Lecture 7
The $\overrightarrow{\Omega} \cdot \overrightarrow{\nabla}$ Operator in Curvilinear Coordinates

The transport leakage operator, $\overrightarrow{\Omega} \cdot \overrightarrow{\nabla}$, is not just the dot-product of $\overrightarrow{\Omega}$ and the spatial gradient of ψ in curvilinear coordinates. However, it represents $\frac{\partial \psi}{\partial s}$ in all geometries, where s is a spatial coordinate in the direction of particle flow. Extra terms occur in curvilinear coordinates because the symmetry preservation in such systems generally requires that the direction coordinate system change as a particle streams. For instance, the directional coordinate system for 1-D spherical geometry is shown in Fig. 1. Note that the particle direction is a function only of $\mu = \overrightarrow{\Omega} \cdot \frac{\overrightarrow{r}}{\| \overrightarrow{r} \|}$. Also note that unless a particle has a direction of $\mu \pm 1$, the particle direction changes as it streams, as illustrated in Fig. 2. Using the chain rule, it follows that

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \psi = \frac{d\psi}{ds} = \frac{dr}{ds}\frac{\partial \psi}{\partial r} + \frac{d\mu}{ds}\frac{\partial \psi}{\partial \mu} \,. \tag{1}$$

We next derive the derivative, $\frac{dr}{ds}$. Using the law of cosines, it follows from Fig. 2 that

$$r^2(s) + \Delta s^2 - 2r(s)\Delta s \cos[\pi - \theta(s)] = r^2(s + \Delta s) \,. \tag{2}$$

Recognizing that $\cos[\pi - \theta(s)] = -\cos\theta(s)$, we manipulate Eq. (2) to obtain,

$$\frac{r^2(s + \Delta s) - r^2(s)}{\Delta s} = \frac{\Delta s^2 + 2r(s)\Delta s \mu(s)}{\Delta s} \,. \tag{3}$$

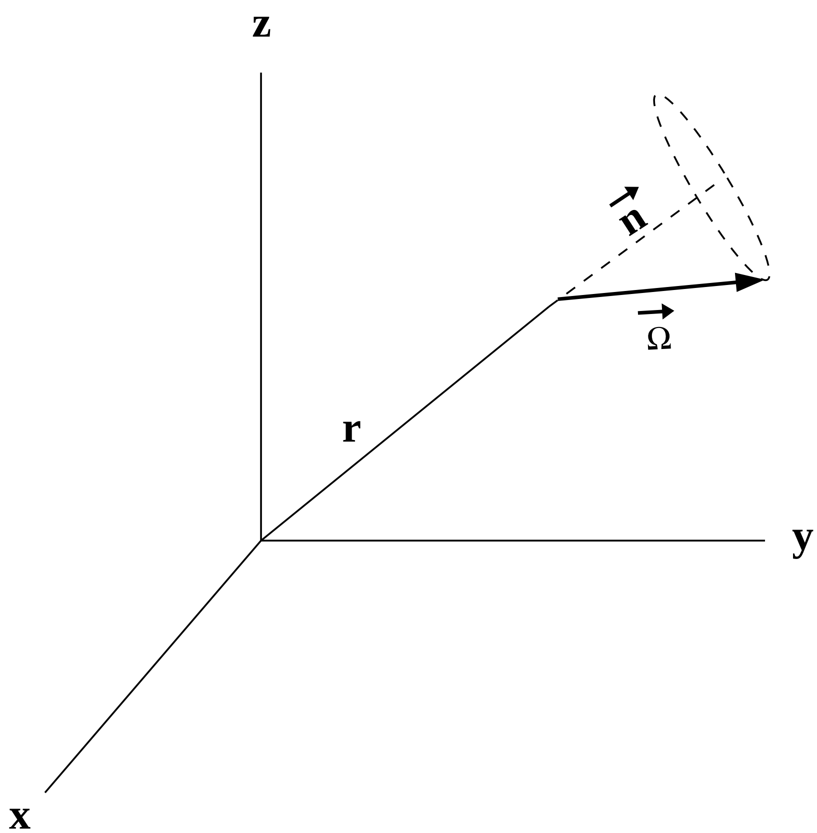

Figure 1: Direction coordinate system for 1-D sherical geometry. Note that there is only one directional variable: $\mu = \overrightarrow{\Omega} \cdot \overrightarrow{n}$, where $\overrightarrow{n} = \frac{\overrightarrow{r}}{\| \overrightarrow{r} \|}$.

