A 1D Spherical S_N Code with Diffusion Synthetic Acceleration

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1.1

The 1D spherical S_N code was used to solve for the scalar flux using an isotropic incident flux. The solution is shown in Fig.(1).

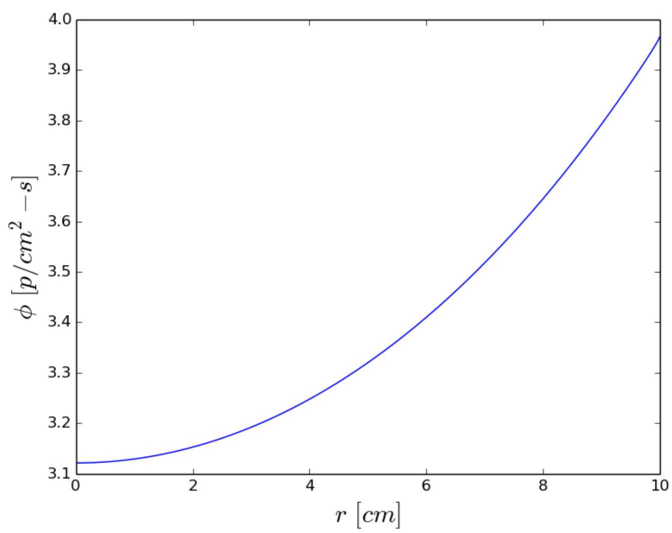

Figure 1: Scalar flux as a function of r for part a

1.2

The 1D spherical S_N code was used to solve for the scalar flux, using a delta-function distribution for the incident angular flux. The solution is shown in Fig.(2). The rapid variation in the solution near the boundary is called the "boundary layer".

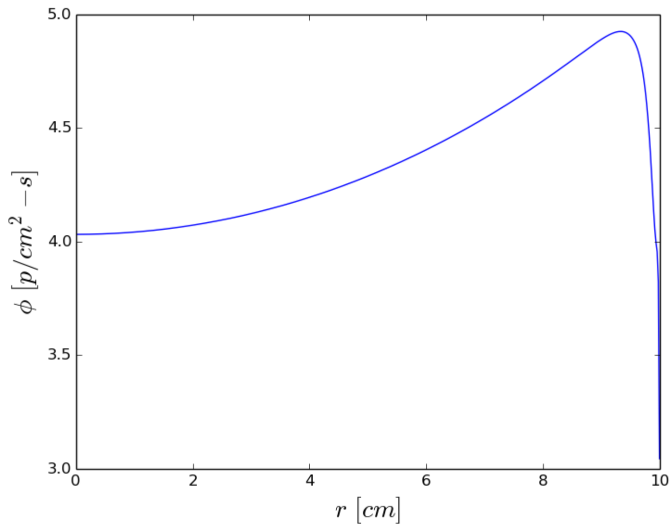

Figure 2: Scalar flux as a function of r for part b

1.3

For the boundary layer, the leading-order diffusion-limit solution is approximately

$$\phi_B = \int_{-1}^{0} d\mu\,(3\mu^2 + 2\mu) f(\mu)$$

where

$$f(\mu) = C\,\delta(\mu - \mu_1)\,.$$

In addition, we know that the incident partial current on the right boundary is 1, thus we can solve for the constant C,

$$J^- = 1 = \int_{-1}^{0} d\mu\,\mu f(\mu)$$

$$1 = \int_{-1}^{0} d\mu\,\mu C \delta(\mu - \mu_1)$$

$$C = \frac{1}{\mu_1}\,.$$

Thus,

$$f(\mu) = \frac{\delta(\mu - \mu_1)}{\mu_1}\,.$$

By plugging our updated $f(\mu)$ into our equation for ϕ_B, we get

$$\phi_B = \int_{-1}^{0} d\mu\,(3\mu^2 + 2\mu) \frac{\delta(\mu - \mu_1)}{\mu_1}$$

$$\phi_B = 3|\mu_1| + 2$$

$$\phi_B = 4.968$$

This is far from the value of $\phi_B = 2.839$ provided by the S_N code. However, by doing a least-squares fit to the flux values between 0 and 9 cm we get the extrapolated value on the boundary would be 5.097, this is within 2.5% of the analytical value. The S_N solution and the least-squares extrapolated solution are both shown in Fig.(3).

