{a}}_w) = \eta(\hat{\mu}(x) + \hat{\mathbf{a}}_w' \hat{\boldsymbol{\gamma}}) \quad \text{and} \quad \hat{\vartheta}_x^{(1)}(x, \hat{\mathbf{a}}_w) = \eta^{(1)}(\hat{\mu}(x) + \hat{\mathbf{a}}_w' \hat{\boldsymbol{\gamma}}) \hat{\mu}^{(1)}(x)$$

## Application: Quantile Semi-Parametric Regression



## Application: Treatment Effect Heterogeneity



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4. Final Remarks

## IMSE-Optimal Partitioning/Binning

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} - \mathbf{w}_i' \boldsymbol{\gamma})^2$$

► Partitioning/Binning:  $\{\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_J\}$ , with  $\hat{\mathcal{B}}_j = [x_{(\lfloor n(j-1)/J \rfloor)}, x_{(\lfloor nj/J \rfloor)}]$ .

► IMSE Expansion:

$$\int \left( \hat{\mu}^{(v)}(x) - \mu^{(v)}(x) \right)^2 f(x) dx \approx_{\mathbb{P}} \frac{J^{1+2v}}{n} \mathcal{V}_n(p, s, v) + J^{-2(p+1-v)} \mathcal{B}_n(p, s, v)$$

► IMSE-optimal choice:

$$J_{\text{IMSE}} = \left\lceil \left( \frac{2(p-v+1) \mathcal{B}_n(p, s, v)}{(1+2v) \mathcal{V}_n(p, s, v)} \right)^{\frac{1}{2p+3}} n^{\frac{1}{2p+3}} \right\rceil$$

► Result handles estimated quantiles. Evenly-Spaced binning also studied.

## IMSE-Optimal Partitioning/Binning

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\boldsymbol{\beta}}, \quad \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\gamma} \end{bmatrix} = \arg \min_{\boldsymbol{\beta}, \gamma} \sum_{i=1}^n (y_i - \hat{\mathbf{b}}_s(x_i)' \boldsymbol{\beta} - \mathbf{w}_i' \gamma)^2$$

- IMSE-optimal choice (fixed  $p$  and  $s$ ):

$$J_{\text{IMSE}}(p, s) = \left[ \left( \frac{2(p-v+1) \mathcal{B}_n(p, s, v)}{(1+2v) \mathcal{V}_n(p, s, v)} \right)^{\frac{1}{2p+3}} n^{\frac{1}{2p+3}} \right]$$

- Alternative: set  $J = \mathbf{J}$  ( $\mathbf{J} = 20$ , say)  $\implies$  choose  $p$  (and  $s$ ):

$$p_{\text{IMSE}} = \arg \min_{p \in \mathbb{N}_0} \left| J_{\text{IMSE}}(p, p) - \mathbf{J} \right|$$

- Implementations: set  $J = \mathbf{J}$  ( $\mathbf{J} = 20$ , say)  $\implies$  choose  $p$  (and  $s$ ):

$$\hat{J}_{\text{IMSE}}(p, s) = \left[ \hat{\mathcal{C}}_n(p, s, v)^{\frac{1}{2p+3}} n^{\frac{1}{2p+3}} \right], \quad \hat{p}_{\text{IMSE}} = \arg \min_{p \in \mathbb{N}_0} \left| \hat{J}_{\text{IMSE}}(p, p) - \mathbf{J} \right|$$

## Pointwise Inference: Confidence Intervals

$$\widehat{T}_p(x) = \frac{\widehat{\mu}^{(v)}(x) - \mu^{(v)}(x)}{\sqrt{\widehat{\Omega}(x)/n}}, \quad 0 \leq v, s \leq p$$

$$\widehat{\Omega}(x) = \widehat{\mathbf{b}}_s^{(v)}(x)' \widehat{\mathbf{Q}}^{-1} \widehat{\Sigma} \widehat{\mathbf{Q}}^{-1} \widehat{\mathbf{b}}_s^{(v)}(x), \quad \widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^n \widehat{\mathbf{b}}_s(x_i) \widehat{\mathbf{b}}_s(x_i)' (y_i - \widehat{\mathbf{b}}_s(x_i)' \widehat{\beta} - \mathbf{w}_i' \widehat{\gamma})^2$$

