PRACTICE PROBLEMS FOR FINITE ELEMENT METHOD

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1 author:

Mahesh Chandra Luintel
Institute of Engineering, Tribhuvan University, Nepal

PRACTICE PROBLEMS FOR FINITE ELEMENT METHOD

Exercise Sheets

Mahesh Chandra Luintel

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EXERCISE 1

DISCRETE ONE DIMENSIONAL ELEMENTS: SPRING AND BAR

1. For the spring assemblages shown in **Figures [P1.1a to P1.1d]**, determine the nodal displacements, the forces in each element, and the reactions. Use the direct stiffness method for all problems.

(a)

(b)

(c)

(d)

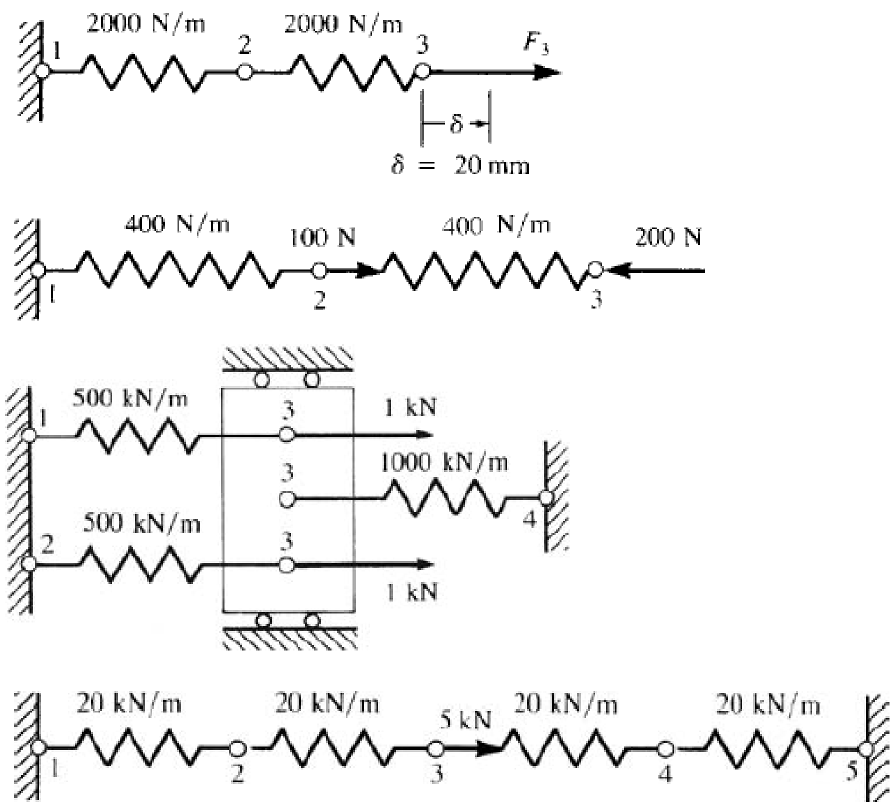


Figure P1.1

2. Number the elements and nodes for the spring system shown in **Figure P1.2**. Compute the global stiffness and force vector, **(b)** Partition the system and solve for the nodal displacements and **(c)** Compute the reaction forces. Take k = 5 kN/m and force in N.

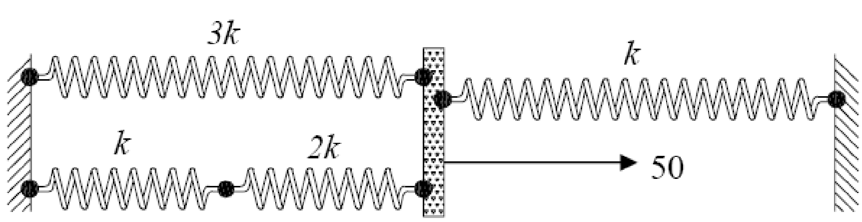


Figure P1.2

3. For the spring assembly shown in **Figure P1.3** solve for the displacements and the reaction force at node 1 if k_1 = 4 kN/m k_2 = 6 kN/m k_3 = 3 kN/m, F_2 = − 30 N and F_4 = 50 N.

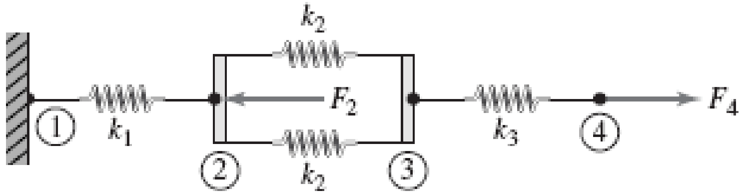


Figure P1.3

4. Four rigid bodies, 1, 2, 3, and 4, are connected by four springs as shown in the **Figure P1.4**. A horizontal force of 1,000 N is applied on Body 1 as shown in the figure. Using FE analysis, **(a)** find the displacements of the two bodies (1 and 3), **(b)** find the element force (tensile/compressive) of spring 1, and **(c)** the reaction force at the right wall (Body 2). Assume the bodies can undergo only translation in the horizontal direction. The spring constants are k_1 = 400 N/mm, k_2 = 500 N/mm, k_3 = 500 N/mm, and k_4 = 300 N/mm.

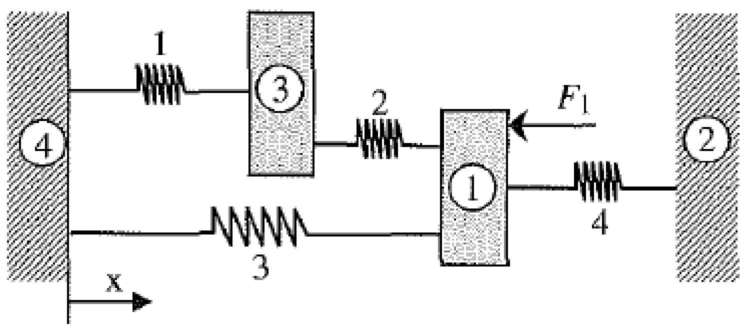


Figure P1.4

5. For the spring assemblage shown in **Figure P1.5**, obtain **(a)** the global stiffness matrix, **(b)** the displacements of nodes *2*, *3* and *4*, **(c)** the global nodal forces, and **(d)** the local external forces. Node *1* is fixed while node *5* is given a fixed known displacement δ = 20 mm. The spring constants are all equal to k = 200 kN/m.

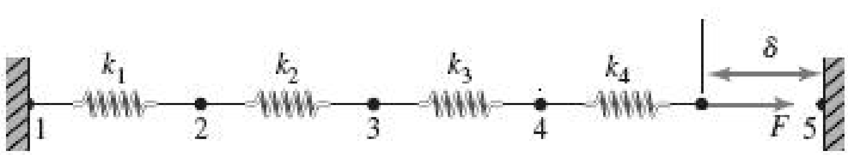


Figure P1.5

6. The stepped bar shown in the **Figure P1.6** is subjected to a force at the center. Use FEM to determine the displacement at the center and reactions. Assume: E = 100 GPa, area of cross sections of the three portions shown are, respectively, 10^{-4}m^2, 2 × 10^{-4}m^2, and 10^{-4}m^2, and F = 10,000N.

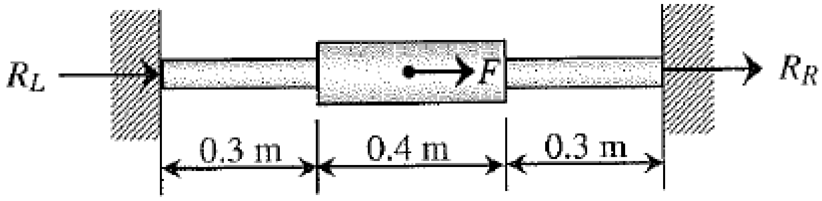


Figure P1.6

7. Determine the support reaction forces at the two ends of the bar shown in **Figure P1.7**, given the following, $P = 6.0 \times 10^4$ N, $E = 2.0 \times 10^4$ N/mm^2, $A = 250$ mm^2, $L = 150$ mm, $\Delta = 1.2$ mm.

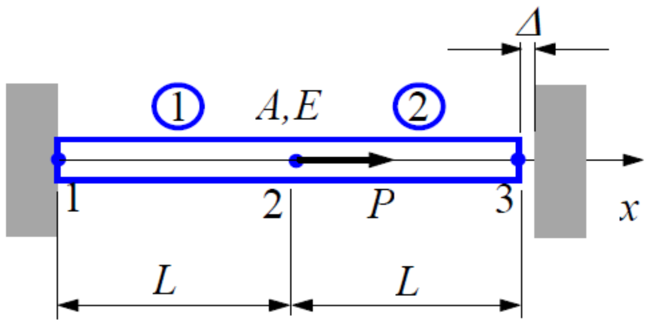


Figure P1.7

8. Determine the field values (nodal displacements and internal forces) for the following structural system shown in **Figure P1.8**. Take F=50,000N, $A_a = 1.5\times10^{-4}$m^2, $A_b = 1.0\times10^{-4}$m^2, $E_a = 203\times10^9$Pa, $E_b = 69\times10^9$Pa, $L_a = L_b = 1$ m.

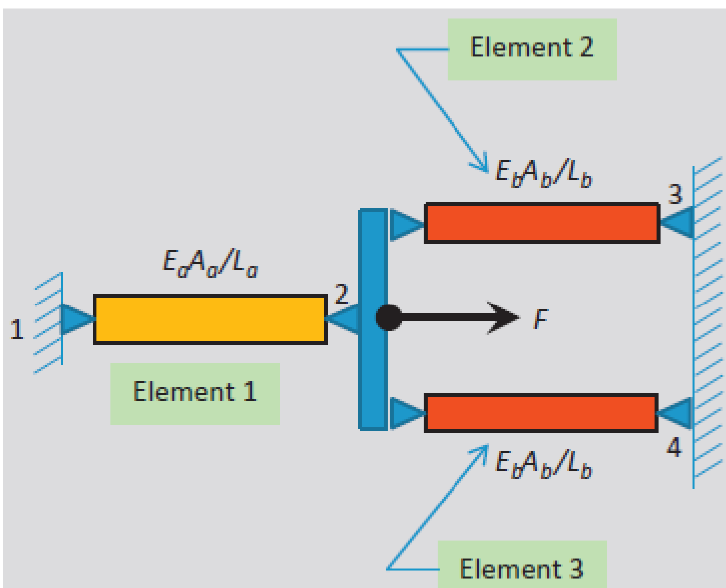


Figure P1.8

9. Derive the finite element equations for a torsion element and analyze the shaft shown in **Figure P1.9**.

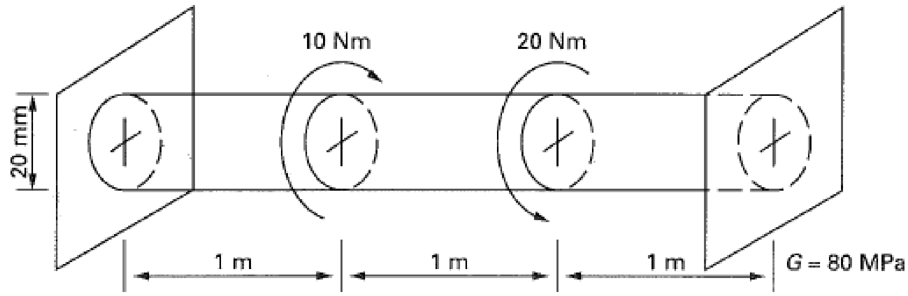


Figure P1.9

10. Consider a tapered bar of circular cross-section shown in **Figure P.10**. The length of the bar is 1 m, and the radius varies as r(x) = 0.050 - 0.040x:, where r and x are in meters. Assume Young's modulus = 100MPa. Both ends of the bar are fixed, and F = 10,000 N is applied at the center. Determine the displacements, axial force distribution, and wall reactions using four elements of equal length.

To approximate the area of cross-section of a bar element, use the geometric mean of the end areas of the element.

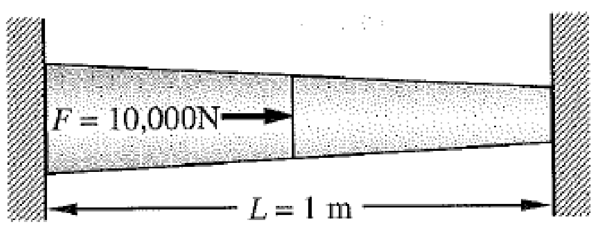


Figure P.10

MATLAB Programming

11. Determine the field values (nodal displacements and internal forces) for the following structural system shown in **Figure P1.11**. Take K = 40,000 N/m, F = 5000 N.

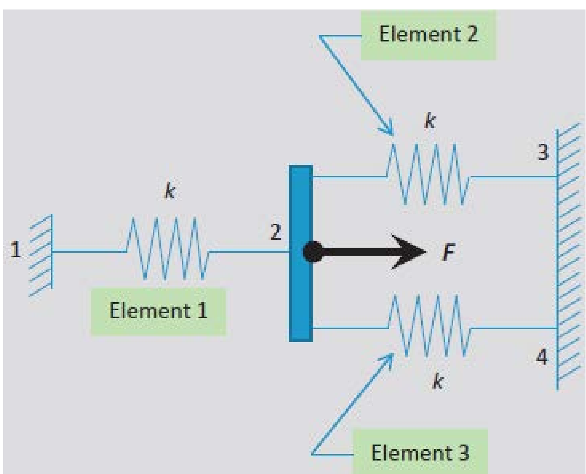


Figure P1.11

12. Find the displacements of the rigid bodies in **Figure P1.12**. Assume that the only non-zero forces is F_3 = 1000 N. Determine the element forces in the springs. What are the reactions at the walls? Assume the bodies can undergo only translation in the horizontal direction. The spring constants are k_1 = 500 N/mm, k_2 = 400 N/mm, k_3 = 600 N/mm, k_4 = 200 N/mm, k_5 = 400 N/mm and k_6 = 300 N/mm.

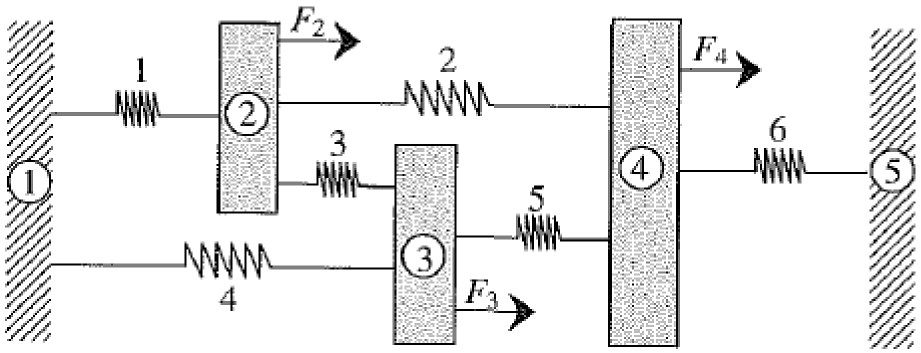


Figure P1.12

13. Determine the field values (nodal displacements and internal forces) for the following structural system shown in **Figure P1.13**. Take k_1 = 1300 N/m, k_2 = 1400 N/m, k_3 = 1500 N/m, k_4 = 1600 N/m, k_5 = 1700 N/m, F = 8000 N.

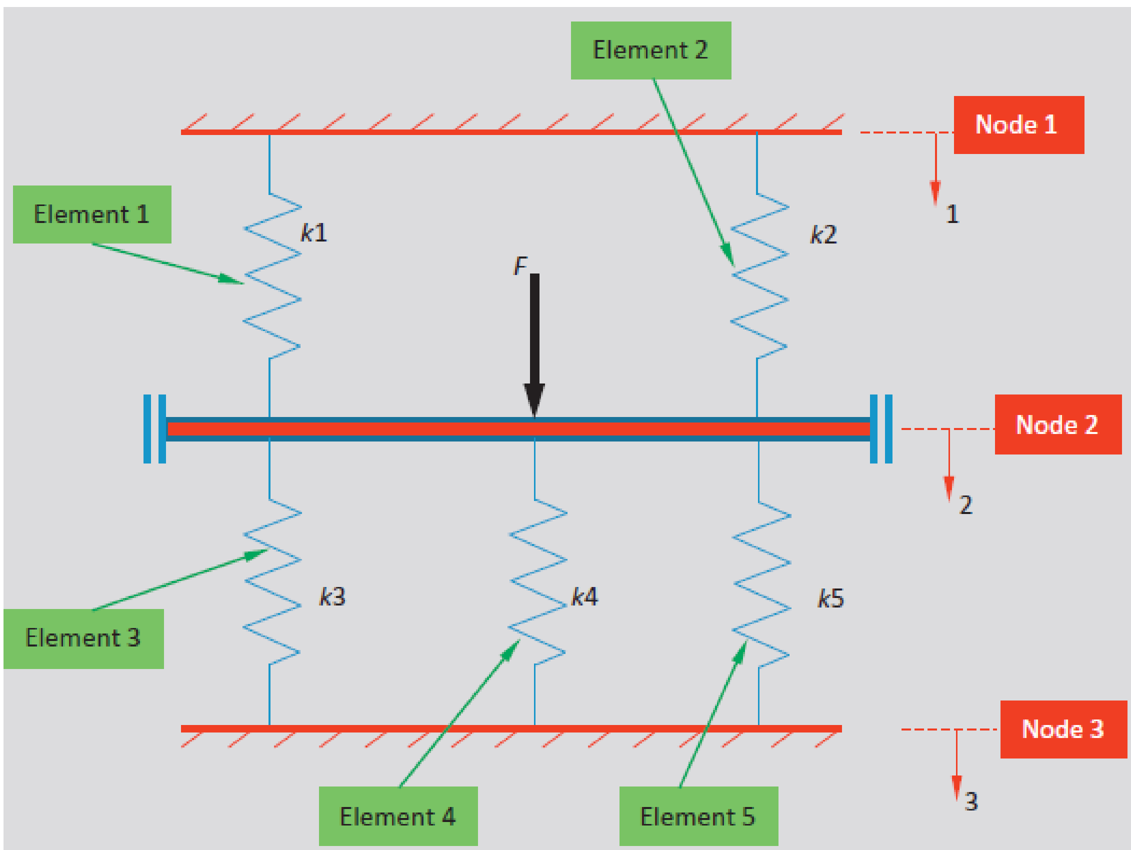


Figure P1.13

14. Model a tapered member of length L circular cross section as stepped shaft with several steps as shown in **Figure P1.14** and develop a discrete element methodology to evaluate the free end elongation. Show how the elongation converges to the exact value as the number of elements increase. Show a graph of the elongation value versus number of elements. D_1 and D_2 are diameter. Take P = 10 kN, E = 200 GPa, D_1 = 10cm, D_2 = 5 cm, L= 50 cm.

