Inverse Problems in Science and Engineering

ISSN: 1741-5977 (Print) 1741-5985 (Online) Journal homepage: https://www.tandfonline.com/loi/gipe20

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**To cite this article:** Jacek Buśkiewicz (2015) A method for optimal path synthesis of four-link planar mechanisms, Inverse Problems in Science and Engineering, 23:5, 818-850, DOI: 10.1080/17415977.2014.939654

**To link to this article:** https://doi.org/10.1080/17415977.2014.939654

*Inverse Problems in Science and Engineering*, 2015
Vol. 23, No. 5, 818–850, http://dx.doi.org/10.1080/17415977.2014.939654

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# A method for optimal path synthesis of four-link planar mechanisms

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(*Received 24 January 2014; accepted 25 June 2014*)

The paper deals with the optimal synthesis of planar mechanisms as path generators. The formulation of the synthesis problem enables decreasing the number of design parameters. The angular position of the floating link (coupling link) connecting input and output links is described by function *sine*. The coupler link is treated as a free body which generates precisely desired motion. The objective is to determine the coupler dimensions for which the coupler joints approximate with a prescribed accuracy the trajectories which these joints trace in the assembled mechanism. The structural error for most typical kinematic pairs was derived in order to apply the method to a wide spectrum of four-link planar mechanisms. Compared with classical path synthesis approach, this technique measures the deviation of the paths of the coupler joints from their paths in the assembled mechanisms, not the deviation of the coupler path from the prescribed trajectory. Various modifications of the method were discussed to widen the range of possible applications. The problem was formulated as an optimization task. An objective function was constructed and minimized in searching for generators of desired motion using an evolutionary algorithm. Test paths were chosen to verify the effectiveness of the method. Numerical tests addressed also practical linkage applications. The technique was verified in the synthesis of four-bar linkage, crank-slider mechanism and mechanism with slotted link. The method was also applied to solve chosen motion synthesis tasks combining the positions and angular orientations of the link.

**Keywords:** four-link planar mechanisms; path synthesis; computer-aided design

**AMS Subject Classifications:** Primary: 70B15, 93B50; Secondary: 93B51

## 1. Introduction

The choice of structure and dimensions of the kinematic system which are desired to generate prescribed motion is a complex problem. Design of mechanisms aimed at meeting required technical functions is dealt with mechanism synthesis. If the number of links and type of kinematic pairs are known, the problem is reduced to dimensional synthesis. The objective of path synthesis is to determine mechanism dimensions for which the coupler point passes through a prescribed set of points lying on a desired trajectory. This problem has to be solved when designing carrier machines, feeders, pushers, shaping machines, etc. In general, a link of one-degree-of-freedom mechanism is incapable of passing precisely through an arbitrarily large set of points. Optimal

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synthesis deals with problems of searching for mechanisms approximating a required motion as precisely as possible in the sense of an introduced distance norm.

There are many tasks of path synthesis – synthesis of open path generators, synthesis of closed path generators and path synthesis with and without timing (time prescription). In path synthesis with timing, each position of the coupler point corresponds to a prescribed angular position of the active link. The open path synthesis is of significant importance as in numerous machines an end-effector does mechanical work when moving along an open trajectory, and the shape of the trajectory in the return motion (e.g. dead movement) to the initial position is not so important. Moreover, in many cases, the correspondence between time and position on the trajectory is of less significance.

Despite of a simple structure, four-link planar mechanisms are mechanisms which have numerous applications. Many of them are related to a variety of curves traced by the coupler. Moreover, the best solution is an one-degree-of-freedom mechanism built of as small number of links as possible. This pursuit is the result of economic and technological reasons. The fewer number of links separates the active link and the working link the smaller motion disturbance, caused by an inaccuracy of assembly, appears. That is why the synthesis of the simplest mechanisms is of vast importance in designing devices for industrial and home use, and a countless number of synthesis methods were elaborated.

The paper presents a method for the optimal synthesis of open path generators without time prescription. The method was worked out for four-bar linkages in paper [1,2] and developed further for other four-link mechanisms. The method is aimed at decreasing the number of design parameters. The mechanism synthesis is a problem of great mathematical complexity. The analytical solutions are possible for simpler cases.[3–5] At present mainly computer methods are developed. The optimization problem is defined by design parameters, an objective function, an optimization algorithm and constraints. The design parameters define a mechanism, the objective function is a measure of how similar the generated and required paths are. The effectiveness of the method depends essentially on the number of design parameters and as a consequence on the structure of the objective function.[2,6–8] The implication being that one of mechanism synthesis branches addresses the minimization of the number of design parameters. This number is important when a synthesis method is based on deterministic-probabilistic algorithms for optimal parameters determination. Every additional parameter increases the dimension of solution space and may increase the computational cost. The advantage of probabilistic methods over deterministic ones is that the former ones search for the optimal solution in the whole defined space, whereas the latter ones, as is the case with gradient methods, converge to the local minimum in the surrounding of the initial parameters. That is why genetic and evolutionary algorithms (EAs) are still developed for synthesis method purposes.[9–15]

In most cases, open and closed path synthesis are dealt with separately, nonetheless, there are methods that draw no distinction between these two synthesis tasks. The synthesis of an open path generator is a complex problem, which requires elaboration of mathematical procedures for recognizing a piece of generated curve that best fits into the desired open path. The geometrical adaptation technique [16] was proposed and used in open-path synthesis of the four-bar linkage. This method reduces the number of optimized parameters to five as an objective function is independent of the position, orientation and size of a given path. Nonetheless, in most techniques presented in the literature, the number of optimized design parameters is not so important. The point is rather to develop and enhance algorithms minimizing an objective function.[9–15,17–25] The methods presented in these works are independent of whether a path is open or close.

The synthesis of linkages approximating straight line has been widely recognized. Many techniques [26–31] were elaborated to synthesize four-bar linkage generating the coupler curve, a part of which is an approximate straight line. Some researchers introduce new type of constraints, the example being synthesis of path/motion generator with prescribed load on the coupler.[32–34]

This concise description of studies shows that numerous synthesis methods were worked out for the four-bar linkage and crank-slider mechanism, but it is difficult to find papers dealing with mechanisms with slotted links as path and motion generators, although they are frequently used for these purposes (e.g. machines with prismatic pairs in hydraulic actuators: rippers, backhole loaders, excavators, machines for reloading, film advance mechanism of a projector, cutting machines, casement window mechanisms, etc.).

The paper presents the method for optimal synthesis of four-link mechanisms generating open paths without time prescription in which the function of angular position of the floating link (coupler) is described by the sine function. Such an approach enables decreasing the number of design parameters describing dimensions, orientation and position of a path generator. Non-optimized geometric parameters are functions of design parameters. Although the number of design parameters is reduced, the method does not require affine transformations to be performed on the synthesized mechanism. The concept of the method was introduced on the example of the four-bar linkage.[1,2] Compared to these works, the method was modified in order to widen the number of geometric input data, i.e. by prescribing the positions of ground pivots. The method is also applied to solve path and motion synthesis problems allowing a designer to have more influence on the expected solution. Moreover, the method was adapted to work for other four-link planar mechanisms: crank-slider mechanism and mechanisms with slotted links.

## 2. The general concept of the synthesis method

The coupler point D traces the open path. For general consideration, we may assume that the path is given parametrically in the coordinate system:

$$x = x(s), \quad y = y(s). \tag{1}$$

Without loss of generality, it can be taken that $s \in \langle 0, 1 \rangle$.

We consider an arbitrary planar mechanism composed of three moveable links and the frame. The links may be connected by means of either revolute or prismatic joints. A dyad of such a mechanism, built of active (input) link $O_1$A and intermediate, floating link (coupler) AB, is shown in Figure 1. The joints of the coupler are denoted as A and B. Point D traces the path. Coupler AB connects input and output links, which are fixed to the frame. We introduce the angle $\theta_{2L}$ that defines the angular position of link AB. The angle is measured anticlockwise from the positive direction of the horizontal axis to arm AD of link AB. For the considered class of mechanisms, $\theta_{2L}$ is a periodical

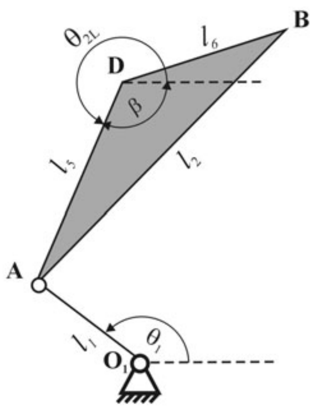


Figure 1. A dyad (active link and coupler) of a four-link mechanism.

function of period $2\pi$ expressed by means of mechanism dimensions and angular position of the active link. A path (1) is traced for a certain range of the input angle $\theta_1$.

When a path is given parametrically, angle $\theta_{2L}$ is the function of parameter $s$ introduced in the angular position of the active link $\theta_1(s)$. The author pointed out [2] that for an arbitrary four-bar mechanism, angle $\theta_{2L}$ can be described by the function of the following form:

$$\theta_{2L} = v_1 \sin(v_2 s + v_3) + v_4. \tag{2}$$

where $\theta_{2L}$ varies in the interval $\langle -\pi, \pi)$ and therefore $v_1 \in \langle -\pi, \pi)$. Let the starting angular position of the active link $\theta_1 = \theta_{10}$ correspond to the position of point D($x$(0), $y$(0)) (i.e. the starting point of the path), and let the angle $\theta_1 = \theta_{11}$ correspond to the ending point ($x$(1), $y$(1)) of the path). The path is traced for the input angle over the range $(\theta_{10}, \theta_{11}) \subset \langle -\pi, \pi)$ and therefore $v_2$ must be included in $\langle -\pi, \pi)$ The phase angle $v_3 \in \langle -\pi, \pi)$ represents the angular position of the active link $\theta_1 = \theta_{10}$ in which the coupler curve starts being drawn. Angle $\theta_{2L}$ is sensitive to the rotation of a mechanism. When the mechanism is rotated by an angle of $v_4 \in \langle -\pi, \pi)$, then angle $\theta_{2L}$ is shifted by $v_4$. Hence, one general formula for angular position of the link coupling input and output links is used for any four-link mechanism. It allows to decrease the number of design parameters, which is the point of the presented synthesis method.

The idea of the method consists in disconnecting the coupler from a mechanism. Point D traces precisely a desired path (Figure 2(a) and (b)). The objective is to determine the dimensions of the coupler (lengths $l_5$, $l_6$ angle $\beta$) and the coefficients of function (2) ($v_1$, $v_2$, $v_3$, $v_4$) in order that the curves traced by the coupler joints A and B are as close as possible to the curves traced by the joints in the assembled mechanism, for example, the moving joints of a four-bar linkage move along circular arcs. In this approach, the deviation of generated paths of the joints from the ideal paths is the measure of the solution fitness, whereas in other path synthesis techniques the generated coupler path is compared to the prescribed path.

The design parameters defining coupler geometry may differ depending on the mechanism type, an example being a mechanism with sliding block, where it suffices to

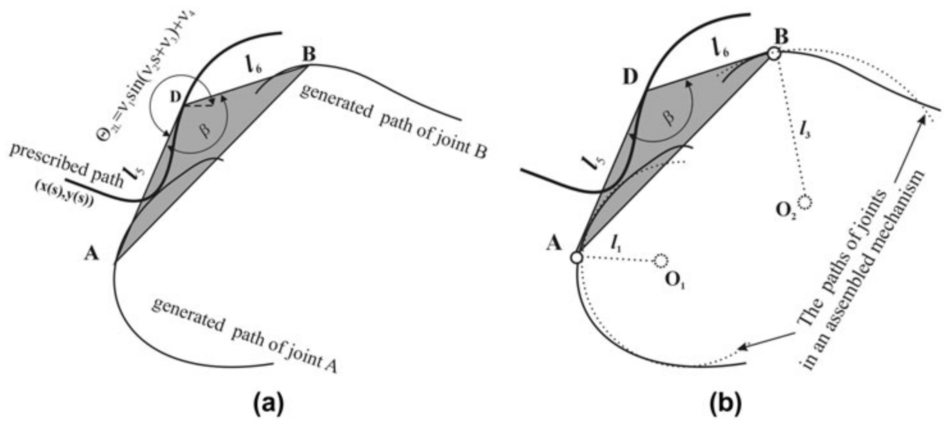


Figure 2. The paths traced by the joints of the free coupler while point D traces a prescribed path (a), and the paths traced by joints after assembling the input and output links with the coupler on the example of a four-bar linkage (b).

take $l_5$ and the angle between arms AD and AB. Nonetheless, the maximum number of design parameters does not exceed 7.

