Lecture 1
Fundamental Concepts of Transport Theory

1 Phase Space

To describe the transport of radiation, we must be able to specify both the position and velocity of particles. A point in phase space corresponds to both a position and a velocity. We will use $\overrightarrow{P}$ to denote a vector of phase-space coordinates. Assuming the use of Cartesian spatial coordinates, the standard set of phase-space coordinates for particle transport is

$$\overrightarrow{P} \equiv (x, y, z, \overrightarrow{\Omega}, E)\,, \tag{1}$$

where x, y, and z are the usual Cartesian coordinates of the particle position, $\overrightarrow{\Omega}$ is a unit Cartesian vector representing the direction of particle flow, and E is the particle energy. The direction coordinates are illustrated in Fig. 1. Note from Fig. 1 that $\overrightarrow{\Omega}$ can be represented in terms of a polar angle, θ, and an azimuthal angle, ω. It can also be represented in terms of any two of its Cartesian components, which are given by

$$\Omega_x = \sin\theta\,\cos\omega\,, \tag{2a}$$

$$\Omega_y = \sin\theta\,\sin\omega\,, \tag{2b}$$

$$\Omega_z = \cos\theta\,. \tag{2c}$$

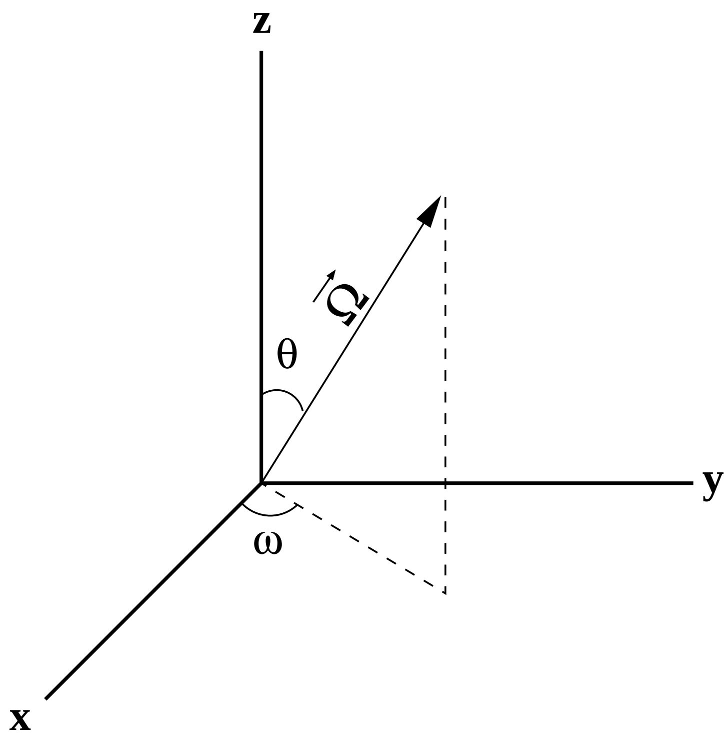

Figure 1: $\overrightarrow{r} = (x, y, z)$ is the particle position vector, $\overrightarrow{\Omega}$ is the particle direction vector, and E is the particle energy

The differential phase-space volume associated with phase-space point $\overrightarrow{P}$ is

$$dP \equiv dV \, d\Omega dE \,, \tag{3a}$$

where

$$dV = dx \, dy \, dz \,, \tag{3b}$$

and

$$d\Omega = \sin\theta \; d\theta \, d\omega \,. \tag{3c}$$

Note that each point on the unit sphere represents a direction, thus $d\Omega$, represents a differential area on the unit sphere. Since the integral over all directions is equal to 4π, i.e.,

$$\int_0^{2\pi} \int_0^{\pi} \sin\theta \; d\theta \, d\omega = 4\pi \,, \tag{4}$$

we say that the unit sphere has a total "area" of 4π *steradians*. It is often convenient to use the variable, μ, in place of the variable, θ, where $\mu = \cos(\theta)$. In this case,

$$d\Omega = \; d\mu \, d\omega \,, \tag{5}$$

and

$$\int_{4\pi} d\Omega = \int_0^{2\pi} \int_{-1}^{+1} \; d\mu \, d\omega \,. \tag{6}$$