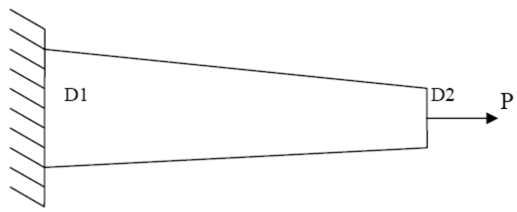


Figure P1.14

EXERCISE 2

DISCRETE ELEMENTS: PLANE AND SPACE TRUSS

1. For each assembly of the bar elements shown in **Figure P2.1**, determine the global stiffness matrix.

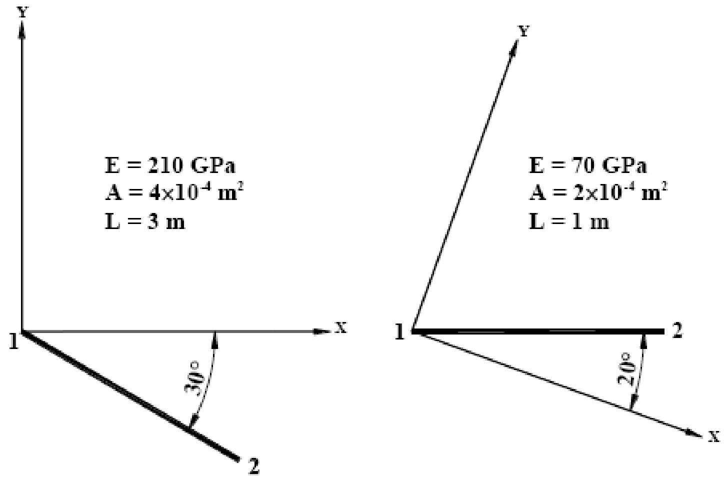


Figure P2.1

2. For the bar shwon in **Figure P2.2**, determine the axial stress. Take A = 10 × 10^{-4} m^2, E = 210 GPa, and L = 3 m. Assume the global displacements have been previously determined to be $\{0mm\ \ 2.5mm\ \ 5.0mm\ \ 3.0mm\}^T$.

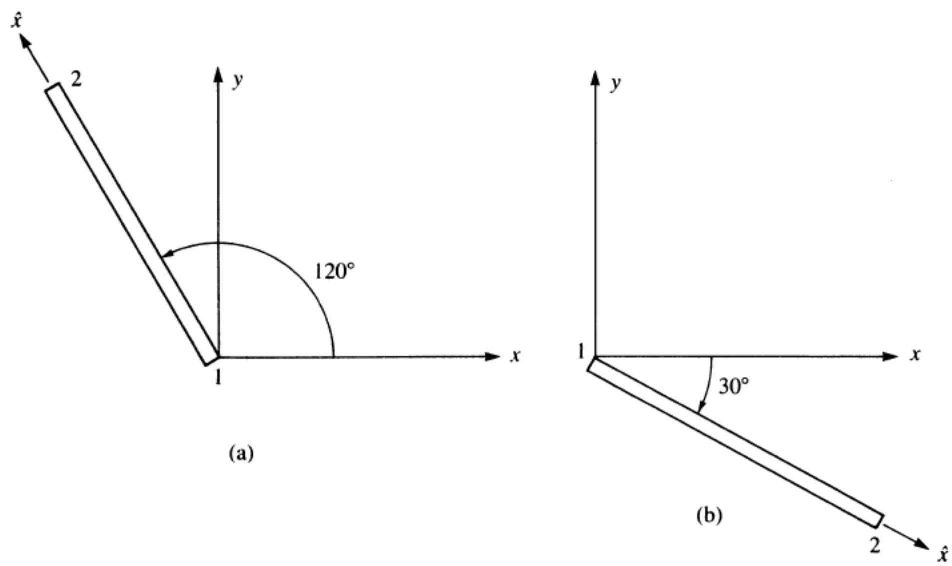


Figure P2.2

3. The plane truss shown in the **Figure P2.3** has two elements and three nodes. What are nodal displacements and the element forces? Assume E = 10^{11} Pa, A = 10^{-4}m^2, L = 1 m, F = 14,142 N.

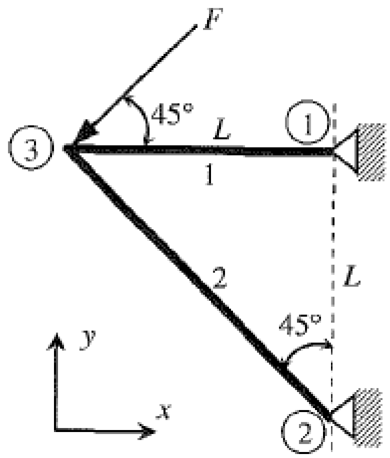
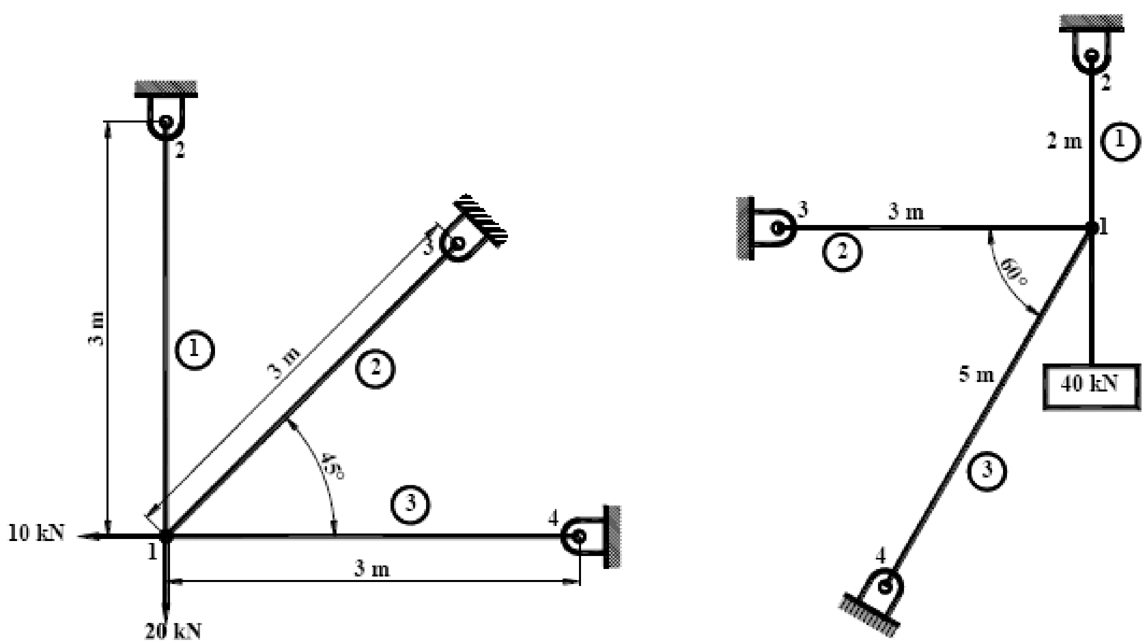


Figure P2.3

4. For the plane trusses shown in **Figure P2.4**, determine the horizontal and vertical displacements of node 1 and the stresses in each element. All the elements have E = 210 GPa and A = 10 × 10^{-4} m^2.



(a) (b)

Figure P2.4

5. For the two bar truss shown in **Figure P2.5**, determine the horizontal displacement of node 1 and the axial force in each element. A force of P = 1000 kN is applied at node 1 while node 1 settles an amount δ = 50 mm as shown. Let E = 210 GPa and A = 6 × 10^{-4} m^2 for each element.

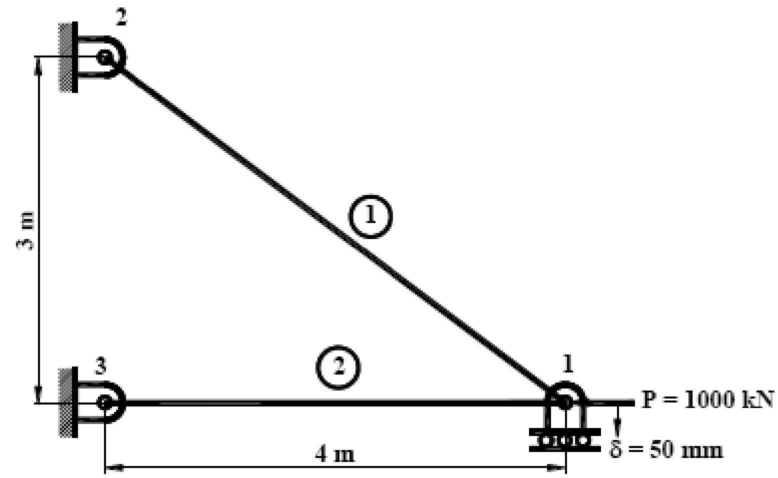
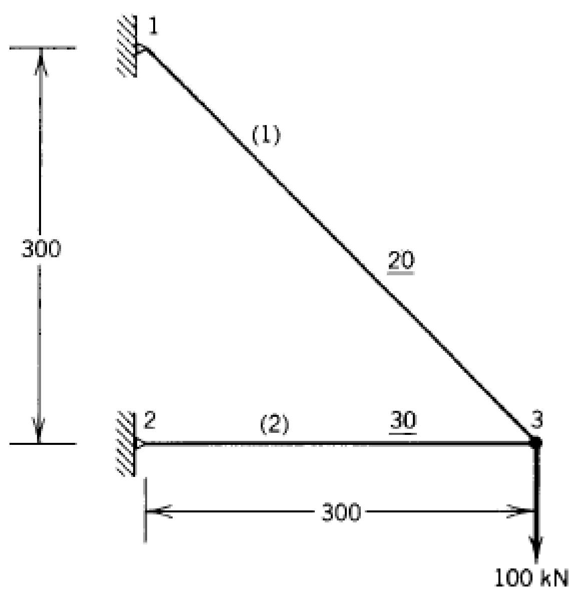
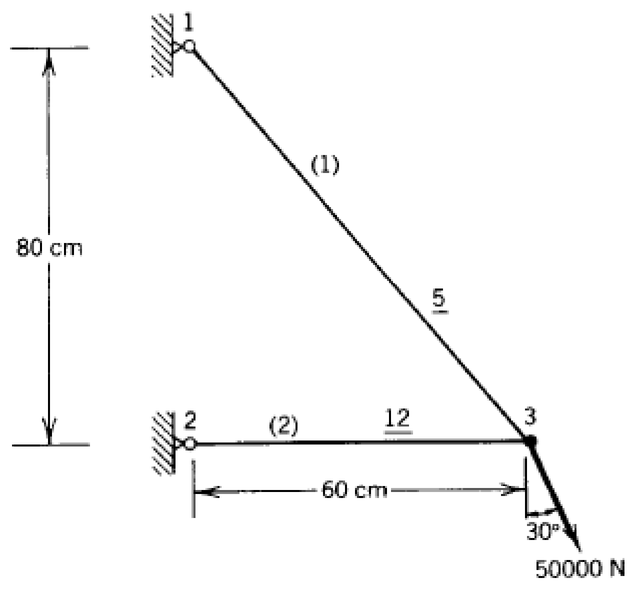


Figure P2.5

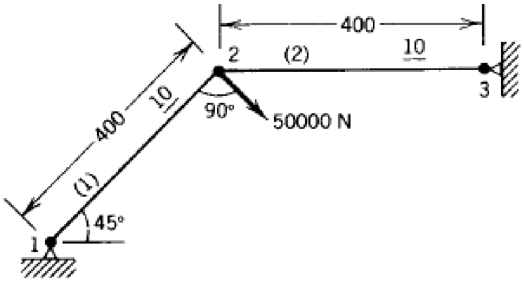
6. Each joint of the structural system shown in **Figure P2.6** is a pinned joint. The cross-sectional are of each member in cm^2 is underlined. Each member is made of steel, E = 20 × 10^6 N/cm^2. All lengths are given in centimeters. Calculate the unknown nodal displacements and the axial force in each member.



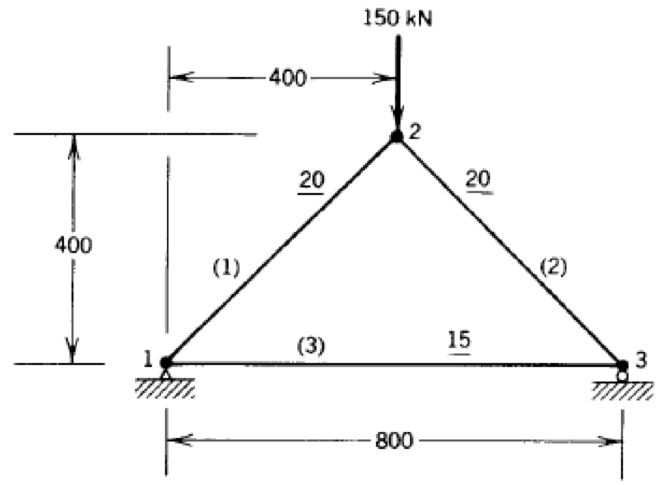
(a)



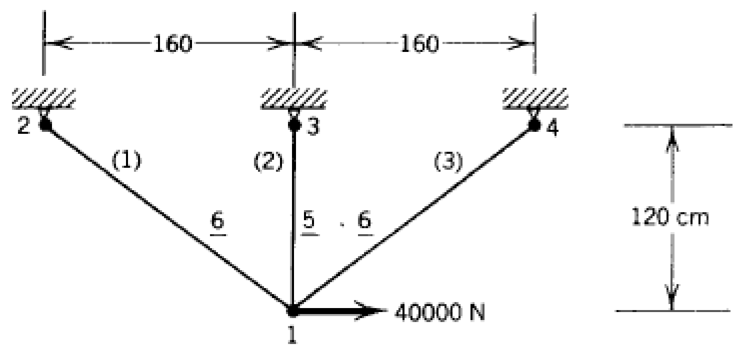
(b)



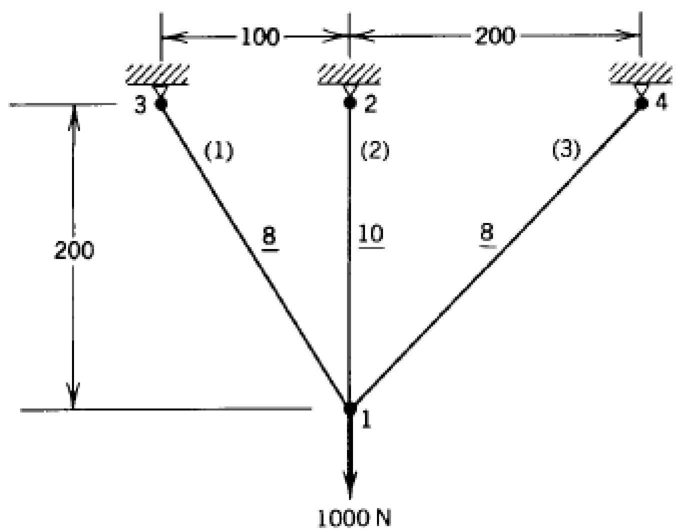
(c)



(d)



(e)



(f)

Figure P2.6

7. For the plane trusses supported by the spring at node 1, as shown in **Figure P2.7**, determine the nodal displacements and the stresses in each element. Let E = 210 GPa and A = 6×10^{-4} m^2 for both truss elements.

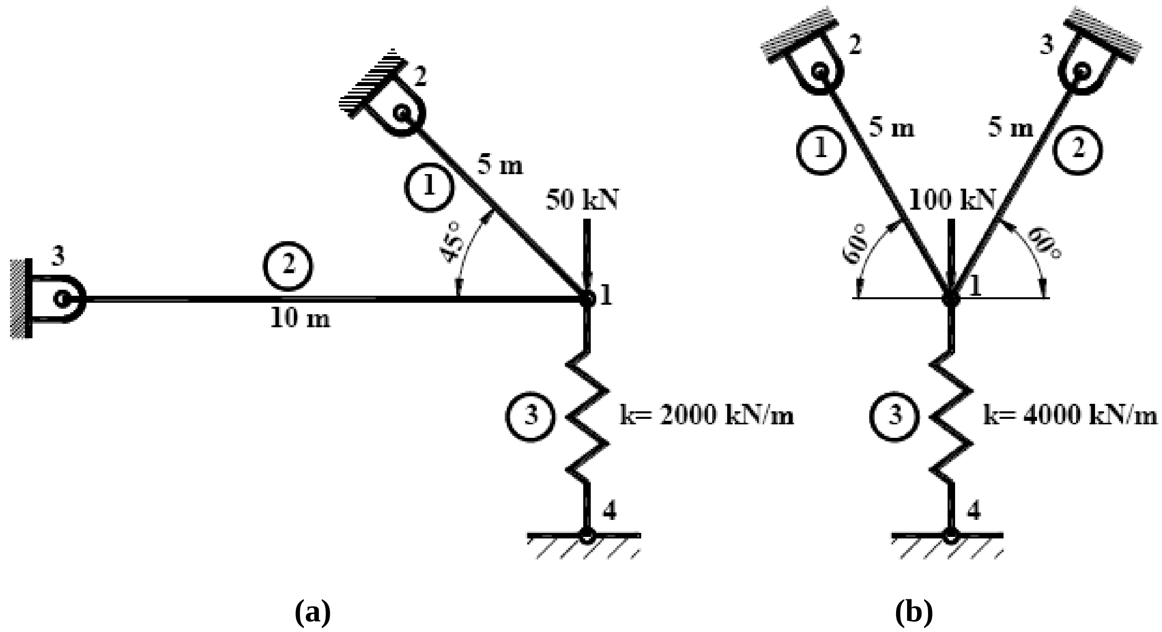


Figure P2.7

8. The plane truss shown in **Figure P2.8** is composed of members having a square 15 mm × 15 mm cross section and modulus of elasticity E = 69 GPa. Determine the nodal displacements and the stresses in each element.