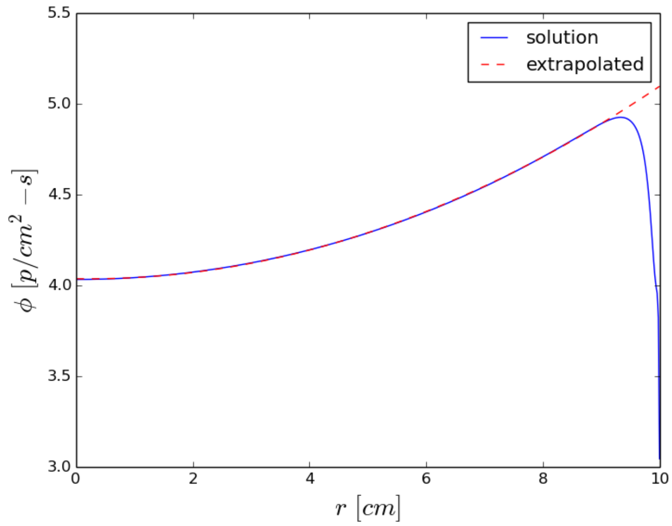

Figure 3: Scalar flux as a function of r for part c

1.4

The S_N code was used to solve for the scalar flux with only 10 cells, using an isotropic incident flux. This solution was compared to the solution for part a, shown in Fig.(4).

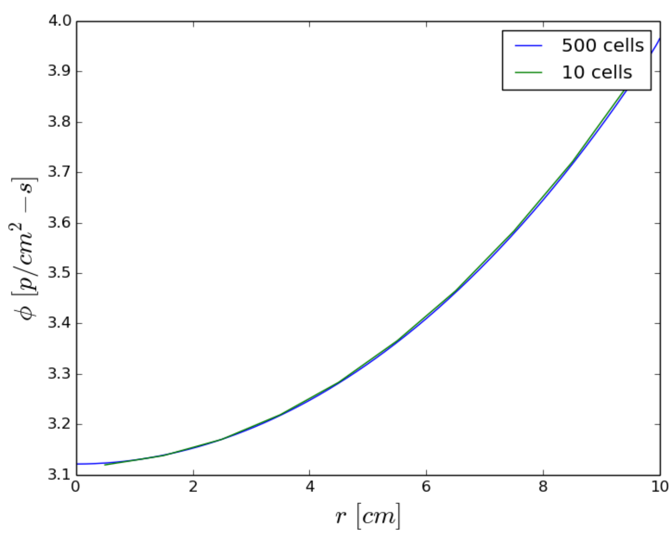

Figure 4: Scalar flux as a function of r for part d

1.5

The S_N code was used to solve for the scalar flux with only 10 cells, using a delta-function distribution for the incident angular flux. This solution was compared to the solution for part b, shown in Fig.(5).

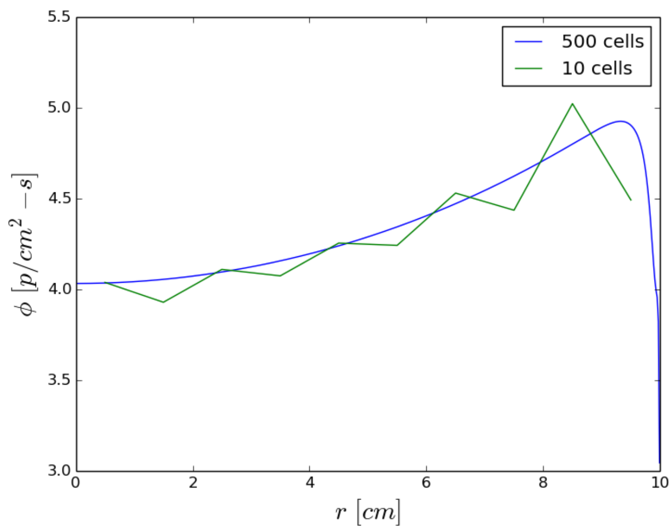

Figure 5: Scalar flux as a function of r for part e

As shown in Fig.(5), making the spatial discretization too coarse results in spatial oscillations. The reason for these spatial oscillations occur can be derived from equation for the starting flux

