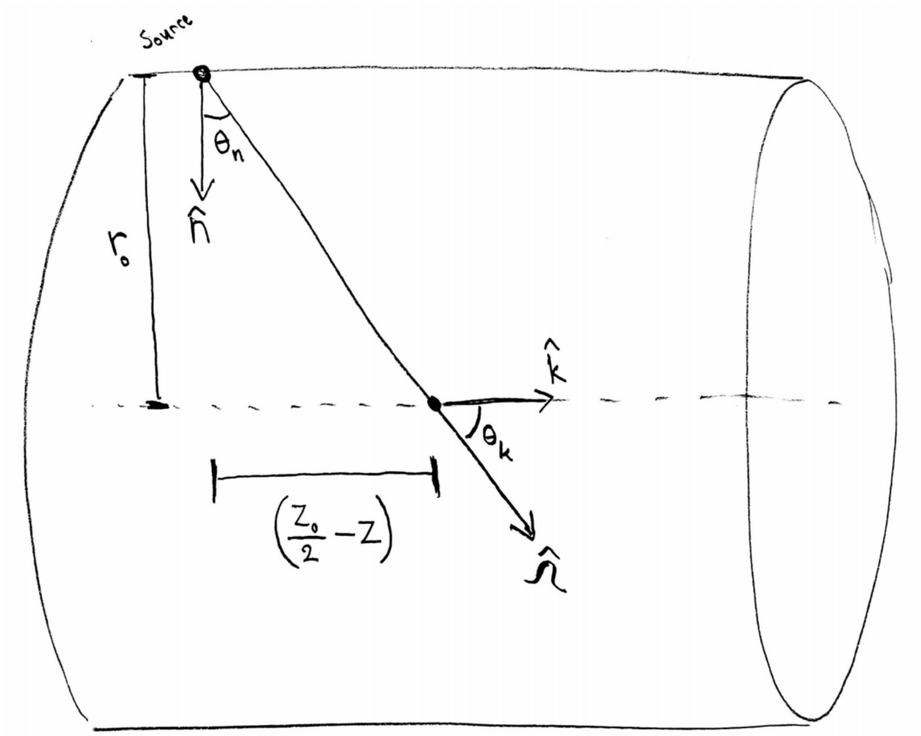
Pablo Vaquer

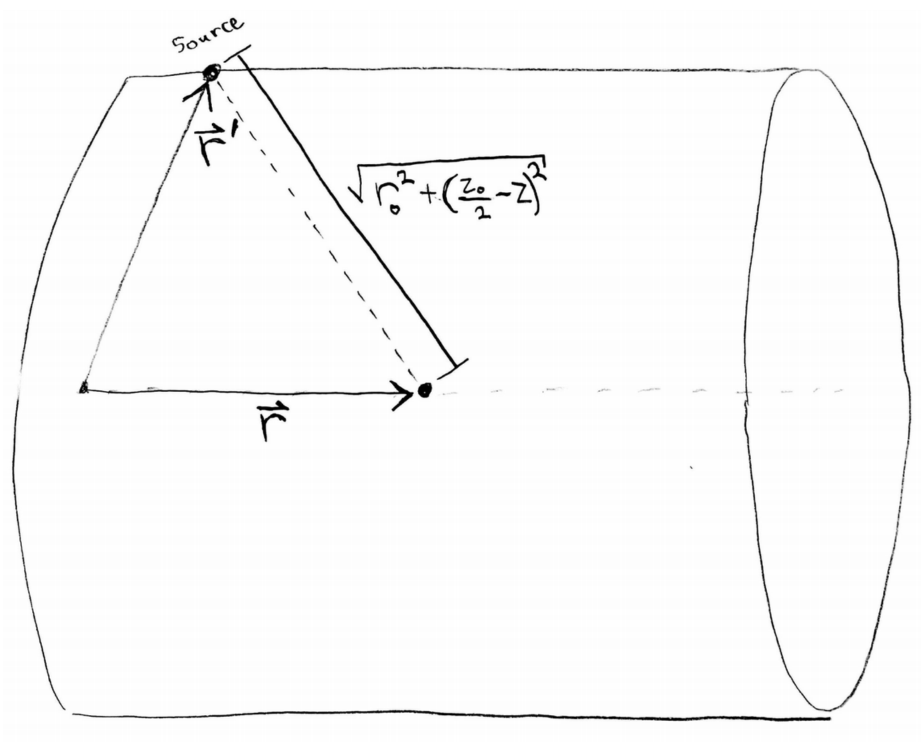
NUEN 625

Midterm

3/18/2016

The following 2 images clarify some of the notation I use for problems 1 and 2.