In general, the equation for the angular position of the coupler depends on the type of the mechanism, on all mechanism dimensions as well as the angular position of the input link. In this concept, only the coupler dimensions of all mechanism dimensions are design parameters. The other dimensions are not optimized, let alone the angle through which the input link rotates when the prescribed path is traced. Therefore, the feature that the formula for the motion of the coupler is not expressed by mechanisms dimensions is very desirable when the synthesis problem is formulated for a larger group of mechanisms. The scope of applicability of the method is not limited to one particular mechanism as it fails to use the equations describing the mechanism geometry. It suffices to know what types of curves (circle, straight line) are the paths traced by the joints by which the coupler is connected to the input link and output link.

The equation for angular position of the coupler depends on the type of a planar mechanism and it is the function of the link lengths and the angle of input link $\theta_1$. For four-bar linkage, this angle is expressed by means of $\tan^{-1}(\cdot)$.[3,4] As the mathematical form of this function is different from Equation (2), it will be denoted as $\theta^*_{2L}$. Hence, $\theta^*_{2L}$ defines how the angular position of the coupler evaluates when a path is traced. The important point, which requires more thorough analysis, is to prove the following theorem:

There exists the function describing the rotation of the input link $\theta_1(s)$ (which is the argument of the formula for angular position of the coupler $\theta^*_{2L}$) such that the following relation is true

$$\theta_{2L} = \theta^*_{2L}(\theta_1(s)) = v_1 \sin(v_2 s + v_3) + v_4.$$

To put it in other words, the active link may rotate clockwise or anticlockwise so that $\theta^*_{2L}$ varies like sin $(\cdot)$. $\theta^*_{2L}(\theta_1)$ is the function of angular position of the active link $\theta_1$. Function $\theta_1(s)$ must be monotonic in order to provide one direction of active link rotation. The explicit form of $\theta_1(s)$ is not required. We indicate that the angle of active link rotation may change in such a way that the coupler position is described by function $\theta_{2L}$. In other words, one can transform $\theta^*_{2L}$ into $\theta_{2L}$ by controlling angular velocity of the active link. In general, $\theta^*_{2L}$ is a periodical function either monotonically decreasing, increasing and decreasing or monotonically increasing, decreasing and increasing. Figure 3(a)–(e) present the graphical construction of how the formula expressing the angular position of the active link must be modified by adjusting parameters $v_1$, $v_2$, $v_3$

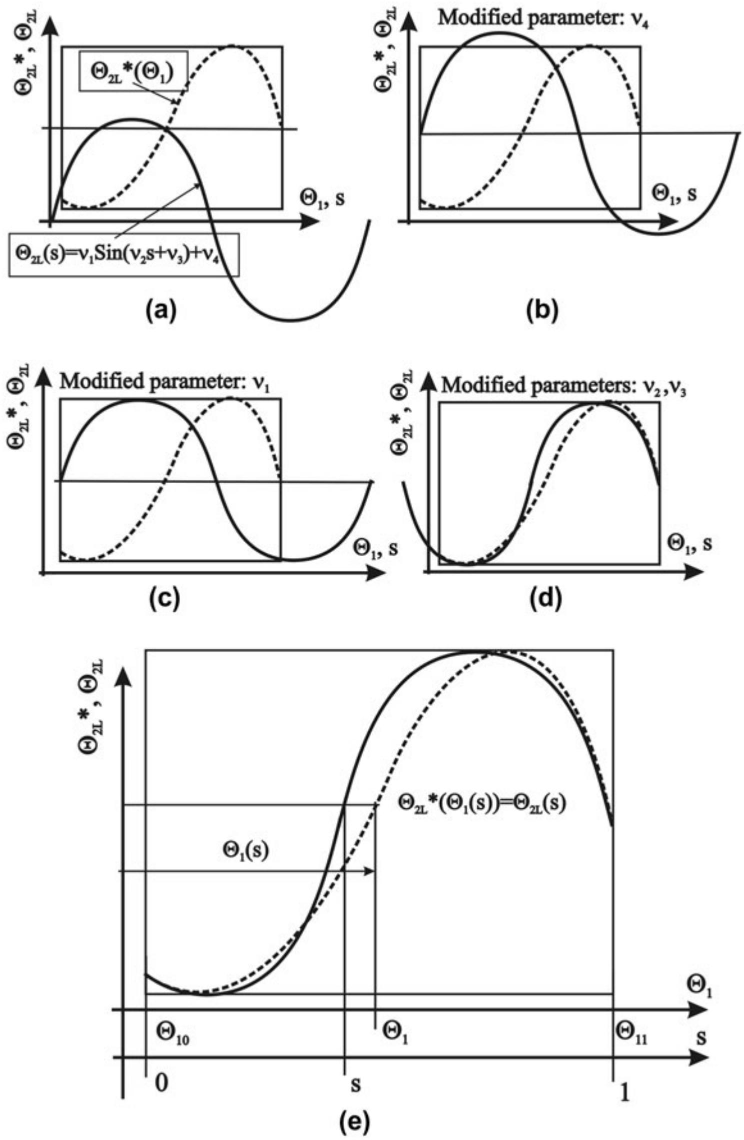


Figure 3. The steps of transformation of $\theta^*_{2L}$ (dotted line) into $\theta_{2L}$ (continuous line).

and $v_4$, to transform function $\theta^*_{2L}$ into $\theta_{2L}$. The dotted curve $\theta^*_{2L}$ is the real angular position of the coupler, the continuous curve is the formula expressed by means of sine function. Firstly, parameter $v_4$ equal to the mean value of maximum and minimum of $\theta^*_{2L}$ is added to the sine function. Subsequently, parameter $v_1$ is taken so that the amplitude of $\theta_{2L}$ is equal to the amplitude of $\theta^*_{2L}$. The beginning and ending values of $\theta_{2L}$ become equal to the boundary values of $\theta^*_{2L}$ by setting $v_2$ and $v_3$. Figure 3(e) shows that the value of $\theta^*_{2L}$ achieved at the position $\theta_1$ should be achieved for the argument $\theta_1(s)$. As a result, the function $\theta^*_{2L}$ is transformed into $\theta_{2L}$, by substituting $\theta_1(s)$ for $\theta_1$ into $\theta^*_{2L}$. Finally, $\theta^*_{2L}(\theta_1(s)) = \theta_{2L}(s)$. The horizontal line $\theta_1(s)$ indicates the angular position of the active link (of argument $s$) for which the angular position of the coupler varies according to Equation (2). In other words, it is the relation between angular position of the active link and the position of coupler point D. It is obvious that function $\theta_1(s)$ is monotonic, as the following relation is met:

$$\text{If } s_i < s_{i+1}, \text{ then } \theta_1(s_i) < \theta_1(s_{i+1}).$$

The parametric equation of a path expressed by Equation (1) and angular position of the coupler given by Equation (2) are functions of the same argument $s$, which provides implicit relation between coupler rotation and the coordinates of the coupler point. It follows that for the same synthesized path the result may depend on the parameterization of this path. This is the fault of the method as long as we consider geometric mechanism synthesis.

## 3. Geometric analysis of trajectories of the coupler joints

We consider a four-link mechanism composed of the coupler and input and output links connected to the frame by revolute or prismatic joints. Let us assume that the coupler is disconnected from the other links and we take arbitrary values of design parameters (coupler dimensions: $l_5$, $l_6$, $\beta$, and coefficients: $v_1$, $v_2$, $v_3$, $v_4$). When point D traces the desired path, the coupler joints trace trajectories depending on these parameters as shown in Figure 2(a). In general, these trajectories differ from the real trajectories which the joints would trace if the coupler was connected to the input link and output link (Figure 2(b)). The measure of the deviation of the computed path of a joint from the real path depends on the type of the kinematic pairs in the dyad coupler-input (or output) link. The three cases met in the most common planar mechanisms are discussed.

### 3.1. *The dyad with two revolute joints – RR*

Let us consider the dyad $O_1AB$ with coupler point D.

Rotating link $O_1A$ is connected to the coupler by means of revolute joint A, and rotates about the ground pin $O_1$. As such joint A traces a circle or circular arc. Let us consider the motion of the coupler only when point D traces a prescribed path, as shown in Figure 4. The design parameters (coupler dimensions and parameters of coupler angular position) are optimal when the trajectory of joint A is as close as possible to a circle, but for arbitrary parameters the trajectory is a certain curve (Figure 4). The objective function must be defined in order to measure the deviation of this curve from

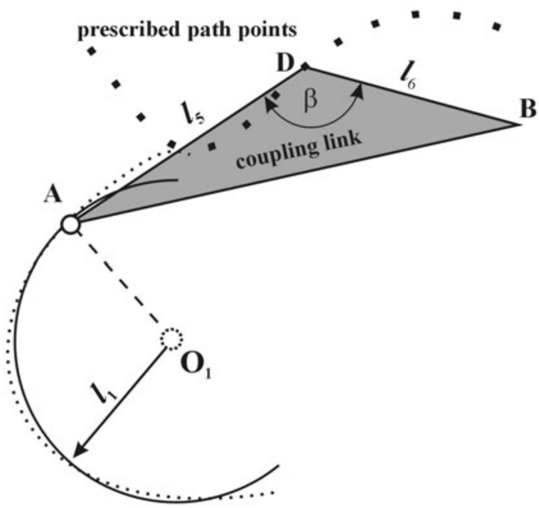


Figure 4. The path traced by joint A (dotted line) while the coupler point traces the prescribed path, and the circular arc (continuous line) approximating this path.

an ideal circle. The deviation of the trajectory of A from the circle that best approximates this trajectory in the sense of a proposed mathematical norm is the measure of solution fitness. Then, the centre and radius of the circle is the ground pin $O_1$ and length $l_1$ of link $O_1A$ in dyad which optimally satisfies prescribed performance. These parameters are functions of the design parameters. As shown in Figure 4, for the sake of the numerical implementation of the method, the prescribed path is discretized for equally spaced values of parameter $s$, and given as a set of $m$ points: $x_i = x(s_i)$, $y_i = y(s_i)$, where $s_i = i\Delta s$, $\Delta s = 1/m$, $i = 1 \ldots m$. The points of the joint trajectory are computed as follows:

$$x_{Ai} = x_i + l_5 \cos\theta_{2L}(s_i), \quad y_{Ai} = y_i + l_5 \sin\theta_{2L}(s_i). \tag{3}$$

Let $l_{1i}$ denote the radius of the circle with centre at $(x_{O_1 i}, y_{O_1 i})$ and passing through three points (Figure 5): $A_{1i}(x_{Ai}, y_{Ai})$, $A_{2i}(x_{Ai'}, y_{Ai'})$, $A_{3i}(x_{Ai''}, y_{Ai''})$, where: $i = 1 \ldots m^*$, $m^* = \lfloor m/2 \rfloor$, $i'' = m^* + i$, $i' = \lfloor (i + i'')/2 \rfloor$.

