# Appendix A
# Position Kinematic Analysis. Trigonometric Method

Chapter 3 shows the kinematic analysis of several mechanisms by using Raven's method. Writing the equations that solve the problem when using this method is easy. However, finding the solution to such equations can be a complicated task. For that reason, we introduce the trigonometric method in this appendix, which is much simpler to write and solve.

## A.1 Position Analysis of a Four-Bar Mechanism

Consider the mechanism shown in Fig. A.1 in which $\overline{O_2O_4}$, $\overline{O_2A}$, $\overline{AB}$ and $\overline{O_4A}$ are the lengths of links 1, 2, 3 and 4 respectively. On the other hand, angles $\theta_2$, $\theta_3$ and $\theta_4$ define the angular position of links 2, 3 and 4 considering the counterclockwise rotations positive.

In order to determine angles $\theta_3$ and $\theta_4$, we need to find the value of distance $\overline{O_4A}$ (Eq. A.1) as well as angles $\beta$ (Eq. A.3), $\delta$ (Eq. A.7) and $\phi$ (Eq. A.5). The value of distance $\overline{O_4A}$ can be determined in triangle $\Delta O_2AO_4$:

$$\overline{O_4A} = \sqrt{\overline{O_2O_4}^2 + \overline{O_2A}^2 - 2\overline{O_2O_4}\,\overline{O_2A}\cos\theta_2} \tag{A.1}$$

The same triangle verifies (Eq. A.2):

$$\overline{O_4A}\sin\beta = \overline{O_2A}\sin\theta_2 \tag{A.2}$$

where:

$$\beta = \arcsin\left(\frac{\overline{O_2A}}{\overline{O_4A}}\sin\theta_2\right) \tag{A.3}$$

Angles $\phi$ and $\delta$ between bars 3 and 4 and diagonal $\overline{O_4A}$ respectively, can be worked out from triangle $\Delta ABO_4$. It verifies (Eq. A.4)



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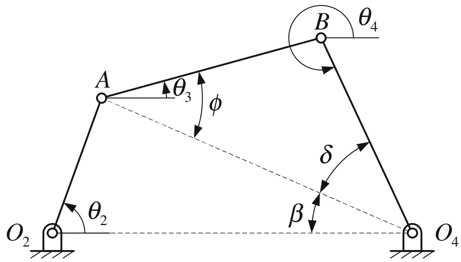



**Fig. A.1** Parameters involved in the calculation of the link positions in a four-bar mechanism by means of the trigonometric method

$$\overline{O_4B}^2 = \overline{AB}^2 + \overline{O_4A}^2 - 2\overline{AB}\overline{O_4A}\cos\phi \tag{A.4}$$

We can clear $\phi$ from Eq. (A.4):

$$\phi = \arccos\frac{\overline{AB}^2 + \overline{O_4A}^2 - \overline{O_4B}^2}{2\overline{AB}\overline{O_4A}} \tag{A.5}$$

In the same triangle its verified (Eq. A.6):

$$\overline{O_4B}\sin\delta = \overline{AB}\sin\phi \tag{A.6}$$

Thus:

$$\delta = \arcsin\left(\frac{\overline{AB}}{\overline{O_4B}}\sin\phi\right) \tag{A.7}$$

Once the values of $\beta$, $\delta$ and $\phi$ have been determined, we can obtain $\theta_3$ (Eq. A.8) and $\theta_4$ (Eq. A.9) in the mechanism (Fig. A.1):

$$\theta_3 = \phi - \beta \tag{A.8}$$

$$\theta_4 = -(\beta + \delta) \tag{A.9}$$

When angle $\theta_2$ takes values between 180° and 360°, angle $\beta$ has a negative value and Eqs. (A.8) and (A.9) are also applicable (Fig. A.2).

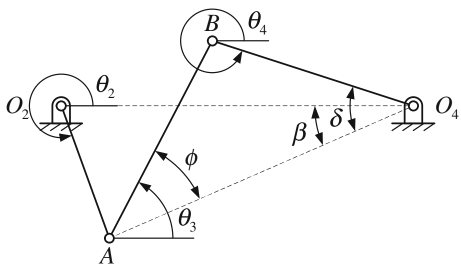



**Fig. A.2** Open four-bar mechanism with link 2 in a position between 180° and 360°

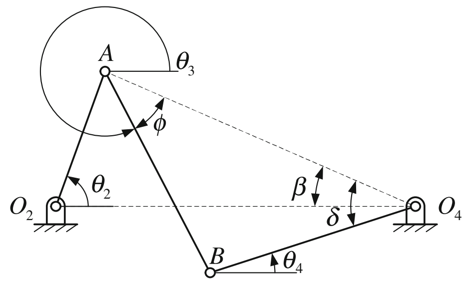



**Fig. A.3** Calculation of the position of links 3 and 4 in a crossed four-bar mechanism by means of the trigonometric method

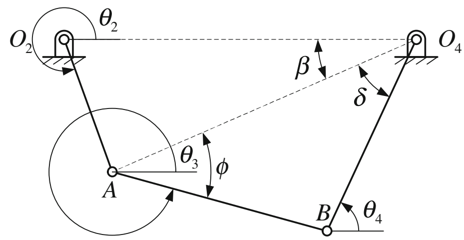



**Fig. A.4** Crossed four-bar mechanism with link 2 in a position between 180° and 360°

For a crossed four-bar mechanism (Fig. A.3) we will use Eqs. (A.10) and (A.11):

$$\theta_3 = -(\phi + \beta) \tag{A.10}$$

$$\theta_4 = \delta - \beta \tag{A.11}$$

Again, when angle $\theta_2$ takes values between 180° and 360°, angle $\beta$ has a negative value and Eqs. (A.10) and (A.11) are also applicable (Fig. A.4).

## A.2 Position Analysis of a Crank-Shaft Mechanism

Figure A.5 shows a crank-shaft mechanism. $x_B$ and $y_B$ are the Cartesian coordinates of point B with respect a system centered on point $O_2$ with its $X$-axis parallel to the piston trajectory. $x_B$ is positive while $y_B$ is negative.

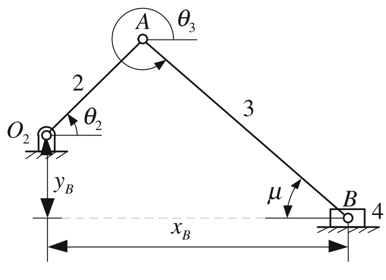



**Fig. A.5** Crank-shaft mechanism and the parameters involved in the position analysis with the trigonometric method

Position of links 3 and 4 can be worked out using Eqs. (A.12)–(A.15):

$$\overline{AB}\sin\mu = \overline{O_2A}\sin\theta_2 - y_B \tag{A.12}$$

$$\mu = \arcsin\frac{\overline{O_2A}\sin\theta_2 - y_B}{\overline{AB}} \tag{A.13}$$

$$\theta_3 = -\mu \tag{A.14}$$

The $x$ position of point $B$ will be given by Eq. (A.15):

$$x_B = \overline{O_2A}\cos\theta_2 + \overline{AB}\cos\theta_3 \tag{A.15}$$

It can easily be verified that Eq. (A.15) works for any position of input link 2. When the trajectory of point $B$ is above $O_2$, the sign of $y_B$ is positive and these equations are also applicable.

## A.3 Position Analysis of a Slider Mechanism

Consider the slider mechanism in Fig. A.6, where link 3 describes a straight trajectory along link 4 that rotates about $O_4$ with offset $\overline{O_4B}$.

Similarly as in previous problems, $\overline{O_2O_4}$ and $\overline{O_2A}$ are the lengths of links 1 and 2 respectively while angles $\theta_2$ and $\theta_4$ define the angular position of links 2 and 4. Assuming that we know $\overline{O_2O_4}$, $\overline{O_2A}$ and $\theta_2$, we can obtain unknown values $\overline{AB}$ (Eq. A.17) and $\theta_4$ (Eq. A.20). To do so, we start by obtaining the value of $\overline{O_4A}$ (Eq. A.16):

$$\overline{O_4A} = \sqrt{\overline{O_2O_4}^2 + \overline{O_2A}^2 - 2\overline{O_2O_4}\,\overline{O_2A}\cos\theta_2} \tag{A.16}$$

We can calculate $\overline{AB}$ as:

$$\overline{AB} = \sqrt{\overline{O_4B}^2 - \overline{O_4A}^2} \tag{A.17}$$

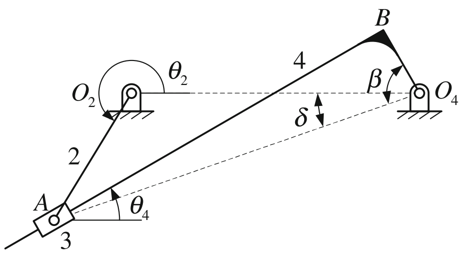



**Fig. A.6** Position analysis of a slider-mechanism by means of the trigonometric method

The value of angle $\delta$ (Eq. A.18) is:

$$\delta = \arctan \frac{\overline{O_2A} \sin \theta_2}{\overline{O_2O_4} + \overline{O_2A} \cos \theta_2} \tag{A.18}$$

Finally, $\theta_4$ can be determined after first computing the value of $\beta$ (Eq. A.19):

$$\beta = \arctan \frac{\overline{AB}}{\overline{O_4B}} \tag{A.19}$$

$$\theta_4 = \delta + (90^\circ - \beta) \tag{A.20}$$

If the offset is opposite, point $B$ is below the $X$-axis and Eq. (A.20) changes to Eq. (A.21):

$$\theta_4 = \delta - (90^\circ - \beta) \tag{A.21}$$

## A.4 Two Generic Bars of a Mechanism

Let us consider that we have carried out the kinematic analysis of links 2, 3 and 4 of the mechanism shown in Fig. A.7. We will continue the position analysis of links 5 and 6 considering that the position of point $C$ of link 3 is known.

To find the position of links 5 and 6 we have to define triangle $\Delta CDO_6$ first (Fig. A.8).

The length of side $\overline{O_6C}$ (Eq. A.22) can be calculated by means of the $x$ and $y$ coordinates of points $C$ and $O_6$:

$$\overline{O_6C} = \sqrt{(x_C - x_{O_6})^2 + (y_C - y_{O_6})^2} \tag{A.22}$$

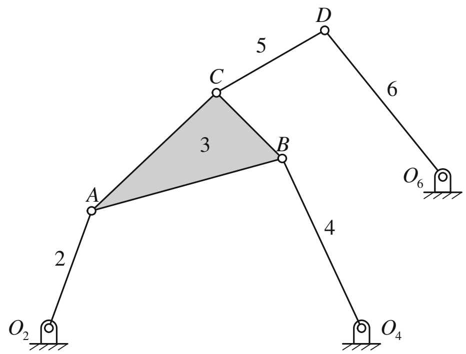
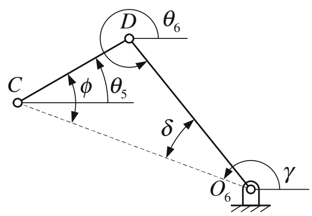



**Fig. A.7** Six-bar mechanism

**Fig. A.8** Position analysis of bars 5 and 6 by means of the trigonometric method




And its angle $\gamma$ (Eq. A.23) is:

$$\gamma = \arctan\frac{y_C - y_{O_6}}{x_C - x_{O_6}} \tag{A.23}$$

Angle $\phi$ (Eq. A.25) can be computed by using the law of cosines (Eq. A.24):

$$\overline{DO_6}^2 = \overline{CD}^2 + \overline{O_6D}^2 - 2\overline{CDO_6D}\cos\phi \tag{A.24}$$

$$\phi = \arccos\frac{\overline{CD}^2 + \overline{O_6D}^2 - \overline{DO_6}}{2\overline{CDO_6D}} \tag{A.25}$$

Finally, angle $\delta$ (Eq. A.26) is determined by using the law of sines:

$$\delta = \arcsin\left(\frac{\overline{CD}}{\overline{O_6D}}\sin\phi\right) \tag{A.26}$$

Therefore, angles $\theta_5$ and $\theta_6$ are (Eqs. A.27 and A.28):

$$\theta_5 = \phi - (180^\circ - \gamma) \tag{A.27}$$

$$\theta_6 = 180^\circ + \gamma - \delta \tag{A.28}$$

# Appendix B
# Freudenstein's Method to Solve the Position Equations in a Four-Bar Mechanism

In Chap. 3 we developed the position analysis of a four-bar mechanism by means of Raven's method. In this appendix we explain Freudenstein's method to solve the obtained equations and calculate the value of angles $\theta_3$ and $\theta_4$.

## B.1 Position Analysis of a Four-Bar Mechanism by Using Raven's Method

We will apply Raven's method to the four-bar mechanism shown in Fig. B.1.