► Distributional Approximation:

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}[\widehat{T}_p(x) \leq u] - \Phi(u) \right| \rightarrow 0, \quad \text{for each } x \in \mathcal{X}$$

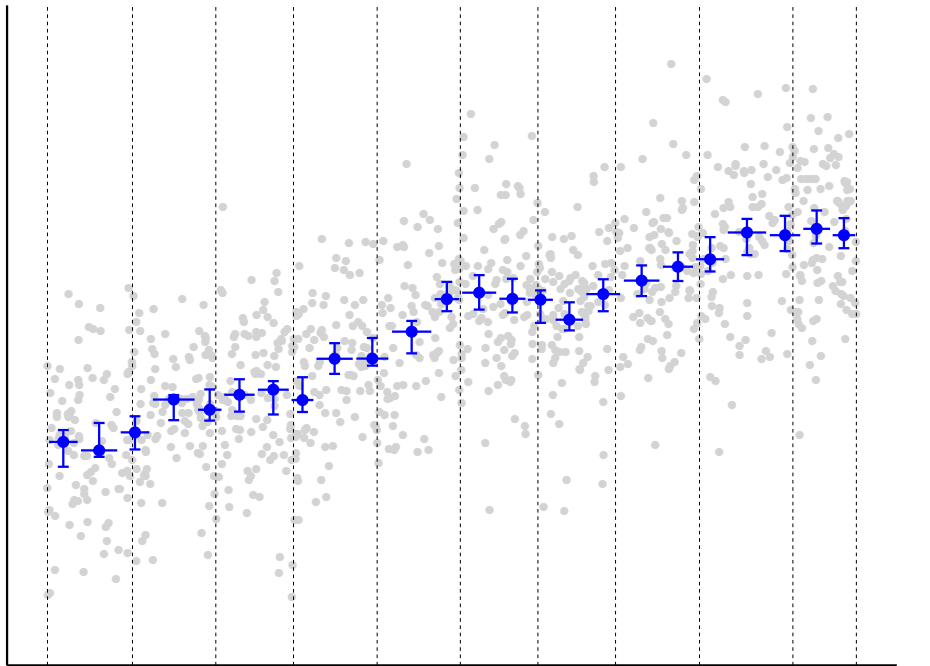
► Valid Confidence Intervals:  $J = J_{\text{IMSE}}$  for  $p$ , then for  $q \geq 1$ ,

$$\mathbb{P}[\mu^{(v)}(x) \in \widehat{I}_{p+q}(x)] \rightarrow 1 - \alpha, \quad \text{for all } x \in \mathcal{X},$$

where

$$\widehat{I}_p(x) = \left[ \widehat{\mu}^{(v)}(x) \pm \mathfrak{c} \cdot \sqrt{\widehat{\Omega}(x)/n} \right], \quad \mathfrak{c} = \Phi^{-1}(1 - \alpha/2).$$





## Uniform Inference

**Main Goal:** Approximate the “distribution” of the stochastic process

$$\left\{ \hat{T}_p(x) = \frac{\hat{\mu}^{(v)}(x) - \mu^{(v)}(x)}{\sqrt{\hat{\Omega}(x)/n}} : x \in \mathcal{X} \right\}, \quad 0 \leq v, s \leq p$$

- Useful to approximate distribution of statistics such as

$$\sup_{x \in \mathcal{X}} |\hat{T}_p(x)|, \quad \sup_{x \in \mathcal{X}} \hat{T}_p(x), \quad \inf_{x \in \mathcal{X}} \hat{T}_p(x), \quad \text{etc.}$$