Taking the limit of Eq. (3) as $\Delta s \to 0$, we get

$$2r(s)\frac{dr}{ds} = 2r(s)\mu \,. \tag{4}$$

Solving Eq. (4) for the desired derivative, we get

$$\frac{dr}{ds} = \mu \,. \tag{5}$$

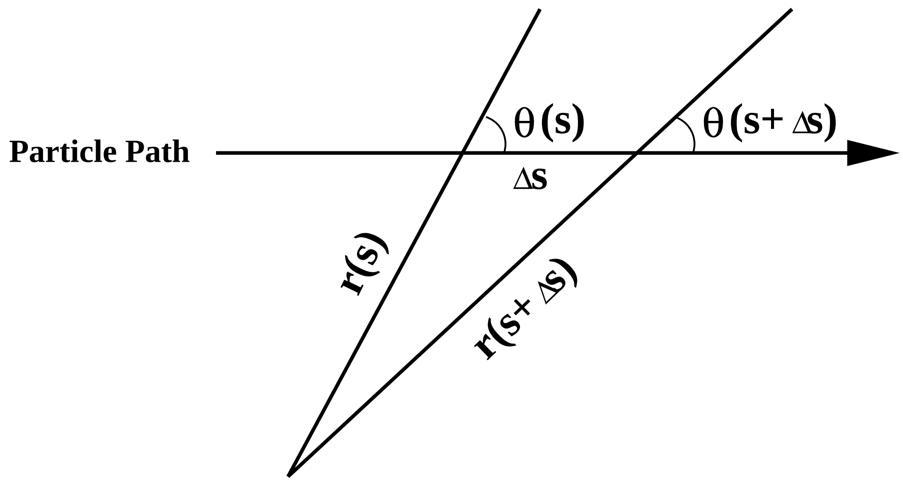

Figure 2: Illustration of change in direction coordinate as a particle streams in 1-D spherical geometry. Note that $\mu = \cos\theta = \overrightarrow{n} = \frac{\overrightarrow{r}}{\| \overrightarrow{r} \|}$.

We next derive the derivative, $\frac{d\mu}{ds}$. Using the law of signs, it follows from Fig. 2 that

$$\frac{r(s+\Delta s)}{\sin[\pi - \theta(s)]} = \frac{r(s)}{\sin[\theta(s+\Delta s)]}\,. \tag{6}$$

Recognizing that $\sin(\pi - x) = \sin x$ for any angle x, we manipulate Eq. (6) to obtain

$$r(s+\Delta s)\sin[\theta(s+\Delta s)] = r(s)\sin[\theta(s)]\,. \tag{7}$$

$$\begin{aligned}
\left[r(s) + \frac{dr}{ds}\Delta s\right]\left[\sin[\theta(s)] + \cos[\theta(s)]\frac{d\theta}{ds}\Delta s\right] &= r(s)\sin[\theta(s)]\,,\\
r(s)\sin[\theta(s)] + r(s)\cos[\theta(s)]\frac{d\theta}{ds}\Delta s + \sin[\theta(s)]\frac{dr}{ds}\Delta s + O(\Delta s^2) &= r(s)\sin[\theta(s)]\,,\\
r(s)\cos[\theta(s)]\frac{d\theta}{ds}\Delta s &= -\sin[\theta(s)]\frac{dr}{ds}\Delta s + O(\Delta s^2)\,, \qquad (8)
\end{aligned}$$

Recognizing that $\frac{dr}{ds} = \cos[\theta(s)]$, we further manipulate Eq. (8) to obtain

$$r(s)\frac{d\theta}{ds} = -\sin[\theta(s)]\,. \tag{9}$$

Since $\mu = \cos\theta$, it follows that $\frac{d\mu}{d\theta} = -\sin\theta$, and further that

$$\frac{d\mu}{ds} = -\sin\theta\frac{d\theta}{ds}\,. \tag{10}$$

Dividing Eq. (9) by $r(s)$ and multiplying it by $-\sin[\theta(s)]$, we obtain

$$-\sin[\theta(s)]\frac{d\theta}{ds} = \frac{1}{r(s)}\sin^2[\theta(s)]\,. \tag{11}$$

Substituting from Eq. (10) into Eq. (11), we obtain the desired quantity:

$$\frac{d\mu}{ds} = \frac{1}{r}(1-\mu^2)\,. \tag{12}$$

Substituting from Eqs. (5) and (12) into Eq. (1), we obtain the leakage operator for 1-D spherical geometry:

$$\overrightarrow{\Omega}\cdot\overrightarrow{\nabla} = \mu\frac{\partial\psi}{\partial r} + \frac{1}{r}(1-\mu^2)\frac{\partial\psi}{\partial\mu}\,. \tag{13}$$