A solid angle, which is illustrated in Fig. 2, is a set of points on the unit sphere centered about some direction, $\overrightarrow{\Omega}_0$, with an associated area in steradians, $\Delta\Omega$. Every direction within the solid angle, $\overrightarrow{\Omega}$, satisfies

$$\overrightarrow{\Omega} \cdot \overrightarrow{\Omega}_0 \geq 1 - \Delta\Omega/2\pi \,. \tag{7}$$

Note that $\overrightarrow{\Omega} \cdot \overrightarrow{\Omega}_0$ represents the cosine of the angle subtended by $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega}_0$. A solid angle of 4π contains all directions, A solid angle of 2π contains half of all the directions, and a solid angle of zero contains only $\overrightarrow{\Omega}_0$.

2 Fundamental Transport Functions

In this section, we define certain fundamental functions associated with transport theory that relate to particle distribution functions. We make a fluid-like approximation in

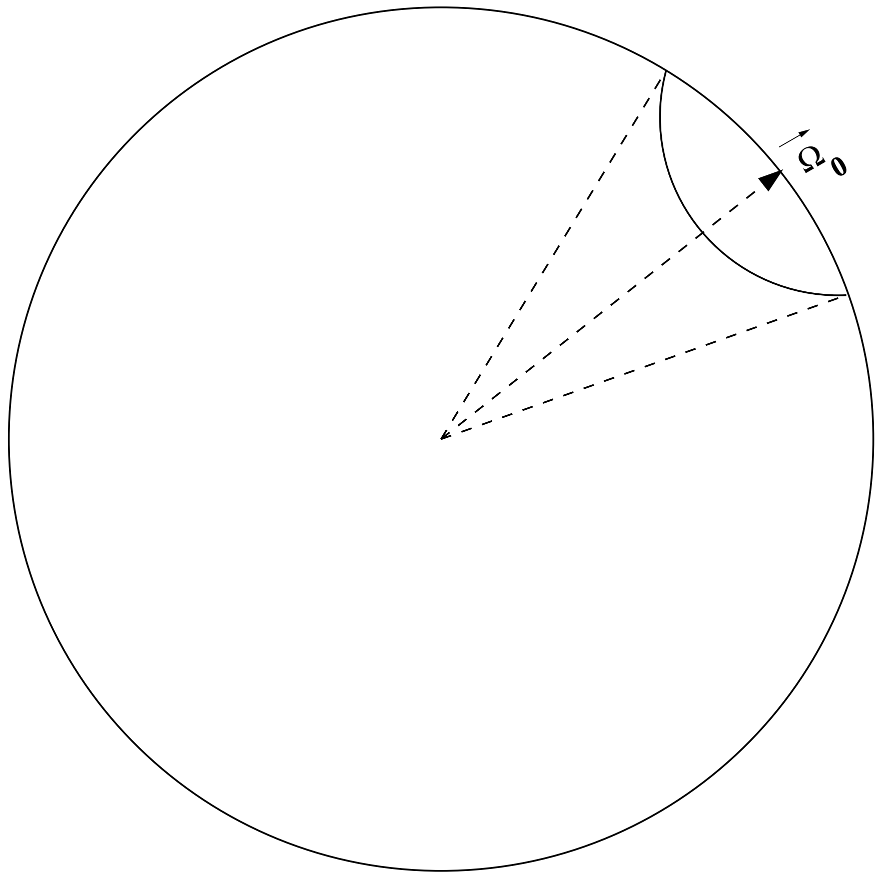

Figure 2: Illustration of solid angle centered about the direction vector, $\overrightarrow{\Omega}_0$.

transport theory in that we assume continuum particle distributions even though real particle distributions are discrete in nature. Nonetheless, this assumption is useful because it enables us to use the language of partial integro-differential equations to formulate an equation for particle transport distributions. Furthermore, the continuum distributions that are obtained accurately predict reality if the time and scales associated with these solutions are large with respect to the time scales associated with particle interactions, and the length scales associated with these solutions are large with respect to particle sizes and interaction lengths. Furthermore, the number of particles in any region in time and space over which a measurement is made must be statistically meaningful to observe agreement with theory.