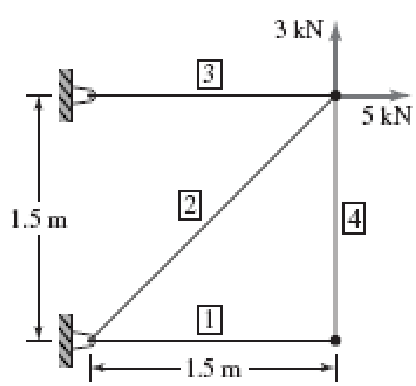
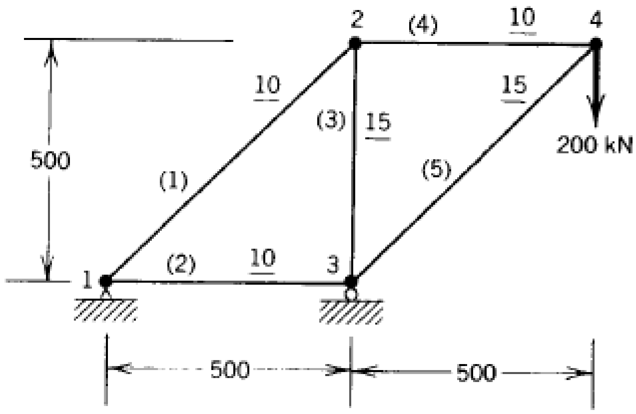


Figure P2.8

9. The displacements for the truss shown in **Figure P2.9** are given below. The cross-sectional are of each member in cm^2 is underlined. Each member is made of steel, E= 20×10^6 N/cm^2. All lengths are given in centimeters. Calculate the axial force in each member.



$U_1 = 0.0$	$U_2 = 0.0$
$U_3 = 1.748$	$U_4 = -0.3333$
$U_5 = -0.5000$	$U_6 = 0.0$
$U_7 = 2.248$	$U_8 = -3.690$

Figure P2.9

10. For the space truss elements shown in **Figure P2.10**, the global displacements at node 2 have been determined to be u_2 = 5 mm, v_2 = 10 mm and w_2 = 15 mm. Determine the displacement along the local axis at node 2 of the elements. The coordinates shown in Figures are in meters.

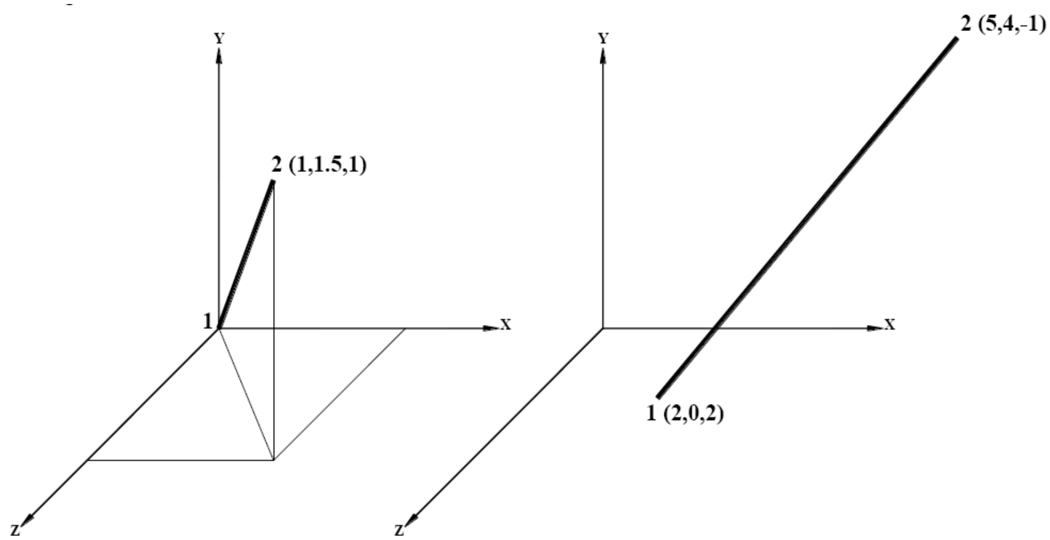


Figure P2.10

11. For the space truss shown in **Figure P2.11**, determine the displacement of node and stresses in each element. Let E = 210 GPa and A = 10 × 10^{-4} m^2 for all elements. A load of 20 kN is applied at node 1 in the global X direction.

12. Determine the normal stress in each member of the truss structure shown in **Figure P2.12**. All joints are ball-joint and the material is steel, whose Young's modulus is E = 210 GPa.

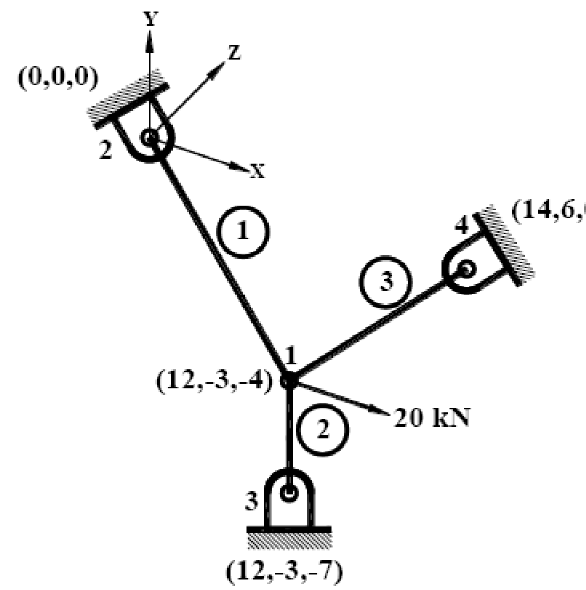


Figure P2.11

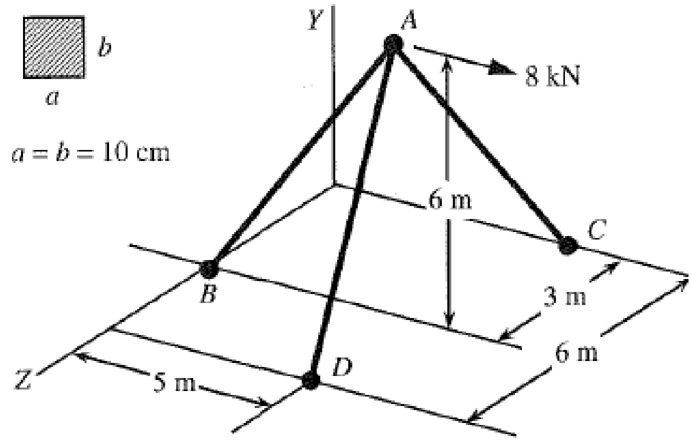


Figure P2.12

Programming

13. Find the nodal displacements developed in the planar truss shown in **Figure P2.13** when a vertically downward load of 1000 N is applied at node 4. The pertinent data are given in Table. Also find the stresses developed in each element.

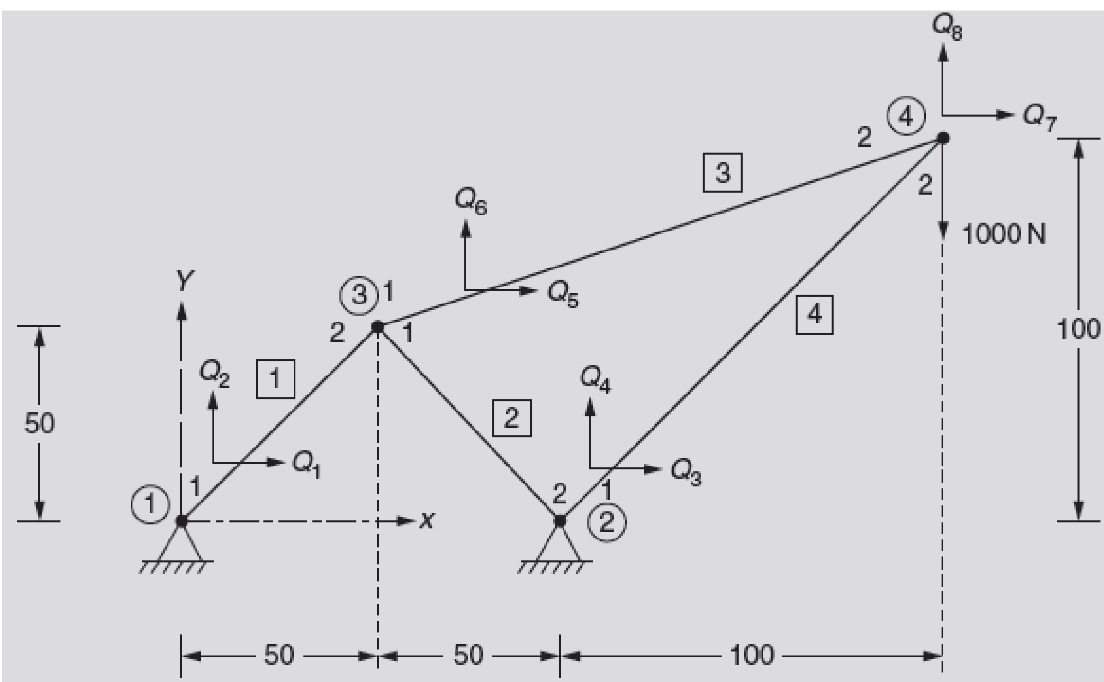


Figure P2.13

TABLE 9.1 Data of Members of Truss

Member Number "e"	Cross-sectional Area $A^{(e)}$ cm^2	Length $l^{(e)}$ cm	Young's Modulus $E^{(e)}$ N/cm^2
1	2.0	$\sqrt{2}$ 50	2×10^6
2	2.0	$\sqrt{2}$ 50	2×10^6
3	1.0	$\sqrt{2.5}$ 100	2×10^6
4	1.0	$\sqrt{2}$ 100	2×10^6

14. Find the stresses of the truss structure shown in **Figure P2.14**. All members have elastic modulus of 200 GPa and cross-sectional area of 2.5×10^{-3} m^2.

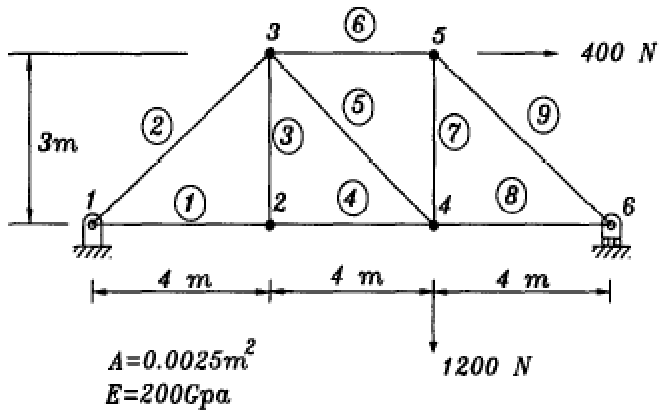


Figure P2.14

15. Each joint of the structural system shown in **Figure P2.15** is a pinned joint. The cross-sectional are of each member in cm^2 is underlined. Each member is made of steel, E = 20 × 10^6 N/cm^2. All lengths are given in centimeters. Calculate the unknown nodal displacements and the axial force in each member.

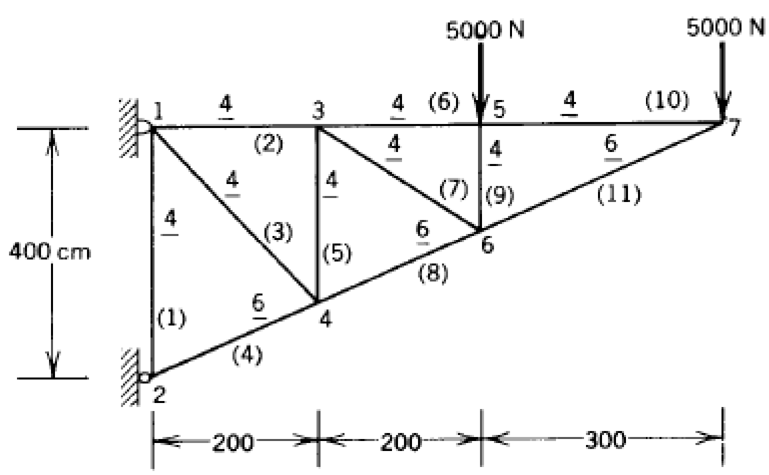


Figure P2.15

16. For the space truss shown in **Figure P2.16**, determine the nodal displacements and the stresses in each element. Let E = 210 GPa, A= 10 × 10^{-4} m^2 for all elements. All supports are ball and socket joints.

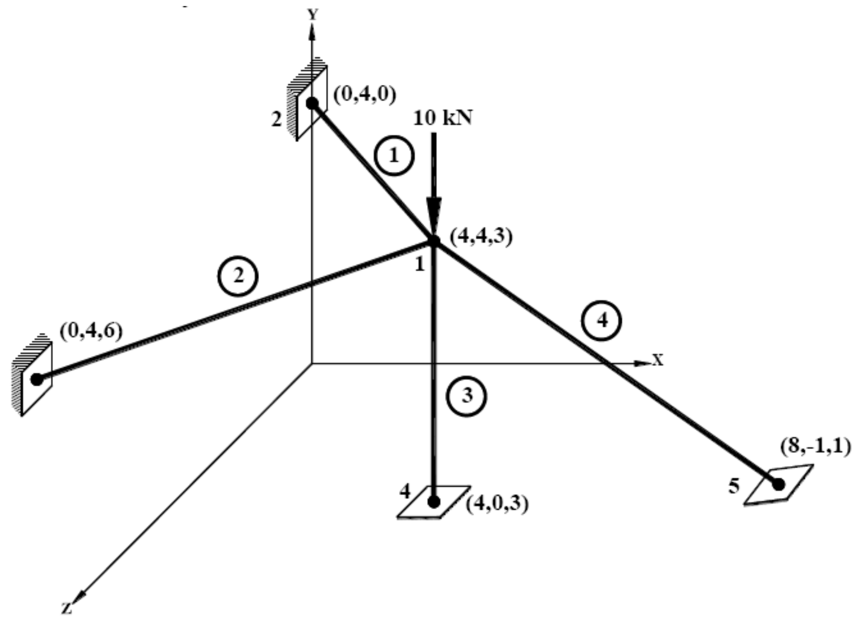


Figure P2.16

17. For the space truss shown in **Figure P2.17,** determine the deflection of upper joint, the stress in each member and the reaction forces.

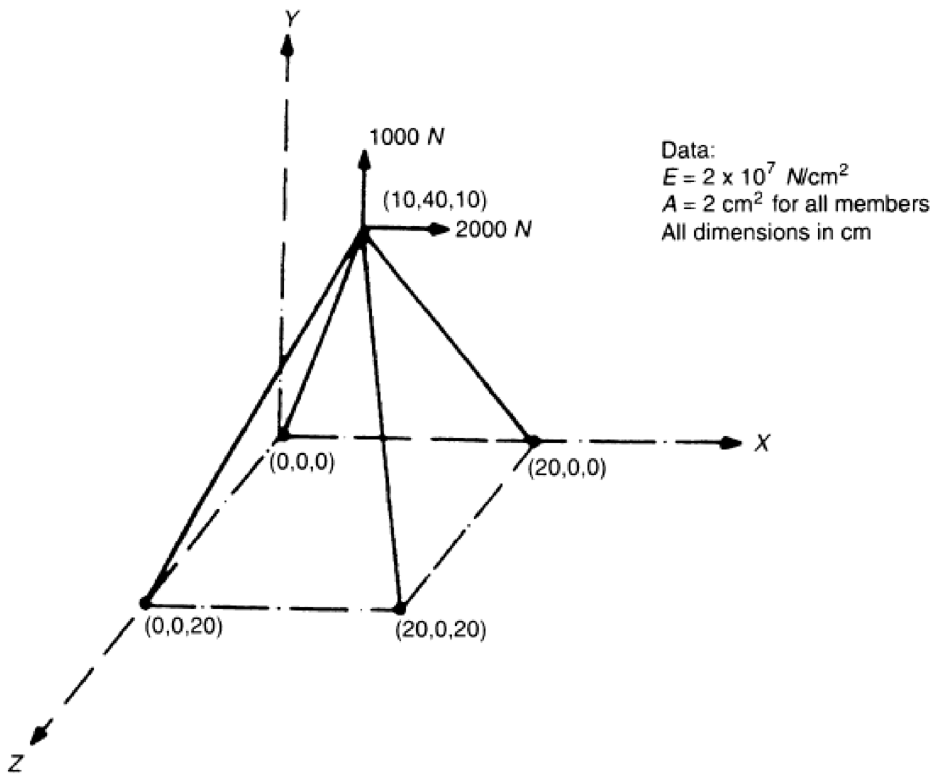


Figure P2.17

18. The space truss shown in **Figure P2.18** has four members. Determine the displacement components of node A and the force in each member. Assume AE = 10^6 N.

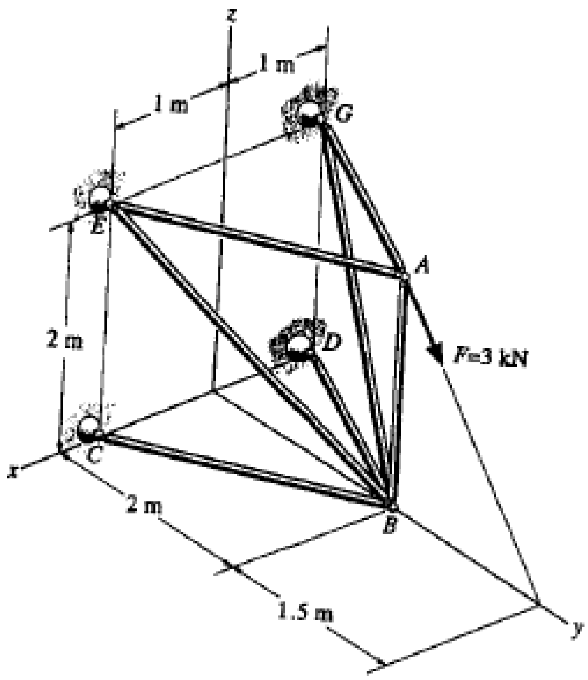


Figure P2.18

EXERCISE 3

DISCRETE ELEMENTS: BEAM AND FRAME

1. For a cantilever beam subjected to a vertical force at its free end, how many elements should be used to obtain the exact solution for the deflection of the beam?