Note that the angular derivative term disappears at $\mu \pm 1$, which is appropriate since particles in these two directions do not change their direction as they stream. Because particles in directions other than $\mu = \pm 1$ change their direction as they stream, the leakage operator contributes both a source and a sink to the differential phase-space volume associated with directions about μ. To see this, we must express Eq. (13) in an equivalent conservative form:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} = \frac{\mu}{r^2}\frac{\partial}{\partial r}r^2\psi + \frac{1}{r}\frac{\partial}{\partial \mu}(1-\mu^2)\psi \, . \tag{14}$$

Using Eq. (14), the 1-D spherical-geometry monoenergetic transport equation with isotropic scattering and an isotropic distributed source becomes

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}r^2\psi + \frac{1}{r}\frac{\partial}{\partial \mu}(1-\mu^2)\psi + \sigma_t\psi = \frac{1}{4\pi}\sigma_s\phi + \frac{1}{4\pi}Q_0 \, . \tag{15}$$

To demonstrate the conservative properties of this equation, it is useful integrate it over a phase-space volume. The differential phase-space volume for 1-D spherical geometry is

$$dP = dV\, d\Omega = 4\pi r^2\, dr\; 2\pi\, d\mu = 8\pi^2 r^2\; dr\, d\mu \, , \tag{16}$$

Integrating Eq. (15) over a finite phase-space volume characterized by radii, r_1 and r_2, and cosines, μ_1 and μ_2, we get

$$\begin{gathered} 4\pi r_2^2 \left\langle\, \mu\psi(r_2,\mu)\,\right\rangle_{\Delta\Omega} - 4\pi r_1^2 \left\langle\, \mu\psi(r_1,\mu)\,\right\rangle_{\Delta\Omega} + \\ 2\pi(1-\mu_2^2) \left\langle\, r^{-1}\psi(r,\mu_2)\,\right\rangle_V - 2\pi(1-\mu_1^2) \left\langle\, r^{-1}\psi(r,\mu_1)\,\right\rangle_V + \\ \left\langle\, \sigma_t\psi\,\right\rangle_{\Delta P} = \left\langle\, \sigma_s\psi\,\right\rangle_V + \left\langle\, Q_0\,\right\rangle_V \; . \end{gathered} \tag{17}$$

where $\langle\cdot\rangle_V$ implies integration over the volume associated with the interval $[r_1, r_2]$, $\langle\cdot\rangle_{\Delta\Omega}$ implies integration over the solid angle associated with the interval, $[\mu_1, \mu_2]$, and $\langle\cdot\rangle_{\Delta P}$ implies integration over both, i.e., over the phase-space volume. Without loss of generality, we assume that μ_1 and μ_2 are both positive, with $\mu_2 < 1$. Under this assumption, we can re-arrange Eq. (17) by placing sinks on the left side of the equation, and sources on the right side:

$$4\pi r_2^2 \left\langle \mu\psi(r_2,\mu) \right\rangle_{\Delta\Omega} + 2\pi(1-\mu_2^2) \left\langle r^{-1}\psi(r,\mu_2) \right\rangle_V + \left\langle \sigma_t\psi \right\rangle_{\Delta P} =$$

$$4\pi r_1^2 \left\langle \mu\psi(r_1,\mu) \right\rangle_{\Delta\Omega} + 2\pi(1-\mu_1^2) \left\langle r^{-1}\psi(r,\mu_1) \right\rangle_V + \left\langle \sigma_s\psi \right\rangle_V + \left\langle Q_0 \right\rangle_V \,. \tag{18}$$

The first term on the left side of Eq. (18) represents the rate at which particles flow out of V through the surface associated with r_2. The second term on the left side of Eq. (18) represents the rate at which particles flow out of $\Delta\Omega$ due to the angular change associated with streaming. The third term on the left side of Eq. (18) represents the rate at which particles are removed from ΔP by absorption and scattering. The first term on the right side of Eq. (18) represents the rate at which particles flow into V through the surface associated with r_1. The second term on the right side of Eq. (18) represents the rate at which particles flow into $\Delta\Omega$ due to the angular change associated with streaming. The third term on the right side of Eq. (18) represents the rate at which particles scatter into ΔP. The fourth term represents the rate at which particles are created within ΔP.

Note that if we integrate Eq. (15) over all directions, the balance equation contains no contributions from the angular derivative term:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2 J + \sigma_a\phi = Q_0 \,. \tag{19}$$

Furthermore, Eq. (19) can be written in a general form as

$$\overrightarrow{\nabla} \cdot \overrightarrow{J} + \sigma_a \phi = Q_0 \, . \tag{20}$$

where $\overrightarrow{\nabla} \cdot$ is the standard spatial divergence operator. This is the case in all geometries.