- New strong approximation approach (based on Hungarian construction):

$$\sup_{x \in \mathcal{X}} \left| \hat{T}_p(x) - Z_p(x) \right| = o_{\mathbb{P}}(r_n), \quad Z_p(x) = \frac{\hat{\mathbf{b}}_0^{(v)}(x)' \mathbf{T}_s' \mathbf{Q}^{-1} \boldsymbol{\Sigma}^{1/2} \mathbf{N}_K}{\sqrt{\hat{\Omega}(x)}},$$

where

$$\mathbf{N}_K \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_K), \quad \hat{\mathbf{Q}} \approx_{\mathbb{P}} \mathbf{Q}, \quad \hat{\mathbf{T}}_s \approx_{\mathbb{P}} \mathbf{T}_s, \quad \hat{\Omega}(x) \approx_{\mathbb{P}} \Omega(x), \quad \text{etc.}$$

## Uniform Inference: Heuristics of Technical Idea (4 Steps)

1. Hats off, except non-uniform-controlled partitioning scheme:

$$\sup_{x \in \mathcal{X}} |\widehat{T}_p(x) - t_p(x)| = o_{\mathbb{P}}(r_n), \quad t_p(x) = \frac{\widehat{\mathbf{b}}_0^{(v)}(x)' \mathbf{T}_s' \mathbf{Q}^{-1} \mathbb{G}_n[\mathbf{b}_s(x_i) \boldsymbol{\epsilon}_i]}{\sqrt{\Omega(x)}}$$

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$$\sup_{x \in \mathcal{X}} |t_p(x) - z_p(x)| = o_{\mathbb{P}}(r_n), \quad z_p(x) = \frac{\widehat{\mathbf{b}}_0^{(v)}(x)' \mathbf{T}_s' \mathbf{Q}^{-1} \mathbb{G}_n[\mathbf{b}_s(x_i) \sigma(x_i) \zeta_i]}{\sqrt{\Omega(x)}}$$

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3. Coupling to unconditional (up to non-uniform partitioning) Gaussian Process:

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4. For example, supremum approximation (with hats back on):

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P} \left[ \sup_{x \in \mathcal{X}} |\widehat{T}_p(x)| \leq u \right] - \mathbb{P}^* \left[ \sup_{x \in \mathcal{X}} |\widehat{Z}_p(x)| \leq u \right] \right| = o_{\mathbb{P}}(1)$$

## Uniform Inference: Confidence Bands

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P} \left[ \sup_{x \in \mathcal{X}} |\widehat{T}_p(x)| \leq u \right] - \mathbb{P}^* \left[ \sup_{x \in \mathcal{X}} |\widehat{Z}_p(x)| \leq u \right] \right| = o_{\mathbb{P}}(1)$$

$$\widehat{Z}_p(x) = \frac{\widehat{\mathbf{b}}_s^{(v)}(x)' \widehat{\mathbf{Q}}^{-1} \widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}(x)}} \mathbf{N}_K, \quad \mathbf{N}_K \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_K)$$

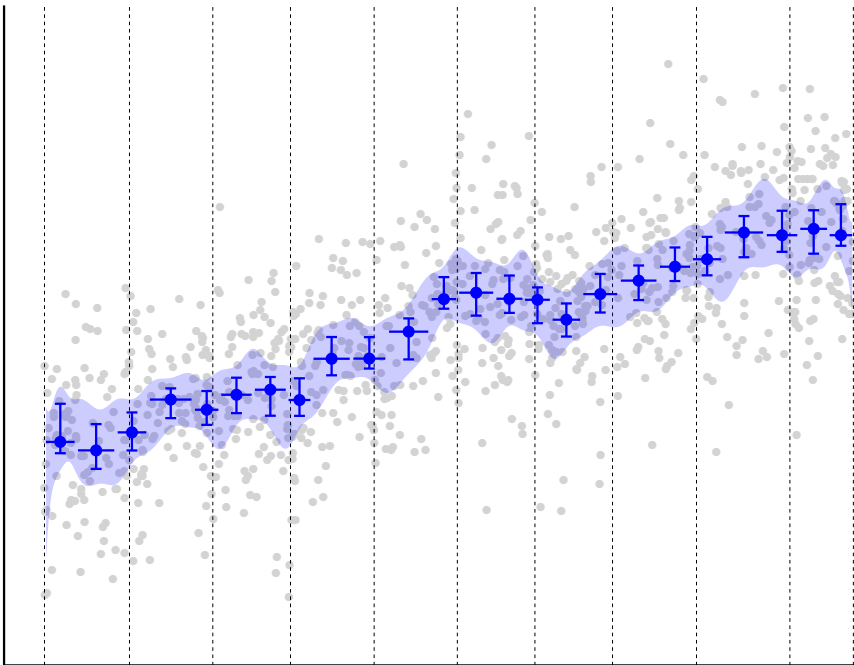
► Valid Confidence Band:  $J = J_{\text{IMSE}}$  for  $p$ , then for  $q \geq 1$ ,