2. Calculate the nodal displacements and the internal member forces for each of the beam loadings shown **Figures P3.2 (a-l)**. Construct the shear force and bending moment diagram for each member. Take $E = 200$ GPa and $I = 0.8 \times 10^{-4}$ m^4.

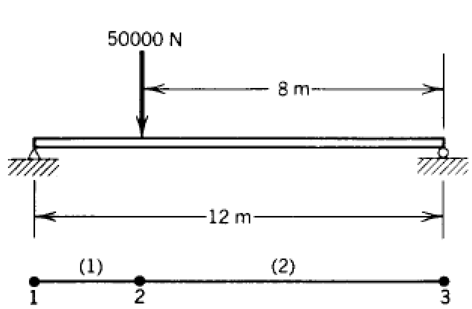


Figure P3.2 (a)

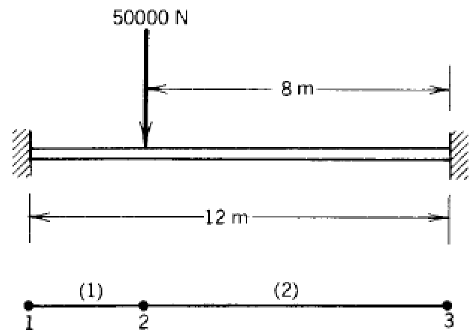


Figure P3.2 (b)

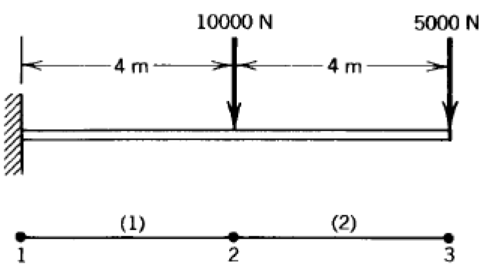


Figure P3.2 (c)

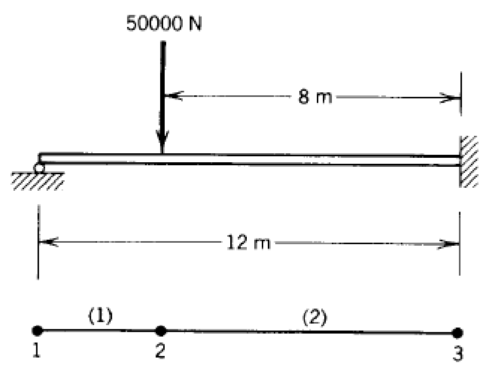


Figure P3.2 (d)

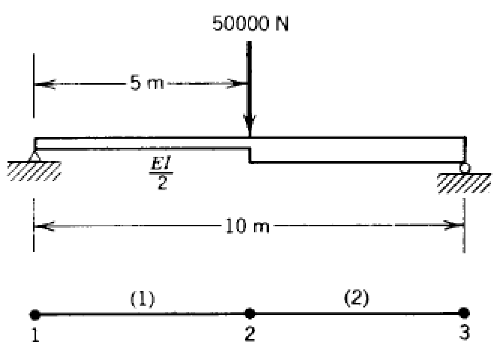


Figure P3.2 (e)

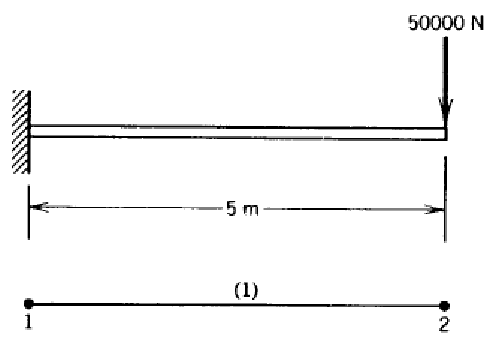


Figure P3.2 (f)

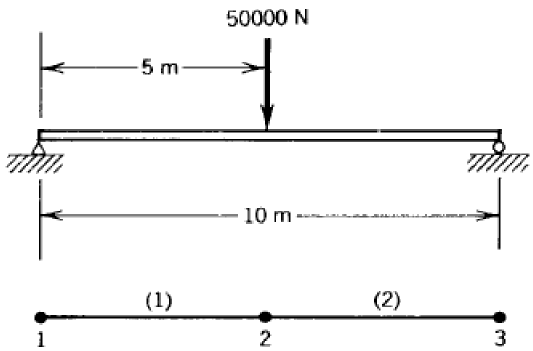


Figure P3.2 (g)

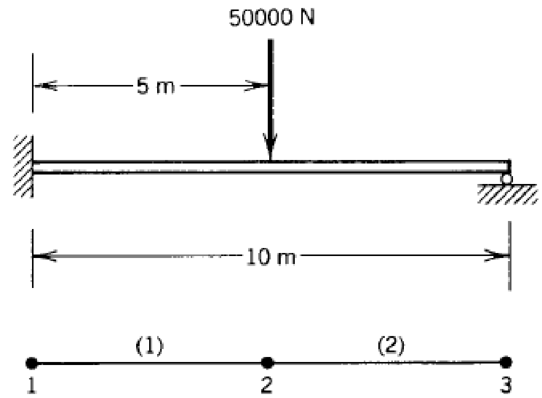


Figure P3.2 (h)

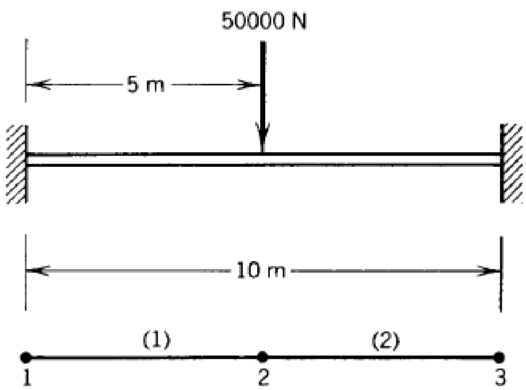


Figure P3.2 (i)

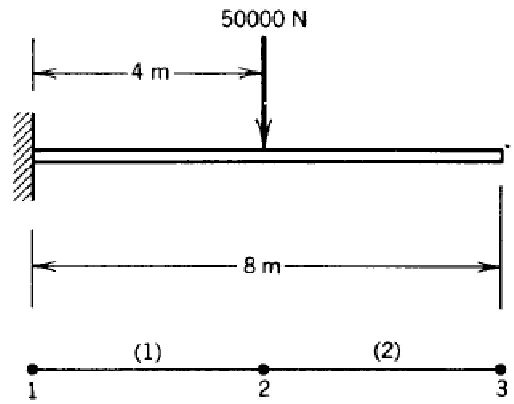


Figure P3.2 (j)

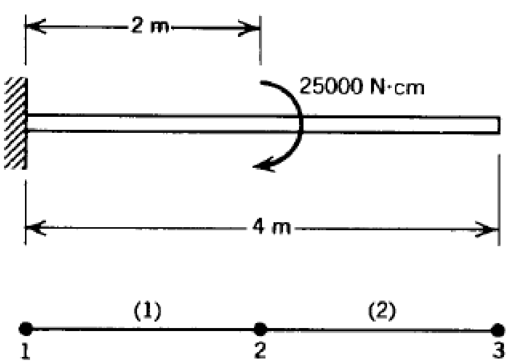


Figure P3.2 (k)

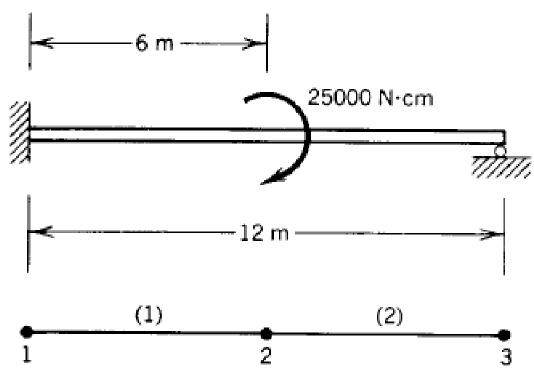


Figure P3.2 (l)

3. Determine the nodal displacements and rotations and the global and element forces for the beam shown in **Figure P3.3**. Take $E = 210$ GPa and $I = 2 \times 10^{-4}$ m^4.

4. For the beam shown in **Figure P3.4**, determine the displacements and the slopes at the nodes, the forces in each element, and the reactions. Take E = 210 GPa and I= 4×10^{-4} m^4.

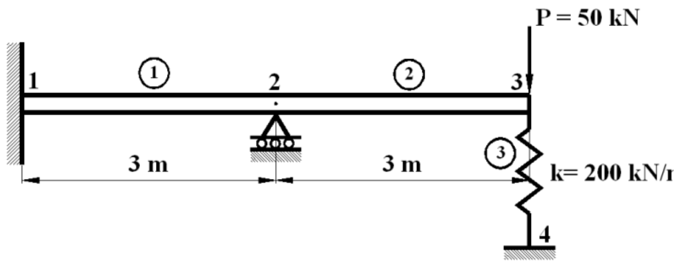


Figure P3.3

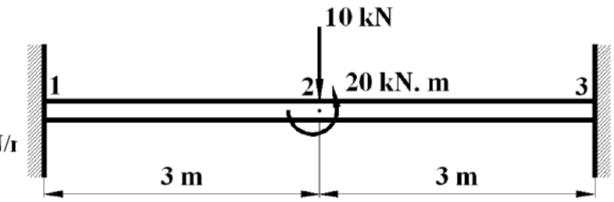


Figure P3.4

5. The bar element 2 is used to stiffen the cantilever beam element 1, as shown in **Figure P3.5**. Determine the displacements at node 1 and the element forces. For the bar, let $A = 1 \times 10^{-3}\ \text{m}^2$. For the beam, let $A = 2 \times 10^{-3}\ \text{m}^2, I = 5 \times 10^{-5}\ \text{m}^4$, and $L = 3\ m$. For both the bar and the beam elements, let $E = 210$ GPa.

6. Assemble element stiffness matrix for the member of plane frame shown in **Figure P3.6**, if it is oriented at angle 30^{o} to the x-axis. Take E = 200 GPa, $I = 4 \times 10^{-6}\ \text{m}^4$ and $A = 4 \times 10^{-3}\ \text{m}^2$.

7. Calculate the displacement at the free end of the inclined beam shown in **Figure P3.7**.Repeat the problem when the applied load is parallel to the *x*- axis and has a magnitude of 15000 N.

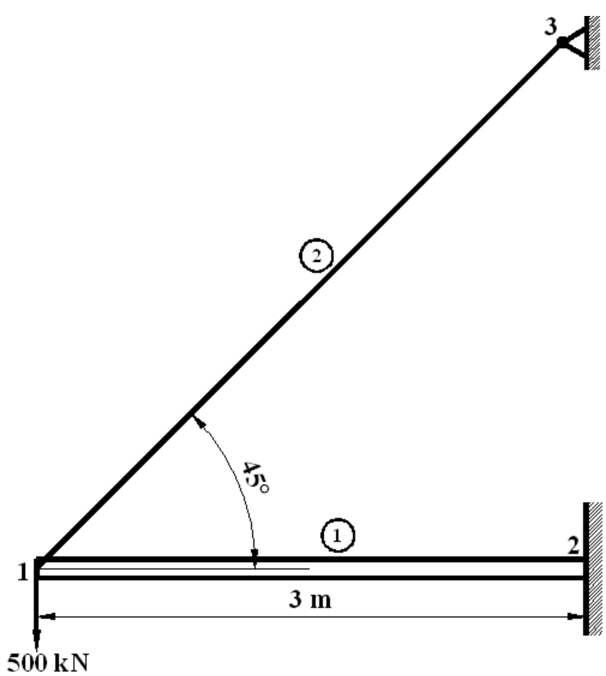


Figure P3.5

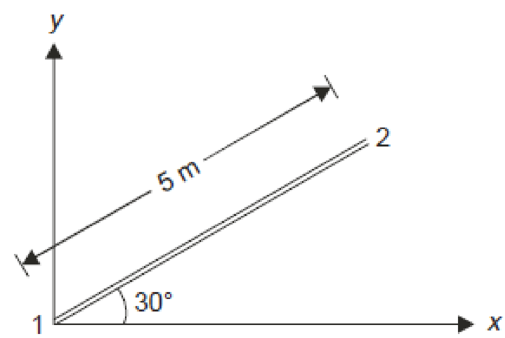


Figure P3.6

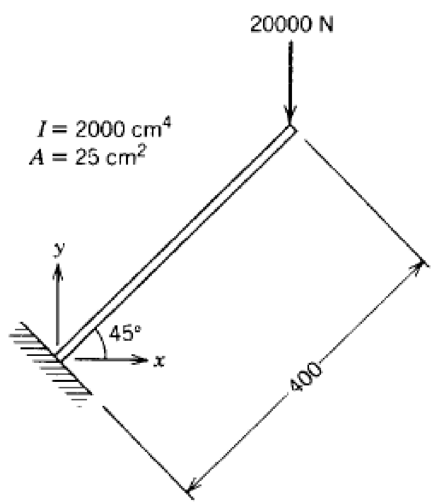


Figure P3.7

8. Determine the stress distribution in the two members of the frame shown in **Figure P3.8**. Use one finite element for each member of the frame.

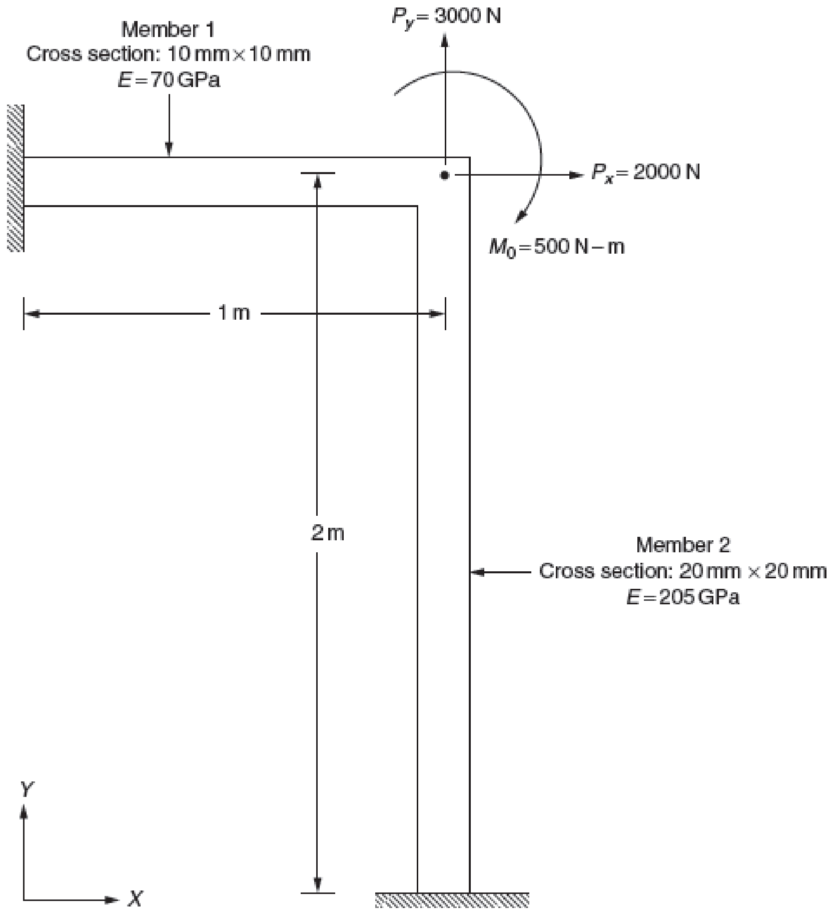


Figure P3.8

9. For the rigid frame shown in **Figure P3.9**, determine the displacements and rotations of the nodes, the element forces, and the reactions. The values of E, A and I to be used are listed next to each figure.

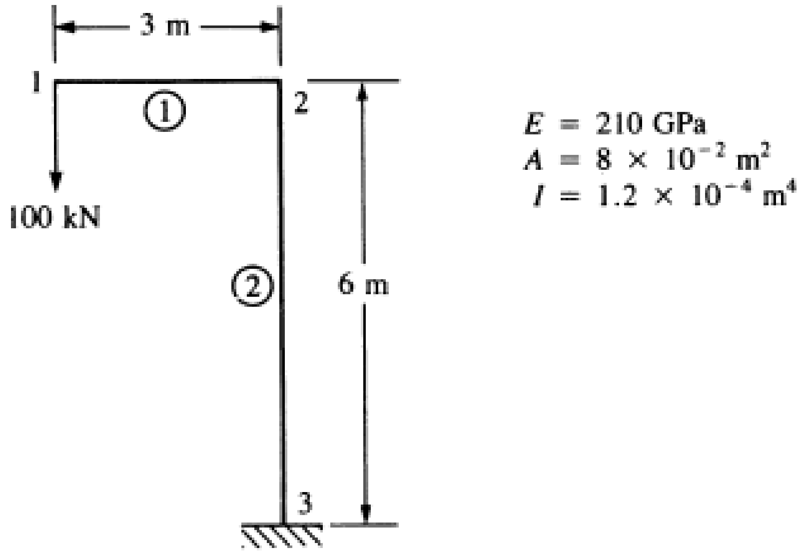


Figure P3.9

Programming

10. For the beam shown in **Figure P3.10**, determine the displacements and the slopes at the nodes, the forces in each element, and the reactions. Also, draw the shear force and bending moment diagrams.

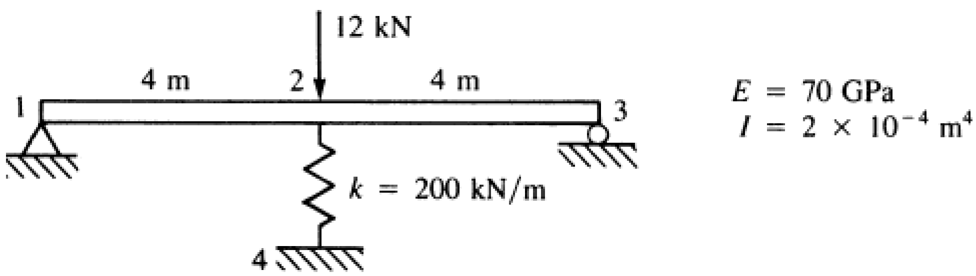


Figure P3.10

11. For the beam shown in **Figure P3.11**, determine the displacements and the slopes at the nodes, the forces in each element, and the reactions. Also, draw the shear force and bending moment diagrams.