In the definitions that follow, we often use the word "about" as in "energy about E." A quantity that is "about" a particular value lies in the differential vicinity of that value. Therefore "a particle with an energy about E" implies a particle that has an energy between E and $E + dE$.

Name: Phase-space Particle Density.

Units: $(particles/(cm^3 - steradian - MeV))$.

Symbol: $\mathcal{N}(\overrightarrow{P})$.

Interpretation: The quantity,

$$\mathcal{N}(\overrightarrow{P})\, dP\,,$$

represents the number of particles in the differential phase-space volume dP.

Name: Angular Flux.

Symbol: $\psi(\overrightarrow{P})$.

Units: $(particles/(cm^2 - sec - steradian - MeV))$.

Equivalence: $\psi(\overrightarrow{P}) \equiv \mathcal{N}(\overrightarrow{P})v$, where v is the particle speed.

Interpretation 1: Consider a differential surface area, dA, located at position, $\overrightarrow{r}$, and oriented such that it is normal to the direction vector, $\overrightarrow{\Omega}$. Then

$$\psi(\overrightarrow{P})\, dA\, d\Omega\, dE\,,$$

represents the number of particles with directions about $\overrightarrow{\Omega}$ and energies about E, passing per second through dA. The normal orientation of dA is very important.

Interpretation 2: Consider a differential surface area, dA, located at position, $\overrightarrow{r}$, that is *arbitrarily* oriented with normal $\overrightarrow{n}$. Then the expression,

$$\psi(\overrightarrow{P}) \left| \overrightarrow{\Omega} \cdot \overrightarrow{n} \right| \, dA\, d\Omega\, dE\,, \tag{8}$$

represents the number of particles with directions about $\overrightarrow{\Omega}$ and energies about E, passing per second through dA. The origin of the factor of $\left| \overrightarrow{\Omega} \cdot \overrightarrow{n} \right|$ is illustrated in Fig. 3. In particular, the area denoted by dA_1 is normal to $\overrightarrow{\Omega}$, while the area denoted by dA_2 is not. It is clear from Fig. 3 that the particle flow rate through both areas is the same, even though the areas differ in magnitude. It is also clear from Fig. 3 that

$$dA_1 = \cos(\gamma) dA_2 = \overrightarrow{\Omega} \cdot \overrightarrow{n_2} dA_2\,.$$

Thus, if the flow rate through dA_1 is

$$\psi(\overrightarrow{P})\, dA_1\, d\Omega\, dE\,,$$

then the flow rate through dA_2 must be

$$\psi(\overrightarrow{P}) \overrightarrow{\Omega} \cdot \overrightarrow{n_2}\, dA_2\, d\Omega\, dE\,.$$

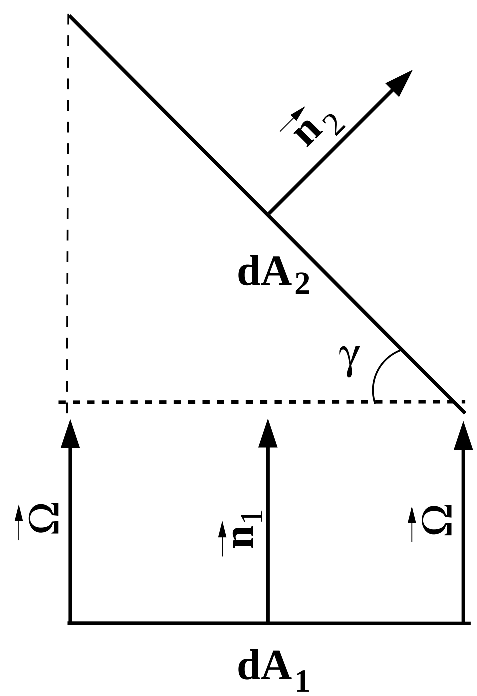

Figure 3: Particle flow through a surface that is normal to the particle direction and a surface that is not normal.

