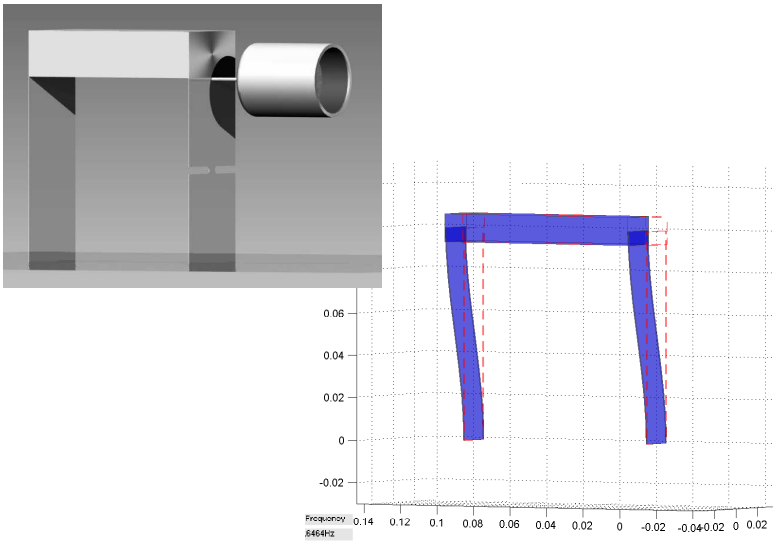
UNIVERSITY OF TWENTE.

Faculty of
Engineering Technology

Mechanical Automation and Mechatronics

Prototype modelling of mechanical systems

An introduction for Project F



R.G.K.M. Aarts, J.B. Jonker, J. van Dijk

edition: 2014
course: Project F (191103250, WB 3.2)
WA-nr.: **1239**

WA

© 2007–2014. Copyright J.B. Jonker, R.G.K.M. Aarts, J. van Dijk, Universiteit Twente, Enschede.

All rights reserved. No part of this work may be reproduced, in any form or by any means, without the explicit and prior permission from the authors.

Acknowledgement: Many of the graphs in this work as well as the data files and some of the text have been created by Jan Bennik and Joost Könemann. Tjeerd van der Poel and Steven Boer have contributed further improvements of the visualisation.

Contents

1 Introduction

Modelling and analysis enable designers to test whether design specifications are met. It is important to be able to analyse a system at different levels - high level analysis in the early, conceptual stage when only a few design details are known, and more detailed towards the end of the design process. The use of prototype models significantly shortens the design time and reduces the cost of design. It further provides the designer with immediate feedback on design decisions, which, in turn, promises a more comprehensive exploration of design alternatives and a better performing final design.

For design of mechatronic systems it is essential to make use of simple prototype models with a few degrees of freedom that capture only the relevant system dynamics. In this tutorial an approach will be outlined to obtain so-called *prototype models* of the mechanical part of a mechatronic system. The applied techniques are taken from a more general *flexible multibody system approach* [8, 9], which is a well-suited method to model the dynamic behaviour of the mechanical (sub)systems. In this approach the mechanical components are considered as rigid bodies that interact with each other through a variety of connections such as hinges and flexible coupling elements like trusses and beams (see figure 1.1). A mathematical description of these models is represented by the equations of motion in terms of the system's degrees of freedom (the Lagrange equations [7]) derived from the multibody systems approach).

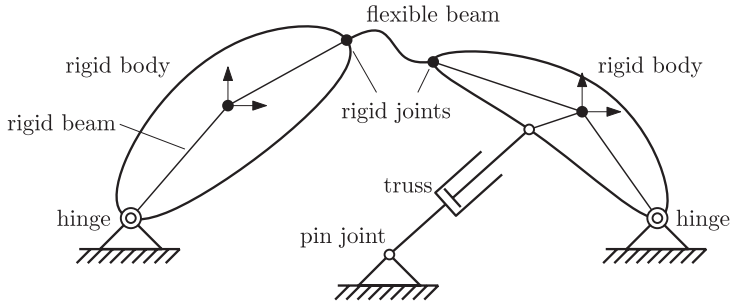


Figure 1.1: Flexible multibody system.

For control synthesis a control engineering based system representation is required. State space formulations are well suited to deal with both Single Input Single Output (SISO) systems as well as with more complicated Multiple Input Multiple Output (MIMO) systems. Linearization of the equations of motion is a well-known technique to obtain the state space matrices of a linearized state-space formulation for flexible multibody systems [1, 10]. Equilibrium solutions are of importance when dealing with elastic structures.

A finite element based modelling approach is presented in which a multibody system is modelled as an assembly of rigid body elements interconnected by joint elements such as hinges and flexible elements like trusses and beams. For each element a fixed number of discrete deformation modes is defined that are invariant for rigid body movements of the element. The method is applicable for flexible multibody systems as well as for flexible structures in which the system members experience only small displacement motions and elastic deformations with respect to an equilibrium position. A software package called SPACAR based on this finite element approach is used for the kinematic and dynamic analysis of flexible multibody systems [8]. It has an interface to MATLAB [12]. Subsequently all MATLAB tools for the analysis of (linear) systems are available including graphical means like Bode plots and s-plane

representations.

Typically for the design of mechatronic systems, the analyses mentioned above, consist of:

- Kinematics
- Natural frequencies and mode shapes
- State space input output formulations
- Static stability (buckling)
- Simulation of the dynamic behaviour

If one considers the following phases in mechatronic system design (not all in chronological order, some are executed in parallel):

- Conceptual design,
- Dimensioning the concepts,
- Computer aided prototyping,
- Final design (fine tuning),

then almost in each phase the dynamics play an important role and therefore modelling the dynamics is of great importance. However, the different phases demand each a more specific analysis type. In the conceptual phase the kinematics are of importance. In this phase it is important to restrict the number of degrees of freedom of the bodies in the concept to the few desired ones, by applying proper constraints. Often the principle of "Exact Constraint Design" [4, 16] is applied. A design is called exactly constraint if it has exactly the required degrees of freedom and constraints so that the system is kinematically and statically determinate. Thinking in terms of degrees of freedom is crucial in this phase and well supported by the underlying modelling method. Especially in the second phase the dynamics play a more dominant role. Mode-shape analysis and analysis of the (MIMO) transfer functions by state space formulations are crucial and also well supported by the modelling method. Computer models are very suitable for the computer aided prototyping if these models are adequate in the sense of incorporating the relevant dynamics but still fast in simulation and easy to change. In the final design phase much more methods and computer tools are needed than the one described here (i.e. ANSYS), although it is still of value in this phase for analysis of buckling and non-linear behaviour.

1.1 Example system

Application of the multibody system approach is illustrated through a detailed model development of an example system, which consists of a one degree of freedom (1-DOF) support mechanism with elastic leaf springs. In the simplest configuration one can consider a bar supported by two "normal" leaf springs [16, Fig. 3-16]. By applying a thicker section in the middle part of the leaf springs, the support stiffness can be increased significantly while the stiffness in the driving direction remains almost the same. Furthermore, as will be discussed in more detail in section 4.1, this system is overconstraint unless an extra bending mode is allowed in one of the leaf springs. In the example shown in figure 1.2 [16, Fig. 3-26] both enhancements are applied.

For the analysis in this tutorial we will consider a somewhat simplified solution as shown in figure 1.3 on page 4. The bar on the top has to be positioned horizontally. The support consisting of two leaf springs should confine all other degrees of freedom. Both springs are fixed at the bottom (clamped support). The extra bending mode is possible as one spring is cut partly, see e.g. Soemers [16, Fig. 2-19] or Koster [11, Fig. i4.22]. The cylinder on the right represents a voice coil motor (VCM) that drives the motion of the system. Table 1.1 summarises the dimensions and mechanical properties of the system. It

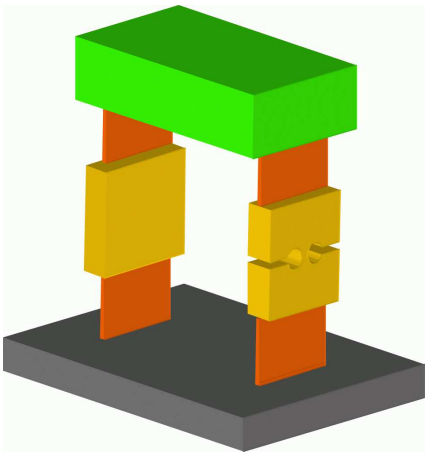


Figure 1.2: Example of a parallel guiding mechanism (Figure 3-26 from [16]).

should be noted that this simplified system has some drawbacks compared to the more advanced design of figure 1.2 as the support stiffness is reduced and the system is more prone to buckling.

This system in figure 1.3 will be analysed with an increasing degree of complexity. In the first and most simplest model only a single mass and spring will be considered. This system will be modelled in two distinct ways and the concepts of *coordinates* and *deformations* as used throughout all analyses will be introduced. Next the bending of the leaf springs will be modelled more accurately with a two-dimensional model. Finally a three-dimensional model will be presented. Even such three-dimensional models can be created easily using only a few number of elements types. As the focus remains on the modelling of only the relevant dynamics, insight is obtained into the relations between component properties and dominant system behaviour.

1.2 SPACAR

In all models the SPACAR software package is applied, which is a toolbox to be used in MATLAB. It is accompanied by a (stand-alone) Graphical User Interface (GUI) to build the models. Appendix A provides download and installation instructions.

The communication from the GUI "front-end" to the MATLAB "back-end" take place via data files, "`dat`"-files, that are readable text files in which the model is defined by means of a series of *keywords*. Experienced users may prefer to edit such `dat`-files manually and the SPACAR manual [3] provides the complete overview of the available keywords and some illustrative example systems. The GUI has its own file format, "`spa`"-files, to store its models.

In this introduction the GUI is used to create and modify the models. The text should be explanatory to redo the simulations while reading the text. In addition the data files (`spa` and `dat` files) can also be downloaded, see Appendix A.

It is essential to recognise that systems are defined in SPACAR by means of a number of *finite elements* interconnected in either two or three dimensions. Each element has *nodal points*. *Translational* and *rotational* nodal points are distinguished of which the *coordinates* describe the *position* and *orientation*, respectively, of the element with respect to a fixed global coordinate frame.

A very important property of the elements are the *deformation modes*. These deformations express the change of the "shape" of the element with respect to some undeformed initial "shape". Formally, the deformations play a role in the mathematical relation between the nodal coordinates of the element. E.g. an elongation of an element will result in a larger distance between translational nodal points. For each element a fixed number of independent (discrete) deformation modes are defined as functions of the nodal coordinates.

The (finite) elements and their properties are introduced in this tutorial when they are used in an

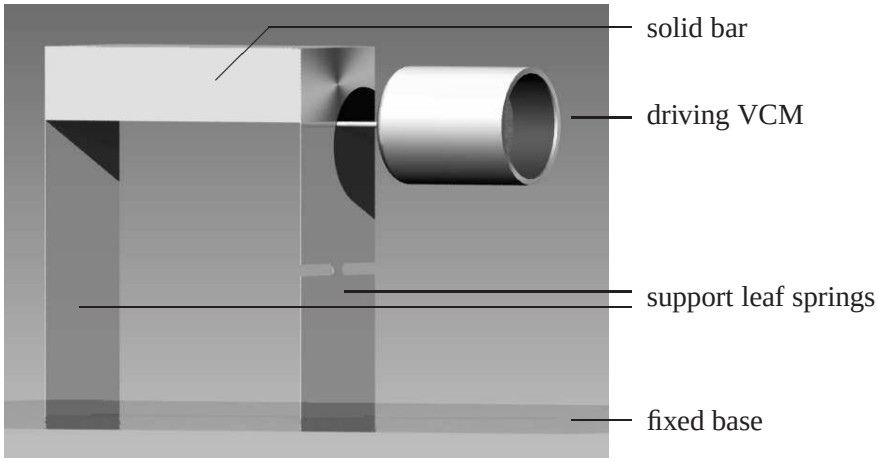


Figure 1.3: Example system driven by a voice coil motor.

Solid bar			
density	ρ_b	7690	kg/m^3
dimensions	$l_b \times w_b \times h_b$	$0.1 \times 0.02 \times 0.013$	m^3
volume	$V_b = l_b w_b h_b$	$2.6 \cdot 10^{-5}$	m^3
mass	$m_b = \rho_b V_b$	0.2	kg
Voice coil motor (VCM)			
moving mass	m_m	0.006	kg
Support leaf springs			
density	ρ_s	7850	kg/m^3
Young's Modulus	E_s	210	GPa
Shear Modulus	G_s	80	GPa
length	l_s	0.1	m
width	w_s	0.018	m
thickness	t_s	0.0005	m
cross-sectional area	$A_s = w_s t_s$	$9.0 \cdot 10^{-6}$	m^2
second moment of inertia	$I_s = \frac{1}{12} w_s t_s^3$	$1.88 \cdot 10^{-13}$	m^4
mass per unit length	$m_{l,s} = \rho_s A_s$	0.07065	kg/m

Table 1.1: Dimensions and mechanical properties of the system in figure 1.3.

example. We will introduce so-called *truss* and *beam* elements. Some more element types are briefly mentioned in appendix A.3 (page 48). For a more detailed introduction and description, the reader is referred elsewhere [3, 9].

2 Mass-spring model

In this first analysis we will model the system of figure 1.3 as a simple one degree-of-freedom mass-spring system, figure 2.1. The voice coil motor gives rise to the input force F that positions the mass in a single direction denoted by coordinate x. The stator part of the voice coil is mounted on the support column which is assumed to be rigid and clamped at the bottom. All mass is lumped in a single equivalent mass m_{eq} and the equivalent stiffness k_{eq} represents the elastic components in actuation direction.

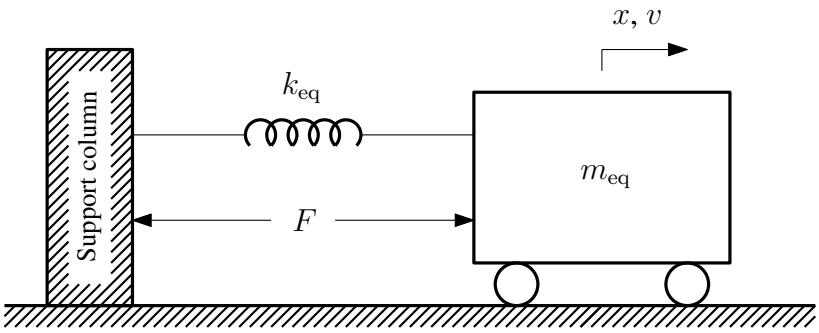


Figure 2.1: Simple 1-DOF mass-spring system.

Using straightforward analysis techniques [2] the input-output relation between the input force F and the position x of the mass can be obtained. A state vector can be defined consisting of the single degree of freedom, the position x, and its time-derivative, the velocity v. Then we can define the system input u, system output y and state vector $\boldsymbol{x}$ to be

$$u = F, \quad y = x, \quad \boldsymbol{x} = \begin{bmatrix} x \\ v \end{bmatrix}. \tag{2.1}$$

As

$$\begin{aligned} \dot{x} &= v \ , \\ \dot{v} &= \frac{1}{m} \left(F - k_{\mathrm{eq}}\, x \right) \ , \end{aligned} \tag{2.2}$$

the state space equations can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 \\ -\frac{k_{\mathrm{eq}}}{m_{\mathrm{eq}}} & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \frac{1}{m_{\mathrm{eq}}} \end{bmatrix} u \ , \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \end{bmatrix} u \ , \end{aligned} \tag{2.3}$$

or in the short hand notation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \ , \\ y &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u \ , \end{aligned} \tag{2.4}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are the system, input, output and direct feedthrough matrices, respectively.

Alternatively, the input output relation can be expressed by means of a transfer function

$$G(s) = \frac{x}{F} = \frac{1}{m_{\text{eq}}\, s^2 + k_{\text{eq}}}. \tag{2.5}$$

In either representation numerical values for the mass m_{eq} and stiffness k_{eq} need to be substituted in order to carry out a numerical analysis. The mass m_{eq} should account for all mass that is translating in the x direction, so e.g. including the moving parts of the VCM. The equivalent bending stiffness of either leaf spring equals

$$k_1 = k_2 = \frac{12 E_s I_s}{l_s^3} = \frac{E_s w_s t_s^3}{l_s^3} = 472.5 \text{ N/m}. \tag{2.6}$$

Note that the cuts in the right leave spring in figure 1.3 do not affect the stiffness in the considered bending direction.

2.1 First SPACAR model: x degree-of freedom

The mass-spring system can be modelled as a one-dimensional SPACAR model. For that purpose we need to identify elements and their coordinates and deformations as illustrated in figure 2.2. The numbers $\vec{1}$ and $\vec{2}$, so numbers with an arrow, indicate the two translational nodal points. The number in the circle ① is the single element needed in this model. To model the spring either a *truss* or a *beam* element can be used. The latter is more complicated and will be introduced later. In this section we want to model this system with a single degree-of-freedom (DOF) being the x-coordinate of the moving mass, denoted $x^{(2)}$ as the mass is in nodal point $\vec{2}$.

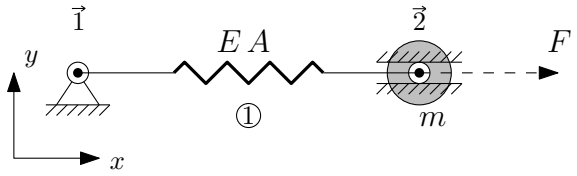


Figure 2.2: Simple 1-DOF mass-spring system with one truss element and two translational nodes.

Planar truss element

The truss element has two translational nodal points, in general denoted p and q as shown in figure 2.3. In two dimensions each translational nodal point has two coordinates, so the truss element is described with four coordinates:

$$\boldsymbol{x}_{\text{truss}}^{(k)} = \begin{bmatrix} \boldsymbol{x}^p \\ \boldsymbol{x}^q \end{bmatrix} = [x^p, y^p, x^q, y^q]^T . \tag{2.7}$$

The superscript in brackets is used to indicate the element number k. For this element a single deformation mode e_1 is defined as the elongation and is written as

$$e_1^{(k)} = l^{(k)} - l_0^{(k)}, \tag{2.8}$$

where the actual length of the element is determined by the instantaneous distance between the nodes p and q

$$l^{(k)} = \sqrt{(x^p - x^q)^2 + (y^p - y^q)^2}, \tag{2.9}$$

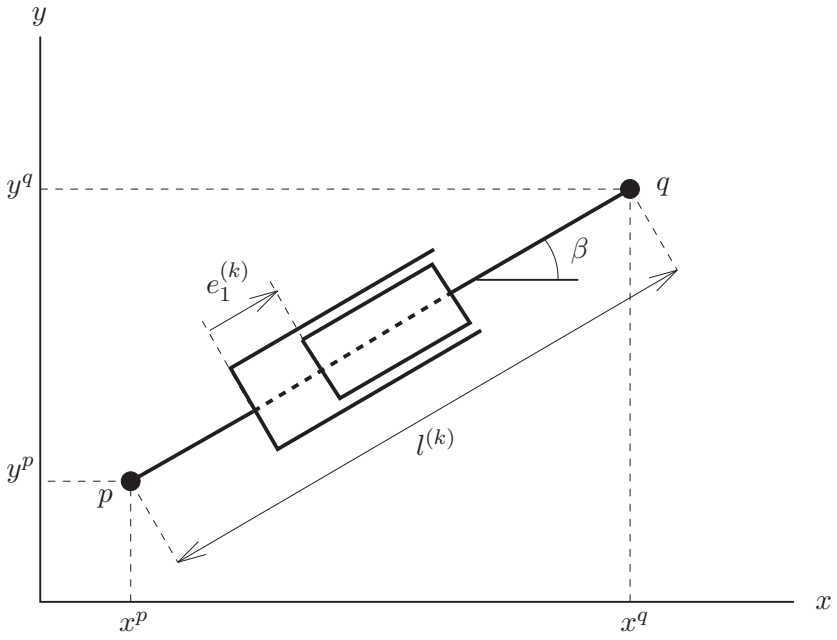


Figure 2.3: Planar slider truss element.

and the length $l_0^{(k)}$ is some reference element length. For a rigid truss the length can not change and the deformation parameter e_1 is zero. In that case equation (2.9) gives a relation between the four coordinates, so then only three coordinates can be chosen independently. This is exactly the number of degrees of freedom for the rigid truss element as it can translate in two directions and rotate in the xy-plane. It will appear for all elements that the number of deformation parameters is equal to the number of independent nodal coordinates minus the number of degrees of freedom of the rigid element.

A truss element can account for elasticity and/or damping in which case a normal force will appear that is modelled by

$$\sigma_1^{(k)} = S_1^{(k)} e_1^{(k)} + S_{d,1} \dot{e}_1^{(k)}, \tag{2.10}$$

with stiffness $S_1^{(k)}$ and damping coefficient $S_{d,1}$. For an elastic beam the stiffness equals

$$S_1^{(k)} = \frac{E^{(k)} A^{(k)}}{l_0^{(k)}}, \tag{2.11}$$

where the *axial rigidity* $E^{(k)}A^{(k)}$ includes the Young's Modulus $E^{(k)}$ and cross-sectional area $A^{(k)}$. Similarly, the damping coefficient can be written as

$$S_{d,1}^{(k)} = \frac{E_d^{(k)} A^{(k)}}{l_0^{(k)}}, \tag{2.12}$$

where coefficient $E_d^{(k)}$ describes the damping properties of the material as will be outlined in more detail following figure 2.10.

To analyse this system with SPACAR we define a new model in the GUI with `File->New`. As our first model is only 2D we select this option in the "Create new model" window and press OK. Although the SPACAR project is still empty, it can already be saved with `File->Save`. The complete project as outlined below has been saved as `pf1dtrx.spa` and can be downloaded, see Appendix A.

