A 1D Spherical S_N Code with Diffusion Synthetic Acceleration

Pablo A. Vaquer

Texas A&M University

April 28, 2016

Contents

1 Introduction

A 1D spherical S_N code is silimilar to a 1D cartesian S_N code, because in both codes you solve the transport equation at a lot of different directions, μ_m, generated by a quadrature set (such as Gauss-Legendre quadrature). The main difference between spherical S_N and cartesian S_N is that in spherical S_N one must deal with the angular derivative term $\partial\psi/\partial\mu$. In S_N, the weights w_m in the quadrature set are used to determine the values of μ_m. You start by setting $\mu_{1/2} = -1$ and then set each $\mu_{m+1/2} = \mu_{m-1/2} + w_m$. The values for $\mu_{m-1/2}$ and $\mu_{m+1/2}$ can then be used for a discrtete approximation of the angular derivative term $\partial\psi_m/\partial\mu$.

Spatial discrtezation in spherical geometries is different than in cartesian geometries because the volume-averaged flux over a cell, ψ_i, is not equal to the arithmetic average of the flux on the two faces, $\psi_{i-1/2}$ and $\psi_{i+1/2}$. In addition, the quadrature points (directions) μ_m are not necessarily equal to the midpoints of $\mu_{m-1/2}$ and $\mu_{m+1/2}$. The 1D spherical S_N equation with weighted-diamond differencing is

$$\begin{aligned}\mu_m(A_{i+1/2}\psi_{i+1/2,m} - A_{i-1/2}\psi_{i-1/2,m}) + \\ \frac{1}{2}(A_{i+1/2} - A_{i-1/2})\frac{(\alpha_{m+1/2}\hat{\psi}_{i,m+1/2} - \alpha_{m-1/2}\hat{\psi}_{i,m-1/2})}{w_m} + \\ \sigma_t V_i \bar{\psi}_{i,m} = Q_{i,m}V_i \end{aligned} \quad (1)$$

where

$$\hat{\psi}_{i,m} = \hat{\gamma}_i\psi_{i+1/2,m} + (1-\hat{\gamma}_i)\psi_{i-1/2,m}$$

$$\bar{\psi}_{i,m} = \gamma_i\psi_{i+1/2,m} + (1-\gamma_i)\psi_{i-1/2,m}$$

$$\hat{\psi}_{i,m} = \beta_m\hat{\psi}_{i,m+1/2} + (1-\beta_m)\hat{\psi}_{i,m-1/2}$$

where $\hat{\gamma}_i$, γ_i, and β_m are used to properly weight the angular flux by r, r^2, and μ, respectively.

In order to solve the 1D spherical S_N equation, we must sweep in angle and space. An important characteristic of neutrons streaming in spherical geometry is that the angle μ with respect to the origin is always increasing as the neutron travels. Thus, when sweeping through the spherical geometry we start with the smallest angle $\mu = -1$ and sweep in space for that angle. Then we move onto the next smallest angle and sweep in space for that angle, and so on.

By sweeping from the smaller angle to larger angle, we know the value for $\hat{\psi}_{i,m-1/2}$ because it comes from the previous-angle's sweep. In addition, when we're sweeping spatially towards $r = 0$ we know the value of $\psi_{i+1/2,m}$, and when we're sweeping spatially towards $r = R$ we know the value of $\psi_{i-1/2,m}$. Thus for any given solve there are two known values, which leaves us with four unknowns and four equations.

1.1 Diamond-Difference with r weighting

To solve for $\hat{\gamma}_i$, we need to make both sides of the following equation equivalent

$$\hat{\gamma}_i \psi_{i+1/2} + (1-\hat{\gamma}_i)\psi_{i-1/2} = \frac{\int_{r_{i-1/2}}^{r_{i+1/2}} dr\, r\tilde{\psi}}{\int_{r_{i-1/2}}^{r_{i+1/2}} dr\, r}$$

where

$$\tilde{\psi} = \psi_{i+1/2}\Big(\frac{r - r_{i-1/2}}{h_i}\Big) + \psi_{i-1/2}\Big(\frac{r_{i+1/2} - r}{h_i}\Big)\,.$$

This yields that

$$\hat{\gamma}_i \psi_{i+1/2} + (1-\hat{\gamma}_i)\psi_{i-1/2} = \frac{1}{\Big(\frac{r_{i+1/2}^2 - r_{i-1/2}^2}{2}\Big)} \int_{r_{i-1/2}}^{r_{i+1/2}} dr\, \Bigg[\psi_{i+1/2}\Big(\frac{r^2 - r_{i-1/2}r}{h_i}\Big) + \Big(\psi_{i-1/2}\frac{r_{i+1/2}r - r^2}{h_i}\Big)\Bigg]$$

$$\hat{\gamma}_i \psi_{i+1/2} + (1-\hat{\gamma}_i)\psi_{i-1/2} = \frac{2}{h_i\Big(r_{i+1/2}^2 - r_{i-1/2}^2\Big)} \Bigg[\Big(\psi_{i+1/2} - \psi_{i-1/2}\Big)\frac{r_{i+1/2}^3 - r_{i-1/2}^3}{3} + \Big(\psi_{i-1/2}r_{i+1/2} - \psi_{i+1/2}r_{i-1/2}\Big)\frac{r_{i+1/2}^2 - r_{i-1/2}^2}{2}\Bigg]\,.$$

Now, by combining the terms that have a $\psi_{i+1/2}$, we can solve for $\hat{\gamma}_i$ and get that

$$\hat{\gamma}_i = \frac{2}{h_i\Big(r^2_{i+1/2} - r^2_{i-1/2}\Big)}\left[\frac{r^3_{i+1/2} - r^3_{i-1/2}}{3} - r_{i-1/2}\frac{r^2_{i+1/2} - r^2_{i-1/2}}{2}\right]$$

$$\hat{\gamma}_i = \frac{\frac{2}{3}r^3_{i+1/2} - r_{i-1/2}r^2_{i+1/2} + \frac{1}{3}r^3_{i-1/2}}{h_i\Big(r^2_{i+1/2} - r^2_{i-1/2}\Big)} \,.$$

Note that

$$h_i = r_{i+1/2} - r_{i-1/2} \,.$$

This ensures that $\hat{\gamma}_i$ is r-weighted.