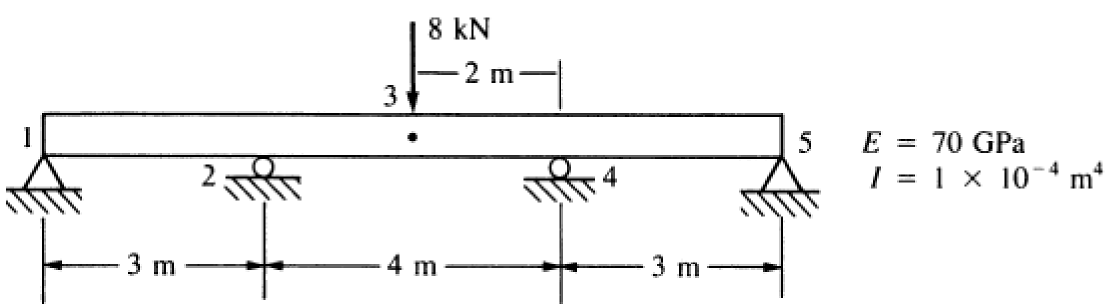


Figure P3.11

12. For the rigid frame shown in **Figure P3.12**, determine the displacements and rotations of the nodes, the element forces, and the reactions. Take E = 210 GPa, A = 1×10^{-2} m^2 and I = 2×10^{-4} m^4.

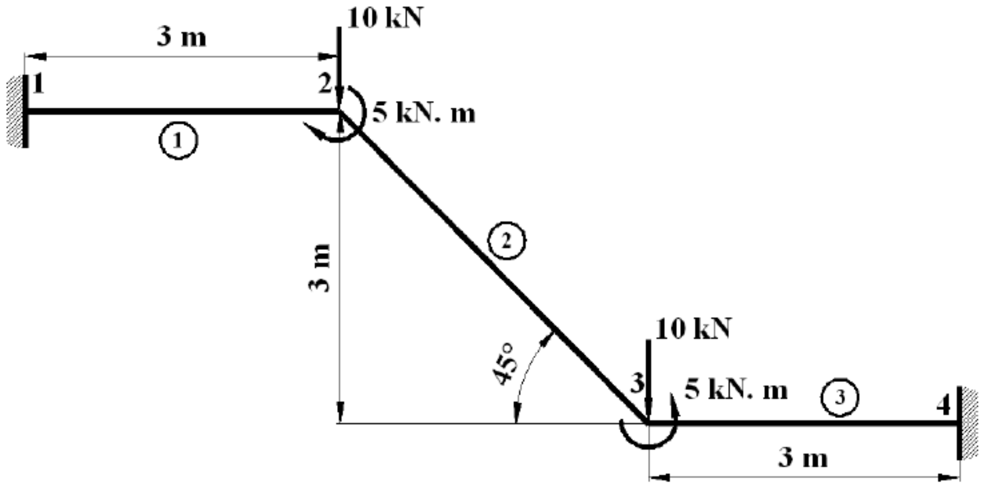


Figure P3.12

13. For the rigid frames shown in **Figure P3.13**, determine the displacements and rotations of the nodes, the element forces, and the reactions. Take $E = 210$ GPa, $A = 1 \times 10^{-2}$ m^2 and $I = 1 \times 10^{-4}$ m^4.

14. Analyze the grid shown in **Figure P3.14**. The grid consists of two elements, is fixed at nodes 1 and 3, and is subjected to a downward vertical load of 22 kN. The global coordinates axes and element length are shown in the figure. Let $E = 210$ GPa, $G = 84$ GPa, $I = 16.6 \times 10^{-5}$ m^4 and $J = 4.6 \times 10^{-5}$ m^4.

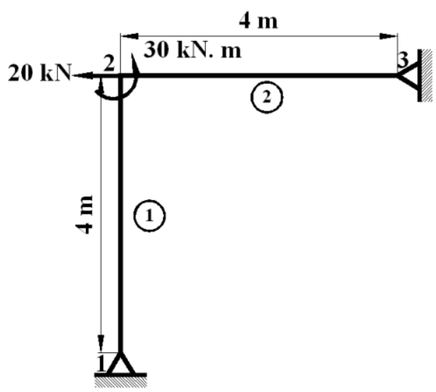


Figure P3.13

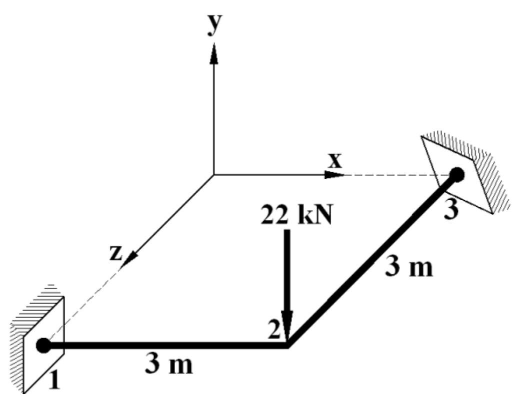


Figure P3.14

EXERCISE 4

CONTINUUM PROBLEMS

1. Find the governing differential equation of a string under tension T subjected to lateral load q for which potential energy is given by

$$\Pi = \frac{T}{2}\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx - \int_0^L q\phi dx$$

with boundary conditions $\phi(0) = \phi(L) = 0$.

2. A 1-Dimensional beam of length , is subjected to a lateral distributed load of intensity q per unit length. The governing differential equation is given by

$$EI\frac{d^4w}{dx^4} + q = 0$$

Derive the variational form of the problem.

3. Obtain an approximate displacement equation (two terms) for the simply supported shown in **Figure P4.3**. Compare the deflection at the center with the theoretical value $y = PH^3/48EI$. The governing differential equation is

$$EI\frac{d^2y}{dx^2} - \frac{Px}{2} = 0 \quad 0 \le x \le \frac{H}{2}$$

$$EI\frac{d^2y}{dx^2} - \frac{P(H-x)}{2} = 0 \quad \frac{H}{2} \le x \le H$$

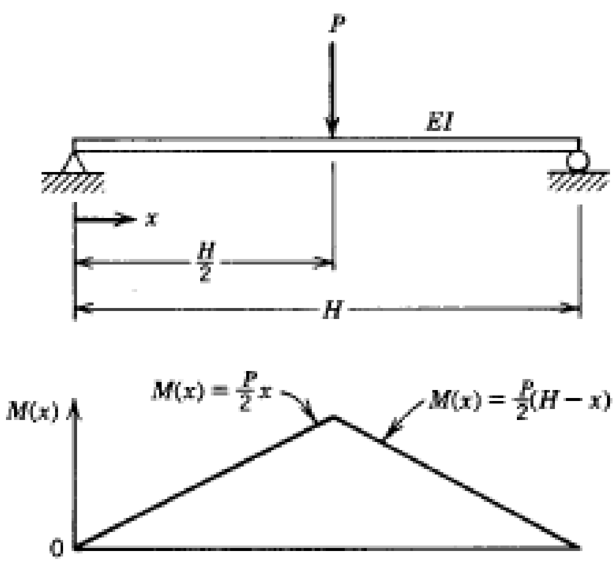


Figure P4.3

Use **(a)** the Galerkin method, (**b**) the Petrov-Galerkin method, (**c**) the least squares method and (**d**) the point collocation method.

4. Derive the variational form of the **Problem 3** and determine two terms approximate solution by using Ritz method.

5. Using Ragleigh–Ritz method determine the expressions for deflection and bending moments in a simply supported beam subjected to uniformly distributed load over entire span as shown in **Figure P4.5**. Find the deflection and moment at midspan and compare with exact solutions.

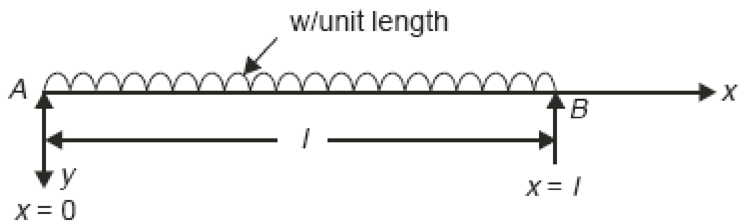


Figure P4.5

6. Find two parameter approximate solution of the equation

$$\frac{d^2\phi}{dx^2} + 3\frac{d\phi}{dx} + 2\phi = 4x \qquad 0 \le x \le 1$$

subjected to boundary conditions

(i) $\phi(0) = 0, \quad \phi(1) = 20$

(ii) $\phi(0) = 5, \quad \phi'(1) = 2$

(iii) $\phi'(0) = 2, \quad \phi'(1) = 20$

Use **(a)** the Galerkin method, (**b**) the Petrov-Galerkin method, (**c**) the least squares method and (**d**) the point collocation method. Compare solution of each case with exact solution.

7. Find two parameter approximate solution of the equation

$$\frac{d^2\phi}{dx^2} - 6\frac{d\phi}{dx} + 5\phi = x^2 \qquad 0 \le x \le 1$$

subjected to boundary conditions $\phi(0) = 0$, $\phi'(1) = 1$.

Use **(a)** the Galerkin method, (**b**) the Petrov-Galerkin method, (**c**) the least squares method and (**d**) the point collocation method. Compare solution of each case with exact solution.

8. Consider the following boundary value problem:

$$\frac{d^2\phi}{dx^2}\sin(x) + \frac{d\phi}{dx}\cos(x) + \phi\sin(x) = 0 \qquad \pi/4 < x < \pi/2$$

$\phi(\pi/4) = 1 \; \phi'(\pi/2) = 2$

Using the Galerkin weighted residual method, obtain two term approximate solution of the boundary value problem.

9. Consider the following boundary value problem:

$$\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} - 2\phi = 1 \qquad\qquad 0 < x < 1$$

$$\phi'(0) - \phi(0) = 1 \text{ and } \phi'(1) = 2$$

Using the Galerkin weighted residual method, obtain two term approximate solution of the boundary value problem.

10. Consider the following boundary value problem:

$$\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} - \phi = 1 \qquad\qquad 0 < x < 1$$

$$\phi'(0) - \phi(0) = 1 \text{ and } \phi'(1) = 1$$

Derive a suitable weak form for use with the Galerkin method. Obtain two term approximate solution of the problem.

11. The boundary value problem for a cantilevered beam can be written as

$$\frac{d^4w}{dx^4} - x = 0 \qquad\qquad 0 < x < 1$$

$$w(0) = w'(0) = 0, w''(1) = 1, w'''(1) = -1$$

Solve for the boundary value problem using the Galerkin method. Compare the approximate solution to the exact solution by plotting the solutions on a graph.

12. Find the three parameter Galerkin solution of the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2 \qquad \text{in } \Omega \qquad \Omega: -3 \le \mathrm{x} \le 3, -2 \le \mathrm{y} \le 2$$

subjected to boundary conditions $\phi = 0$ on Γ.

13. Solve the problem of steady state heat condition in a material of unit thermal conductivity and occupying the square region $: -1 \le \mathrm{x} \le 1, -1 \le \mathrm{y} \le 1$. The sides $y = \pm 1$ are maintained at zero temperature, while heat is supplied at the rate $\cos(\pi y/2)$ per unit length on the sides $x = \pm 1$.

14. Set up the equations for 2 parameters Ritz approximation of the following equations associated with a string under tension T subjected to lateral load q for which potential energy is given by

$$\Pi = \frac{T}{2}\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx - \int_0^L q\phi dx$$

with boundary conditions $\phi(0) = \phi(L) = 0$.

a) Use algebraic polynomials.
b) Use trigonometric functions.

MAPLE Application

15. Find one, two, three and four parameter approximate solution of the equation

$$2x\frac{d^2\phi}{dx^2} + 3\frac{d\phi}{dx} = 4 \qquad 1 < x < 2$$

$\phi'(1) = 1$ and $\phi(2) = 2$

Use **(i)** the Galerkin method, (**ii**) the Petrov-Galerkin method, (**iii**) the least squares method and (**iv**) the point collocation method.

- **(a)** Compare one, two, three and four parameter solution obtained by the Galerkin method with the exact solution.
- **(b)** Compare one, two, three and four parameter solution obtained by the Petrov-Galerkin method with the exact solution.
- **(c)** Compare one, two, three and four parameter solution obtained by the least squares method with the exact solution.
- **(d)** Compare one, two, three and four parameter solution obtained by the point collocation method with the exact solution.
- **(e)** Compare one term solution of the Galerkin method, the Petrov-Galerkin method, the least squares method and the point collocation method with the exact solution.
- **(f)** Compare two term solution of the Galerkin method, the Petrov-Galerkin method, the least squares method and the point collocation method with the exact solution.
- **(g)** Compare three term solution of the Galerkin method, the Petrov-Galerkin method, the least squares method and the point collocation method with the exact solution.
- **(h)** Compare four term solution of the Galerkin method, the Petrov-Galerkin method, the least squares method and the point collocation method with the exact solution.

16. Find the four parameter Galerkin solution of the equation for the deflection of a membrane subjected to uniform transverse load q and tension T

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{q}{T} \qquad \text{in } \Omega \qquad \Omega: -2 \le \mathrm{x} \le 2, -1 \le \mathrm{y} \le 1$$

$\phi = 0 \qquad$ on Γ.

- **a)** Use algebraic polynomials.
- **b)** Use trigonometric functions.

EXERCISE 5

INTERPOLATION FUNCTIONS FOR FINITE ELEMENT FORMULATION

1. The nodal temperatures of nodes *i* and *j* (same as local nodes 1 and 2) of an element in a one dimensional fin are known to be T_i = 120°C and T_j = 80°C with the *x*-coordinates x_i = 30 cm and x_j = 50 cm. Find the following:
   (a) Shape functions associated with the nodal values T_i and T_j.
   (b) Interpolation model for the temperature inside the element, *T(x)*.
   (c) Temperature in the element at x = 45 cm.

2. A one-dimensional tapered fin element has the nodal coordinates x_i = 20 mm and x_j = 60 mm with the area of cross section changing linearly from a value of A_i = 20 mm^2 at x_i to a value of A_j = 10 mm^2 at x_j as shown in **Figure P5.2**.
   (a) Determine the shape functions.
   (b) Express the area of cross section of the fin element in terms of the shape functions.

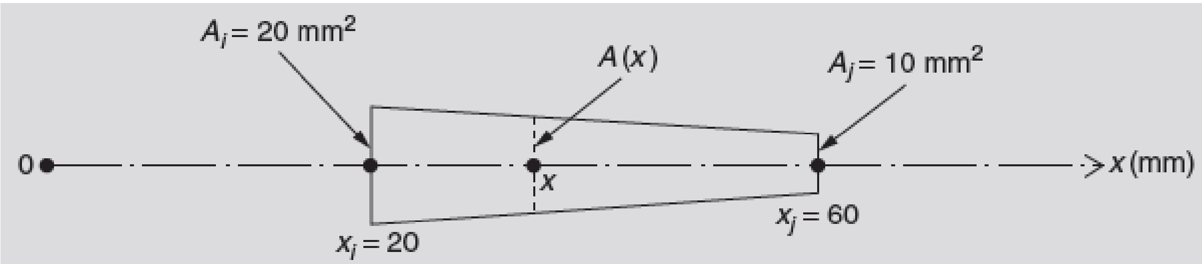


Figure P5.2

3. The nodal coordinates and nodal temperatures of a triangular simplex element follow:
   Node *i*: (x_i, y_i) = (10, 10) mm, T_i =100°C
   Node *j*: (x_j, y_j) = (20, 50) mm, T_j = 80°C
   Node *k*: (x_k, y_k) = (–10, 30) mm, T_k = 130°C

Express the temperature variation in the element, *T(x, y)*, in terms of the shape functions.

4. The nodal coordinates and its functional value of a triangular linear element is given below. Calculate the value at (20, 6).

Node	Coordinate (*x,y*)	Value
1	(13,1)	190
2	(25,6)	160
3	(13,13)	185

5. Two-dimensional triangular elements have been used to determine the stress distribution in a machine part. The nodal stresses and their corresponding positions for a triangular element

are shown in **Figure P5.5**. What is the value of stress at $x = 2.15$ cm and $y = 1.1$ cm? Plot the 8.0 GPa and 7.86 GPa stress contour lines for an element.

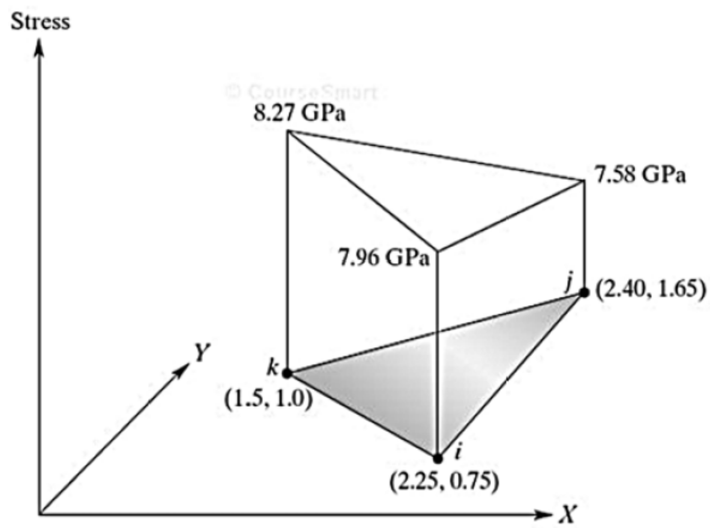


Figure P5.5

6. The (x, y) coordinates of the nodes i, j and k of a triangular element are (1,1), (4, 2) and (3, 5) respectively. The shape functions of a point ***P*** located inside the element are given by $N_1 = 0.15$ and $N_2 = 0.25$. Determine the x and y coordinates of the point ***P***.

7. The nodal coordinates of the triangular element are (1,2), (5,3) and (4,6). At the interior point ***P***, the x-coordinate is 3.3 and the shape function at node 1 is 0.3. Determine the shape functions at nodes 2 and 3 and also the y-coordinate at the point ***P***.

8. A bilinear rectangular element has coordinates as shown in **Figure P5.8** and the nodal temperatures are $T_1 = 100^0$C, $T_2 = 60^0$C, $T_3 = 50^0$C and $T_4 = 90^0$C. Compute the temperature at the point (2.5, 2.5). Also determine the 80^0C isotherm.