It is easy to show that for each triple points of trajectory of joint A, the centre coordinates and the radius of the $i$th circle are:

$$x_{O_1 i} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad y_{O_1 i} = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}},$$
$$l_{1i} = \sqrt{(x_A(s_{2i}) - x_{O_1 i})^2 + (y_A(s_{2i}) - y_{O_1 i})^2}, \tag{4}$$

where

$$a_{11} = 2(x_{Ai} - x_{Ai'}), \quad a_{12} = 2(y_{Ai} - y_{Ai'}),$$

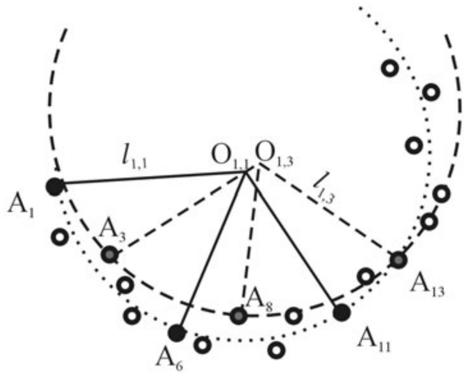


Figure 5. Two chosen circles passing through points $A_1$, $A_6$, $A_{11}$ and $A_3$, $A_8$, $A_{13}$ of the radiuses $l_{1,1}$, $l_{1,3}$, and centres at $O_{1,1}$, $O_{1,3}$, respectively (number of points $m = 20$).[1]

$$a_{21} = 2(x_{Ai} - x_{Ai''}), \quad a_{22} = 2(y_{Ai} - y_{Ai''}),$$

$$b_1 = x_{Ai}^2 + y_{Ai}^2 - x_{Ai'}^2 - y_{Ai'}^2, \quad b_2 = x_{Ai}^2 + y_{Ai}^2 - x_{Ai''}^2 - y_{Ai''}^2$$

Averaging the expressions (4), one gets the coordinates of the ground pin and the length of the rotating link:

$$x_{O_1} = \frac{\sum_{i=1}^{m^*} x_{O_1 i}}{m^*}, \quad y_{O_1} = \frac{\sum_{i=1}^{m^*} y_{O_1 i}}{m^*}, \quad l_1 = \frac{\sum_{i=1}^{m^*} l_{1_i}}{m^*}. \tag{5}$$

The measure of the deviation of the path of joint A from ideal circle is:

$$\delta_1 = \frac{\max|O_1 A_i| - \min|O_1 A_i|}{l_1}. \tag{6}$$

This error for joint B is derived analogically. The only difference are equations for the coordinates of the path of joint, which in case of B are as follows:

$$x_{Bi} = x_i + l_6 \cos(\theta_{2L}(s_i) + \beta), \quad y_{Bi} = y_i + l_6 \sin(\theta_{2L}(s_i) + \beta). \tag{7}$$

### 3.2. *The dyad with prismatic and revolute joints – PR*

Let us consider dyad ***PR*** consisting of the coupler and slider (Figure 6). The first symbol (***P*** – prismatic) refers to the type of the joint by which the dyad is connected to the frame, and the other (***R*** – revolute) refers to the type of joint which connects the moving links of the dyad. As opposed to the previous dyad, let the coupler be connected to the slider (moving along straight guiding link) by revolute joint B. The desired trajectory of point B is a straight line.

We analyse the motion of the coupler only when point D traces the prescribed path. The determination of the straight line which approximates the positions of the slider in

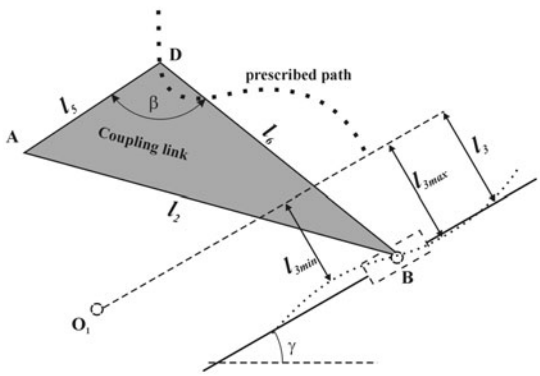


Figure 6. The path traced by joint B (dotted line) while the coupler point traces the prescribed path (dotted line), and the straight line which approximates this path (continuous line).

subsequent instants in the sense of least square error is presented. Then, this straight line, given as $ax + by - 1 = 0$, minimizes the expression:

$$\Delta = \sum_{i=1}^{m} (ax_{Bi} + by_{Bi} - 1)^2.$$

Point B for parameter $s_i$ has coordinates$(x_{Bi}, y_{Bi})$ computed from Equations (7). The optimal values of parameters $a$ and $b$ meet the linear equations obtained after differentiating $\Delta$ with respect to these parameters. After transformations it yields:

$$\sum_{i=1}^{m} x_{Bi} = a\sum_{i=1}^{m} x_{Bi}^2 + b\sum_{i=1}^{m} x_{Bi}y_{Bi}, \tag{8}$$

$$\sum_{i=1}^{m} y_{Bi} = b\sum_{i=1}^{m} y_{Bi}^2 + a\sum_{i=1}^{m} x_{Bi}y_{Bi}.$$

The coefficients $a$ and $b$ are determined by solving the foregoing system of linear equations. The straight line given as $ax + by - 1 = 0$ is transformed to the form $(l)$ $x\cos\alpha + y\sin\alpha - k = 0$, by means of the following substitutions: $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, $k = \frac{1}{\sqrt{a^2+b^2}}$. The guiding link is inclined at $\gamma = \alpha - \frac{\pi}{2}$ to the horizontal axis. To describe a mechanism geometry, it is desirable to define the guiding link by the inclination angle and the distance from a reference point, instead of parameters $a$ and $b$. The example being a crank-slider mechanism where the offset – the distance from the guiding link to the parallel axis passing through input link ground pivot $O_1$ – is given. For this purpose, we take point $O_1(x_{O_1}, y_{O_1})$, and $l_3$ equals to the distance of $O_1$ from straight line $(l)$. Unit vector $\boldsymbol{n} = [\cos\alpha, \sin\alpha]$ is perpendicular to the line $(l)$. Vector $\boldsymbol{r}_{O_1} = [x_{O_1} - x, y_{O_1} - y]$ points from an arbitrary point of line $(l)$ $(x, y)$ to $O_1$. The expression:

$$\boldsymbol{n} \circ \boldsymbol{r}_{O_1} = 1 r_{O_1} \cos(\boldsymbol{n}, \boldsymbol{r}_{O_1}) = l_3, \tag{9}$$

is the length of the projection of $\boldsymbol{r}_{O_1}$ onto direction perpendicular to line $(l)$.

The optimal dyad minimizes the expression:

$$\delta_2 = \frac{\max(l_{3i}) - \min(l_{3i})}{l_3}, \tag{10}$$

where $l_{3i}$ is the distance from point $B_i$ to the line parallel to the guiding link and passing through point $O_1$ at instant corresponding to the value $s_i$.

### 3.3. *The dyad with revolute and prismatic joints – RP*

We consider a dyad $ABO_2$ in which the floating link is connected by prismatic joint B to the rotating link pinned to the ground at $O_2$.

For kinematic analysis, it suffices to consider the dyad with link AC perpendicular to link $O_2B$. Every dyad in which the angle between the yoke AC and link $O_2B$ equals to $\alpha$ is kinematically equivalent to the dyad in which the yoke is perpendicular to the rotating link. The relation between the links lengths of these two cases is $l_3 = l_3' \sin\alpha$, as is illustrated in Figure 7. The motion of the coupler only with point D tracing the prescribed path is considered. The design parameters (coupler dimensions: $l_5$, $\theta_4$, and the parameters of the coupler angular position) are optimal when the envelope of arm AC is as close as possible to a circle, but for arbitrary parameters the envelope of AC is a certain curve (Figure 7). We present the method for determination of the circle which approximates the envelope of straight lines in the sense of least square error. Let versor $\boldsymbol{n}$ indicate the direction inclined at angle $\alpha$ to the horizontal line. The straight line perpendicular to versor $\boldsymbol{n}$ and passing through point $(x_0, y_0)$ is as follows:

$$\boldsymbol{m} \circ \boldsymbol{n} = [x - x_0, y - y_0] \circ [\cos\alpha, \sin\alpha] = x\cos\alpha + y\sin\alpha - x_0\cos\alpha - y_0\sin\alpha = 0. \tag{11}$$

The product of vectors $\boldsymbol{r} = [x_0, y_0]$ and $\boldsymbol{n} = [\cos\alpha, \sin\alpha]$ is the distance of the straight line from the origin:

$$\boldsymbol{r} \circ \boldsymbol{n} = x_0\cos\alpha + y_0\sin\alpha = r\cos(\boldsymbol{r}, \boldsymbol{n}) = k.$$

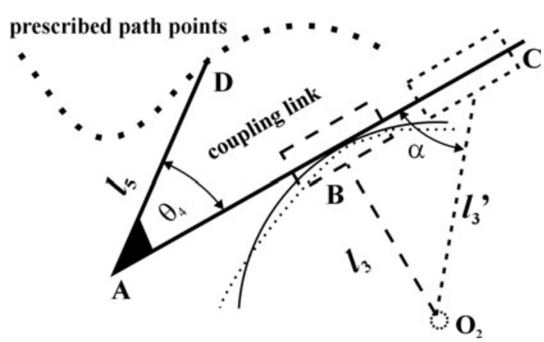


Figure 7. The envelope on which link AC moves (dotted line) while the coupler point traces the prescribed path, and the circular arc approximating this envelope (continuous line).

Finally, the expression

$$x \cos \alpha + y \sin \alpha - k = 0, \tag{12}$$

represents the straight line perpendicular to versor $\boldsymbol{n}$ and distant by $|k|$ from the origin, which is illustrated in Figure 8.

Let $p$ stand for the distance from a straight line to point $(x_{O_2}, y_{O_2})$. The equation of such a line is given by

$$(x - x_{O_2}) \cos \alpha + (y - y_{O_2}) \sin \alpha - p = 0,$$

which can be rewritten as follows:

$$x \cos \alpha + y \sin \alpha - x_{O_2} \cos \alpha - y_{O_2} \sin \alpha - p = 0.$$

Hence, $p' = x_{O_2} \cos \alpha + y_{O_2} \sin \alpha + p$ is the distance of a straight line to the origin.

The positions of arm AC in successive instants, specified by parameter $s = s_i$, $i = 1 \dots m$, are given as $a_i x + b_i y - 1 = 0$, which we transform to the form:

$$x \cos \alpha_i + y \sin \alpha_i - k_i = 0, \quad \text{where } \cos \alpha_i = \frac{a_i}{\sqrt{a_i^2 + b_i^2}}, \quad \sin \alpha_i = \frac{b_i}{\sqrt{a_i^2 + b_i^2}},$$
$$k_i = \frac{1}{\sqrt{a_i^2 + b_i^2}}.$$

The family of straight lines distant by $p$ from the point $(x_{O_2}, y_{O_2})$ can be written as follows:

$$x \cos \alpha_i + y \sin \alpha_i - (x_{O_2} \cos \alpha_i + y_{O_2} \sin \alpha_i + p) = 0.$$

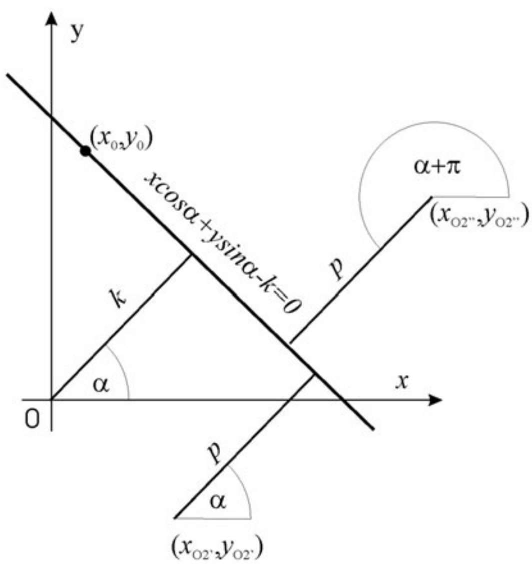


Figure 8. Illustration of two cases: points $O_2$ and O lie on the same side of the straight line and the points lie on the half planes divided by the straight line.

The least square method allows to find the circle which approximates the envelope of the straight lines $x\cos\alpha_i + y\sin\alpha_i - k_i = 0$. The coordinates of the circle centre $(x_{O_2}, y_{O_2})$ and the radius $p$ minimize the expression:

$$\Delta_2 = \sum_{i=1}^{n} (k_i - x_{O_2}\cos\alpha_i - y_{O_2}\sin\alpha_i - p)^2. \tag{13}$$

Differentiating $\Delta_2$ with respect to $x_{O_2}, y_{O_2}$, and $p$ we get the following system of linear equations:

$$\begin{aligned}
\sum_{i=1}^{m} k_i\cos\alpha_i &= x_{O_2}\sum_{i=1}^{m}\cos^2\alpha_i + y_{O_2}\sum_{i=1}^{m}\sin\alpha_i\cos\alpha_i + p\sum_{i=1}^{m}\cos\alpha_i,\\
\sum_{i=1}^{m} k_i\sin\alpha_i &= x_{O_2}\sum_{i=1}^{m}\cos\alpha_i\sin\alpha_i + y_{O_2}\sum_{i=1}^{m}\sin^2\alpha_i + p\sum_{i=1}^{m}\sin\alpha_i,\\
\sum_{i=1}^{m} k_i &= x_{O_2}\sum_{i=1}^{m}\cos\alpha_i + y_{O_2}\sum_{i=1}^{m}\sin\alpha_i + p\sum_{i=1}^{m}1.
\end{aligned} \tag{14}$$

The foregoing system is solved with respect to $x_{O_2}$, $y_{O_2}$, and $p$.