1.2 Diamond-Difference with r^2 weighting

To solve for γ_i, we need to make both sides of the following equation equivalent

$$\gamma_i\psi_{i+1/2} + (1-\gamma_i)\psi_{i-1/2} = \frac{\int_{r_{i-1/2}}^{r_{i+1/2}} dr\, r^2\tilde{\psi}}{\int_{r_{i-1/2}}^{r_{i+1/2}} dr\, r^2}$$

where

$$\tilde{\psi} = \psi_{i+1/2}\Big(\frac{r - r_{i-1/2}}{h_i}\Big) + \psi_{i-1/2}\Big(\frac{r_{i+1/2} - r}{h_i}\Big) \,.$$

This yields that

$$\gamma_i\psi_{i+1/2}+(1-\gamma_i)\psi_{i-1/2} = \frac{1}{\Big(\frac{r^3_{i+1/2}-r^3_{i-1/2}}{3}\Big)}\int_{r_{i-1/2}}^{r_{i+1/2}} dr\,\left[\psi_{i+1/2}\Big(\frac{r^3 - r_{i-1/2}r^2}{h_i}\Big)+\Big(\psi_{i-1/2}\frac{r_{i+1/2}r^2 - r^3}{h_i}\Big)\right]$$

$$\begin{aligned}\gamma_i\psi_{i+1/2} + (1-\gamma_i)\psi_{i-1/2} = \frac{3}{h_i\Big(r^3_{i+1/2} - r^3_{i-1/2}\Big)}\Bigg[&\Big(\psi_{i+1/2} - \psi_{i-1/2}\Big)\frac{r^4_{i+1/2} - r^4_{i-1/2}}{4} + \\ &\Big(\psi_{i-1/2}r_{i+1/2} - \psi_{i+1/2}r_{i-1/2}\Big)\frac{r^3_{i+1/2} - r^3_{i-1/2}}{3}\Bigg] \,.\end{aligned}$$

Now, by combining the terms that have a $\psi_{i+1/2}$, we can solve for γ_i and get that

$$\gamma_i = \frac{3}{h_i\Big(r^3_{i+1/2} - r^3_{i-1/2}\Big)}\left[\frac{r^4_{i+1/2} - r^4_{i-1/2}}{4} - r_{i-1/2}\frac{r^3_{i+1/2} - r^3_{i-1/2}}{3}\right]$$

$$\gamma_i = \frac{\frac{3}{4}r^4_{i+1/2} - r_{i-1/2}r^3_{i+1/2} + \frac{1}{4}r^4_{i-1/2}}{h_i\left(r^3_{i+1/2} - r^3_{i-1/2}\right)} \, .$$

This ensures that γ_i is r^2-weighted.

2 Sweeping and Source Iteration

Sweeping in spherical geometry involves three steps. First, one must first sweep for the starting direction flux $\mu = -1$, eventhough $\mu = -1$ is not in the quadrature set. The purpose of doing this is that the 1D spherical S_N equation simplifies to the just the equation for a slab, and it provides the values for $\hat{\psi}_{i,1/2}$ which is the angular inflow term which is needed for the first quadrature direction sweep μ_1. Afterwards, one must sweep for the inwards for quadrature directions $\mu_m < 0$, and outward for quadrature directions $\mu_m > 0$. Each sweep for a particular direction μ_m updates the angular inflow term $\hat{\psi}_{i,m-1/2}$ that is needed for the next direction sweep.

After sweeping for all quadrature directions we have completed one iteration and we can update the flux moments ϕ_k which is needed for the scattering source term,

$$\phi_k = \sum_m^M P_k(\mu_m)\psi_m w_m.$$

In addition, one can using diffusion synthetic acceleration (DSA) to get an improved esimate of ϕ_k before moving to the next iteration,

$$\phi_k = \delta\phi_k + \sum_m^M P_k(\mu_m)\psi_m w_m \, .$$

This is explained in more detail in sections 4 and 5 of this report.

2.1 Starting-Direction Flux Sweep

The first step in solving the S_N equation is to sweep for the starting direction flux $\mu = -1$. We start with the outermost cell (because we know what the incoming flux is) and sweep towards the innermost cells of the sphere. Thus we need to obtain an equation for $\psi_{i-1/2,1/2}$ in terms of $\psi_{i+1/2,1/2}$

$$-\psi_{i+1/2,1/2} + \psi_{i-1/2,1/2} + \sigma_t \bar{\psi}_{i,1/2} \Delta r = Q_{i,1/2} \Delta r$$

$$-\psi_{i+1/2,1/2} + \psi_{i-1/2,1/2} + \frac{\sigma_t \Delta r}{2} (\psi_{i+1/2,1/2} + \psi_{i-1/2,1/2}) = Q_{i,1/2} \Delta r$$

$$\psi_{i-1/2,1/2} = \frac{\left(1 - \frac{\sigma_t \Delta r}{2}\right) \psi_{i+1/2,1/2} + Q_{i,1/2} \Delta r}{1 + \frac{\sigma_t \Delta r}{2}} .$$

From here we can calculate angular derivate term (which is needed to solve the equation for the next direction, μ_1),

$$\hat{\psi}_{i,1/2} = \hat{\gamma}_i \psi_{i+1/2,1/2} + (1 - \hat{\gamma}_i) \psi_{i-1/2,1/2} .$$

2.2 Inward Flux Sweep

Once we know the starting flux in all spatial cells, we can sweep for the first direction in the quadrature set, μ_1. Once again, we start with the outermost cell (because we know what the incoming flux is) and sweep towards the innermost cells of the sphere. We already know $\hat{\psi}_{i,m-1/2}$ from the previous sweep so we can derive an equation for $\psi_{i-1/2,m}$ in terms of $\psi_{i+1/2,m}$ for any $\mu_m < 0$,

$$\mu_m (A_{i+1/2} \psi_{i+1/2,m} - A_{i-1/2} \psi_{i-1/2,m}) + \frac{1}{2} (A_{i+1/2} - A_{i-1/2}) \frac{(\alpha_{m+1/2} \hat{\psi}_{i,m+1/2} - \alpha_{m-1/2} \hat{\psi}_{i,m-1/2})}{w_m} + \sigma_t V_i \bar{\psi}_{i,m} = Q_{i,m} V_i$$

where

$$\hat{\psi}_{i,m+1/2} = \frac{1}{\beta_m} \left[\hat{\gamma}_i \psi_{i+1/2,m} + (1 - \hat{\gamma}_i) \psi_{i-1/2,m} - (1 - \beta_m) \hat{\psi}_{i,m-1/2} \right]$$

$$\bar{\psi}_{i,m} = \gamma_i \psi_{i+1/2,m} + (1 - \gamma_i) \psi_{i-1/2,m} .$$

By expanding all terms we get that,

$$\mu_m A_{i+1/2}\psi_{i+1/2,m} - \mu_m A_{i-1/2}\psi_{i-1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}\hat{\gamma}_i\psi_{i+1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\hat{\gamma}_i)\psi_{i-1/2,m} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m)\hat{\psi}_{i,m-1/2} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m}\alpha_{m-1/2}\hat{\psi}_{i,m-1/2} + \sigma_t V_i\gamma_i\psi_{i+1/2,m} + \sigma_t V_i(1-\gamma_i)\psi_{i-1/2,m} = Q_{i,m}V_i\ .$$