Every surface has two normal vectors that are oppositely directed. The normal, $\overrightarrow{n}_2$ in Fig. 3 is the choice that makes $\overrightarrow{\Omega} \cdot \overrightarrow{n}$ positive. However, this quantity would be negative if we had instead chosen the normal to be $-\overrightarrow{n}_2$. Thus the absolute value of $\overrightarrow{\Omega} \cdot \overrightarrow{n}$ must be used in Eq. (8) to obtain a flow-rate expression that is independent of the choice of surface normal.

Interpretation 3: The expression,

$$\psi(\overrightarrow{P})\, dP\,,$$

represents the total pathlength traveled per second by particles within the differential phase-space volume, dP. This interpretation can be understood from the equivalence between

$\mathcal{N}v$ and ψ. By definition, $\mathcal{N}(\overrightarrow{P})\,dP$ is the total number of particles in dP. Each of those particles is traveling with speed v. Speed represents the pathlength traveled per second by each particle in dP. The total pathlength traveled per second by all particles is the pathlength traveled per second by each particle multiplied by the total number of particles: $\mathcal{N}(\overrightarrow{P})\,dP\,v = \psi(\overrightarrow{P})\,dP$.

Name: Scalar Flux.

Symbol: $\phi(\overrightarrow{r}, E)$.

Units: $(particles/(cm^2 - sec - MeV))$.

Definition: $\phi(\overrightarrow{r}, E) \equiv \int_{4\pi} \psi(\overrightarrow{P})\,d\Omega$.

Interpretation: The expression,

$$\phi(\overrightarrow{r}, E)\,dV\,dE\,,$$

represents the total pathlength traveled per second by particles with energies about E within the differential phase-space volume, dP. The quantity referred to as *fluence* simply represents scalar flux integrated over some time interval. It has units of $(particles/(cm^2 - MeV))$

Name: Net Current Vector.

Symbol: $\overrightarrow{J}(\overrightarrow{r}, E)$.

Units: $(particles/(cm^2 - sec - MeV))$.

Definition: $\overrightarrow{J}(\overrightarrow{r}, E) \equiv \int_{4\pi} \psi(\overrightarrow{P}) \overrightarrow{\Omega}\, d\Omega$.

Interpretation: Consider a differential surface area, dA, located at position $\overrightarrow{r}$, that is part of a closed surface and has an outward-directed normal, $\overrightarrow{n}$. Further let $\Delta\Omega_-$ and $\Delta\Omega_+$ denote those directions satisfying $\overrightarrow{\Omega} \cdot \overrightarrow{n} < 0$ and $\overrightarrow{\Omega} \cdot \overrightarrow{n} > 0$, respectively. Then the expression,

$$\overrightarrow{J}(\overrightarrow{r}, E) \cdot \overrightarrow{n}\, dA\, dE \,,$$

represents the *net* number of particles with energies about E, *exiting* the enclosed volume per second through dA. By this we mean the number of particles with energies about E, *exiting* the enclosed volume per second through dA *minus* the number of particles with energies about E, *entering* the enclosed volume per second through dA. To see why this is so, consider that

$$\begin{aligned}
\overrightarrow{J}(\overrightarrow{r}, E) \cdot \overrightarrow{n}\, dA\, dE &= \int_{4\pi} \psi(\overrightarrow{P}) \overrightarrow{\Omega} \cdot \overrightarrow{n}\, d\Omega\, dA\, dE \,, \\
&= \left\{ \int_{\Delta\Omega_-} \psi(\overrightarrow{P}) \overrightarrow{\Omega} \cdot \overrightarrow{n}\, d\Omega + \int_{\Delta\Omega_+} \psi(\overrightarrow{P}) \overrightarrow{\Omega} \cdot \overrightarrow{n}\, d\Omega \right\} dA\, dE \,.
\end{aligned}$$

The first partial integral is a negative quantity, but its absolute value represents the number of particles with energies about E, entering the enclosed volume per second through dA. The second partial integral represents the number of particles with energies about E, exiting the enclosed volume per second through dA. Given any normal vector, $\overrightarrow{n}$, it is useful to define partial currents, denoted by j^+ and j^-, where

$$j^- = -\int_{\Delta\Omega_-} \psi(\overrightarrow{P})\overrightarrow{\Omega} \cdot \overrightarrow{n}\; d\Omega\,, \tag{9}$$

$$j^+ = \int_{\Delta\Omega_+} \psi(\overrightarrow{P})\overrightarrow{\Omega} \cdot \overrightarrow{n}\; d\Omega\,. \tag{10}$$