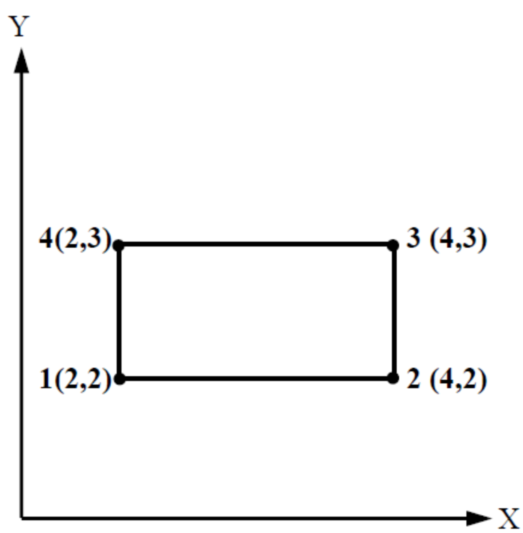


Figure P5.8

EXERCISE 6

ONE DIMENSIONAL APPLICATION

1. Solve the problem given below using Galerkin's method and piecewise linear functions. Use two elements.

$$\frac{d^2\phi}{dx^2} = 1 \qquad 0 < x < 2 \qquad \text{subjected to boundary conditions} \qquad \phi(0) = 0 \text{ and } \phi(2) = 0$$

2. Consider the following boundary value problem:

$$2\frac{d^2\phi}{dx^2} + 3\phi = 0 \qquad 0 < x < 2 \qquad \text{subjected to boundary conditions} \qquad \phi(0) = 1 \text{ and } \phi'(2) = 1$$

Using equal-length two finite elements, calculate unknown ϕ (x) and its derivative. Compare the finite element solution with the exact solution.

3. Consider the following boundary value problem

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) = \frac{2}{x^2} \qquad 1 < x < 2 \qquad \text{subjected to boundary conditions} \qquad \phi(1) = 2 \text{ and } \phi'(2) = -1/4$$

Calculate the approximate nodal values using two equal-length finite elements.

4. Use three elements of equal length, solve the differential equation given below.

$$\frac{d^2\phi}{dx^2} + x = 0 \qquad 0 < x < 1 \qquad \text{subjected to boundary conditions} \qquad \phi(0) = 0 \text{ and } \phi(1) = 0$$

5. A differential equation with boundary conditions is given below:

$$x^2\frac{d^2\phi}{dx^2} + 2x\frac{d\phi}{dx} + 2 = 0 \quad 1 < x < 4 \qquad \text{subjected to boundary conditions} \qquad \phi(1) = 1 \text{ and } \phi(4) = 0$$

   - **a)** Derive the weak formulation.
   - **b)** Compute element matrices and vectors for the given discretization using three linear elements.
   - **c)** Assemble them into the global matrix and vector.
   - **d)** Solve for the unknown nodal values.

6. Solve the differential equation for the mixed boundary conditions

$$\frac{d^2\phi}{dx^2} + \phi - x^2 = 0 \qquad 0 < x < 1 \qquad \text{subjected to boundary conditions} \qquad \phi(0) = 0 \text{ and } \phi'(1) = 1$$

   Use the uniform mesh of three linear elements.

7. Solve the differential equation for the natural boundary conditions

$$\frac{d^2\phi}{dx^2} + \cos\pi x = 0 \qquad 0 < x < 1 \qquad \text{subjected to boundary conditions} \qquad \phi'(0) = 1 \text{ and } \phi'(1) = 0$$

Use the uniform mesh of three linear elements.

8. A vertical rod shown in **Figure P6.8** of elastic material is fixed at both ends with constant cross-sectional area *A*, Young's modulus *E*, and height of *L* under the distributed load f per unit length. The vertical deflection *u(x)* of the rod is governed by the following differential equation

$$AE\frac{d^2u}{dx^2} + f = 0$$

Using three elements of equal length, solve for u(x) and compare it with the exact solution. Use the following numerical values: $A = 10^{-4}$ m^2, $E = 10$ GPa, $L = 0.3$ m, $f = 10^6$ N/m.

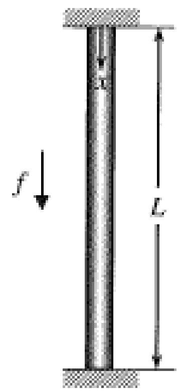


Figure P6.8

Programming

9. Write a program to use the finite element method to solve the problem described by the following differential equation and boundary conditions

$$\frac{d^2\phi}{dx^2} + \phi - x^2 = 0 \quad 0 < x < 1 \qquad \text{subjected to boundary conditions} \qquad \phi(0) = 0 \text{ and } \phi(1) = 0$$

(a) Solve using two linear elements.
(b) Solve using four linear elements.
(c) Compare solution obtained in **(a)** and **(b)** with the exact solution by plotting them.

10. A tapered bar with circular cross-section as shown in **Figure P6.10** is fixed at $x = 0$, and an axial force of 0.3×10^6 N is applied at the other end. The length of the bar (L) is 0.3 m, and the radius varies as $r(x) = 0.03 - 0.07\,x$, where r and x are in meters. Use $E = 10^{10}$ Pa.

(a) Solve using two linear elements.
(b) Solve using four linear elements.
(c) Compare solution obtained in **(a)** and **(b)** with the solution obtained by Galerkin method by plotting them.

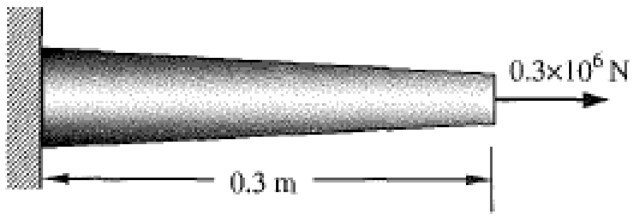


Figure P6.10

EXERCISE 7

APPLICATIONS IN HEAT TRANSFER

1. A wall of an industrial oven consists of three different materials, as shown in **Figure P7.1**. The first layer is composed of 10 cm of insulating cement with a thermal conductivity of 12 *W/m*· K. The second layer is made from 20 cm of S-ply asbestos board with a thermal conductivity of 0.071 *W/m*.K. The exterior consists of 12-cm cement mortar with a thermal conductivity of 0.72 W/m^2.K. The inside wall temperature of the oven is 250°C, and the outside air is at a temperature of 35°C with a convection coefficient of 45 W/m^2.K. Determine the temperature distribution along the composite wall.

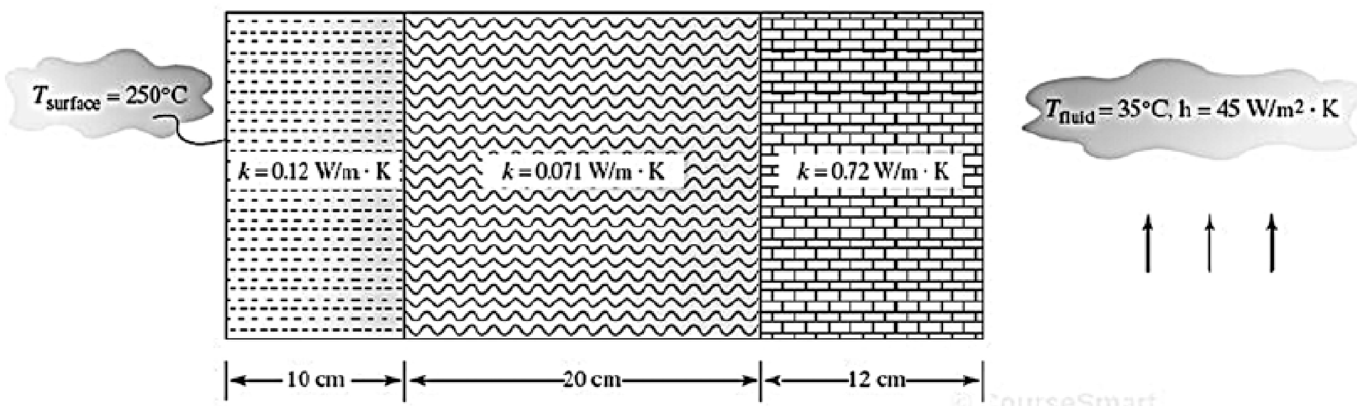


Figure P7.1

Replace the temperature boundary condition of the inside wall of the oven with air temperature of 400°C and an associated convection coefficient of 100 W/m^2.K. Show the contribution of this boundary condition to the conductance matrix and the forcing matrix of element (1). Also determine the temperature distribution along the composite wall.

2. Consider a heat chamber in which the temperature inside is maintained at a constant value of 200^{o}C (see **Figure P7.2**). The chamber is covered by a 1.0-m-thick metal wall and outside is insulated so that no heat flows to the outer surface of the wall. There is a uniform heat source inside the wall generating Q = 400 W/m^3. The thermal conductivity of the wall is k = 25 W/m^{0}C. Assuming that the temperature only varies in the x-direction, find the temperature distribution in the wall.

3. Use the system analysis procedure described in this chapter and constructs the discrete system for heat conduction through the composite wall shown in **Figure P7.3**. Also, from the following data, calculate the temperature distribution in the composite wall.

Areas: A_1 = 2.0 m^2, A_2 = 1.0 m^2 and A_3 = 1.0 m^2.
Thermal conductivity: k_1 = 2.00 W⁄mK, k_2 = 2.5 W⁄mK and k_3 = 1.5 W⁄mK.
Heat transfer coefficient: h = 0.1 W⁄m^{2}K.

Atmospheric temperature: T_a = 30^{0}C.
Temperature at the left face of wall: T_1 = 75.0 ^{0}C.

4. A metallic fin, with thermal conductivity 360 *W/m*.K, 0.1 cm thick and 10 cm long extends from a plane wall whose temperature is 235°C. Determine the temperature distribution along the fin if heat is transferred to ambient air at 20°C with heat transfer coefficient of 9 W/m^2K. Take width of the fin as 1 m. Use three elements.

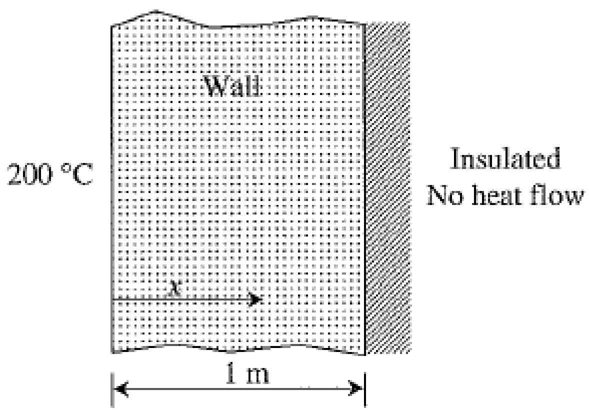


Figure P7.2

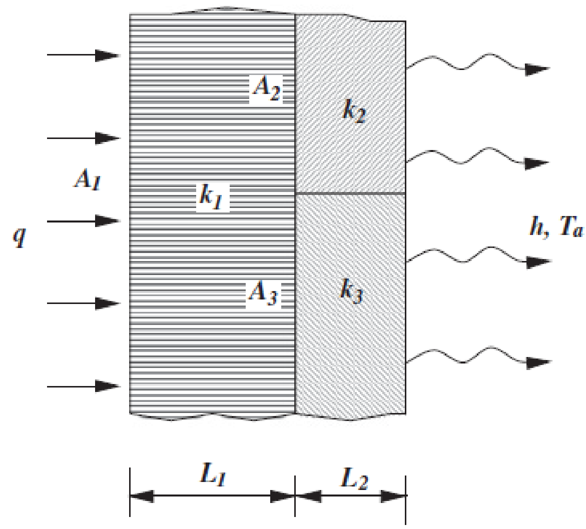


Figure P7.3

5. Find the temperature distribution in the one-dimensional fin shown in **Figure P7.5**
   (a) using one finite element.
   (b) using two finite elements.

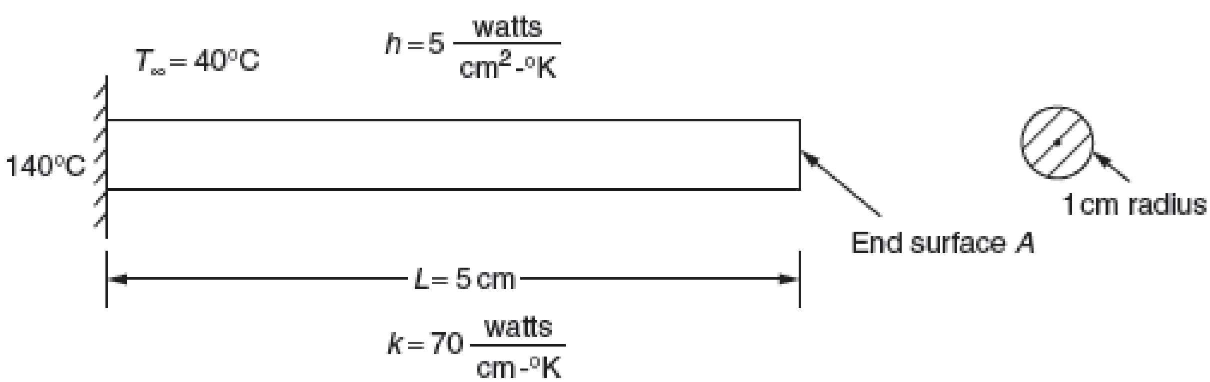


Figure P7.5

6. The fin shown in **Figure P7.6** is insulated on the perimeter. The left end has a constant temperature of 100^{0}C. A positive heat flux of q = 5000W/m^2 acts on the right end. Let K = 6 W/ m.^{0}C and cross-sectional area A = 0.1m^2. Determine the temperatures at L/4, L/2, 3L/4, and L; where L = 0.4 m.

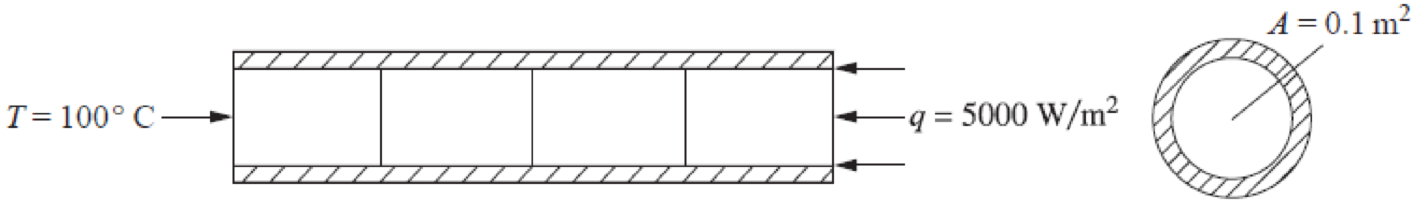


Figure P7.6

7. Aluminum fins with rectangular profiles shown in **Figure P7.7** are used to remove heat from a surface whose temperature is 150°C. The temperature of the ambient air is 20°C. The thermal conductivity of aluminum is 168 *W/m*.K. The natural convective coefficient associated with the surrounding air is 35 W/m^2.K. The fins are 150 mm long. 5 mm wide, and 1 mm thick. **(a)** Determine the temperature distribution along a fin using five equally spaced elements. **(b)** Approximate the total heat loss for an array of 50 fins. **(c)** Determine the temperature of a point on the fin 45 mm from the base. Also compute the fraction of the total heat that is lost through this section of the fin.

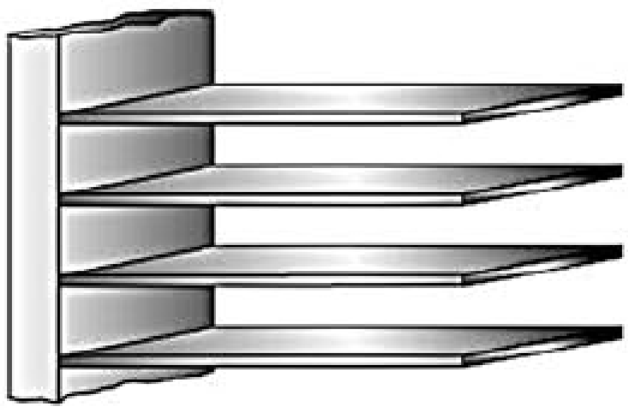


Figure P7.7

8. A rectangular aluminum fin is used to remove heat from a surface whose temperature is *80°C*. The temperature of the ambient air varies between 18°C and 25°C. The thermal conductivity of aluminum is 168 W/*m*.K. The natural convective coefficient associated with the surrounding air is 25 W/m^2. K. The fin is 100 mm long, 5 mm wide, and 1 mm thick. (a) Determine the temperature distribution along a fin using five equally spaced elements for both ambient conditions. (b) Approximate the total heat loss for an array of 50 fins for each ambient condition. (c) Compare your finite element results to the exact results.

9. Find the temperature distribution in the tapered fin shown in **Figure P7.9**
(a) using one finite element.
(b) using two finite elements.

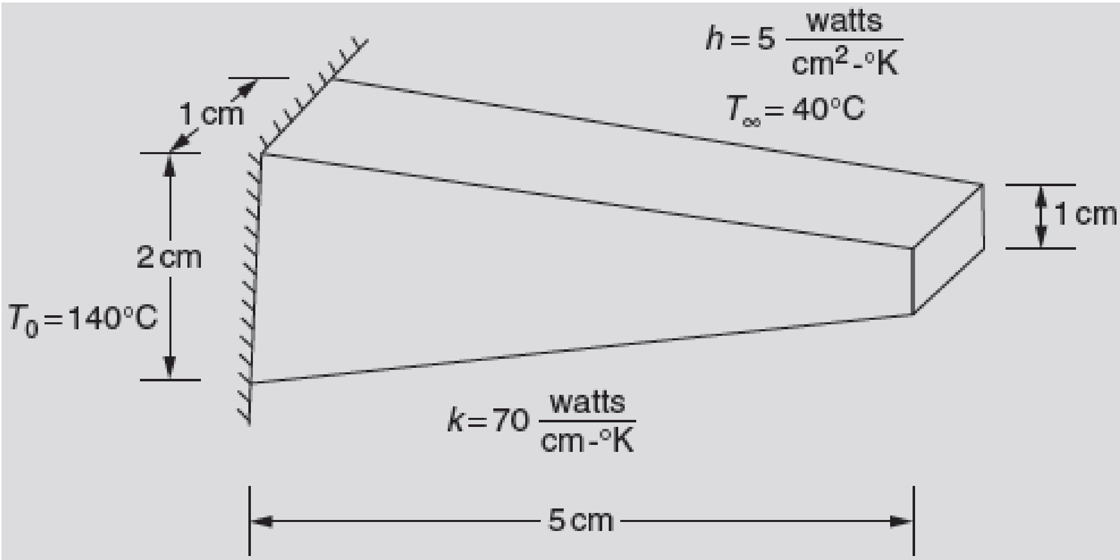


Figure P7.9

10. Compute the element matrices and vectors for the element shown in **Figure P7.10** when the edges *jk* and *ki* experience convection heat loss.