Creating a truss. To define a system one has to start by adding elements. To create an element its type and its nodal points must be specified. In this example a (planar) truss element is defined with

`Build->Truss`. In the “Create Truss” window the name can be changed in e.g. “truss” and the default group is kept. Next for the truss the two translational nodal points at each end must be specified. Being our first element these nodes are `new` in which case the `X` and `Y` coordinates must be given. For the system in figure 2.2 nodal point `P` is positioned in the origin and nodal point `Q` is positioned at 0.1 m on the positive x-axis, see figure 2.4.

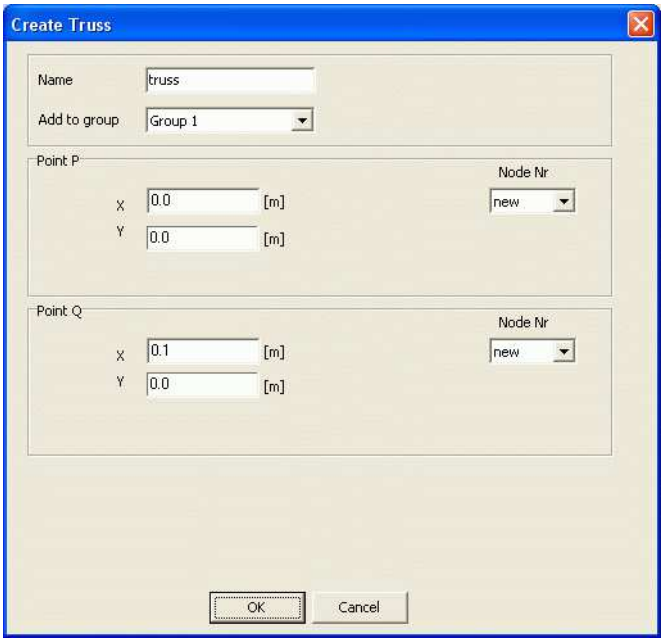


Figure 2.4: Creating a truss element.

After pressing OK, the truss is shown in the GUI with default setting for dimensions, figure 2.5. All element and nodal point properties can be modified by double clicking on the respective element or nodal point in the navigation pane.

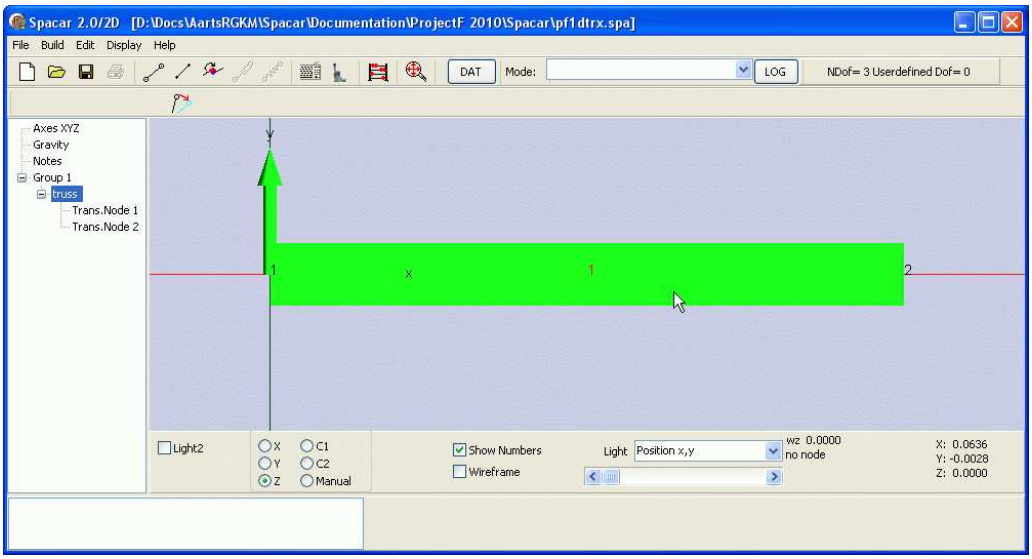


Figure 2.5: A newly created truss element in the GUI.

Counting DOF’s. Note that the GUI shows the number of degrees of freedom (DOF’s) in the system at any time: `NDof = 3  Userdefined Dof = 0`

The first expression `NDof = 3` means that the system with the single truss has three degrees of freedom at the moment. This can be understood as the newly defined element is rigid by default, i.e. all deformations are prescribed zero, and all nodal points can move freely. The planar rigid truss thus has two DOF's for its translational motion in the xy-plane and one DOF to describe its orientation. Although the number of the DOF's is well-defined, the actual selection of the quantities that represent the DOF's is not unique as will be outlined in this chapter. It is the user's responsibility to select meaningful DOF's in the system. The second expression `Userdefined Dof = 0` indicates that so far no DOF's have been defined.

Specifying constraint coordinates. Reconsidering figure 2.2 it is clear that the truss should not move freely in the xy plane. More precisely, both x and y coordinates of nodal point $\vec{1}$ and the y coordinate of nodal point $\vec{2}$ are *fixed* to the support. This can be accomplished by editing the properties of the nodal points, i.e. double clicking on the nodal point in the list shown in the navigation pane. For nodal point $\vec{1}$ the `Coordinate Type` of all coordinates can be changed into `Fixed`. For nodal point $\vec{2}$ the coordinates have to be treated different, i.e. clear the flag `X,Y,Z identical`, and next the `Y` coordinate is set to `Fixed` as well, see figure 2.6.

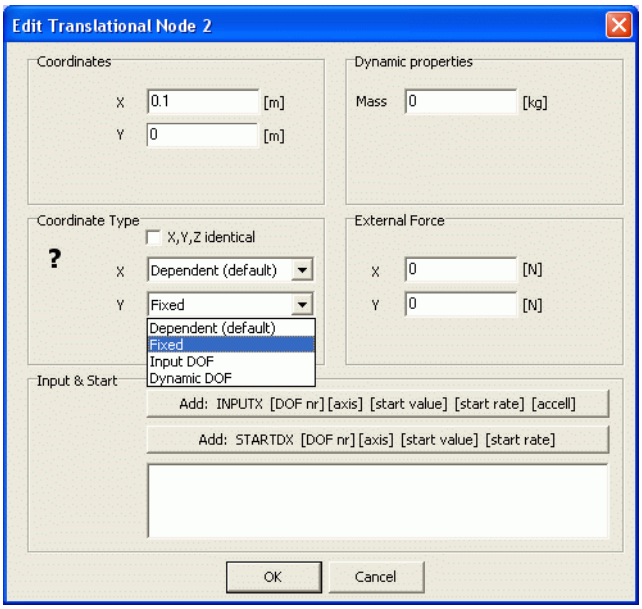


Figure 2.6: Editing a nodal point.

The overview with the number of DOF's now displays: `NDof = 0  Userdefined Dof = 0` Clearly, by fixing the coordinates this way, the truss can not move at all anymore and there are no DOF's left.

Releasing a deformation. The element deformations are prescribed zero unless defined otherwise. To allow horizontal motion (`X` coordinate) of nodal point $\vec{2}$ again, it is necessary to *release* the deformation of the truss, namely the elongation. The `Deformation type` can be changed by editing the element, i.e. double clicking the element in the navigation pane, and selecting the `Deformations` tab. After defining the `Elongation` to be `Released` as in figure 2.7 and accepting the settings (press `OK`), it can be verified that the system indeed has a DOF: `NDof = 1  Userdefined Dof = 0`

Note that although the system has indeed one DOF (`NDof = 1`), we still didn't define this degree-of-freedom explicitly.

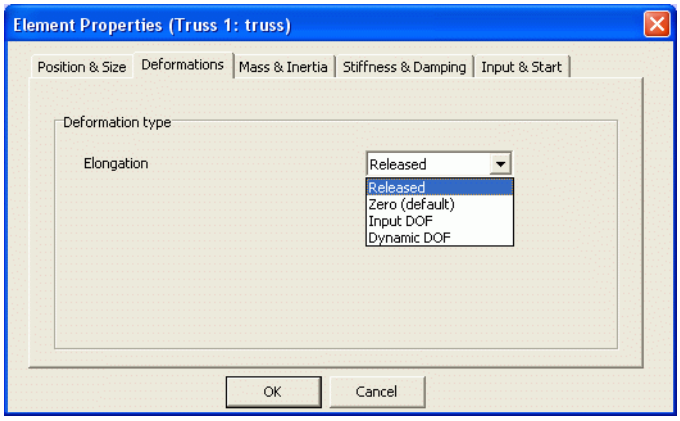


Figure 2.7: Editing the deformation type of a truss.

Defining the DOF. SPACAR computes the number of degrees of freedom NDOF in the system from

$$\texttt{NDOF} = \texttt{NX} - \texttt{NXO} - \texttt{NEO}, \tag{2.13}$$

which is the number of nodal coordinates NX minus the numbers of *absolute* and *relative constraints*, NXO and NEO respectively. Absolute constraints are defined by setting the `Coordinate Type` to `Fixed`. The relative constraints are the remaining unreleased element deformation parameters, so with the `Deformation type` equal to the (default) setting `Zero`. An overview of all coordinate and deformation types can be obtained by pressing F5, see figure 2.8.

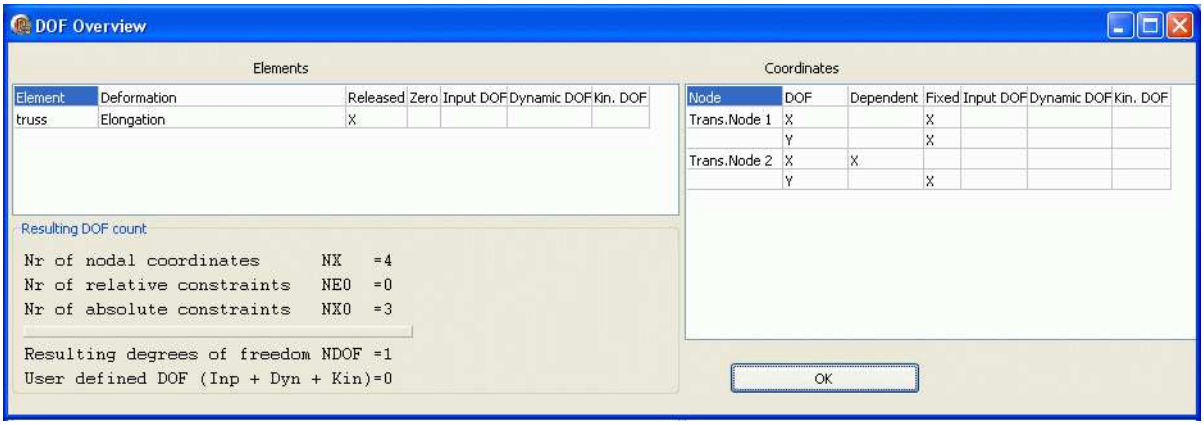


Figure 2.8: Overview of the DOF's before the correct DOF is defined.

As the overview shows, there are $\texttt{NX} = 2 \times 2 = 4$ nodal coordinates of which $\texttt{NXO} = 3$ coordinates are `Fixed`. The single truss element has only one deformation parameter which is released, so $\texttt{NEO} = 0$ relative constraints remain. Then the system has $\texttt{NDOF} = 4 - 3 - 0 = 1$ degree of freedom, which still has to be defined. In this section we will consider a so-called *absolute degree of freedom*, which is a coordinate that is defined as DOF, namely the x coordinate of nodal point $\vec{2}$. As in figure 2.6 the properties of nodal point $\vec{2}$ need to be edited once more. Now the `Coordinate Type` of the X coordinate is set to `Dynamic DOF` and then the DOF overview will show `NDof = 1  Userdefined Dof = 1`, reflecting that the user has indeed defined one DOF and the system definition is valid regarding the number of DOF's.

It must be noted that it is a *necessary* condition for a valid system that the computed number of DOF's according to equation (2.13) equals the number of user defined DOF's. Yet it is still possible to define invalid systems of which these numbers match. Consider e.g. the present system in which by mistake the Y coordinate of nodal point $\vec{2}$ is defined as `Dynamic DOF` and the X coordinate is `Fixed`.

The number of DOF's match, but the system is invalid as the Y coordinate cannot change and therefore is not a valid DOF. In a later stage of the SPACAR analysis this will result in an error.

Truss properties: Position & Size. There are a number of element properties that can be defined as is illustrated by the tabs in figure 2.7. Next we consider the `Position & Size`. For a truss several `Cross section type`s are available like `Rectangular` and `Cylinder`. The settings are used for the visualisation of the element. Selecting a `Cylinder` type, decreasing the `Radius` to 0.002 and accepting the new setting with `OK` will result in a somewhat leaner truss than the initial drawing of figure 2.5.

The `Cross section type` and dimensions of an element *can* be used to compute physical properties of the element as discussed below. However, in this example the dimensions of the truss have no physical meaning and hence there is no relation with these physical properties. Such relation will be illustrated in the example of the next chapter.

Adding mass. In SPACAR there are basically two ways to add mass (and inertia) to the system: A *lumped* mass is attached at a nodal point and a *distributed* mass is present along an element. In this example we can use both methods. The system's masses are specified in table 1.1.

Comparing the system drawing in figure 1.3 and the schematic drawing in figure 2.2 we note that the point mass in nodal point $\vec{2}$ should represent the masses of the solid bar, $m_b = 0.2$ kg, and the moving mass of the VCM, $m_m = 0.006$ kg. The total mass can be added to the `Dynamic properties` of the nodal point, i.e. the `Mass` in figure 2.6.

The mass of the leaf springs can be added to this lumped mass, but then one has to determine the *equivalent* mass of the clamped leaf springs as only a fraction of the total mass has to be taken into account. Alternatively, the distributed mass formalism can be used which accounts implicitly for this behaviour. Then we need to specify the element's mass per unit length. For one leaf spring it is $m_{l,s} = 0.07065$ kg/m, so twice this value is entered as the `Mass per Length` on the `Mass & Inertia` tab of the element properties, see figure 2.9. Note that the option to `Calculate Inertia` properties automatically is not used in this chapter.

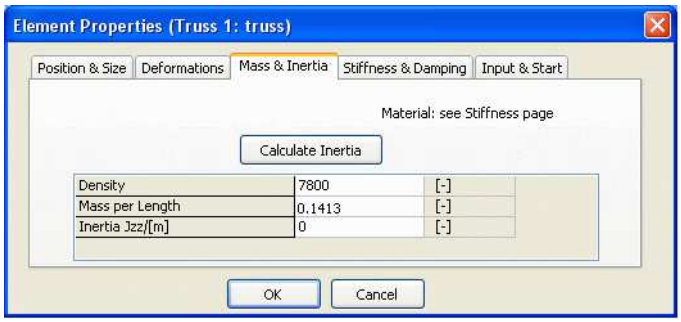


Figure 2.9: Editing the mass and inertia properties of a truss.

Adding stiffness and damping. Next we would like to specify the system's stiffness and damping properties. The truss can represent the spring damper combination as shown in figure 2.10. In our system of figure 2.2 the truss is clamped on one side and attached to a mass on the other side. Figure 2.11 shows the lumped model that will be used to analyse the stiffness and damping properties. Figure 2.12 shows the tab to edit the `Stiffness & Damping` properties of the truss.

The stiffness is specified as the axial rigidity `EA` of the truss element that represents the spring, where E is the Young's modulus and A the cross-sectional area of the truss. In the case the elasticity is described by an equivalent longitudinal stiffness k_{eq} we recognise that

$$(EA)_{\text{eq}} = k_{\text{eq}} l_0 = 94.5 \text{ N},$$

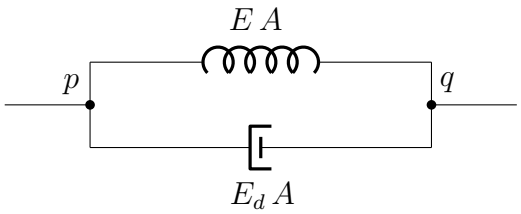


Figure 2.10: Spring damper combination represented by a truss.

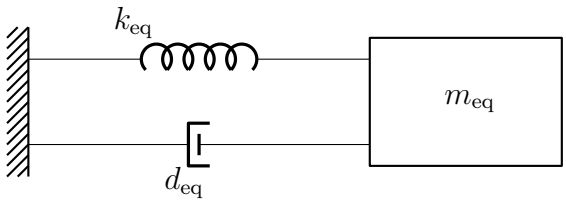


Figure 2.11: Lumped model for internal damping.

where l_0 is the reference length of the truss element and, as in this example $l_0 = 0.1$ m, the equivalent stiffness equals

$$k_{\text{eq}} = k_1 + k_2 = 945 \text{ N/m},$$

making use of the equivalent bending stiffness of both leaf springs described by equation (2.6).

Similarly the `Longitudinal damping/[m]` specifies the element's internal viscous damping. The damping of the spring leaves is not exactly known. For metals a typical *relative damping* ζ in the range of 0.01 to 0.001 is expected. Then each spring leave can be modelled as the mass-spring-damper system of figure 2.11. For this system the (equivalent) damping d_{eq} is related to the relative damping ζ, the (equivalent) stiffness k_{eq} and (equivalent) mass m_{eq} as

$$d_{\text{eq}} = 2\zeta\sqrt{k_{\text{eq}} m_{\text{eq}}}\ . \tag{2.14}$$

In this tutorial the relative damping is taken: $\zeta = 0.005$. For the truss, an equivalent damping for the two leaf springs can be calculated

$$d_{eq} = 2 \cdot 0.005 \cdot \sqrt{945 \cdot 0.01413} = 0.0365 \text{ Ns/m}. \tag{2.15}$$

Multiplication with reference length l_0 gives the value of the damping to be entered, namely

$$(E_d A)_{\text{eq}} = d_{\text{eq}} l_0 = 0.00365 \text{ Ns}.$$

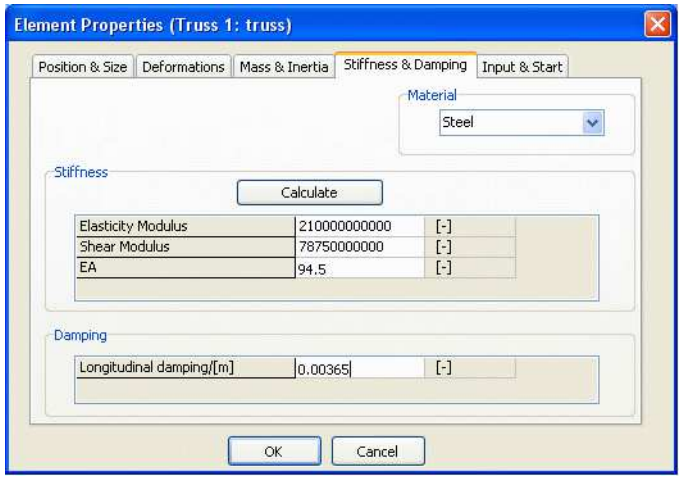


Figure 2.12: Editing the stiffness and damping properties of a truss.

Adding an external force. The effect of an external force on the system can be investigated by applying a force in a nodal point. The (constant) size of the force vector can be specified by editing the nodal point. Figure 2.13 shows an update of figure 2.6 in which the mass has been entered as outlined previously and furthermore a horizontal force with a magnitude of 1 N is applied at node 2 in the positive x-direction.

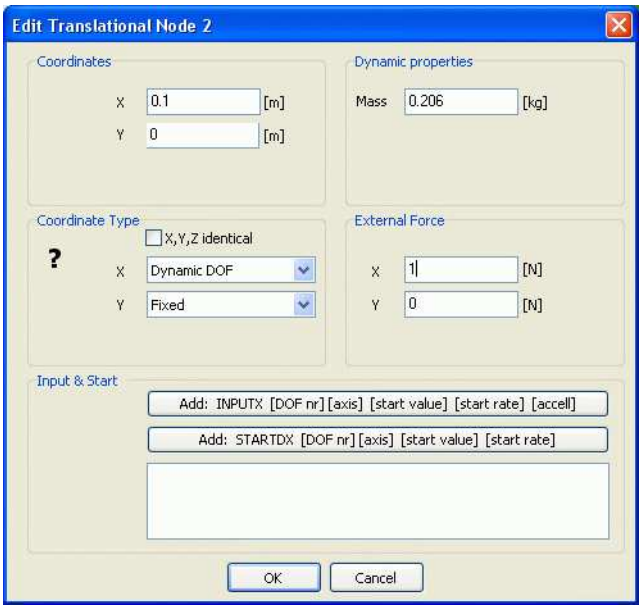


Figure 2.13: Editing a nodal point with mass and an external force.

