Lecture 2
The Linearized Boltzmann Transport Equation

1 Transport Assumptions

1. Continuum point particles.

2. No particle-particle interactions.

3. Binary particle-target interactions.

4. Instantaneous collisions.

5. No macroscopic force fields.

2 The Boltzmann Equation

The Boltzmann equation with absorption and scattering can be expressed as follows:

$$\frac{1}{v}\frac{\partial \psi}{\partial t} = Q + \int_0^\infty \int_{4\pi} \sigma_s \left(E' \rightarrow E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega} \right) \psi \left(E', \overrightarrow{\Omega}' \right) \, d\Omega' \, dE' - \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi - \sigma_t \psi \quad (1)$$

where $\psi(t, \overrightarrow{r}, \overrightarrow{\Omega}, E)$ is the angular flux, $Q(t, \overrightarrow{r}, \overrightarrow{\Omega}, E)$ is the inhomogeneous source, $\sigma_s \left(\overrightarrow{r}, E' \rightarrow E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega} \right)$ is the scattering kernel expressed in terms of the differential macroscopic scattering cross section, and $\sigma_t \left(\overrightarrow{r}, E \right)$ is the total macroscopic cross-section.

We next interpret each term. The term,

$$\frac{1}{v}\frac{\partial \psi}{\partial t}\, dP\,, \quad (p/sec),$$

represents the time rate of change of the number of particles in the differential phase-space volume, dP. The term,

$$Q\, dP, \quad (p/sec),$$

represents the rate at which particles are created in the differential phase-space volume, dP. The term,

$$\int_0^\infty \int_{4\pi} \sigma_s \left(E' \rightarrow E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega} \right) \psi \left(E', \overrightarrow{\Omega}' \right)\, d\Omega'\, dE'\, dP\,, \quad (p/sec),$$

represents the rate at which particles scatter into the differential phase-space volume, dP. The term,

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi\, dP\,, \quad (p/sec),$$

represents the rate at which particles advect out of the differential phase-space volume, dP, minus the rate at which particles advect into it. It is easier to understand this interpretation if one integrates this term over an arbitrary spatial volume, V_0, and uses the divergence theorem to re-express that integral as a surface integral.

$$\begin{aligned} \int_{V_0} \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi\, dV &= \oint \psi \overrightarrow{\Omega} \cdot \overrightarrow{n}\, dA\,, \\ &= \int_{\Gamma_+} \psi \overrightarrow{\Omega} \cdot \overrightarrow{n}\, dA + \int_{\Gamma_-} \psi \overrightarrow{\Omega} \cdot \overrightarrow{n}\, dA\,, \end{aligned}$$

where Γ_+ denotes that portion of the surface on which $\overrightarrow{\Omega} \cdot \overrightarrow{n} > 0$, and Γ_- denotes that portion of the surface on which $\overrightarrow{\Omega} \cdot \overrightarrow{n} < 0$. The integral over Γ_+ represents the rate at which particles are advecting out of the volume, while the integral over Γ_- represents the negative of the rate at which particles are advecting into the volume. If we divide these integrals by V_0, and take the limit as $V_0 \rightarrow 0$, we recover the original gradient term. The term,

$$\sigma_t \psi \, dP \,, \quad (p/sec),$$

represents the rate at which particles are removed from the differential phase-space volume, dP, through either absorption or scatter.

Thus we see that the Boltzman equation simply states that the time rate of change in the number of particles in dP is equal to the rate at which particles come into dP minus the rate at which particles leave dP. It is therefore a particle conservation equation in phase-space.

3 The Balance Equation

Integrating the Boltzmann equation over all angles and energies, we get

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = Q_0 + S_0 - \overrightarrow{\nabla} \cdot \overrightarrow{J} - \sigma_t \phi \,, \tag{2}$$

where ϕ and $\overrightarrow{J}$ are the energy-integrated scalar flux and current, respectively,

$$Q_0 = \int_0^\infty \int_{4\pi} Q \; d\Omega \, dE \, , \tag{3}$$

and

$$S_0 = \int_0^\infty \int_{4\pi} S \; d\Omega \, dE \, . \tag{4}$$

We can further reduce Eq. (2) by noting that if $S \, dP$ represents the rate at which particles in the differential spatial dV scatter into directions about $\overrightarrow{\Omega}$ and energies about E, then S_0 must represent the rate at which particles are scattering within dV. Thus,

$$S_0 = \sigma_s \phi \, . \tag{5}$$

Substituting from Eq. (5) into Eq. (2), and recognizing that $\sigma_t = \sigma_a + \sigma_s$, we obtain