The vector loop equation (Eq. B.1) for the position analysis of the mechanism is:

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} + r_3 e^{i\theta_3} + r_4 e^{i\theta_4} \tag{B.1}$$

By converting this equation into its trigonometric form (Eq. B.2):

$$\begin{aligned} r_1(\cos\theta_1 + i\sin\theta_1) = r_2(\cos\theta_2 + i\sin\theta_2) + r_3(\cos\theta_3 + i\sin\theta_3) \\ + r_4(\cos\theta_4 + i\sin\theta_4) \end{aligned} \tag{B.2}$$

And by separating its real and imaginary parts, we obtain the system (Eq. B.3) with two unknowns ($\theta_3$ and $\theta_4$):

$$\left.\begin{aligned} r_1\cos\theta_1 = r_2\cos\theta_2 + r_3\cos\theta_3 + r_4\cos\theta_4 \\ r_1\sin\theta_1 = r_2\sin\theta_2 + r_3\sin\theta_3 + r_4\sin\theta_4 \end{aligned}\right\} \tag{B.3}$$

## B.2 Freudenstein's Method

We substitute $\theta_1 = 0$ in Eq. (B.3) and isolate $\theta_3$ (Eq. B.4):



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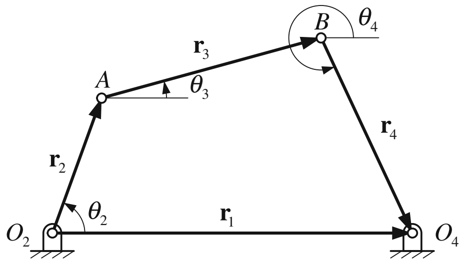


**Fig. B.1** Position analysis of a four-bar mechanism by means of Raven's method

$$\left.\begin{aligned} r_1 - r_2\cos\theta_2 - r_4\cos\theta_4 &= r_3\cos\theta_3 \\ -r_2\sin\theta_2 - r_4\sin\theta_4 &= r_3\sin\theta_3 \end{aligned}\right\} \tag{B.4}$$

We raise each equation to the second power and add them term by term (Eq. B.5):

$$r_1^2 + r_2^2 + r_4^2 - 2r_1r_2\cos\theta_2 - 2r_1r_4\cos\theta_4 + 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4) = r_3^2 \tag{B.5}$$

By dividing all terms by the coefficient of term $\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4$, $2r_2r_4$, it yields Eq. (B.6):

$$\frac{r_1^2 + r_2^2 - r_3^2 + r_4^2}{2r_2r_4} - \frac{r_1}{r_4}\cos\theta_2 - \frac{r_1}{r_2}\cos\theta_4 + (\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4) = 0 \tag{B.6}$$

In order to simplify Eq. (B.6), we use the following coefficients (Eq. B.7):

$$\left.\begin{aligned} k_1 &= \frac{r_1}{r_2} \\ k_2 &= \frac{r_1}{r_4} \\ k_3 &= \frac{r_1^2 + r_2^2 - r_3^2 + r_4^2}{2r_2r_4} \end{aligned}\right\} \tag{B.7}$$

Thus, Eq. (B.6) remains Eq. (B.8):

$$k_3 - k_2\cos\theta_2 - k_1\cos\theta_4 + (\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4) = 0 \tag{B.8}$$

We substitute $\cos\theta_4$ and $\sin\theta_4$ for their expressions in terms of the half angle tangent (Eq. B.9):

$$k_3 - k_2 \cos\theta_2 - k_1 \frac{1 - \tan^2\frac{\theta_4}{2}}{1 + \tan^2\frac{\theta_4}{2}} + \left(\cos\theta_2 \frac{1 - \tan^2\frac{\theta_4}{2}}{1 + \tan^2\frac{\theta_4}{2}} + \sin\theta_2 \frac{2\tan\frac{\theta_4}{2}}{1 + \tan^2\frac{\theta_4}{2}}\right) = 0 \tag{B.9}$$

Next, we remove the denominators and group the terms for tan, $\tan^2$ and the independent term (Eq. B.10), all in the same member.

$$\begin{aligned}(k_3 - k_2\cos\theta_2 - k_1 - \cos\theta_2)\tan^2\frac{\theta_4}{2} + 2\sin\theta_2\tan\frac{\theta_4}{2} \\ + (k_3 - k_2\cos\theta_2 - k_1 + \cos\theta_2) = 0\end{aligned} \tag{B.10}$$

Again, we rename the different coefficients (Eq. B.11) of the second degree equation:

$$\left.\begin{array}{l} A = k_3 - k_2\cos\theta_2 - k_1 - \cos\theta_2 \\ B = 2\sin\theta_2 \\ C = k_3 - k_2\cos\theta_2 - k_1 + \cos\theta_2 \end{array}\right\} \tag{B.11}$$

Thus, Eq. (B.10) can be written as Eq. (B.12):

$$A\tan^2\frac{\theta_4}{2} + B\tan\frac{\theta_4}{2} + C = 0 \tag{B.12}$$

Hence, $\theta_4$, which is the unknown that defines the angular position of link 4, is (Eq. B.13):

$$\theta_4 = 2\arctan\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{B.13}$$

where the + and − signs indicate two possible solutions for the open and crossed configurations of the four-bar mechanism respectively.

Similarly, but in this case isolating $\theta_4$ in one of the members, we reach (Eq. B.14) for $\theta_3$, which defines the angular position of link 3. Again, there are two possible solutions depending on the configuration of the four-bar mechanism:

$$\theta_3 = 2\arctan\frac{-E \pm \sqrt{E^2 - 4DF}}{2D} \tag{B.14}$$

where the different coefficients (Eq. B.15) of the second degree equation (Eq. B.14) are:

$$\left.\begin{array}{l} D = k_1 - k_4 \cos\theta_2 + k_5 - \cos\theta_2 \\ E = 2\sin\theta_2 \\ F = -k_1 - k_4 \cos\theta_2 + k_5 + \cos\theta_2 \end{array}\right\} \tag{B.15}$$

And $k_4$ and $k_5$ (Eq. B.16) are:

$$\left.\begin{array}{l} k_1 = \dfrac{r_1}{r_2} \\ k_4 = \dfrac{r_1}{r_3} \\ k_5 = \dfrac{r_1^2 + r_2^2 + r_3^2 - r_4^2}{2r_2r_3} \end{array}\right\} \tag{B.16}$$

# Appendix C
# Kinematic and Dynamic Analysis of a Mechanism

The conveyor transfer mechanism shown in Fig. C.1 pushes boxes with a mass of 8 kg from one conveyor belt to another. The motor link turns at a constant speed of 40 rpm in counter clockwise direction.

In order to make a complete kinematic and dynamic analysis of the mechanism, we will use all the analysis methods described in this book. We will carry out the analysis at a given position. In general, the most interesting one for dynamic analysis is the position at which the acceleration of the piston is maximum. This way we can determine the forces that act on the links in extreme conditions. The position chosen for this study is $\theta_2 = 350°$.

This analysis includes the following sections:

- Kinematic chain. Study and identification of the kinematic pairs. Number of D.O.F of the mechanism. Kinematic inversion that results from fixing link 4.
- Kinematic graph of slider displacement versus crank rotation.
- Velocity analysis by means of the relative velocity method.
- Velocity analysis by means of the method of Instantaneous Centers of Rotation.
- Acceleration analysis by means of the relative acceleration method.
- Velocity and acceleration analysis by means of Raven's method.
- Calculation of the inertial force and inertial torque of each of the links in the mechanism.
- Dynamic analysis by means of the graphical method.
- Dynamic analysis by means of the matrix method.

## C.1 Kinematic Chain

We begin the study of the mechanism by drawing its kinematic diagram as shown in Fig. C.2. This figure also shows the nomenclature that will be used along this study.

Table C.1 shows the different types of kinematic pairs in the mechanism and the degrees of freedom of each pair.



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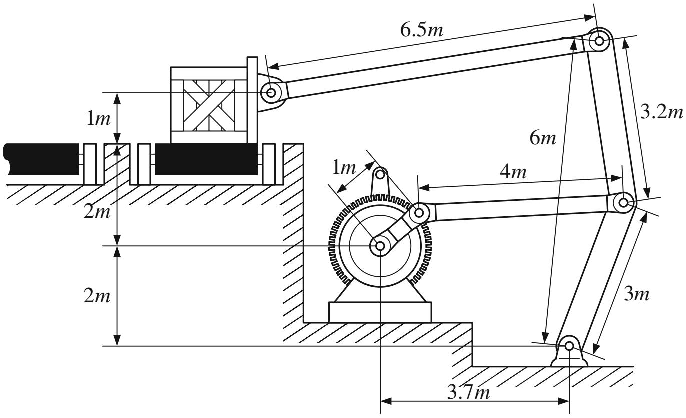



**Fig. C.1** Conveyor transfer mechanism

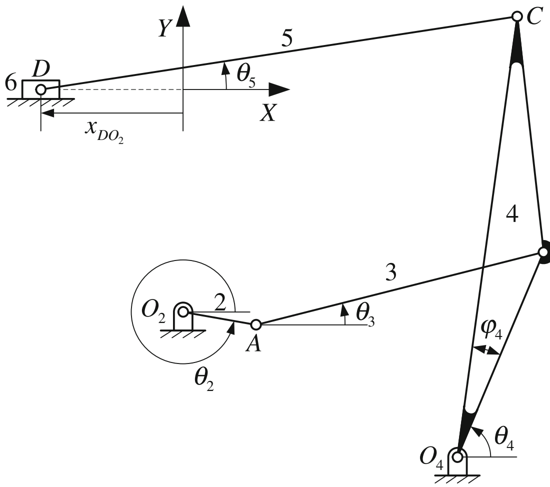



**Fig. C.2** Kinematic diagram of the mechanism

**Table C.1** Type of kinematic pairs in the mechanism

| PAIR | Type | Number of D.O.F. |
|---|---|---|
| 1–2 | Rotation | 1 |
| 2–3 | Rotation | 1 |
| 3–4 | Rotation | 1 |
| 1–4 | Rotation | 1 |
| 4–5 | Rotation | 1 |
| 5–6 | Rotation | 1 |
| 1–6 | Prismatic | 1 |

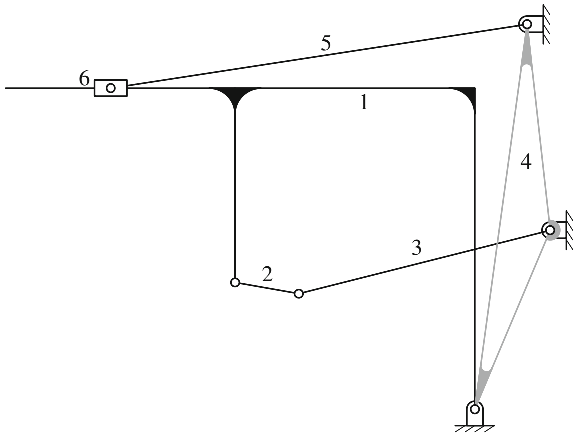



**Fig. C.3** Kinematic inversion of the mechanism when link 4 is fixed

We use Kutzbach's equation to calculate the number of degrees of freedom of the mechanism (Eq. C.1):

- Number of links: $N = 6$
- Kinematic pairs with 1 DOF: $J_1 = 7$
- Kinematic pairs with 2 DOF: $J_2 = 0$

$$DOF = 3(N - 1) - 2J_1 - J_2 = 3(6 - 1) - 2 \cdot 7 - 0 = 1 \qquad \text{(C.1)}$$

To better understand the mechanism, we will draw the kinematic diagram of one of its inversions. In this case we will consider link 4 as the frame. This is shown in Fig. C.3.

## C.2 Slider Displacement Versus Crank Rotation

We will draw the kinematic graph of point D displacement versus crank rotation by means of the graphical method. To do so, we divide the whole turn of the crank in 12 positions starting from position 0°. This way, we find the 12 positions of point A which correspond to 12 angular positions of the crank in steps of 30°. Knowing the length of the links, we can find the equivalent 12 positions for points B, C and D (Fig. C.4).

We can graph the position of point D versus the crank position. This is shown in Fig. C.5. We can see that the stroke end positions of the piston are close to positions $\theta_2 = 10°$ and $\theta_2 = 195°$. In these positions, the velocity of the piston has to be null. As the velocity is the first time-derivative of displacement, this can be verified by tracing a line tangent to the curve at the end-of-stroke position. If the line is horizontal, the velocity is null.

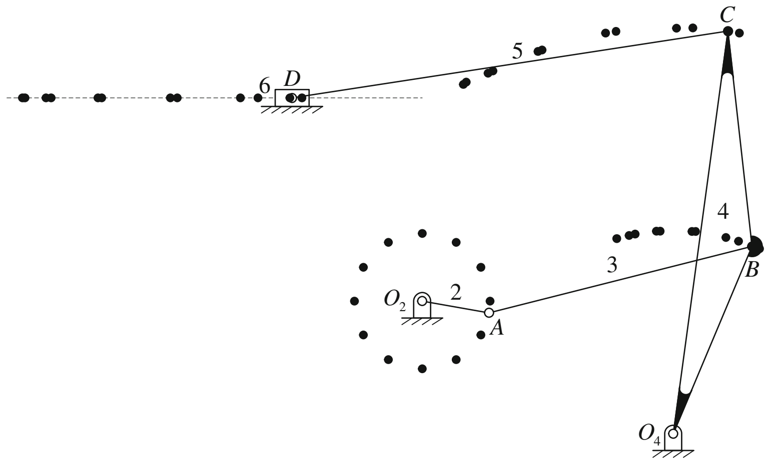



**Fig. C.4** Kinematic diagram of the mechanism in a complete turn of the crank in steps of 30°

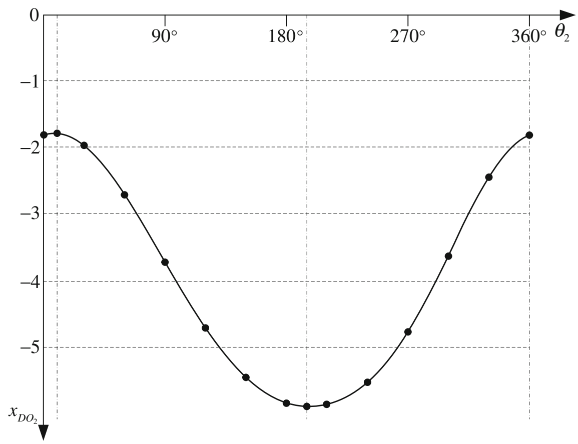



**Fig. C.5** Kinematic graph of the slider displacement versus the crank rotation

## C.3 Velocity Analysis by Relative Velocity Method

Before starting the velocity analysis, the positions of the links have to be determined. To do so, we will use the trigonometric method explained in Appendix A. Figure C.6 shows the angles and distances used to solve the position problem.