This leaves us with an equation with only one unknown $\psi_{i-1/2}$. By manipulating the equation above we get

$$-\mu_m A_{i-1/2}\psi_{i-1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\hat{\gamma}_i)\psi_{i-1/2,m} + \sigma_t V_i(1-\gamma_i)\psi_{i-1/2,m} = -\mu_m A_{i+1/2}\psi_{i+1/2,m} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}\hat{\gamma}_i\psi_{i+1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m)\hat{\psi}_{i,m-1/2} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m}\alpha_{m-1/2}\hat{\psi}_{i,m-1/2} - \sigma_t V_i\gamma_i\psi_{i+1/2,m} + Q_{i,m}V_i$$

$$\psi_{i-1/2,m} = \frac{1}{-\mu_m A_{i-1/2} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\hat{\gamma}_i) + \sigma_t V_i(1-\gamma_i)} \times \left\{\left[-\mu_m A_{i+1/2} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}\hat{\gamma}_i - \sigma_t V_i\gamma_i\right]\psi_{i+1/2,m} + \left[\frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m) + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m}\alpha_{m-1/2}\right]\hat{\psi}_{i,m-1/2} + Q_{i,m}V_i\right\}.$$

Afterwards, we can solve for $\hat{\psi}_{i,m+1/2}$ by plugging in the appropriate values into

$$\hat{\psi}_{i,m+1/2} = \frac{1}{\beta_m}\left[\hat{\gamma}_i\psi_{i+1/2,m} + (1-\hat{\gamma}_i)\psi_{i-1/2,m} - (1-\beta_m)\hat{\psi}_{i,m-1/2}\right].$$

Once sweep all the way to the first cell, we realize we have an overconstrianed system because $\psi_{3/2,m}$ is known and $\psi_{1/2,m}$ (because we set it to be equal to the starting direction flux $\psi_{1/2,1/2}$). However, we can relax the condition that $\psi_{1,m}$ should be determine weighted-diamond difference, and instead use the balance equation to solve for $\psi_{1,m}$

$$\mu_m(A_{3/2}\psi_{3/2,m} - A_{1/2}\psi_{1/2,m}) + \frac{1}{2}(A_{3/2}-A_{1/2})\frac{(\alpha_{m+1/2}\hat{\psi}_{1,m+1/2} - \alpha_{m-1/2}\hat{\psi}_{1,m-1/2})}{w_m} + \sigma_t V_1\psi_{1,m} = Q_{1,m}V_1$$

where

$$\psi_{1/2,m} = \psi_{1/2,1/2}$$

$$\hat{\psi}_{1,m+1/2} = \frac{1}{\beta_m}\left[\hat{\psi}_{1,m} - (1-\beta_m)\hat{\psi}_{1,m-1/2}\right]$$

$$\hat{\psi}_{1,m} = \left[\frac{\gamma_{1,i}\psi_{3/2,m} + (1-\gamma_{1,i})\psi_{1/2,m}}{\gamma_{2,i}\psi_{3/2,m} + (1-\gamma_{2,i})\psi_{1/2,m}}\right]\psi_{1,m}$$

and

$$A_{1/2} = 0\,.$$

This time since we know $\psi_{3/2,m}$ and $\psi_{1/2,m}$ for $\mu_m < 0$, we will instead solve for $\psi_{1,m}$. Thus, by manipulating the equation above we get

$$\mu_m A_{3/2}\psi_{3/2,m} + \frac{A_{3/2}}{2w_m\beta_m}\alpha_{m+1/2}\left[\frac{\gamma_{1,m}\psi_{3/2,m} + (1-\gamma_{1,m})\psi_{1/2,m}}{\gamma_{2,m}\psi_{3/2,m} + (1-\gamma_{2,m})\psi_{1/2,m}}\right]\psi_{1,m} - \\ \frac{A_{3/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m)\hat{\psi}_{1,m-1/2} - \frac{A_{3/2}}{2w_m}\alpha_{m-1/2}\hat{\psi}_{1,m-1/2} + \sigma_t V_1\psi_{1,m} = Q_{1,m}V_1$$

$$\psi_{1,m} = \frac{\left[-\mu_m A_{3/2}\psi_{3/2,m}\right] + \left[\frac{A_{3/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m) + \frac{A_{3/2}}{2w_m}\alpha_{m-1/2}\right]\hat{\psi}_{1,m-1/2} + Q_{1,m}V_1}{\sigma_t V_1 + \frac{A_{3/2}}{2w_m\beta_m}\alpha_{m+1/2}\left[\frac{\gamma_{1,m}\psi_{3/2,m}+(1-\gamma_{1,m})\psi_{1/2,m}}{\gamma_{2,m}\psi_{3/2,m}+(1-\gamma_{2,m})\psi_{1/2,m}}\right]}\,.$$

2.3 Outward Flux Sweep

Once we know the flux in all spatial cells for directions $\mu < 0$, we can sweep for the directions $\mu > 0$. This time we start with the innermost cell in the sphere and sweep outwards. We set $\psi_{1/2,m}$ equal to the starting direction flux $\psi_{1/2,1/2}$. Also, we already know $\hat{\psi}_{i,m-1/2}$ from the previous sweep so we can derive an equation for $\psi_{i+1/2,m}$ in terms of $\psi_{i-1/2,m}$,

$$\mu_m A_{i+1/2}\psi_{i+1/2,m} - \mu_m A_{i-1/2}\psi_{i-1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}\hat{\gamma}_i\psi_{i+1/2,m} +$$
$$\frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\hat{\gamma}_i)\psi_{i-1/2,m} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m)\hat{\psi}_{i,m-1/2} -$$
$$\frac{A_{i+1/2}-A_{i-1/2}}{2w_m}\alpha_{m-1/2}\hat{\psi}_{i,m-1/2} + \sigma_t V_i\gamma_i\psi_{i+1/2,m} + \sigma_t V_i(1-\gamma_i)\psi_{i-1/2,m} = Q_{i,m}V_i\,.$$

This leaves us with an equation with only one unknown $\psi_{i+1/2}$. By manipulating the equation above we get

$$\mu_m A_{i+1/2}\psi_{i+1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}\hat{\gamma}_i\psi_{i+1/2,m} + \sigma_t V_i\gamma_i\psi_{i+1/2,m} =$$
$$\mu_m A_{i-1/2}\psi_{i-1/2,m} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\hat{\gamma}_i)\psi_{i-1/2,m} -$$
$$\sigma_t V_i(1-\gamma_i)\psi_{i-1/2,m} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m)\hat{\psi}_{i,m-1/2} +$$
$$\frac{A_{i+1/2}-A_{i-1/2}}{2w_m}\alpha_{m-1/2}\hat{\psi}_{i,m-1/2} + Q_{i,m}V_i$$

$$\psi_{i+1/2,m} = \frac{1}{\mu_m A_{i+1/2} + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}\hat{\gamma}_i + \sigma_t V_i\gamma_i}\times$$
$$\left\{\left[\mu_m A_{i-1/2} - \frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\hat{\gamma}_i) - \sigma_t V_i(1-\gamma_i)\right]\psi_{i-1/2,m} + \right.$$
$$\left.\left[\frac{A_{i+1/2}-A_{i-1/2}}{2w_m\beta_m}\alpha_{m+1/2}(1-\beta_m) + \frac{A_{i+1/2}-A_{i-1/2}}{2w_m}\alpha_{m-1/2}\right]\hat{\psi}_{i,m-1/2} + Q_{i,m}V_i\right\}.$$