$$\mathbb{P} \left[ \mu^{(v)}(x) \in \widehat{I}_{p+q}(x), \text{ for all } x \in \mathcal{X} \right] \rightarrow 1 - \alpha,$$

where

$$\widehat{I}_p(x) = \left[ \widehat{\mu}^{(v)}(x) \pm \mathfrak{c} \cdot \sqrt{\widehat{\Omega}(x)/n} \right],$$

$$\mathfrak{c} = \inf \left\{ c \in \mathbb{R}_+ : \mathbb{P}^* \left[ \sup_{x \in \mathcal{X}} |\widehat{Z}_p(x)| \leq c \right] \geq 1 - \alpha \right\}$$





# Uniform Inference: Parametric Specification Testing

$$\begin{array}{ll} \ddot{H}_0 : \sup_{x \in \mathcal{X}} |\mu^{(v)}(x) - m^{(v)}(x, \boldsymbol{\theta})| = 0 & \text{vs.} \quad \ddot{H}_A : \sup_{x \in \mathcal{X}} |\mu^{(v)}(x) - m^{(v)}(x, \boldsymbol{\theta})| > 0 \\ \text{for some } \boldsymbol{\theta} \in \boldsymbol{\Theta} & \text{for all } \boldsymbol{\theta} \in \boldsymbol{\Theta} \end{array}$$

- Test statistic: for  $\hat{\boldsymbol{\theta}}$  and  $m(\cdot)$  “well-behaved” under  $\ddot{H}_0$  and  $\ddot{H}_A$ ,

$$\ddot{T}_p(x) = \frac{\hat{\mu}^{(v)}(x) - m^{(v)}(x, \hat{\boldsymbol{\theta}})}{\sqrt{\hat{\Omega}(x)/n}}, \quad 0 \leq v, s \leq p,$$

- For given  $p$  set  $J = J_{\text{IMSE}}$ , and for  $q \geq 1$  set

$$\mathfrak{c} = \inf \left\{ c \in \mathbb{R}_+ : \mathbb{P}^* \left[ \sup_{x \in \mathcal{X}} |\hat{Z}_{p+q}(x)| \leq c \right] \geq 1 - \alpha \right\}$$

- Under  $\ddot{H}_0$ , then

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \sup_{x \in \mathcal{X}} |\ddot{T}_{p+q}(x)| > \mathfrak{c} \right] = \alpha,$$

- Under  $\ddot{H}_A$ , then

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \sup_{x \in \mathcal{X}} |\ddot{T}_{p+q}(x)| > \mathfrak{c} \right] = 1.$$

# Uniform Inference: Shape Restriction Testing

$$\dot{H}_0 : \sup_{x \in \mathcal{X}} \mu^{(v)}(x) \leq 0 \quad \text{vs.} \quad \dot{H}_A : \sup_{x \in \mathcal{X}} \mu^{(v)}(x) > 0$$

► Test statistic:

$$\dot{T}_p(x) = \frac{\hat{\mu}^{(v)}(x)}{\sqrt{\hat{\Omega}(x)/n}}, \quad 0 \leq v, s \leq p,$$