We start with the four-bar mechanism formed by links 1, 2, 3 and 4. Distance $\overline{O_2O_4}$ (Eq. C.2) and angles $\theta_1$ (Eq. C.3) and $\gamma$ (Eq. C.4) can be calculated as:

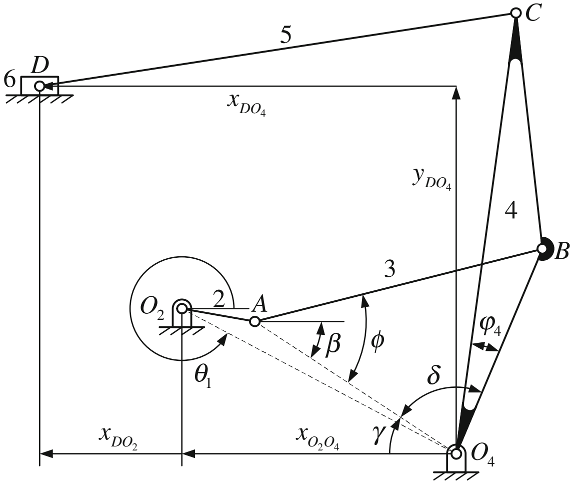



**Fig. C.6** Calculation of the position of links 3, 4, 5 an 6 by means of the trigonometric method

$$\overline{O_2O_4} = \sqrt{\left(x_{O_4} - x_{O_2}\right)^2 + \left(y_{O_4} - y_{O_2}\right)^2} = 4.206\,\text{m} \tag{C.2}$$

$$\theta_1 = 270^\circ + \arctan\frac{3.7}{2} = 331.6^\circ \tag{C.3}$$

$$\gamma = 180^\circ - 90^\circ - \arctan\frac{3.7}{2} = 28.4^\circ \tag{C.4}$$

The application of the cosine rule to triangle $\Delta O_2AO_4$ yields (Eq. C.5):

$$\begin{aligned}\overline{O_4A} &= \sqrt{\overline{O_2O_4}^2 + \overline{O_2A}^2 - 2\overline{O_2O_4}\,\overline{O_2A}\cos(\theta_2 - \theta_1)} \\ &= \sqrt{4.206^2 + 1^2 - 2\cdot 4.206\cos 18.4^\circ} = 3.272\,\text{m}\end{aligned} \tag{C.5}$$

And the sine rule on the same triangle yields (Eqs. C.6–B.7):

$$\overline{O_4A}\sin(\beta - \gamma) = \overline{O_2A}\sin(\theta_2 - \theta_1) \tag{C.6}$$

$$\beta = \gamma + \arcsin\frac{\overline{O_2A}\sin(\theta_2 - \theta_1)}{\overline{O_4A}} = 28.4^\circ + \arcsin\frac{\sin 18.4}{3.272} = 33.9^\circ \tag{C.7}$$

The application of the cosine rule to triangle $\Delta ABO_4$ yields (Eqs. C.8 and C.9):

$$\overline{O_4B}^2 = \overline{AB}^2 + \overline{O_4A}^2 - 2\overline{AB}\,\overline{O_4A}\cos\phi \tag{C.8}$$

$$\begin{aligned}\phi &= \arccos \frac{\overline{AB}^2 + \overline{O_4A}^2 - \overline{O_4B}^2}{2\overline{AB}\overline{O_4A}} \\ &= \arccos \frac{4^2 + 3.272^2 - 3^2}{2 \cdot 4 \cdot 3.272} = 47.4^\circ\end{aligned} \tag{C.9}$$

Thus, the positions of the link 3 (Eq. C.10) is:

$$\theta_3 = \phi - \beta = 47.4^\circ - 33.9^\circ = 13.5^\circ \tag{C.10}$$

The application of the sine rule to triangle $\Delta ABO_4$ yields (Eqs. C.11 and C.12):

$$\overline{O_4B} \sin \delta = \overline{AB} \sin \phi \tag{C.11}$$

$$\begin{aligned}\delta &= \arcsin\left(\frac{\overline{AB}}{\overline{O_4B}} \sin \phi\right) \\ &= \arcsin\left(\frac{4 \sin 47.4^\circ}{3}\right) = 79.12^\circ\end{aligned} \tag{C.12}$$

Therefore, the position of link 4 (Eq. C.13) is:

$$\theta_4 = 180^\circ - \beta - \delta = 180^\circ - 33.9^\circ - 79.12^\circ = 67^\circ \tag{C.13}$$

We continue with the position analysis of the crank-shaft mechanism formed by links 4, 5 and 6 in Fig. C.6.

Although angle $\varphi_4$ formed by $\overline{O_4B}$ and $\overline{O_4C}$ has a fixed value and it could be part of the data of the mechanism, in this case we have the length of the sides of triangle $\Delta O_4BC$ instead of angle $\varphi_4$ itself. We can easily obtain its value (Eq. C.15) by means of the rule of cosines (Eq. C.14):

$$\overline{BC} = \sqrt{\overline{O_4B}^2 + \overline{O_4C}^2 - 2\overline{O_4B}\overline{O_4C} \cos \varphi_4} \tag{C.14}$$

$$\begin{aligned}\varphi_4 &= \arccos\left(\frac{\overline{O_4B}^2 + \overline{O_4C}^2 - \overline{BC}^2}{2\overline{O_4B}\overline{O_4C}}\right) \\ &= \arccos\left(\frac{3^2 + 6^2 - 3.2^2}{2 \cdot 3 \cdot 6}\right) = 15.1^\circ\end{aligned} \tag{C.15}$$

The projection of triangle $\Delta O_4CD$ over a direction perpendicular to the trajectory of the piston yields the trigonometric equation (Eq. C.16):

$$y_{DO_4} + \overline{CD} \sin \theta_5 = \overline{O_4C} \sin(\theta_4 + \varphi_4) \tag{C.16}$$

And clearing $\theta_5$ (Eq. C.17), we obtain its value:

$$\begin{aligned}\theta_5 &= \arcsin\frac{\overline{O_4C}\sin(\theta_4+\varphi_4)-y_{DO_4}}{\overline{CD}}\\ &= \arcsin\frac{6\sin(67^\circ+15.1^\circ)-5}{6.5} = 8.34^\circ\end{aligned} \tag{C.17}$$

The projection of the sides of triangle $\Delta O_4CD$ over de piston trajectory yields (Eq. C.18):

$$\begin{aligned}x_{DO_4} &= \overline{O_4C}\cos(\theta_4+\varphi_4)-\overline{CD}\cos\theta_5\\ &= 6\cos(67^\circ+15.1^\circ)-6.5\cos 8.34^\circ = -5.607\,\mathrm{m}\end{aligned} \tag{C.18}$$

Hence, the horizontal component of the distance between $D$ and $O_2$ (Eq. C.19) is:

$$x_{DO_2} = x_{DO_4} - x_{O_2O_4} = -5.607\,\mathrm{m} - (-3.7\,\mathrm{m}) = -1.907\,\mathrm{m} \tag{C.19}$$

Therefore, the positions of the links (Eq. C.20) corresponding to crank position $\theta_2 = 350^\circ$ are:

$$\left.\begin{aligned}\theta_3 &= 13.5^\circ\\ \theta_4 &= 67^\circ\\ \theta_5 &= 8.34^\circ\\ x_{DO_2} &= -1.907\,\mathrm{m}\end{aligned}\right\} \tag{C.20}$$

The following step is to find the velocity of the links when link 2 rotates at an angular speed of 40 rpm counterclockwise. We have to use the velocity of link 2 in radians per second: 4.19 rad/s.

The velocity of point $A$ (Eq. C.21) can be calculated as:

$$\begin{aligned}\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0 & 4.19\\ 1\cos 350^\circ & 1\sin 350^\circ & 0\end{vmatrix} = 0.73\hat{\mathbf{i}} + 4.13\hat{\mathbf{j}}\\ &= 4.19\,\mathrm{cm/s}\angle 80^\circ\end{aligned} \tag{C.21}$$

To calculate the angular velocity of links 3 and 4 we have to use the relative velocity vector equation: $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$.

Vectors $\mathbf{v}_B$ and $\mathbf{v}_{BA}$ can be obtained the following way (Eqs. C.22 and C.23):

$$\mathbf{v}_{BA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ 4\cos 13.5^\circ & 4\sin 13.5^\circ & 0 \end{vmatrix}$$
$$= -4\omega_3(\sin 13.5^\circ \hat{\mathbf{i}} - \cos 13.5^\circ \hat{\mathbf{j}}) \quad \text{(C.22)}$$

$$\mathbf{v}_B = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ 3\cos 67^\circ & 3\sin 67^\circ & 0 \end{vmatrix} = -3\omega_4(\sin 67^\circ \hat{\mathbf{i}} - \cos 67^\circ \hat{\mathbf{j}}) \quad \text{(C.23)}$$

By introducing the three velocity vectors, $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_{BA}$, in the relative velocity equation and projecting them on the $X$ and $Y$ Cartesian axles, we reach the system of equations (Eq. C.24):

$$\left.\begin{aligned} 0.73 - 4\omega_3 \sin 13.5^\circ &= -3\omega_4 \sin 67^\circ \\ 4.13 + 4\omega_3 \cos 13.5^\circ &= 3\omega_4 \cos 67^\circ \end{aligned}\right\} \quad \text{(C.24)}$$

The solution to the system of equations (Eq. C.24) yields the velocities of links 3 and 4 (Eq. C.25):

$$\left.\begin{aligned} \omega_3 &= -1.27\ \text{rad/s} \\ \omega_4 &= -0.69\ \text{rad/s} \end{aligned}\right\} \quad \text{(C.25)}$$

Using these values, we can calculate velocities $\mathbf{v}_B$ (Eq. C.26) and $\mathbf{v}_{BA}$ (Eq. C.27):

$$\mathbf{v}_B = 1.91\hat{\mathbf{i}} - 0.81\hat{\mathbf{j}} = 2.08\ \text{m/s}\angle 336.96^\circ 13.5 \quad \text{(C.26)}$$

$$\mathbf{v}_{BA} = 1.185\hat{\mathbf{i}} - 4.939\hat{\mathbf{j}} \quad \text{(C.27)}$$

The velocity of point $C$ (Eq. C.28) can be determined by using the value of $\omega_4$:

$$\mathbf{v}_C = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{CO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ 6\cos(67^\circ + 15.1^\circ) & 6\sin(67^\circ + 15.1^\circ) & 0 \end{vmatrix} \quad \text{(C.28)}$$
$$= 4.12\hat{\mathbf{i}} - 0.58\hat{\mathbf{j}} = 4.16\ \text{m/s}\angle 352.1^\circ$$

We use vector equation (C.29) to calculate the angular velocity of link 5 and the linear velocity of link 6.

$$\mathbf{v}_D = \mathbf{v}_C + \mathbf{v}_{DC} \tag{C.29}$$

Since points $C$ and $D$ are two points of the same link, their relative velocity is given by Eq. (C.30):

$$\mathbf{v}_{DC} = \boldsymbol{\omega}_5 \wedge \mathbf{r}_{DC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_5 \\ 6.5\cos(\theta_5 + 180°) & 6.5\sin(\theta_5 + 180°) & 0 \end{vmatrix} \\ = -6.5\omega_5(\sin(\theta_5 + 180°)\hat{\mathbf{i}} - \cos(\theta_5 + 180°)\hat{\mathbf{j}}) \tag{C.30}$$

The velocity of point $D$ (Eq. C.31) has the same direction as the trajectory. Therefore, its vertical component is null:

$$\mathbf{v}_D = v_D\hat{\mathbf{i}} \tag{C.31}$$

By substituting the three velocity vectors, $\mathbf{v}_C$, $\mathbf{v}_D$ and $\mathbf{v}_{DC}$, in Eq. (C.29) we obtain the system of equations (Eq. C.32):

$$\left.\begin{aligned} 4.12 - 6.5\omega_5 \sin 188.3° &= v_D \\ -0.58 + 6.5\omega_5 \cos 188.3° &= 0 \end{aligned}\right\} \tag{C.32}$$

Hence, the values of the velocities of links 5 and 6 (Eq. C.33) are:

$$\left.\begin{aligned} \omega_5 &= -0.09\,\text{rad/s} \\ v_6 = v_D &= 4.04\,\text{m/s} \end{aligned}\right\} \tag{C.33}$$

And the vector velocity of point $D$ (Eq. C.34) is:

$$\mathbf{v}_D = 4.04\hat{\mathbf{i}} = 4.04\,\text{m/s}\angle 0° \tag{C.34}$$

Figure C.7 shows the velocity polygon of the mechanism. We can see how absolute velocities start at velocity pole $O$ and relative velocities connect the end points of the absolute velocity vectors. It can also be seen that triangle $\Delta obc$ in the polygon is similar to $\Delta O_4BC$ in the mechanism since their sides are perpendicular.

