A review of simple analytical methods for the kinematic synthesis of four-bar and slider-crank function generators for two and three prescribed finite positions

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A review of simple analytical methods for the kinematic synthesis of four-bar and slider-crank function generators for two and three prescribed finite positions

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1 Introduction

Kinematic synthesis is an useful tool mainly used in the first stages of mechanisms design. The possibility to obtain through computation the dimensions of a linkage whose parts reproduce a prescribed motion allows to spare time and increase accuracy.

It is well known that time consuming trial-and-error kinematic synthesis techniques are often adopted at the industry level.

This situation is caused by different reasons such as:

- Many useful kinematic synthesis methods are scattered in technical literature which spans a time interval of decades. These methods are not always available on line and their retrieval often requires a manual library search.

- Most of the university textbooks concentrate on the theory of few *classical* methods of kinematic synthesis. Considered their didactic purposes, the description of the methods is usually made with abundance of details. Hence, the style of description is far to be concise and essential, as the one usually adopted in manuals dedicated to industrial designers.

The main purpose of this paper is to provide industrial designers with ready-to-use kinematic synthesis methods. For this reason, the description format adopted is consistent with this purpose. In fact, the methods have been summarized in an algorithmic way and with explicative numerical examples provided.

Each method has been also programmed in the scripting language `Ch` [1].

This work has been written to provide the engineer with simple and ready-to-use kinematic synthesis procedures which can assist him in the early stages of mechanisms design.

2 The transmission angle

No design problem admits a unique solution. Therefore, quality indices must be adopted for screening out non optimal solutions. Often a compromise solution should be searched in order to satisfy simultaneously more than one optimality criterion.

It is of paramount importance the comparison of design alternatives on the basis of the quality of motion transmitted. One of the oldest performances index is the *transmission angle* and has been introduced by H. Alt in 1930.

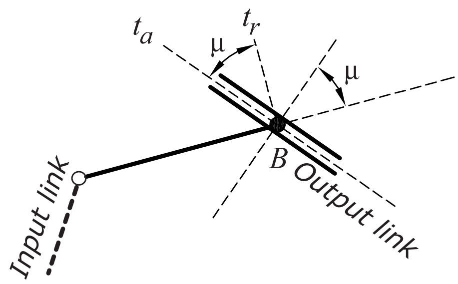

Figure 1: Definition of transmission angle

The definition of the transmission angle is as follows (see Figure 1)

> Let B be the point where the force is transmitted to the output link. The transmission angle μ is the angle between the direction t_a of the velocity of B of the output link and the direction t_r of the relative velocity of B with respect to the driving link.

It can be observed that:

- $\mu = 90°$ is the most favorable value of the transmission angle;
- the transmission of motion is theoretically impossible when $\mu = 0°$ and $\mu = 180°$ (locking condition);

[1] `Ch` is a C/C++ interpreter for cross-platform scripting, shell programming, 2D/3D plotting, numerical computing. `Ch` is an alternative to C/C++ compilers. It is free for academic use and `Ch` can be downloaded from www.softintegration.com.

The listings of the Ch programs developed are available for download at the following web page http://www.ingegneriameccanica.org/mechanisms.htm

- it is therefore significant the difference $|90° - \mu|$ instead of the absolute value of μ (in other words, according to the discussed criterion, a mechanism with $\mu = 50°$ has the same merit as one with a value of $\mu = 130°$);
- the location of μ depends on the way in which the mechanism is driven.

With reference to Figure 2, let μ be the inner angle between the coupler and output lever.

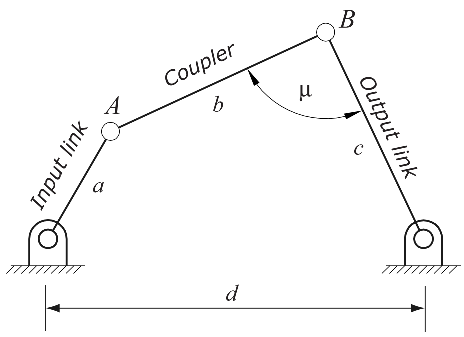

Figure 2: Transmission angle in a four-bar linkage

In the absence of friction, gravity and inertia:

- the coupler can transmit only a tensile force F;
- the torque transmitted to the output link of length L is $FL \sin \mu$;
- a locking condition occurs when the coupler and the output lever are aligned.

This suggest that, for satisfactory operation, the value of μ should not get values smaller than 30°. In addition to force transmission there is another reason to comply with the rule above. A poor transmission angle makes the position of the output lever very sensitive to clearance in the joints, manufacturing tolerances on link lenghhs, and changes in dimension due to thermal expansions [8, 9].

