Lecture 8

Spatially Analytic Transport Solutions Via Separation-of-Variables

The 1-D slab-geometry monoenergetic tranport operator with isotropic scattering can be solved by a separation-of-variables technique in a homogeneous region. This approach yields a set of singular functions that are referred to as "singular eigenfunctions" even though they are really not eigenfunctions of the transport operator, but rather are a complete basis for homogeneous solutions to the transport equation. We shall describe these eigenfunctions in detail, but we shall not construct any specific transport solutions due to the complexity of the process. Nonetheless, it is useful to understand the nature of these eigenfunctions since any discretization of the transport equation should lead to eigenvectors that mimic these eigenfunctions in some sense.

We begin with the transport equation:

$$\mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \frac{\sigma_s}{4\pi} \phi \,. \tag{1}$$

It is useful at this point to transform the spatial variable from x to $z = \sigma_t x$, i.e., to measure distance in mean-free-paths rather than centimeters. dividing Eq. (1) by σ_t, and recognizing that $\frac{\partial}{\partial z} = \frac{1}{\sigma_t}\frac{\partial}{\partial x}$, we get

$$\mu \frac{\partial \psi}{\partial z} + \psi = \frac{c}{4\pi} \phi \,. \tag{2}$$

where $c = \sigma_s/\sigma_t$. We next assume an exponential spatial dependence for the solution:

$$\psi(z,\mu) = \psi_\nu(\mu)\exp^{-\frac{z}{\nu}}\,, \tag{3}$$

where ν is a relaxation length measured in mean-free-paths. Substituting from Eq. (3) into Eq. (2), we get

$$-\frac{\mu}{\nu}\psi_\nu(\mu)\exp^{-\frac{z}{\nu}} + \psi_\nu(\mu)\exp^{-\frac{z}{\nu}} = \frac{c}{4\pi}\phi_\nu\exp^{-\frac{z}{\nu}}\;. \tag{4}$$

Solving Eq. (4) for ψ, we obtain

$$\psi_\nu(\mu) = \frac{c}{4\pi}\frac{\nu\phi_\nu}{\nu-\mu}\,. \tag{5}$$

Assuming that $\nu \notin [-1,+1]$, we integrate Eq. (5) over all directions to obtain a consistency criterion for the values of ν:

$$\begin{aligned} 1 &= \frac{c}{2}\int_{-1}^{+1}\frac{\nu}{\nu-\mu}\,d\mu\,, \\ &= \frac{c}{2}\nu\ln\left(\frac{\nu+1}{\nu-1}\right)\,. \end{aligned} \tag{6}$$

Note that if this consistency criterion is satisfied, one can choose any positive value for ϕ_ν. We choose a scalar flux value of 2π, which yields the following explicit expression for the angular flux:

$$\psi_\nu(\mu) = \frac{c}{2}\frac{\nu}{\nu-\mu}\,. \tag{7}$$

There are two solutions to Eq. (6):

$$\nu = \pm\nu_0\,, \tag{8}$$

where ν_0 is real and $\nu_0 \notin [-1, +1]$ if $c \in (0, 1]$. If $c > 1$, ν_0 is imaginary, but we are concerned with infinite-medium steady-state transport solutions here, and such a solution will not exist if $c > 1$. The two modes corresponding to $\pm\nu_0$ are referred to as the asymptotic modes because they dominate the solution at points many mean-free-paths away from material interfaces and sources. The reason for this dominance is discussed later.

We next consider modes for which $\nu \in [-1, +1]$. In this case, there exists a value of μ at which the angular flux given by Eq. (5) is singular. To proceed, we must first define the scalar flux for such angular fluxes. This is achieved simply by using the Cauchy principal value for the following integral:

$$\begin{aligned} \mathbf{P}\int_{-1}^{+1} \frac{1}{\nu - \mu}\, d\nu &= \lim_{\delta \to 0} \int_{-1}^{\nu-\delta} \frac{1}{\nu - \mu}\, d\nu + \int_{\nu+\delta}^{+1} \frac{1}{\nu - \mu}\, d\nu\,, \\ &= \ln\left(\frac{1+\nu}{1-\nu}\right) . \end{aligned} \tag{9}$$