► For given  $p$  set  $J = J_{\text{IMSE}}$ , and for  $q \geq 1$  set

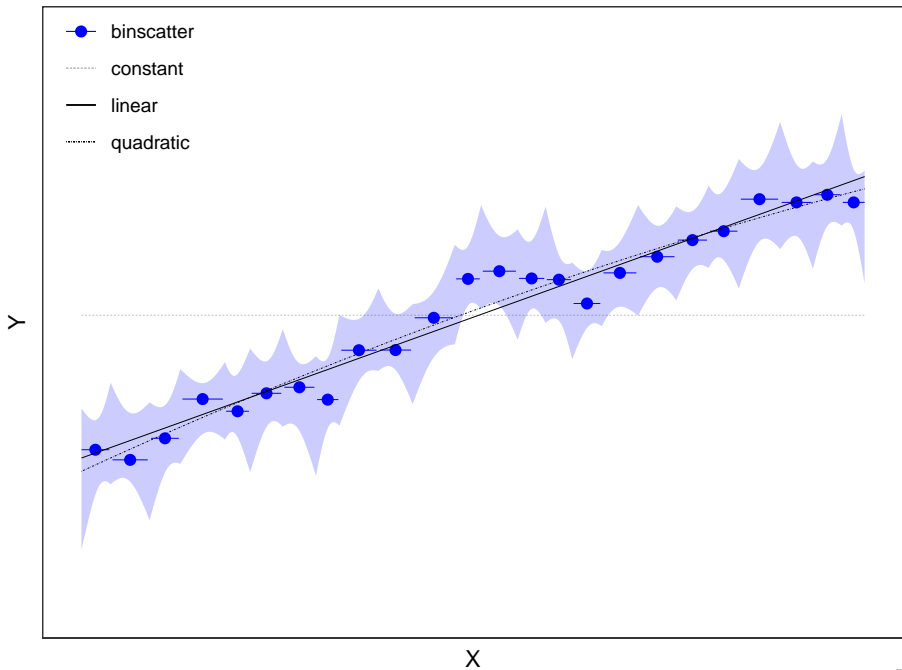
$$\mathfrak{c} = \inf \left\{ c \in \mathbb{R}_+ : \mathbb{P}^* \left[ \sup_{x \in \mathcal{X}} \hat{Z}_{p+q}(x) \leq c \right] \geq 1 - \alpha \right\}$$

► Under  $\dot{H}_0$ , then

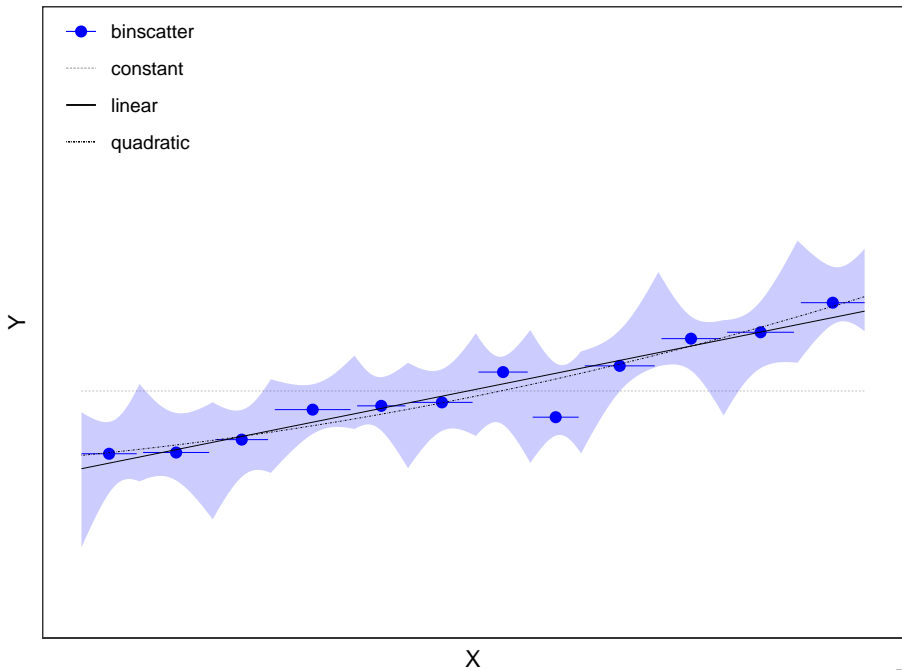
$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \sup_{x \in \mathcal{X}} \dot{T}_{p+q}(x) > \mathfrak{c} \right] \leq \alpha,$$