The expression

$$\cos \mu = \frac{\left(b^2 + c^2\right) - \left(a^2 + d^2\right) + 2ad \cos \phi}{2bc} \tag{1}$$

is valid for all configurations of the mechanism. If the mechanism can reach the configurations shown in Figure 3, then the minimum and maximum values of the transmission angle can be computed as follows:

$$\mu_{\min} = \operatorname{acos}\left[\frac{b^2 + c^2 - (1-a)^2}{2bc}\right] , \tag{2a}$$

$$\mu_{\max} = \operatorname{acos}\left[\frac{b^2 + c^2 - (1+a)^2}{2bc}\right] . \tag{2b}$$

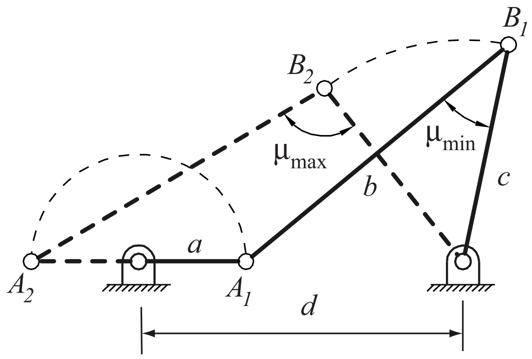

Figure 3: Configurations of minimum and maximum transmission angle

The transmission angle of a slider crank (see Figure 4):

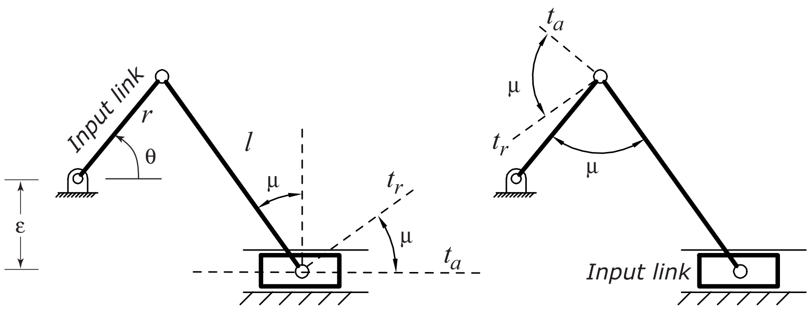

Figure 4: Locations of transmission angle in a slider crank

- with the crank as the input link, is the angle between the coupler and the line perpendicular to the velocity of the slider;
- with the slider as the input link (*e.g.* internal combustion engine) is the angle between the coupler and the crank.

Let us denote with r, the crank length, l the coupler length, ε, the slider-crank offset, θ the crank angle. In the first case, the value of the transmission angle at a generic position is given by

$$\mu = \arctan\left[\frac{\varepsilon + r\sin\theta}{\sqrt{l^2 - (\varepsilon + r\sin\theta)^2}}\right] - \frac{\pi}{2} \tag{3}$$

The minimum value of the transmission angle, attained for θ=90°, is

$$\mu_{\min} = \arctan\sqrt{\left(\frac{l}{r+\varepsilon}\right)^2 - 1} \tag{4}$$

There is not a general consensus on the most appropriate quality index of transmission motion in the presence of forces. The index should account for the true forces acting on the mechanism.

Here are some more general definitions of the transmission angle [7]:

- the angle whose tangent is the ratio of the tangential force to the normal force on the output link (proposed by W. Rössner);

- the angle between the direction of the force on the output link and the direction of the absolute motion of the point of application of the force on the output link (proposed by E. Lenk).

3 Quick-return crank-rocker four-bar linkages

In the design of mechanisms it is often necessary to design a linkage that will convert a rotating input motion into an oscillating output motion.

The methods herein described will permit the design of four-bar linkages of this nature with a considerable accuracy.

These methods, however, are limited to rocker oscillation angles less than 180°.

With the acronym **FOURB-CR** we will denote all the procedures for the design of a plane crank-rocker mechanism for prescribed values of the oscillating angle ψ of the output link and angle $180° + \theta$ of the input crank. (see Figure 6). The angle θ is positive when c.c.w. The crank-rocker can provide a quick-return motion. Assuming uniform angular speed of the crank, *time ratio* TR is defined as the ratio of the time taken for the forward stroke to the time taken for the return stroke. Its value is given by

$$TR = \frac{\text{Duration of forward stroke}}{\text{Duration of return stroke}} = \frac{180° + \theta}{180° - \theta}\ . \tag{5}$$

The reader may find useful the plot in Figure 5 to quickly obtain θ as a function of TR.

The optimal design of these type of four-bar linkages can be stated as follows [5]:

> " Determine the crank-rocker proportions of a four-bar mechanism with a given swing angle ψ and corresponding crank rotation θ+180°, or time ratio, such that the maximum deviation of the transmission angle from 90° is minimum."