$$\psi_{i-1/2,1/2} = \frac{\left(1 - \frac{\sigma_t \Delta r}{2}\right)\psi_{i+1/2,1/2} + Q_{i,1/2}\Delta r}{1 + \frac{\sigma_t \Delta r}{2}} .$$

By plugging in our values for σ_t and Δr (which are 5 and 1 respectively) we get

$$\psi_{i-1/2,1/2} = -\frac{3}{7}\psi_{i+1/2,1/2} + \frac{2}{9}Q_{i,1/2} .$$

Thus, if $\psi_{i+1/2,1/2} > \frac{27}{14}Q_{i,1/2}$ we will get spatial oscillations for ψ. For this problem, these spatial oscillations will be strongest for $\mu = -1$ and μ_1 because of the delta function incident flux. The simple way to get rid of these oscillations is to satisfy the following condition:

$$\left(1 - \frac{\sigma_t \Delta r}{2}\right) > 0$$

or a slightly different version of this condition for a spherical geometry.

2

2.1

The response for the forward method is

$$R = \oint dA \int_{\mu>0} d\mu \, \psi(1, \mu)\mu$$

or using the S_N code, it can be evaluated by

$$R = A \sum_{m=M/2}^{M} \psi(1, \mu_m)\mu_m w_m \, .$$

Using the forward method, the S_N code gives $R = 4.93447$.

2.2

The adjoint is defined as

$$\langle q^\dagger, \psi \rangle = \langle q, \psi^\dagger \rangle - \int_0^\infty dE \oint dA \int d\Omega \, \psi\psi^\dagger \hat{\Omega} \cdot \hat{n} \, .$$

For this problem the adjoint simplifies to just

$$\langle q^\dagger, \psi \rangle = \langle q, \psi^\dagger \rangle - \oint dA \int d\mu \, \psi\psi^\dagger \mu \, .$$

By splitting up the boundary integral into negative and positive μ values we get

$$\langle q^\dagger, \psi \rangle = \langle q, \psi^\dagger \rangle - \oint dA \int_{\mu<0} d\mu \, \psi\psi^\dagger \mu - \oint dA \int_{\mu>0} d\mu \, \psi\psi^\dagger \mu \, .$$

Since the response for this problem is not volumetric, $q^\dagger = 0$, and since there is no source, $q = 0$. In addition, since we do not automatically know what the forward flux is for $\mu > 0$, we can set $\psi^\dagger(\mu > 0) = 0$ for simplification. This leaves us with

$$R = -\oint dA \int_{\mu<0} d\mu \, \psi \psi^\dagger \mu$$

which we solve by doing an adjoint calculation.

In the S_N code, an adjoint calculation is performed by setting the extraneous source equal to $q^\dagger(-\mu)$. The solution we get from this S_N calculation is $\hat{\psi}$. We can then calculate the adjoint flux by setting $\psi^\dagger(\mu) = \hat{\psi}(-\mu)$.

Thus, our response can be determine via the following inner product

$$R = -A \sum_{m=1}^{M/2} \psi(1, \mu_m) \hat{\psi}(1, -\mu_m) \mu_m w_m \, .$$

Using adjoints, the S_N code gives $R = 4.93447$. The same answer as with the forward method! The reason both calculations provide the same answer is that $\psi(1, \mu_m)$ is proportional to $\hat{\psi}(1, -\mu_m)$ (the source term is zero for both, and both have an isotropic boundary condition). Also, by taking the inner product between $\psi(1, \mu_m)$ and $\hat{\psi}(1, -\mu_m)$ the solutions are renormalized to give the same result as the forward solution.