Afterwards, we can solve for $\hat{\psi}_{i,m+1/2}$ by plugging in the appropriate values into

$$\hat{\psi}_{i,m+1/2} = \frac{1}{\beta_m}\left[\hat{\gamma}_i\psi_{i+1/2,m} + (1-\hat{\gamma}_i)\psi_{i-1/2,m} - (1-\beta_m)\hat{\psi}_{i,m-1/2}\right].$$

3 Deriving the spatially-analytic P_1 equations

Now we will shift gears and derive the spatially-analytic P_1, which will be used later to derive our DSA equations. The 0^{th} moment equation can be derived by integrating the S_N equation over all quadrature directions

$$\sum_{m=1}^{N} w_m \left[\frac{\mu_m}{r^2}\frac{\partial(r^2\psi_m)}{\partial r} + \frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2})}{w_m} + \sigma_t\psi_m = \sum_{k=0}^{K}\frac{2k+1}{2}(\sigma_k\phi_k + q_k)P_k(\mu_m)\right]$$

$$\frac{1}{r^2}\frac{\partial(r^2 J)}{\partial r} + \sum_{m=1}^{N}\Big[\frac{1}{r}(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2})\Big] + \sigma_t\phi = \sum_{m=1}^{N} w_m P_0(\mu_m)\Big[\sum_{k=0}^{K}\frac{2k+1}{2}(\sigma_k\phi_k + q_k)P_k(\mu_m)\Big]$$

$$\frac{1}{r^2}\frac{\partial(r^2 J)}{\partial r} \;+\; \frac{1}{r}(\alpha_{N+1/2}\psi_{N+1/2} - \alpha_{1/2}\psi_{1/2}) \;+\; \sigma_t\phi \;=\; \sum_{k=0}^{K}\frac{2\delta_{0,k}}{2k+1}\frac{2k+1}{2}(\sigma_k\phi_k + q_k)$$

$$\alpha_{1/2} = \alpha_{N+1/2} = 0$$

$$\frac{1}{r^2}\frac{\partial(r^2 J)}{\partial r} + \sigma_t\phi = \sigma_s\phi + q$$

$$\boxed{\frac{1}{r^2}\frac{\partial(r^2 J)}{\partial r} + \sigma_a\phi = q}$$

The 1^{st} moment equation can be derived by integrating the S_N equation over all directions and weighting it by the quadrature direction

$$\sum_{m=1}^{N} w_m\mu_m \left[\frac{\mu_m}{r^2}\frac{\partial(r^2\psi_m)}{\partial r} + \frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2})}{w_m} + \sigma_t\psi = \sum_{k=0}^{K}\frac{2k+1}{2}(\sigma_k\phi_k + q_k)P_k(\mu_m)\right]$$

$$\frac{1}{r^2}\frac{\partial(r^2\vec{E}\phi)}{\partial r} + \sum_{m=1}^{N}\Big[\frac{\mu_m}{r}(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2})\Big] + \sigma_t J = \sum_{m=1}^{N} w_m P_1(\mu_m)\Big[\sum_{k=0}^{K}\frac{2k+1}{2}(\sigma_k\phi_k + q_k)P_k(\mu_m)\Big]$$

$$\frac{1}{3r^2}\frac{\partial(r^2\phi)}{\partial r} \;+\; \sum_{m=1}^{N}\Big[\frac{\mu_m}{r}(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2})\Big] \;+\; \sigma_t J \;=\; \sigma_{s,1}J + q_1$$

$$\psi_m = \frac{1}{2}\phi + \frac{3}{2}\mu_m J$$

$$\sigma_{tr} = \sigma_t - \sigma_{s,1}$$

$$\frac{1}{3}\frac{\partial\phi}{\partial r} \;+\; \frac{2}{3r}\phi \;+\; \sum_{m=1}^{N}\Big\{\frac{\mu_m}{r}\Big[\alpha_{m+1/2}(\frac{1}{2}\phi + \frac{3}{2}\mu_{m+1/2}J) - \alpha_{m-1/2}(\frac{1}{2}\phi + \frac{3}{2}\mu_{m-1/2}J)\Big]\Big\} \;+\; \sigma_{tr}J \;=\; q_1$$

$$\frac{1}{3}\frac{\partial\phi}{\partial r} + \frac{2}{3r}\phi + \sum_{m=1}^{N}\frac{\mu_m}{r}(\frac{1}{2}\phi)(\alpha_{m+1/2} - \alpha_{m-1/2}) - \sum_{m=1}^{N}\frac{\mu_m}{r}(\frac{3}{2}\mu_{m+1/2}J)(\alpha_{m+1/2} - \alpha_{m-1/2}) + \sigma_{tr}J \;=\; q_1$$

$$\frac{1}{3}\frac{\partial\phi}{\partial r} \;+\; \frac{2}{3r}\phi \;+\; \sum_{m=1}^{N}\frac{\mu_m}{r}(\frac{1}{2}\phi)(-2\mu_m w_m) \;-\; \sum_{m=1}^{N}\frac{\mu_m}{r}(\frac{3}{2}\mu_{m+1/2}J)(-2\mu_m w_m) \;+\; \sigma_{tr}J \;=\; q_1$$

$$\frac{1}{3}\frac{\partial\phi}{\partial r} \;+\; \frac{2}{3r}\phi \;-\; \frac{2}{3r}\phi \;+\; 3J\sum_{m=1}^{N}\frac{\mu_m^2}{r}\mu_{m+1/2}w_m \;+\; \sigma_{tr}J \;=\; q_1$$

$$\mu_{m+1/2} = \mu_m + w_m(1-\beta_m)$$

$$\frac{1}{3}\frac{\partial\phi}{\partial r} \;+\; \frac{3J}{r}\sum_{m=1}^{N}\mu_m^3 w_m \;+\; \frac{3J}{r}\sum_{m=1}^{N}\mu_m^2 w_m^2(1-\beta_m) \;+\; \sigma_{tr}J \;=\; q_1$$

Note that μ_m and β_m are both odd functions, and w_m is an even function. Also, the integral of an odd function is zero, thus

$$\frac{3J}{r}\sum_{m=1}^{N}\mu_m^3 w_m \;=\; \frac{3J}{r}\sum_{m=1}^{N}\mu_m^2 w_m^2(1-\beta_m) \;=\; 0$$

and thus the first moment equation is

$$\boxed{\frac{1}{3}\frac{\partial\phi}{\partial r} \;+\; \sigma_{tr}J \;=\; q_1}$$