Specification of input and output. For some of the SPACAR analyses one has to explicitly define the system's inputs and outputs. By clicking on the `Edit additional Dynamic commands` button a screen appears in which these inputs and outputs can be defined. Currently, this interface is still a bit crude as commands need to be entered manually. Figure 2.14 shows an example with in the lower right `Input Output` section the commands:

```
INPUTF  1  2 1
OUTX    1  2 1
```

The keywords `INPUTF` and `OUTX` define a single input force `F` and a single output coordinate `X`, respectively. Both keywords are followed by three integer numbers. The first number indicates the input or output number that corresponds with its position in the input vector $\boldsymbol{u}$ or output vector $\boldsymbol{y}$ respectively. The inputs and outputs of a system should be numbered consistently, i.e. no numbers should be skipped

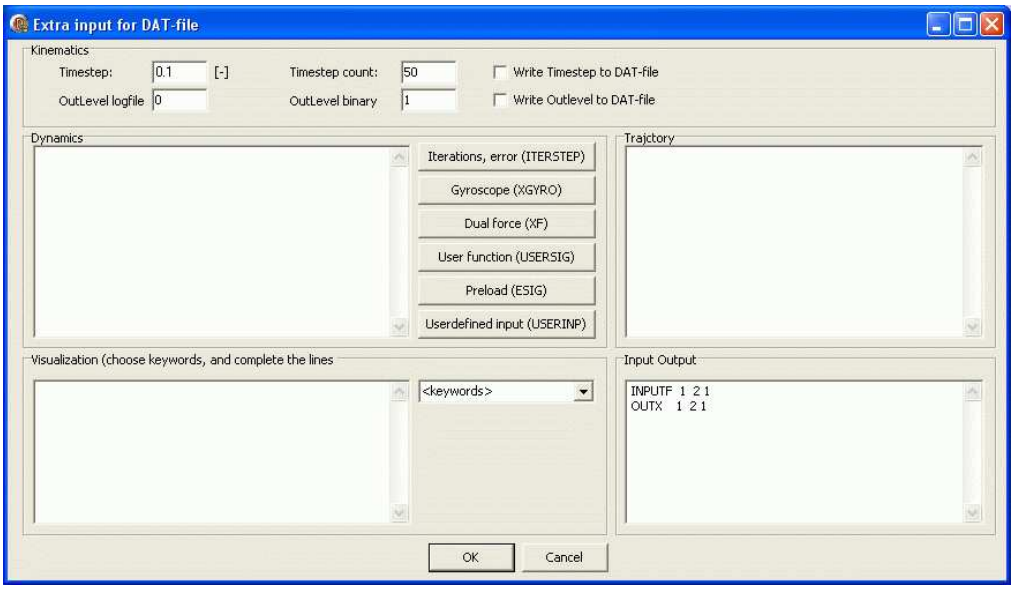


Figure 2.14: Entering extra input for the DAT-file.

or used more than once. The second and third integers indicate the nodal point number and the corresponding Cartesian nodal coordinate. So the `OUTX` keyword defines the x coordinate of nodal point `2` to be the first (and only) component of the output vector $\boldsymbol{y}$. Similarly the force on this point is the first (and only) component of the input vector $\boldsymbol{u}$. The third number `1` indicates that the component of the force in the x direction is considered. A complete overview of all input and output commands is given in the SPACAR manual [3].

SPACAR results: Mode 7 for natural frequencies and mode shapes

The goal of a typical SPACAR analysis could be a static analysis and the computation of the natural frequencies of the system. For this purpose mode `7: Natural freq. and modeshapes` is well-suited. It can be selected from the `Mode:` pull-down menu. The actual SPACAR analysis will be done in MATLAB. The software needs the description of the system in the `dat` input file in a special syntax. This file is created or overwritten by pressing the `Create the DAT file, and launch Matlab` button or simply `F9`. This will indeed update the `dat` file and launch MATLAB if the software is not already active. In the case MATLAB is launched it will be started in the correct folder with the data files. If MATLAB was already running, it will not be started again and the user has to verify that the current folder is correct. Of course the `spacar.mexw32` has to be installed and should be in the MATLAB `path`, see appendix A.

Assuming the system has been named `pf1dtrx` as mentioned on page 7, the SPACAR analysis is started by typing manually at the MATLAB command prompt

```
>> spacar(7,'pf1dtrx')
```

or simply pressing `Ctrl-V` in MATLAB should also work as this command is stored in the copy-and-paste buffer by the GUI.

The SPACAR analysis opens a MATLAB plot window that shows the system in its equilibrium configuration, where a green line indicates the truss element. The generated `log` file `pf1dtrx.log` is a plain text file in which the processing of the input file `pf1dtrx.dat` and some of the analysis results can be read. It is advisable to check this file always after a SPACAR run to look for any unexpected messages that can indicate mistakes in the input file. Data is written to the binary files `pf1dtrx.sbd` and `pf1dtrx.sbm`. Immediately after the SPACAR run, the variables stored in these files are also available in MATLAB. We can check the actual horizontal position of node 2 with the command:

```
>> x(lnp(2,1))
ans =
    0.1011
```

Location matrix `lnp(i,j)` denotes the location of the jth coordinate, or associated force component, of node i, see Appendix A. The normal force in the truss element is obtained by:

```
>> sig(le(1,1))
ans =
    1.0000
```

Location matrix `le(i,j)` denotes the location of the jth deformation, or associated stress resultant component, of element i, see Appendix A.

Furthermore, we can e.g. check the mass and stiffness “felt” by the degree of freedom that we defined:

```
>> m0
m0 =    0.2107

>> k0
k0 =    945
```

From these values the natural frequency (in Hz) follows as:

```
>> sqrt(eig(k0,m0))/2/pi
ans =
    10.6584
```

SPACAR results: Mode 9 for state space matrices

With mode `9: State space matrices` the state space matrices can be computed:

```
>> spacar(9,'pf1dtrx')
```

As a result of this run an additional data file `pf1dtrx.ltv` is written in which the state space matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are stored. These are not immediately available in MATLAB, but can be read with the `getss` command:

```
>> getss('pf1dtrx')

a =
            x1       x2
   x1        0        1
   x2    -4485  -0.1732

b =
          u1
   x1      0
   x2  4.746

c =
       x1  x2
   y1   1   0

d =
       u1
   y1   0

Continuous-time model.
```

Note that in this (unmodified) MATLAB output the variables `a`, `b`, `c` and `d` denote the system, input, output and direct feedthrough state space matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively, as in equation (2.4) with a small extra term due to the damping. Also the names for the states `x1` and `x2`, the input `u1` and output `y1` are generated by MATLAB. The input, output and direct feedthrough matrices depend on the previously defined input(s) and output(s) of course.

The state space representation can be easily transformed into a transfer function (equation (2.5)) using MATLAB's `tf` command:

```
>> G=tf(getss('pf1dtrx'))

Transfer function:
        4.746
---------------------
s^2 + 0.1732 s + 4485

>> bode(G)
```

where the latter command is used to create the Bode plot in figure 2.15.

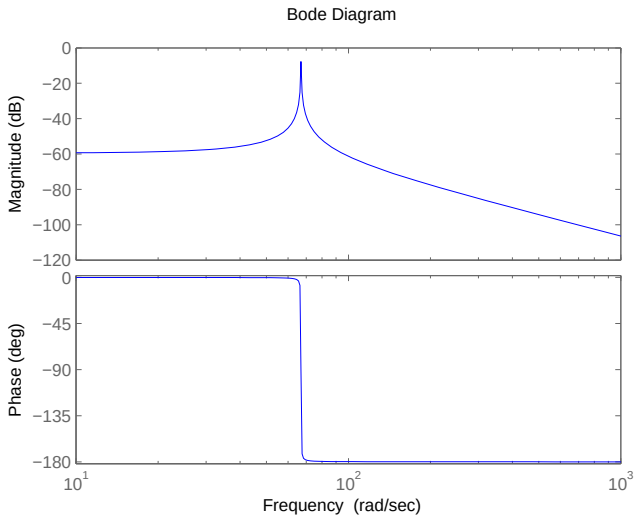


Figure 2.15: Bode plot of the 1-DOF mass-spring system.

2.2 Second SPACAR model: e degree-of freedom

As was mentioned in the previous section, the selection of the degree of freedom is not unique. Another possibility is to define the deformation parameter e_1 of the truss element, the elongation, as a *relative* degree of freedom. This can be done by changing the `Coorinate type` of the `X` coordinate of nodal point $\vec{2}$ into its default type `Dependent`. The properties of the truss element are edited to change the `Deformation type` of the `Elongation` into a `Dynamic DOF`. Figure 2.16 shows the `DOF overview` and illustrates that the number of user defined DOF's satisfy again the necessary condition of equation (2.13) for a valid system definition. The new system can be saved with a different name as `pf1dtre.spa` and it can be used for similar SPACAR analyses as in the previous section.

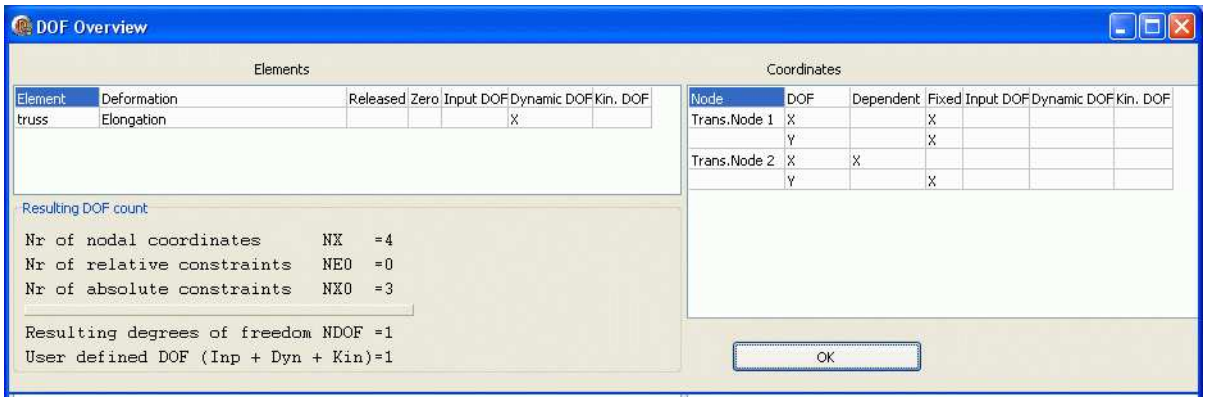


Figure 2.16: Overview of the DOF's with the elongation of the truss as DOF.

2.3 SPACAR models with more degrees of freedom

So far we only analysed the simple 1-DOF mass-spring system of figure 2.1. While still considering only motion in one dimension, the presence of additional degrees of freedom can be modelled straightforwardly. In the rest of this section, these models will be outlined shortly. Modelling and analysing these systems with SPACAR is left as exercises for the reader.

2.3.1 Flexible support

So far, the support was assumed to infinitely stiff and heavy and therefore the effect of the reaction force was negligible. However, in general the support is neither infinitely heavy, nor infinitely stiff. Then the support will exhibit a resonance that is excited by the reaction forces. The effect of such a resonance can have a significant impact on the set-point response and closed-loop stability of the system. Assuming all motions are in one direction, figure 2.17 gives a schematic overview of this system. The support with mass m_s can move in the direction x_s. Its flexibility is modelled with the stiffness k_s. Note that the VCM force is no longer described with an external force F working on the system, but instead it generates an internal stress σ.

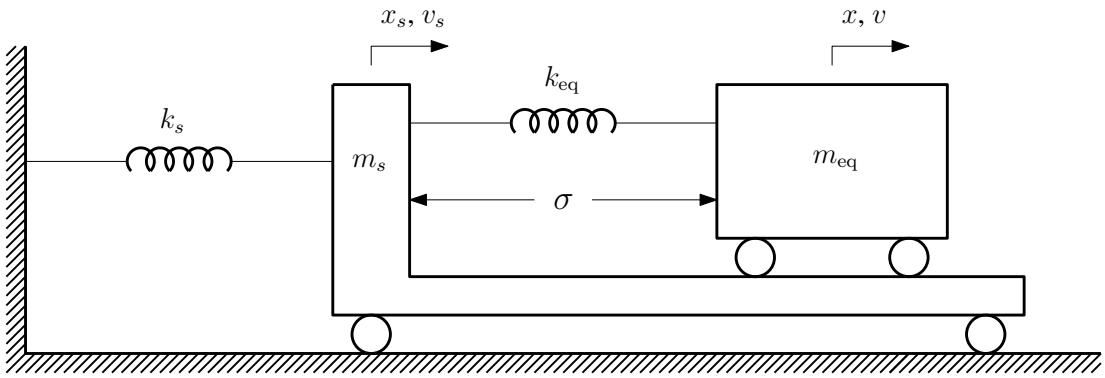


Figure 2.17: 2-DOF mass-spring system with flexible support.

2.3.2 Flexible support (2)

In the previous model the base of the VCM and the fixation of the leaf springs are connected rigidly and this combined support is mounted on a support column of which elasticity k_s is taken into account. It might as well be the case that the connection between the VCM base and leaf spring fixation is not rigid. Figure 2.18 gives a schematic overview of this system. Now the support mass only represents the support of the VCM with position x_s relative to the overall support including the fixation of the leaf springs.

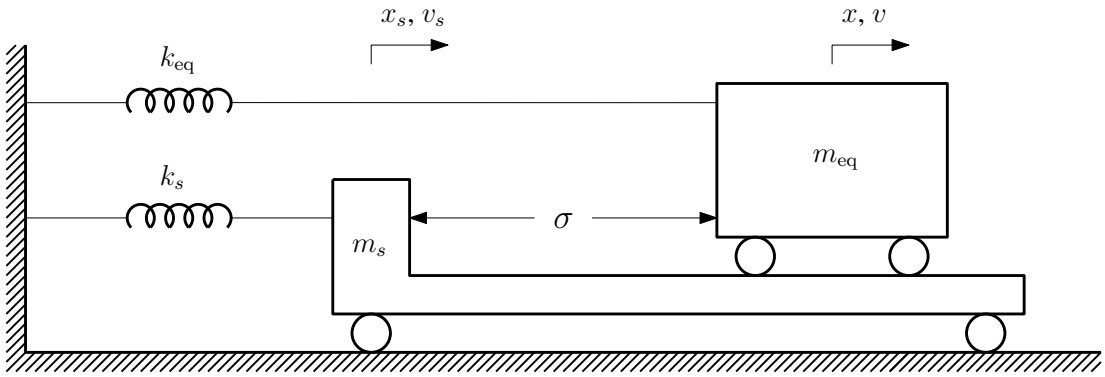


Figure 2.18: 2-DOF mass-spring system with another flexible support.

2.3.3 Internal mode

The moving mass does not have to be infinitely stiff as well. Figure 2.19 shows a system in which this mass is divided into two parts with an intermediate stiffness k_{int}.

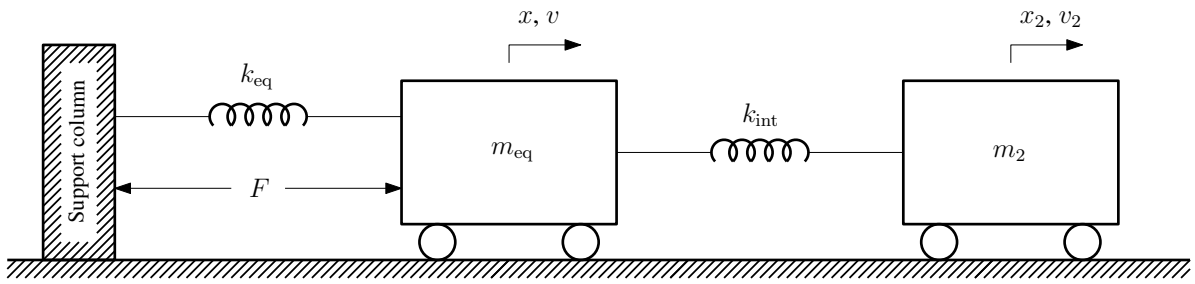


Figure 2.19: 2-DOF mass-spring system with an internal mode.

2.3.4 And more ...

The previous models can be combined into a single model with three degrees of freedom representing both a flexible support and the internal mode, see figure 2.20. In this figure the flexible support of section 2.3.1 and the internal mode of section 2.3.3. Modelling and analysing them with SPACAR is also left as an exercise for the reader,

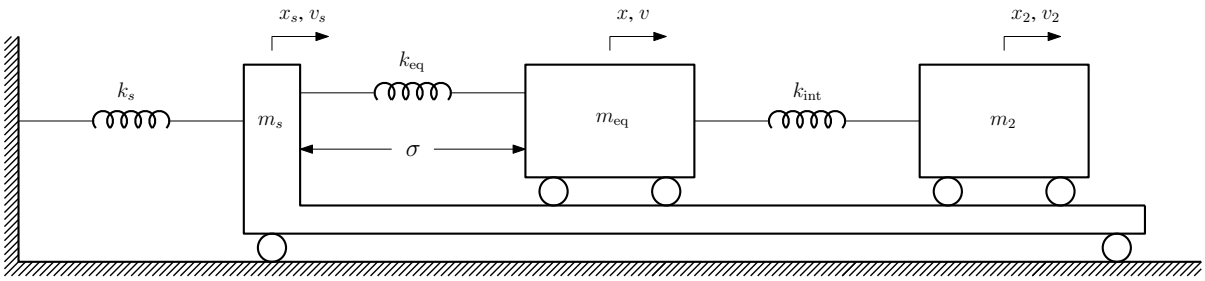


Figure 2.20: 3-DOF mass-spring system with flexible support and internal mode.

3 Two-dimensional model

Next, a more complicated model will be used in which the bending of the leaf springs will be modelled more accurately using a so-called planar beam element. Figure 3.1 shows a planar frame model of the example system of figure 1.3. It is modelled using three beam elements. The solid bar is modelled by a rigid beam element (2). Both support leaf springs are modelled with the flexible beam elements (1) and (3) as will be explained in more detail below.

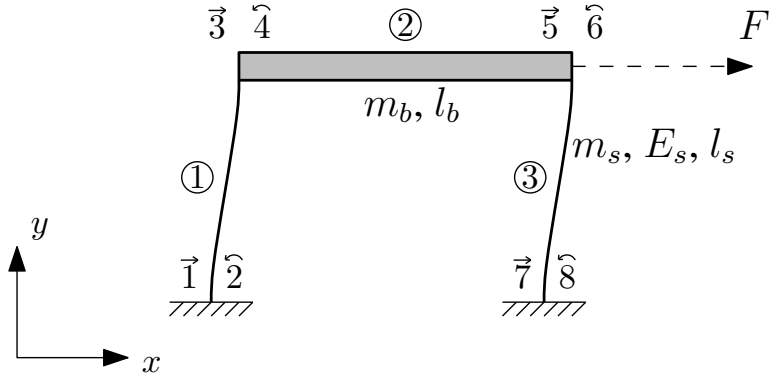


Figure 3.1: Planar frame model with elastic beams.

Like the truss element, on either side of a beam a translational node is attached. In figure 3.1 the nodes 1, 3, 5 and 7 represent these translational nodes, where nodes 3 and 5 are common nodes between two beams to indicate that these nodes are shared. In addition, the nodes 2, 4, 6 and 8 indicate *rotational* nodes. These are of importance in the description of the bending deformations of the flexible beam elements. Note that also the rotational nodes 4 and 6 are common between the two beams on either side. Sharing a translational node between two or more beams implies that these beams are connected to each other in that point, but they can still rotate relative to each other. If these beams also share a rotational node, then relative rotations are no longer possible and the beams are completely fixed.

Planar beam element

The configuration of the beam element is described by the position vectors $\boldsymbol{x}^p$ and $\boldsymbol{x}^q$ of the end nodes p and q and the angular orientation of orthonormal triads rigidly attached to the element nodes, as shown in figure 3.2. The rotation part of the motion of the beam element is described by the rotation of the triads which are described by planar rotation matrices $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$, defined by

$$\boldsymbol{R}^p \equiv \begin{bmatrix} \cos\phi^p & -\sin\phi^p \\ \sin\phi^p & \cos\phi^p \end{bmatrix} \quad \text{and} \quad \boldsymbol{R}^q \equiv \begin{bmatrix} \cos\phi^q & -\sin\phi^q \\ \sin\phi^q & \cos\phi^q \end{bmatrix} . \tag{3.1}$$

As nodal coordinates for the beam element we have four Cartesian coordinates (x^p, y^p), (x^q, y^q) describing the position of the beam in the (x, y)-coordinate system and two rotation angles ϕ^p and ϕ^q representing the angular orientation of the triads $(\boldsymbol{R}^p\boldsymbol{n}_x, \boldsymbol{R}^p\boldsymbol{n}_y)$ and $(\boldsymbol{R}^q\boldsymbol{n}_x, \boldsymbol{R}^q\boldsymbol{n}_y)$ at the nodes p

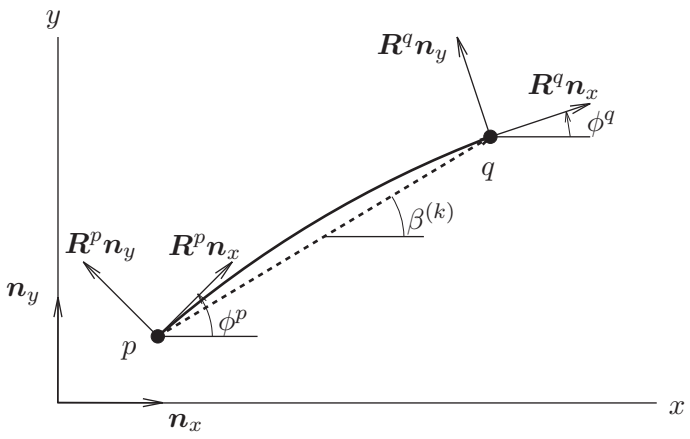


Figure 3.2: Planar flexible beam element.

and q respectively. Hence the vector of nodal coordinates is given by

$$\boldsymbol{x}^{(k)}_{\text{beam}} = \begin{bmatrix} \boldsymbol{x}^p \\ \phi^p \\ \boldsymbol{x}^q \\ \phi^q \end{bmatrix} = [x^p, y^p, \phi^p, x^q, y^q, \phi^q]^T \,, \tag{3.2}$$

where $\boldsymbol{x}^p, \boldsymbol{x}^q$ and ϕ^p, ϕ^q are the translational and rotational nodal points respectively. In the undeflected state the vectors $\boldsymbol{R}^p\boldsymbol{n}_x$ and $\boldsymbol{R}^q\boldsymbol{n}_x$ are directed along the beam axis pq and the vectors $\boldsymbol{R}^p\boldsymbol{n}_y$ and $\boldsymbol{R}^q\boldsymbol{n}_y$ are perpendicular to the beam axis.

