Week 1 • Problem Set 4 • Deviations and acyclic closures

Quick Problem:

Let $(Q, \mathfrak{n}, k)$ be a regular local ring and let $R = Q/I$ with $I \subseteq \mathfrak{n}^2$.

(a) From how the acyclic closure is constructed, identify $\varepsilon_2(R)$ as a commonly known quantity involving some kind of homology.

(b) Explain what the correspondence between the acyclic closure of $R \to k$ and the minimal model of $Q \to R$ implies about the deviation $\varepsilon_2(R)$.

Problems:

1. Consider the running example from the lectures:

$$\text{Let } R = k[[x, y]]/(f) \text{ with } f = x^3 + y^3, \text{ and set } S = k = R/(x, y).$$

(a) Determine some or all of the deviations of R.

(b) Determine some or all of the complex $\operatorname{Ind}^{\gamma}_{R} R\langle X\rangle$ of Γ-indecomposables where $R\langle X\rangle$ is an acyclic closure of k over R.

(c) Let e be the variable in degree 1 which maps to x, and ϑ be the derivation sending e to 1 and the rest of X_1 to zero. Using the equation $\vartheta(\partial(t)) = \partial(\vartheta(t))$, determine the values of ϑ on X_2.

2. Consider the example you computed in #1 of Problem Set 2:

$$R = k[[x, y]]/(x^2, xy) \longrightarrow S = k = R/(x, y)$$

Running the Macaulay2 code from Wednesday's M2 session yields the output:

$$\{\mathtt{x}, \mathtt{y}, \mathtt{xT_1}, \mathtt{yT_1}, -\mathtt{xT_1T_2}, -\mathtt{yT_1T_3}, \mathtt{xT_1T_2T_3}, \mathtt{xT_1T_2T_4}\}$$

giving a list of the images of the variables under the differential of the variables denoted by $T_1, T_2, \dots$, numbered in the order that they were adjoined. Translating, we see that the acyclic closure looks like:

$$R\langle T\rangle = \langle T_1, T_2, \dots\rangle$$

with

$$\begin{gathered}\partial(T_1) = x, \partial(T_2) = y,\\ \partial(T_3) = xT_1, \partial(T_4) = yT_1,\\ \partial(T_5) = -xT_1T_2, \partial(T_6) = -yT_1T_3,\\ \partial(T_7) = xT_1T_2T_3, \partial(T_8) = xT_1T_2T_4\end{gathered}$$

(a) Find the degree of each T_i above.

(b) Determine some or all of the deviations of R.

(c) Determine some or all of the complex $\operatorname{Ind}^{\gamma}_{R} R\langle T\rangle$ of Γ-indecomposables.

3. (a) Here is an example of a homomorphism whose acyclic closure is not a minimal complex. Let k be a field of characteristic zero. Consider the canonical surjection.

$$R = k[[x, y]] \longrightarrow S = R/(x^2, xy)$$

Compute a few steps of the acyclic closure until you see the non-minimality appear.

(b) Show that a surjective map of finite projective dimension $\varphi\colon R \to S$ is complete intersection if and only if the acyclic closure is minimal.

Lastly, here are two proofs that we skipped in the lecture for you to try.

Let $\varphi\colon R \to S$ be a surjective homomorphism of local rings. (Much does not actually require surjectivity, but we assume this for simplicity here.)

5. Let $R \to R\langle X\rangle$ be a semi-free Γ-extension resolving S with $X_0 = \emptyset$.

For any $x \in X_n$ such that the class of x generates a free summand of the cokernel of $\partial^I_{n+1}\colon SX_{n+1} \to SX_n$, there exists an R-linear chain Γ-derivation $\theta_x\colon R\langle X\rangle \to R\langle X\rangle$ of degree -n such that $\theta_x(x) = 1$ and $\theta_x(X_n \backslash \{x\}) = 0$.

Hint: Note that $Sx \cap \operatorname{im} \partial^I_{n+1} = 0$. Define a map from I to $R\langle X\rangle$. Why is it a chain map? Why can you extend it to a derivation from $R\langle X\rangle$?

6. Let $R \to R\langle X\rangle$ be a semi-free Γ-extension resolving S with $X_0 = \emptyset$.

Prove that $R\langle X\rangle$ is an acyclic closure of S over R if and only if the complex $I = \operatorname{Ind}^{\gamma}_R R\langle X\rangle$ of Γ-indecomposables is minimal ($\partial(I) \subseteq \mathfrak{m}I$).