## C.4 Instantaneous Center Method for Velocities

To calculate the ICRs in the mechanism, we start by identifying the ICRs which correspond to real joints. In this case, the known ICRs are: $I_{12}$, $I_{23}$, $I_{34}$, $I_{14}$, $I_{45}$, $I_{16}$, and $I_{56}$ (Fig. C.8).

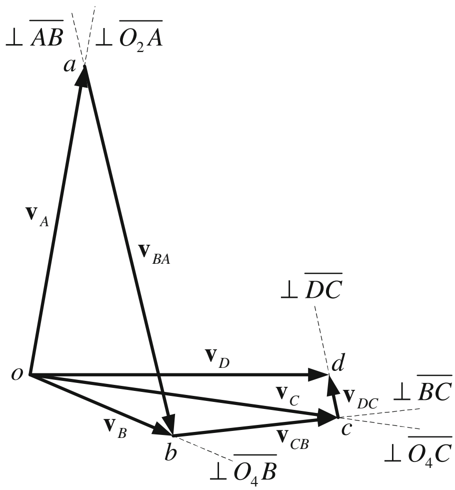



**Fig. C.7** Velocity polygon

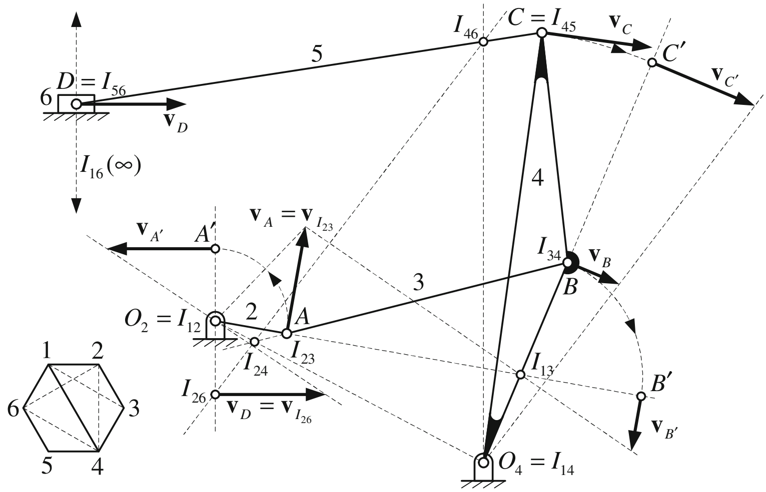



**Fig. C.8** Velocity calculation by means of the ICR method

Then we draw a polygon with as many vertexes as links in the mechanism. Each of the sides or diagonals of the polygon represents one ICR. A solid line is used to draw those ICRs that are already known while those which are unknown are drawn as dotted lines.

In this case, to calculate the velocity of points $B$, $C$ and $D$ (Eqs. C.35–C.39), we have to obtain ICRs $I_{16}$, $I_{24}$ and $I_{46}$ by using Kennedy's theorem.

$$I_{26}\begin{cases} I_{16}I_{12} \\ I_{24}I_{46} \end{cases} \quad I_{24}\begin{cases} I_{23}I_{34} \\ I_{14}I_{12} \end{cases} \quad I_{46}\begin{cases} I_{45}I_{56} \\ I_{14}I_{16} \end{cases}$$

Figure C.8 shows the graphical development of the method and the vector obtained for each velocity.

$$\left.\begin{array}{l} v_{23} = v_A \\ v_{23} = \overline{I_{13}I_{23}}\omega_3 = 3.33\,\mathrm{m}\cdot\omega_3 \end{array}\right\} \rightarrow \omega_3 = 1.26\,\mathrm{rad/s} \tag{C.35}$$

$$v_B = \overline{BI_{13}}\omega_3 = 2.08\,\mathrm{m/s} \tag{C.36}$$

$$\left.\begin{array}{l} v_{24} = \overline{I_{12}I_{24}}\omega_2 = 0.58\,\mathrm{m}\cdot\omega_2 \\ v_{24} = \overline{I_{14}I_{24}}\omega_4 = 3.52\,\mathrm{m}\cdot\omega_4 \end{array}\right\} \rightarrow \omega_4 = 0.69\,\mathrm{rad/s} \tag{C.37}$$

$$v_C = \overline{I_{14}C}\omega_4 = 4.14\,\mathrm{m/s} \tag{C.38}$$

$$\left.\begin{array}{l} v_{26} = \overline{I_{12}I_{26}}\omega_2 = 0.96\,\mathrm{m}\cdot\omega_2 \\ v_{26} = v_D \end{array}\right\} \rightarrow v_D = 4.04\,\mathrm{m/s} \tag{C.39}$$

## C.5 Acceleration Analysis with the Relative Acceleration Method

We know that the motor link turns at a constant rate of 40 rpm. Therefore, its angular acceleration is null ($\alpha_2 = 0$). In order to calculate the acceleration of the links, we start with the acceleration of point $A$ (Eq. C.40). The tangential component will be zero as it depends on the angular acceleration value. Therefore, it will have only one normal component:

$$\begin{aligned} \mathbf{a}_A = \mathbf{a}^n_{AO_2} = \boldsymbol{\omega}_2 \wedge \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 4.19 \\ 0.73 & 4.13 & 0 \end{vmatrix} \\ = -17.3\hat{\mathbf{i}} + 3.06\hat{\mathbf{j}} = 17.55\,\mathrm{m/s^2}\angle 170^\circ \end{aligned} \tag{C.40}$$

To calculate the angular acceleration of links 3 and 4, we use the vectors (Eqs. C.41 and C.42):

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \tag{C.41}$$

$$\mathbf{a}^n_B + \mathbf{a}^t_B = \mathbf{a}^n_A + \mathbf{a}^t_A + \mathbf{a}^n_{BA} + \mathbf{a}^t_{BA} \tag{C.42}$$

$$\mathbf{a}_B^n = \boldsymbol{\omega}_4 \wedge \mathbf{v}_B = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.69 \\ 1.91 & -0.81 & 0 \end{vmatrix} = -0.559\hat{\mathbf{i}} - 1.318\hat{\mathbf{j}} \tag{C.43}$$

$$\mathbf{a}_B^t = \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_4 \\ 3\cos 67^\circ & 3\sin 67^\circ & 0 \end{vmatrix} = -3\alpha_4 \sin 67^\circ \hat{\mathbf{i}} + 3\alpha_4 \cos 67^\circ \hat{\mathbf{j}} \tag{C.44}$$

$$\mathbf{a}_{BA}^n = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -1.27 \\ 1.185 & -4.939 & 0 \end{vmatrix} = -6.272\hat{\mathbf{i}} - 1.506\hat{\mathbf{j}} \tag{C.45}$$

$$\begin{aligned} \mathbf{a}_{BA}^t = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{BA} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_3 \\ 4\cos 13.5^\circ & 4\sin 13.5^\circ & 0 \end{vmatrix} \\ &= -4\alpha_4 \sin 13.5^\circ \hat{\mathbf{i}} + 4\alpha_3 \cos 13.5^\circ \hat{\mathbf{j}} \end{aligned} \tag{C.46}$$

Substituting these vectors (Eqs. C.43–C.46) in Eq. (C.42) and projecting them on the Cartesian axles, we reach the system of equations (Eq. C.47):

$$\left.\begin{aligned} -0.559 - 3\alpha_4 \sin 67^\circ &= -17.3 - 6.272 - 4\alpha_3 \sin 13.5^\circ \\ -1.311 + 3\alpha_4 \cos 67^\circ &= +3.06 - 1.506 + 4\alpha_3 \cos 13.5^\circ \end{aligned}\right\} \tag{C.47}$$

The solution yields the angular speed of links 3 and 4 (Eq. C.48).

$$\left.\begin{aligned} \alpha_3 &= 1.98\,\mathrm{rad/s^2} \\ \alpha_4 &= 9\,\mathrm{rad/s^2} \end{aligned}\right\} \tag{C.48}$$

Once $\alpha_4$ is known, we can calculate the acceleration of point $B$ (Eq. C.49):

$$\mathbf{a}_B = -25.4\hat{\mathbf{i}} + 9.24\hat{\mathbf{j}} = 27.03\,\mathrm{m/s} \angle 160^\circ \tag{C.49}$$

The acceleration of point $C$ can be calculated by means of (Eq. C.50):

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{CB} \tag{C.50}$$

As points $B$ and $C$ belong to the same link, the components of the relative acceleration (Eq. C.51) are:

$$\left.\begin{aligned} \mathbf{a}_{CB}^n &= \boldsymbol{\omega}_4 \wedge \mathbf{v}_{CB} \\ \mathbf{a}_{CB}^t &= \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{CB} \end{aligned}\right\} \tag{C.51}$$

Substituting the values previously obtained in Eq. (C.51), we calculate the acceleration vector of point $C$ (Eq. C.52):

$$\mathbf{a}_C = -53.89\hat{\mathbf{i}} + 4.59\hat{\mathbf{j}} = 54.06\,\mathrm{m/s}\angle 175^\circ \tag{C.52}$$

To determine the angular acceleration of link 5 and the linear acceleration of link 6, we use the vector equation (C.53):

$$\mathbf{a}_D^n + \mathbf{a}_D^t = \mathbf{a}_C^n + \mathbf{a}_C^t + \mathbf{a}_{DC}^n + \mathbf{a}_{DC}^t \tag{C.53}$$

where:

$$\left.\begin{array}{l} \mathbf{a}_D^n = 0 \\ \mathbf{a}_D^t = a_D\hat{\mathbf{i}} \end{array}\right\} \tag{C.54}$$

$$\left.\begin{array}{l} \mathbf{a}_{DC}^n = \boldsymbol{\omega}_5 \wedge \mathbf{v}_{DC} \\ \mathbf{a}_{DC}^t = \boldsymbol{\alpha}_5 \wedge \mathbf{r}_{DC} \end{array}\right\} \tag{C.55}$$

Substituting vectors $\mathbf{a}_C$ (Eq. C.52), $\mathbf{a}_D$ (Eq. C.54) and $\mathbf{a}_{DC}$ (Eq. C.55) in Eq. (C.53) and projecting them onto the Cartesian axles, we reach to the system of equations (Eq. C.56):

$$\left.\begin{array}{r} a_D = -53.89 + 0.052 - 6.5\alpha_5 \sin 188.3^\circ \\ 0 = 4.59 + 0.0076 + 6.5\alpha_5 \cos 188.3^\circ \end{array}\right\} \tag{C.56}$$

The solution of the system yields the accelerations of links 5 and 6 (Eq. C.57):

$$\left.\begin{array}{r} \alpha_5 = 0.72\,\mathrm{rad/s^2} \\ a_D = -53.14\,\mathrm{m/s^2} \end{array}\right\} \tag{C.57}$$

Thus, the vector acceleration of point $D$ (Eq. C.58) is:

$$\mathbf{a}_D = -53.14\hat{\mathbf{i}} = 53.14\,\mathrm{m/s^2}\angle 180^\circ \tag{C.58}$$

Figure C.9 shows the acceleration polygon of the mechanism. It can be noticed that triangle $\Delta obc$ of the acceleration polygon is similar to triangle $\Delta O_4BC$ of the mechanism. In the acceleration polygon, the sides of triangle $\Delta obc$ are not perpendicular to the sides of triangle $\Delta O_4BC$ like in the velocity polygon. Angle $\phi_4$ (Eq. C.59) between the sides of both triangles can be calculated as:

$$\phi_4 = \arctan\frac{a_C^t}{a_C^n} = \arctan\frac{a_B^t}{a_B^n} = \arctan\frac{a_{CB}^t}{a_{CB}^n} = \arctan\frac{\alpha_4}{\omega_4^2} \tag{C.59}$$

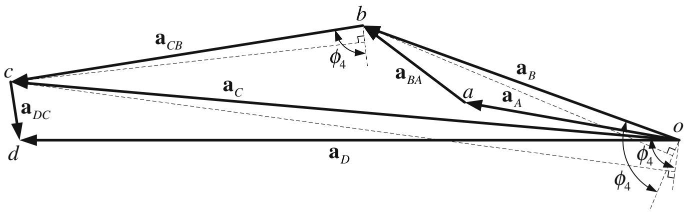



**Fig. C.9** Acceleration polygon

Therefore, triangle $\Delta obc$ in the polygon is similar to triangle $\Delta O_4BC$ in the mechanism and rotated angle $\phi_4$.