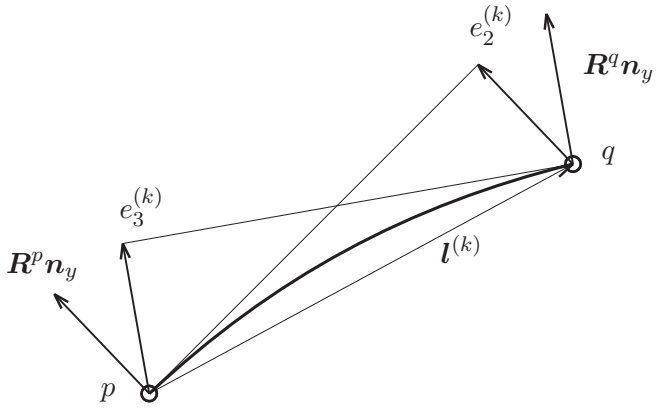


Figure 3.3: Visualisation of bending deformations $e_2^{(k)}$ and $e_3^{(k)}$ of the planar beam element.

In addition to the deformation $e_1^{(k)}$ representing the elongation of the element, two deformation parameters $e_2^{(k)}$ and $e_3^{(k)}$, associated with the flexible deformation of the beam element, can be defined:

elongation: $$e_1^{(k)} = l^{(k)} - l_0^{(k)}, \tag{3.3}$$

bending: $$e_2^{(k)} = -(\boldsymbol{R}^p\boldsymbol{n}_y, \boldsymbol{l}^{(k)}), \tag{3.4}$$

$$e_3^{(k)} = \quad (\boldsymbol{R}^q\boldsymbol{n}_y, \boldsymbol{l}^{(k)}), \tag{3.5}$$

where the notation $(...,...)$ denotes a scalar vector product, vector $\boldsymbol{n}_y = [\,0\,,\,1\,]^T$ and the vector $\boldsymbol{l}^{(k)}$ is defined by

$$\boldsymbol{l}^{(k)} \equiv \boldsymbol{x}^q - \boldsymbol{x}^p = [(x^q - x^p), (y^q - y^p)]^T \quad . \tag{3.6}$$

The geometric meaning of the bending deformations $e_2^{(k)}$ and $e_3^{(k)}$ is visualised in figure 3.3.

Note that the deformations are independent for rigid body movements of the element. Furthermore, they have a clear physical meaning. If the deformations remain sufficiently small, then in the elastic range they are linearly related to known beam quantities as normal force ($\sigma_1^{(k)}$) and bending moments ($\sigma_2^{(k)} l^{(k)}$, $\sigma_3^{(k)} l^{(k)}$) by the element stiffness equations:

$$\begin{bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \sigma_3^{(k)} \end{bmatrix} = \begin{bmatrix} \frac{E^{(k)}A^{(k)}}{l^{(k)}} & 0 & 0 \\ 0 & 4\frac{E^{(k)}I^{(k)}}{(l^{(k)})^3} & -2\frac{E^{(k)}I^{(k)}}{(l^{(k)})^3} \\ 0 & -2\frac{E^{(k)}I^{(k)}}{(l^{(k)})^3} & 4\frac{E^{(k)}I^{(k)}}{(l^{(k)})^3} \end{bmatrix} \begin{bmatrix} e_1^{(k)} \\ e_2^{(k)} \\ e_3^{(k)} \end{bmatrix}, \tag{3.7}$$

where $E^{(k)}$ is the modulus of elasticity, $A^{(k)}$ the cross section area and $I^{(k)}$ the second moment of inertia. In figure 3.4 the stress resultant components are visualised.

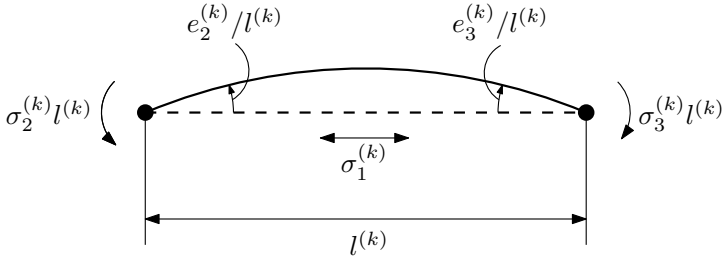


Figure 3.4: Normal force and bending moments of the planar beam element.

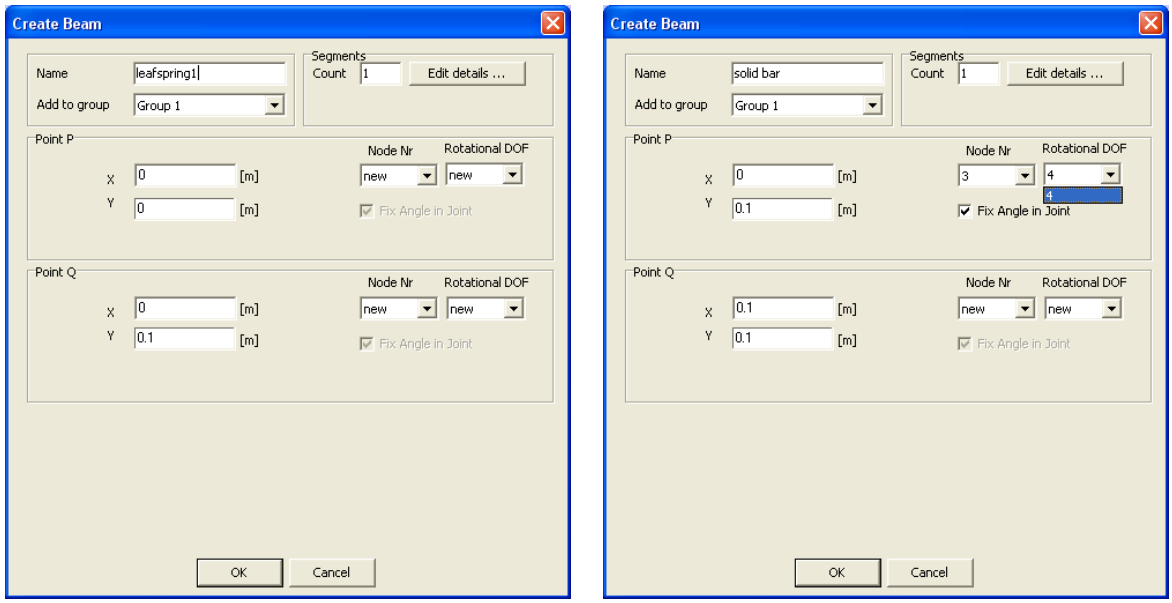
3.1 One degree of freedom model

In this section a two-dimensional SPACAR model will be outlined for the planar frame model with elastic beams in figure 3.1. This model `pf2d.spa` is also available for download.

Starting with three beams. In a new 2D model the first beam is created with the first nodal point `P` in the origin and the second nodal point `Q` in $(0, 0.1)$, see figure 3.5(a). Being the first beam al `Node Nr`'s and `Rotational DOF`'s are set to `New`. In this model the dimensions of the elements do have a physical meaning, so the newly created beam element is `Edit`ed and the height and width of the leaf spring are specified in agreement with table 1.1, see figure 3.6.

Next the second beam is added to model the solid bar, see figure 3.5(b). The first nodal point `P` of this beam is connected to the end point `Q` of the first beam. This is accomplished by selecting the `Node Nr` 3 in the pull-down menu. Note that the pull-down list only show the existing *translational* nodes and the option `New`. The beams should be rigidly connected, so also the rotational node needs to be shared. In the pull-down list for the `Rotational DOF` number 4 can be selected. In this list only rotational nodes defined so far in combination with the selected translational node 3 can be selected or the option `New`. The end point `Q` of this beam is `New` and positioned in $(0.1, 0.1)$ to assure the correct length of the bar. Also for this beam the height and width of the solid bar from table 1.1 can be specified by editing the element properties. Note that SPACAR offers also a *rigid beam* that could be used to model the rigid solid bar. We will not use this element type in this tutorial.

Similar the third beam is added where point `P` is connected to `Node Nr` 5 and `Rotational DOF` 6. Point `Q` is `New` and in $(0.1, 0)$. The height and width are identical to the other leaf spring. Figure 3.8 (page 24) shows the model in the GUI.



(a) (b)

Figure 3.5: Creating the first and second planar beam elements.

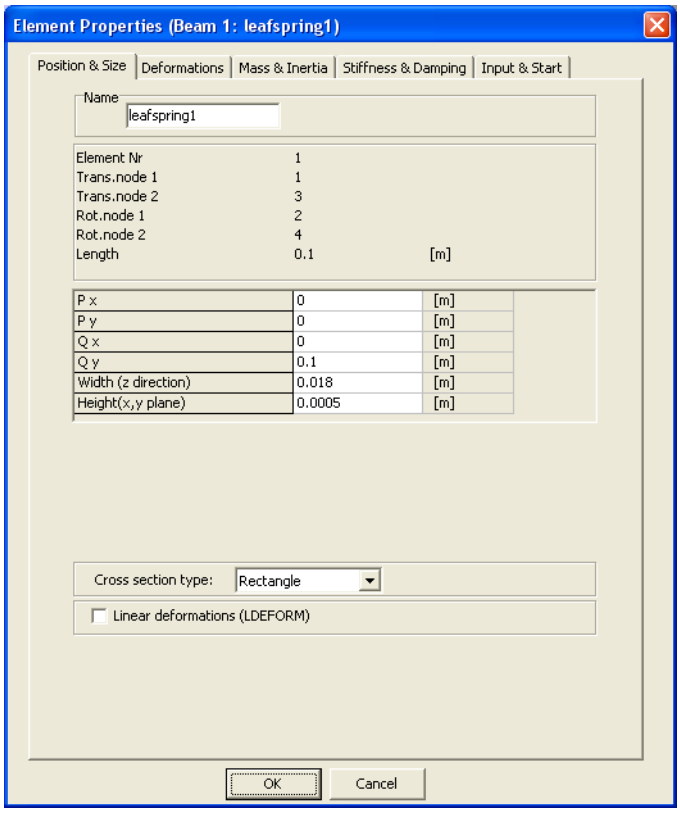


Figure 3.6: Editing the dimensions of a planar beam element.

Counting the DOF's, adding constraints and releasing deformations. So far we have created a system with three rigid beams that are rigidly attached to each other and can move and rotate freely. Hence the DOF counter displays `NDof = 3  Userdefined Dof = 0`.

First we add the constraints to the leaf springs. To clamp the beams rigidly on one end, both the translational and the rotational nodes need to be fixed. For the first beam this implies that in node 1 both `X` and `Y` coordinates are `Fixed` and in node 2 the `Coordinate type` of the rotation is also set to `Fixed`. This procedure is repeated for the third beam with nodes 7 and 8, respectively. After defining these six constraints the DOF counter displays `NDof = -3  Userdefined Dof = 0`, which clearly shows that the system is now heavily overconstraint.

Clearly, deformations need to be released. Both leaf springs have to bend while elongation of beams is assumed to be negligible at first. So for both beams representing the leaf springs the `Deformations` are edited and the `Deformation type`s of `Bending e2` and `Bending e3` are `Released`. With these four released deformations the DOF counter displays `NDof = 1  Userdefined Dof = 0`, so the correct number of DOF's is obtained.

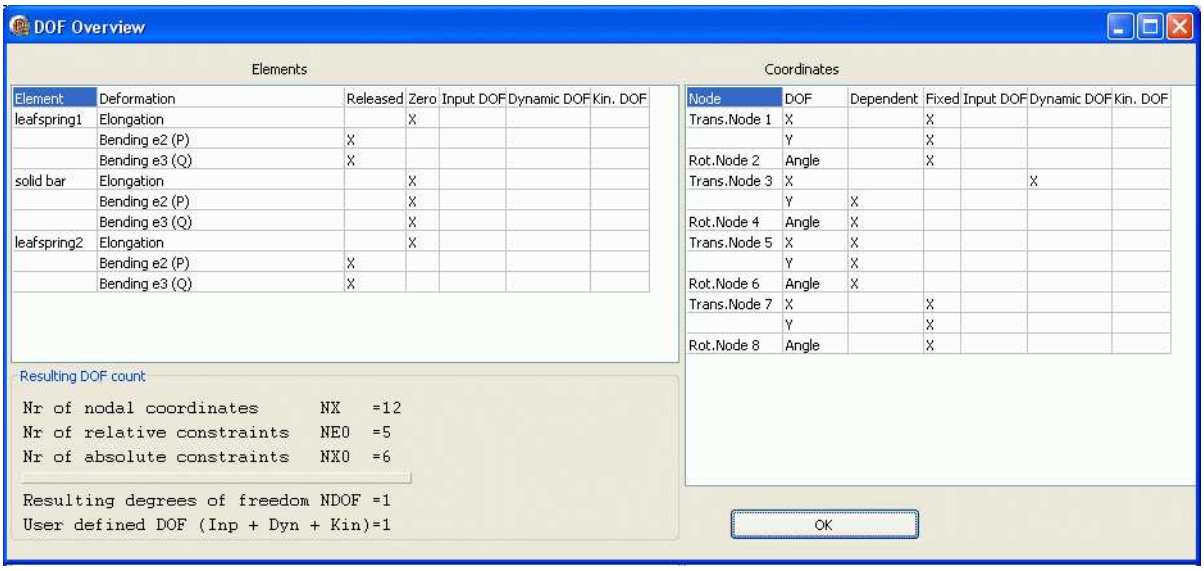


Figure 3.7: Overview of the DOF's of the planar frame model with elastic beams.

The desired DOF still has to be defined by the user. E.g. the `X` coordinate of a nodal point on either side of the solid bar reflects the horizontal motion of the solid bar and is a correct choice. Figure 3.7 shows the `DOF overview` in which the `X` coordinate of nodal point 3 is taken as DOF. Figure 3.8 shows the model in the GUI with the DOF counter indicating that the user defined DOF's satisfy the necessary condition of equation (2.13) for a valid system definition.

Specifying the mass and inertia. For the leaf springs (1) and (3) the mass can be defined as before by manually specifying the mass per unit length $m_{l,s}$ from table 1.1 for each of the leaf springs, compare with figure 2.9 except now for each leaf spring apart. Instead we can use the automatic `Calculate Inertia` button to do the job. Provided the beam dimensions and density are correct, clicking on this button will also show the correct value for both leaf springs.

Similarly the mass (and inertia) of the solid bar (2) can be computed as well and the moving mass of the VCM m_m is added manually. In the text below we will show a (more complicated) alternative approach in which the *distributed* inertia of the solid bar is modelled by means of a lumped mass representation. In this idealisation rigid bodies with equivalent mass and rotational inertia properties are attached to the end nodes of the element. Three conditions have to be satisfied to obtain the equivalent masses and rotational inertias [14]:

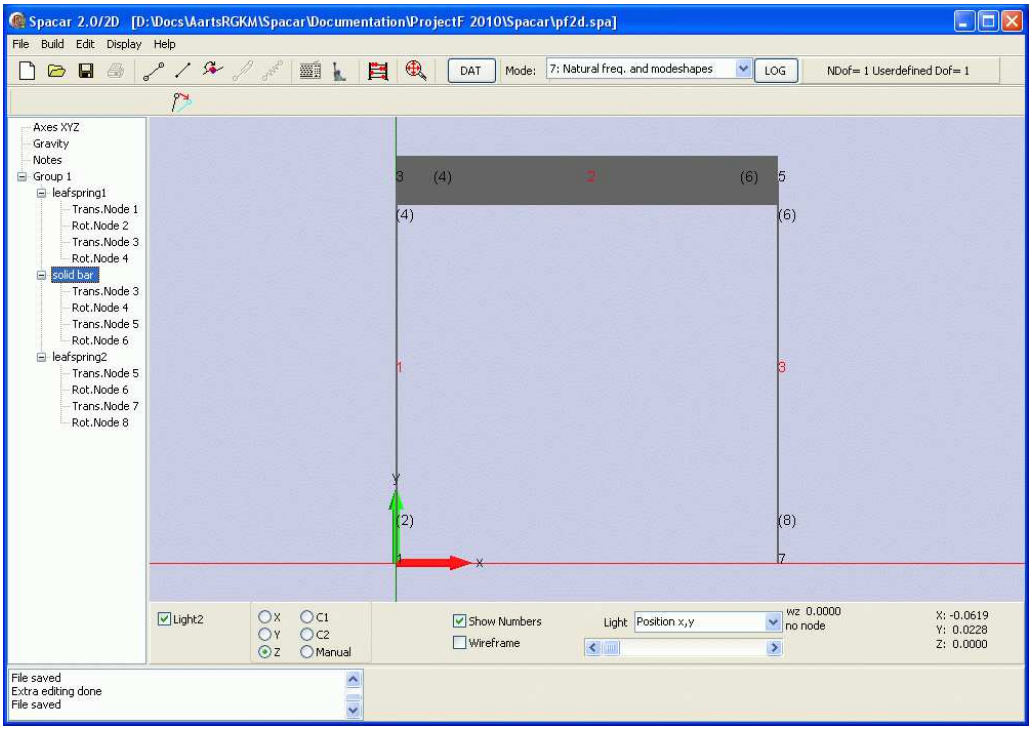


Figure 3.8: Planar frame model with elastic beams in the GUI in which the DOF's have been defined correctly.

1. The mass of the element should be equal to the sum of the lumped masses.
2. The center of mass of the element and that of the discrete mass model should coincide.
3. The rotational inertia of the element and that of the lumped system should be equal.

For the solid bar in the left graph of figure 3.9 with mass m and length l the rotational inertia relative to the centre of mass c equals $J_c = \frac{1}{12}ml^2$. The equivalent lumped masses and rotational inertias are then shown in the right graph and can be written as

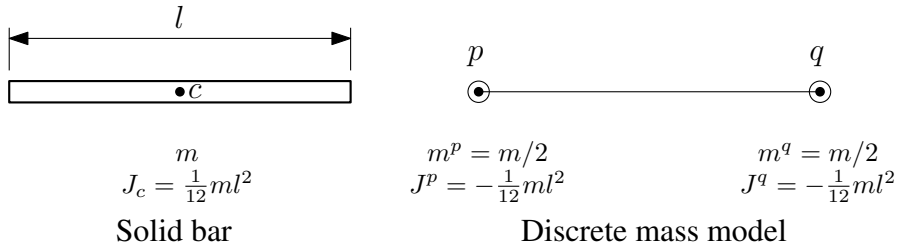


Figure 3.9: Solid bar and equivalent lumped mass system.

$$m^p = m^q = \frac{1}{2}m, \tag{3.8}$$

and

$$J^p = J^q = -\frac{1}{12}ml^2. \tag{3.9}$$

In the SPACAR input of our example system the masses $m^p = m^q = 0.103$ kg are defined in the `Translational Nodes` 3 and 5. The rotational inertias $J^p = J^q = -0.0001716$ kgm^2 are defined in the `Rotational Nodes` 4 and 6. Note that in this 1-DOF example the solid bar, element `2`, experiences a translation only and hence the rotational inertia will not affect the analysis.

Stiffness and damping. In the `Stiffness & Damping` tab of the planar beam `Element Properties` window the stiffness and damping properties can be entered. The `Calculate` button offers again a user-friendly automatic calculation of the stiffness properties. The parameter `EA` represents the axial rigidity EA, where E is the modulus of elasticity and A is the cross-sectional area of the beam. The parameter `EI` represents the flexural rigidity EI, where I is the second moment of inertia of the cross-section with respect to the neutral axis. For the leaf springs this moment of inertia equals

$$I_s = \frac{w_s t_s^3}{12} = 1.88 \cdot 10^{-13} \text{ m}^4,$$

as was used previously in equation (2.6). With a click on the `Calculate` button the numerical values are found to be $EA = 1890000$ N and $EI = 0.039375$ Nm^2.

The damping values can be calculated with equation (2.14). As was shown before, first the longitudinal stiffness EA/l_0 is computed from the axial rigidity EA. This stiffness and the total mass of the spring are substituted in equation (2.14) to obtain the damping which is related to the required parameter $E_d A = 0.3654$ Ns. The procedure for the flexural rigidity EI is similar, except that the cube of the length l_0^3 appears in the expressions, e.g. to compute the bending stiffness EI/l_0^3. The damping is then $E_d I = 5.3 \cdot 10^{-6}$ Nsm^2.

Note that in the 1-DOF example the longitudinal deformation is not released and then the specifications for the longitudinal stiffness and damping are of no account.

Specifying input and outputs. As in figure 2.14 we want to specify the system's input and output. As before the input is the horizontal VCM force applied in nodal point 5 and the desired horizontal position of nodal point 3 is an output. To extend the analysis in chapter 2 five other translational and rotational nodal coordinates are defined in the output vector as shown in figure 3.10. Both x and y coordinates of the `Translational Node`s 3 and 5 are the first four output signals. The fifth and sixth output are the rotations of the `Rotational Node`s 4 and 6.

