Lecture 6
Using the Point-Kernel Formula

The purpose of this lecture is to illustrate the application of the point-kernel formula in the contexts of general anisotropic point sources, line sources, surface sources, and volumetric sources for the purposes of computing both the scalar flux and the current. For purposes of discussion, we will assume monoenergetic particles and spatially-constant cross sections. The extension to space-dependent cross sections and energy-dependent particles is straightforward.

1 Point Sources

A general point source at position $\overrightarrow{r}_0$ is denoted by $Q_p(\overrightarrow{r}_0, \overrightarrow{\Omega})$ and has units of $particles/sec-steradian$. The contributions to the uncollided scalar flux and current at position $\overrightarrow{r}$ from a point source at position $\overrightarrow{r}_0$ are respectively given by

$$\phi(\overrightarrow{r}) = Q_p(\overrightarrow{r}_0, \overrightarrow{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}, \tag{1}$$

and

$$\overrightarrow{J}(\overrightarrow{r}) = \overrightarrow{\Omega}_0 Q_p(\overrightarrow{r}_0, \overrightarrow{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}, \tag{2}$$

where

$$\overrightarrow{\Omega}_0 = \frac{\overrightarrow{r} - \overrightarrow{r}_0}{\|\overrightarrow{r} - \overrightarrow{r}_0\|}\,, \tag{3}$$

and

$$s = \|\overrightarrow{r} - \overrightarrow{r}_0\|\,. \tag{4}$$

2 Line Sources

A general line source at position $\overrightarrow{r}_0$ is denoted by $Q_l(\overrightarrow{r}_0, \overrightarrow{\Omega})$ and has units of $particles/cm-sec-steradian$. It is uniquely associated with a line that passes through the point $\overrightarrow{r}_0$. The contributions to the uncollided scalar flux and current at position $\overrightarrow{r}$ from a line source at position $\overrightarrow{r}_0$ are differential and respectively given by:

$$d\phi(\overrightarrow{r}) = Q_l(\overrightarrow{r}_0, \overrightarrow{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}\,dl_0\,, \tag{5}$$

and

$$d\overrightarrow{J}(\overrightarrow{r}) = \overrightarrow{\Omega}_0\,Q_l(\overrightarrow{r}_0, \overrightarrow{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}\,dl_0\,, \tag{6}$$

where

$$\overrightarrow{\Omega}_0 = \frac{\overrightarrow{r} - \overrightarrow{r}_0}{\|\overrightarrow{r} - \overrightarrow{r}_0\|} \tag{7}$$

$$s = \|\overrightarrow{r} - \overrightarrow{r}_0\|\,. \tag{8}$$

and dl_0 corresponds to a differential length located at $\overrightarrow{r}_0$ oriented along the associated line passing through $\overrightarrow{r}_0$.

3 Surface Sources

A general surface source at position $\vec{r}_0$ is denoted by $Q_s(\vec{r}_0, \vec{\Omega})$ and has units of $particles/cm^2 - sec - steradian$. It is uniquely associated with a surface that passes through the point $\vec{r}_0$. The contributions to the uncollided scalar flux and current at position $\vec{r}$ from a surface source at position $\vec{r}_0$ are differential and respectively given by:

$$d\phi(\vec{r}) = Q_s(\vec{r}_0, \vec{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}\, dA_0\,, \tag{9}$$

and

$$d\vec{J} = \vec{\Omega}_0\, Q_s(\vec{r}_0, \vec{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}\, dA_0\,, \tag{10}$$

where

$$\vec{\Omega}_0 = \frac{\vec{r} - \vec{r}_0}{\|\vec{r} - \vec{r}_0\|} \tag{11}$$

$$s = \|\vec{r} - \vec{r}_0\|\,. \tag{12}$$

and dA_0 corresponds to a differential surface area located at $\vec{r}_0$ that is parallel to the associated surface passing through $\vec{r}_0$.

4 Volumetric Sources

As discussed in Lecture 5, a general volumetric source at position $\vec{r}_0$ is denoted by $Q(\vec{r}_0, \vec{\Omega})$ and has units of $particles/cm^3 - sec - steradian$. The contributions to the uncollided scalar flux and current at position $\overrightarrow{r}$ from a volumetric source at position $\overrightarrow{r}_0$ are differential and respectively given by:

$$d\phi(\overrightarrow{r}) = Q(\overrightarrow{r}_0, \overrightarrow{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}\, dV_0\,, \tag{13}$$

and

$$d\overrightarrow{J}(\overrightarrow{r}) = \overrightarrow{\Omega}_0\, Q(\overrightarrow{r}_0, \overrightarrow{\Omega}_0)\frac{\exp(-\Sigma_t s)}{s^2}\, dV_0\,, \tag{14}$$

where

$$\overrightarrow{\Omega}_0 = \frac{\overrightarrow{r} - \overrightarrow{r}_0}{\|\overrightarrow{r} - \overrightarrow{r}_0\|} \tag{15}$$

$$s = \|\overrightarrow{r} - \overrightarrow{r}_0\|\,. \tag{16}$$

and dV_0 corresponds to a differential surface area located at $\overrightarrow{r}_0$.