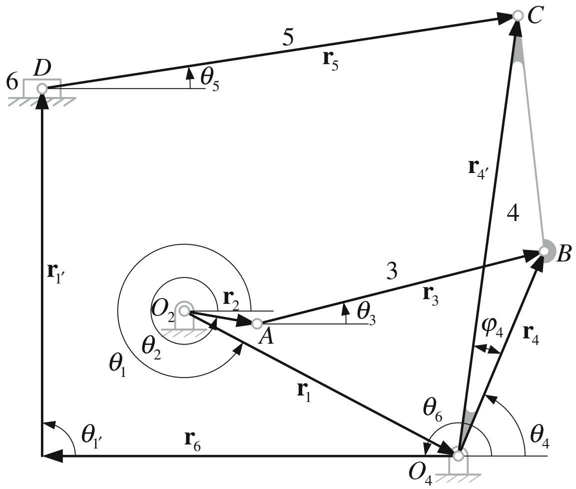
## C.6 Raven's Method

The number of needed vector loop equations depends on the number of unknowns. In this case, the position unknowns are $\theta_3$, $\theta_4$, $\theta_5$ and $r_6$. As each vector equation allows solving two unknowns and we have four, we will need 2 vector equations (Eq. C.60) (Fig. C.10).

$$\left.\begin{aligned} \mathbf{r}_1 + \mathbf{r}_4 = \mathbf{r}_2 + \mathbf{r}_3 \\ \mathbf{r}_{1'} + \mathbf{r}_5 + \mathbf{r}_6 = \mathbf{r}_{4'} \end{aligned}\right\} \tag{C.60}$$

Using the complex exponential form for the vectors, vector equation (Eq. C.60) can be written as (Eq. C.61):

**Fig. C.10** Kinematic diagram of the mechanism with the two vector loop equations used to solve the problem



$$\left.\begin{aligned} r_1 e^{i\theta_1} + r_4 e^{i\theta_4} &= r_2 e^{i\theta_2} + r_3 e^{i\theta_3} \\ r_{1'} e^{i\theta_{1'}} + r_5 e^{i\theta_5} + r_6 e^{i\theta_6} &= r_{4'} e^{i(\theta_4 + \varphi_4)} \end{aligned}\right\} \tag{C.61}$$

By separating the real and imaginary parts we obtain a system (Eq. C.62) with four equations and four unknowns: $\theta_3$, $\theta_4$, $\theta_5$ and $r_6$:

$$\left.\begin{aligned} r_1 \cos\theta_1 + r_4 \cos\theta_4 &= r_2 \cos\theta_2 + r_3 \cos\theta_3 \\ r_1 \sin\theta_1 + r_4 \sin\theta_4 &= r_2 \sin\theta_2 + r_3 \sin\theta_3 \\ r_{1'} \cos\theta_{1'} + r_5 \cos\theta_5 + r_6 \cos\theta_6 &= r_{4'} \cos(\theta_4 + \varphi_4) \\ r_{1'} \sin\theta_{1'} + r_5 \sin\theta_5 + r_6 \sin\theta_6 &= r_{4'} \sin(\theta_4 + \varphi_4) \end{aligned}\right\} \tag{C.62}$$

Using Freudenstein's equation, explained in Appendix B of this book, the first two equations of the system yields (Eqs. C.63 and C.64):

$$\theta_3 = 2\arctan\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{C.63}$$

$$\theta_4 = 2\arctan\frac{-E \pm \sqrt{E^2 - 4DF}}{2D} \tag{C.64}$$

where $A$, $B$, $C$, $D$, $E$ and $F$ coefficients (Eq. C.65) are:

$$\begin{aligned} &\left.\begin{aligned} A &= k_3 \cos\theta_1 - k_2 \cos(\theta_2 - \theta_1) + k_1 - \cos\theta_2 \\ B &= 2(\sin\theta_2 - k_3 \sin\theta_1) \\ C &= k_3 \cos\theta_1 - k_2 \cos(\theta_2 - \theta_1) + k_1 + \cos\theta_2 \end{aligned}\right\} \\ &\left.\begin{aligned} D &= k_3 \cos\theta_1 - k_5 \cos(\theta_2 - \theta_1) + k_4 + \cos\theta_2 \\ E &= 2(-\sin\theta_2 + k_3 \sin\theta_1) \\ F &= k_3 \cos\theta_1 - k_5 \cos(\theta_2 - \theta_1) + k_4 - \cos\theta_2 \end{aligned}\right\} \end{aligned} \tag{C.65}$$

And where $k_1$, $k_2$, $k_3$, $k_4$ and $k_5$ geometrical data (Eq. C.66) are:

$$\left.\begin{aligned} k_1 &= \tfrac{r_1^2 + r_2^2 + r_3^2 - r_4^2}{2 r_2 r_3} \\ k_2 &= \tfrac{r_1}{r_3} \\ k_3 &= \tfrac{r_1}{r_2} \\ k_4 &= \tfrac{r_1^2 + r_2^2 + r_4^2 - r_3^2}{2 r_2 r_4} \\ k_5 &= \tfrac{r_1}{r_4} \end{aligned}\right\} \tag{C.66}$$

Using Freudenstein's equation, explained in Appendix B of this book, the last two equations of the system (Eq. C.62) yields (Eqs. C.67 and C.68):

$$\theta_5 = \arcsin \frac{r_{4'} \sin(\theta_4 + \varphi_4) - r_{1'}}{r_5} \tag{C.67}$$

$$r_6 = r_5 \cos\theta_5 - r_{4'} \cos(\theta_4 + \varphi_4) \tag{C.68}$$

Using Eqs. (C.63), (C.64)–(C.67), (C.68), we can plot the position of the links relative to the positions of link 2 along one full turn. Figure C.11a shows angles $\theta_3$, $\theta_4$ and $\theta_5$ and Fig. C.11b shows distance $r_6$.

These figures illustrate the benefits of mathematical methods over graphical ones. The latter would only yield the solution to one of the points in such curves and the problem has to be solved again when there are any changes in the geometric parameters of the mechanism. Conversely, the expressions in Raven's method yield a solution for all the points in the curve and they do not need to be modified whenever geometrical data are modified.

The solution of the obtained equations for $\theta_2 = 350°$ yield (Eq. C.69) the following values for the position unknowns:

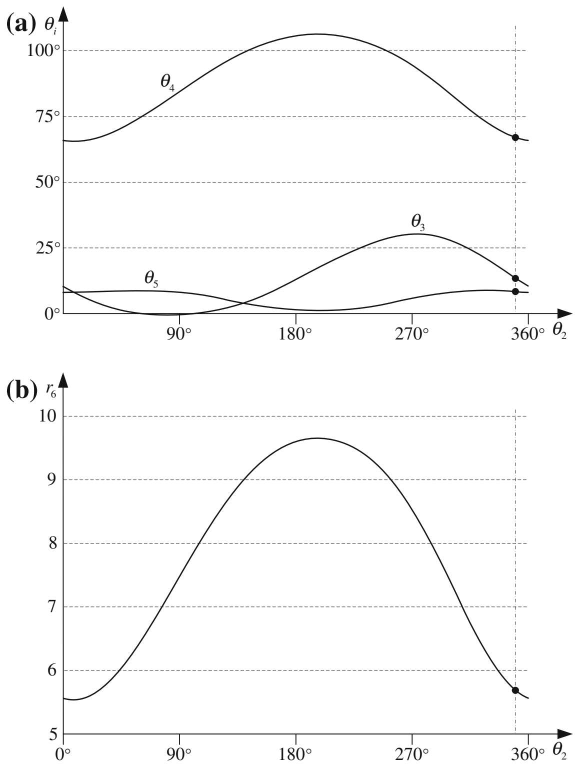



**Fig. C.11** **a** Angular position of links 3, 4 and 5 in terms of $\theta_2$, **b** Plot of the linear position of link 6 versus $\theta_2$

$$\left.\begin{aligned}\theta_3 &= 13.5^\circ\\ \theta_4 &= 67.03^\circ\\ \theta_5 &= 8.3^\circ\\ r_6 &= 5.607\,\mathrm{m}\end{aligned}\right\} \tag{C.69}$$

The position along the horizontal path of link 6 with respect to the coordinate system origin (Eq. C.71) will be given by the position of point D (Eq. C.70):

$$r_6 = -x_{DO_4} = -x_{DO_2} - x_{O_2O_4} = -x_{DO_2} - (-3.7\,\mathrm{m}) = 5.607\,\mathrm{m} \tag{C.70}$$

So for $\theta_2 = 350^\circ$ the $X$ coordinate of link 6 is:

$$x_{DO_2} = -1.907\,\mathrm{m} \tag{C.71}$$

By differentiating with respect to time (Eq. C.61), we obtain (Eq. C.72):

$$\left.\begin{aligned} ir_2\omega_2 e^{i\theta_2} + ir_3\omega_3 e^{i\theta_3} &= ir_4\omega_4 e^{i\theta_4}\\ v_6 e^{i\pi} + ir_5\omega_5 e^{i\theta_5} &= ir_{4'}\omega_4 e^{i(\theta_4+\varphi_4)}\end{aligned}\right\} \tag{C.72}$$

We separate the real and imaginary parts in Eq. (C.72), which yields the equation system (Eq. C.73) with four unknowns: $\omega_3$, $\omega_4$, $\omega_5$ and $v_6$:

$$\left.\begin{aligned} -r_2\omega_2\sin\theta_2 - r_3\omega_3\sin\theta_3 &= -r_4\omega_4\sin\theta_4\\ r_2\omega_2\cos\theta_2 + r_3\omega_3\cos\theta_3 &= r_4\omega_4\cos\theta_4\\ -v_6 - r_5\omega_5\sin\theta_5 &= -r_{4'}\omega_4\sin(\theta_4+\varphi_4)\\ r_5\omega_5\cos\theta_5 &= r_{4'}\omega_4\cos(\theta_4+\varphi_4)\end{aligned}\right\} \tag{C.73}$$

From the first two algebraic equations in Eq. (C.73) we can obtain the expressions for $\omega_3$ (Eq. C.74) and $\omega_4$ (Eq. C.75):

$$\omega_3 = \frac{r_2}{r_3}\frac{\sin(\theta_4-\theta_2)}{\sin(\theta_3-\theta_4)}\omega_2 \tag{C.74}$$

$$\omega_4 = \frac{r_2}{r_4}\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_3-\theta_4)}\omega_2 \tag{C.75}$$

Finally, from the third and fourth algebraic equation we reach expressions for $\omega_5$ (Eq. C.76) and $v_6$ (Eq. C.77):

$$\omega_5 = \frac{r_{4'}}{r_5}\frac{\cos(\theta_4+\varphi_4)}{\cos\theta_5}\omega_4 \tag{C.76}$$

$$v_6 = -r_5\omega_5\sin\theta_5 + r_{4\prime}\omega_4\sin(\theta_4+\varphi_4) \tag{C.77}$$

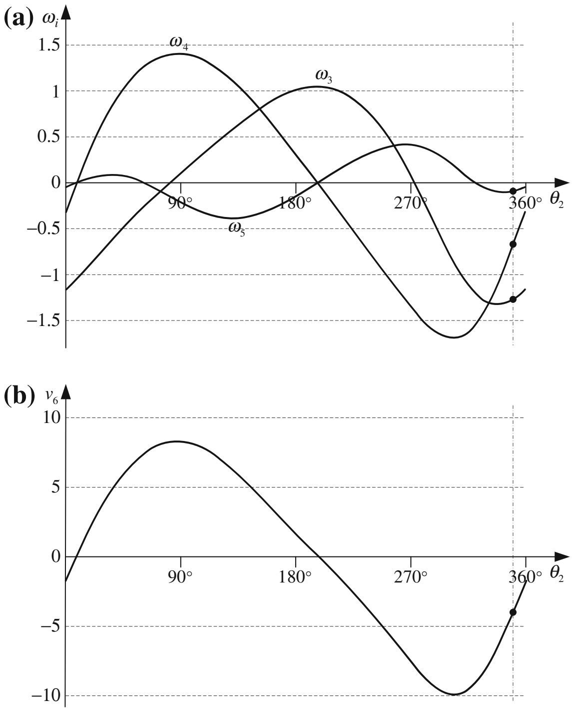



**Fig. C.12** **a** Angular velocities of links 3, 4 and 5 in terms of $\theta_2$, **b** Plot of the linear velocity of link 6 versus $\theta_2$

Using Eqs. (C.74) and (C.75), we can plot the kinematic curve of the link velocity versus the position of link 2. These curves are shown in Fig. C.12a, b.

Again, equations (Eqs. C.74–C.77) can be particularized for $\theta_2 = 350°$ yielding the values for the velocity unknowns (Eq. C.78):

$$\left.\begin{aligned} \omega_3 &= -1.27\,\text{rad/s} \\ \omega_4 &= -0.69\,\text{rad/s} \\ \omega_5 &= -0.09\,\text{rad/s} \\ v_6 &= -4.04\,\text{m/s} \end{aligned}\right\} \tag{C.78}$$

Once more, Eq. (C.72) can be time-differentiated again in order to find accelerations (Eq. C.79):

$$\left.\begin{aligned} (-r_2\omega_2^2 + ir_2\alpha_2)\mathrm{e}^{i\theta_2} + (-r_3\omega_3^2 + ir_3\alpha_3)\mathrm{e}^{i\theta_3} &= (-r_4\omega_4^2 + ir_4\alpha_4)\mathrm{e}^{i\theta_4} \\ (-r_5\omega_5^2 + ir_5\alpha_5)\mathrm{e}^{i\theta_5} + a_6\mathrm{e}^{i\theta_6} &= (-r_{4'}\omega_4^2 + ir_{4'}\alpha_4)\mathrm{e}^{i(\theta_4+\varphi_4)} \end{aligned}\right\} \tag{C.79}$$

By separating real and imaginary parts we reach, once more, a system (Eq. C.80) with four equations and four unknowns: $\alpha_3$, $\alpha_4$, $\alpha_5$ and $a_6$.

$$\left.\begin{aligned}-r_2\omega_2^2\cos\theta_2 - r_2\alpha_2\sin\theta_2 - r_3\omega_3^2\cos\theta_3 - r_3\alpha_3\sin\theta_3 &= -r_4\omega_4^2\cos\theta_4 - r_4\alpha_4\sin\theta_4\\ -r_2\omega_2^2\sin\theta_2 + r_2\alpha_2\cos\theta_2 - r_3\omega_3^2\sin\theta_3 + r_3\alpha_3\cos\theta_3 &= -r_4\omega_4^2\sin\theta_4 + r_4\alpha_4\cos\theta_4\\ -r_5\omega_5^2\cos\theta_5 - r_5\alpha_5\sin\theta_5 + a_6\cos\theta_6 &= -r_{4'}\omega_4^2\cos\theta_{4'} - r_{4'}\alpha_4\sin(\theta_4+\varphi_4)\\ -r_5\omega_5^2\sin\theta_5 + r_5\alpha_5\cos\theta_5 + a_6\sin\theta_6 &= -r_{4'}\omega_4^2\sin\theta_{4'} + r_{4'}\alpha_4\cos(\theta_4+\varphi_4)\end{aligned}\right\} \tag{C.80}$$