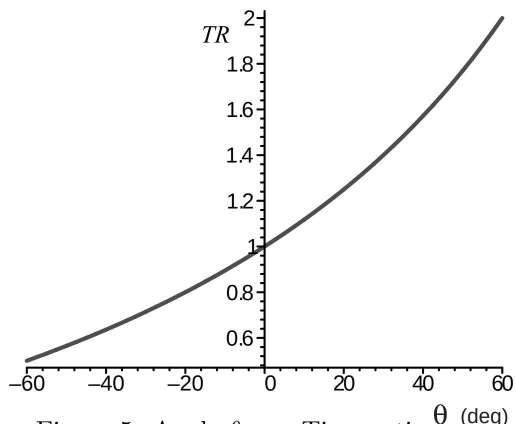

Figure 5: Angle θ .vs. Time ratio

3.1 Algorithm FOURB-CR-1

Purpose: Design of a plane crank-rocker mechanism for prescribed values of the oscillating angle ψ of the output link, of angle $\pi + \theta$ of the input crank and initial crank angle θ_0. (see Figure 6 for the nomenclature).

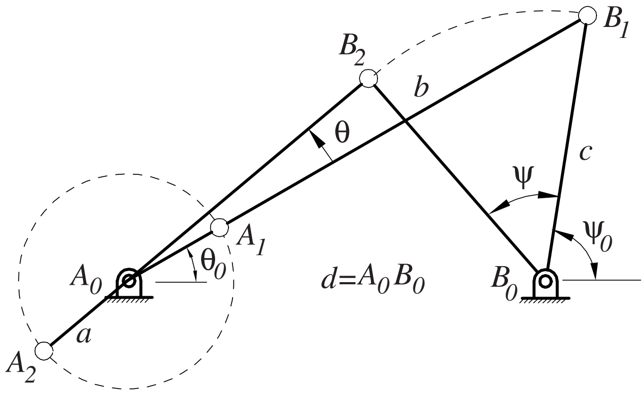

Figure 6: Nomenclature

Data: Angles ψ and θ
Unknowns: Link lengths ratios a/d, b/d and c/d
Source: [2, 11]
Computational procedure

1. Assume a consistent value for the angle θ_0

2. Let

$$
\begin{aligned}
n_1 &= \sin\theta_0 \\
n_2 &= \cos\theta_0 \\
n_3 &= \sin(\theta+\theta_0) \\
n_4 &= \sin(\psi-\theta-\theta_0) \\
n_5 &= \cos(\psi-\theta-\theta_0)
\end{aligned}
$$

3. Compute

$$
\psi_0 = \operatorname{atan}\left[\frac{n_1(n_3+n_4)}{n_2 n_3 - n_1 n_5}\right]
$$

4. Let

$$
\begin{aligned}
p_1 &= \frac{\sin\psi_0}{\sin\theta_0} \\
p_2 &= \frac{\sin(\psi+\psi_0)}{\sin(\theta+\theta_0)} \\
p_3 &= \sin(\psi_0-\theta_0)
\end{aligned}
$$

5. Compute

$$
\begin{aligned}
d &= 1 \\
c &= \frac{\sin\theta_0}{p_3} \\
a &= \frac{c(p_1-p_2)}{2} \\
b &= \frac{c(p_1+p_2)}{2}
\end{aligned}
$$

The extremes values $\mu_{\min}$ e $\mu_{\max}$ of the transmission angle for the synthesized mechanism follow from the equations

$$
\begin{aligned}
\mu_{\min} &= \operatorname{acos}\left[\frac{b^2+c^2-(1-a)^2}{2bc}\right] , \\
\mu_{\max} &= \operatorname{acos}\left[\frac{b^2+c^2-(1+a)^2}{2bc}\right] .
\end{aligned}
$$

The procedure can be repeated for different values of the angle θ_0 until satisfactory extremes values of the transmission angle are obtained.

Numerical example
Let θ=10°, ψ=50° and θ_0=30°. The previous formulas give ψ_0=81.05° and

$$
\begin{aligned}
a &= 0.257961 \quad & b &= 1.012188 \\
c &= 0.642896 \quad & d &= 1
\end{aligned}
$$

The extreme values of the transmission angle are

$$\mu_{\min} = 47.022^\circ \quad \mu_{\max} = 96.380^\circ$$

Program: `FOURB-CR-1.ch`

3.2 Algorithm FOURB-CR-2

Purpose: Design of a plane crank-rocker mechanism for prescribed values of the oscillating angle ψ of the output link, angle $\pi + \theta$ of the input crank and initial rocker angle ψ_0 (see Figure 6 for the nomenclature).
Data: Angles ψ and θ
Unknowns: Link lengths ratios a/d, b/d and c/d
Source: [2, 11]
Computational procedure

1. Assume a consistent value for the angle ψ_0
2. Let

$$
\begin{aligned}
\theta_f &= \theta + 180^\circ \\
y &= \psi_0 + \psi - \theta_f \\
m_2 &= -\sin\psi_0 \sin\theta_f \\
m_1 &= \sin y + \cos\psi_0 \sin\theta_f - \sin\psi_0 \cos\theta_f \\
m_0 &= \cos\psi_0 \cos\theta_f - \cos y
\end{aligned}
$$