Optional visualisation settings. The `Extra input for DAT-file` dialog can also be used for settings of the visualisation. With `VIBRATIONMODE 1` the default mode is set to the first natural frequency. `VIBREND 7.854` sets the period of the sine function for the vibration mode. With

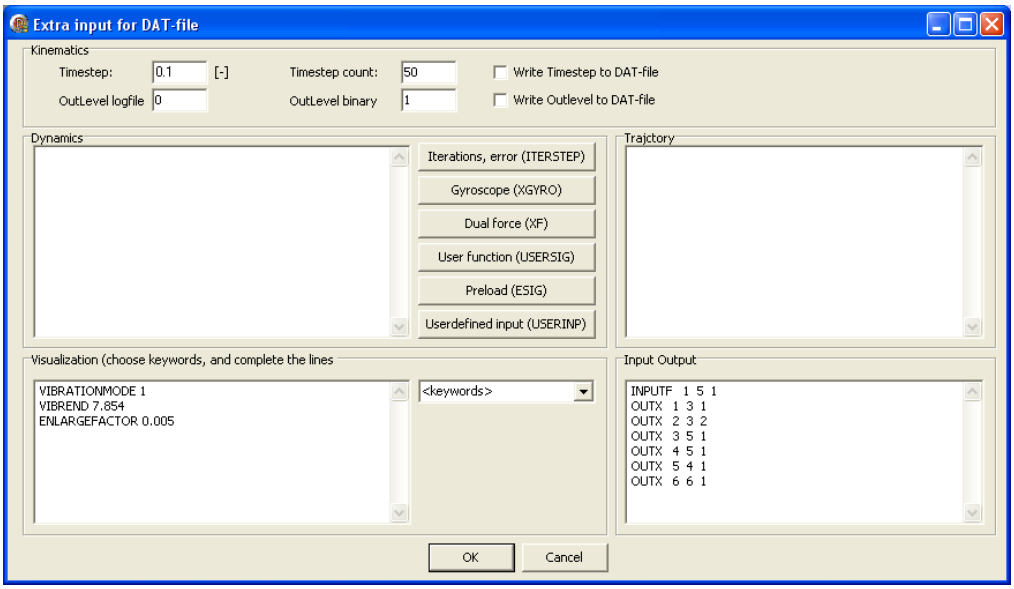


Figure 3.10: Entering extra input for the DAT-file, including visualisation settings.

`ENLARGEFACTOR 0.005` the default setting for the amplitude of the vibration mode is overruled. A more detailed explanation of these and other keywords can be found in the SPACAR manual [3].

SPACAR results: Mode 7 for natural frequencies and mode shapes

Assuming the system has been named `pf2d` as mentioned on page 21 and the `pf2d.dat` file has been generated, a mode 7 analysis can be executed to find the natural frequency of the mechanism by typing at the MATLAB command prompt:

```
>> spacar(7,'pf2d');
>> sqrt(eig(k0,m0))/2/pi

ans =
   10.6448
```

The vibration mode is visualised using SPAVISUAL, see figure 3.11.

```
>> spavisual('pf2d');
```

SPACAR results: Mode 8 for static stability (buckling)

By means of mode 8 a linear buckling analysis can be carried out for a static equilibrium configuration. A buckling load $\boldsymbol{f}_i = \boldsymbol{\lambda}_i \boldsymbol{f}_0$ is computed, where $\boldsymbol{\lambda}_i$ is a critical loading parameter and $\boldsymbol{f}_0$ represents a static reference loading vector of nodal forces.

An external force component has to be applied to the system in figure 3.1, for which a downward vertical force with a magnitude of $1\,\mathrm{N}$ in `Translational Node` 3 is taken. This is accomplished by editing this node and specifying the `External Force` similar as in figure 2.13, but

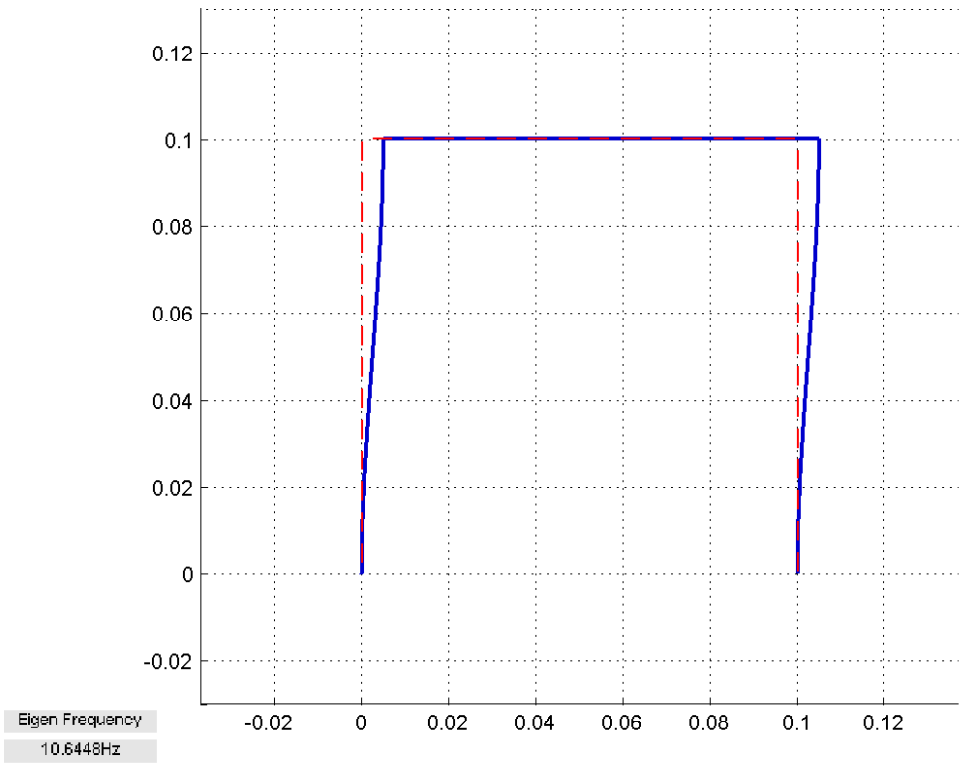


Figure 3.11: First vibration mode of the two-dimensional model (10.6448 Hz).

then of course for node 3 and with a force equal to -1 N in the `Y` direction. The `Mode` is set to "`8: Static stability (buckling)`" and the model is saved as `pf2db.spa`. The accompanying `pf2db.dat` is generated and a linear buckling analysis is executed by typing at the MATLAB command prompt

```
>> spacar(8,'pf2db')
```

Before the buckling analysis is executed, the equilibrium configuration is determined. The critical loading parameter λ_1 is computed using the command:

```
>> eig(-k0,g0)
ans =
   78.7500
```

The buckling mode is visualised with

```
>> spavisual('pf2db');
```

In figure 3.12 this mode is shown.

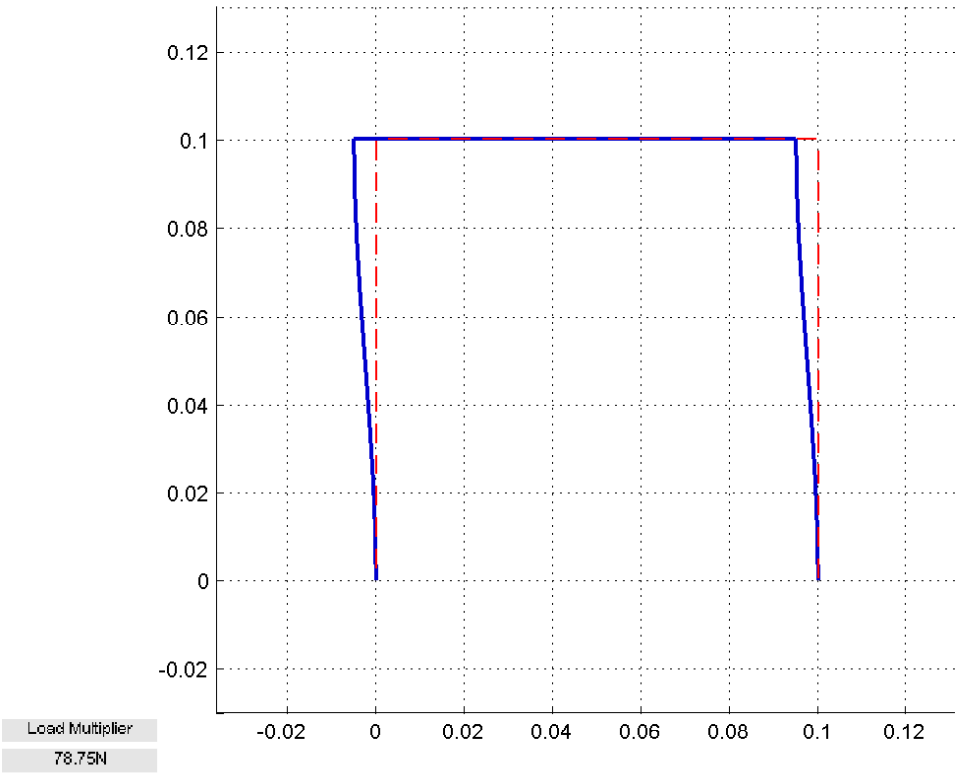


Figure 3.12: First buckling mode of the two-dimensional model ($f_1 = 78.75$ N).

SPACAR results: Mode 9 for state space matrices

Next, the state space matrices are computed (reverting to the earlier `pf3d.dat`-file without the static reference load):

```
>> spacar(9,'pf2d');
```

Again, the matrices must be read from the `ltv`-file with the `getss` command (where small numbers in the actual MATLAB output have been replaced by zeros for readability):

```
>> getss('pf2d')

a =
            x1       x2
   x1        0        1
   x2    -4473  -0.6021

b =
          u1
   x1      0
   x2  4.734

c =
       x1  x2
   y1   1   0
   y2   0   0
   y3   1   0
   y4   0   0
   y5   0   0
   y6   0   0

d =
       u1
   y1   0
   y2   0
   y3   0
   y4   0
   y5   0
   y6   0

Continuous-time model.
```

Note that the first and third rows in the output matrix C are equal. These rows are associated with the first and third output, so with the horizontal motion of nodes 3 and 5, respectively. Apparently, these nodes move identical. Note also that all other rows in the output matrix C are zero, indicating that in the initial, unloaded equilibrium configuration the linearised model shows no vertical or angular motion of the solid bar. It is left as an exercise for the reader to investigate the behaviour in different configurations.

The standard MATLAB tools can be used for further analysis of the state space model. E.g. for each input - output combination a transfer function can be generated:

```
>> tf(getss('pf2d'))

Transfer function from input to output...
              4.734
 #1:  ---------------------
      s^2 + 0.6021 s + 4473

 #2:  0

              4.734
 #3:  ---------------------
      s^2 + 0.6021 s + 4473
```

```
#4:  0

#5:  0

#6:  0
```

The Bode plot of the first transfer function can be found in figure 3.13.

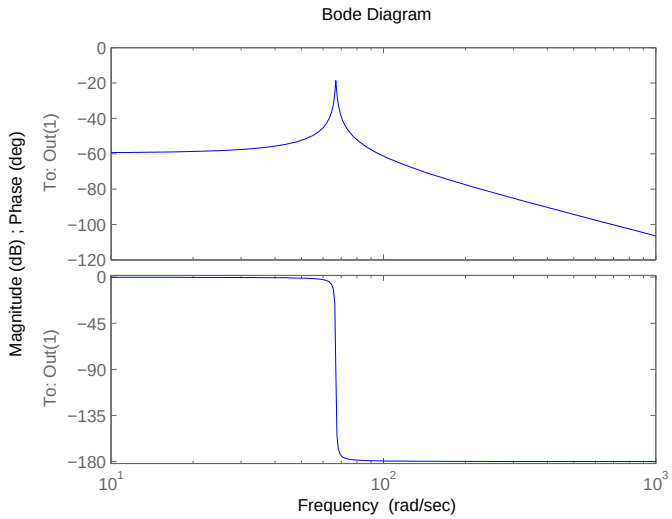


Figure 3.13: Bode plot of transfer from input to first output of the two-dimensional model.

3.2 Three degrees of freedom model

It is the responsibility of the user to select the proper degrees of freedom and release deformations. Releasing the elongations of both leaf springs as well leads to a system in which only the three deformations of the solid bar are relative constraints, so the the number of degrees of freedom according to equation (2.13) becomes

$$\texttt{NDOF} = \texttt{NX} - \texttt{NXO} - \texttt{NEO} = 12 - 6 - 3 = 3. \tag{3.10}$$

indicating that two extra degrees of freedom have to be defined. The coordinates of `Translational Node` 3 and the single coordinate of `Rotational node` 4 are chosen as dynamic degrees of freedom. After all changes are applied by editing the respective elements and nodes, the DOF counter displays `NDof = 3  Userdefined Dof = 3`, indicating that the number of defined degrees of freedom is correct. Note that the previously specified stiffness and damping for the elongation now are needed for a valid analysis of the system.

The new model is saved as `pf2d3dof.spa`, the file `pf2d3dof.dat` is generated and the analyses in MATLAB can be repeated for the more complicated 3-DOF model. The extra degrees of freedom result in mass and stiffness matrices of dimension 3×3 and a total of three natural frequencies. The matrices are available in the MATLAB workspace as row vectors. With MATLAB's `reshape` command they can be presented as (symmetric) square matrices:

```
>> reshape(k0,nddof,nddof)

  1.0e+007 *
```

```
    0.0001   -0.0000    0.0000
   -0.0000    3.7800    0.1890
    0.0000    0.1890    0.0189

>> reshape(m0,nddof,nddof)

    0.2112         0    0.0001
         0    0.2112    0.0106
    0.0001    0.0106    0.0007
```

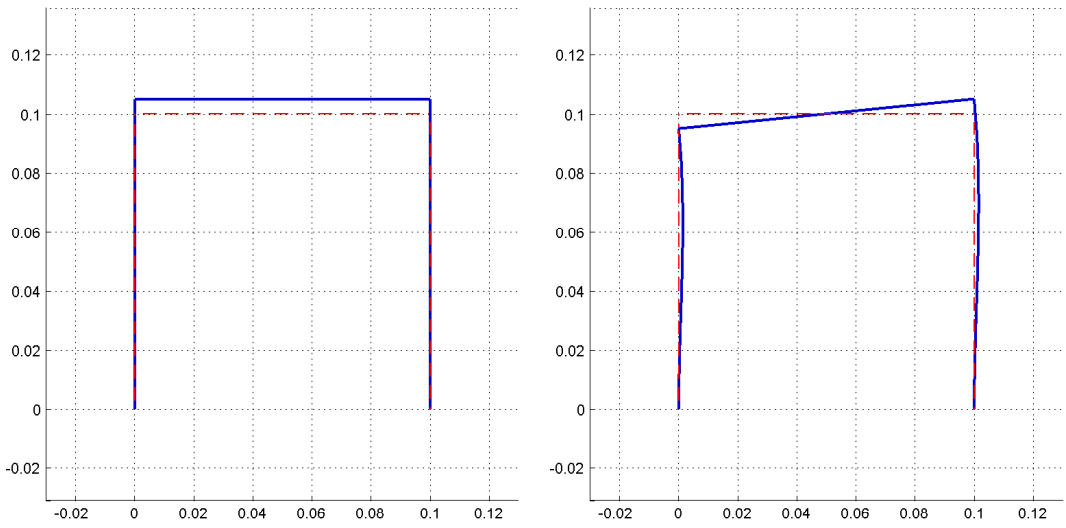
The natural frequencies are:

```
>> sort(sqrt(eig(reshape(k0,nddof,nddof),...
                 reshape(m0,nddof,nddof)))/2/pi)

  1.0e+003 *

    0.0106
    2.1290
    3.5833
```

In figure 3.14 the mode shapes of the second (2129 Hz) and third (3583 Hz) natural frequency are displayed. These natural frequencies are much higher than the first natural frequency, so in most cases the 1-DOF approximation of section 3.1 is justified.



(a) Second natural frequency (2129 Hz). (b) Third natural frequency (3583 Hz).

Figure 3.14: Mode shapes of the second and third natural frequency.

4 Three-dimensional model

Finally, a three-dimensional model will be described in which the mass-spring system will be modelled using spatial beam elements. Figure 4.1 shows a spatial frame model of the sample system of figure 1.3. It is modelled using three spatial beam elements. The element numbers and the nodal numbers correspond with the two-dimensional frame model in figure 3.1.

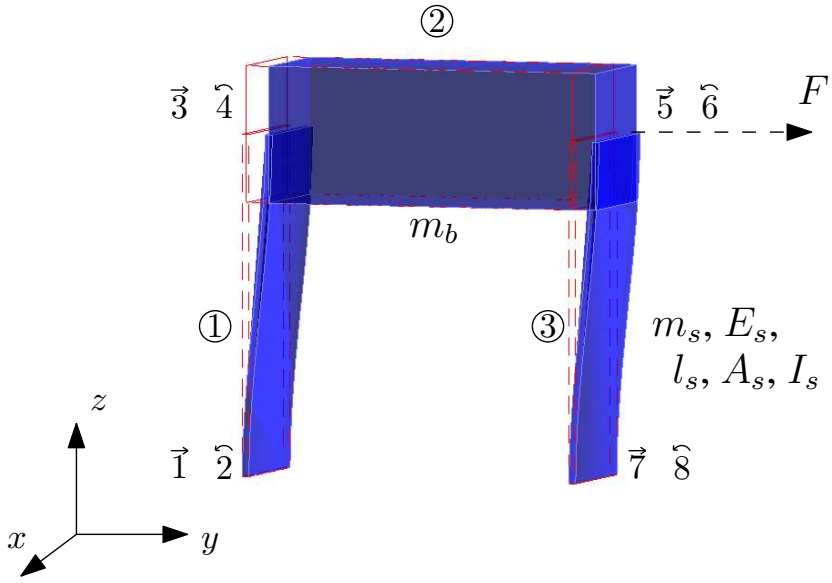


Figure 4.1: Spatial frame model with elastic beams.

Spatial beam element (BEAM)

Figure 4.2 shows a spatial beam element in an (x, y, z) inertial coordinate system. The configuration of the element is determined by the position vectors $\boldsymbol{x}^p$ and $\boldsymbol{x}^q$ of the end nodes and the angular orientation of orthogonal triads $(\boldsymbol{n}_{x'}, \boldsymbol{n}_{y'}, \boldsymbol{n}_{z'})$ rigidly attached to each end point. In the undeflected state the triads coincide with the axis pq and the principle axes of its cross section. The rotation part of the motion of the (flexible) beam is described by the rotation of the triads $(\boldsymbol{n}_{x'}, \boldsymbol{n}_{y'}, \boldsymbol{n}_{z'})$ which are determined by rotation matrices $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$. If the beam is rigid then the rotation matrices are identical and in the initial undeflected state they are equal to the identity matrix.

The nodal coordinates for the spatial beam element consist of translational and rotational coordinates. The six translational coordinates are from two position vectors $\boldsymbol{x}^p$ and $\boldsymbol{x}^q$ describing the position of the undeflected beam in the fixed inertial coordinate system. There are also six independent rotational coordinates as the orientation of each rotational node in three dimensions is given by three independent rotation coordinates collected in the vectors $\boldsymbol{\lambda}^p$ and $\boldsymbol{\lambda}^q$ respectively. These coordinates describe the angular orientation of the triads $(\boldsymbol{n}_{x'}, \boldsymbol{n}_{y'}, \boldsymbol{n}_{z'})$ at the nodes p and q. Hence the spatial beam has a total of twelve nodal coordinates. As a rigid body the spatial beam has six degrees of freedom, so we can define $12 - 6 = 6$ independent deformation parameters. In addition to the deformation mode coordinate $e_1^{(k)}$

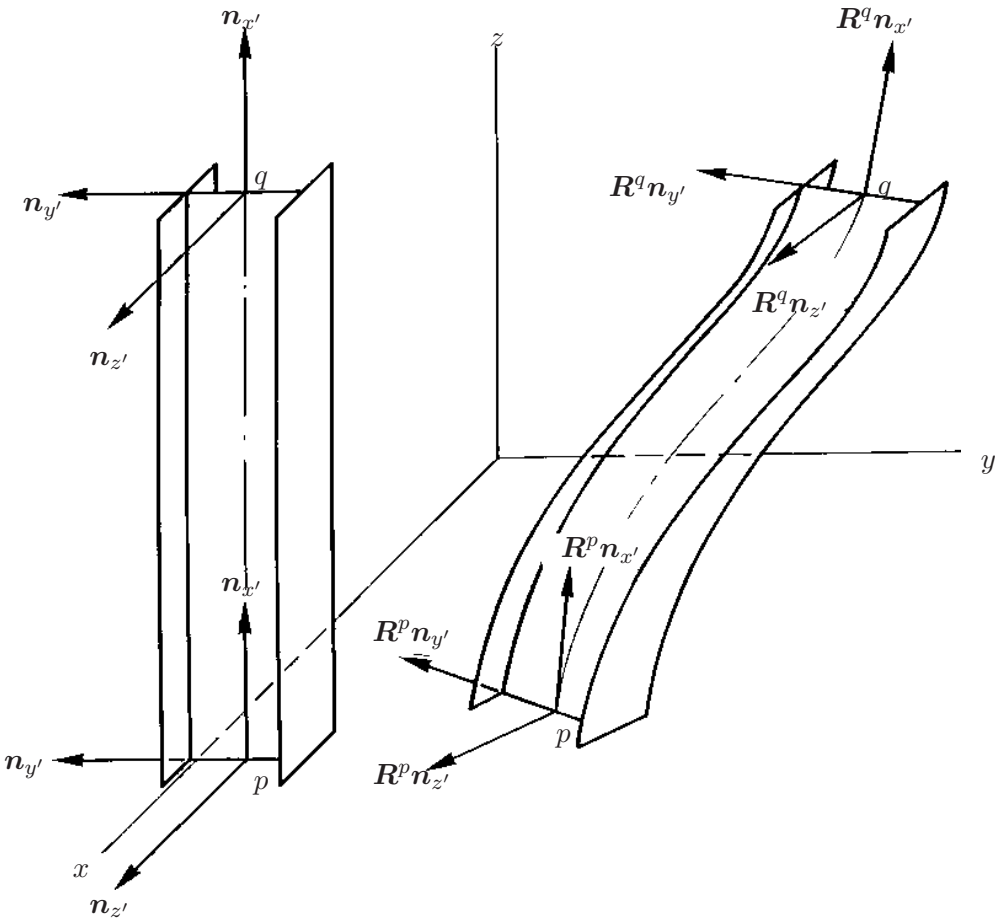


Figure 4.2: Beam element, initial and deformed state.

representing the elongation, five deformation parameters can be defined for the beam element in terms of the relative rotations at the end nodes p and q:

elongation:
$$e_1^{(k)} = l^{(k)} - l_0^{(k)}, \tag{4.1}$$

torsion:
$$e_2^{(k)} = \frac{1}{2} l_0^{(k)} \left[(\boldsymbol{R}^p \boldsymbol{n}_{z'}, \boldsymbol{R}^q \boldsymbol{n}_{y'}) - (\boldsymbol{R}^p \boldsymbol{n}_{y'}, \boldsymbol{R}^q \boldsymbol{n}_{z'}) \right], \tag{4.2}$$

bending:
$$e_3^{(k)} = -(\boldsymbol{R}^p \boldsymbol{n}_{z'}, \boldsymbol{l}^{(k)}), \tag{4.3}$$

$$e_4^{(k)} = \quad (\boldsymbol{R}^q \boldsymbol{n}_{z'}, \boldsymbol{l}^{(k)}), \tag{4.4}$$

$$e_5^{(k)} = \quad (\boldsymbol{R}^p \boldsymbol{n}_{y'}, \boldsymbol{l}^{(k)}), \tag{4.5}$$

$$e_6^{(k)} = -(\boldsymbol{R}^q \boldsymbol{n}_{y'}, \boldsymbol{l}^{(k)}), \tag{4.6}$$

where the vector $\boldsymbol{l}^{(k)}$ is defined as

$$\boldsymbol{l}^{(k)} = \boldsymbol{x}^q - \boldsymbol{x}^p = [\, x^q - x^p \,,\, y^q - y^p \,,\, z^q - z^p \,]\,. \tag{4.7}$$

The geometric meaning of the bending deformations $e_3^{(k)}$, $e_4^{(k)}$, $e_5^{(k)}$, and $e_6^{(k)}$ is visualised in figure 4.3. Note that the deformations are invariant with respect to rigid body movements of the element.