Again, we start by considering the first two algebraic equations in the system (Eq. C.80), which yield the angular accelerations of links 3 and 4 (Eq. C.81).

$$\left.\begin{aligned}\alpha_3 &= \frac{-r_2\alpha_2\sin\theta_2 + r_4\alpha_4\sin\theta_4 - r_2\omega_2^2\cos\theta_2 - r_3\omega_3^2\cos\theta_3 + r_4\omega_4^2\cos\theta_4}{r_3\sin\theta_3}\\ \alpha_4 &= \frac{-r_2\alpha_2\sin(\theta_3-\theta_2) + r_2\omega_2^2\cos(\theta_3-\theta_2) + r_3\omega_3^2 - r_4\omega_4^2\sin(\theta_4-\theta_3)}{r_4\sin(\theta_4-\theta_3)}\end{aligned}\right\} \tag{C.81}$$

Finally, $a_6$ and $\alpha_5$ (Eq. C.82) are obtained from the last two algebraic equations in the system (Eq. C.80):

$$\left.\begin{aligned}\alpha_5 &= \frac{r_{4'}\alpha_4\cos(\theta_4+\varphi_4) - r_{4'}\omega_4^2\sin(\theta_4+\varphi_4) + r_5\omega_5^2\sin\theta_5}{r_5\cos\theta_5}\\ a_6 &= r_{4'}\alpha_4\sin(\theta_4+\varphi_4) + r_{4'}\omega_4^2\cos(\theta_4+\varphi_4) - r_5\alpha_5\sin\theta_5 - r_5\omega_5^2\cos\theta_5\end{aligned}\right\} \tag{C.82}$$

These expressions (Eqs. C.81 and C.82) can be particularized for $\theta_2 = 350°$ yielding the values for the unknowns (Eq. C.83):

$$\left.\begin{aligned}\alpha_3 &= 1.98\,\text{rad/s}^2\\ \alpha_4 &= 9\,\text{rad/s}^2\\ \alpha_5 &= 0.72\,\text{rad/s}^2\\ a_6 &= 53.14\,\text{m/s}^2\end{aligned}\right\} \tag{C.83}$$

## C.7 Mass, Inertia Moments, Inertia Forces and Inertia Pairs

We assume that we know the value of the mass and the moment of inertia of the links. Their values are included in Table C.2.

Figure C.13 shows the center of mass of each link. Their position (Eq. C.84) is given by the following distances:

**Table C.2** Mass and moment of inertia of the links

| Link | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $M_k$ (kg) | 15.31 | 61.26 | 154.75 | 99.54 | 85 |
| $I_k$ (kg m$^2$) | 1.278 | 81.68 | 495.76 | 358.479 | – |

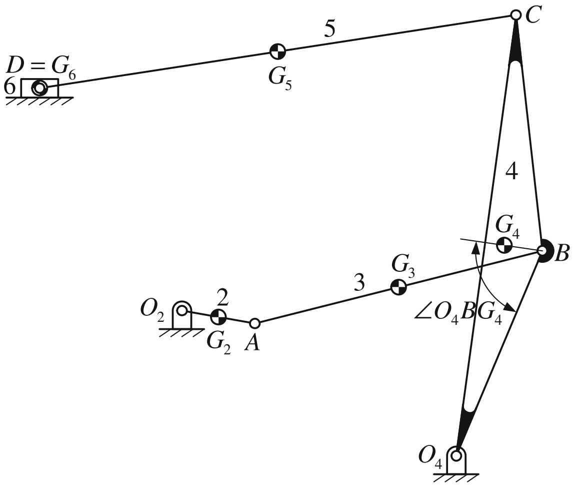



**Fig. C.13** Position of the centers of mass of the links

$$\left.\begin{aligned} \overline{O_2G_2} &= 0.5\,\mathrm{m} \\ \overline{AG_3} &= 2\,\mathrm{m} \\ \overline{BG_4} &= 0.52\,\mathrm{m} \\ \angle O_4BG_4 &= 75.4^\circ \\ \overline{DG_5} &= 3.25\,\mathrm{m} \end{aligned}\right\} \tag{C.84}$$

The acceleration of the center of mass of each link (Eq. C.85) has been determined by Raven's Method yielding the following results:

$$\left.\begin{aligned} \mathbf{a}_{G_2} &= -8.64\hat{\mathbf{i}} + 1.52\hat{\mathbf{j}} = 8.78\,\mathrm{m/s^2}\angle 170^\circ \\ \mathbf{a}_{G_3} &= -21.35\hat{\mathbf{i}} + 6.15\hat{\mathbf{j}} = 22.22\,\mathrm{m/s^2}\angle 163.94^\circ \\ \mathbf{a}_{G_4} &= -25.84\hat{\mathbf{i}} + 4.57\hat{\mathbf{j}} = 26.24\,\mathrm{m/s^2}\angle 170^\circ \\ \mathbf{a}_{G_5} &= -53.55\hat{\mathbf{i}} + 2.16\hat{\mathbf{j}} = 53.59\,\mathrm{m/s^2}\angle 177.69^\circ \\ \mathbf{a}_{G_6} &= -53.14\hat{\mathbf{i}} = 53.14\,\mathrm{m/s^2}\angle 180^\circ \end{aligned}\right\} \tag{C.85}$$

Once the masses, moments of inertia and accelerations of each center of mass have been determined, we can calculate the forces (Eq. C.86) and moments (Eq. C.87) due to inertia:

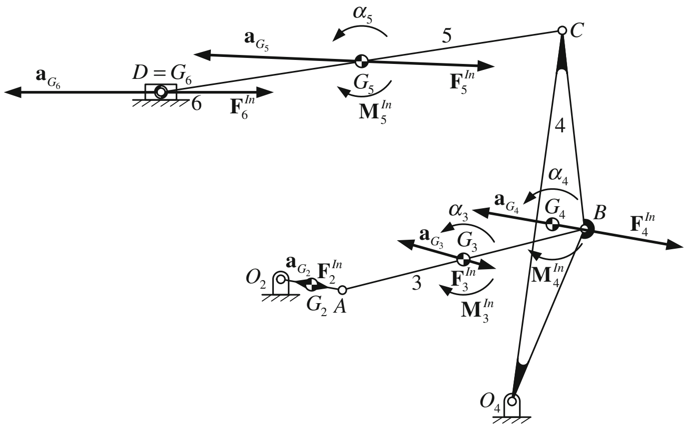



**Fig. C.14** Forces and moments due to inertia in the mechanism

$$\left.\begin{aligned}
F_2^{In} &= 15.31 \cdot 8.78 = 134.42\,\text{N} \\
F_3^{In} &= 61.26 \cdot 22.22 = 1361.2\,\text{N} \\
F_4^{In} &= 154.75 \cdot 26.24 = 4060.64\,\text{N} \\
F_5^{In} &= 99.54 \cdot 53.59 = 5334.35\,\text{N} \\
F_6^{In} &= 85 \cdot 53.14 = 4516.9\,\text{N}
\end{aligned}\right\} \tag{C.86}$$

$$\left.\begin{aligned}
M_2^{In} &= 1{,}278 \cdot 0 = 0 \\
M_3^{In} &= 81.68 \cdot 1.98 = 161.73\,\text{Nm} \\
M_4^{In} &= 495.76 \cdot 9 = 4461.84\,\text{Nm} \\
M_5^{In} &= 358.48 \cdot 0.72 = 258.1\,\text{Nm} \\
M_6^{In} &= 0
\end{aligned}\right\} \tag{C.87}$$

Figure C.14 shows the force and moment that acts on each link due to inertia. We can see that the force is opposite to the linear acceleration of the center of mass and the moment is opposite to the angular acceleration of the link.

## C.8 Force Analysis. Graphical Method

In order to calculate the torque that is acting on link 2 to equilibrate the mechanism, we will consider the inertia of the links as well as the force needed to move the 80 kg box. We will consider a friction coefficient of 0.4 and will neglect the inertia force of the box. Obviously, in a real problem the inertia of the box would have to be considered.

So, in this example the force that has to be exerted by link 6 to move the box (Eq. C.88) is:

$$F_R = \mu \mathrm{N} = 0.4(80\,\mathrm{kg} \cdot 9.81\,\mathrm{m/s^2}) = 314.2\,\mathrm{N} \tag{C.88}$$

We study the forces on the mechanism starting with link 6 (Eq. C.89).

$$\mathbf{F}_{56} + \mathbf{F}_{16} + \mathbf{F}_R = 0 \tag{C.89}$$

In order to simplify the problem, we will consider that force $\mathbf{F}_R$ acts on point D. This will affect the position of reaction force $\mathbf{F}_{16}$ but as force $\mathbf{F}_R$ is quite small compared to $\mathbf{F}_{i6}$ and the distance from point D to the base of the piston is also small compared to the mechanism dimensions, the error is very small.

Since the direction of force $\mathbf{F}_{56}$ is unknown, it will be broken into a vertical and horizontal component, $\mathbf{F}_{56}^V$ and $\mathbf{F}_{56}^H$. We know that the direction of force $\mathbf{F}_{16}$ is perpendicular to the slider trajectory. So, this force will be equilibrated by $\mathbf{F}_{56}^V$. The value of $\mathbf{F}_{56}^H$ (Eq. C.91) can be calculated by means of the force equilibrium of the horizontal components of the forces acting on the link (Eq. C.90) as shown in Fig. C.15b.

$$\left.\begin{aligned} 0 = \Sigma F_x = F_R + F_6^{In} + F_{56}^H \\ 0 = \Sigma F_y = F_{16} + F_{56}^V \end{aligned}\right\} \tag{C.90}$$

$$\mathbf{F}_{56}^H = 4831.14\,\mathrm{N}\angle 180^\circ \tag{C.91}$$

Figure C.16 shows the free body diagram of link 5. As we already know, force $\mathbf{F}_{56}^H$ is equal to force $\mathbf{F}_{65}^H$ but with opposite direction.

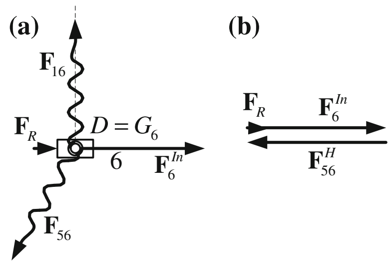



**Fig. C.15** **a** Free body diagram of link 6, **b** horizontal components acting on link 6

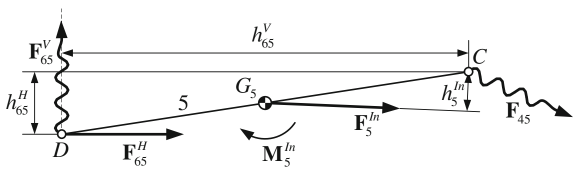



**Fig. C.16** Free body diagram of link 5

Next, we analyze the moment equilibrium at point $C$ (Eqs. C.92–C.94) assuming the direction of $\mathbf{F}_{65}^{V}$ to be upwards:

$$\sum_j M_{j_z}^{C} = 0 \rightarrow h_5^{In} F_5^{In} - h_{65}^{V} F_{65}^{V} + h_{65}^{H} F_{65}^{H} - M_5^{In} = 0 \tag{C.92}$$

$$0.6\,\text{m} \cdot 5334.35\,\text{N} - 6.43\,\text{m} \cdot F_{65}^{V} + 0.94\,\text{m} \cdot 4831.14\,\text{N} - 258.1\,\text{Nm} = 0 \tag{C.93}$$

$$\mathbf{F}_{56}^{V} = 1163.88\,\text{N}\angle 90^\circ \tag{C.94}$$

Then, $\mathbf{F}_{65}$ will be (Eq. C.95):

$$\left.\begin{aligned} F_{65} &= \sqrt{\left(F_{65}^{V}\right)^2 + \left(F_{65}^{H}\right)^2} \\ &= \sqrt{4831.14^2 + 1163.88^2} = 4969.32\,\text{N} \\ \theta_{F_{65}} &= \arctan\frac{F_{65}^{V}}{F_{65}^{H}} \\ &= \arctan\frac{1163.88}{4831.14} = 13.5^\circ \end{aligned}\right\} \tag{C.95}$$

where distances $h_5^{In}$, $h_{65}^{V}$ and $h_{65}^{H}$ (Eq. C.96) are:

$$\left.\begin{aligned} h_5^{In} &= \overline{G_5C}\sin(360^\circ - \theta_{F_5^{In}} + \theta_5) = 3.25\sin 10.65^\circ \\ h_{65}^{V} &= \overline{DC}\cos\theta_5 = 6.5\cos 8.34^\circ \\ h_{65}^{H} &= \overline{DC}\sin\theta_5 = 6.5\sin 8.34^\circ \end{aligned}\right\} \tag{C.96}$$

Back to link 6, we know the vertical forces acting on it, as $\mathbf{F}_{65}^{V} = -\mathbf{F}_{56}^{V} = \mathbf{F}_{16}$. The equilibrium of forces acting on link 5 (Eq. C.97) yields the value of force $\mathbf{F}_{45}$ (Eq. C.98) as shown in Fig. C.17.

$$\mathbf{F}_{56}^{H} + \mathbf{F}_{56}^{V} + \mathbf{F}_5^{In} + \mathbf{F}_{45} = 0 \tag{C.97}$$

$$\mathbf{F}_{45} = 10{,}207.9\,\text{N}\angle 185.36^\circ \tag{C.98}$$

Similarly, the equilibrium equations of links 3 and 4 yield the value of the forces transmitted by the links.