Problem 1

(a)

The general point kernal formula is

$$\phi(\vec{r}) = \int_V dV' \, \frac{Q(\vec{r}\,', \hat{\Omega})}{||\vec{r}\,' - \vec{r}||^2} \exp\Big[- \tau\big(\vec{r}\,', \vec{r}\big)\Big] \, .$$

Since there is no material attenuation and the source is a surface, the point kernal formula simplifies to

$$\phi(\vec{r}) = \int_A dA' \, \frac{Q_o r_o}{4\pi ||\vec{r}\,' - \vec{r}||^2} \tag{1}$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}$$

and

$$dA' = r_o d\varphi dz \, .$$

Thus,

$$\phi\Big(\frac{z_o}{2}\Big) = \int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{Q_o r_o}{4\pi \Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2 \Big]}$$

$$\phi\Big(\frac{z_o}{2}\Big) = \int_0^{z_o} dz \, \frac{Q_o r_o}{2 \Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2 \Big]}$$

$$\phi\Big(\frac{z_o}{2}\Big) = \frac{Q_o}{2r_o} \int_0^{z_o} dz \, \frac{1}{1 + \Big(\frac{\frac{z_o}{2} - z}{r_o}\Big)^2} \, .$$

Now we can use u-substitution to solve the integral where

$$u = \Big(\frac{\frac{z_o}{2} - z}{r_o}\Big)$$

$$\frac{du}{dz} = -\frac{1}{r_o}$$

$$dz = -r_o \, du \, .$$

This gives us that

$$\phi\Big(\frac{z_o}{2}\Big) = -\frac{Q_o}{2} \arctan\Big(\frac{\frac{z_o}{2} - z}{r_o}\Big)\Big|_0^{z_o}$$

$$\phi\Big(\frac{z_o}{2}\Big) = -\frac{Q_o}{2} \Bigg[\arctan\Big(-\frac{z_o}{2r_o}\Big) - \arctan\Big(\frac{z_o}{2r_o}\Big) \Bigg]$$

$$\boxed{\phi\Big(\frac{z_o}{2}\Big) = Q_o \arctan\Big(\frac{z_o}{2r_o}\Big)} \, .$$

(b)

Eq.(1) can be modified to calculate for the current by simply including a $(\hat{\Omega}\cdot\hat{k})$ term in the integrand,

$$J(\vec{r}) = \int_A dA' \, \frac{Q_o}{4\pi||\vec{r}\,' - \vec{r}||^2}(\hat{\Omega}\cdot\hat{k})$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}$$

$$(\hat{\Omega}\cdot\hat{k}) = \cos(\theta_k) = \frac{\frac{z_o}{2} - z}{\sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}}$$

and

$$dA' = r_o d\varphi dz \, .$$

Thus,

$$J\Big(\frac{z_o}{2}\Big) = \frac{Q_o r_o}{4\pi}\int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{\frac{z_o}{2} - z}{\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]^{\frac{3}{2}}}$$

$$J\Big(\frac{z_o}{2}\Big) = \frac{Q_o r_o}{2}\int_0^{z_o} dz \, \frac{\frac{z_o}{2} - z}{\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]^{\frac{3}{2}}}$$