It should be noted, however, that the straight line $x\cos\alpha + y\sin\alpha - k = 0$ may divide the plane so that the origin and point $O_2$ fall on different half planes, as illustrated in Figure 8. In such a case, we obtain the straight line equation distant by $k$ from the origin and by $p$ from point $O_2$, after substituting $\alpha + \pi$ for angle $\alpha$ into equation (13). Then, the formula for $E_2$ should be modified taking into account that

$$k = (x_{O_2}\cos(\alpha+\pi) + y_{O_2}\sin(\alpha+\pi) + p).$$

The circle with centre at $O_2$ and radius $p$ is an approximation of the envelope of the lines describing the positions of link AC. The ground pin is located at point $O_2$. In general case, the actual distance of a straight line from point $O_2$ is not equal $p$. Determining the distance of each straight line AC from point $O_2$, one can express the approximation error. For this purpose, we consider the line $(l)$ passing through points $(x, y)$ and $(x_0, y_0)$. Let us denote:

$$\mathbf{m} = \frac{[x_0 - x, y_0 - y]}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2}} \text{ and } \boldsymbol{r}_{O_2} = [x_{O_2} - x, y_{O_2} - y].$$

The length of the projection of vector $\boldsymbol{r}_{O_2}$ onto line $(l)$ is equal to $\boldsymbol{m} \circ \boldsymbol{r}_{O_2} = 1 r_{O_2}\cos(n, r_{O_2}) = \lambda$. The distance between line $(l)$ and point $(x_{O_2}, y_{O_2})$ equals to:

$$p = \sqrt{r_{O_2}^2 - \lambda^2}. \tag{15}$$

Using Equation (15), one can calculate the distance $p_i = |O_2B_i|$ for every position of link AC, $i = 1 \ldots m$. The deviation of the motion of link AC expressed by design parameters from the motion which occurs after coupling link AC to $O_2B$ is as follows:

$$\delta_3 = \frac{\max|O_2B_i| - \min|O_2B_i|}{l_3}, \tag{16}$$

where $l_3 = \frac{1}{m}\sum_{i=1}^{m}|O_2B_i|$ can be taken as the length of the link pinned at point $O_2$ $(x_{O_2}, y_{O_2})$ instead of $p$ determined from the system (14).

## 4. The objective functions for chosen planar mechanisms

For the most common planar mechanisms, the objective function measuring the structural error in path synthesis is a combination of functions $\delta_1$, $\delta_2$ and $\delta_3$. It is illustrated in the example of three mechanisms: four-bar linkage, crank-slider mechanism and a mechanism with sliding block.

### 4.1. *Four-bar linkage*

The geometry of the four-bar linkage as a closed path generator is defined by nine parameters (Figure 9):

- coordinates of the fixed ground pins – $x_{O_1}$, $y_{O_1}$, $x_{O_2}$, $y_{O_2}$,
- links lengths – $l_1$, $l_2$, $l_3$, $l_5$,
- the angle between arms AD and AB of the coupler – $\theta_4$.

This number can be reduced to five ($l_2$, $l_3$, $l_4 = x_{O_2}$, $l_5$, $\theta_4$) when the generator of a path obtained by means of affine transformation (of a path of the same shape) is searched for [2, 6–8, 16]. In consequence of non-dimensionality of such a path definition, one can take that:

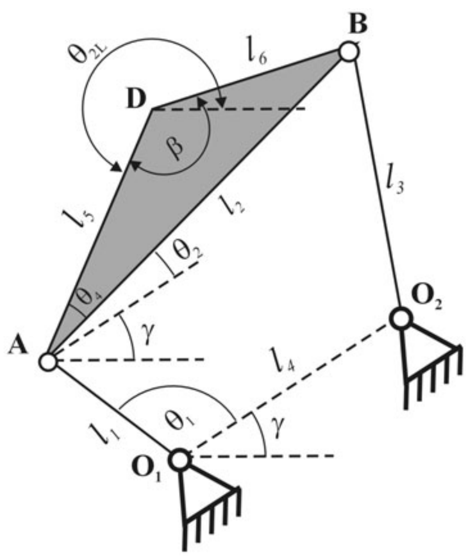


Figure 9. The geometric scheme of a four-bar linkage.

- $x_{O_1} = 0$ and $y_{O_1} = 0$, as the description is invariant with respect to translation. The active link can be fixed at the system origin.
- $y_{O_2} = 0$, as the description is invariant with respect to the rotation and the immoveable link can be directed horizontally.
- The active link length $l_1 = 1$, as the description is invariant with respect to scaling.

The four-bar linkage as an open path generator is described by 11 parameters. The two extra parameters are the boundaries of the angle through which the active link rotates when a path is traced – starting angle $\theta_{10}$ and ending angle $\theta_{11}$.

To solve the synthesis problem for the four-bar linkages as open path generator by means of the method introduced in this paper, one has to determine the following:

- the dimensions of the coupler: lengths $l_5$, $l_6$ and angle $\beta$,
- the coefficients of function (2): $v_1$, $v_2$, $v_3$, $v_4$,

for which the coupler moves in a way constrained by the four-bar linkage geometry when point D traces required trajectory.

It follows that the number of optimized (design) parameters is by two greater than the minimum number of design parameters in four-bar linkage synthesis, but it is by four less than the number of geometric parameters describing four-bar linkage as path generator.

The objective function combines the functions $\delta_1$ (6) for the both joints of the coupler, and is given by

$$E_1 = \frac{\max|O_1A_i| - \min|O_1A_i|}{l_1} + \frac{\max|O_2B_i| - \min|O_2B_i|}{l_3}. \tag{17}$$

### 4.2. *Crank-slider mechanism*

The crank-slider mechanism is built up of the crank, the connecting rod and the slider. In case of this mechanism, the trajectory of point A is a circle, whereas the trajectory

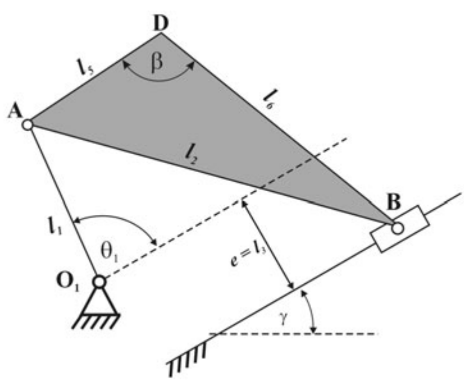


Figure 10. The geometric scheme of a crank-slider mechanism.

of point B is the straight line distant by $e = l_3$ from joint $O_1$ and inclined at angle $\gamma$ to the horizontal axis.

The geometry of the considered mechanism as an open path generator is defined by 10 parameters (Figure 10):

- coordinates of the fixed ground pin – $x_{O_1}$, $y_{O_1}$,
- links lengths – $l_1$, $l_2$, $l_5$,
- the angle between arms AD and DB of the connecting rod – $\beta$,
- the angle of the guiding link inclination – $\gamma$,
- the offset $e = l_3$,
- the range of the angle through which the crank rotates when a path is drawn – $\theta_{10}$, $\theta_{11}$.

According to the synthesis method, the connecting rod is disconnected from the mechanism, and point D traces the desired path. The objective is to determine the coefficients of function (2) and lengths $l_5$, $l_6$ as well as angle $\beta$ between arms AD and BD for which the trajectory of joint A is approximated by a circle (or circular arc), and the trajectory of joint B is approximated by a straight line as closely as possible (in the sense of elaborated mathematical measures). The number of the design parameters is reduced to seven: $l_5$, $l_6$, $\beta$, $v_1$, $v_2$, $v_3$, $v_4$.

The objective function is the sum of $\delta_1$ (6) for joint A and $\delta_2$ (10) for joint B, and has the following form:

$$E_2 = \frac{\max|O_1A_i| - \min|O_1A_i|}{l_1} + \frac{\max(l_{3i}) - \min(l_{3i})}{l_3}. \tag{18}$$

### 4.3. *Mechanism with slotted link*

We pass on a variation of the quick return mechanism (Figure 11). The external links are connected to the frame by revolute joints. The slotted link AC (yoke) is connected

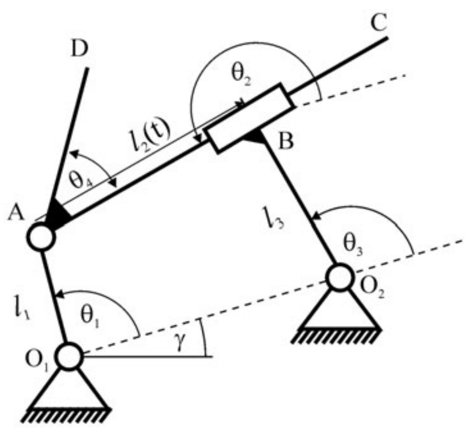


Figure 11. The geometric scheme of a mechanism with slotted link.

to active link $O_1A$ by revolute joint A. The sliding block (output link, rocker) $O_2B$ slides along the slotted link. Links AC and $O_2B$ constitute a prismatic pair.

The geometry of the considered mechanism as an open path generator is defined by 10 parameters:

- coordinates of the fixed ground pins – $x_{O_1}$, $y_{O_1}$, $x_{O_2}$, $y_{O_2}$,
- links lengths – $l_1$, $l_3$, $l_5$,
- the angle between arms AD and AC of the slotted link – $\theta_4$,
- the range of the angle through which the crank rotates when a path is drawn – $\theta_{10}$, $\theta_{11}$.

When the input link rotates around point $O_1$, the trajectory of point A is a circle (or circular arc). Simultaneously, link AC rolls on the circumference of the other circle (i.e. link AC is tangent to this circle in successive instants), which means that the envelope of the positions of arm AC is a circular arc as well. The circle has the centre at point $O_2$ and the radius equal to $l_3$. We repeat the same procedure as in case of the four-bar linkage and crank-slider mechanism. The link AC is disconnected from the mechanism, and point D moves along the desired path. The aim is to determine the optimal dimensions of link AC and the coefficients describing angular motion of its own such that the trajectory of point A and the envelope of the positions of the arm AC approximate circles. The sought parameters are: the length AD and the angle between arms AD and AC – $\theta_4$ (Figure 11). The number of all the design parameters is reduced to six: $l_5$, $\theta_4$, $v_1$, $v_2$, $v_3$, $v_4$.

The objective function combines the functions $\delta_1$ (6) for the first joint and $\delta_3$ (16) for the second joint of the slotted link, and is given by

$$E_3 = \frac{\max|O_1A_i| - \min|O_1A_i|}{l_1} + \frac{\max|O_2B_i| - \min|O_2B_i|}{l_3}. \tag{19}$$

## 5. The optimization technique

### 5.1. *The EA description*

The EA is employed to minimize the objective functions (17)–(19). In accordance with the nomenclature of the EAs mechanism is an individual, its optimized features $\chi$ are the design parameters, e.g. for the four-bar linkage $\chi \in \{l_5, l_6, \beta, v_1, v_2, v_3, v_4\}$.

The population is a set of all individuals. $N_{\max}$ stands for the size of the population, $l_{cr}$ is the number of individuals to be crossed over in each generation.

The best-fitted individual (the one which gives the lowest value of the objective function) in the generation is the first parent (I) for all new individuals of the successive generation. $l_{cr}$ individuals are randomly selected from the population and become the second parents (II). Then, $l_{cr}$ crossovers are executed in each generation, as the consequence $l_{cr}$ new individuals ($N$) come into being. The probability of drawing out the second parent is the same for all the remaining individuals and equal to $1/(N_{\max} - 1)$.

Let us take a pair of parents which produce a new individual. During a crossover, a $c$th feature $\chi_{i,c}^{j+1}$ of the new $i$th individual (N) in the generation $j+1$ is inherited either from the best fitted individual with a probability $p_b$ or from parent (II) with the probability $1-p_b$.