3. Solve w.r.t. x

$$m_2 x^2 + m_1 x + m_0 = 0$$

4. If the roots are complex the initial data are not consistent.
5. For each root x, let $\theta_0 = \operatorname{atan}(x)$
6. Let

$$
\begin{aligned}
p_1 &= \frac{\sin\psi_0}{\sin\theta_0} \\
p_2 &= \frac{\sin(\psi + \psi_0)}{\sin(\theta + \theta_0)} \\
p_3 &= \sin(\psi_0 - \theta_0)
\end{aligned}
$$

7. Compute link legths

$$
\begin{aligned}
d &= 1 \\
c &= \frac{\sin\theta_0}{p_3} \\
a &= \frac{c\,(p_1 - p_2)}{2} \\
b &= \frac{c\,(p_1 + p_2)}{2}
\end{aligned}
$$

8. Accept only the solution with all positive values of link lengths.

The extremes values μ_{min} e μ_{max} of the transmission angle for the synthesized mechanism follow from the equations

$$
\begin{aligned}
\mu_{\text{min}} &= \text{acos}\left[\frac{b^2 + c^2 - (1-a)^2}{2bc}\right] \; , \\
\mu_{\text{max}} &= \text{acos}\left[\frac{b^2 + c^2 - (1+a)^2}{2bc}\right] \; .
\end{aligned}
$$

The procedure can be repeated for different values of the angle θ_0 until satisfactory extremes values of the transmission angle are obtained.
Numerical example
Let θ=36.5°, ψ=85° and ψ_0=70°. The previous formulas give θ_0=33.89° and

$$
\begin{aligned}
a &= 0.585014 \quad b = 1.009510 \\
c &= 0.946189 \quad d = 1.000000
\end{aligned}
$$

The extreme values of the transmission angle are

$$
\mu_{\text{min}} = 24.223^\circ \quad \mu_{\text{max}} = 108.238^\circ
$$

Program: `FOURB-CR-2.ch`

3.3 Algorithm FOURB-CR-3

Purpose: Determine the crank-rocker proportions of a four-bar mechanism with a given swing angle ψ and corresponding crank rotation $\theta + 180^\circ$, or time ratio, such that the maximum deviation Δ of the transmission angle from 90° is minimized (see Figure 6 for the nomenclature).
Data: Angles ψ and $\theta \neq 0$
Unknowns: Link lengths ratios a/d, b/d, c/d, maximum deviation Δ of the transmission angle from 90°.
Source: [5, 4]
Computational procedure

1. Let $\theta_f = \theta + 180°$

$$t = \tan\frac{\theta_f}{2}$$
$$u = \tan\left(\frac{\theta_f - \psi}{2}\right)$$

2. Solve the cubic equation w.r.t. x

$$x^3 + 2x^2 - t^2x - \frac{t^2\left(1+t^2\right)}{u^2} = 0$$

3. Let

$$\lambda = \frac{t}{\sqrt{x}}$$

and, if you want a crank-rocker mechanism, accept only those solutions which satisfy the inequality

$$1 \leq \lambda^2 \leq u^2t^2$$

4. For all the feasible values of λ compute

$$a = \sin\frac{\psi}{2}$$
$$b = |\lambda a|$$
$$c = \sqrt{\sin^2\frac{\theta_f}{2} + \lambda^2\cos^2\frac{\theta_f}{2}}$$
$$d = \sqrt{\sin^2\frac{\theta_f - \psi}{2} + \lambda^2\cos^2\frac{\theta_f - \psi}{2}}$$

5. Among all the feasible values of λ choose the solution with the least value of Δ

$$\sin\Delta = \frac{\left|(a \pm d)^2 - c^2 - b^2\right|}{2bc}$$

where $0° \leq \Delta \leq 90°$, $+$ sign if $\theta < 0$, $-$ sign if $\theta > 0$.

Numerical example
Let θ=-10°, ψ=45°. The previous formulas give

$$a = 0.253031 \quad b = 0.671266$$
$$c = 0.676194 \quad d = 1$$

The extreme values of the transmission angle are [2]:

$$\mu_{\min} = 67.331° \quad \mu_{\max} = 136.844°$$

[2] Use (2)

Thus, the maximum deviation of the transmission angle from the optimal value of 90° is

$$\Delta = 46.844°$$

For the case of $TR = 1$ (*i.e.* $\theta = 0$) the method does not give feasible results and algorithm FOURB-CR-4 should be used.
Program: `FOURB-CR-3.ch`

3.4 Algorithm FOURB-CR-4

Purpose: Design a crank-rocker four-bar linkage with prescribed unit time ratio $TR = 1$, rocker oscillating angle ψ and minimum value $\mu_{\min}$ of the transmission angle. A mechanism with unit time ratio is said *centric four-bar linkage* (see Figure 7).
Data: Angles ψ and $\theta = 0°$, $0 < \mu_{\min} < \left(90° - \frac{\psi}{2}\right)$