Thus

$$\overrightarrow{J}(\overrightarrow{r}, E) \cdot \overrightarrow{n} = j^+(\overrightarrow{r}, E) - j^-(\overrightarrow{r}, E)\,. \tag{11}$$

We refer to j^- is the counter-flowing partial current and to j^+ as the co-flowing partial current. Note that unlike $\overrightarrow{J} \cdot \overrightarrow{n}$, the partial currents must always be positive.

3 Fundamental Properties of Transport Media

Particles can undergo a wide variety of interactions as they propagate through matter. For simplicity, we consider only absorption and scattering here. However, more types of interactions can be accomodated within the basic framework associated with absorption and scattering. The purpose of this section is to define certain basic functions that describe the transport properties of materials through which the radiation propagates. A statistical viewpoint is taken with respect to interactions, i.e., over a given pathlength, the number of interactions that a particle will undergo is probabilistic rather than deterministic.

Name: Microscopic Interaction Cross-Section.

Symbol: $\hat{\sigma}(\overrightarrow{r}, E)$.

Units: cm^2.

Interpretation: This is the effective cross-sectional area of a target atom for a particular type of interaction seen at position, $\overrightarrow{r}$, by a transport particle with an energy about E.

Name: Macroscopic Interaction Cross-Section.

Symbol: $\sigma(\overrightarrow{r}, E)$.

Units: cm^{-1}.

Equivalence: $\sigma(\overrightarrow{r}, E) = \rho_a(\overrightarrow{r})\hat{\sigma}(\overrightarrow{r}, E)$, where ρ_a is the atomic density ($target-atoms/cm^3$).

Interpretation 1: The expression,

$$\sigma(\overrightarrow{r}, E)\, ds\,,$$

represents the probability of an interaction by a particle with energy, E, that starts at position, $\overrightarrow{r}$, and travels a differential distance, ds.

Interpretation 2: Assuming a space-independent value of σ, the quantity,

$$\frac{1}{\sigma}\,,$$

represents the average distance that a particle will travel between interactions. This distance is called the *mean-free-path*.

Name: Differential Scattering Distribution Function.

Symbol: $f(\overrightarrow{r}, E' \rightarrow E, \mu_s)$.

Units: $(steradian - MeV)^{-1}$.

Interpretation: Consider the scattering frame coordinate system illustrated in Fig. 4. Given that a particle with initial energy, E', has scattered at position, $\overrightarrow{r}$, the expression,

$$f(\overrightarrow{r}, E' \rightarrow E, \mu_s)\, d\mu_s\, d\omega_s\, dE\,,$$

represents the probability that the particle will scatter into the differential solid angle, $d\Omega_s = d\mu_s\, d\omega_s$, with a final energy about E. Note that f is independent of the azimuthal scattering angle. Since f is a probability distribution function, it follows that

$$\int_0^\infty 2\pi \int_0^\pi f(\overrightarrow{r}, E' \rightarrow E, \mu_s)\, d\mu_s\, dE = 1\,.$$

Name: The Macroscopic Differential Scattering Cross-Section.

Symbol: $\sigma_s(\overrightarrow{r}, E' \rightarrow E, \mu_s)$.

Units: $(cm - steradian - MeV)^{-1}$.

Equivalence: $\sigma_s(\overrightarrow{r}, E' \rightarrow E, \mu_s) = \sigma_s(\overrightarrow{r}, E') f(\overrightarrow{r}, E' \rightarrow E, \mu_s)$.