The features are also mutated and randomly disturbed. For these purposes, two individuals are randomly selected with equal probability to mutate the feature of the new individual. The numbers of these two individuals are $l, k$, and the values of their mutated features are $\chi^j_{l,c}$, $\chi^j_{k,c}$. The difference $\left(\chi^j_{k,c} - \chi^j_{l,c}\right)$ is computed, and multiplied by $p_1\lambda(1+\mu p_2)$, where $p_1 \in (0,1)$ and $p_2 \in (-1,1)$ are random numbers, $\lambda$ is the mutation coefficient, $\mu$ is the disturbance coefficient.

To sum it up, the crossover and mutation for one feature may be written as follows:

$$\chi^{j+1}_{i,c} = \left(\chi^j_{h,c} + p_1\lambda\left(\chi^j_{k,c} - \chi^j_{l,c}\right)\right)(1+\mu p_2).$$

The superscript $j$ indicates the generation number, the subscript $i$ indicates the individual number, the subscript $c$ indicates the feature number.

$h$ – the number of the parent from which the individual ($N$) inherits the feature: $h=1$ with probability $p_b$ or $h=i$ with probability $1-p_b$.

This procedure is executed for all features and for all new individuals.

If the individual ($N$) is better fitted than parent (II), it becomes a member of the successive population, otherwise parent (II) remains and the individual ($N$) is removed. The size of each population is constant. The initial population is generated for preset intervals of all the features: $l_{5d} \le l_5 \le l_{5g}$, $l_{6d} \le l_6 \le l_{6g}$, $0 \le v_i \le 2\pi$, $i=1 \dots 4$, $0 \le \beta \le 2\pi$.

The algorithm includes no special improvements and as a tool widely used in mechanism theory is not described in depth.[35] The only enhancement consists in introducing to each population $l_{rnd}$ ($l_{rnd} < N_{\max} - l_{cr}$) randomly generated individuals in place of the worst fitted ones. The whole solution space is still searched through, along with the improvement of the best individual. It prevents the algorithm from getting stuck in local minima. Moreover, the mutation and disturbance coefficients are decreased when the fitness of the best solution fails to increase in a prescribed number of iterations.

The following parameters of the EA were taken in the presented examples to minimize the objective functions:

- The size of the population $N_{\max} = 100$.
- The number of individuals to be crossed over in each generation $l_{cr} = 55$.
- The number of randomly generated individuals introduced to each population in place of the worst fitted ones $l_{rnd} = 25$.
- A feature of the new individual ($N$) is inherited from the parent (I) with a probability $p_b = 0.75$ (from parent (II) with the probability 0.25).
- The initial value of the mutation coefficient $\lambda$=0.5 (the value is gradually decreased to 0.005), the initial disturbance coefficient $\mu = 0.25$ (the coefficient is gradually decreased to 0.0025).
- The minimum value of the error function (objective function) $E_{\min} = 0.03$.
- The maximum number of iterations executed (generations) in case of not achieving prescribed minimum value of error function $L_g = 1000$.

To establish the values of the parameters controlling the EA, preliminary numerical simulations were carried out. For this purpose, the results of syntheses of a four-bar linkage generating a prescribed path were analysed. The number of iterations ($L_g = 1000$) and the size of the population ($N_{\max} = 100$) were assumed. The values of the remaining parameters were determined in two iterations. The steps of each iteration are as follows:

(1) The initial values of $l_{cr}$, $l_{rnd}$ and $p_b$ were chosen arbitrarily and assumed to be the same in all executions of the algorithm in this step. The values of $\lambda$ and $\mu$ were being changed, but the algorithm was executed 10 times for each set of the same values, and the average value of the errors (objective functions) of the best solutions in this 10 executions was computed. The values of $\lambda$ and $\mu$ were changed 20 times and for all new sets of the parameters the average error was computed. The parameters $\lambda$ and $\mu$ corresponding to the minimum average error were read and taken as inputs in the next step.
(2) $p_b$ and $\lambda$, $\mu$ (obtained in step 1) were assumed, whereas $l_{cr}$, $l_{rnd}$ were being changed. For each set of the parameters, the algorithm was executed 10 times. Analogically as in step 1, the parameters $l_{cr}$, $l_{rnd}$ corresponding to the minimum average error were read and taken as inputs in the next step.
(3) $\lambda$, $\mu$ (obtained in step 1), $l_{cr}$, $l_{rnd}$ (obtained in step 2) were assumed, and $p_b$ was being changed. The procedure was repeated to establish the value $p_b$.

Steps 1–3 were carried out twice. The outputs of the first iteration were taken as the input parameters in the second iteration. The resultant EA parameters were assumed for computations presented in the paper.

The graphical scheme of the optimization process is presented in Figure 12.

### 5.2. *Number of points defining the curve*

In most cases, it suffices to discretize a desired path by 10 points. If it is necessary to improve the accuracy of the solution, this number may be increased. Nonetheless, the number of points should be taken carefully. An excessive number of points not only fails to improve a solution but also increases the computational costs.

### 5.3. *The control of the input link motion*

The input link cannot come to a halt and start rotating in the opposite direction when the coupler point D traces the prescribed path. In such a case the solution is rejected. The input link rotates in one direction if the sign of the vector product for each pair of adjacent vectors representing input link in subsequent positions is of the same sign:

$$sgn(\boldsymbol{O}_1\boldsymbol{A}_i \times \boldsymbol{O}_1\boldsymbol{A}_{i+1}) = sgn(\boldsymbol{O}_1\boldsymbol{A}_j \times \boldsymbol{O}_1\boldsymbol{A}_{j+1}), \quad \text{for each } i,j = 1 \ldots m-1.$$

Such a control of direction of the input link rotation ensures that the mechanism can be assembled at worst as a double-rocker mechanism approximating the prescribed path. Moreover, the control of the input link motion rejects the solutions in which the path is traced by two conjugate mechanisms (branching effect).

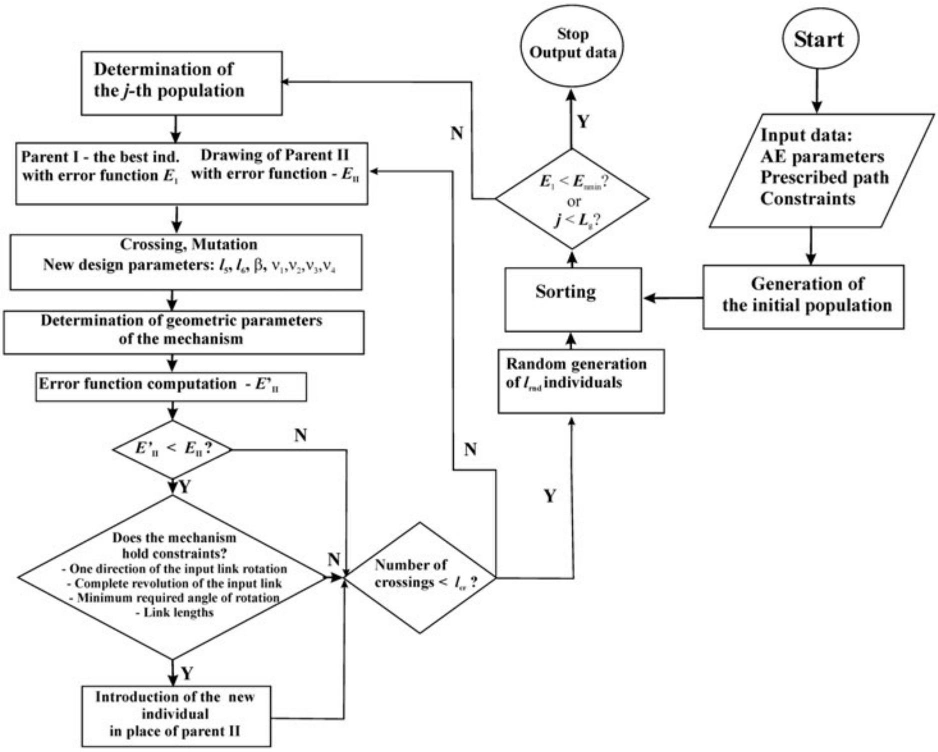


Figure 12. Flowchart illustrating optimization process.

### 5.4. *The minimum required angle of rotation of the input link*

The minimum angle $\Delta\theta_1 = \theta_{11} - \theta_{10}$, through which the active link rotates when the coupler point traces a path, is imposed to reduce the workspace of a mechanism obtained. The observations show that in some cases the algorithm converges to the mechanism which traces a prescribed path for a small rotation of the input link. The implication being that the size of the mechanism is very large in comparison with the length of the path.

In general, the accuracy with which coupler point approximates a prescribed path may be improved by decreasing $\Delta\theta_1$. The maximum values of the coupler dimensions as well as of the lengths of the input and output links are specified for the same purpose, i.e. to reduce the workspace of a mechanism obtained.

### 5.5. *Mobility analysis*

An approximating path can be a section of either a closed coupler curve obtained during complete rotation of the input link and best fitted to the desired path or an open curve obtained during incomplete rotation of the input link and best fitted to the desired path. The four-bar linkage may work as double-crank or crank-rocker mechanisms (when the active link can make complete revolution), or as double-rocker mechanism (it has limited revolutions of both links connected to the ground). For technical reasons, a designer may not accept double-rocker mechanism as a path generator. In this case, a penalty function is added to the objective function if Grashof conditions are not met. Similarly, one can verify whether the active link can rotate continuously through 360° when the crank-slider mechanism and mechanisms with slotted link are synthesized. The control of the direction of rotation (Section 5.3) ensures that the mechanism obtained can be assembled as a double-rocker mechanism at worst as, even in this case, the prescribed path is approximated over the permissible range of the motion of the input link.

If the full revolution of the input link is required, the following conditions are checked:

- Grashof conditions for a four-bar mechanism.
- For the crank-slider mechanism:

$$|l_3| + l_1 \leq l_2,$$

where $l_1, l_2, l_3$ are the length of the input link, the length of the connecting rod and the offset, respectively.

- For the mechanism with slotted link:

$$||O_1O_2| - l_3| \geq l_1,$$

where $l_1, l_3$ are the lengths of the input and output links.

The constraints for the crank-slider mechanism and the mechanism with slotted link can be formulated on the basis of geometric schemes or can be derived from analytical equations for links positions.

### 5.6. *Comparison measures*

The basic idea of the method is to find the motion of the coupler such that a coupler point traces precisely desired path, and the joints of free (unconstrained) coupler approximate either circular arcs or straight lines. When the coupler is connected to the input link and output link, determined by means of the algorithms presented in Section 5, the coupler point trajectory approximates the desired path. The introduced objective functions fail to directly measure the accuracy of this approximation and hence more natural errors should be introduced. An example being the variation $\Delta y$ in $y$-value between the desired path and its approximation. It is obvious that for determined dimensions the real angular position of the coupler $\theta^*_{2L}$ expressed in terms of computed mechanism dimensions and angular position of the input link varies from $\theta_{2L}$, expressed in terms of design parameters $v_1$, $v_2$, $v_3$ and $v_4$. Therefore, the other proposition of the error has the form $E_{\theta_2} = \theta_{2L} - \theta^*_{2L}$.

## 6. Numerical tests

The algorithms are employed to determine the crank-slider mechanism and mechanism with sliding block as generators of chosen paths. The four-bar linkage as path generator is presented in Section 7 to illustrate modifications of the synthesis method. The following paths are taken to verify the effectiveness of the method:

- *Straight line*: $x = 4s$, $y = 0$, $0 \leq s \leq 1$.
- *Square angle*: $x = 2s$, $y = 0$, $0 \leq s \leq 0.5$, $x = 1$, $y = 2s - 1$, $0.5 \leq s \leq 1$.
- *Parabola*: $x = s$, $y = 3s(s - 1)$, $0 \leq s \leq 1$.
- *Test*: the path traced by the four-bar linkage: $l_1 = 1$, $l_2 = 2$, $l_3 = 2$, $l_4 = 2$, $l_5 = 3$, $\theta_4 = 1$ rad, for prescribed input link revolution $1.5 \leq \theta_1 \leq 4.5$ rad.

The cases comprise the paths given parametrically and the path traced by the coupler point of the prescribed four-bar linkage. With the parameter $s$ non-dimensional, the links lengths are also non-dimensional. The mechanisms presented in this section were localized in the first 200 iterations of the EA realizing synthesis problem.