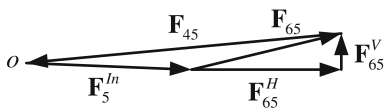



**Fig. C.17** Polygon of forces acting on link 5

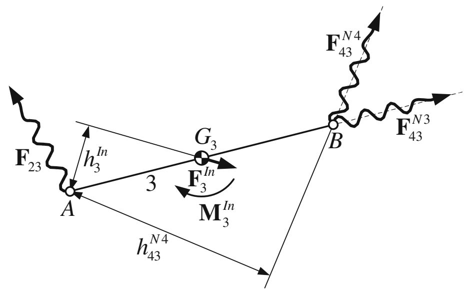



**Fig. C.18** Free body diagram of link 3

In link 3, we break force $\mathbf{F}_{43}$ into components $\mathbf{F}_{43}^{N3}$ and $\mathbf{F}_{43}^{N4}$ (Eq. C.101), with directions $\overline{AB}$ and $\overline{O_4B}$ respectively (Fig. C.18). We consider the equilibrium of moments about point $A$ (Eqs. C.99 and C.100) assuming that $\mathbf{F}_{43}^{N4}$ (Eq. C.101) goes upwards.

$$\sum_j M_{j_z}^A = 0 \rightarrow -h_3^{In} F_3^{In} + h_{43}^{N4} F_{43}^{N4} - M_3^{In} = 0 \tag{C.99}$$

$$-0.986\,\mathrm{m} \cdot 1361.2\,\mathrm{N} + 3.215 \cdot F_{43}^{N4} - 161.73\,\mathrm{Nm} = 0 \tag{C.100}$$

$$F_{43}^{N4} = 467.77\,\mathrm{N} \tag{C.101}$$

Where distances $h_3^{In}$ and $h_{43}^{N4}$ can be measured on the drawing of the mechanism or be determined as (Eq. C.102):

$$\left.\begin{aligned} h_3^{In} &= \overline{G_3A}\cos(90^\circ + \theta_{F_3^{In}} - \theta_3) = 2\cos 60.44^\circ \\ h_{43}^{N4} &= \overline{BA}\sin(\theta_4 - \theta_3) = 4\sin 53.5^\circ \end{aligned}\right\} \tag{C.102}$$

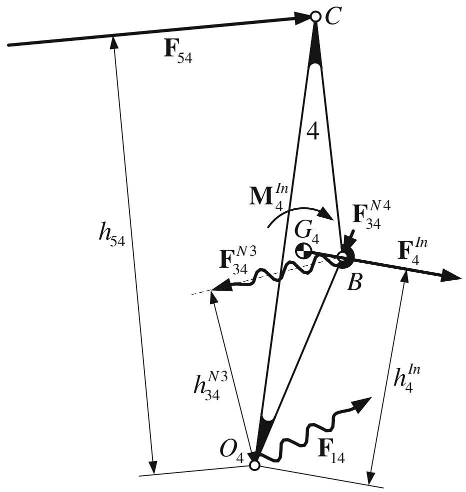



**Fig. C.19** Free body diagram of link 4

Figure C.19 shows the free body diagram of link 4. We can calculate $\mathbf{F}_{43}^{N3}$ (Eq. C.105) by means of the equilibrium equation of moments with respect to point $O_4$ (Eqs. C.103 and C.104). We assume the direction of $F_{34}^{N3}$ to be oriented to the left.

$$\sum_j M_{j_z}^{O_4} = 0 \rightarrow -h_4^{In} F_4^{In} + h_{34}^{N3} F_{34}^{N3} - h_{54} F_{54} - M_4^{In} = 0 \tag{C.103}$$

$$\begin{aligned} 0 = &-2.923\,\mathrm{m} \cdot 4060.64\,\mathrm{N} + 2.412\,\mathrm{m} \cdot F_{34}^{N3} \\ &- 5.84\,\mathrm{m} \cdot 10{,}207.9\,\mathrm{N} - 4461.84\,\mathrm{Nm} \end{aligned} \tag{C.104}$$

$$F_{34}^{N3} = 31{,}486\,\mathrm{N} \tag{C.105}$$

where distances $h_4^{In}$, $h_{34}^{N3}$ and $h_{54}$ (Eq. C.106) can be determined as:

$$\left.\begin{aligned} h_4^{In} &= \overline{O_4B}\cos(90^\circ + \theta_{F_4^{In}} - \theta_4) = 3\cos 13^\circ \\ h_{54} &= \overline{O_4C}\cos(90^\circ + \theta_{F_{54}} - \theta_{4'}) = 6\cos 13.26^\circ \\ h_{34}^{N3} &= \overline{O_4B}\sin(\theta_4 - \theta_3) \end{aligned}\right\} \tag{C.106}$$

The force equilibrium analysis of the forces acting on link 4 (Eq. C.107) yields $\mathbf{F}_{14}$ (Eq. C.108). Figure C.20 shows the force polygon.

$$\mathbf{F}_{54} + \mathbf{F}_4^{In} + \mathbf{F}_{34}^{N4} + \mathbf{F}_{34}^{N3} + \mathbf{F}_{14} = 0 \tag{C.107}$$

$$\mathbf{F}_{14} = 18{,}454\,\mathrm{N}\angle 24^\circ \tag{C.108}$$

Also, the analysis of the force equilibrium of link 3 (Eq. C.109) yields force $\mathbf{F}_{23}$ (Eq. C.110). The force polygon of forces acting on link 3 is shown in Fig. C.21.

$$\mathbf{F}_3^{In} + \mathbf{F}_{43} + \mathbf{F}_{23} = 0 \tag{C.109}$$

$$\mathbf{F}_{23} = 31{,}520\,\mathrm{N}\angle 193.5^\circ \tag{C.110}$$

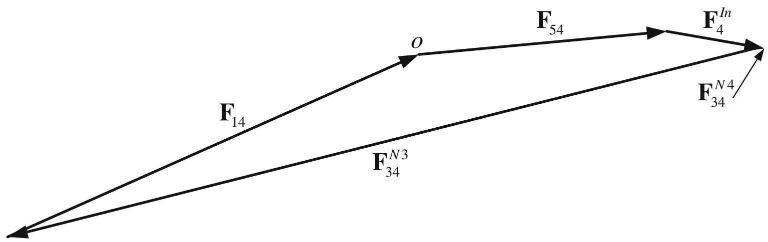



**Fig. C.20** Polygon of forces acting on link 4

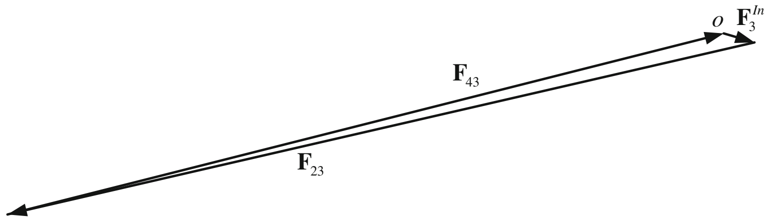
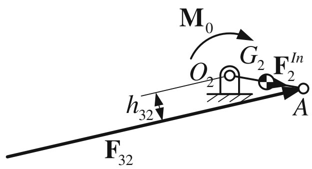



**Fig. C.21** Polygon of forces acting on link 3

**Fig. C.22** Free body diagram of link 2




Finally, Fig. C.22 shows the equilibrium analysis of link 2, from which we find the value of the equilibrating torque (Eq. C.114) as well as force $\mathbf{F}_{12}$ (Eq. C.112). Since the value of $\mathbf{F}_2^{In}$ is very small compared to $\mathbf{F}_{32}$, we will neglect it in Eq. (C.111). Therefore:

$$\mathbf{F}_{32} + \mathbf{F}_{12} = 0 \tag{C.111}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{32} = \mathbf{F}_{23} \tag{C.112}$$

Figure C.22 shows the free body diagram of link 2.

Torque $M_0$, which acts on link 2 to equilibrate the mechanism, can be obtained with (Eq. C.113):

$$\sum_j M_{j_z}^{O_2} = 0 \rightarrow h_{32}F_{32} + M_0 = 0 \tag{C.113}$$

$$M_0 = -0.398\,\text{m} \cdot 31{,}520\,\text{N} = -12{,}568.6\,\text{Nm} \tag{C.114}$$

where distance $h_{32}$ (Eq. C.115) can be calculated as follows:

$$h_{32} = \overline{O_2A}\sin(360^\circ - \theta_{F_{32}} - \theta_2) = 1 \cdot \sin 23.5^\circ \tag{C.115}$$

## C.9 Dynamic Analysis. Matrix Method

We start the dynamic analysis of the mechanism by writing the equations of the force and moment equilibrium of each link (Eqs. C.116–C.20). Figure C.23 shows radius vectors $\mathbf{p}_i$, $\mathbf{q}_i$ and $\mathbf{r}_i$ used in the moment equations:

- Link 2:

$$\left.\begin{aligned} \mathbf{F}_{32} - \mathbf{F}_{21} &= m_2 \mathbf{a}_{G_2} \\ \mathbf{p}_2 \wedge \mathbf{F}_{32} - \mathbf{q}_2 \wedge \mathbf{F}_{21} + \mathbf{M}_0 &= I_{G_2} \boldsymbol{\alpha}_2 \end{aligned}\right\} \tag{C.116}$$

- Link 3:

$$\left.\begin{aligned} \mathbf{F}_{43} - \mathbf{F}_{32} &= m_3 \mathbf{a}_{G_3} \\ \mathbf{p}_3 \wedge \mathbf{F}_{43} - \mathbf{q}_3 \wedge \mathbf{F}_{32} &= I_{G_3} \boldsymbol{\alpha}_3 \end{aligned}\right\} \tag{C.117}$$

- Link 4:

$$\left.\begin{aligned} \mathbf{F}_{54} - \mathbf{F}_{43} + \mathbf{F}_{14} &= m_4 \mathbf{a}_{G_4} \\ \mathbf{p}_4 \wedge \mathbf{F}_{54} - \mathbf{q}_4 \wedge \mathbf{F}_{43} + \mathbf{r}_4 \wedge \mathbf{F}_{14} &= I_{G_4} \boldsymbol{\alpha}_4 \end{aligned}\right\} \tag{C.118}$$

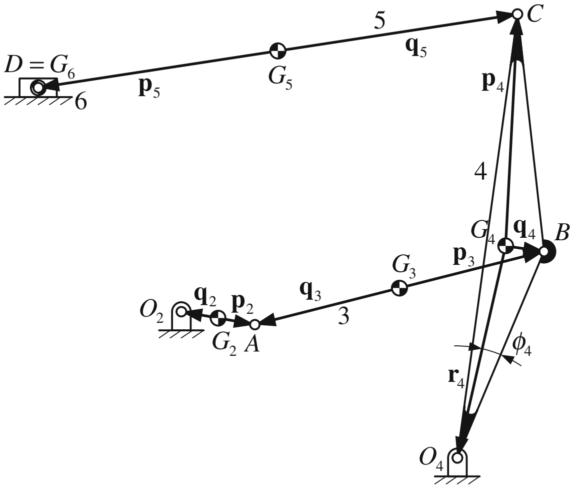



**Fig. C.23** Radius vectors used in the moment equilibrium equations

- Link 5:

$$\left.\begin{aligned} \mathbf{F}_{65} - \mathbf{F}_{54} &= m_5 \mathbf{a}_{G_5} \\ \mathbf{p}_5 \wedge \mathbf{F}_{65} - \mathbf{q}_5 \wedge \mathbf{F}_{54} &= I_{G_5} \boldsymbol{\alpha}_5 \end{aligned}\right\} \tag{C.119}$$

- Link 6:

$$\mathbf{F}_{16} - \mathbf{F}_{65} + \mathbf{F}_R = m_6 \mathbf{a}_{G_6} \tag{C.120}$$

The moment equilibrium equation of link 6 is not necessary as we suppose the forces are concurrent at point $D$ and $\alpha_6 = 0$.

Equations C.116–C.20 yield a system of 14 algebraic equations (Eq. C.121) and 14 unknowns where $F_{16_x} = 0$ as the direction of force $\mathbf{F}_{16}$ has to be perpendicular to the sliding trajectory. So, in this example it will only have a vertical component.