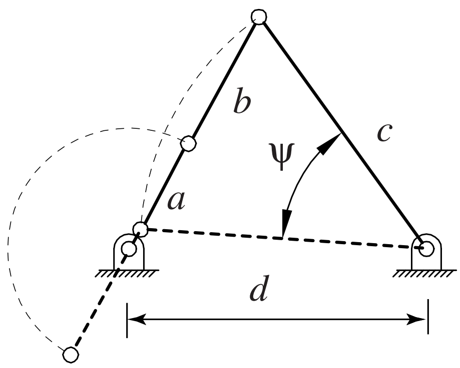

Figure 7: Centric four-bar linkage

Unknowns: Link lengths ratios a/d, b/d and c/d
Source: [13]
Computational procedure

1. Compute

$$\frac{b}{d} = \sqrt{\frac{1 - \cos\psi}{2\cos^2\mu_{\min}}}$$

$$\frac{c}{d} = \sqrt{\frac{1 - \left(\frac{b}{d}\right)^2}{1 - \left(\frac{b}{d}\right)^2 \cos^2\mu_{\min}}}$$

$$\frac{a}{d} = \sqrt{\left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 - 1}$$

Numerical example
Let ψ=50° and $\mu_{\min} = 30°$. The previous formulas give

$$a = 0.407014 \quad b = 0.487998$$
$$c = 0.963078 \quad d = 1$$

The maximum value of the transmission angle is (use last of eqs. (2))

$$\Delta = 149.99°$$

Program: `FOURB-CR-4.ch`
Design charts for this procedure have been prepared and included in another section.

Alternatively, the problem can be solved by means of the formulas deduced by J.G. Volmer[3].

$$\frac{d}{a} = \sqrt{\frac{\cos^2\mu_{\min}\cot^2\frac{\psi}{2}}{\cos^2\mu_{\min} - \sin^2\frac{\psi}{2}}}\ , \tag{6a}$$

$$\frac{b}{a} = \sqrt{\frac{\cos^2\frac{\psi}{2}}{\cos^2\mu_{\min} - \sin^2\frac{\psi}{2}}}\ , \tag{6b}$$

$$\frac{c}{a} = \frac{1}{\sin\frac{\psi}{2}}\ . \tag{6c}$$

4 Drag-link four-bar linkages

One of the most common applications of the drag-link four-bar mechanism is as a coupling between two parallel shafts. It provides 1 to 1 overall speed ratio while generating, over the cycle, a nonuniform rotation of the output shaft. These mechanisms usually have favorable motion transmission characteristics.

4.1 Algorithm DRAGLINK-1

Purpose: Design a drag-link mechanism with design positions corresponding at input angles ϕ_1=0° and ϕ_2=180°:

- with the same minimum value ($\cos\mu_{\min} = -\cos\mu_{\max}$) of transmission angle at design positions;
- with a prescribed rotation $\Delta\psi_0$ of the output link.

The prescribed rotation of the input link is $\Delta\phi = \phi_2 - \phi_1$=180° (see Figure 8).
Data: $\Delta\psi_0$, $\mu_{\min}$.
Unknowns: Link lengths ratios a/d, b/d and c/d.
Source: [1]
Computational procedure

[3]A modern source which reports the formulas is [10], p.198-199

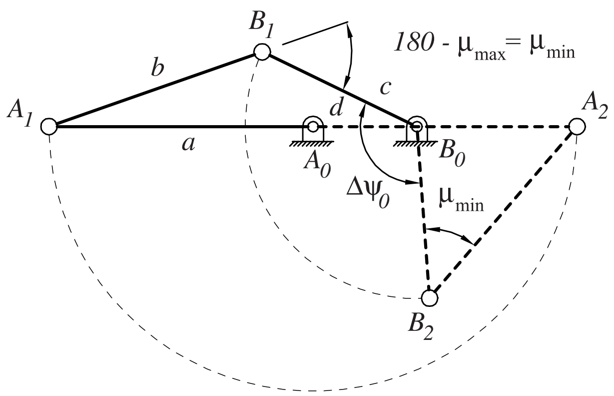

Figure 8: Nomenclature

1. Let c=1;
2. Compute[4]

$$b = c\sqrt{\frac{\sin\Delta\psi_0}{\sin(\Delta\psi_0 - \mu_{\min})}}$$

$$\lambda = \operatorname{atan}\left(\frac{b\sin\mu_{\min}}{c + b\cos\mu_{\min}}\right)$$

$$a_1 = b\frac{\sin\mu_{\min}}{\sin\lambda}$$

$$b_1 = \frac{-c\cos\Delta\psi_0 + b\cos(\Delta\psi_0 - \mu_{\min})}{\cos\lambda}$$

$$a = \frac{a_1 + b_1}{2}$$

$$d = a_1 - a$$

Numerical example
Design a drag-link four-bar for $\Delta\psi_0$=120° and $\mu_{\min}$=45°.
The formulas listed above give ($c = 1$)

$$b = 1.316074 \quad \lambda = 25.7353^\circ$$
$$a_1 = 2.143189 \quad b_1 = 0.933189$$
$$a = 1.538189 \quad b = 1.316074$$

Therefore, the link ratios are

$$a/d = 2.542460 \quad b/d = 2.175328 \quad c/d = 1.652892$$

[4]If the term under square root is negative no solution exists.