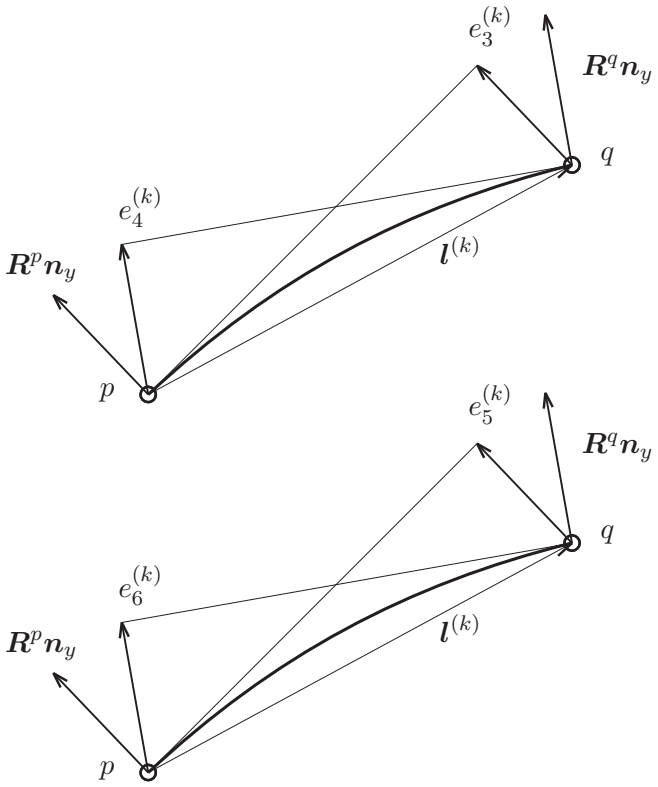


Figure 4.3: Visualisation of bending deformations e_3, e_4, e_5, and e_6 of the spatial beam element.

There may arise some confusion about the number of coordinates of a rotational node. In the description of the spatial beam it was mentioned that each rotational node is described by three independent coordinates. However, in the computations SPACAR uses internally 4 orientational coordinates (called Euler parameters denoted by λ_i) for each rotational node which are related by $\sum_{i=0}^{3}(\lambda_i)^2 = 1$. From this point of view each rotational node adds 4 nodal coordinates (instead of 3) to `NX` and one (extra) relative constraint to `NEO`. Effectively, this does not affect the outcome of equation (2.13) for `NDOF` as both `NX` and `NEO` are incremented with the number of rotational nodes. However, to avoid misunderstanding it has to be specified in which way the nodal coordinates and deformations are counted. In the SPACAR `log` file both values can be found, i.e. 4 nodal coordinates and 1 relative constraint or just 3 nodal coordinates per rotational node. In this tutorial the latter approach is preferred and this is also the way in which the `NDOF`'s are counted by the GUI while building and editing models.

4.1 One degree of freedom model

In this section a three-dimensional SPACAR model will be outlined for the spatial frame model with elastic beams in figure 4.1. This model `pf3d.spa` is also available for download.

Starting with three beams. The model is built analogously to the 2D model in section 3.1. Of course now we start with a new 3D model and the first beam is created with the first nodal point `P` in the origin and the second nodal point `Q` in $(0, 0.0, 0.1)$, see figure 4.4. Compared to the dialog for the planar beam in figure 3.5(a) there is a third `Z` coordinate and furthermore an `Orientationvector (local y-axis)` has to be specified. As shown in figure 4.2 the beam's principal x'-axis coincides with the undeformed element axis and is directed from end point p to end point q, so it is defined from the initial position of the beam. When the principal y'-axis is defined, the principal z'-axis is also known.

Any vector may be used to define the principal y'-axis as long as it is not (almost) parallel with the local x' axis. In this example the local x' axis is along the global z axis, so it is fine to set the local y' axis equal to the global y axis. Being the first beam al `Node Nr`'s and `Rotational DOF`'s are set to `New`. Once the beam has been defined, it can be `Edit`ed to set its height and width to the values specified in table 1.1 for the leaf spring. Next the second and third beam can be added to the model, their dimensions can be set and figure 4.6 (page 36) shows the model in the GUI. Note that the local y' axis of the solid bar can not be equal to the global y axis as these axes are parallel. Instead the local y' axis is set equal to the global z axis. The yellow lines in figure 4.6 show the local y' axes of the beams.

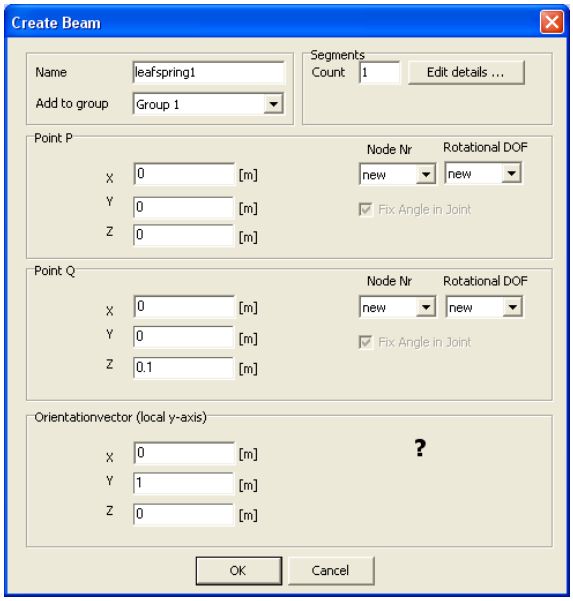


Figure 4.4: Creating the first spatial beam element.

Counting the DOF's, adding constraints and releasing deformations. So far we have created a system with three rigid beams that are rigidly attached to each other and can move and rotate freely. Hence the DOF counter displays `NDof = 6  Userdefined Dof = 0`.

As in the 2D case the beams representing the leaf springs are clamped rigidly on the bottom end, so all coordinates of both translational and rotational nodes (1, 7, 2, and 8) need to be fixed. Then we obtain an overconstraint system with `NDof = -6  Userdefined Dof = 0`. Next, deformations are released. A three-dimensional leaf spring allows bending motion in its slender direction as well as torsion. For leaf spring 1 this means that its bending modes in the local y' direction will deform, i.e. the `Deformation type`s of `Bending e5 (Rp.ny')` and `Bending e6 (Rq.ny')` are `Released`. In addition, `Torsion` is also `Released`. Next this procedure is repeated for the second leaf spring and the DOF is defined to be the `Y` coordinate of `Translational node` 3. The the DOF counter will show `NDof = 0  Userdefined Dof = 1`, indicating that the system is not well defined. More specifically, the system is overconstraint as both leaf springs restrict the rotation of the solid bar around its longitudinal axis. An analysis with SPACAR `mode=0` can detect such overconstraint (and underconstraint) modes in a system. For an overconstraint system a statically indeterminate stress distribution can be calculated with SPACAR. SPAVISUAL can visualise this stress distribution in the flexible spatial beams to show the overconstraint mode in the system. Figure 4.5 illustrates this for the current system with two normal leaf springs. The bending stresses found in both leaf springs can be avoided by allowing a rotation around the local y' axis in one of the beams.

For this reason it was proposed to cut one spring partly as visualised in figure 1.3. In this way also bending in the local z' direction is allowed. This bending motion cannot be represented with the deformation modes of the single beam that models the cut leaf spring. For the 1-DOF motion

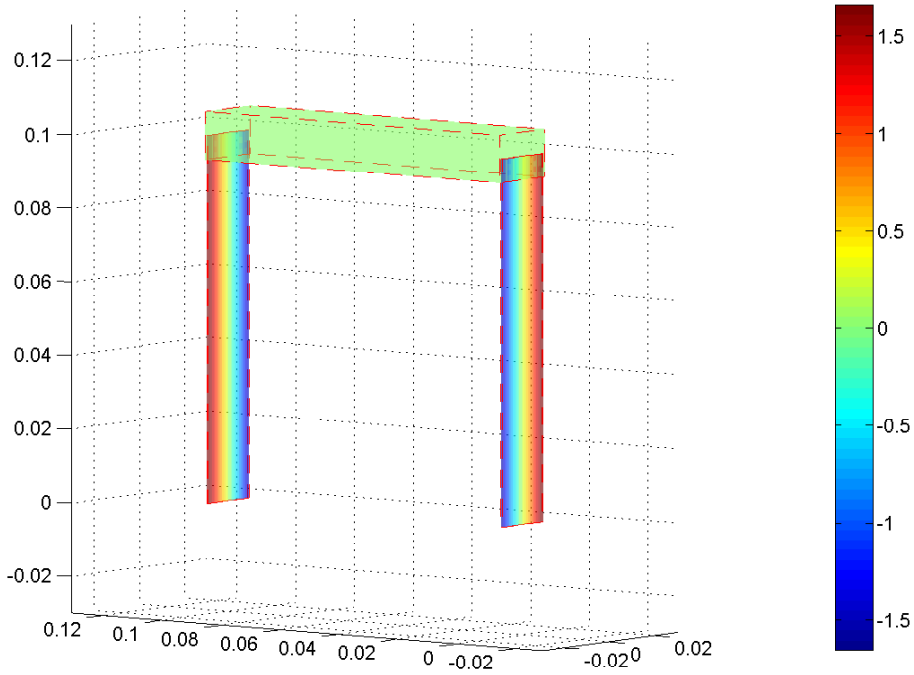


Figure 4.5: Longitudinal component of the statically indeterminate stress distribution in the system with two normal leaf springs illustrating that the overconstraint mode causes a bending stress in both leaf springs.

analysed in this section the details of this bending mode are not important and we only take into account that the system is no longer overconstraint by releasing one of the z' bending deformations, e.g. `Bending e3 (Rp.nz')`. This gives the valid system definition with `NDof = 1 Userdefined Dof = 1` satisfying the necessary condition of equation (2.13). The model is shown in figure 4.6.

Specifying the mass and inertia. For this example the element masses and inertia properties are calculated with the GUI using the correct densities from table 1.1. The moving mass of the VCM m_m is added in `Translation node` 5. Note that in a `Rotational node` all inertia components J_{xx}, J_{xy}, J_{xz}, J_{yy}, J_{yz}, and J_{zz} can be specified [3, Sect. 2.3].

Stiffness and damping. For the spatial beam element four real parameters specify the axial rigidity EA, the torsional rigidity GI_t and the flexural rigidities $EI_{y'}$ and $EI_{z'}$ [3, Sect. 2.3]. These can be computed in the GUI from the Young's Modulus E, Shear Modulus G and beam dimensions. Note that the torsional stiffness S_t is not straightforward for beams with a non-circular cross section. The torsional stiffness in SPACAR is defined as

$$S_t = \frac{G\,I_t}{l^3}, \tag{4.8}$$

where G is the shear modulus of the material and I_t is Saint-Venant's torsion constant which is the polar moment of inertia for a circular cross section. For elements having a rectangular cross section it can be determined by [15]

$$I_t = wt^3 \left(\tfrac{1}{3} - 0.21 \frac{t}{w} \left(1 - \frac{(t/w)^4}{12} \right) \right) , \tag{4.9}$$

where w is the width and t is the thickness of the cross section. This approximation only holds for a beam with both ends free. The twisting angle θ_l of a prismatic beam clamped at one end can be described

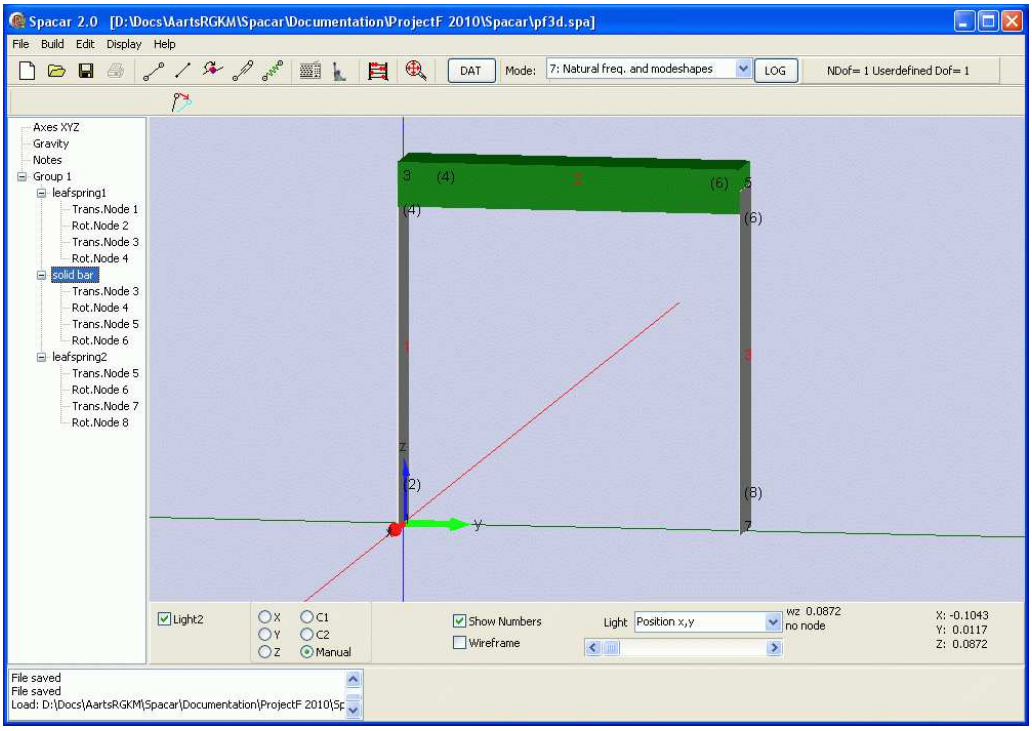


Figure 4.6: Spatial frame model with elastic beams in the GUI in which the DOF's have been defined correctly.

with

$$\theta_l = \frac{T}{G\,I_t}\left(l - \frac{1}{\alpha}\right), \tag{4.10}$$

where T is the twisting moment, GI_t is the torsional rigidity, l represents the length of the prismatic beam and $\frac{1}{\alpha}$ represents the influence of the clamped end on the twisting angle. For a narrow rectangular cross section where the width w is large in comparison to the thickness t, it can be written as

$$\frac{1}{\alpha} = \frac{w}{\sqrt{24(1-\nu)}}. \tag{4.11}$$

Here ν denotes the Poisson's ratio of the material, which is approximately 0.3 for steel. Now, $\frac{1}{\alpha}$ can be written as a fraction of the width w of the beam

$$\frac{1}{\alpha} = 0.244\,w. \tag{4.12}$$

Equation (4.10) can be rewritten with an equivalent torsional rigidity $(GI_t)_{\text{eq}}$ as

$$\theta_l = \frac{T}{(GI_t)_{\text{eq}}}, \tag{4.13}$$

with

$$(GI_t)_{\text{eq}} = \frac{1}{1-\frac{1}{\alpha l}}GI_t = \frac{1}{1-0.244\frac{w}{l}}GI_t. \tag{4.14}$$

If $l = 0.244w$, the denominator of the fraction in the righthand side of equation (4.14) becomes zero, implying an infinitely large torsional rigidity. Equation (4.14) is valid for a beam which is clamped at one end and subjected to a torque T at the free end. From the theory of elasticity it is known that

warping is only constrained near the clamped end. A good estimate is obtained if the adjoining element is modelled rigidly for torsion over a length $l = 0.244w$.

In the current model torsion does not play a dominant role and only one element is used for each leaf spring. The cut in the second leaf spring decreases the torsional stiffness somewhat and in particular z'-bending stiffness. For the 1-DOF analysis in this section these deformations should not play a role and the computed stiffnesses of these deformation modes are overruled by setting the values of the respective moments of inertia to zero.

The longitudinal, torsional and bending damping values (EDAMP) can be calculated analogously as for the planar beam with equation (2.14): $E_dA = 0.3654$ Ns, $G_dI_t = 0.0000065$ Nsm^2, $E_dI_y = 0.00019$ Nsm^2, and $E_dI_z = 0.0000053$ Nsm^2.

Specifying input and outputs. The `Extra input for DAT-file` dialog is used to define the system's input (force on node 5 in the y direction) and outputs (all coordinates of translational nodes 3 and 5):

```
INPUTF  1  5  2

OUTX  1  3  1
OUTX  2  3  2
OUTX  3  3  3
OUTX  4  5  1
OUTX  5  5  2
OUTX  6  5  3
```

Optional visualisation settings. In the `Extra input for DAT-file` dialog the settings of the visualisation are set as in the previous chapter. In addition the TRANSPARENCY is set to 0.6 [3].

SPACAR results

For this three-dimensional model the natural frequency, vibration mode (figure 4.7), state space matrices, transfer functions and the Bode plot (figure 4.8) from input force to y-displacement of node 3 are generated.

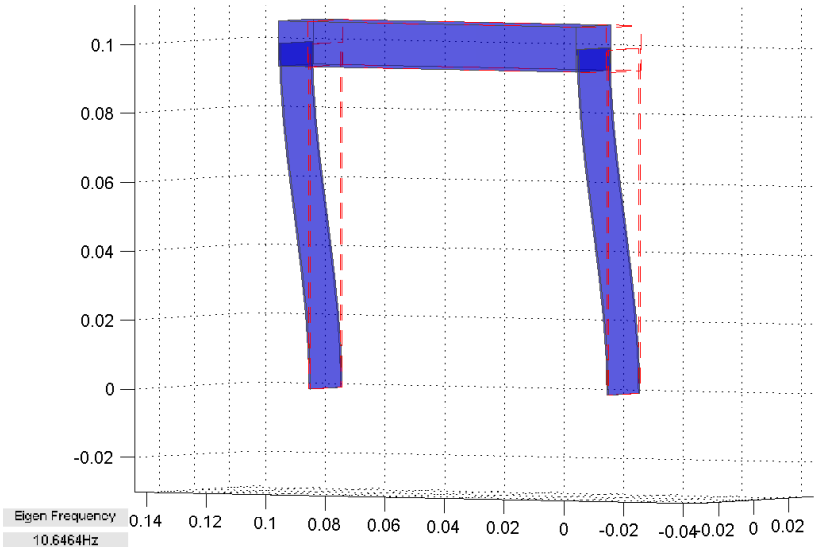


Figure 4.7: First vibration mode of the three-dimensional model.