Now we can use u-substitution to solve the integral where

$$u = r_o^2 + \big(\frac{z_o}{2} - z\big)^2$$

$$\frac{du}{dz} = -2\big(\frac{z_o}{2} - z\big)$$

$$dz = -\frac{du}{2\big(\frac{z_o}{2} - z\big)} \, .$$

This gives us that

$$J\Big(\frac{z_o}{2}\Big) = -\frac{Q_o r_o}{4}\int_{r_o^2 + (\frac{z_o}{2})^2}^{r_o^2 + (-\frac{z_o}{2})^2} dz \, u^{-\frac{3}{2}}$$

$$J\Big(\frac{z_o}{2}\Big) = \frac{Q_o r_o}{2}\left[\frac{1}{\sqrt{r_o^2 + \big(-\frac{z_o}{2}\big)^2}} - \frac{1}{\sqrt{r_o^2 + \big(\frac{z_o}{2}\big)^2}}\right]$$

$$\boxed{J\Big(\frac{z_o}{2}\Big) = 0}\, .$$

(c)

The general point kernal formula is

$$\phi(\vec{r}) = \int_V dV' \, \frac{Q(\vec{r}\,', \hat{\Omega})}{||\vec{r}\,' - \vec{r}||^2} \exp\Big[-\tau\big(\vec{r}\,', \vec{r}\big)\Big] \, .$$

Since there is no material attenuation and the source is a surface, the point kernal formula simplifies to

$$\phi(\vec{r}) = \int_A dA' \, \frac{Q_o r_o}{4\pi ||\vec{r}\,' - \vec{r}||^2} \tag{2}$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \big(z_o - z\big)^2}$$

and

$$dA' = r_o d\varphi dz \, .$$

Thus,

$$\phi(z_o) = \int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{Q_o r_o}{4\pi \Big[r_o^2 + \big(z_o - z\big)^2 \Big]}$$

$$\phi(z_o) = \int_0^{z_o} dz \, \frac{Q_o r_o}{2 \Big[r_o^2 + \big(z_o - z\big)^2 \Big]}$$

$$\phi(z_o) = \frac{Q_o}{2r_o} \int_0^{z_o} dz \, \frac{1}{1 + \Big(\frac{z_o - z}{r_o}\Big)^2} \, .$$

Now we can use u-substitution to solve the integral where

$$u = \Big(\frac{z_o - z}{r_o}\Big)$$

$$\frac{du}{dz} = -\frac{1}{r_o}$$

$$dz = -r_o \, du \, .$$

This gives us that

$$\phi(z_o) = -\frac{Q_o}{2} \arctan\Big(\frac{z_o - z}{r_o}\Big)\Big|_0^{z_o}$$

$$\phi(z_o) = -\frac{Q_o}{2} \Bigg[\arctan(0) - \arctan\Big(\frac{z_o}{r_o}\Big) \Bigg]$$

$$\boxed{\phi(z_o) = \frac{Q_o}{2} \arctan\Big(\frac{z_o}{r_o}\Big)} \, .$$

(d)

Eq.(2) can be modified to calculate for the current by simply including a $(\hat{\Omega}\cdot\hat{k})$ term in the integrand,

$$J(\vec{r}) = \int_A dA' \, \frac{Q_o r_o}{4\pi||\vec{r}\,' - \vec{r}||^2}(\hat{\Omega}\cdot\hat{k})$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \left(z_o - z\right)^2}$$

$$(\hat{\Omega}\cdot\hat{k}) = \cos(\theta_k) = \frac{z_o - z}{\sqrt{r_o^2 + \left(z_o - z\right)^2}}$$

and

$$dA' = r_o d\varphi dz \,.$$

Thus,

$$J\left(z_o\right) = \frac{Q_o r_o}{4\pi}\int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{z_o - z}{\left[r_o^2 + \left(z_o - z\right)^2\right]^{\frac{3}{2}}}$$

$$J\left(z_o\right) = \frac{Q_o r_o}{2}\int_0^{z_o} dz \, \frac{z_o - z}{\left[r_o^2 + \left(z_o - z\right)^2\right]^{\frac{3}{2}}}$$

Now we can use u-substitution to solve the integral where

$$u = r_o^2 + \left(z_o - z\right)^2$$

$$\frac{du}{dz} = -2\left(z_o - z\right)$$

$$dz = -\frac{du}{2\left(z_o - z\right)} \,.$$

This gives us that

$$J\left(z_o\right) = -\frac{Q_o r_o}{4}\int_{r_o^2+z_o^2}^{r_o^2} dz \, u^{-\frac{3}{2}}$$

$$J\left(z_o\right) = \frac{Q_o r_o}{2}\left(\frac{1}{\sqrt{r_o^2}} - \frac{1}{\sqrt{r_o^2 + z_o^2}}\right)$$

$$\boxed{J\left(z_o\right) = \frac{Q_o}{2}\left(1 - \frac{r_o}{\sqrt{r_o^2 + z_o^2}}\right)}$$