Design charts for this problem are given in [1] and [12]. These charts have been also included in this paper.
Program: `DRALINK1.ch`

4.2 Algorithm DRAGLINK-2

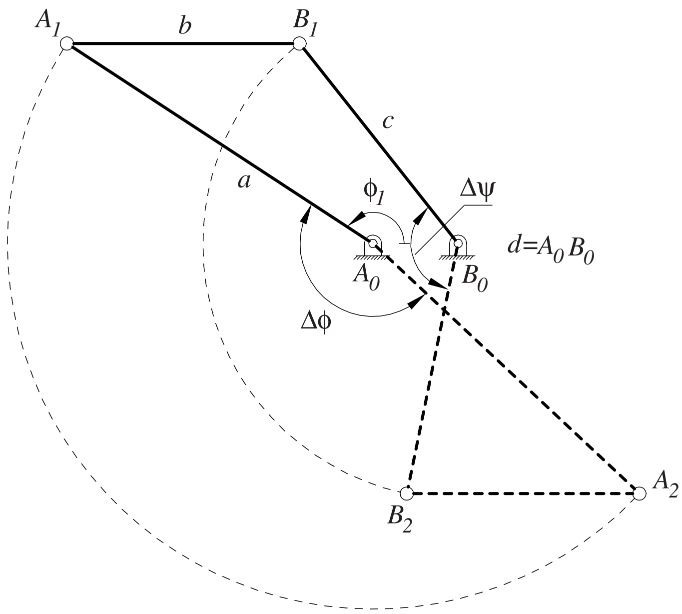

Figure 9: Drag-link mechanism at design positions

Purpose: Design a drag-link mechanism in its two unity velocity ratio positions for prescribed crank rotations, $\Delta\phi$ and $\Delta\psi$, and minimized maximum deviation $\Delta\mu_{\max}$ of the transmission angle from 90° (see Figure 9).

The method of drag-link design previously described often introduces excessive velocity and acceleration. This is due to the 180° input link rotation arbitrarily chosen for the design. The mechanisms so achieved are not used for their maximum capacity. Therefore, this method approaches the problem with a different design criterion.
Data: Input link rotation $\Delta\phi$, Output link rotation $\Delta\psi$.
Unknowns: Link lengths ratios a/d, b/d and c/d and input crank angle ϕ_1 at the first design position.
Source: [15, 16]
Computational procedure(without transmission angle optimization)

1. Let[5]

$$\Delta L = \Delta\phi - \Delta\psi$$
$$t = \tan\left(\frac{\Delta\phi}{2}\right)$$
$$u = \tan\left(\frac{\Delta\psi}{2}\right)$$
$$v = \tan\left(\frac{\Delta L}{2}\right)$$

2. Assume

$$\lambda = \frac{\text{Length of coupler}}{\text{Length of input crank}}$$

3. Compute link lengths

$$d = \sqrt{\frac{v^2}{1+v^2}}$$
$$a = \sqrt{\frac{u^2+\lambda^2}{1+u^2}}$$
$$b = \sqrt{\frac{\lambda^2 v^2}{1+v^2}}$$
$$c = \sqrt{\frac{t^2+\lambda^2}{1+t^2}}$$

Computational procedure (with transmission angle optimization)

1. Let

$$\Delta L = \Delta\phi - \Delta\psi$$
$$t = \tan\left(\frac{\Delta\phi}{2}\right)$$
$$u = \tan\left(\frac{\Delta\psi}{2}\right)$$
$$v = \tan\left(\frac{\Delta L}{2}\right)$$

2. Solve the cubic equation w.r.t. x

$$x^3 + 2x^2 - t^2 x - \frac{t^2\left(1+t^2\right)}{u^2}$$

[5]For drag-link proportions the limits on $\Delta\phi$ and ΔL are

$$0^\circ \le \Delta L \le 180^\circ$$
$$90^\circ + \frac{\Delta L}{2} \le \Delta\phi \le 270^\circ + \frac{\Delta L}{2}$$

The algorithm fails when $\phi = 180^\circ$.