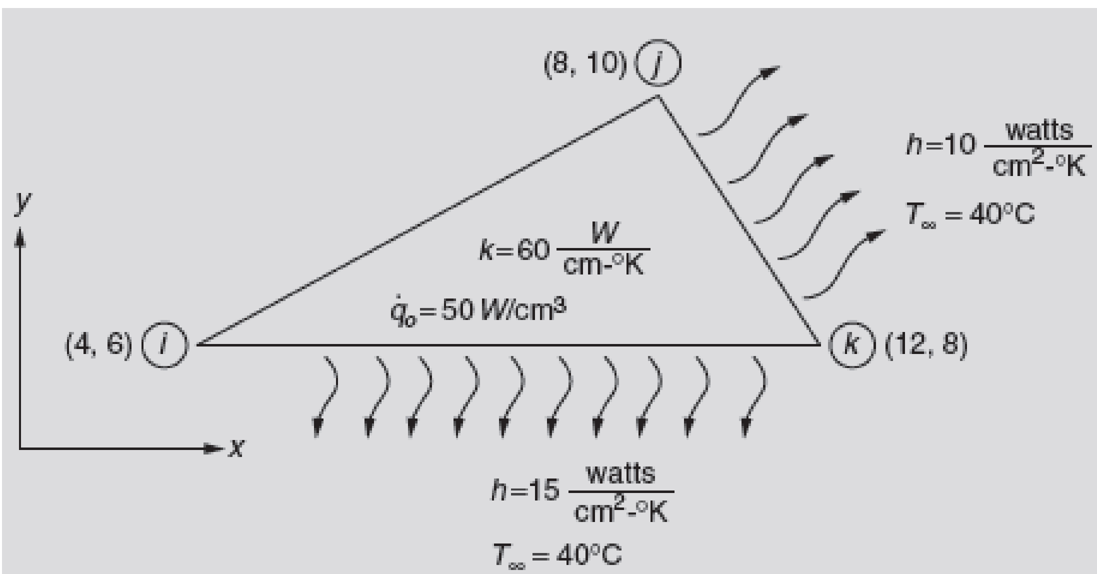


Figure P7.10

11. A line source $Q^* = 52$ W/cm is located at (5,2) in the element shown in **Figure P7.11**. Determine the Q^* allocated to each node.

12. For the square plate shown in **Figure P7.12**, determine the temperature distribution. Let k = 10 W/(m ^{0}C) and h = 20 W/(m^2 ^{0}C). The temperature along the left side is maintained at 100 ^{0}C and that along the top side is maintained at 200 ^{0}C.

(a) Use four triangular elements.
(b) Use two rectangular elements.

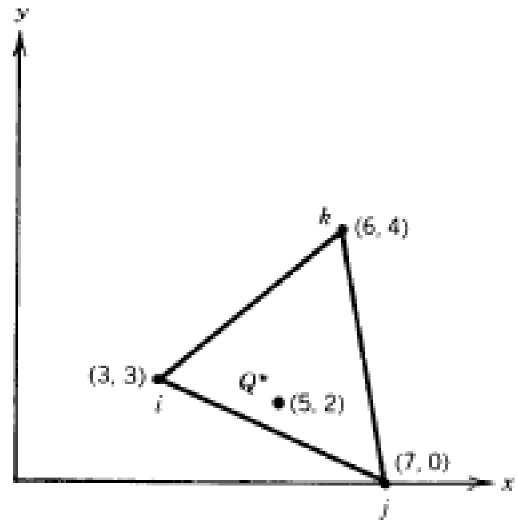


Figure P7.11

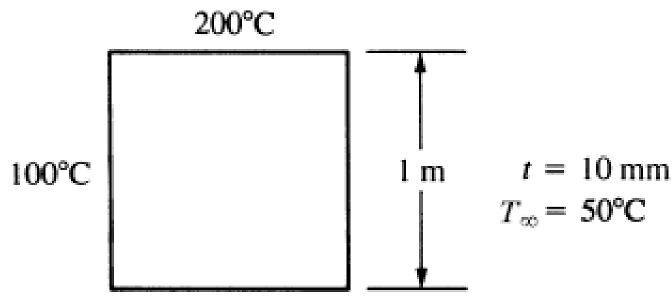


Figure P7.12

Programming

13. For the two-dimensional body shown in **Figure P7.13**, determine the temperature distribution. The temperature of the top side of the body is maintained at 100^0C. The body is insulated on the other edges. A uniform heat source of Q = 1000 W/m^3 acts over the whole plate. Assume a constant thickness of 1 m. Let k = 25 W/(m ^{0}C).

14. For the two-dimensional body shown in **Figure P7.14**, determine the temperature distribution. The top and bottom sides are insulated. The right side is subjected to heat transfer by convection. Let k = 10 W/(m °C).

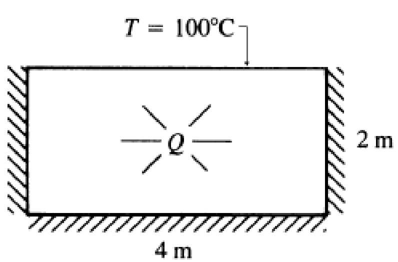


Figure P7.13

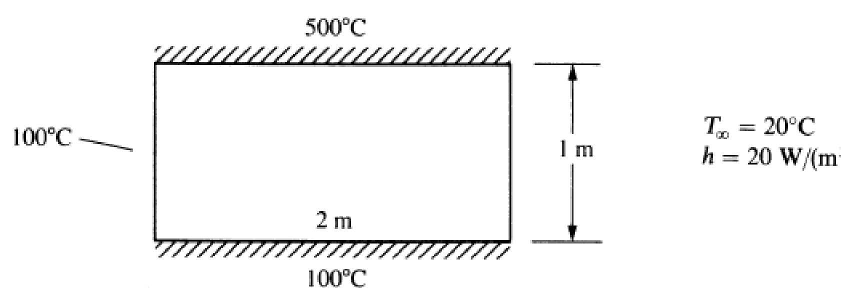


Figure P7.14

ANSYS

15. Aluminum fins with triangular profiles, shown in the accompanying **Figure P7.15**, are used to remove heat from a surface whose temperature is 150°C. T he temperature of the surrounding air is 20°C. T he natural heat transfer coefficient associated with the surrounding air is 30 W/m^2.K. The thermal conductivity of aluminum is k = 168 W/m.K. Using manual calculations, determine the temperature distribution along a fin. Approximate the heat loss for one such fin.

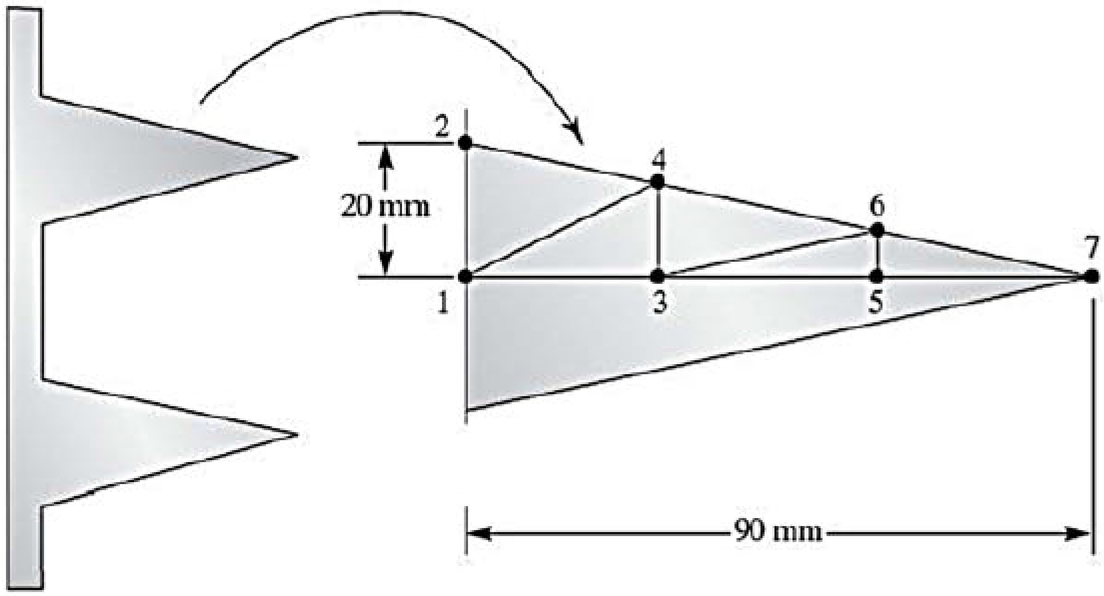


Figure P7.15

16. For the two-dimensional body shown in **Figure P7.16**, determine the temperature distribution. The left and right sides are insulated. The top surface is subjected to heat transfer by convection. The bottom and internal portion surfaces are maintained at 300 °C.

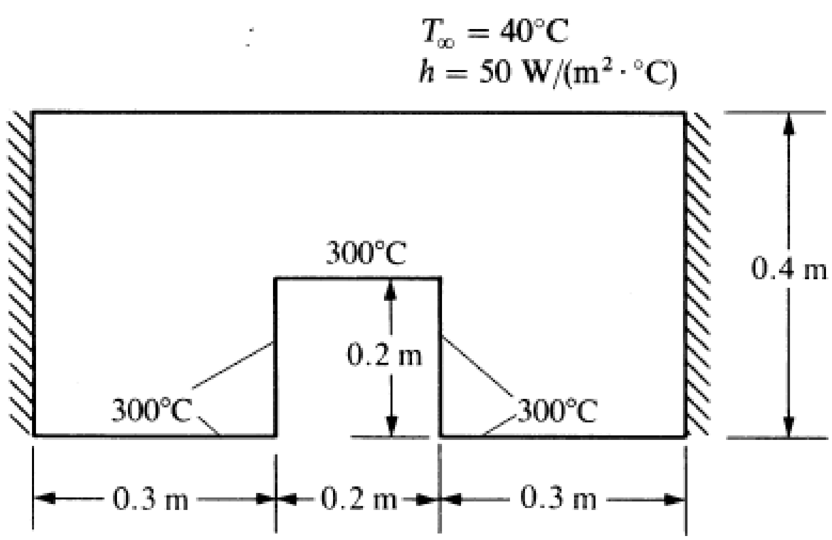


Figure P7.16

EXERCISE 8

APPLICATIONS IN SOLID MECHANICS

1. Determine the nodal forces for **(a)** a linearly varying pressure p_x on the edge of the triangular element shown in **Figure P8.1(a)** and **(b)** the quadratic varying pressure shown in **Figure P81.1(b)**. Assume the element thickness is equal to *t*.

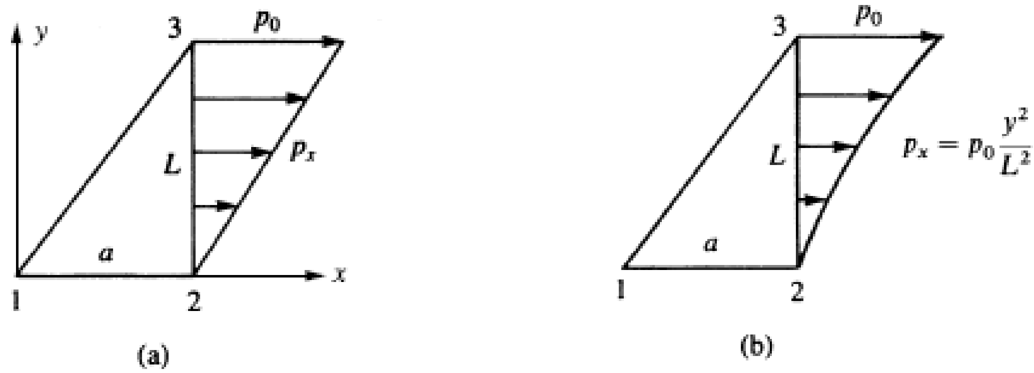


Figure P8.1

2. A two-dimensional triangular plane stress element made of steel, with modulus of elasticity $E = 200$ GPa and Poisson's ratio $v = 0.32$, is shown in **Figure P8.2**. The element is 3 mm thick, and the coordinates of nodes *i*, *j* and *k* are given in centimeters in Figure. Determine the stiffness and load matrices under the given conditions.

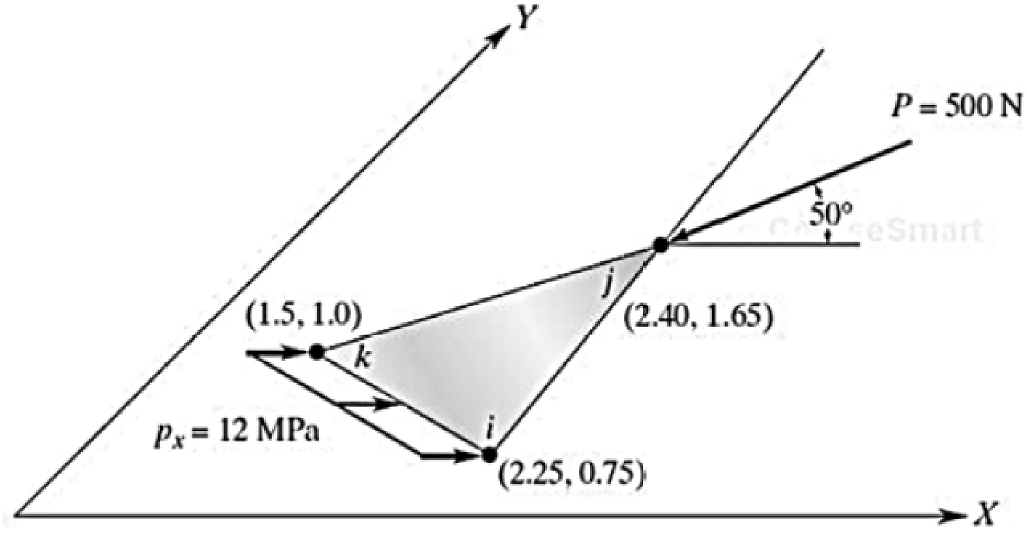
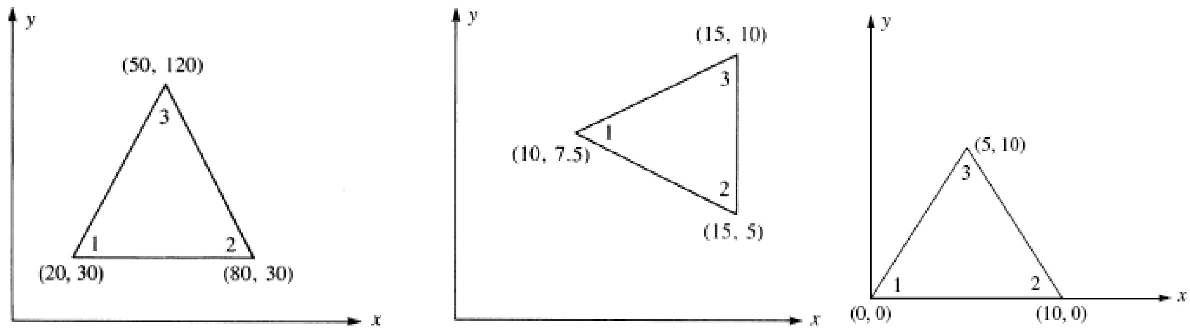


Figure P8.2

3. Evaluate the stiffness matrix for the elements shown in **Figure P8.3**. The coordinates are given in units of millimeters. Assume plane stress condition. Take E= 210 GPa, υ = 0.25 and t = 10 mm.



(a) **(b)** **(c)**

Figure P8.3

4. The nodal displacements for the simplex two-dimensional element shown in the **Figure P8.4** given below are u_1 = 2 mm, u_2 = 6 mm, u_3 = -1 mm, v_1 = 4 mm , v_2 = 5 mm and v_3 = 8 mm. Determine the displacement components at an interior point B (10,10). The nodal coordinates (in mm) are given in parenthesis.

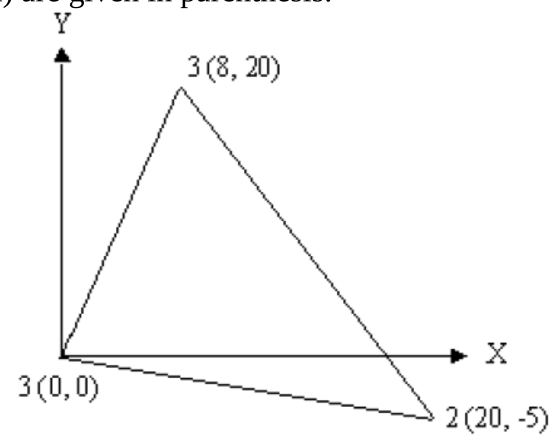
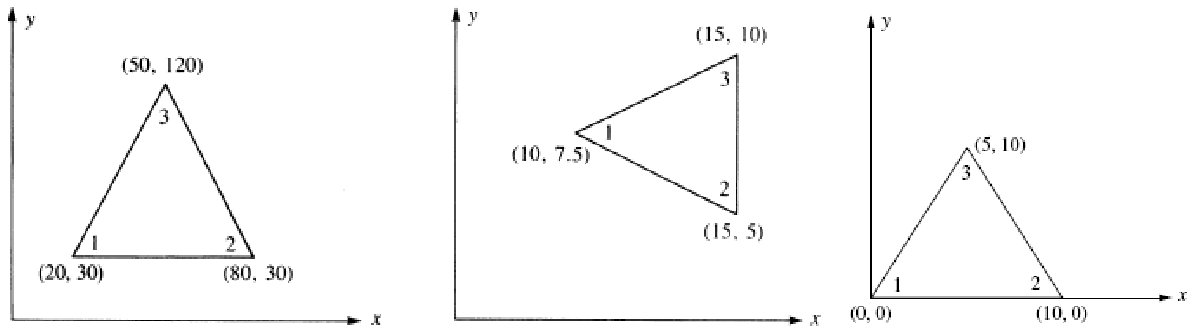


Figure P8.4

5. For the elements shown in **Figure P8.5**, the nodal displacements are given as u_1 = 2 mm, u_2 = 0.5 mm, u_3 = 3 mm, v_1 = 1 mm , v_2 = 0 mm and v_3 = 1 mm. Determine the element stresses. Take E = 210 GPa, υ = 0.25 and t = 10 mm.