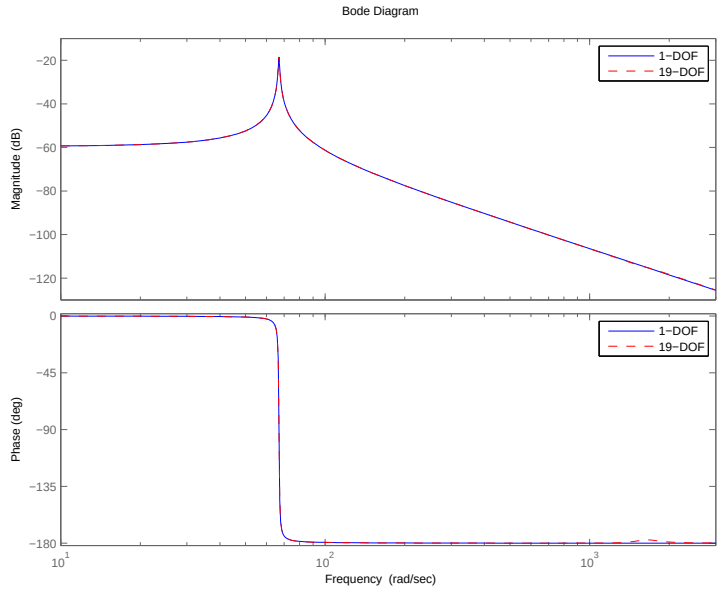


Figure 4.8: Bode plot of transfer from input force to y-displacement of node 3 of the 1-DOF and 19-DOF three-dimensional models, sections 4.1 and 4.2 respectively.

```
>> spacar(7,'pf3d');
>> sqrt(eig(k0,m0))/2/pi

   10.6464

>> spavisual('pf3d');

>> spacar(9,'pf3d');
>> getss('pf3d')

a =
            x1       x2
   x1        0        1
   x2    -4475  -0.6023

b =
         u1
   x1     0
   x2  4.735

c =
       x1  x2
   y1   0   0
   y2   1   0
   y3   0   0
   y4   0   0
   y5   1   0
   y6   0   0

d =
       u1
```

```
   y1   0
   y2   0
   y3   0
   y4   0
   y5   0
   y6   0

Continuous-time model.

>> G=tf(getss('pf3d'))

Transfer function from input to output...
 #1:  0

              4.735
 #2:  ---------------------
      s^2 + 0.6023 s + 4475

 #3:  0

 #4:  0

              4.735
 #5:  ---------------------
      s^2 + 0.6023 s + 4475

 #6:  0

>> bode(G(2))
```

4.2 Higher order model

It is the responsibility of the user to select the degrees of freedom carefully. In the previous example only one degree of freedom was defined and hence only one vibrational mode (with one natural frequency) was be found. A more or less straightforward extension is to release all 12 deformation modes of the leaf springs. Then there are only $\mathtt{NEO} = 18 - 12 = 6$ relative constraints and the number of degrees of freedom according to equation (2.13) is

$$\mathtt{NDOF} = \mathtt{NX} - \mathtt{NXO} - \mathtt{NEO} = 24 - 12 - 6 = 6. \tag{4.15}$$

However, this model will not give meaningful results as a single beam cannot describe the deformations of the cut leaf spring accurately. Therefore a more complicated model is considered as shown in figure 4.9. Three beams are used for the cut leaf spring and two beams for the normal leaf spring. There are three additional translational and rotational nodes connecting the beams. Releasing all deformations results in a system with

$$\mathtt{NDOF} = \mathtt{NX} - \mathtt{NXO} - \mathtt{NEO} = 2 * 7 * 3 - 12 - 6 = 24, \tag{4.16}$$

so 24 degrees of freedom. Some reduction is possible as the longitudinal stiffness of the beams is relatively high. Keeping the elongation deformation elongation of the beams fixed to zero gives

$$\mathtt{NDOF} = \mathtt{NX} - \mathtt{NXO} - \mathtt{NEO} = 2 * 7 * 3 - 12 - (6 + 5) = 19. \tag{4.17}$$

To obtain a correct set of degrees of freedom, the 1-DOF model of section 4.1 is taken as the starting point with seven released deformations and the horizontal motion of the solid bar as the single degree of freedom. Next all 18 other non-zero deformations are defined as relative degrees of freedom. Of course all masses, inertias, stiffness and damping parameters need to be set. This model is saved as `pf3d19dof.spa` and `pf3d19dof.dat`.

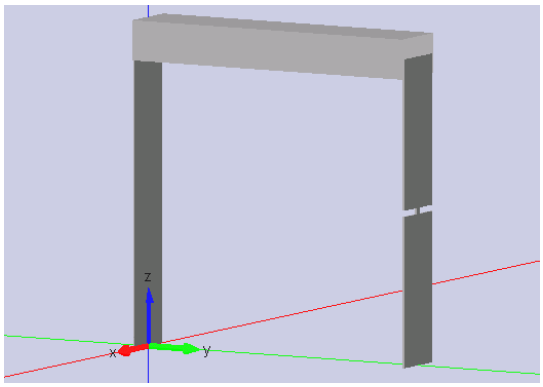


Figure 4.9: Spatial frame model with elastic beams representing the cut leaf spring more accurately.

SPACAR results

The result of the extra degrees of freedom is that more natural frequencies and their vibration modes can be found. After a mode 7 run, the natural frequencies can be obtained similarly as before, but the matrices `k0` and `m0` have to be reshaped first.

```
>> spacar( 7,'pf3d19dof')

>> eigf=sort(sqrt(eig(reshape(k0,nddof,nddof),...
                      reshape(m0,nddof,nddof)))/2/pi); eigf(1:5)

   10.6469
  148.5680
  252.1481
  270.9347
  619.9492
```

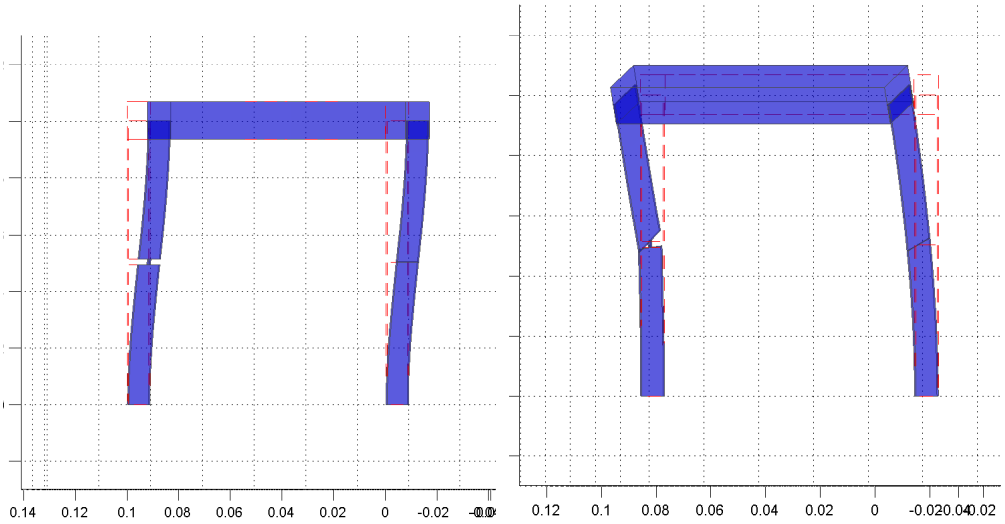
Only the lowest five natural frequencies are shown. The first natural frequency is identical to the result in section 4.1. In figure 4.10 four mode shapes of the 19-DOF system can be found. The first mode is the intended motion as expected. The second mode shows a bending motion that is possible due to the cut in the leaf spring. The third and (not shown) fourth mode are internal vibrations of the leaf springs.

Using mode 9, the state space matrices and consequently the transfer functions for the system can be determined. Only the transfer function from input force to y-displacement of node 3 is given, in figure 4.8 the Bode plot of this higher order transfer function is included.

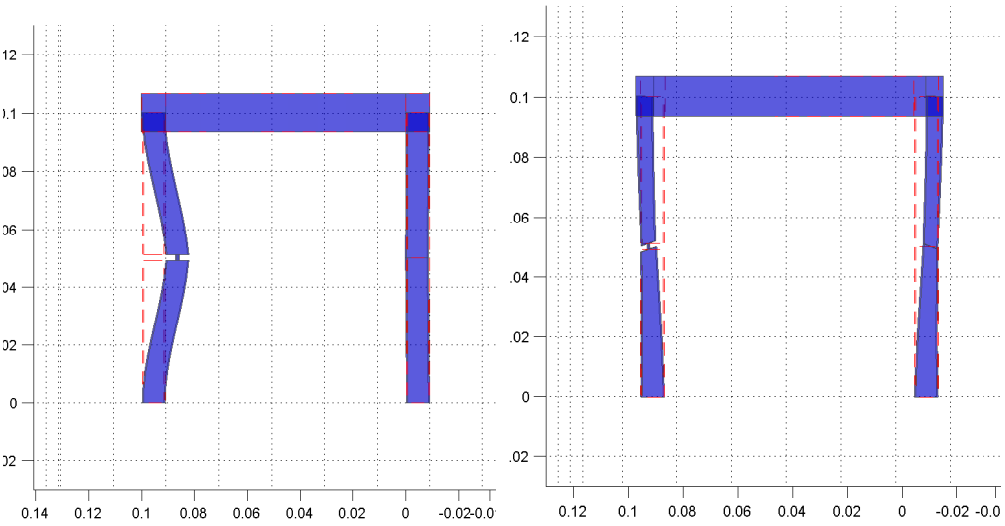
```
>> spacar(9,'pf3d19dof');
>> G=tf(getss('pf3d6dof')); G(2)

Transfer function:

4.81 s^12 + 3.64e7 s^11 + 2.371e12 s^10 + ... + 9.54e41 s^2 + 7.15e44 s + 8.85e47
-------------------------------------------------------------------------------
   s^14 + 7.57e6 s^13 + 4.93e11 s^12 + ... + 1.88e47 s^2 + 7.89e47 s + 8.38e50
```



(a) First natural frequency (10.6 Hz). (b) Second natural frequency (149 Hz).



(c) Third natural frequency (252 Hz). (d) Fifth natural frequency (620 Hz).

Figure 4.10: Mode shapes 1, 2, 3 and 5 of the 19-DOF three dimensional model.

In the plotted frequency range, the Bode plot of this 14^{th} order transfer function can hardly be distinguished from the second order transfer function from the 1-DOF system of section 4.1. Some differences in the phase near 1600 rad/s reveal the higher order poles and zeros. Apparently, most of the poles and zeros cancel, which can be verified with MATLAB's `minreal` command:

```
>> minreal(G(2), 1e-2)

Transfer function:
        4.811
---------------------
s^2 + 0.6024 s + 4475
```

When the "tolerance" parameter 10^{-2} is set sufficiently large, indeed a second order transfer function is obtained that is close the corresponding transfer function from the 1-DOF model in the previous section.

Finally, SPAVISUAL is used to visualise the local stresses in the (spatial) beams that occur during operation of the system. For this purpose a force of 1 N is applied in the y direction at node 5. This leads to a horizontal displacement of slightly more that 1 mm. In this configuration the Von Mises stresses are shown in figure 4.11. The MATLAB commands to create this figure are

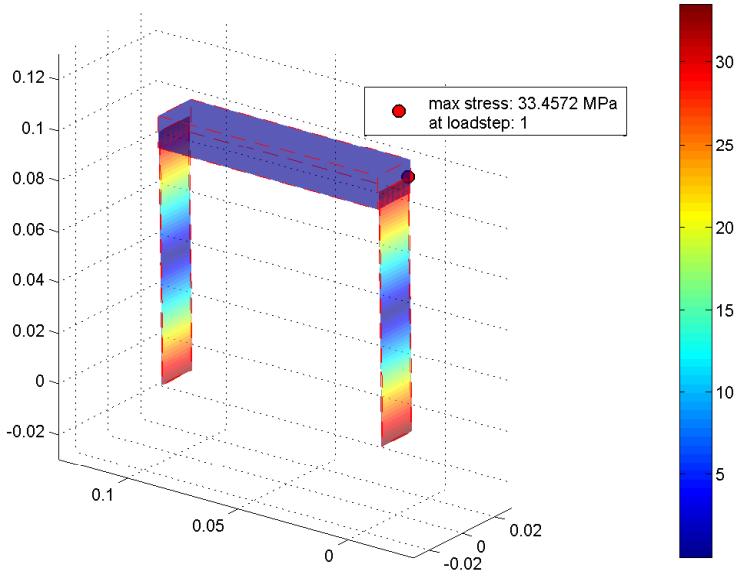


Figure 4.11: Von Mises stresses in the mechanism subject to a horizontal load of 1 N.

```
>> beamopts.vibr = 0;
>> visopts.stress ='mises';
>> spavisual('pf3df',beamopts,visopts);
```

In section 1.1 it was mentioned that the simplified system of figure 1.3 has some drawbacks compared to a more advanced design as in figure 1.2. Comparing support stiffnesses of both system and the buckling performance is left as an exercise for the reader.

A SPACAR software

A.1 Introduction

The computer program SPACAR is based on the non-linear finite element theory for multi-degree of freedom mechanisms as described in Jonker's lecture notes on the Dynamics of Machines and Mechanisms [9]. The program is capable of analysing the dynamics of planar and spatial mechanisms and manipulators with flexible links and treats the general case of coupled large displacement motion and small elastic deformation. The motion can be simulated by solving the complete set of non-linear equations of motion or by using the so-called perturbation method. The computational efficiency of the latter method can be improved further by applying modal techniques.

In this chapter an outline of the SPACAR package is given. It starts with the GUI to build and edit models. Analysing the models in MATLAB is outlined next. A special visualisation tool, SPAVISUAL, is described in Section A.4. Installation notes for SPACAR are given in Section A.5.

A.2 SPACAR GUI

The SPACAR "kernel" discussed in the next section can analyse any system described in a text file, the "`dat`" file. To create and edit such models one has to be familiar with the syntax of these files. A more user-friendly approach is provided by a (stand-alone) Graphical User Interface (GUI) available for any current Microsoft Windows operating system. This is a commercial product, of which a time-limited educational license ia available for students participating in the "Project F" course. With this GUI both two-dimensional and three-dimensional systems can be defined which consist of a number of elements, see also appendix A.3.

Installation of this part of the software is rather straightforward. All necessary files are provided in a `zip` archive. Extract the files in some folder, start `spacar.exe` and that's it. You may prefer to add a link to the application on the desktop or the start menu. The application has a built-in facility to check for updates, to download and install any updates. On-line help is provided. The GUI stores its models in files with extension `spa` and generates the `dat` input files for the analysis in MATLAB. Note however that there are restrictions on the filenames that can be used as outlined in the section A.3.

A.3 SPACAR & MATLAB

SPACAR can run different types of analysis called modes. Only mode 7, 8 and 9 will be used in this tutorial. For information about the other modes, see the SPACAR manual [3].

In `mode=7` eigenvalues (frequencies) and corresponding eigenvectors are computed for a static equilibrium configuration. The associated frequency equation of the undamped system is given by

$$\det\left(-\omega_i^2 \bar{\boldsymbol{M}}_0^{dd} + \boldsymbol{K}_0^{dd} + \boldsymbol{N}_0^{dd} + \boldsymbol{G}_0^{dd}\right) = 0\ , \tag{A.1}$$

where the quantities ω_i are the natural frequencies of the system. $\boldsymbol{M}_0^{dd}$ is the mass matrix and $\boldsymbol{K}_0^{dd}$, $\boldsymbol{N}_0^{dd}$ and $\boldsymbol{G}_0^{dd}$ are the structural, dynamic and geometric stiffness matrices respectively.

In `mode=8` a linear buckling analysis is carried out for a static equilibrium configuration. Critical load parameters λ_i are determined by solving the eigenvalue problem.

$$\det(\boldsymbol{K}_0^{dd} + \lambda_i \boldsymbol{G}_0^{dd}) = 0 \,, \tag{A.2}$$

where,

$$\lambda_i = \boldsymbol{f}_i / \boldsymbol{f}_0. \tag{A.3}$$

Here, $\boldsymbol{K}_0^{dd}$ is the structural stiffness matrix and $\boldsymbol{G}_0^{dd}$ is the geometric stiffness matrix due to the reference load $\boldsymbol{f}_0$ giving rise to the reference stresses $\boldsymbol{\sigma}_0$. $\boldsymbol{f}_i$ represents the bucking load that corresponds with λ_i.

In `mode=9` locally linearized equations for control system analysis are computed for a static equilibrium configuration. The linearized equations can be transformed into the linearized state space form:

$$\begin{aligned} \delta\dot{\boldsymbol{z}} &= \boldsymbol{A}\delta\boldsymbol{z} + \boldsymbol{B}\delta\boldsymbol{u}, \\ \delta\boldsymbol{y} &= \boldsymbol{C}\delta\boldsymbol{z} + \boldsymbol{D}\delta\boldsymbol{u}, \end{aligned} \tag{A.4}$$

where $\boldsymbol{A}$ is the state matrix, $\boldsymbol{B}$ the input matrix, $\boldsymbol{C}$ the output matrix and $\boldsymbol{D}$ the feed through matrix. The state vector $\delta\boldsymbol{z}$ is defined by $\delta\boldsymbol{z} = [\delta\boldsymbol{q}^{dT}, \delta\dot{\boldsymbol{q}}^{dT}]^T$, where $\delta\boldsymbol{q}^d$ is the vector of dynamic degrees of freedom. The matrices $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ depend on the chosen input vector $\delta\boldsymbol{u}$ and the output vector $\delta\boldsymbol{y}$. Details of the linearization are discussed in [9, chapter 12].

Definition of a mechanism model

A model of a mechanism must be defined in an input file of file type (or file name extension) `dat`. This input file consists of a number of keywords with essential and optional parameters. The input file can be generated with the GUI (section A.2) or any text editor. For a complete overview of the available keywords and their parameters the reader is referred to the SPACAR manual [3]. Some of these keywords define the elements of which the system is composed, see also page 48.

Running SPACAR in the MATLAB environment

Once the mechanism is defined and this information is saved to a `dat` input file, SPACAR can be activated with the MATLAB command

```
>> spacar(mode,'filename')
```

Here, `filename` is the name of the input file, without the extension `dat`. The `filename` is limited to 20 characters from the set “0”–“9”, “a”–“z”, “A”–“Z” and “_”, so it can not include drive or path specifications.

During the computation a plot of the mechanism is shown in a separate window. While the simulation is running an `Abort` button is activated in the plot area. Pressing this button will terminate the simulation (possibly after some delay). To speed up the computation, the plot can be disabled by specifying the mode with a minus sign, e.g. `mode=-9`. The plotting utility `spadraw` can also be used after the simulation to visualise the results, see page 48.

During the computations the results are stored in one or more data files and in MATLAB arrays. A `log` file is always created when SPACAR starts processing the input `dat` file. This `log` file contains an analysis of the input and possible errors and warnings. It is described in more detail on page 47. Many errors in the input file do not lead to an early termination of the SPACAR computation, but nevertheless give unusable results. Therefore it is advisable to check the `log` file for unexpected messages.

All other data files are so-called SPACAR Binary data Files (SBF), which implies that these are in a binary format and can not be read easily by a user. Therefore, utilities are provided to read and modify data in these files, see page 48. Depending on the `mode` up to three binary output files may be created.

For all `mode`s a SPACAR Binary Data file with filename identical to the input file and extension `sbd` is written. The contents of this file is also stored in MATLAB arrays, that are of course immediately available in the MATLAB workspace e.g. to be visualised with the standard MATLAB graphics commands, such as `plot`. The following variables are created or overwritten:

`mode`	SPACAR mode number	
`ndof`	number of kinematic DOFs	
`nddof`	number of dynamic DOFs	
`nx`	number of coordinates	
`ne`	number of deformation parameters	
`nxp`	number of fixed, calculable, input and dynamic coordinates	
`nep`	number of fixed, calculable, input and dynamic deformation parameters	
`lnp`	location matrix for the nodes	*1
`le`	location matrix for the elements	*1
`ln`	connection matrix for the nodes in the elements	*2
`it`	list of element types	*2
`time`	time column vector	
`x`	coordinates (nodal displacements)	
`xd`	nodal velocities	
`xdd`	nodal accelerations	
`fx`	prescribed nodal forces/moments	
`fxtot`	reaction forces/moments	
`e`	generalized deformations	
`ed`	velocities of generalized deformations	
`edd`	accelerations of generalized deformations	
`sig`	generalized stress resultants	
`de`	first order geometric transfer function for the deformations $\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)}$	*3
`dx`	first order geometric transfer function for the coordinates $\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}$	*3
`d2e`	second order geometric transfer function for the deformations $\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e)}$	*3
`d2x`	second order geometric transfer function for the coordinates $\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}$	*3
`xcompl`	location vector for directional nodal compliances	*4
`rxyz`	initial orientations of elements	*2

Notes:

$*1$ The two location matrices provide information to find the location of a specific quantity in the data matrices:

`lnp`	location matrix for the nodes. The matrix element `lnp(i,j)` denotes the location of the `j`$^{\text{th}}$ coordinate (`j`=1..4) of node `i`.
`le`	location matrix for the elements. The matrix element `le(i,j)` denotes the location of the `j`$^{\text{th}}$ generalised deformation (`j`=1..6) of element `i`.

The locations of undefined or unused coordinates and deformations equal zero.

For example, the x- and y-coordinates of node 7 can be shown as function of time in a graph by typing

```
>> plot(time,x(:,lnp(7,1:2)))
```

and the first generalised stresses in elements 1, 2 and 3 can be plotted by typing

```
>> plot(time,sig(:,le(1:3,1)))
```

Obviously, storage in the x, xd, xdd, fx, e, ed, edd and sig matrices is like x(t,k) where t is the time step and k ranges from 1 to nx for x, xd, xdd and fx, fxtot and from 1 to ne for e, ed, edd and sig, respectively.

∗2 The variables ln, it and rxyz are mainly intended for internal use in the drawing tool spadraw. More user-friendly information is available in the log file, page 47.

∗3 The (large) variables de, dx, d2e and d2x are only created if the parameters of the LEVELLOG are set accordingly, [3].

∗4 After a linearisation run (mode=8) directional nodal compliances (inverse stiffnesses) are computed. Using the location matrix, xcompl(lnp(i,j)) gives this quantity for the j^{th} coordinate (j=1..4) of node i.

After a linearization run (mode=3, 4, 7, 8 or 9) the coefficient matrices are stored in a SPACAR Binary Matrix file with extension sbm. The accompanying MATLAB matrices are:

m0	reduced mass matrix $\boldsymbol{M}_0$	*5
b0	input matrix $\boldsymbol{B}_0$	*5, *6
c0	velocity sensitivity matrix $\boldsymbol{C}_0$	*5
d0	damping matrix $\boldsymbol{D}_0$	*5
k0	structural stiffness matrix $\boldsymbol{K}_0$	*5
n0	geometric stiffness matrix $\boldsymbol{N}_0$	*5
g0	geometric stiffness matrix $\boldsymbol{G}_0$	*5

Notes:

∗5 Storage of the time-varying matrices is in a row for each time step, so in m0(t,k) index t is the time step and k ranges from 1 to ndof×ndof. To restore the matrix structure at some time step type e.g. reshape(m0(t,:),ndof,ndof).