3. Let
$$\lambda = \frac{t}{\sqrt{x}}$$
and accept only those solutions which satisfy the inequality
$$1 \leq \lambda^2 \leq u^2 t^2$$

4. For each real feasible solution of the cubic let
$$\epsilon = \frac{\lambda}{ut}$$

5. Compute link lengths ratios
$$\begin{aligned}
\frac{a}{d} &= \frac{\sin(\Delta\psi/2)}{\sin(\Delta L/2)}\sqrt{1+t^2\epsilon^2} \\
\frac{b}{d} &= ut\epsilon \\
\frac{c}{d} &= \frac{\sin(\Delta\phi/2)}{\sin(\Delta L/2)}\sqrt{1+u^2\epsilon^2}
\end{aligned}$$

6. Compute maximum deviation $\Delta\mu_{\max}$ of transmission angle from 90°
$$\begin{aligned}
\sin\Delta\mu_1 &= \sin\frac{a^2+d^2-b^2-c^2+2da}{2bc} \\
\sin\Delta\mu_2 &= \sin\frac{a^2+d^2-b^2-c^2-2da}{2bc} \\
\Delta\mu_{\max} &= \max(\mu_1, \mu_2)
\end{aligned}$$

7. Among all the feasible solutions choose the one with the minimum value of $\Delta\mu_{\max}$.

8. Compute the initial angular position of input crank
$$\begin{aligned}
r &= -\frac{b}{a}\frac{\cos(\Delta\psi/2)}{\sin(\Delta L/2)} \\
s &= -\frac{d}{a}\frac{\sin(\Delta\psi/2)}{\sin(\Delta L/2)} \\
\phi_1 &= \mathrm{ATAN2}(r, s) - \frac{\Delta\phi}{2}
\end{aligned}$$

F. Freudenstein[6] demonstrated, by means of a kinematic inversion, the correspondence between the design of a drag-link mechanism and the one of a crank-rocker mechanism.

[6] See his discussion in the paper authored by L.W. Tsai [15].

Numerical example
Design a drag-link four-bar for $\Delta\phi$=170° and $\Delta\psi$=130°.

The formulas listed above give $\lambda_{\text{opt}} = 2.727403$ and

$$\frac{a}{d} = 4.287136 \quad \frac{b}{d} = 2.727403 \quad \frac{c}{d} = 2.994452$$

The maximum deviation of the transmission angle is $\Delta\mu_{\max}$=44.91°. The input crank angle at first design position is ϕ_1=146.82°.

This design procedure can also be carried out by means of design charts.
Program: `DRALINK2.ch`

5 Function generator four-bar:2 positions

In the previous sections have been presented specialized algorithms for the coordination of rotation of input and output links of crank-rocker and drag-link mechanisms. Both input and output links are adjacent to the frame.

In this section a similar problem will be approached, but for a general four-bar linkage. The type of four-bar linkage is not predetermined, as in the cases previously discussed.

5.1 Algorithm FOURB-FG2-1

Purpose: Design a four-bar linkage such that the absolute angular displacements ϕ_j and ψ_j of input and output links are coordinated. The initial angular positions of driving and driven links are denoted by ϕ_0 and ψ_0, respectively.(see Figure 10).
Data: ϕ_j and ψ_j with $\phi_j \neq \psi_j$, ϕ_0, a/d, c/d
Unknowns:Link lengths ratio b/d and ψ_0.
Source: [3]
Computational procedure

1. Let

$$\xi = \frac{\phi_j}{2} \qquad \epsilon_j = \frac{\psi_j}{2}$$
$$t = \tan\phi_0 \qquad q = \tan\psi_0$$
$$d = 1$$

2. Compute

$$r_{aj} = \frac{\sin\epsilon_j}{\sin(\xi_j - \epsilon_j)}$$
$$r_{bj} = \frac{\sin\xi_j}{\sin(\xi_j - \epsilon_j)}$$

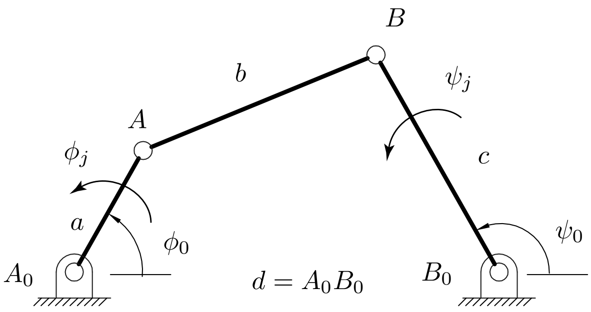

Figure 10: Nomenclature

3. Choose a value for a and ϕ_0 and compute

$$X = \frac{1}{a\cos\phi_0}$$

and

$$t = \tan\phi_0$$

4. Compute

$$\begin{aligned}
a_1 &= \frac{r_{aj}\cos\epsilon_j}{a\cos\phi_0} - t\sin(\xi-\epsilon) + \cos(\xi-\epsilon) \\
b_1 &= \frac{r_{aj}\sin\epsilon_j}{a\cos\phi_0} - t\cos(\xi-\epsilon) - \sin(\xi-\epsilon) \\
c_1 &= -\frac{r_{bj}}{c}\left(t\cos\xi + \sin\xi\right)
\end{aligned}$$