Interpretation: The expression,

$$\sigma_s(\overrightarrow{r}, E' \rightarrow E, \mu_s)\, ds\, d\mu_s\, d\omega_s\, dE\,,$$

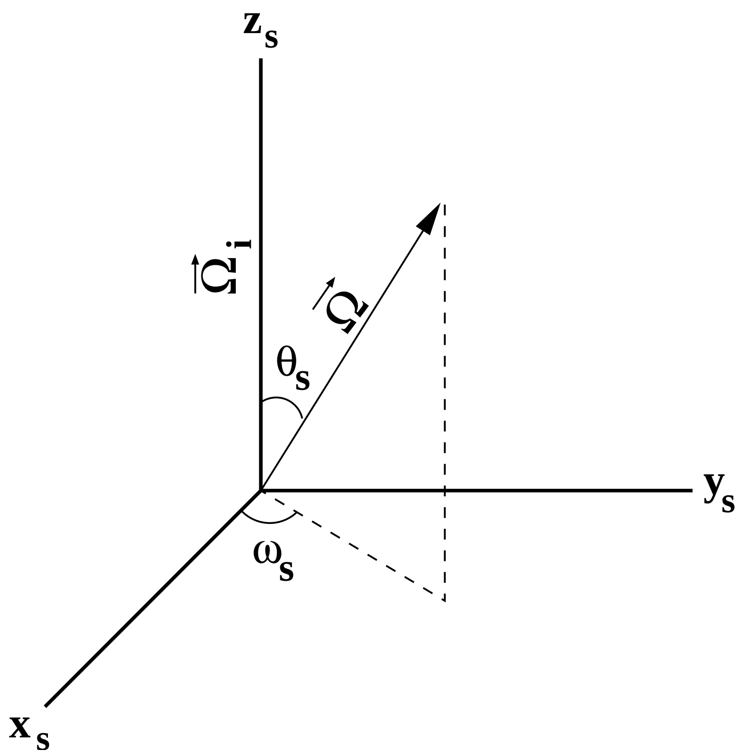

Figure 4: The scattering frame coordinate system. This coordinate system is obtained via two sequential rotations of the lab-frame coordinate system. Assume $\overrightarrow{\Omega}_i$ is characterized by an azimuthal angle of ϕ_i and a polar cosine of θ_i. Then the first is a rotation about the z-axis through an angle ϕ_i, resulting in in intermediate coordinate system. The second is a rotation about the intermediate y-axis through an angle θ_i. These rotations are discussed in detail in Lecture 3. The z_s-axis lies along the initial direction of the scattering particle as shown in this figure. The angle, θ_s, is called the polar scattering angle. The angle, ω_s, is called the azimuthal scattering angle, and the vector, $\overrightarrow{\Omega}_s$, represents the direction into which the particle scatters. It is convenient to use the variable $\mu_s = \cos\theta_s$ in place of θ_s.

represents the probability of a scattering event by a particle with initial energy, E', that starts at position, $\overrightarrow{r}$, travels a differential distance, ds, and scatters into the differential solid angle, $d\Omega_s = d\mu_s \, d\omega_s$, with a final energy about, E. Note that this quantity expresses scattering probabilities in terms of a local directional scattering frame that is oriented along the initial particle direction. Further note that the materials that we consider are isotropic in that the probability of interaction does not depend upon the initial direction and the scattering probabilities are azimuthally symmetric in the scattering frame.

Name: Differential Scattering Kernel.

Symbol: $\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega})$.

Units: $(steradian - MeV)^{-1}$.

Equivalence: $\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega}) = \sigma_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega})$.

Interpretation: The expression,

$$\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega})\, ds\, d\Omega\, dE\,,$$

represents the probability of a scattering event by a particle with initial energy, E', and initial direction, $\overrightarrow{\Omega}'$, that starts at position, $\overrightarrow{r}$, travels a differential distance, ds, and scatters into a direction about $\overrightarrow{\Omega}$, with a final energy about, E.

4 Reaction Rates and Sources

We use fundamental transport functions together with fundamental transport medium properties to obtain particle reaction rates and certain solution-dependent sources.

Name: Total Interaction Rate.

Representation: $\psi(\overrightarrow{P})\sigma_t(\overrightarrow{r}, E)$.

Units: $(p/(cm^3 - sec - steradian - MeV))$.

Interpretation: The expression,

$$\psi(\overrightarrow{P})\,\sigma_t(\overrightarrow{r}, E)\; dP\,,$$

represents the number of particles per second interacting (being absorbed or scattered) within the differential volume dV with directions about $\overrightarrow{\Omega}$ and energies about, E.