Problem 2

(a)

The general point kernal formula is

$$\phi(\vec{r}) = \int_V dV' \, \frac{Q(\vec{r}\,', \hat{\Omega})}{||\vec{r}\,' - \vec{r}||^2} \exp\Big[-\tau\big(\vec{r}\,', \vec{r}\big)\Big] \, .$$

Since there is no material attenuation and the source is a surface, the point kernal formula simplifies to

$$\phi(\vec{r}) = \int_A dA' \, \frac{Q_o}{4\pi||\vec{r}\,' - \vec{r}||^2}(\hat{\Omega}\cdot\hat{n}) \tag{3}$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}$$

$$(\hat{\Omega}\cdot\hat{n}) = \cos(\theta_n) = \frac{r_o}{\sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}}$$

and

$$dA' = r_o d\varphi dz \, .$$

Thus,

$$\phi\Big(\frac{z_o}{2}\Big) = \int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{Q_o r_o^2}{4\pi\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]^{\frac{3}{2}}}$$

$$\phi\Big(\frac{z_o}{2}\Big) = \int_0^{z_o} dz \, \frac{Q_o r_o^2}{2\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]^{\frac{3}{2}}}$$

$$\phi\Big(\frac{z_o}{2}\Big) = \frac{Q_o\big(z - \frac{z_o}{2}\big)}{2\sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}}\Bigg|_0^{z_o}$$

$$\phi\Big(\frac{z_o}{2}\Big) = \frac{Q_o\frac{z_o}{2}}{2\sqrt{r_o^2 + \big(-\frac{z_o}{2}\big)^2}} - \frac{Q_o\big(-\frac{z_o}{2}\big)}{2\sqrt{r_o^2 + \big(\frac{z_o}{2}\big)^2}}$$

$$\phi\Big(\frac{z_o}{2}\Big) = \frac{Q_o\frac{z_o}{2} + Q_o\frac{z_o}{2}}{2\sqrt{r_o^2 + \big(\frac{z_o}{2}\big)^2}}$$

$$\boxed{\phi\Big(\frac{z_o}{2}\Big) = \frac{Q_o z_o}{2\sqrt{r_o^2 + \big(\frac{z_o}{2}\big)^2}}} \, .$$

(b)

Eq.(3) can be modified to calculate for the current by simply including a $(\hat{\Omega}\cdot\hat{k})$ term in the integrandm,

$$J(\vec{r}) = \int_A dA' \, \frac{Q_o}{4\pi||\vec{r}\,' - \vec{r}||^2}(\hat{\Omega}\cdot\hat{n})(\hat{\Omega}\cdot\hat{k})$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}$$

$$(\hat{\Omega}\cdot\hat{n}) = \cos(\theta_n) = \frac{r_o}{\sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}}$$

$$(\hat{\Omega}\cdot\hat{k}) = \cos(\theta_k) = \frac{\frac{z_o}{2} - z}{\sqrt{r_o^2 + \big(\frac{z_o}{2} - z\big)^2}}$$

and

$$dA' = r_o d\varphi dz\,.$$

Thus,

$$J\Big(\frac{z_o}{2}\Big) = \int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{Q_o r_o^2\big(\frac{z_o}{2} - z\big)}{4\pi\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]^2}$$

$$J\Big(\frac{z_o}{2}\Big) = \int_0^{z_o} dz \, \frac{Q_o r_o^2\big(\frac{z_o}{2} - z\big)}{2\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]^2}$$