4 Deriving the spatially-analytic P_1 equations for DSA

In order to derive the 0^{th} and 1^{st} moment equation for DSA, we must first derive the transport equation for the error in iteration $\ell + 1/2$. We start with the spatially-analytic transport equation with isotropic scattering

$$\frac{\mu_m}{r^2}\frac{\partial(r^2\psi_m)}{\partial r} + \frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2})}{w_m} + \sigma_t\psi_m = \frac{1}{2}\sigma_s\phi + \sum_{k=0}^{K}\frac{2k+1}{2}q_k\ .$$

Then we express the transport equation for source iteration $\ell + 1/2$, which is

$$\frac{\mu_m}{r^2}\frac{\partial(r^2\psi_m^{\ell+1/2})}{\partial r} + \frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2}^{\ell+1/2} - \alpha_{m-1/2}\psi_{m-1/2}^{\ell+1/2})}{w_m} + \sigma_t\psi_m^{\ell+1/2} = \frac{1}{2}\sigma_s\phi^\ell + \sum_{k=0}^{K}\frac{2k+1}{2}q_k\ .$$

By subtracting the equation for $\ell + 1/2$ from the analytic equation we get a transport equation for the error

$$\frac{\mu_m}{r^2}\frac{\partial(r^2\delta\psi_m^{\ell+1/2})}{\partial r} + \frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2}^{\ell+1/2} - \alpha_{m-1/2}\delta\psi_{m-1/2}^{\ell+1/2})}{w_m} + \sigma_t\delta\psi_m^{\ell+1/2} = \frac{1}{2}\sigma_s\delta\phi^\ell\ .$$

where

$$\delta\psi^\ell = \psi - \psi^\ell$$

By subtracting $\frac{1}{2}\sigma_s\delta\phi^{\ell+1/2}$ from both sides we get

$$\frac{\mu_m}{r^2}\frac{\partial(r^2\delta\psi_m^{\ell+1/2})}{\partial r}+\frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2}^{\ell+1/2}-\alpha_{m-1/2}\delta\psi_{m-1/2}^{\ell+1/2})}{w_m}+\sigma_t\delta\psi_m^{\ell+1/2}-\frac{1}{2}\sigma_s\delta\phi^{\ell+1/2}=\frac{1}{2}\sigma_s(\delta\phi^\ell-\delta\phi^{\ell+1/2})$$

$$\frac{\mu_m}{r^2}\frac{\partial(r^2\delta\psi_m^{\ell+1/2})}{\partial r}+\frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2}^{\ell+1/2}-\alpha_{m-1/2}\delta\psi_{m-1/2}^{\ell+1/2})}{w_m}+\sigma_t\delta\psi_m^{\ell+1/2}-\frac{1}{2}\sigma_s\delta\phi^{\ell+1/2}=\frac{1}{2}\sigma_s(\phi^{\ell+1/2}-\phi^\ell).$$

$$\frac{\mu_m}{r^2}\frac{\partial(r^2\delta\psi_m^{\ell+1/2})}{\partial r}+\frac{1}{r}\frac{(\alpha_{m+1/2}\psi_{m+1/2}^{\ell+1/2}-\alpha_{m-1/2}\delta\psi_{m-1/2}^{\ell+1/2})}{w_m}+\sigma_t\delta\psi_m^{\ell+1/2}-\frac{1}{2}\sigma_s\delta\phi^{\ell+1/2}=\frac{1}{2}\sigma_s(\phi^{\ell+1/2}-\phi^\ell).$$

Notice how the equation above for the error is just like the spatially-analytic tranpsort equation except the source term has no higher moments. Thus, the 0^{th} and 1^{st} moment equation for the error are same as for the angular flux, except that $q_0 = \sigma_s(\phi^{\ell+1/2}-\phi^\ell)$ and $q_1 = 0$:

$$\boxed{\frac{1}{r^2}\frac{\partial(r^2\delta J^{\ell+1/2})}{\partial r}+\sigma_a\delta\phi^{\ell+1/2}=R^{\ell+1/2}}$$

$$\boxed{\frac{1}{3}\frac{\partial\delta\phi^{\ell+1/2}}{\partial r}\;+\;\sigma_{tr}\delta J^{\ell+1/2}\;=\;0}$$

where

$$R^{\ell+1/2} = \sigma_s(\phi^{\ell+1/2}-\phi^\ell)\,. \tag{2}$$

4.1 DSA Interior Mesh Equation

The discrete form of the 0^{th} moment equation can be derived for cell i integrating over volume element i:

$$(A_{i+1/2}\delta J_{i+1/2}^{\ell+1/2}-A_{i-1/2}\delta J_{i-1/2}^{\ell+1/2})+\sigma_a[\gamma_i\delta\phi_{i+1/2}^{\ell+1/2}+(1-\gamma_i)\delta\phi_{i-1/2}^{\ell+1/2}]V_i=R_i^{\ell+1/2}V_i$$

In addition, for cell $i+1$ the discrete 0^{th} moment equation is

$$(A_{i+3/2}\delta J_{i+3/2}^{\ell+1/2}-A_{i+1/2}\delta J_{i+1/2}^{\ell+1/2})+\sigma_a[\gamma_{i+1}\delta\phi_{i+3/2}^{\ell+1/2}+(1-\gamma_{i+1})\delta\phi_{i+1/2}^{\ell+1/2}]V_{i+1}=R_{i+1}^{\ell+1/2}V_{i+1}$$

by combining both of these equation we get that

$$(A_{i+3/2}\delta J^{\ell+1/2}_{i+3/2} - A_{i-1/2}\delta J^{\ell+1/2}_{i-1/2}) + \sigma_a[\gamma_i \delta\phi^{\ell+1/2}_{i+1/2} + (1-\gamma_i)\delta\phi^{\ell+1/2}_{i-1/2}]V_i +$$
$$\sigma_a[\gamma_{i+1}\delta\phi^{\ell+1/2}_{i+3/2} + (1-\gamma_{i+1})\delta\phi^{\ell+1/2}_{i+1/2}]V_{i+1} = R^{\ell+1/2}_i V_i + R^{\ell+1/2}_{i+1} V_{i+1}$$

However we do not have an equation for J at the cell edges, but we can make the approximation that

$$A_{i+3/2}\delta J^{\ell+1/2}_{i+3/2} - A_{i-1/2}\delta J^{\ell+1/2}_{i-1/2} \approx 2(A_{i+1}\delta J^{\ell+1/2}_{i+1} - A_i \delta J^{\ell+1/2}_i)$$

and

$$\frac{1}{3}(\delta\phi^{\ell+1/2}_{i+1/2} - \delta\phi^{\ell+1/2}_{i-1/2}) \ + \ \sigma_{tr}\delta J^{\ell+1/2}_i h_i \ = \ 0$$
$$\frac{1}{3}(\delta\phi^{\ell+1/2}_{i+3/2} - \delta\phi^{\ell+1/2}_{i+1/2}) \ + \ \sigma_{tr}\delta J^{\ell+1/2}_{i+1} h_{i+1} \ = \ 0$$