Crank-slider mechanisms generating paths *Square angle* and *Parabola* are presented. Table 1 presents numbers of points defining the paths and constraints imposed, i.e. maximum permissible links lengths and minimum required angle through which the active link has to rotate when a path is drawn. The values of the design parameters are gathered in Table 2. Given the design parameters, one can compute the remaining dimensions of crank-slider mechanisms using equations derived in Section 3.2. These dimensions as well as the intervals of angle $\theta_1$ for which paths are traced are also gathered in Table 2. Figure 13 presents the mechanism approximating *Square angle*, comparison of paths generated and desired, differences between $y$ coordinates of these paths and error $E_{\theta_2}$. The mechanisms in which the active link cannot complete full revolution were also accepted. Figure 14 presents the graphs characterizing the generator of path *Parabola*.

The method was employed to find the mechanisms with slotted link approximating test paths *Straight Line* and *Test*. The design parameters obtained are gathered in Table 2. Given the design parameters, one can compute the remaining dimensions of the mechanisms using equations derived in Section 3.3. Table 2 contains also both the mechanism dimensions, the angles through which the active link rotates when paths are traced, the numbers of points defining the paths and constraints imposed. Figure 15 shows the mechanism that may work as a film advance mechanism of a movie camera or cinema projector. Film frame is advanced when the coupler point is in rectilinear motion. The crank rotates through 90° in this phase of motion. At a constant rotational speed of the crank, the ratio $t_2/t_1 = 3$, where $t_1$ is the time taken to advance a frame, and $t_2$ is the time taken for return motion. It follows that the motion of the frame takes

Table 1. Constraints imposed in the synthesis of crank-slider mechanism and mechanism with slotted link.

| Path | Square angle | Parabola | Straight line | Test |
|---|---|---|---|---|
| $m$ | | 10 | | |
| $l_{5g}$ | 20 | 20 | 10 | 20 |
| $l_{6g}$ | 30 | 30 | – | – |
| Max($l_1$) | | 5 | | |
| Max($l_3/l_1$) | | 5 | | |
| $\Delta\theta_1$ | $\pi/4$ | $\pi/4$ | $\pi/2$ | $\pi/4$ |

Table 2. The design parameters and remaining geometric parameters obtained in the synthesis of crank-slider mechanism and mechanism with slotted link.

| Test path | Square angle | Parabola | Straight line | Test |
|---|---|---|---|---|
| Mechanism | Crank-slider mechanism | Crank-slider mechanism | Mechanism with slotted link | Mechanism with slotted link |
| Error | $E_2 = 0.1$ | $E_2 = 0.0291$ | $E_3 = 0.011$ | $E_3 = 0.0177$ |
| *Design parameters* | | | | |
| $l_5$ | 10.84 | 13.784 | 5.716 | 5.104 |
| $l_6$ | 8.37 | 7.34 | – | |
| $\beta$ [rad] | 0.936 | 2.04 | 3.153 | 3.127 |
| $v_1$ | 0.292 | 3.382 | 1.546 | 1.124 |
| $v_2$ [rad] | 1.984 | 0.0926 | −0.347 | −0.836 |
| $v_3$ [rad] | −4.611 | 2.623 | 0.062 | −2.595 |
| $v_4$ [rad] | 0 | 0 | −1.388 | 2.717 |
| *Geometric parameters* | | | | |
| $l_1$ | 3.222 | 4.997 | 0.698 | 2.022 |
| $l_3$ | 6.00 | 5.598 | 0.011 | 1.387 |
| $x_{O_1}$ | 8.641 | 1.1039 | 2.096 | −0.936 |
| $y_{O_1}$ | 0.421 | 18.023 | −5.016 | 2.127 |
| $x_{O_2}$ | – | – | 2.068 | 3.032 |
| $y_{O_2}$ | – | – | −7.036 | 1.857 |
| $\gamma$ [rad] | 2.9176 | 4.637 | −1.583 | −0.068 |
| $\theta_{10}$ [rad] | −3.224 | −0.456 | −0.815 | −5.129 |
| $\theta_{11}$ [rad] | −1.93 | 0.551 | 0.757 | −2.483 |
| | The initial positions of coupler points for $\theta_1 = \theta_{10}$ | | | |
| – | (Angle $\theta_1$ is measured from the guiding link as is shown in Figure 10) | | (Angle $\theta_1$ is measured from line $O_1O_2$ as is shown in Figure 11) | |
| $x_A$ | 10.396 | −1.429 | 1.583 | 0.006 |
| $y_A$ | 3.123 | 13.715 | −5.489 | 3.916 |
| $x_B$ | 2.839 | −6.165 | 2.079 | 1.875 |
| $y_B$ | 7.9 | −3.98 | −7.357 | 1.091 |
| $x_D$ | 0.007 | 0.002 | 0.051 | 2.76 |
| $y_D$ | 0.0266 | 0.006 | 0.0189 | −0.381 |

approximately 25% of the total cycle time. The required straight line and the line approximated by the mechanism are shown in Figure 15(b). Figure 15(c) presents the error $E_{\theta_2}$.

The solution for path *Test* proves that the mechanism of this type with high accuracy approximates the coupler curve of a four-bar linkage. The appropriate graphs are presented in Figure 16.

## 7. Examples of practical applications

### 7.1. *Modification I – the mechanism with prescribed ground pins coordinates*

An engineer frequently cannot arbitrarily locate the ground pins when machine designing. With respect to the housing conditions, the coordinates of the mechanism fixation

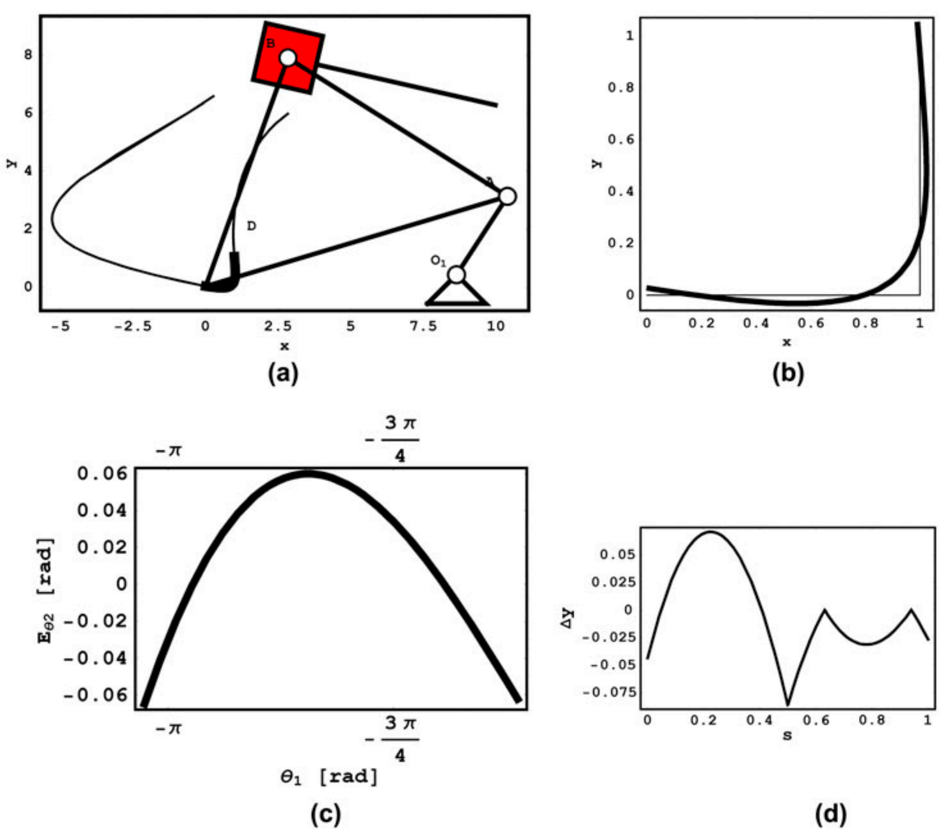


Figure 13. The crank-slider mechanism approximating path *Square angle* (a), comparison of paths desired and generated (b), error $E_{\theta_2}$ (c), error $\Delta y$ (d).

must be prescribed or, at least, must be specified an area in which the ground pins are located. Let us take the harbour crane as an example. It transports a cargo at constant height during shipment and disembarkation. The structure of the crane may be based on the double-rocker four-bar linkage, but the location of the ground pins cannot be arbitrary. One can demand, for example, that the rotating links are pinned to the ground.

An inconvenience of the presented method is that the coordinates of the ground pins cannot be specified at the stage of the problem formulation and derivation of the objective functions. The reason is that the objective functions measure the deviation of the joints trajectories from the circles approximating these trajectories and centred at computed (not at prescribed) points. It means that an objective function does not depend on these coordinates, and at best they can be taken into account in the form of constraint equations. But the prescription of the permissible ranges of ground pins coordinates may extend the time of calculations especially when the ranges are small or a specific location of a ground pin is required. It makes the method in the current form less favourable for practical applications.

A modification of the synthesis method is presented to regard the ground pins coordinates as input data and overcome the discussed deficiency. In the modified form, the

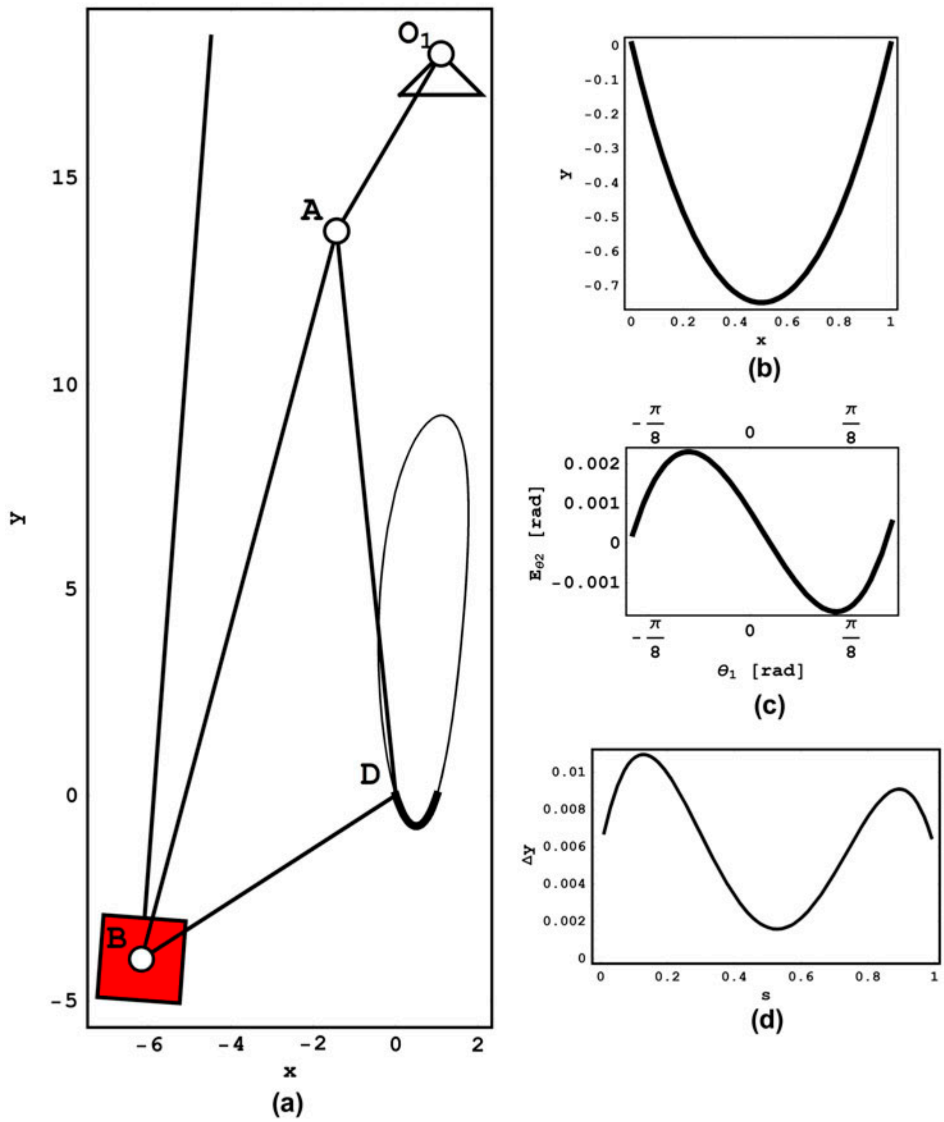


Figure 14. The crank-slider mechanism approximating path *Parabola* (a), comparison of paths desired and generated (b), error $E_{\theta_2}$ (c), error $\Delta y$ (d).

objective function measures the deviation of the joints trajectories from the circles with prescribed centres or centres lying on prescribed lines.