Now we can use u-substitution to solve the integral where

$$u = r_o^2 + \big(\frac{z_o}{2} - z\big)^2$$

$$\frac{du}{dz} = -2\big(\frac{z_o}{2} - z\big)$$

$$dz = -\frac{du}{2\big(\frac{z_o}{2} - z\big)}\,.$$

This gives us that

$$J\Big(\frac{z_o}{2}\Big) = \frac{Q_o r_o^2}{4\Big[r_o^2 + \big(\frac{z_o}{2} - z\big)^2\Big]}\Bigg|_0^{z_o}$$

$$J\Big(\frac{z_o}{2}\Big) = \frac{Q_o r_o^2}{4\Big[r_o^2 + \big(-\frac{z_o}{2}\big)^2\Big]} - \frac{Q_o r_o^2}{4\Big[r_o^2 + \big(\frac{z_o}{2}\big)^2\Big]}$$

$$\boxed{J\Big(\frac{z_o}{2}\Big) = 0}\,.$$

(c)

The general point kernal formula is

$$\phi(\vec{r}) = \int_V dV' \, \frac{Q(\vec{r}\,', \hat{\Omega})}{||\vec{r}\,' - \vec{r}||^2} \exp\Big[-\tau\big(\vec{r}\,', \vec{r}\big)\Big] \,.$$

Since there is no material attenuation and the source is a surface, the point kernal formula simplifies to

$$\phi(\vec{r}) = \int_A dA' \, \frac{Q_o}{4\pi||\vec{r}\,' - \vec{r}||^2} (\hat{\Omega} \cdot \hat{n}) \tag{4}$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \big(z_o - z\big)^2}$$

$$(\hat{\Omega} \cdot \hat{n}) = \cos(\theta_n) = \frac{r_o}{\sqrt{r_o^2 + \big(z_o - z\big)^2}}$$

and

$$dA' = r_o d\varphi dz \,.$$

Thus,

$$\phi(z_o) = \int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{Q_o r_o^2}{4\pi\Big[r_o^2 + \big(z_o - z\big)^2\Big]^{\frac{3}{2}}}$$

$$\phi(z_o) = \int_0^{z_o} dz \, \frac{Q_o r_o^2}{2\Big[r_o^2 + \big(z_o - z\big)^2\Big]^{\frac{3}{2}}}$$

$$\phi(z_o) = \frac{Q_o\big(z - z_o\big)}{2\sqrt{r_o^2 + \big(z_o - z\big)^2}}\Bigg|_0^{z_o}$$

$$\boxed{\phi(z_o) = \frac{Q_o z_o}{2\sqrt{r_o^2 + z_o^2}}} \,.$$

(d)

Eq.(4) can be modified to calculate for the current by simply including a $(\hat{\Omega}\cdot\hat{k})$ term in the integrand,

$$J(\vec{r}) = \int_A dA' \, \frac{Q_o}{4\pi||\vec{r}\,' - \vec{r}||^2}(\hat{\Omega}\cdot\hat{n})(\hat{\Omega}\cdot\hat{k})$$

where

$$||\vec{r}\,' - \vec{r}|| = \sqrt{r_o^2 + \left(z_o - z\right)^2}$$

$$(\hat{\Omega}\cdot\hat{n}) = \cos(\theta_n) = \frac{r_o}{\sqrt{r_o^2 + \left(z_o - z\right)^2}}$$

$$(\hat{\Omega}\cdot\hat{k}) = \cos(\theta_k) = \frac{z_o - z}{\sqrt{r_o^2 + \left(z_o - z\right)^2}}$$

and

$$dA' = r_o d\varphi dz\,.$$

Thus,

$$J\left(\frac{z_o}{2}\right) = \int_0^{z_o} dz \int_0^{2\pi} d\varphi \, \frac{Q_o r_o^2\left(z_o - z\right)}{4\pi\left[r_o^2 + \left(z_o - z\right)^2\right]^2}$$

$$J\left(\frac{z_o}{2}\right) = \int_0^{z_o} dz \, \frac{Q_o r_o^2\left(z_o - z\right)}{2\left[r_o^2 + \left(z_o - z\right)^2\right]^2}$$