► Under  $\dot{H}_A$ , then

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \sup_{x \in \mathcal{X}} \dot{T}_{p+q}(x) > \mathfrak{c} \right] = 1.$$



	Half Support ( $n = 482$ )			Full Support ( $n = 1000$ )		
	Test Statistic	P-value	J	Test Statistic	P-value	J
<b>Parametric Specification</b>						
Constant	11.716	0.000	12	11.607	0.000	24
Linear	2.994	0.092	12	4.968	0.000	24
Quadratic	2.392	0.384	12	4.300	0.002	24
<b>Shape Restrictions</b>						
Negativity	4.069	0.000	12	12.226	0.000	24
Increasing	-1.964	0.536	13	-2.168	0.394	13
Concavity	2.269	0.316	14	2.544	0.180	14



## Uniform Inference: Generalized Binscatter

### Generalized Binscatter:

$$\hat{\mu}^{(v)}(x) = \hat{\mathbf{b}}_s^{(v)}(x)' \hat{\beta}, \quad \begin{bmatrix} \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \arg \min_{\beta, \gamma} \sum_{i=1}^n \rho(y_i - \eta(\hat{\mathbf{b}}_s(x_i)' \beta + \mathbf{w}_i' \gamma)).$$

$$\hat{\vartheta}(x, \hat{\mathbf{a}}_w) = \eta(\hat{\mu}(x) + \hat{\mathbf{a}}_w' \hat{\gamma}) \quad \hat{\vartheta}_x(x, \hat{\mathbf{a}}_w) = \eta^{(1)}(\hat{\mu}(x) + \hat{\mathbf{a}}_w' \hat{\gamma}) \hat{\mu}^{(1)}(x)$$

### Uniform Bahadur Representation (up to bias of order $J^{-m}$ ):

$$\sup_{x \in \mathcal{X}} \left| \hat{\mu}^{(v)}(x) - \mu_0^{(v)}(x) + \hat{\mathbf{b}}_s^{(v)}(x)' \bar{\mathbf{Q}}^{-1} \mathbb{E}_n[\hat{\mathbf{b}}_s(x_i) \eta_{i,1} \psi(\epsilon_i)] \right| \lesssim_{\mathbb{P}} J^v \left( \frac{J \log n}{n} \right)^{3/4} \sqrt{\log n}$$

$$\eta_i = \eta(\mu_0(x_i) + \mathbf{w}_i' \gamma_0), \quad \psi(u) = \text{weak derivative of } \rho(u), \quad \epsilon_i = y_i - \eta_i$$

**Key condition:**  $J^2 \log(n)/n = o(1)$  — even  $J \log(n)/n = o(1)$  when  $s = 0$ ).

# Outline

1. Introduction

2. Overview

3. Theoretical Contributions

4. Final Remarks

## Overview

- ▶ Binscatter is widely used across disciplines.
- ▶ Methodological and formal results lagging behind its popularity.
- ▶ We offer a through treatment of canonical binscatter and its generalizations.
  - ▶ Formal framework: covariate-adjustment, smoothness restrictions, and more.
  - ▶ Optimal choice of partitioning/binning.
  - ▶ Confidence intervals and confidence bands.
  - ▶ Hypothesis testing for shape restrictions and for parametric specifications.
  - ▶ Quantile, non-linear least squares, and other QMLE estimation methods.
- ▶ New theoretical results for linear and non-linear partitioning-based estimators with random partitions.
- ▶ Binsreg package for Python, R, and Stata.

<https://nppackages.github.io/binsreg/>