This gives us a discrete form of the diffusion equation,

$$-\frac{2A_{i+1}(\delta\phi^{\ell+1/2}_{i+3/2} - \delta\phi^{\ell+1/2}_{i+1/2})}{3\sigma_{tr}h_{i+1}} + \frac{2A_i(\delta\phi^{\ell+1/2}_{i+1/2} - \delta\phi^{\ell+1/2}_{i-1/2})}{3\sigma_{tr}h_i} +$$
$$\sigma_a[\gamma_i \delta\phi^{\ell+1/2}_{i+1/2} + (1-\gamma_i)\delta\phi^{\ell+1/2}_{i-1/2}]V_i + \sigma_a[\gamma_{i+1}\delta\phi^{\ell+1/2}_{i+3/2} + (1-\gamma_{i+1})\delta\phi^{\ell+1/2}_{i+1/2}]V_{i+1} =$$
$$R^{\ell+1/2}_i V_i + R^{\ell+1/2}_{i+1} V_{i+1}$$

$$\Big\{-\frac{2A_i}{3\sigma_{tr}h_i} + \sigma_a(1-\gamma_i)V_i\Big\}\delta\phi^{\ell+1/2}_{i-1/2} + \Big\{\frac{2A_{i+1}}{3\sigma_{tr}h_{i+1}} + \frac{2A_i}{3\sigma_{tr}h_i} + \sigma_a[\gamma_i V_i + (1-\gamma_{i+1})V_{i+1}]\Big\}\delta\phi^{\ell+1/2}_{i+1/2} +$$
$$\Big\{-\frac{2A_{i+1}}{3\sigma_{tr}h_{i+1}} + \sigma_a\gamma_{i+1}V_{i+1}\Big\}\delta\phi^{\ell+1/2}_{i+3/2} = R^{\ell+1/2}_i V_i + R^{\ell+1/2}_{i+1} V_{i+1}$$

where

$$R^{\ell+1/2} = \sigma_s(\phi^{\ell+1/2} - \phi^{\ell})$$

4.2 DSA Left Boundary Equation

The discrete form of the 0^{th} moment equation can be derived for cell 1 integrating over volume element 1:

$$(A_{3/2}\delta J_{3/2}^{\ell+1/2} - A_{1/2}\delta J_{1/2}^{\ell+1/2}) + \sigma_a\big[\gamma_1\delta\phi_{3/2}^{\ell+1/2} + (1-\gamma_1)\delta\phi_{1/2}^{\ell+1/2}\big]V_1 = R_1^{\ell+1/2}V_1$$

but we know that $A_{1/2} = 0$ thus

$$(A_{3/2}\delta J_{3/2}^{\ell+1/2} - A_{1/2}\delta J_{1/2}^{\ell+1/2}) = A_{3/2}\delta J_{3/2}^{\ell+1/2}\ .$$

In addition, since we don't know what $J_{3/2}$ we can make the assumption that

$$A_{3/2}\delta J_{3/2}^{\ell+1/2} = 2A_1\delta J_1^{\ell+1/2}$$

and

$$\frac{1}{3}(\delta\phi_{3/2}^{\ell+1/2} - \delta\phi_{1/2}^{\ell+1/2}) \ + \ \sigma_{tr}\delta J_1^{\ell+1/2}h_1 \ = \ 0\ .$$

This gives us a discrete form of the diffusion equation,

$$-\frac{2A_1(\delta\phi_{3/2}^{\ell+1/2} - \delta\phi_{1/2}^{\ell+1/2})}{3\sigma_{tr}h_1} + \sigma_a[\gamma_1\delta\phi_{3/2}^{\ell+1/2} + (1-\gamma_1)\delta\phi_{1/2}^{\ell+1/2}]V_1 = R_1^{\ell+1/2}V_1$$

$$\Big\{\frac{2A_1}{3\sigma_{tr}h_1} + \sigma_a(1-\gamma_1)V_1\Big\}\delta\phi_{1/2}^{\ell+1/2} + \Big\{-\frac{2A_1}{3\sigma_{tr}h_1} + \sigma_a\gamma_1V_1\Big\}\delta\phi_{3/2}^{\ell+1/2} = R_1^{\ell+1/2}V_1$$

where

$$R^{\ell+1/2} = \sigma_s(\phi^{\ell+1/2} - \phi^{\ell})$$

4.3 DSA Right Boundary Equation

The discrete form of the 0^{th} moment equation can be derived for cell N integrating over volume element N:

$$(A_{N+1/2}\delta J_{N+1/2}^{\ell+1/2} - A_{N-1/2}\delta J_{N-1/2}^{\ell+1/2}) + \sigma_a\big[\gamma_1\delta\phi_{N+1/2}^{\ell+1/2} + (1-\gamma_1)\delta\phi_{N-1/2}^{\ell+1/2}\big]V_1 = R_1^{\ell+1/2}V_1$$

Since we don't know what $J_{N-1/2}$ we can make the assumption that

$$A_{N+1/2}\delta J^{\ell+1/2}_{N+1/2} - A_{N-1/2}\delta J^{\ell+1/2}_{N-1/2} \; = \; 2(A_{N+1/2}\delta J^{\ell+1/2}_{N+1/2} - A_N \delta J^{\ell+1/2}_{N})$$

and after plugging in the boundary condition this is equivalent to

$$2(A_{N+1/2}\delta J^{\ell+1/2}_{N+1/2} - A_N \delta J^{\ell+1/2}_{N}) \; = \; 2A_{N+1/2}\delta\phi^{\ell+1/2}_{N+1/2}\langle\mu\rangle - 2A_N\delta J^{\ell+1/2}_{N} \, .$$