Now we can use u-substitution to solve the integral where

$$u = r_o^2 + \left(z_o - z\right)^2$$

$$\frac{du}{dz} = -2\left(z_o - z\right)$$

$$dz = -\frac{du}{2\left(z_o - z\right)}\,.$$

This gives us that

$$J\left(\frac{z_o}{2}\right) = \frac{Q_o r_o^2}{4\left[r_o^2 + \left(z_o - z\right)^2\right]}\Bigg|_0^{z_o}$$

$$J\left(\frac{z_o}{2}\right) = \frac{Q_o r_o^2}{4r_o^2} - \frac{Q_o r_o^2}{4\left(r_o^2 + z_o^2\right)}$$

$$\boxed{J\left(\frac{z_o}{2}\right) = \frac{Q_o}{4}\left(1 - \frac{r_o^2}{r_o^2 + z_o^2}\right)}\,.$$

Problem 3

(a)

The following diffusion equation

$$-\frac{\partial}{\partial x}\frac{1}{3\sigma_t}\frac{\partial\phi}{\partial x}+\sigma_a\phi=0$$

can be rewritten as

$$\frac{\partial^2\phi}{\partial x^2}-3\sigma_t\sigma_a\phi=0\,.$$

The solution to this equation for a semi-infinite medium is

$$\phi(x)=\phi_o e^{-\sqrt{3\sigma_t\sigma_a}\,x} \tag{5}$$

with a Mark boundary at $x=0$. Note that a Mark boundary condition is an approximation of the half-range current on a boundary using alinearly-anisotropic flux with the value of $\mu=-1/\sqrt{3}$ or $\mu=1/\sqrt{3}$ (depending on the boundary), such that

$$J^+=2\pi\int_0^1 d\mu\,\mu\psi\approx 2\pi\Big(\frac{1}{\sqrt{3}}\Big)\left[\frac{1}{4\pi}\phi+\frac{3}{4\pi}\Big(\frac{1}{\sqrt{3}}\Big)J\right].$$

Thus, if $J^+=1$ then

$$2\pi\Big(\frac{1}{\sqrt{3}}\Big)\left[\frac{1}{4\pi}\phi+\frac{3}{4\pi}\Big(\frac{1}{\sqrt{3}}\Big)J\right]=1$$

$$\frac{1}{4\pi}\phi+\frac{\sqrt{3}}{4\pi}J=\frac{\sqrt{3}}{2\pi}\,.$$

$$\phi+\sqrt{3}J=2\sqrt{3}$$

and therefore

$$\phi(0)-\sqrt{3}D\frac{\partial\phi(0)}{\partial x}=2\sqrt{3}$$

$$\phi_o-\frac{\sqrt{3}}{3\sigma_t}\big(-\sqrt{3\sigma_t\sigma_a}\big)\phi_o=2\sqrt{3}$$

$$\phi_o\Big(1+\sqrt{\frac{\sigma_a}{\sigma_t}}\Big)=2\sqrt{3}$$

$$\phi_o=\frac{2\sqrt{3}}{1+\sqrt{\frac{\sigma_a}{\sigma_t}}}\,. \tag{6}$$

By plugging in Eq.(6) into Eq.(5),

$$\phi(x)=\frac{2\sqrt{3}\exp(-\sqrt{3\sigma_t\sigma_a}\,x)}{1+\sqrt{\frac{\sigma_a}{\sigma_t}}}\,.$$

The fraction reflected is equal to $\alpha=J^-/J^+$ at $x=0$, where

$$J^+=1$$

$$J^-=\frac{1}{2\sqrt{3}}\big[\phi(0)-\sqrt{3}J(0)\big]\,.$$

$$\alpha=\frac{1}{2\sqrt{3}}\Big[\phi_o+\frac{\sqrt{3}}{3\sigma_t}\big(-\sqrt{3\sigma_t\sigma_a}\big)\phi_o\Big]$$

$$\alpha = \frac{1}{2\sqrt{3}}\phi_o\Big(1 - \sqrt{\frac{\sigma_a}{\sigma_t}}\Big)$$

and by combining this with Eq.(6) we get

$$\boxed{\alpha = \left(\frac{1 - \sqrt{\frac{\sigma_a}{\sigma_t}}}{1 + \sqrt{\frac{\sigma_a}{\sigma_t}}}\right)}$$

(b)

$$\boxed{\lim_{\sigma_a \to 0} \left(\frac{1 - \sqrt{\frac{\sigma_a}{\sigma_t}}}{1 + \sqrt{\frac{\sigma_a}{\sigma_t}}}\right) = 1}$$