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = Q_0 - \overrightarrow{\nabla} \cdot \overrightarrow{J} - \sigma_a \phi \, . \tag{6}$$

Equation (6) is called the balance equation, and represents a statement of particle balance for particles of all directions and energies. The term,

$$\frac{1}{v} \frac{\partial \phi}{\partial t} \, dV \, ,$$

represents the time rate of change of particles in dV. The term,

$$Q_0 \, dV \, ,$$

represents the rate at which particles are being created within dV. The term,

$$\overrightarrow{\nabla} \cdot \overrightarrow{J}\ dV\,,$$

represents the net rate at which particles leave dV, i.e., the rate at which they leave minus the rate at which they enter. The term,

$$\sigma_a \phi\ dV\,,$$

represents the rate at which particles are absorbed within dV.

4 Neutronics Calculation Types

In this section we discuss the various types of calculations that are routinely performed in reactor engineering analysis.

4.1 Source Calculations

Source calculations can be either time-dependent or steady-state calculations with or without fission. The most general form of the equation solved is:

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla}\psi + \sigma_t \psi = \int_0^\infty \int_{4\pi} \sigma_s \left(E' \rightarrow E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega} \right) \psi \left(E', \overrightarrow{\Omega}' \right)\ d\Omega'\, dE' + \frac{\chi(E)}{4\pi} \int_0^\infty \nu\sigma_f(E')\phi(E')\, dE' + Q\,, \tag{7}$$

where ν is the average number of neutrons per fission, χ is the normalized fission spectrum, and σ_f is the macroscopic fission cross section.

4.2 k-Eigenvalue Calculations

This type of eigenvalue calculation relates to the criticality of a multiplying system. The most general form of the equation solved is:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla}\psi + \sigma_t\psi - \int_0^\infty \int_{4\pi} \sigma_s \left(E' \to E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right) \psi \left(E', \overrightarrow{\Omega}'\right) \, d\Omega' \, dE' = \frac{1}{k} \frac{\chi(E)}{4\pi} \int_0^\infty \nu\sigma_f(E')\phi(E') \, dE' \tag{8}$$

where k is the desired eigenvalue.

4.3 α-Eigenvalue Calculations

This type of eigenvalue calculation relates to the asymptotic temporal behavior of multiplying system. In particular, for asymptotically large times the neutron flux is given by $\psi_0 \exp(\alpha t)$ where ψ_0 is the eigenfunction associated with α. The most general form of the equation solved is:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla}\psi + \sigma_t\psi - \int_0^\infty \int_{4\pi} \sigma_s \left(E' \to E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right) \psi \left(E', \overrightarrow{\Omega}'\right) \, d\Omega' \, dE' - \frac{\chi(E)}{4\pi} \int_0^\infty \nu\sigma_f(E')\phi(E') \, dE' = -\frac{\alpha}{v}\psi \, . \tag{9}$$

4.4 Critical Parameter Search Calculations

Critical parameter search calculations generally relate to reactor design. One chooses solves for certain parameters, such as system dimensions, isotope concentrations, etc., to obtain a critical system. The k-eigenvalue calculation is used for critical parameter searches in conjunction with a search algorithm. There are many variations on this type of calculation that relate to the full spectrum of design activities.

5 Other Forms of the Boltzmann Equation

In this section we discuss the Boltzmann equation for charged-particles and the Boltzmann equation for thermal radiation transport.

5.1 The Charged-Particle Transport Equation

The charged-particle transport equation is actually identical to that for neutrons except that for Coulomb interactions (as opposed to nuclear interactions), the mean-free-paths are extremely small and very small scattering angles are highly probable. The Coulomb interactions are inelastic, so small scattering angles necessarily mean small energy losses as well. The Coulomb interactions are usually treated with an asymptotic approximation to the Boltzmann scattering operator called the Fokker-Planck approximation, while the

standard Boltzmann scattering operator is used for the nuclear interactions. This strategy leads to the Boltzmann-Fokker-Planck (BFP) equation.

$$\mu\frac{\partial\psi}{\partial x}+\sigma_t\psi=S+\frac{\alpha}{2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]+\frac{\partial}{\partial E}(\beta\psi)+Q\,, \tag{10}$$