Projecting each force equation on the X and Y axles and finding the vector products in the torque equations, we reach the following system of equations:

$$\left.\begin{aligned}
F_{32_x} - F_{21_x} &= m_2 a_{G_{2x}} \\
F_{32_y} - F_{21_y} &= m_2 a_{G_{2y}} \\
(p_{2_x} F_{32_y} - p_{2_y} F_{32_x}) - (q_{2_x} F_{21_y} - q_{2_y} F_{21_x}) + M_0 &= I_{G_2} \alpha_2 \\
F_{43_x} - F_{32_x} &= m_3 a_{G_{3x}} \\
F_{43_y} - F_{32_y} &= m_3 a_{G_{3y}} \\
(p_{3_x} F_{43_y} - p_{3_y} F_{43_x}) - (q_{3_x} F_{32_y} - q_{3_y} F_{32_x}) &= I_{G_3} \alpha_3 \\
F_{54_x} - F_{43_x} + F_{14_x} &= m_4 a_{G_{4x}} \\
F_{54_y} - F_{43_y} + F_{14_y} &= m_4 a_{G_{4y}} \\
(p_{4_x} F_{54_y} - p_{4_y} F_{54_x}) - (q_{4_x} F_{43_y} - q_{4_y} F_{43_x}) + (r_{4_x} F_{14_y} - r_{4_y} F_{14_x}) &= I_{G_4} \alpha_4 \\
F_{65_x} - F_{54_x} &= m_5 a_{G_{5x}} \\
F_{65_y} - F_{54_y} &= m_5 a_{G_{5y}} \\
(p_{5_x} F_{65_y} - p_{5_y} F_{65_x}) - (q_{5_x} F_{54_y} - q_{5_y} F_{54_x}) &= I_{G_5} \alpha_5 \\
F_{16_x} - F_{65_x} &= m_6 a_{G_{6x}} - F_R \\
F_{16_y} - F_{65_y} &= m_6 a_{G_{6y}}
\end{aligned}\right\} \tag{C.121}$$

The system (Eq. C.121) can be written in matrix form (Eq. C.122):

$$[L]\mathbf{q} = \mathbf{F} \tag{C.122}$$

where:

$$[L] = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_{2_y} & -q_{2_x} & -p_{2_y} & p_{2_x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{3_y} & -q_{3_x} & -p_{3_y} & p_{3_x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{4_y} & -q_{4_x} & -p_{4_y} & p_{4_x} & -r_{4_y} & r_{4_x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{5_y} & -q_{5_x} & 0 & 0 & -p_{5_y} & p_{5_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \tag{C.123}$$

$$\mathbf{q} = \begin{pmatrix} F_{21_x} \\ F_{21_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{54_x} \\ F_{54_y} \\ F_{14_x} \\ F_{14_y} \\ F_{65_x} \\ F_{65_y} \\ F_{16_y} \\ M_0 \end{pmatrix} \tag{C.124}$$

$$\mathbf{F} = \begin{pmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2} \alpha_2 \\ m_3 a_{G_{3x}} \\ m_3 a_{G_{3y}} \\ I_{G_3} \alpha_3 \\ m_4 a_{G_{4x}} \\ m_4 a_{G_{4y}} \\ I_{G_4} \alpha_4 \\ m_5 a_{G_{5x}} \\ m_5 a_{G_{5y}} \\ I_{G_5} \alpha_5 \\ m_6 a_{G_{6x}} - F_R \\ m_6 a_{G_{6y}} \end{pmatrix} \tag{C.125}$$

The analytical expressions of the radius vectors in matrix $[L]$ are defined in Eqs. (C.126)–(C.129). Angles $\theta_2$, $\theta_3$, $\theta_4$, $\varphi_4$ and $\theta_5$ are the ones defined in Sect. 3.6 for the analysis with Raven's Method. Angle $\phi_4$ is defined in Fig. C.23.

- Link 2:

$$\left.\begin{aligned} \mathbf{p}_2 &= \overline{AG_2}(\cos\theta_2\hat{\mathbf{i}} + \sin\theta_2\hat{\mathbf{j}}) \\ \mathbf{q}_2 &= -\overline{O_2G_2}(\cos\theta_2\hat{\mathbf{i}} + \sin\theta_2\hat{\mathbf{j}}) \end{aligned}\right\} \tag{C.126}$$

- Link 3:

$$\left.\begin{aligned} \mathbf{p}_3 &= \overline{BG_3}(\cos\theta_3\hat{\mathbf{i}} + \sin\theta_3\hat{\mathbf{j}}) \\ \mathbf{q}_3 &= -\overline{AG_3}(\cos\theta_3\hat{\mathbf{i}} + \sin\theta_3\hat{\mathbf{j}}) \end{aligned}\right\} \tag{C.127}$$

- Link 4:

$$\left.\begin{aligned} \mathbf{p}_4 &= \overline{CO_4}(\cos(\theta_4+\varphi_4)\hat{\mathbf{i}} + \sin(\theta_4+\varphi_4)\hat{\mathbf{j}}) - \overline{O_4G_4}(\cos(\theta_4+\phi_4)\hat{\mathbf{i}} + \sin(\theta_4+\phi_4)\hat{\mathbf{j}}) \\ \mathbf{q}_4 &= \overline{BO_4}(\cos\theta_4\hat{\mathbf{i}} + \sin\theta_4\hat{\mathbf{j}}) - \overline{O_4G_4}(\cos(\theta_4+\phi_4)\hat{\mathbf{i}} + \sin(\theta_4+\phi_4)\hat{\mathbf{j}}) \\ \mathbf{r}_4 &= -\overline{O_4G_4}(\cos(\theta_4+\phi_4)\hat{\mathbf{i}} + \sin(\theta_4+\phi_4)\hat{\mathbf{j}}) \end{aligned}\right\} \tag{C.128}$$

- Link 5:

$$\left.\begin{aligned} \mathbf{p}_5 &= -\overline{DG_5}(\cos\theta_5\hat{\mathbf{i}} + \sin\theta_5\hat{\mathbf{j}}) \\ \mathbf{q}_5 &= \overline{CG_5}(\cos\theta_5\hat{\mathbf{i}} + \sin\theta_5\hat{\mathbf{j}}) \end{aligned}\right\} \tag{C.129}$$

If we find the values of these vectors (Eqs. C.126–C.129) for $\theta_2 = 350°$ we obtain (Eq. C.130):

$$\left.\begin{aligned} \mathbf{p}_2 &= 0.4924\hat{\mathbf{i}} - 0.0868\hat{\mathbf{j}}\,\mathrm{m} \\ \mathbf{q}_2 &= -\mathbf{p}_2 \\ \mathbf{p}_3 &= 1.9447\hat{\mathbf{i}} + 0.4672\hat{\mathbf{j}}\,\mathrm{m} \\ \mathbf{q}_3 &= -\mathbf{p}_3 \\ \mathbf{p}_4 &= 0.1692\hat{\mathbf{i}} + 3.1052\hat{\mathbf{j}}\,\mathrm{m} \\ \mathbf{q}_4 &= 0.5144\hat{\mathbf{i}} - 0.0766\hat{\mathbf{j}}\,\mathrm{m} \\ \mathbf{r}_4 &= -0.6598\hat{\mathbf{i}} - 2.8373\hat{\mathbf{j}}\,\mathrm{m} \\ \mathbf{p}_5 &= -3.2157\hat{\mathbf{i}} - 0.4712\hat{\mathbf{j}}\,\mathrm{m} \\ \mathbf{q}_5 &= -\mathbf{p}_5 \end{aligned}\right\} \tag{C.130}$$

The analytical expressions of the acceleration vector of the center of mass of each link (Eqs. C.131–C.34) are:

$$\mathbf{a}_{G_2} = \overline{O_2G_2}\alpha_2(-\sin\theta_2\hat{\mathbf{i}} + \cos\theta_2\hat{\mathbf{j}}) - \overline{O_2G_2}\omega_2^2(\cos\theta_2\hat{\mathbf{i}} + \sin\theta_2\hat{\mathbf{j}}) \tag{C.131}$$

$$\begin{aligned}\mathbf{a}_{G_3} = {} & \overline{O_2A}\alpha_2(-\sin\theta_2\hat{\mathbf{i}} + \cos\theta_2\hat{\mathbf{j}}) - \overline{O_2A}\omega_2^2(\cos\theta_2\hat{\mathbf{i}} + \sin\theta_2\hat{\mathbf{j}}) \\ & + \overline{G_3A}\alpha_3(-\sin\theta_3\hat{\mathbf{i}} + \cos\theta_3\hat{\mathbf{j}}) - \overline{G_3A}\omega_3^2(\cos\theta_3\hat{\mathbf{i}} + \sin\theta_3\hat{\mathbf{j}})\end{aligned} \tag{C.132}$$

$$\begin{aligned}\mathbf{a}_{G_4} = {} & \overline{O_4G_4}\alpha_4(-\sin(\theta_4+\phi_4)\hat{\mathbf{i}} + \cos(\theta_4+\phi_4)\hat{\mathbf{j}}) \\ & - \overline{O_4G_4}\omega_4^2(\cos(\theta_4+\phi_4)\hat{\mathbf{i}} + \sin(\theta_4+\phi_4)\hat{\mathbf{j}})\end{aligned} \tag{C.133}$$

$$\begin{aligned}\mathbf{a}_{G_5} = {} & \overline{CO_4}\alpha_4(-\sin(\theta_4+\varphi_4)\hat{\mathbf{i}} + \cos(\theta_4+\varphi_4)\hat{\mathbf{j}}) \\ & - \overline{CO_4}\omega_4^2(\cos(\theta_4+\varphi_4)\hat{\mathbf{i}} + \sin(\theta_4+\varphi_4)\hat{\mathbf{j}}) \\ & + \overline{G_5C}\alpha_5(-\sin(\theta_5+180^\circ)\hat{\mathbf{i}} + \cos(\theta_5+180^\circ)\hat{\mathbf{j}}) \\ & - \overline{G_5C}\omega_5^2(\cos(\theta_5+180^\circ)\hat{\mathbf{i}} + \sin(\theta_5+180^\circ)\hat{\mathbf{j}})\end{aligned} \tag{C.134}$$

$$\begin{aligned}\mathbf{a}_{G_6} = {} & \overline{CO_4}\alpha_4(-\sin(\theta_4+\varphi_4)\hat{\mathbf{i}} + \cos(\theta_4+\varphi_4)\hat{\mathbf{j}}) \\ & - \overline{CO_4}\omega_4^2(\cos(\theta_4+\varphi_4)\hat{\mathbf{i}} + \sin(\theta_4+\varphi_4)\hat{\mathbf{j}}) \\ & + \overline{DC}\alpha_5(-\sin(\theta_5+180^\circ)\hat{\mathbf{i}} + \cos(\theta_5+180^\circ)\hat{\mathbf{j}}) \\ & - \overline{DC}\omega_5^2(\cos(\theta_5+180^\circ)\hat{\mathbf{i}} + \sin(\theta_5+180^\circ)\hat{\mathbf{j}})\end{aligned} \tag{C.135}$$

If we find the values of all the elements in the vector $\mathbf{F}$ for $\theta_2 = 350^\circ$ and solve the system (Eq. C.122), we obtain the values (Eq. C.136) for the unknowns (Eq. C.124):

$$\left.\begin{aligned}\mathbf{F}_{21} &= 32{,}239\hat{\mathbf{i}} + 7386\hat{\mathbf{j}}\,\mathrm{N} \\ \mathbf{F}_{32} &= 32{,}106\hat{\mathbf{i}} + 7409\hat{\mathbf{j}}\,\mathrm{N} \\ \mathbf{F}_{14} &= 16{,}636\hat{\mathbf{i}} + 7550\hat{\mathbf{j}}\,\mathrm{N} \\ \mathbf{F}_{54} &= 10{,}161\hat{\mathbf{i}} + 944\hat{\mathbf{j}}\,\mathrm{N} \\ \mathbf{F}_{65} &= 4833\hat{\mathbf{i}} + 1173\hat{\mathbf{j}}\,\mathrm{N} \\ \mathbf{F}_{16} &= 1173\hat{\mathbf{j}}\,\mathrm{N} \\ M_0 &= -12{,}872\,\mathrm{Nm}\end{aligned}\right\} \tag{C.136}$$

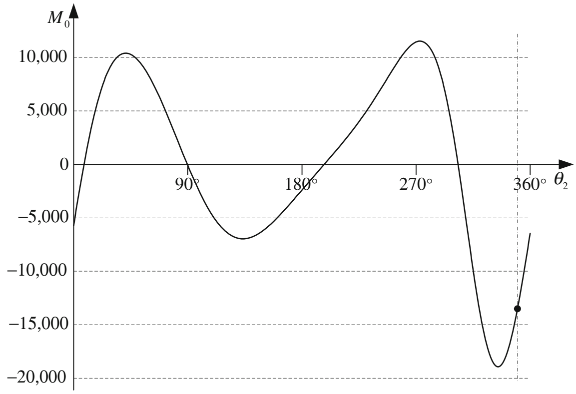



**Fig. C.24** Instantaneous motor torque $M_0$ versus crank angle $\theta_2$


**Fig. C.25** Shaking force curve versus crank angle $\theta_2$

One of the main advantages of the Matrix Method for the dynamic analysis when compared to the graphical method is the ability to calculate the value of the unknowns along a complete cycle.

With the latter we can only find one solution for one position of the crank in the curve and the expressions cannot be used again if there are any changes in the geometrical data of the mechanism.

Figure C.24 shows a curve with the value of the instantaneous motor torque for the different positions of the crank, $\theta_2$. The value of the motor torque is given by $M_0$ and it is the torque that is necessary to apply to motor link 2 in order to obtain the desired speed and acceleration. In this case, $\omega_2 = 4.19\,\text{rad/s}$ and $\alpha_2 = 0$.

Moreover, we can obtain the curve of the magnitude of the shaking force versus the positions of the crank (Fig. C.25). The shaking force is given by Eq. (C.137):

$$\mathbf{F}_S = \mathbf{F}_{21} + \mathbf{F}_{41} + \mathbf{F}_{61} \tag{C.137}$$

In Fig. C.25 we can see that the maximum value for the shaking force is at a position close to $\theta_2 = 350°$. In this case, this position coincides with the maximum acceleration of link 6.