(c)

$$\boxed{\lim_{\sigma_a \to \sigma_t} \left(\frac{1 - \sqrt{\frac{\sigma_a}{\sigma_t}}}{1 + \sqrt{\frac{\sigma_a}{\sigma_t}}}\right) = 0}$$

Problem 4

In the weighted residual method

$$\int_{i-1/2}^{i+1/2} dx\, R(x) W_n(x) = 0\,.$$

In this case the residual is

$$R(x) = \frac{\partial \tilde{f}}{\partial x} + \sigma \tilde{f}$$

and thus

$$\int_{x_{i-1/2}}^{x_{i+1/2}} dx \left(\frac{\partial \tilde{f}}{\partial x} + \sigma \tilde{f}\right) W_n(x) = 0\,.$$

The weight space generates two different equations. For W_1,

$$\int_{x_{i-1/2}}^{x_i} dx \left(\frac{\partial \tilde{f}}{\partial x} + \sigma \tilde{f}\right) = 0$$

and for W_2

$$\int_{x_i}^{x_{i+1/2}} dx \left(\frac{\partial \tilde{f}}{\partial x} + \sigma \tilde{f}\right) = 0\,.$$

Now, for W_1 we get

$$\tilde{f}(x_i) - \tilde{f}(x_{i-1/2}) + \int_{x_{i-1/2}}^{x_i} dx\, \sigma \tilde{f} = 0\,.$$

$$\frac{f_L + f_R}{2} - 1 + \sigma f_L \frac{h_i}{2} = 0$$

$$f_L + f_R - 2 + \sigma f_L h_i = 0$$

$$(1 + \sigma h_i) f_L + f_R = 2\,. \tag{7}$$

Now, for W_2 we get

$$\tilde{f}(x_{i+1/2}) - \tilde{f}(x_i) + \int_{x_i}^{x_{i+1/2}} dx\, \sigma \tilde{f} = 0\,.$$

$$f_R - \frac{f_L + f_R}{2} + \sigma f_R \frac{h_i}{2} = 0$$

$$f_R - f_L + \sigma f_R h_i = 0$$

$$-f_L + (1 + \sigma h_i) f_R = 0\,. \tag{8}$$

From Eq.(8) we get

$$f_L = (1 + \sigma h_i) f_R\,. \tag{9}$$

By plugging in Eq.(9) into Eq.(7) we get

$$(1 + \sigma h_i)^2 f_R + f_R = 2$$

$$f_R = \frac{2}{1 + (1 + \sigma h_i)^2}$$

$$f_R = \frac{2}{2 + 2\sigma h_i + \sigma^2 h_i^2}$$

which gives us that

$$f_L = \frac{2(1 + \sigma h_i)}{2 + 2\sigma h_i + \sigma^2 h_i^2}$$

$$f_L = \frac{2 + 2\sigma h_i}{2 + 2\sigma h_i + \sigma^2 h_i^2}\,.$$

Thus, the discrete solution is

$$\boxed{\begin{array}{lcl} \tilde{f} \;= & 1 & \text{for } x \in x_{i-1/2} \\ & \frac{2+2\sigma h_i}{2+2\sigma h_i+\sigma^2 h_i^2} & \text{for } x \in (x_{i-1/2}, x_i) \\ & \frac{2}{2+2\sigma h_i+\sigma^2 h_i^2} & \text{for } x \in (x_i, x_{i+1/2}] \\ & \frac{2+\sigma h_i}{2+2\sigma h_i+\sigma^2 h_i^2} & \text{for } x = x_i \end{array}}\,.$$

Problem 5

(a)

Using asymptotic scaling,

$$\frac{\epsilon}{v}\frac{\partial \psi}{\partial t} + \mu\frac{\partial \psi}{\partial x} + \frac{\sigma}{\epsilon}\psi = \frac{\sigma}{4\pi\epsilon}g$$

$$\frac{\epsilon}{v}\frac{\partial g}{\partial t} = \frac{\sigma}{\epsilon}(\phi - g)\,.$$

