Week 1 • Problem Set 5 • Homotopy Lie algebra and cotangent complex

Quick Problem:

Consider the running example $R = k[[x, y]]/(x^3 + y^3)$.

Compute $\pi(R)$ and the cotangent complex $\mathcal{L}_{R|k} = \mathbb{L}_{R|k} = L_{R|k}$. .

Problems:

1. Let k be a field of characteristic zero. Consider again the running example from the lectures:

$$\text{Let } R = k[[x, y]]/(f) \text{ with } f = x^3 + y^3, \text{ and set } S = k = R/(x, y).$$

Recall that in the quick problem at the end of class today, we showed that all the brackets of $\pi(R)$ are zero. What does this imply about $\operatorname{Ext}_R(k, k) = U(\pi(R))$?

2. Let $R = k[x, y]/(x^2, xy)$ with k of characteristic 0. Then from Macaulay2 we see that a minimal model for R over k has the form $k[T] = k[x, y, T_1, T_2, \ldots]$ with

$$\partial(T_1) = x^2,\ \partial(T_2) = xy,\ \partial(T_3) = xT_2 - yT_1,$$

$$\partial(T_4) = -T_1T_2 + xT_3,\ \partial(T_5) = -T_2T_3 + yT_4,\ \partial(T_6) = -T_1T_3 + xT_4,$$

$$\partial(T_7) = -T_2T_4 + xT_5,\ \partial(T_8) = -T_3^2 - 2T_2T_4 + 2yT_6,\ \partial(T_9) = -T_1T_4 + xT_6, \ldots$$

(a) Write ordered bases for the sets T_0, T_1, T_2, T_3, T_4, and T_5.

(b) Using lexicographic ordering, order the sets T_0^2, $T_1 * T_0$, $T_1^2 \oplus T_2T_0$, $T_2T_1 \oplus T_3T_0$, and $T_2T_2 \oplus T_3T_1 \oplus T_4T_0$. Use the resulting bases to write matrices for the quadratic part $\partial^{[2]} \colon kT \to kT^2$. *hint: Based on the formulas for the differentials, how do ∂ and $\partial^{[2]}$ compare in this example?*

(c) Transposing these matrices, one gets matrices describing brackets of $\pi(R)$. Use this to determine some of the relations of $\pi(R)$.

(d) Based on your work above, it $\operatorname{Ext}_R(k, k)$ graded-commutative?

3. Let $R \to R[Y]$ be a semi-free extension with no variables of degree 0. Define its module of indecomposables as

$$\operatorname{Ind}_R R[Y] = R[Y]/(R + IY + Y^2).$$

Assume that $R[Y]$ resolves S.

(a) Prove that $R[Y]$ is a minimal model for $R \to S$ if and only if the differential is *decomposable:* setting Y_0 equal to a minimal set of generators for $\mathfrak{m}$, one has $\partial(Y_{n+1}) \subseteq \sum_{i+j=n} RY_iY_j$ for all $n \geq 0$.

(b) Deduce that $R[Y]$ is a minimal model for $R \to S$ if and only if the complex $\operatorname{Ind}_R R[Y]$ is minimal.

(c) Use part (a) and lifting lemmas to show that minimal models are unique up to isomorphism of dg algebras.

Note 1: One can also define $\operatorname{Ind}_R R[Y]$ *when* $Y_0 \neq \emptyset$, *by also modding out by the entire degree 0 piece of* $R[Y]$, *but that complicates the notation introduced in part (a) above.*

Note 2: Some people define both $\operatorname{Ind}^{\gamma}_R R\langle X\rangle$ *and* $\operatorname{Ind}_R R[Y]$ *by going modulo the maximal ideal* $\mathfrak{m}$ *of* R *(or* $\mathfrak{n}$ *of* S*, that is, applying* $-\otimes_S k$*), so then the differentials in these complexes would vanish for an acyclic closure* $R\langle X\rangle$*, respectively a minimal model* $R[Y]$*.*