Assume that the position of the first ground pin is given, e.g. $O_1(0, 0)$. The other ground pin $O_2(x_{O2}, 0)$ has to lie on the line $y = 0$, and the interval of the horizontal coordinate is prescribed: $x_{\min} < x_{O_2} < x_{\max}$ (Figure 17). The successive locations of joints A and B, corresponding to the locations of coupler point D on a desired path, are denoted as: $A_i$ and $B_i$ $(i = 1 \ldots m)$. The length of the active link is calculated from

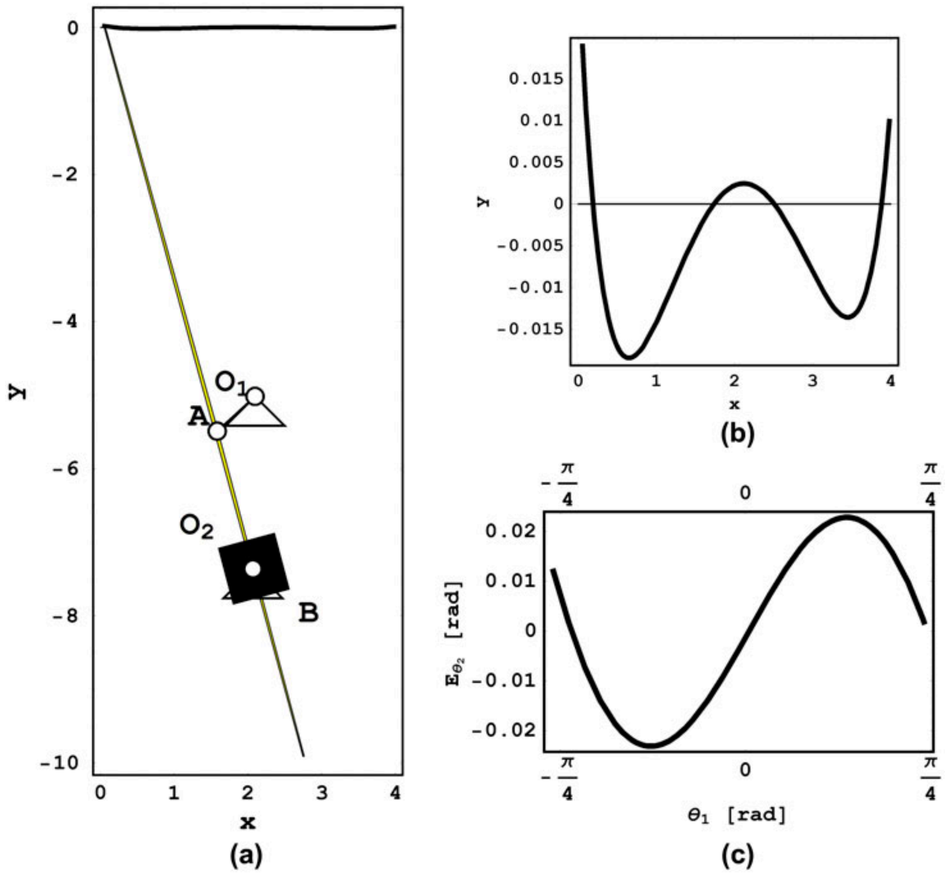


Figure 15. The mechanism approximating path *Straight Line* (a), comparison of paths desired and generated (b), error $E_{\theta_2}$ (c).

Equations (4) and (5). One can determine the circle that best approximates the trajectory of joint B, regarding the fact that the centre of this circle lies on the line $y = 0$. The proposed approach for the problem solution consists in determining centre coordinates $x$ of the circles passing through two points through which joint B passes at instants corresponding to the iterations $i$ and $j = m^* + i$, where $m^* = \lfloor m/2 \rfloor$:

$$(x_{Bi} - x_{O_2 i})^2 + y_{Bi}^2 = (x_{Bj} - x_{O_2 i})^2 + y_{Bj}^2,$$

from which:

$$x_{O_2 i} = \frac{x_{Bi}^2 + y_{Bi}^2 - (x_{Bj}^2 + y_{Bj}^2)}{2(x_{Bi} - x_{Bj})},$$

where $i = 1 \ldots m^*$, $j = m^* + i$.

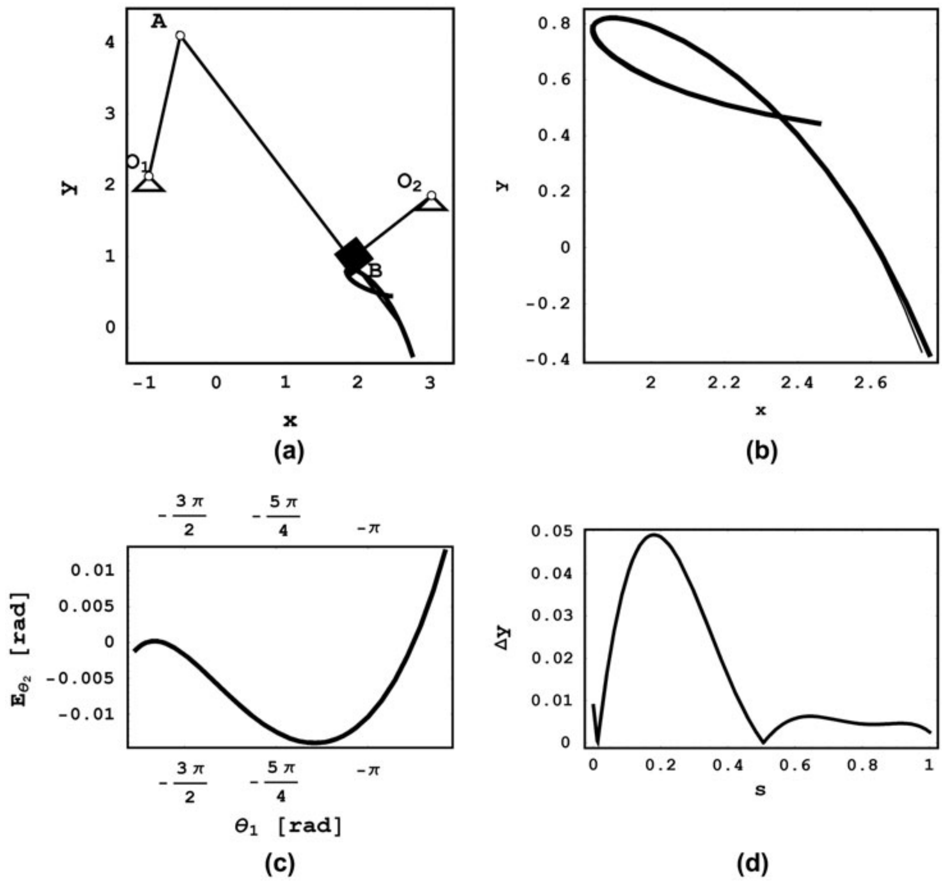


Figure 16. The mechanism approximating path *Test* (a), comparison of paths desired and generated (b), error $E_{\theta_2}$ (c), error $\Delta y$ (d).

The coordinate of joint $O_2$ is given by:

$$x_{O_2} = \frac{\sum_{i=1}^{m^*} x_{O_2 i}}{m^*}. \tag{20}$$

The solution is accepted if the horizontal coordinate meets the relations $x_{\min} < x_{O_2} < x_{\max}$. The objective function has the form (17). Obviously, the coordinates of point $O_2$ can be precisely prescribed, as is the case with joint $O_1$. For illustrative purposes, the following problem is solved using this method.

**Example 1.** Design the crane (four-bar linkage) for transporting cargo at constant height of 12 m between the points of disembarkation and shipment. The first point is located at the vertical reference line $x = 0$, the other is distant by 5 m from this line. Take that the active link is attached 2 m away from the reference line, and the ground pin of the passive link is distant less than 5 m from the first support. The height of the

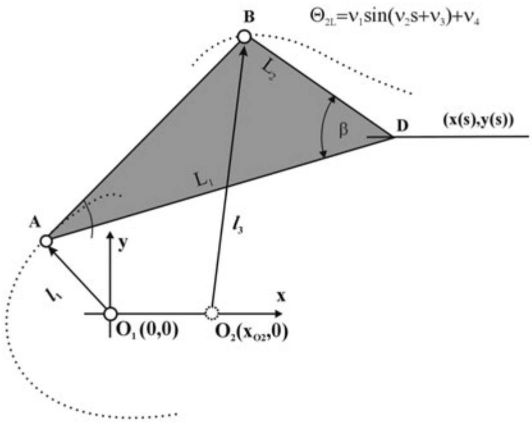


Figure 17. The illustration of the method with fixed ground pins coordinates.

crane must be less than 15 m. The minimum required angle $\Delta\theta_1$ through which the active link has to rotate to transport the cargo along the prescribed line is $\pi/10$.

**Solution**

As was the case with the numerical tests presented in the previous section, the synthesis process is executed according to the scheme shown in Figure 12. The straight line is discretized into 10 equally spaced points. The minimized objective function is given by (17) as the four-bar linkage is synthesized. The coordinates of point $O_1$ and link lengths are determined from Equations (5), whereas the horizontal coordinate of joint $O_2$ is obtained from (20).

The design parameters of an exemplary solution are: rotating links lengths [m]: $l_1 = 13.864$, $l_3 = 14.442$; lengths of the arms of the coupler link [m]: $l_5 = 2.256$, $l_6 = 2.425$; the angle between coupler arms $\beta = 5.464$ rad; ground pins coordinates [m] $O_1(-2, 0)$, $O_2(2.165, 0)$; and the starting and ending angles of active link position: $\theta_{10} = 1.205$ rad, $\theta_{11} = 1.518$ rad. The position of coupler points at $\theta_1 = \theta_{11}$: A(−1.292, 13.846), B(0.51, 14.347), D(−0.22, 11.98).

The crane and the trajectory of the cargo fixation are shown in Figure 18. The maximum deviation of the projected coupler curve from straight line is 2.5 cm. The deviation from the straight line referred to the length of the line equals to 0.005.

### 7.2. *Modification II – motion along prescribed path with prescribed initial and final angular orientations of the coupler*

One of important problems of mechanism synthesis is to design machines in which a link is guided through a series of specified positions. The translational and rotational motions of this link are prescribed. An example of such a device is the feeder transporting a load. The mass centre of the load is to move along a straight line, and the load has to be rotated through the angle equal to the difference between the inclination

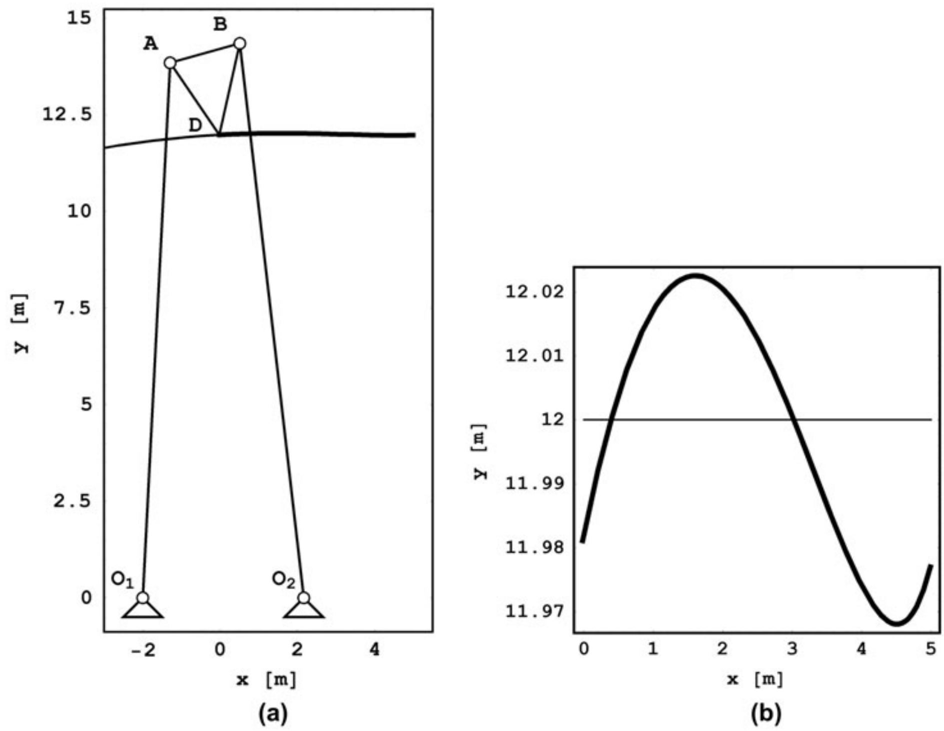


Figure 18. The geometric scheme of the harbour crane (a), the deviation of the cargo suspension point from the straight line (b).

angles at the feeder and receiver. The method can be easily adapted to this synthesis task. The difference $\Delta\theta_{2L}$ between the extreme positions of the coupler is given as:

$$\begin{aligned}\Delta\theta_{2L} &= \theta_{2L}(1) - \theta_{2L}(0) = v_1 \sin(v_2 + v_3) + v_4 - (v_1 \sin(v_3) + v_4)\\ &= v_1(\sin(v_2 + v_3) - \sin(v_3)).\end{aligned}$$

The parameter $v_1$ is calculated from the above equation:

$$v_1 = \frac{\Delta\theta_{2L}}{\sin(v_2 + v_3) - \sin(v_3)}, \tag{21}$$

which is no longer a design parameter. The following example is presented to illustrate the adaptation of the method to this type of motion synthesis problem.