After multiplying by ϵ,

$$\frac{\epsilon^2}{v}\frac{\partial \psi}{\partial t} + \mu\epsilon\frac{\partial \psi}{\partial x} + \sigma\psi = \frac{\sigma}{4\pi}g$$

$$\frac{\epsilon^2}{v}\frac{\partial g}{\partial t} = \sigma(\phi - g)\,.$$

Now by using a power series for ψ,

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + ...$$

we get that

$$\frac{\epsilon^2}{v}\frac{\partial\big(\psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + ...\big)}{\partial t} + \mu\epsilon\frac{\partial\big(\psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + ...\big)}{\partial x} + \sigma\big(\psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + ...\big) = \frac{\sigma}{4\pi}\big(g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + ...\big)$$

$$\frac{\epsilon^2}{v}\frac{\partial\big(g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + ...\big)}{\partial t} = \sigma\Big[\big(\psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + ...\big) - \big(g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + ...\big)\Big]\,.$$

Now by keeping only the terms of order 1,

$$\boxed{\psi^{(0)} = \frac{1}{4\pi}g^{(0)}} \tag{10}$$

(b)

Note that $g(x,t)$ is not a function of μ therefore using from Eq.(10) we get,

$$J^{(0)} = 0\,. \tag{11}$$

Now, by keeping only the terms of order ϵ,

$$\mu\frac{\partial\psi^{(0)}}{\partial x} + \sigma\psi^{(1)} = \frac{1}{4\pi}\sigma g^{(1)} \tag{12}$$

$$0 = \sigma(\phi^{(1)} - g^{(1)})$$

By substituting Eq.(10) into Eq.(12),

$$\frac{\mu}{4\pi}\frac{\partial g^{(0)}}{\partial x} + \sigma\psi^{(1)} = \frac{1}{4\pi}\sigma g^{(1)}$$

$$\psi^{(1)} = -\frac{\mu}{4\pi\sigma}\frac{\partial g^{(0)}}{\partial x} + \frac{1}{4\pi}g^{(1)}$$

by multiplying by μ and integrating over all 4π steradians,

$$J^{(1)} = -\frac{1}{3\sigma}\frac{\partial g^{(0)}}{\partial x}\,. \tag{13}$$

Now, by keeping only the terms of order ϵ^2,

$$\frac{1}{v}\frac{\partial\psi^{(0)}}{\partial t} + \mu\frac{\partial\psi^{(1)}}{\partial x} + \sigma\psi^{(2)} = \frac{\sigma}{4\pi}g^{(2)} \tag{14}$$

$$\frac{1}{v}\frac{\partial g^{(0)}}{\partial t} = \sigma(\phi^{(2)} - g^{(2)})\,. \tag{15}$$

Now by integrating Eq.(14) over all 4π steradians

$$\frac{1}{v}\frac{\partial\phi^{(0)}}{\partial t} + \frac{\partial J^{(1)}}{\partial x} + \sigma\phi^{(2)} = \sigma g^{(2)}$$

$$\frac{1}{v}\frac{\partial\phi^{(0)}}{\partial t} + \frac{\partial J^{(1)}}{\partial x} + \sigma(\phi^{(2)} - \sigma g^{(2)}) = 0$$

and combining it with Eq.(15)

$$\frac{1}{v}\frac{\partial\phi^{(0)}}{\partial t} + \frac{\partial J^{(1)}}{\partial x} + \frac{1}{v}\frac{\partial g^{(0)}}{\partial t} = 0\,.$$

By integrating equation Eq.(10) and substituting for $\phi^{(0)}$,

$$\frac{1}{v}\frac{\partial g^{(0)}}{\partial t} + \frac{\partial J^{(1)}}{\partial x} + \frac{1}{v}\frac{\partial g^{(0)}}{\partial t} = 0$$

$$\frac{2}{v}\frac{\partial g^{(0)}}{\partial t} + \frac{\partial J^{(1)}}{\partial x} = 0$$

and by combining this with Eq.(13) we get a diffusion equation for $g^{(0)}$,

$$\boxed{\frac{2}{v}\frac{\partial g^{(0)}}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{3\sigma}\frac{\partial g^{(0)}}{\partial x}\right) = 0}\,.$$