2.3

In 1^{st}-order perturbation theory we start with the forward transport equation,

$$\hat{L}\psi = q$$

and assume each term gets perturbed some small amount

$$(\hat{L} + \delta\hat{L})(\psi + \delta\psi) = q + \delta q$$

$$\hat{L}\psi + \psi\delta\hat{L} + \hat{L}\delta\psi + \delta\hat{L}\delta\psi = q + \delta q\,.$$

Now we eliminate 2^{nd}-order terms for simplication

$$\hat{L}\psi + \psi\delta\hat{L} + \hat{L}\delta\psi = q + \delta q$$

and take an inner product to get

$$\langle\psi^\dagger, \hat{L}\psi\rangle + \langle\psi^\dagger, \psi\delta\hat{L}\rangle + \langle\psi^\dagger, \hat{L}\delta\psi\rangle = \langle\psi^\dagger, q\rangle + \langle\psi^\dagger, \delta q\rangle\,. \tag{1}$$

Similary we can start with adjoint equation,

$$\hat{L}^\dagger\psi^\dagger = q^\dagger$$

and take an inner product with $\psi + \delta\psi$ to get

$$\langle\hat{L}^\dagger\psi^\dagger, \psi\rangle + \langle\hat{L}^\dagger\psi^\dagger, \delta\psi\rangle = \langle\hat{L}^\dagger\psi^\dagger, q^\dagger\rangle \tag{2}$$

Next by subtracting Eq.(1) from Eq.(2) we get

$$\langle q^\dagger, \delta\psi\rangle = \langle\delta q, \psi^\dagger\rangle - \langle\delta\hat{L}\psi, \psi^\dagger\rangle - \int_0^\infty dE \oint dA \int d\Omega\, \delta\psi\psi^\dagger\hat{\Omega}\cdot\hat{n}$$

$$\delta R = \langle\delta q, \psi^\dagger\rangle - \langle\delta\hat{L}\psi, \psi^\dagger\rangle - \int_0^\infty dE \oint dA \int d\Omega\, \delta\psi\psi^\dagger\hat{\Omega}\cdot\hat{n} \tag{3}$$

For this problem since there is only a change in $\delta\hat{L}$, Eq.(3) simplifies to

$$\delta R = -\langle\delta\hat{L}\psi, \psi^\dagger\rangle\,.$$

Which can be approximated with S_N as

$$\delta R = -\delta\sigma_a \sum_{m=1}^{M}\sum_{i=1}^{I} \psi(1,\mu_m)\hat{\psi}(1,-\mu_m)\frac{4\pi}{3}(r_{i+1/2}^3 - r_{i-1/2}^3)w_m\,.$$

Thus,

$$\frac{\partial R}{\partial\sigma_a} = -\sum_{m=1}^{M}\sum_{i=1}^{I} \psi(1,\mu_m)\hat{\psi}(1,-\mu_m)\frac{4\pi}{3}(r_{i+1/2}^3 - r_{i-1/2}^3)w_m\,.$$

Using this forward adjoint inner product, the S_N code gives $\frac{\partial R}{\partial\sigma_a} = 3.32628$.

2.4

The 1$^{\text{st}}$-order estimate of δR can be computed as

$$\delta R = \frac{\partial R}{\partial \sigma_a} \delta \sigma_a \,.$$

Thus, the 1$^{\text{st}}$-order perturbation theory estimate is $\delta R = -0.16631$. By using two seperate S_N calculations and finding the difference in the response we get that $\delta R = -0.16158$. The difference between the perturbation theory estimate and true δR is only 2.85%. The reason this perturbation theory estimate isn't exact is because it is only a 1$^{\text{st}}$-order perturbation theory estimate. In other words, we're attempting to approximate a curve using a line. Higher-order perturbation theory would have to be used in order to obtain a more accurate result for δR.