∗6 Only available for mode=4 and 9.

In mode=2, 3, 4 and 9 a so-called ltv file is created. The contents of this file varies and is not automatically imported to the MATLAB workspace. From a mode=2 run the following data is available (the names indicate the identities of the data used in the file; identities marked with "*" are available at each time step):

NNOM	number of (actuator) inputs	
NY	number of outputs	
T	time	*
U0	nominal input for the desired motion	*
Y0	reference output of the desired motion	*

In the addition the linearization runs yield additional setpoints, state space matrices and other data in the ltv file (not all identities are always present):

NNOM	number of (actuator) inputs	
NX	number of states (2×ndof)	
NU	number of inputs (length of U0)	
NY	number of outputs (length of Y0)	
NRBM	number of rigid body DOF's	
NYS	number of outputs with 2^{nd} order expression	
DFT	direct feedthrough flag (D≠0)	
X0	initial state vector	
T	time	*
A	state space system matrix	*
B	state space input matrix	*

`C`	state space output matrix	*
`D`	state space direct feedthrough matrix	*
`G`	second order output tensor	*
`M0`	mass matrix $\boldsymbol{M}_0$	*
`C0B`	combined damping matrix $\boldsymbol{C}_0 + \boldsymbol{D}_0$	*
`K0B`	combined stiffness matrix $\boldsymbol{K}_0 + \boldsymbol{N}_0 + \boldsymbol{G}_0$	*
`SIG0`	generalized stress resultants	*

The `getss` tool can be used to read the state space matrices from the `ltv` file, see page 48. Other utilities are available to use parts of these data in a SIMULINK environment, e.g. to read setpoints or to simulate a Linear Time-Varying (LTV) system, as is described in the manual.

The `log` file

The `log` file contains an analysis of the input and possible errors and warnings that are encountered. The error and warning messages are explained in more detail in the SPACAR manual. The other output can be separated into a number of blocks.

The first lines indicate the version and release date of the software and a copyright note.

Next the lines from the input file read by the KIN module are shown (not showing comments present in the input file) [3]. From the analysis is written:

- The elements used in this model. The deformations of all elements are shown with the internal numbers according to the `le` array and the classification of each deformation: `O` = fixed, `C` = calculable and `M` = DOF.
- The nodal point information with the internal numbers of the coordinates according to the `lnp` array and the classification as above.
- Finally a list shows the degrees of freedom. Dynamic degrees of freedom are indicated.

The DYN module reads the next data block and processed input lines are shown. From the analysis we get

- The numbers `NEO`, `NEMM`, `NEM` and `NEC` indicate the numbers of deformations in each class as explained in the lecture notes [9].
- The numbers `NXO`, `NXC`, `NXMM` and `NXM` indicate the numbers of position coordinates in each class as explained in the lecture notes [9].
- The stiffness, damping and mass of the elements.
- The nodal point forces, mass and gyroscopic terms.
- The total mass of the system.

The zeroth, first, second and third order transfer functions are shown next, each for the position parameters and deformation parameters, respectively. The amount of output can be controlled by the keyword `OUTLEVEL` in the input file.

Next for a forward analysis (`mode=1` and `mode=4`) the name of the integrator and accuracy settings are shown. Finally a list with all time steps and the number of internal iterations are given. For an inverse dynamics analysis the trajectories and input/output definitions are read and analysed. In case of `mode=3` the name of the data file of the previous `mode=2` is shown. In case of `mode=7` the eigenvalues (frequencies) and normalized eigenvectors of the state system matrix are shown. In case of `mode=8` load multipliers and normalized buckling modes are presented. In addition the vector of directional nodal compliances is shown.

SPACAR Binary data Files

Some utilities are available to show, check, load or replace the data in SPACAR Binary data Files (SBF) files. These are files with extensions `sbd`, `sbm` and `ltv`.

`checksbf` check and shows the contents of a SPACAR Binary data File. The output for each variable is the name ("Id"), the type (1 for integer, 2 for real, 3 for text) and the size (number of rows and columns). First the "header" variables are shown with their value. Long vectors may be truncated. Between `TDEF` and `TDAT` the time-varying data is given. The number of time steps equals the number of rows specified for `TDEF`.

`getfrsbf` extract a variable from a SPACAR Binary data File. The "Id" must be specified and for time-varying data the time step as well.

`getss` extracts the state space formulation from a SPACAR `ltv`-file.

`repinsbf` replaces the value of a variable in a SPACAR Binary data File. The "Id" must be specified and for time-varying data the time step as well.

`loadsbd` loads all data from a SPACAR Binary Data (`sbd`) file into MATLAB's workspace.

`loadsbm` loads all data from a SPACAR Binary Matrix data (`sbm`) file into MATLAB's workspace.

`combsbd` combines data from two or more SPACAR Binary Data (`sbd`) files into a single output file. The specified output file is overwritten without a warning.

`spadraw` is the plotting utility used internally by SPACAR to show a schematic drawing of the analysed system. It can also be used to present the results after a simulation has been completed.

`spavisual` is a visualisation tool that is described in more detail in section A.4.

For all utilities additional online help is available by typing `help command` at the MATLAB prompt.

SPACAR element types

SPACAR can analyse two-dimensional and three-dimensional systems. In either case a number of elements types are available. The planar truss and (flexible) beam elements are introduced in chapters 2 and 3 of this tutorial. Chapter 4 presents a spatial system with (flexible) beam elements. A spatial truss element is available as well. This element type and some others are described in the SPACAR manual [3]. In this section a few (planar) examples are given.

The model of the planar elevator in figure A.1 is made with a hinge (`PLTOR`), a truss (`PLTRUSS`), a rigid beam (`PLRBEAM`) and two belts (`PLBELT`). Each of the belts models the vertical connections between the pulley in the upper part of the system and the point masses in the nodes 3 and 5 in the lower part. These point masses can be moved in vertical direction by rotating the pulley. The stiffness of each belt depends on its length, so the stiffnesses vary in the case the masses move. In this system the rotation of the pulley is driven by the hinge. The rigid beam and truss element do not affect the system's dynamic properties, but are included to improve the visualisation with `spadraw` during the SPACAR simulation. For more detailed information, the reader is advised to download the datafiles `elevator.spa` and/or `elevator.dat` that define this system.

For this simulation the system is released in an initial configuration with the hinge (and the pulley) rotated 1 rad. In the hinge a stiffness and damping are included which mimic a PD-controller that drives the hinge to its undeformed configuration as can be simulated with a SPACAR `mode=1` analysis. Figure A.2(a) shows that the hinge rotation returns indeed from the initially rotated state to an undeformed state. In fact, this is the closed-loop system's step response. Note however that the system is non-linear as the lengths of the belt are not constant and hence the stiffnesses and natural frequencies vary. This can

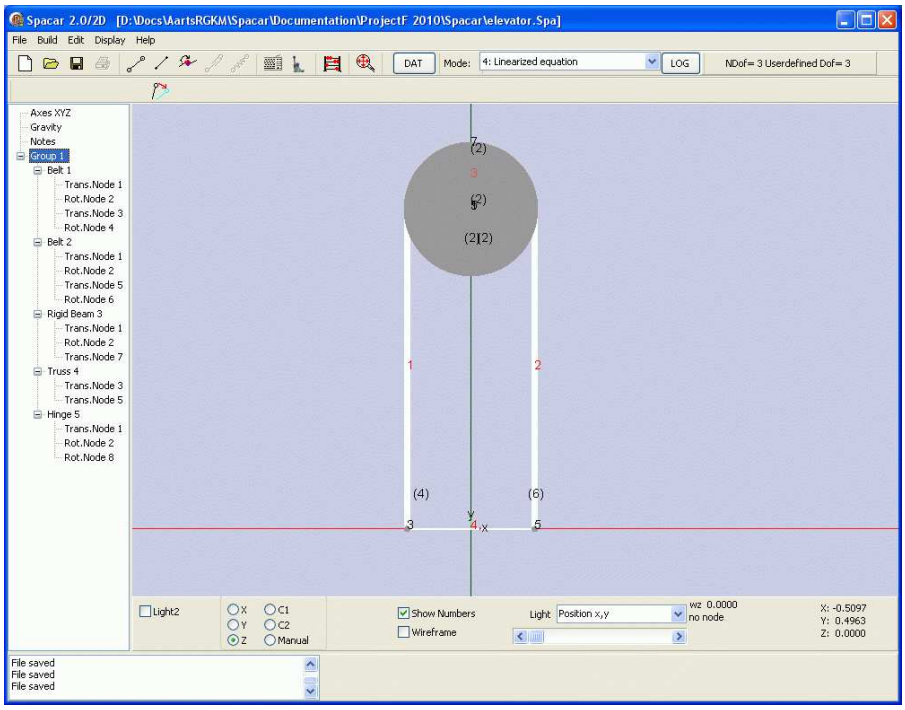
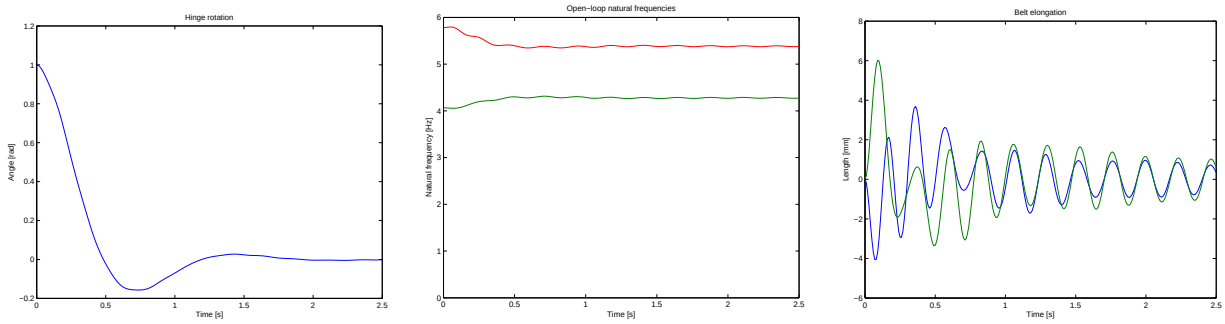


Figure A.1: Planar elevator model in the GUI.

be shown with a SPACAR `mode=4` analysis which computes mass, damping and stiffness matrices. The natural frequencies of the open-loop system (so without the PD-controller in the hinge) are shown in figure A.2(b). From the observed natural frequencies it is concluded that it is save to tune the PD-controller to a cross-over frequency of about 0.8 Hz or 5 rad/s. Finally, figure A.2(c) shows the elongation of the belt during the motion. Apparently, the damping of these higher frequent modes is less compared to the main motion of the hinge rotation.



(a) (b) (c)

Figure A.2: Elevator simulation: (a) hinge rotation, (b) varying natural frequencies, (c) belt elongation.

For spatial systems a `SCREW` element is available. It can be used to simulate e.g. spindle driven systems. At the time of writing this tutorial, this element can not be defined in the GUI, but the keyword is recognised in the `dat` file. It has nine parameters of which the first five are similar to the flexible beam element, i.e. the element number followed by the position and orientation nodes of the first and second node. The next three parameters describe the initial direction of the local x' screw axis. The ninth parameter expresses the pitch in displacement per radian (not per full turn). In the SPACAR manual an example is given of an elastic beam that is deformed by a spindle motion, figure A.3. To define a

SCREW element in a system using the GUI, one can define a BEAM element instead and save the dat file. Before running SPACAR, one should then replace the keyword and add the extra four parameters by editing the dat file manually. Note that the SCREW element can only be used to describe a kinematic relation, so no dynamic properties like mass and stiffness should be associated with this element.

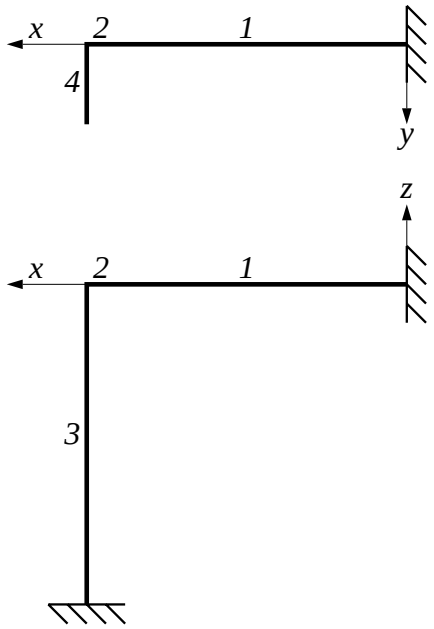


Figure A.3: Screw moving a flexible beam.

Limitations

The SPACAR package has some built-in limitations on the size of the manipulators that can be analysed. Table A.1 shows the limits for the so-called "Student version" that can be downloaded as describes in Appendix A.5. In case your requirements are larger, you need to contact the authors. The licence for the freely downloadable software is time limited.

Maximum number of coordinates/deformations	175
Maximum number of DOFs	20
Maximum number of elements/nodal points	50
Maximum number of inputs	12
Maximum number of outputs	25

Table A.1: Built-in limitations of the "Student version" of the SPACAR package.

A.4 SPAVISUAL

SPAVISUAL is the visualisation tool for SPACAR. It can visualise deformation, vibration and buckling modes. SPAVISUAL shows e.g. beams, trusses, and hinges in 2D as well as in 3D. It works with default settings which can be adjusted by the user. The only required input parameter for SPAVISUAL is a filename. This file has to be a dat file which has been analysed with SPACAR. This is necessary because SPAVISUAL needs the sbd files for the deformation modes and also the sbm files for the vibration and the buckling modes. There are a couple of keywords that can adjust the default settings [3].

SPAVISUAL is a stand-alone function in MATLAB. To run SPAVISUAL the user has to type

```
>> spavisual(filename)
```

Here `filename` refers to the `dat`-file that is analysed by SPACAR. The visualisation mode can be controlled interactively from the GUI window that the tool opens or from the command-line by specifying additional parameters. Furthermore, the appearance depends on keywords specified at the end of the `dat` file. A separate manual describes the extensive possibilities. In this tutorial a few examples are given.

An optional (second) parameter may be used to override the default vibration or buckling mode.

```
>> spavisual(filename,mode)
```

More advanced visualisation options are available by calling the tool with three parameters as in the example of section 4.2 to show the Von Mises stresses in spatial beams:

```
>> beamopts.vibr = 0;
>> visopts.stress ='mises';
>> spavisual(filename,beamopts,visopts);
```

With `help spavisual` an overview is presented with the options that are available to overrule the default visualisation settings or the settings specified in the `dat`-file.

Normally, the SPAVISUAL plot windows are protected from unwanted modifications from the command line, e.g. from other `plot` commands. Unfortunately this also disables e.g. the `print` command with which a plot window can be saved in various graphical formats. Access from the command line can be enabled with

```
>> set(1,'HandleVisibility','on')
```

assuming that the the SPAVISUAL plot window is number `1`. Similarly, it can be disabled again by setting this property to `off`.

A.5 Download & install

Prerequisites

Before installing SPACAR on a computer system it is advisable to check that the system is suitable of running the software and to have MATLAB installed. This SPACAR version has been developed and tested with MATLAB release R2013a (64-bit). It is expected to work with any modern version of MATLAB/SIMULINK since R12, but in case of problems we can offer only limited support. Some problems have been reported with the most recent 32-bit MATLAB releases R2011a and newer. It appears that SPACAR can be used with these MATLAB versions after installing a missing library that can be downloaded separately. Check the web site for information on known issues.

The system requirements depend heavily on the version of MATLAB you are using. Consult the accompanying Installation Guide or check The Mathworks. You may expect that SPACAR will run on any Microsoft Windows PC on which MATLAB/SIMULINK (32-bit or 64-bit) are running. Only the base systems of MATLAB and SIMULINK are required to run SPACAR, but additional toolboxes like the Control System Toolbox may be helpful to develop and analyse control systems. The installation of SPACAR uses less than 4 MB extra disk space. The SPACAR files are stored in ZIP-archives or, in Microsoft Windows XP, a compressed folder. In Windows XP you can easily open such archives, but of course you may chose to use your favourite unzipper. The ZIP-archives can be downloaded from
`http://www.wa.ctw.utwente.nl/software/spacar/`
In addition to the software there is a ZIP-archive with the data files that are used in this tutorial available on the SPACAR-site. The installation of the SPACAR GUI has to be done separately as mentioned in section A.2. The rest of this section is about the analysis software to be used in MATLAB.

Installation

First of all, you should create a subdirectory e.g. `\Matlab\Toolbox\Spacar`. Next, you extract the files from the SPACAR software ZIP-archive `spacar2013_bin.zip` into this subdirectory. There are three types of files:

- Files with the extension `.mexw32`/`.mexw64` are the actual executables of the SPACAR package. The original SPACAR-code (not provided) is written in C and FORTRAN77, compiled and linked into so-called MEX-modules, that are executables for use within the MATLAB-environment. The following files must exist:

  ```
  checksbf.mexw64  combsbd.mexw64   getfrsbf.mexw64  loadsbd.mexw64
  loadsbm.mexw64   ltv.mexw64       mrltv.mexw64     repinsbf.mexw64
  spacar.mexw64    spacntrl.mexw64  spasim.mexw64
  ```

  as well as their `.mexw32` versions.

- Files with extension `.m` are the MATLAB-files necessary to use the SPACAR and SPAVISUAL program. The following files must exist:

  ```
  getss.m
  spadraw.m
  spavisual.m
  ```

  Other `.m`-files provide help text for the MEX-modules. These files are:

  ```
  checksbf.m      combsbd.m       getfrsbf.m      loadsbd.m
  loadsbm.m       ltv.m           mrltv.m         repinsbf.m
  spacar.m        spacntrl.m      spasim.m
  ```

- Files with extension `.mdl` are SIMULINK models. There is only one file which is actually a library from which the SPACAR modules for use in SIMULINK can be copied:

  ```
  spacar_lib.mdl
  ```

- A folder with the name `private` with a lot of `.p`-files that are used by SPAVISUAL.

The (optional) data files can be extracted in a separate working directory.

The files in the SPACAR subdirectory should be in the MATLAB path when MATLAB is running. There are two ways to accomplish this:

1. Make sure that the SPACAR subdirectory is the local directory. You can verify this by typing `pwd`. If necessary, change your local directory by typing

   ```
   cd \Matlab\Toolbox\Spacar
   ```

   or whatever directory you chose to store your files.

2. Another possibility is to change the settings of the MATLAB environment by adding the SPACAR subdirectory to the MATLAB path. This modification is either temporary or permanent. The path can be modified from the pulldown menu with `File`|`Set Path...`, or by using the MATLAB commands `path` or `addpath`.

Now you are ready to run SPACAR in MATLAB and SIMULINK.

Bibliography

[1] R.G.K.M. Aarts and J.B. Jonker, *Dynamic simulation of planar flexible link manipulators using adaptive modal integration*, Multibody Systems Dynamics, **7**, pp. 3150, 2002.

[2] R.G.K.M. Aarts and J. van Dijk, *Inleiding Systeem- en Regeltechniek* (in Dutch), Lecture notes University of Twente, 2009.

[3] R.G.K.M. Aarts, J.P. Meijaard, J.B. Jonker, SPACAR *manual*, University of Twente, 2011.

[4] D. Blanding, *Exact Constraint: Machine Design Using Kinematic Principles*, ASME Press, New York, 1999.

[5] J. van Dijk, *Systeem- en Regeltechniek 2* (in Dutch), Lecture notes University of Twente, 2013.

[6] Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback control of dynamic systems*, 5^{th} edition, Pearson Education, ISBN 0-13-149930-0, 2006.

[7] P.J.M. van der Hoogt, *Dynamica 2*, Lecture notes University of Twente, 2006.

[8] J.B. Jonker and J.P. Meijaard, SPACAR-*computer program for dynamic analysis of flexible spational mechanisms and manipulators*. In *Multibody Systems Handbook*, W. Schiehlen (ed.), 123–143. Springer-Verlag, Berlin, 1990.

[9] J.B. Jonker, R.G.K.M. Aarts and J.P. Meijaard, *Flexible multibody dynamics for (control) design purposes: A Finite Element Approach*, Lecture notes University of Twente, 2014.

[10] J.B. Jonker, R.G.K.M. Aarts, J. van Dijk, *A linearized input-output representation of flexible multibody systems for control synthesis*, Multibody System Dynamics, **21** (2), 99-122, 2009.

[11] M.P. Koster, *Constructieprincipes voor het nauwkeurig bewegen en positioneren*, 3^{e} druk, Twente University Press, ISBN: 90-365-1456-8 or 90-365-1455-X, 2000.

[12] The Math Works Inc., *Getting Started with* MATLAB, version 7, Revised for MATLAB 7.3 (Release 2006b), September 2006.
WWW*: `http://www.mathworks.com/.../matlab/getstart.pdf`

[13] The Math Works Inc., SIMULINK — *Getting Started*, version 6, Revised for SIMULINK 6.5 (Release 2006b), September 2006.
WWW*: `http://www.mathworks.com/.../simulink/sl_gs.pdf`

[14] J.L. Meriam and L.G. Kraige, *Engineering Mechanics, Dynamics*, 4^{th} edition, SI version, John Wiley & Sons, ISBN 0-471-24167-9, 1998.

[15] Robert L. Mott, *Machine elements in mechanical design*, Fourth edition in SI units, Pearson Education, ISBN 0-13-197644-3, 2005.

[16] Herman Soemers, *Design Principles for precision mechanisms*, Lecture notes, University of Twente, 2009.

* The full path to the PDF documents of The Math Works starts with
`http://www.mathworks.com/access/helpdesk/help/pdf_doc/`