5 Examples

In this section, we give some simple examples of various types of sources and solution. We begin by assuming a point source at the origin and a solution point at an arbitrary position, as illustrated in Fig. 1. If we assume a total macroscopic cross section, Σ_t, and an isotropic point source, $Q_p = \frac{q_0}{4\pi}$ $(p/sec - steradian)$, the uncollided scalar flux at an arbitrary point is given by

$$\phi(x, y, z) = \frac{q_0}{4\pi}\frac{\exp\left(-\Sigma_t\sqrt{x^2+y^2+z^2}\right)}{x^2+y^2+z^2}\,, \tag{17}$$

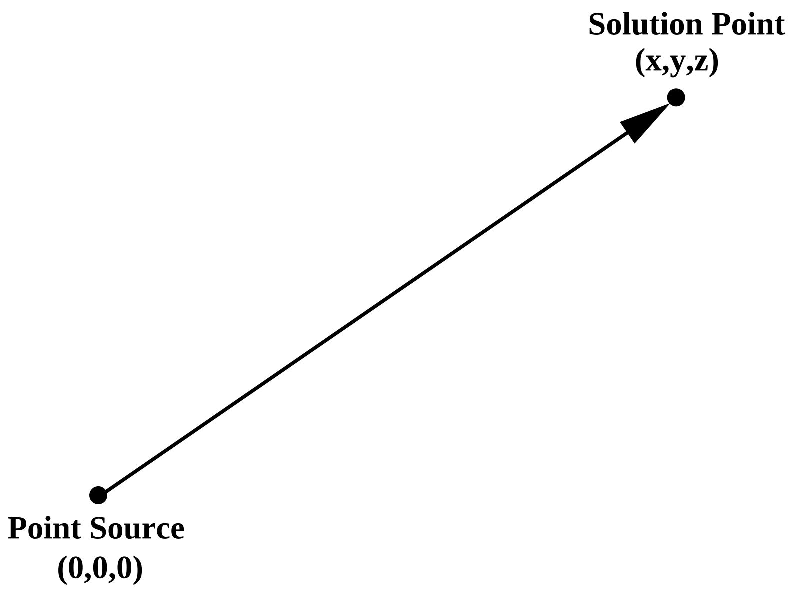

Figure 1: Illustration of a point source and the point at which the corresponding scalar flux and current solutions are desired.

and the uncollided current is given by

$$\overrightarrow{J}(x,y,z) = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{\sqrt{x^2+y^2+z^2}} \, \frac{q_0}{4\pi} \frac{\exp\left(-\Sigma_t \sqrt{x^2+y^2+z^2}\right)}{x^2+y^2+z^2}, \tag{18}$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ denote the standard unit vectors associated with the x-axis, y-axis, and z-axis, respectively. If we replace the isotropic point source with an anisotropic point source, $Q_p = \frac{q_0}{4\pi} + \frac{3q_1}{4\pi}\mu \; (p/sec - steradian)$, where μ denotes the x-axis direction cosine. The uncollided scalar flux is given by

$$\phi(x,y,z) = \left(\frac{q_0}{4\pi} + \frac{3q_1}{4\pi} \frac{x}{\sqrt{x^2+y^2+z^2}} \right) \frac{\exp\left(-\Sigma_t \sqrt{x^2+y^2+z^2}\right)}{x^2+y^2+z^2}, \tag{19}$$

and the uncollided current is given by

$$\overrightarrow{J} = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{\sqrt{x^2+y^2+z^2}} \left(\frac{q_0}{4\pi} + \frac{3q_1}{4\pi} \frac{x}{\sqrt{x^2+y^2+z^2}} \right) \frac{\exp\left(-\Sigma_t \sqrt{x^2+y^2+z^2}\right)}{x^2+y^2+z^2}, \tag{20}$$

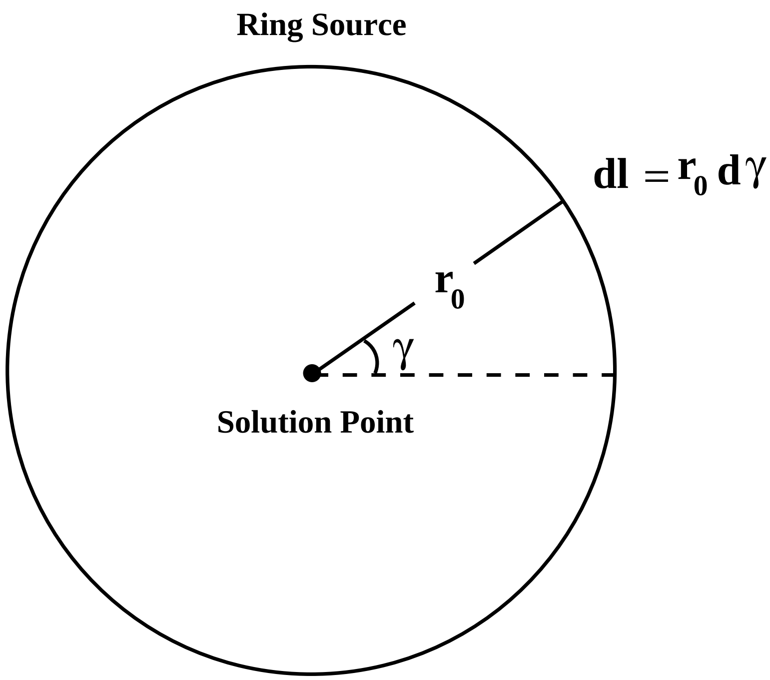

Figure 2: Illustration of a Ring Source.

