Lecture 12
The Zeroth and First Moment Equations

The scalar flux and the current are generally the two physical quantities of greatest interest in transport calculations. In this section we show that rigorous equations for these two quantities can be derived. However, these equations also contain a third transport unknown, thus additional information is required to close them. We briefly discuss a closure that leads to the diffusion equation, and include the effect of anisotropic scattering and anisotropic inhomogeneous sources. We begin our derivation with the time-dependent transport equation in Cartesian geometry:

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi + \sigma_t \psi = \int_{4\pi} \sigma_s(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega})\, \psi(\overrightarrow{\Omega}')\, d\Omega' + Q(\overrightarrow{\Omega})\,. \tag{1}$$

Next we expand the angular flux in the scattering source and the inhomogeneous source in spherical harmonics:

$$\begin{aligned} \frac{1}{v}\frac{\partial \psi}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi + \sigma_t \psi \;\; &= \;\; \int_{4\pi} \sigma_s(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}) \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \frac{2k+1}{4\pi} \psi_k^m Y_k^m(\overrightarrow{\Omega}')\, d\Omega' + \\ & \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \frac{2k+1}{4\pi} Q_k^m Y_k^m(\overrightarrow{\Omega})\,. \end{aligned} \tag{2}$$

where the definition of the spherical harmonics is given in Appendix C, and

$$\psi_k^m = \int_{4\pi} \psi(\overrightarrow{\Omega}) Y_k^m(\overrightarrow{\Omega})\, d\Omega\,, \tag{3}$$

$$Q_k^m = \int_{4\pi} Q(\overrightarrow{\Omega}) Y_k^m(\overrightarrow{\Omega})\, d\Omega\,. \tag{4}$$

It is shown in Appendix C that

$$\int_{4\pi} \sigma_s(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega})\, Y_k^m(\overrightarrow{\Omega}')\, d\Omega' = \sigma_k Y_k^m(\overrightarrow{\Omega})\,, \tag{5}$$

where σ_k is the k'th Legendre expansion coefficient for the scattering cross-section:

$$\sigma_k = 2\pi \int_{-1}^{+1} \sigma_s(\mu_0) P_k(\mu_0)\, d\mu_0\,, \tag{6}$$

and P_k is the Legendre polynomial of degree k, or equivalently, the spherical harmonic, Y_k^0.

Substituting from Eq. (5) into Eq. (2), we get

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi + \sigma_t \psi = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \frac{2k+1}{4\pi} \left[\sigma_k \psi_k^m + Q_k^m\right] Y_k^m(\overrightarrow{\Omega})\,. \tag{7}$$

Recognizing that $Y_0^0 = 1$ and taking the the orthogonality of the spherical harmonics into account, we integrate Eq. (7) over all directions and obtain

$$\frac{1}{v}\frac{\partial \phi}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{J} + \sigma_t \phi = \sigma_0 \phi_0^0 + Q_0^0\,. \tag{8}$$

Since $P_0 = 1$, it follows from Eq. (6) that

$$\sigma_0 = \sigma_s\,. \tag{9}$$

Since $Y_0^0 = 1$, it follows from Eqs. (3) and (4), respectively, that

$$\phi_0^0 = \phi\,. \tag{10}$$

and

$$Q_0^0 = \int_{4\pi} Q \, d\Omega \, . \tag{11}$$

Thus, as required, Eq. (8) reduces to the balance equation:

$$\frac{1}{v}\frac{\partial \phi}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{J} + \sigma_a \phi = Q_0 \, , \tag{12}$$

where Q_0 simply denotes $Q(\overrightarrow{\Omega})$ integrated over all angles. This equation is also called the zeroth moment equation because it corresponds to taking a moment of the transport equation with respect to $\overrightarrow{\Omega}^0$.

Multiplying Eq. (7) by $\overrightarrow{\Omega}$ and integrating over all angles, we get

$$\frac{1}{v}\frac{\partial \overrightarrow{J}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{P}} + \sigma_t \overrightarrow{J} = \int_{4\pi} \overrightarrow{\Omega} \left\{ \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \frac{2k+1}{4\pi} \left[\sigma_k \psi_k^m + Q_k^m \right] Y_k^m(\overrightarrow{\Omega}) \right\} d\Omega \, , \tag{13}$$

where $\overrightarrow{\mathbf{P}}$ is the radiation pressure tensor. Specifically,

$$\mathbf{P}_{i,j} = \int_{4\pi} \Omega_i \Omega_j \, \psi(\overrightarrow{\Omega}) \, d\Omega \, , \tag{14}$$

where i and j are component indices, e.g., in Cartesian coordinates,

$$\begin{aligned} \Omega_x &\equiv x\text{-component of } \overrightarrow{\Omega}, \\ \Omega_y &\equiv y\text{-component of } \overrightarrow{\Omega}, \\ \Omega_z &\equiv z\text{-component of } \overrightarrow{\Omega}, \end{aligned} \tag{15}$$

Note from Eq. (14) that the radiation pressure tensor is symmetric. Furthermore, the divergence of a tensor is a vector:

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{P}} = \begin{bmatrix} \frac{\partial}{\partial x} P_{x,x} + \frac{\partial}{\partial y} P_{x,y} + \frac{\partial}{\partial z} P_{x,z} \\ \frac{\partial}{\partial x} P_{y,x} + \frac{\partial}{\partial y} P_{y,y} + \frac{\partial}{\partial z} P_{y,z} \\ \frac{\partial}{\partial x} P_{z,x} + \frac{\partial}{\partial y} P_{z,y} + \frac{\partial}{\partial z} P_{z,z} \end{bmatrix} . \tag{16}$$

To evaluate the right side of Eq. (13), we must first recognize that

$$\begin{aligned} \Omega_x &= Y_1^1 \,, \\ \Omega_y &= Y_1^{-1} \,, \\ \Omega_z &= Y_1^0 \,. \end{aligned} \tag{17}$$

Substituting from Eq. (17) into Eq. (13), and taking the orthogonality of the spherical harmonics into account, we obtain

$$\frac{1}{v} \frac{\partial}{\partial t} \overrightarrow{J} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{P}} + \sigma_t \overrightarrow{J} = \sigma_1 \overrightarrow{J} + \overrightarrow{Q}_1 \,, \tag{18}$$

where

$$\overrightarrow{Q}_1 = \left(Q_1^1, Q_1^{-1}, Q_1^0 \right) \,. \tag{19}$$

We refer to Eq. (18) as the first moment equation because it corresponds to taking a moment of the transport equation with respect to $\overrightarrow{\Omega}^1$.

The zeroth and first moment equations constitute a system of equations for the scalar flux and the current, but the system is not closed due to the presence of the radiation pressure tensor. There are many closures for these equations that form the basis of approximate transport theories as well as exact closures that form the basis of numerical transport solution techniques. We will discuss some of these in detail at a later point. For the moment, we simply note that almost all closures are defined in terms of the Eddington tensor. This tensor is defined as follows:

$$E_{i,j} = P_{i,j}/\phi = \left[\int_{4\pi} \Omega_i \Omega_j \, \psi(\overrightarrow{\Omega}) \, d\Omega\right] \Big/ \left[\int_{4\pi} \psi(\overrightarrow{\Omega}) \; d\Omega\right] . \tag{20}$$

Substituting from Eq. (20) into Eq. (18), we get the Eddingtron form of the momentum equation:

$$\frac{1}{v}\frac{\partial}{\partial t}\overrightarrow{J} + \overrightarrow{\nabla}\cdot \left[\overrightarrow{\mathbf{E}}\,\phi\right] + \sigma_t \overrightarrow{J} = \sigma_1 \overrightarrow{J} + \overrightarrow{Q}_1 , \tag{21}$$

Defining the Eddington tensor closes the system of equations for the scalar flux and current. Note that the elements of the Eddington tensor relate purely to the angular shape of the angular flux. Knowledege of its magnitude is not required. Thus closing the system requires information on the angular flux shape. This information can either be obtained via certain physical assumptions or via a computational procedure. The latter approach will be discussed at a later point.

If we assume that the angular flux is isotropic linear in angle, i.e.,

$$\psi = \frac{1}{4\pi}\phi + \frac{3}{4\pi}\overrightarrow{J}\cdot\overrightarrow{\Omega} , \tag{22}$$

the Eddington tensor reduces to a scalar, and Eq. (21) becomes

$$\frac{1}{v}\frac{\partial}{\partial t}\overrightarrow{J} + \frac{1}{3}\overrightarrow{\nabla}\phi + \sigma_{tr}\overrightarrow{J} = \overrightarrow{Q}_1\,, \tag{23}$$

where

$$\sigma_{tr} = \sigma_t - \sigma_1\,. \tag{24}$$

The quantity σ_{tr} is called the transport-corrected total cross-section. This is equal to the sum of the absorption cross-section and the transport-corrected scattering cross-section, $\sigma_{sr} \equiv \sigma_s - \sigma_1$. Equations (12) and (23) form a closed system for the scalar flux and current, and as one would expect, yield the diffusion approximation in steady-state. Note that with this simple closure, the effect of anisotropic simply takes the form of a modifed scattering cross section. In particular, the scattering cross-section is replaced by the transport-corrected scattering cross-section. It is insightful to note that

$$\sigma_{sr} = \sigma_s(1 - \bar{\mu}_0)\,, \tag{25}$$

where $\bar{\mu}_0$ is the average scattering cosine, i.e.

$$\bar{\mu}_0 = \frac{2\pi}{\sigma_s}\int_{-1}^{+1} \sigma_s(\mu_0)\mu_0\,d\mu_0\,. \tag{26}$$

Thus, if $\bar{\mu}_0 > 0$, the scattering is forward-peaked, and the effective scattering cross section is smaller than the true scattering cross-section. Conversely, if $\bar{\mu}_0 < 0$, the effective scattering cross section is larger than the true scattering cross section. This clearly makes physical sense. Also note that $\bar{\mu}_0 = 1$, implies that $\mu_0 = 1$ for all scattering events. For this case, Eq. (25) yields an effective scattering cross section of zero. This is correct even for exact transport theory, because the particle direction is not changed with a scattering cosine of unity. Thus, in the monoenergetic case, such an event is equivalent to no scattering at all.