This gives us a discrete form of the diffusion equation,

$$2A_{N+1/2}\delta\phi^{\ell+1/2}_{N+1/2}\langle\mu\rangle + \frac{2A_N(\delta\phi^{\ell+1/2}_{N+1/2} - \delta\phi^{\ell+1/2}_{N-1/2})}{3\sigma_{tr}h_N} + \sigma_a[\gamma_N\delta\phi^{\ell+1/2}_{N+1/2} + (1-\gamma_N)\delta\phi^{\ell+1/2}_{N-1/2}]V_N = R^{\ell+1/2}_N V_N$$

$$\Big\{-\frac{2A_N}{3\sigma_{tr}h_N} + \sigma_a(1-\gamma_N)V_N\Big\}\delta\phi^{\ell+1/2}_{N-1/2} + \Big\{2A_{N+1/2}\langle\mu\rangle + \frac{2A_N}{3\sigma_{tr}h_N} + \sigma_a\gamma_N V_N\Big\}\delta\phi^{\ell+1/2}_{N+1/2} = R^{\ell+1/2}_N V_N$$

where

$$R^{\ell+1/2} = \sigma_s(\phi^{\ell+1/2} - \phi^{\ell})$$

5 DSA is conservative

To show that DSA is conservative after each iteration, we can integrate the spatially-analytic S_N transport equation over all angles

$$\sum_{m=1}^{N} w_m \left[\frac{\mu_m}{r^2}\frac{\partial(r^2\psi^{\ell+1/2}_m)}{\partial r} + \frac{1}{r}\frac{(\alpha_{m+1/2}\psi^{\ell+1/2}_{m+1/2} - \alpha_{m-1/2}\psi^{\ell+1/2}_{m-1/2})}{w_m} + \sigma_t\psi^{\ell+1/2}_m = \sum_{k=0}^{K}\frac{2k+1}{2}(\sigma_k\phi^{\ell}_k + q_k)P_k(\mu_m)\right]$$

$$\frac{1}{r^2}\frac{\partial(r^2J^{\ell+1/2})}{\partial r} + \sum_{m=1}^{N}\Big[\frac{1}{r}(\alpha_{m+1/2}\psi^{\ell+1/2}_{m+1/2} - \alpha_{m-1/2}\psi^{\ell+1/2}_{m-1/2})\Big] + \sigma_t\phi = \sum_{m=1}^{N} w_m P_0(\mu_m)\Big[\sum_{k=0}^{K}\frac{2k+1}{2}(\sigma_k\phi^{\ell}_k + q_k)P_k(\mu_m)\Big]$$

$$\frac{1}{r^2}\frac{\partial(r^2J^{\ell+1/2})}{\partial r} \; + \; \frac{1}{r}(\alpha_{N+1/2}\psi^{\ell+1/2}_{N+1/2} - \alpha_{1/2}\psi^{\ell+1/2}_{1/2}) \; + \; \sigma_t\phi \; = \; \sum_{k=0}^{K}\frac{2\delta_{0,k}}{2k+1}\frac{2k+1}{2}(\sigma_k\phi^{\ell}_k + q_k)$$

$$\alpha_{1/2} = \alpha_{N+1/2} = 0$$

$$\frac{1}{r^2}\frac{\partial(r^2J^{\ell+1/2})}{\partial r} + \sigma_t\phi^{\ell+1/2} = \sigma_s\phi^{\ell} + q$$

and subtract $\sigma_s \phi^{\ell+1/2}$ from both sides to get

$$\frac{1}{r^2}\frac{\partial(r^2 J^{\ell+1/2})}{\partial r} + \sigma_a \phi^{\ell+1/2} = \sigma_s(\phi^{\ell} - \phi^{\ell+1/2}) + q \,.$$

Afterwards we add the DSA equation,

$$\frac{1}{r^2}\frac{\partial(r^2 \delta J^{\ell+1/2})}{\partial r} + \sigma_a \delta\phi^{\ell+1/2} = \sigma_s(\phi^{\ell+1/2} - \phi^{\ell}) \,,$$

to get

$$\boxed{\frac{1}{r^2}\frac{\partial(r^2 J)}{\partial r} + \sigma_a \phi = q}$$

This proves that DSA allows us to maintain balance after each iteration.

6 Test Problems

6.1 Problem 1

Problem 1 is a sphere of radius = 1 cm, $\sigma_t = 1$ cm^{-1}, and $\sigma_a = 1$ cm^{-1} with an incoming partial current of 1 $[p/cm^2 - s]$. We solved for the scalar flux using S_4 quadrature and 50, 100, and 200 spatial cells. The balance tables generated by the S_N code for 50, 100, and 200 cells are shown below, respectively.

```
>> ITERATION #  0

--------------------------------
inflow      =  12.5663706144
outflow     =  4.13045979258
absorption  =  8.43591082178
source rate =  0.0
--------------------------------
balance     =  4.9475295045e-16
--------------------------------
```

```
>> ITERATION #  0

--------------------------------
inflow      =  12.5663706144
outflow     =  4.13050150758
absorption  =  8.43586910677
source rate =  0.0
--------------------------------
balance     =  0.0
--------------------------------
```

```
>> ITERATION #  0

--------------------------------
inflow      =  12.5663706144
outflow     =  4.13051193699
absorption  =  8.43585867737
source rate =  0.0
--------------------------------
balance     =  4.24073957528e-16
--------------------------------
```

The scalar flux was compared for 50, 100, and 200 spatial cells. As shown in Fig.(1) all three discretization solutions are overlapping and it is difficult to see the difference between each solution.

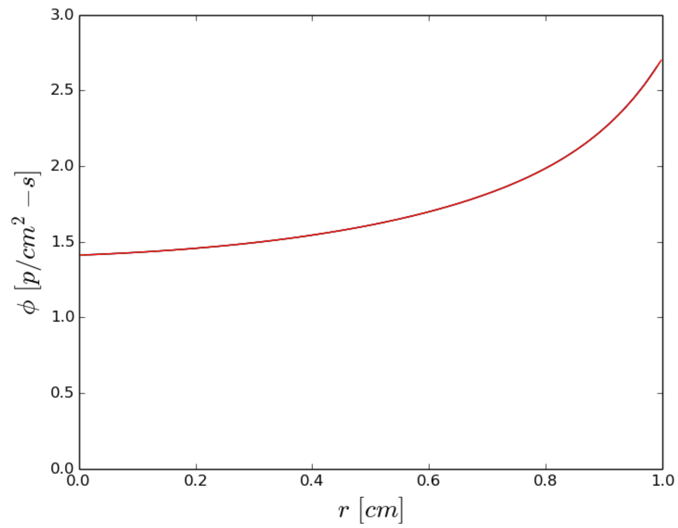

Figure 1: Scalar flux as a function of r for problem 1.

The spatial-covergence order was determined by

$$\xi = \frac{L_{50} - L_{100}}{L_{100} - L_{200}}$$

where L_k denotes the outflow from the outer boundary of the sphere [p/s]. The spatial-convergence order was found to be 3.99974 (which is close to the expected value of 4).

6.2 Problem 2

Problem 2 is a sphere of radius = 1 cm, $\sigma_t = 3$ cm^{-1}, and $\sigma_a = 1$ cm^{-1} with an incident right boundary flux $\phi_o = q_o/\sigma_a$, and a uniformly distributed source of $q_o = 1[p/cm^3 - s]$. We solved for the scalar flux using S_8 quadrature and 100 spatial cells with a convergence tolerance of 10^{-4}. The balance table generated by the S_N code for this problem is shown below.

```
>> ITERATION #  16

--------------------------------
inflow      =  3.17780913292
outflow     =  3.17772871305
absorption  =  4.18866690379
source rate =  4.18879020479
--------------------------------
balance     =  2.765466957e-05
--------------------------------
```

We compared the analytical flux $\phi = q/\sigma_s$ to the S_N solution. As shown in Fig.(2) both solution are overlapping, and thus the S_N is behaving as expected.