**Example 2.** Design a carrier machine which transports a load in such a way that the mass centre D of the load moves along straight line between two conveyors distant by 4 m. The load is required to match the correct positions at initial and final instants of transporting. The left conveyor is horizontal, whereas the other one is inclined at 45°. The load has to be rotated through $\Delta\theta_{2L} = 45^\circ$ to be put onto the right conveyor. The minimum required angle through which the active link has to rotate to transport the load along the prescribed line is $\pi$.

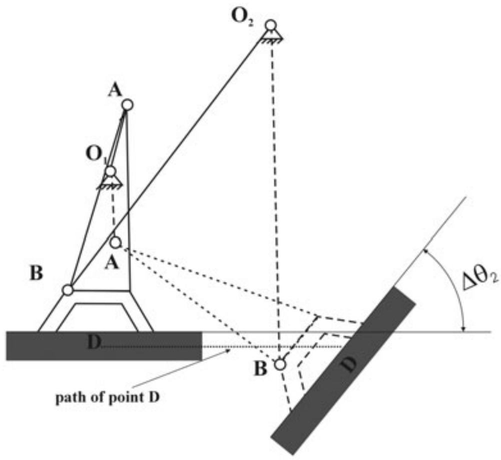


Figure 19. Extreme positions of the carrier machine.

**Solution**

The synthesis of the four-bar linkage is also executed according to the scheme shown in Figure 12. The only difference is that $v_1$ is not a design parameter. From Equation (21)

$$v_1 = \frac{\pi/4}{\sin(v_2 + v_3) - \sin(v_3)}.$$

The straight line is discretized into 10 equally spaced points. The minimized objective function is given by (17). A solution is the four-bar linkage shown in Figure 19.

Coordinates of ground pins are: $O_1(0.221, 2.887)$, $O_2(2.832, 5.344)$ [m]. Link lengths [m]: $|O_1A| = 1.141$, $|AB| = 3.238$, $|O_2B| = 5.543$, $|O_1O_2| = 3.586$, $|AD| = 4.024$, $\theta_4 = 0.195$ rad. The starting and ending angles of active link position: $\theta_{10} = 0.5479$ rad, $\theta_{11} = 4.0961$ rad. The initial positions of coupler points at $\theta_1 = \theta_{10}$: A(0.523, 3.989), B (−0.501, 0.916), D(0.017, −0.004).

The maximum deviation of the projected coupler curve from straight line of the length 4 m is 7 mm.

It should be noted that the angular positions of the coupler and associated locations of the coupler point can be prescribed. In this case, the approximating function (2) is not introduced and only geometric design parameters are optimized. Then the problem is reduced to fundamental motion synthesis problem.

## 8. Conclusions

The new technique for path synthesis of planar mechanisms composed of four links was presented. The distinctive features of the method are as follows:

- One general formula for angular position of the floating link, which couples input and output links, is used for any four-link mechanism. This allows to decrease the number of design parameters. The formula does not depend on the mechanism dimensions, let alone the range of the angle through which the input link rotates when the prescribed path is traced.
- The effectiveness of the method was proven in synthesis of the four-bar linkage, crank-slider mechanism as well as of the mechanism with slotted link approximating open paths. The methodology worked out for the presented mechanisms can be easily applied to the remaining four-link mechanisms with prismatic and revolute joints. To define appropriate objective functions for most of the four-link planar mechanisms, it suffices to combine appropriate terms derived in Section 3.
- Small number of design parameters yields significant advantages. The EA is more accurate for smaller number of optimized parameters. This not only accelerates convergence, but also diminishes the requirement for a large population.
- The method may be also easily modified to solve the problems combining the angular orientation and position of the link.
- The presented method deals with dimensional synthesis, but using the method with all possible combinations of $\delta_1$, $\delta_2$ and $\delta_3$ for each joint of the floating link one can solve the problem of both structural and dimensional synthesis. The term structural refers to any four-link planar mechanism with dyads for which the errors $\delta_i$ were derived. Then, the problem will be not limited to a mechanism pointed out by designer, but the mechanism will be chosen from a group of the four-link planar mechanisms.

The method can be applied only to problems without time prescription. Therefore, further studies are aimed at adapting the method to synthesis of path generators with prescribed timing. The author hopes that the idea enables elaborating analytical methods for specific paths or mechanisms.

## References

[1] Buśkiewicz J. Synthesis method of planar mechanisms approximating open paths. In: Visa I, editor. Proceedings of the 10th IFToMM International Symposium on Science of Mechanisms and Machines SYROM'09; XXII. London: Springer Science+Business Media B.V, ISBN: 978-90-481-3521-9; 2009. p. 597–609.
[2] Buśkiewicz J. Minimization of the dimension of the solution space in optimal mechanism synthesis [Publishing House of Poznan University of Technology, series Dissertations, no 442]; 2009.
[3] Suh CH, Radcliffe CW. Kinematics and mechanism design. New York (NY): Wiley; 1978.
[4] Erdman AG, Sandor GN, Kota SS. Mechanism design: analysis & synthesis. 4th ed. (Web Enhanced), Vol. I. New Jersey (NJ): Prentice-Hall; 2001.
[5] McCarthy JM. Geometric design of linkages. New York (NY): Springer Verlag; 2000.
[6] Buśkiewicz J. Use of shape invariants in optimal synthesis of geared five-bar linkage. Mech. Mach. Theory. 2010;45:273–290.
[7] McGarva J, Mullineux G. Harmonic representation of closed curves. Appl. Math. Model. 1993;17:213–218.

[8] Ullah I, Kota S. Optimal synthesis of mechanisms for path generation using fourier descriptors and global search methods. J. Mech. Design. 1997;119:504–510.
[9] Lin Wen-Yi. A GA–DE hybrid evolutionary algorithm for path synthesis of four-bar linkage. Mech. Mach. Theory. 2010;45:1096–1107.
[10] Peñuñuri F, Peón-Escalante R, Villanueva C, Pech-Oy D. Synthesis of mechanisms for single and hybrid tasks using differential evolution. Mech. Mach. Theory. 2011;46:1335–1349.
[11] Cabrera JA, Ortiz A, Nadal F, Castillo JJ. An evolutionary algorithm for path synthesis of mechanisms. Mech. Mach. Theory. 2011;46:127–141.
[12] Cabrera JA, Nadal F, Muñoz JP, Simon A. Multiobjective constrained optimal synthesis of planar mechanisms using a new evolutionary algorithm. Mech. Mach. Theory. 2007; 42:791–806.
[13] Shiakolas PS, Koladiya D, Kebrle J. On the optimum synthesis of four-bar linkages using differential evolution and the geometric centroid of precision positions. Inv. Probl. Eng. 2002;10:485–502.
[14] Shiakolas PS, Koladiya D, Kebrle J. On the optimum synthesis of six-bar linkages using differential evolution and the geometric centroid of precision positions technique. Mech. Mach. Theory. 2005;40:319–335.
[15] Kunjur A, Krishnamurty S. Genetic algorithms in mechanical synthesis. J. Appl. Mech. Robotics. 1997;4:18–24.
[16] Sanchez Marin FT, Gonzalez AP. Open-path synthesis of linkages through geometrical adaptation. Mech. Mach. Theory. 2004;39:943–955.
[17] Da Lio MD, Cossalter V, Lot R. On the use of natural coordinates in optimal synthesis of mechanisms. Mech. Mach. Theory. 2000;35:1367–1389.
[18] Akçali ID, Dittrich G. Path generation by subdomain method. Mech. Mach. Theory. 1989;24:45–52.
[19] Lio MD. Robust synthesis of linkages – synthesis by solving non-linear optimization problem. Mech. Mach. Theory. 1997;32:921–932.
[20] Matekar SB, Gogate GR. Optimum synthesis of path generating four-bar mechanisms using differential evolution and a modified error function. Mech. Mach. Theory. 2012;52:158–179.
[21] Gogate GR, Matekar SB. Optimum synthesis of motion generating four-bar mechanisms using alternate error functions. Mech. Mach. Theory. 2012;54:41–61.
[22] Avilés R, Vallejo J, de Bustos I, Aguirrebeitia J, Ajuria G. Optimum synthesis of planar linkages using a strain–energy error function under geometric constraints. Mech. Mach. Theory. 2010;45:65–79.
[23] Sancibrian R, Viadero F, García P, Fernández A. Gradient-based optimization of path synthesis problems in planar mechanisms. Mech. Mach. Theory. 2004;39:839–856.
[24] Bulatović RR, Dordević SR. On the optimum synthesis of a four-bar linkage using differential evolution and method of variable controlled deviations. Mech. Mach. Theory. 2009;44:235–246.
[25] Sancibrian R, García P, Viadero F, Fernández A. A general procedure based on exact gradient determination in dimensional synthesis of planar mechanisms. Mech. Mach. Theory. 2006;41:212–229.
[26] Vidosic JP, Tesar D. Selections of four-bar mechanisms having required approximate straight-line outputs, Part I, II, III. J. Mech. 1967;2:23–44. 45–59, 61–76.
[27] Karelin VS. On the synthesis of the inverted slider-crank mechanisms for approximate straight line motion. Mech. Mach. Theory. 1985;21:13–18.
[28] Hodges PH, Pisano AP. On the synthesis of straight line, constant velocity scanning mechanisms. J. Mech. Design. 1991;113:464–472.
[29] Dijksman EA, Smals ATJM. $\lambda$-formed 4-bar linkages set in a translation-position to design mechanisms approximating a straight line. Mech. Mach. Theory. 1996;31:1033–1042.

[30] Dijksman EA., Smals ATJM. How to exchange centric inverted slider cranks with $\lambda$-formed four-bar linkages. Mech. Mach. Theory. 2000;35:305–327.
[31] Jianyou H, Weixiang Q, Huishe Z. Study on synthesis method of $\lambda$-formed 4-bar linkages approximating a straight line. Mech. Mach. Theory. 2009;44:57–65.
[32] Shen Q, Russell K, Sodhi RS, He Y. Spherical four-bar motion generation with a prescribed rigid-body load. Trans. Can. Soc. Mech. Eng. 2008;32:401–410.
[33] Shen Q, Al-Smadi YM, Russell K, Sodhi RS. On planar five-bar motion generation with a driver torque constraint. J. Adv. Mech. Design System Manufactur. 2008;2:408–416.
[34] Shen Q, Al-Smadi YM, Martin PJ, Russell K, Sodhi RS. An extension of mechanism design optimization for motion generation. Mech. Mach. Theory. 2009;44:1759–1767.
[35] Goldberg DE. Genetic algorithms in search, optimization, and machine learning. Reading (MA): Addison-Wesley; 1994.