Name: Scattering Source.

Symbol: $S(\overrightarrow{P})$.

Representation: $\int_0^\infty \int_{4\pi} \mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega})\,\psi(\overrightarrow{r}, \overrightarrow{\Omega}', E')\; d\Omega'\, dE'\,.$

Units: $(p/(cm^3 - sec - steradian - MeV))$.

Interpretation: The expression,

$$S(\overrightarrow{P})\; dP\,,$$

represents the number of particles per second within the differential volume dV being scattered into directions about $\overrightarrow{\Omega}$ and energies about, E.

Name: Inhomogeneous Source.

Symbol: $Q(\overrightarrow{P})$.

Units: $(p/(cm^3 - sec - steradian - MeV))$.

Interpretation: The expression,

$$Q(\overrightarrow{P})\; dP\,,$$

represents the number of particles being created per second within the differential volume dV with directions about $\overrightarrow{\Omega}$ and energies about, E.

5 Cross-Section Functionals

Name: Momentum Transfer/Transport-Corrected Scattering Cross Section.

Symbol: $\alpha(\overrightarrow{r}, E)$.

Units: $(steradian/cm)$.

Definition: $\alpha(\overrightarrow{r}, E) = \int_0^\infty 2\pi \int_{-1}^{+1} \sigma_s(\overrightarrow{r}, E \to E', \mu_s)(1 - \mu_s)\, d\mu_s\, dE'$.

Interpretation: The expression,

$$\alpha(\overrightarrow{r}, E')\, ds\,,$$

represents the expected change in direction due to scattering of a particle starting at position $\overrightarrow{r}$ with initial energy E, and traveling a differential distance ds.

Name: Stopping Power.

Symbol: $\alpha(\overrightarrow{r}, E)$.

Units: (MeV/cm).

Definition: $\beta(\overrightarrow{r}, E) = 2\pi \int_{-1}^{+1} \int_0^\infty \sigma_s(\overrightarrow{r}, E \to E', \mu_s)(E - E')\, dE'\, d\mu_s$.

Interpretation: The expression,

$$\beta(\overrightarrow{r}, E)\, ds\,,$$

represents the expected energy loss due to scattering of a particle starting at position $\overrightarrow{r}$ with initial energy E and traveling a differential distance ds.

6 Derivation of the Macroscopic Cross Section

The microscopic cross section is easily understood, but the origins of the macroscopic cross section are not obvious. Here we give a very simple derivation of the macroscopic cross section that will hopefully be instructive. Consider a rectangular box having dimensions, Δx, Δy, and Δz, where $\Delta x = \Delta y$ and Δz is small relative to the other two dimensions. Assume that:

- this box contains randomly positioned target nucleii with an average density denoted by ρ_t ($targets/cm^3$),
- that fluctuations in the target density are negligible on a scale length that is small with respect to the box dimensions,
- that each target nucleus has an effective microscopic interaction cross section denoted by $\hat{\sigma}$,
- and that the average distance between nucleii, $\bar{\rho} = \rho_t^{-\frac{1}{3}}$ is much larger than the effective interaction cross sectional radius, $\hat{r} = (\hat{\sigma}/\pi)^{-\frac{1}{2}}$.

Denote one of the two faces of the box having area $\Delta A = \Delta x \Delta y$ as the "entry" face. If one peers into the box through the entry face along a line of sight normal to that face, the total target area that will be seen is given by

$$A_t = \rho_t \hat{\sigma} \Delta A \, \Delta x \, , \tag{12}$$

assuming that Δx is chosen sufficiently small that no target nucleus is shadowed by another. The total target area seen through the entry face divided by the area of the entry face represents the probability that a transport particle entering the box at a random point on the entry face in a direction normal to that face will interact in the box. This probability is given by

$$\begin{aligned} A_t/\Delta A &= \rho_t \hat{\sigma} \Delta x \, , \\ &= \sigma \Delta x \, , \end{aligned} \tag{13}$$

where σ denotes the macroscopic interaction cross section. Note that Eq. (13) is consistent with the definition of the macroscopic cross section in that $A_t/\Delta A$ represents the expected number of interactions for a particle that travels a distance Δx in the transport medium.