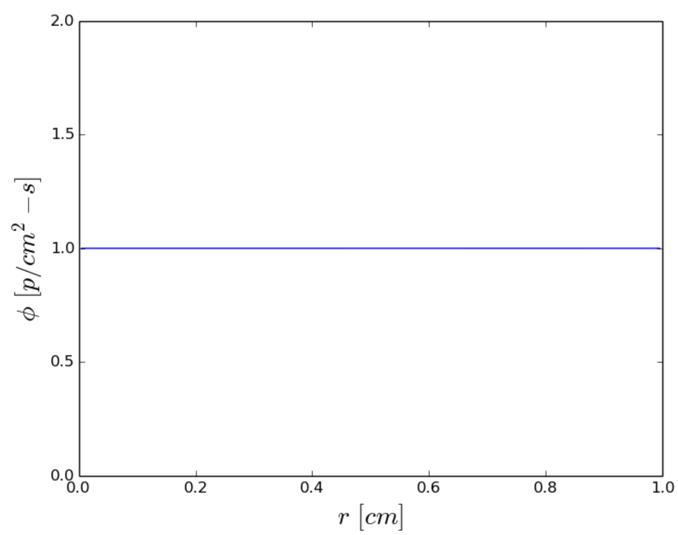

Figure 2: Scalar flux as a function of r for problem 2.

6.3 Problem 3

Problem 3 is a sphere of radius = 1 cm, $\sigma_t = 2$ cm^{-1}, P_3 anisotropic scattering with all Legendre coefficients euqal to $\sigma_{s,\ell} = 1$ cm^{-1} an incoming partial current of 1 $[p/cm^2 - s]$. We solved for the scalar flux using S_4 quadrature and 100 spatial cells with a convergence tolerance of 10^{-4}. The balance table for this problem is shown below.

```
>> ITERATION #  7

--------------------------------
inflow      =  12.5663706144
outflow     =  4.12984021188
absorption  =  8.43572163056
source rate =  0.0
--------------------------------
balance     =  6.43600240304e-05
--------------------------------
```

In Fig.(3) we compared the flux for problem 1 and problem 3. Both solutions are almost overlapping, thus the solution for problem 3 is behaving as expected.

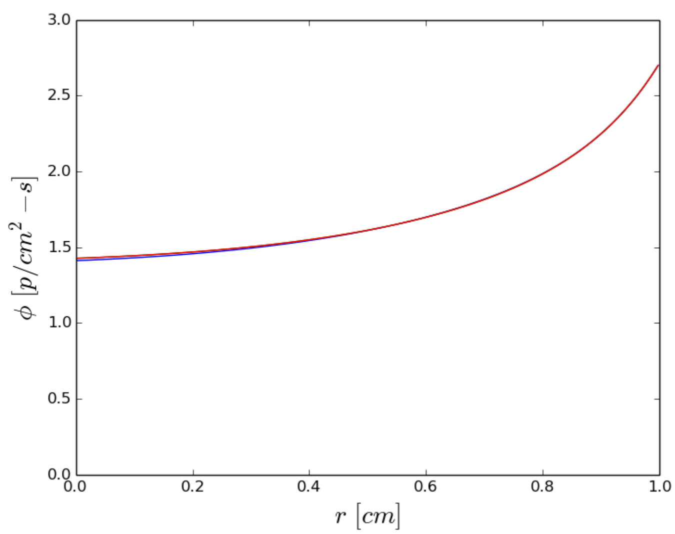

Figure 3: Scalar flux as a function of r for problem 3.

The spatial-convergence order for this problem was 4.00017 (close to the expected value of 4).

6.4 Problem 4

Problem 4 is a sphere of radius = 1 cm, $\sigma_t = 30$ cm^{-1}, and $\sigma_a = 0$ cm^{-1} with isotropic scattering, a vaccum right boundary condition, and a uniformly distributed source of $q_o = 1[p/cm^3 - s]$. The analytical solution for this problem can be derived using diffusion theory (which is accurate everywhere except at the boundary layer),

$$-D\nabla^2\phi + \sigma_a\phi = q_o$$

$$-\frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = q_o$$

$$\int dr\,\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \int dr\, -\frac{q_o r^2}{D}$$

$$r^2\frac{\partial\phi}{\partial r} = -\frac{q_o r^3}{3D} + C_1$$

$$\int dr\,\frac{\partial\phi}{\partial r} = \int dr\, -\frac{q_o r}{3D} + \frac{C_1}{r^2}$$

$$\phi = -\frac{q_o r^2}{6D} - \frac{C_1}{r} + C_2$$

Now by applying the boundary condition, we can solve for C_1 and C_2. We know that ϕ must be finite at $r = 0$, therefore

$$C_1 = 0\,.$$

We can use a Marshak vacuum boundary condition for the right boundary,

$$\phi(R + 2D) = 0$$

$$-\frac{q_o(R + 2D)^2}{6D} + C_2 = 0$$

$$\phi = \frac{q}{6D}\left(R^2 - r^2\right) + \frac{2q}{3}\left(R + D\right)$$

Next, we solved for the scalar flux using S_8 quadrature and 100 spatial cells with a convergence tolerance of 10^{-4}. The balance table for this problem is shown below.

```
>> ITERATION #  4

--------------------------------
inflow      =  0.0
outflow     =  4.18879020479
absorption  =  0.0
source rate =  4.18879020479
--------------------------------
balance     =  3.79546191988e-14
--------------------------------
```

We compared the analytic flux to the S_N solution, shown in Fig.(4). Notice that both solutions overlap, thus the solution from the S_N code is behaving as expected.

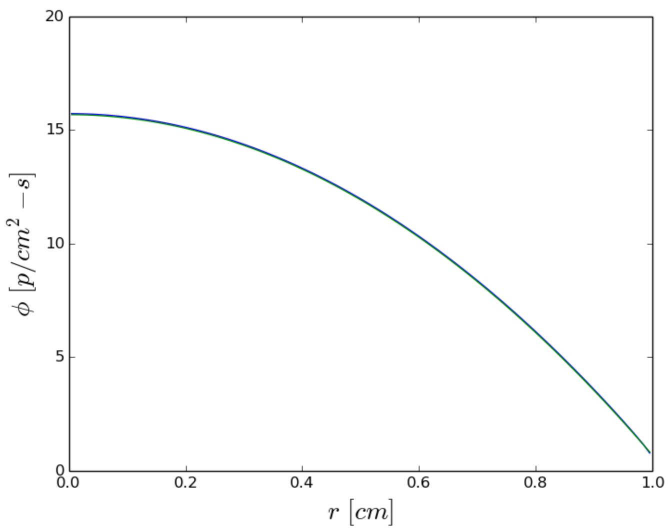

Figure 4: Scalar flux as a function of r for problem 4.