Using this result, we next integrate Eq. (5) over all directions to obtain a consistency criterion for the values of ν:

$$\begin{aligned} 1 &= \frac{c}{2}\int_{-1}^{+1} \frac{\nu}{\nu - \mu}\, d\mu\,, \\ &= \frac{c}{2}\nu \ln\left(\frac{1+\nu}{1-\nu}\right) . \end{aligned} \tag{10}$$

In analogy with Eq. (6), there are always two solutions for Eq. (10):

$$\nu = \pm\nu_1\,. \tag{11}$$

However, it is clear that the four solutions we have at this point cannot possibly form a complete set. Fortunately, we can obtain a weak solution of Eq. (1) for *every* value of $\nu \in [-1, +1]$ by adding add an angular delta-function to Eq. (5) as follows:

$$\psi_\nu(\mu) = \left[\frac{c}{4\pi} \frac{\nu}{\nu - \mu} + \frac{\lambda(\nu)}{2\pi} \delta(\mu - \nu) \right] \phi_\nu \,. \tag{12}$$

Re-expressing Eq. (4) as follows

$$(\nu - \mu)\, \psi_\nu = \frac{c}{4\pi} \nu \phi_\nu \,, \tag{13}$$

and substituting from Eq. (12) into Eq. (13), we obtain

$$(\nu - \mu) \left[\frac{c}{4\pi} \frac{\nu}{\nu - \mu} + \frac{\lambda(\nu)}{2\pi} \delta(\mu - \nu) \right] \phi_\nu = \frac{c}{4\pi} \nu \phi_\nu \,. \tag{14}$$

If we multiply Eq. (14) by any arbitrary non-singular angular function and integrate over all angles, we find that the resultant equation is satisfied because the contribution from the delta-function is identically zero. This means that Eq. (14) is satisfied in a weak sense. We next obtain a consistency equation by integrating Eq. (12) over all angles:

$$1 = \frac{c}{2} \nu \ln \left(\frac{1 + \nu}{1 - \nu} \right) + \lambda(\nu) \,. \tag{15}$$

This equation defines $\lambda(\nu)$:

$$\lambda(\nu) = 1 - \frac{c}{2} \nu \ln \left(\frac{1 + \nu}{1 - \nu} \right) \,. \tag{16}$$

Finally, assuming a scalar flux value of 2π in Eq. (12), we obtain an explicit form for ψ_ν that is analogous to Eq. (7):

$$\psi_\nu(\mu) = \frac{c}{2}\frac{\nu}{\nu - \mu} + \lambda(\nu)\delta(\mu - \nu)\,. \tag{17}$$

The modes corresponding to $\nu \in [-1, +1]$ are called the *transient* modes because the relaxation lengths of these modes are always smaller than the asymptotic relaxation length if $c \in (0, 1]$. In particular, the transient relaxation lengths are always a mean-free-path or less, while the asymptotic relaxation lengths are always greater than a mean-free path if $c \in (0, 1]$. Consequently, the transient modes are attenuated more quickly than the asymptotic modes, which causes the asymptotic modes to dominate many mean-free-paths from material interfaces and sources. As one might expect, the asymptotic relaxation length becomes equal to the diffusion length in the limit as $c \to 1$. In the limit as $c \to 0$, the asymptotic modes effectively disappear, leaving only the continuum modes. The asymptotic plus transient modes form a complete set of functions.

In conclusion, the 1-D slab-geometry monoenergetic transport solution can be expressed in a homogeneous sourceless region as a combination of asymptotic modes and transient continuum modes as follows:

$$\psi(x, \mu) = a_0^+ \frac{c}{2}\frac{\nu_0}{\nu_0 - \mu} \exp^{\frac{-\sigma_t x}{\nu_0}} +$$

$$a_0^- \frac{c}{2}\frac{\nu_0}{\nu_0 + \mu} \exp^{\frac{+\sigma_t x}{\nu_0}} +$$

$$\int_{-1}^{+1} A(\nu) \left[\frac{c}{2} \frac{\nu}{\nu - \mu} + \lambda(\nu) \delta(\mu - \nu) \right] \exp^{\frac{-\sigma_t x}{\nu}} \; d\nu \,, \tag{18}$$

where a_0^+, a_0^-, and $A(\nu)$ are chosen to meet boundary conditions. Here we will not explain exactly how they are chosen. Our main purpose is simply to give insight into the fundamental nature of transport solutions in 1-D slab geometry.