We next replace the point source with a particular type of line source corresponding to a ring. This type of source is illustrated in Fig. 2. Assuming an isotropic line source, $Q_l = \frac{q_0}{4\pi}$ $(particles/cm - sec - steradians)$, the uncollided flux at the center of the ring is given by

$$\begin{aligned} \phi &= \int_0^{2\pi} \frac{q_0}{4\pi} \frac{\exp(-\Sigma_t r_0)}{r_0^2}\, r_0\, d\gamma\,, \\ &= \frac{q_0}{2} \frac{\exp -\Sigma_t r_0}{r_0}\,. \end{aligned} \tag{21}$$

Orienting the coordinate system so that $x = r_0 \cos(\gamma)$ and $y = r_0 \sin(\gamma)$, we find that the

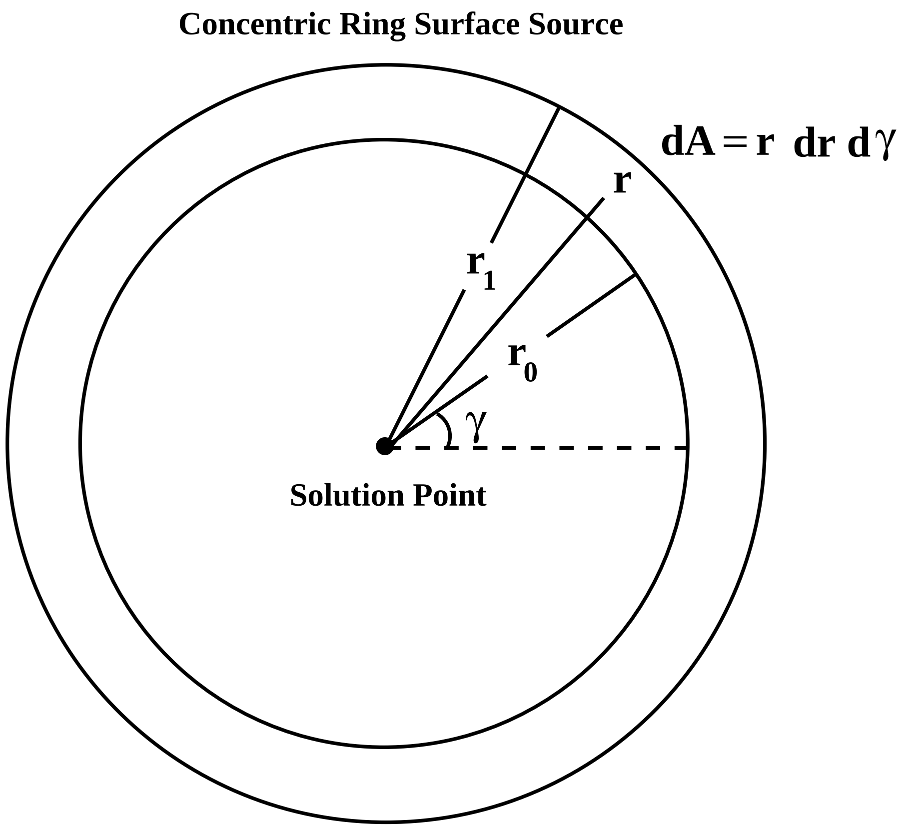

Figure 3: Illustration of a surface source defined by two concentric rings.

uncollided current at the center of the sphere is

$$\begin{aligned}\overrightarrow{J} &= -\int_0^{2\pi} \left(\hat{i}\,\cos(\gamma) + \hat{j}\,\sin(\gamma)\right) \frac{q_0}{4\pi} \frac{\exp(-\Sigma_t r_0)}{r_0^2}\, r_0\, d\gamma\,, \\ &= \overrightarrow{0}\,. \end{aligned} \tag{22}$$

Next we replace the ring source with a surface source defined on a domain defined by two concentric rings having inner and outer radii of r_0 cm and r_1 cm, respectively. This source is illustrated in Fig. 3. Assuming an isotropic surface source, $Q_s = \frac{q_0 r}{4\pi}$ $(particles/cm^2 -$

$sec - steradians$, the uncollided scalar flux at the center of the concentric rings is given by

$$
\begin{aligned}
\phi &= \int_0^{2\pi} \int_{r_0}^{r_1} \frac{q_0 r}{4\pi} \frac{\exp(-\Sigma_t r)}{r^2} \, r \, dr \, d\gamma \, , \\
&= \frac{q_0}{2\Sigma_t} \left[\exp(-\Sigma_t r_1) - \exp(-\Sigma_t r_2) \right] \, .
\end{aligned} \tag{23}
$$