The first Fokker-Planck operator is called the continuous scattering operator. It causes particles to continuously scatter with a change in direction per unit pathlength equal to α $(steradians/cm)$. When one is approximating the Boltzmann scattering operator with the Fokker-Planck operator, α is calculated in terms of the cross-section as follows:

$$\alpha(\overrightarrow{r},E)=2\pi\int_0^E\int_{-1}^{+1}\sigma_s(\overrightarrow{r},E\rightarrow E',\mu_0)\,(1-\mu_0)\,d\mu_0\,dE'\,. \tag{11}$$

Note from the upper integration limit in energy that we have assumed that particles only downscatter, i.e., they lose energy in scattering events, but do not gain energy. This is appropriate for relativistic charged-particle transport because particles that thermalize are considered to be "absorbed". Upscatter does occur at thermal energies. The second Fokker-Planck operator is called the continuous-slowing-down operator. It causes particles to continuously slow down with an energy loss per unit pathlength equal to β (MeV/cm). When one is approximating the Boltzmann scattering operator with the Fokker-Planck operator, β is calculated in terms of the cross-section as follows:

$$\beta(\overrightarrow{r},E)=2\pi\int_0^E\int_{-1}^{+1}\sigma_s(\overrightarrow{r},E\rightarrow E',\mu_0)\,(E-E')\,d\mu_0\,dE'\,. \tag{12}$$

Use of the continuous-slowing-down operator is not adequate at thermal energies. One must also use a higher order term to obtain a Maxwellian equilibrium solution. We will not consider thermal charged-particle transport here, but it does arise in plasma physics modeling.

The same techniques used to solve the Boltzmann can be applied to solve the Boltzmann-Fokker-Planck equation. In general, charged-particle transport is usually much more computationally demanding than neutral particle transport because of the extreme anisotropy of the scattering and the huge magnitude of the scattering cross-sections. Thus better numerical methods must generally be used for charged-particle transport. We will later describe solution techniques for the BFP equation in detail, and discuss Fokker-Planck operators within the context of asymptotic transport approximations.

5.2 The Thermal Radiation Transport Equation

The equations of thermal radiation transport consist of two equations: a transport equation and a material temperature equation. In particular, they are

$$\frac{1}{c}\frac{\partial \mathcal{I}}{\partial t} = \sigma_s \int_{4\pi} \mathcal{I}\, d\Omega + \sigma_a B(T,E) - \overrightarrow{\Omega} \cdot \overrightarrow{\nabla}\mathcal{I} - \sigma_t \mathcal{I}\,, \tag{13}$$

and

$$C_v \frac{\partial T}{\partial t} = \int_0^\infty \int_{4\pi} \sigma_a \left(\mathcal{I}(\overrightarrow{\Omega'}, E') - B(T,E') \right)\, d\Omega'\, dE' + q\,, \tag{14}$$

where $\mathcal{I}$ is the angular intensity , σ_t is the total (absorption plus scattering) cross section, σ_s is the scattering cross section, σ_a is the absorption cross-section, $B(T,E) = \frac{2E^3}{h^3c^2}\left[\exp\left(\frac{E}{kT}\right)-1\right]^{-1}$ is the Planck emission function, C_v is the heat capacity, and T is the material temperature, and q is an inhomogeneous material energy source. The angular intensity is equal to the angular flux multiplied by the particle energy, i.e., $\mathcal{I}(\overrightarrow{P}) = \psi(\overrightarrow{P})E$. The equation is cast in terms of $\mathcal{I}$ rather than ψ because photon and material internal energy are conserved in the thermal radiation transport process, but photon number is not preserved. These equations simply express the conservation of photon and material internal energy within each differential phase-space volume in the problem domain. Photons are emitted isotropically (uniformly in all directions) in proportion to the energy-averaged absorption cross section and the material temperature raised to the fourth power. The emission energy spectrum is defined by the Planck function (the black-body spectrum). Photons are absorbed by the material as a function of the absorption cross section. The photon emission process tends to decrease the material temperature and the photon absorption process tends to increase it. The absorption-emission process is very much like scattering since the emission and absorption are both proportional to the absorption cross section. In equilibrium, the intensity is equal to the Planck function.

In the infrared regime, infrared transport can generally be performed using relatively simple numerical methods. However, in the stellar regime, thermal radiation transport

requires very sophisticated methods. At sufficiently high intensities, the radiation energy and momentum deposition in the material can cause it to move, in which case one must use the equations of radiation-hydrodynamics to describe the physics. The equations of radiation-hydrodynamics express the conservation of material mass, the conservation of total energy (radiation, material-internal, and material-kinetic), and the conservation of total momentum (radiation plus material).