(a) **(b)** **(c)**

Figure P8.5

6. The nodal coordinates and the nodal displacements of a triangular element, under a specific load condition are given below: $x_i = 0$, $y_i = 0$, $x_j = 1$ mm , $y_j = 3$ mm, $x_k = 4$ mm, $y_k = 1$, u_1=1 mm, $u_2 = -0.05$ mm, $u_3 = 2$ mm, $v_1 = 0.5$ mm , $v_2 = 1.5$ mm and $v_3 = -1$ mm. If $E = 2 \times 10^5$ N/mm^2 and $\mu = 0.3$. Find the stresses in the element.

7. The coordinates of the nodes 1, 2 and 3 of a triangular element are (1, 1), (8, 4) and (2, 7) in mm. The displacements at the nodes are $u_1 = 1$ mm, $u_2 = 3$ mm, $u_3 = -2$ mm, $v_1 = -4$ mm, $v_2 = 2$ mm and $v_3 = 5$ mm. Obtain the strain-displacement relation matrix B and determine the strains ε_x, ε_y and γ_{xy}.

8. For the plane strain elements shown in **Figure P8.8**, the nodal displacements are given as $u_1 = 0.005$ mm $v_1 = 0.002$ mm $u_2 = 0.0$ mm $v_2 = 0.0$ mm $u_3 = 0.005$ mm $v_3 = 0.0$ mm
Determine the element stresses σ_x; σ_y; τ_{xy}; σ_1, and σ_2 and the principal angle θ_p. Let $E = 70$ GPa and $\upsilon = 0.3$, and use unit thickness for plane strain. All coordinates are in millimeters.

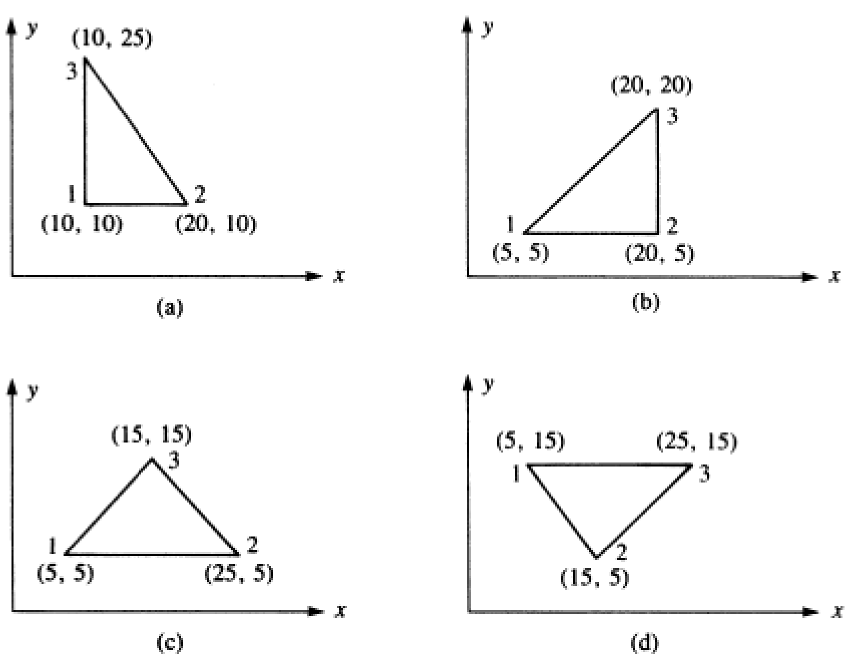


Figure P8.8

9. Consider a one-element triangular mesh shown in **Figure P8.9**. The boundary conditions are as follows. The edge BC is constrained in y and traction free in x, whereas the edge AB is constrained in x and traction free in y. The edge AC is subject to traction normal to the edge. Assume Young's modulus $E = 3 \times 10^{11}$Pa and Poisson's ratio $\upsilon = 0.3$.

(a) Construct the weak form corresponding to the generalized boundary conditions.
(b) Construct the stiffness matrix.
(c) Calculate the global force matrix.
(d) Solve for the unknown displacement matrix and calculate the stress at (1.5, 1.5).

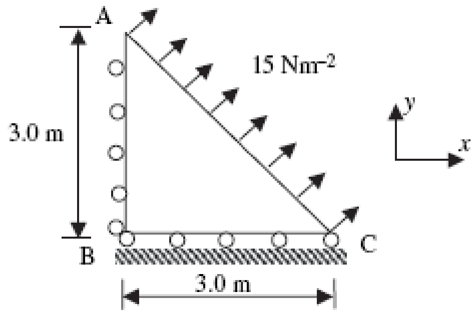


Figure P8.9

10. For the configuration shown in **Figure P8.10**, determine the deflection at the point of load application using a one element model. If a mesh of several triangular elements is used, comment on the stress values in the elements close to the tip.

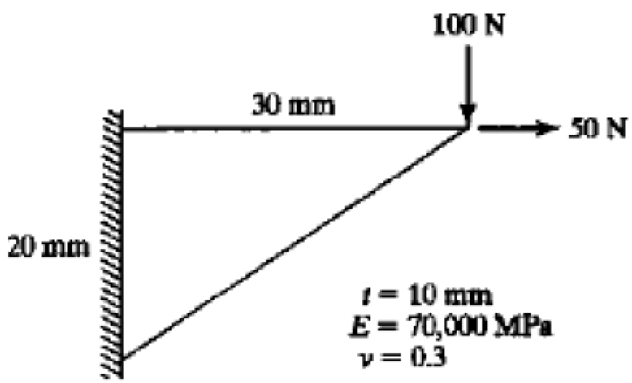


Figure P8.10

11. Find the nodal displacements in **Figure P8.11**. Use two linear triangular elements.

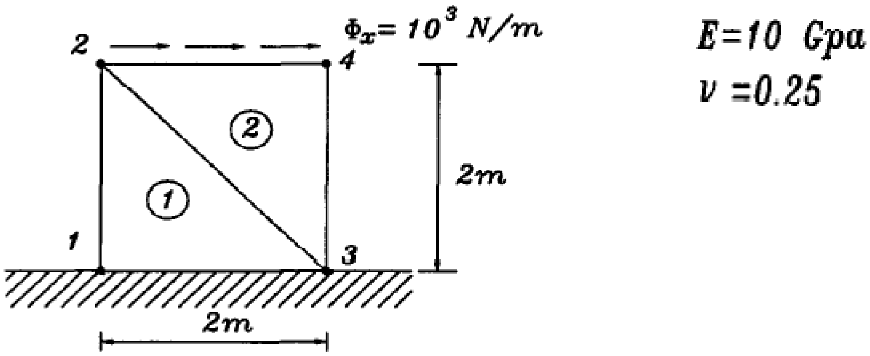


Figure P8.11

12. Consider a rectangular panel as shown in **Figure P8.12**. The panel is modeled using a plane stress linear elastic material with the following properties: Young's modulus $E = 3 \times 10^{11}$ Pa and Poisson's ratio $\upsilon = 0.3$. The essential boundary conditions are $u_{Ax} = u_{Ay} = u_{By} = 0$.
The natural boundary conditions are as follows. Along each edge of the panel, the prescribed traction consists of normal and lateral components, both equal to 10^3N/m.
Discretize the panel using a single rectangular element as shown below. For convenience, use identical global and local numberings as shown in *Figure*. Calculate nodal displacements and stresses at the element Gauss points.

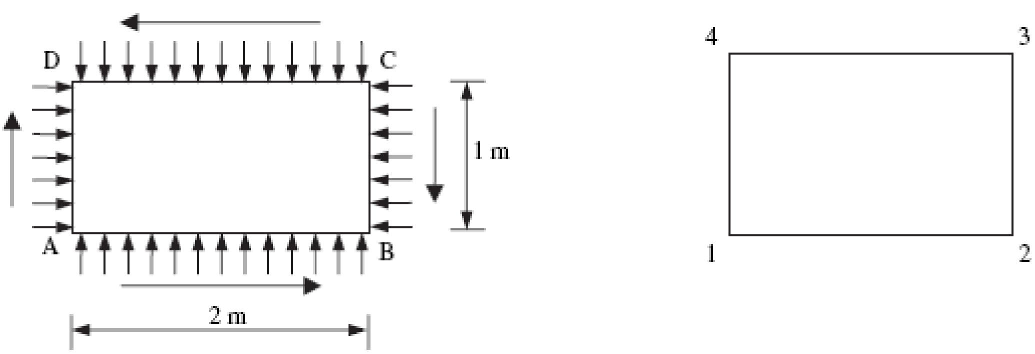


Figure P8.12

13. On a four nodal quadrilateral plane stress element the nodes are (0, 0), (6, 2), (6, 6) and (1, 5). A concentrated load whose x and y components are 10 kN, 16 kN respectively is applied at a point (4, 5). Find the equivalent nodal forces and the displacement of nodes.

14. Solve the plane stress problem in **Figure P8.14**.

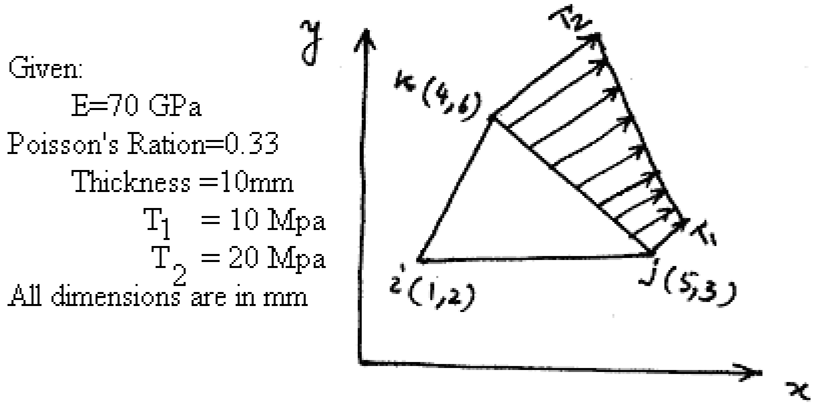


Figure P8.14

15. Determine the nodal displacements and the element stresses, including principal stresses, due to the loads shown for the thin plates in **Figure P8.15**. Use E = 210 GPa, υ = 0.3, and t = 5 mm. Assume plane stress conditions apply. The recommended discretized plates are shown in the figures.

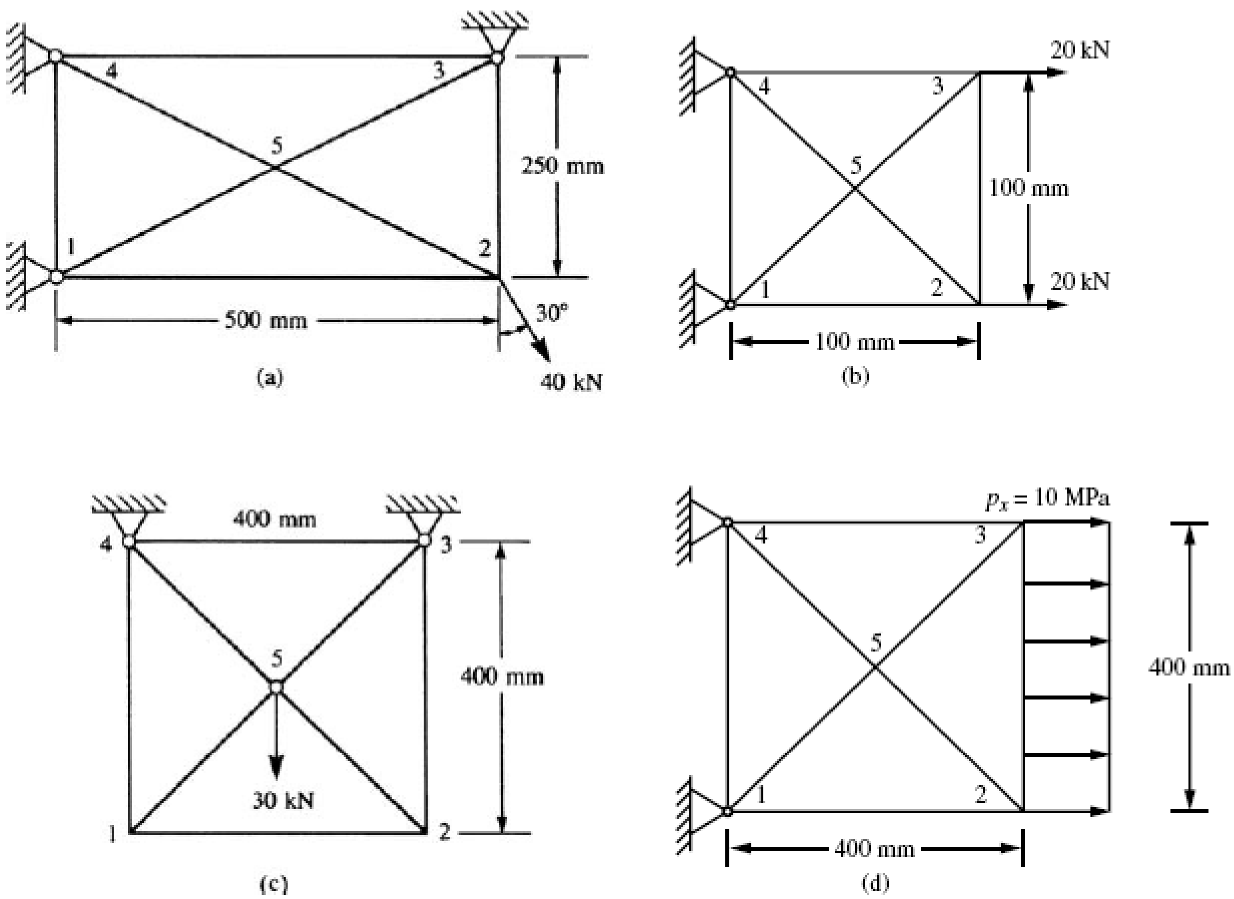


Figure P8.15

16. Solve the stress problem shown in **Figure P8.16**.

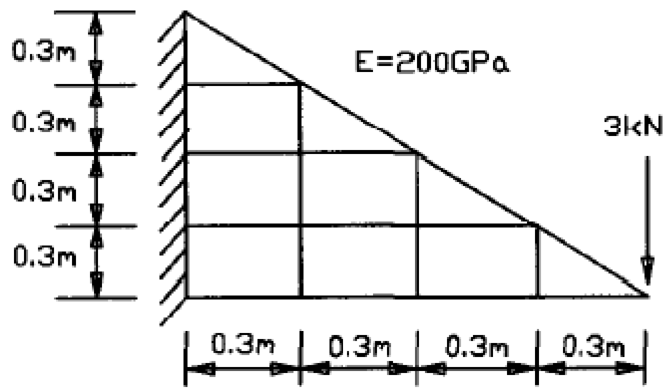


Figure P8.16

17. Determine the stresses in the plate with the hole subjected to the tensile stress shown in **Figure P8.17**. Graph the stress variation σx versus the distance y from the hole. Let *E* = 200 GPa, *υ* = 0.25, and *t* = 25 mm. Use symmetry as appropriate.

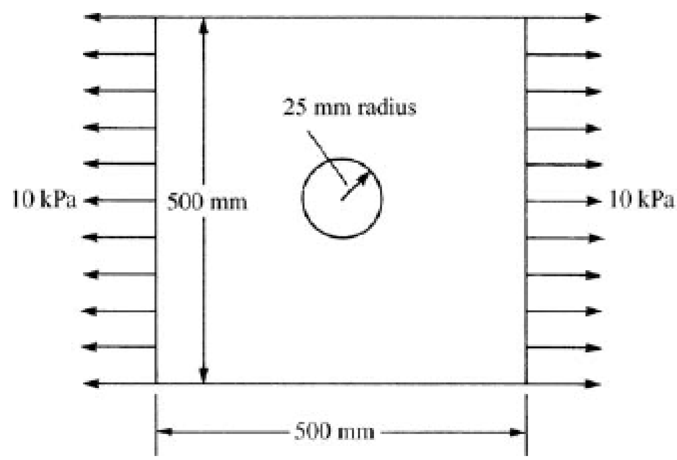


Figure P8.17

18. Determine the stresses in the plates with the round and square holes subjected to the tensile stresses shown in **Figure P8.18**. Compare the largest principal stresses for each plate. Let *E* = 210 GPa, *υ* = 0.25, and *t* = 5 mm.

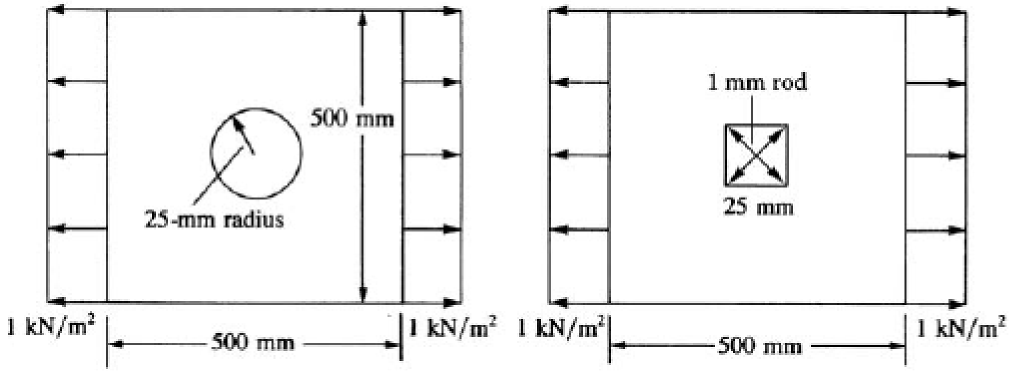

500 mm
25-mm radius
1 kN/m²
500 mm
1 kN/m²
1 mm rod
25 mm
1 kN/m²
500 mm
1 kN/m²

Figure P8.18