5. Solve w.r.t.ψ_0

$$a_1\sin\psi_0 + b_1\cos\psi_0 + c_1 = 0$$

6. Compute

$$\begin{aligned}
x_A &= a\cos\phi_0 & y_A &= a\sin\phi_0 \\
x_B &= d + c\cos\psi_0 & y_B &= c\sin\psi_0
\end{aligned}$$

$$b = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Numerical example

Design a four-bar linkage such that:

- Rotation of driving link: ϕ_j=60°
- Rotation of driven link: ψ_j=30°

- Initial position of driving link: ϕ_0=50°
- Link length ratios: $a/d = 4.000000$, $c/d = 2.000000$

The procedure gives two different solutions:
Solution n.1: $b/d = 2.695481$ and ψ_0=114.260208°.
Solution n.2: $b/d = 2.4.441984$ and ψ_0=169.186625°.
Program: `FOURB-FG2-1.ch`

6 Function generator four-bar:3 positions

6.1 Algorithm FOURB-FG3-1

Purpose: Design a four-bar linkage so that the output angles ϕ_1, ϕ_2 and ϕ_3 are coordinated with the input angles ψ_1, ψ_2 and ψ_3, respectively. (see Figure 11 for the nomenclature).
Data: ϕ_1, ϕ_2, ϕ_3 and ψ_1, ψ_2, ψ_3

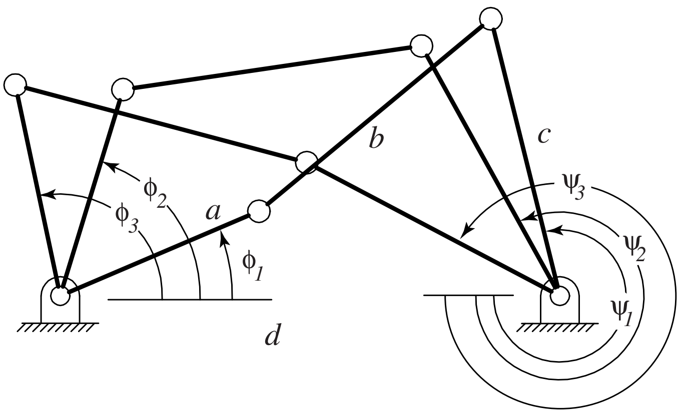

Figure 11: Freudenstein's equation:Nomenclature

Unknowns: Link lengths ratios a/d, b/d and c/d. The values of the transmission angles at ech position are also computed.
Source: Reported in almost all books on kinematics synthesis published after 1960. The algorithm is due to F. Freudenstein. The book [6] reports a detailed algebraic treatment and an interesting analysis of errors.
Computational procedure

1. For $i = 1, 2, 3$, compute

$$\begin{aligned}
A_{i1} &= \cos\phi_i \\
A_{i2} &= \cos\psi_i \\
A_{i3} &= -1 \\
B_i &= \cos\left(\phi_i - \psi_i\right)
\end{aligned}$$

2. Solve w.r.t. k the system

$$[A]\{k\} = \{B\}$$

3. Compute

$$\begin{aligned}
d &= 1 \\
\frac{c}{d} &= \frac{1}{k_1} \\
\frac{a}{d} &= \frac{1}{k_2} \\
b &= \sqrt{a^2 + c^2 + d^2 - 2ack_3}
\end{aligned}$$

4. Compute transmission angle values at each design position

$$\mu_i = \text{acos}\left(\frac{b^2 + c^2 - \left(a^2 + d^2\right) + 2ad\cos\phi_i}{2bc}\right)$$

Numerical example
The values of input-output angles are summarized in Table 1

i	ϕ_i	ψ_i
1	30°	195°
2	45°	220°
3	60°	245°

Table 1: Four-bar function generator: Input-output angles.

The set of equations $[A]\{k\} = \{B\}$ is

$$\begin{bmatrix} 0.8480 & -0.9613 & -1.0000 \\ 0.7071 & -0.7660 & -1.0000 \\ 0.5000 & -0.4226 & -1.0000 \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} = \begin{Bmatrix} -0.9659 \\ -0.9962 \\ -0.9962 \end{Bmatrix} \tag{7}$$

Solving the system we have

$$k_1 = 0.7887 \quad k_2 = 0.4756 \quad k_3 = 1.1895$$

Therefore, the proportions of the four-bar are

$$a/d = 2.102450$$
$$b/d = 0.828241$$
$$c/d = 1.267905$$

The values of the transmission angle in the design positions are

$$\mu_1 = 75.81^\circ \quad \mu_2 = 94.19^\circ \quad \mu_3 = 119.19^\circ$$

Program: `FOURB-FG3-1.ch`

7 Function generator slider-crank: 2 and 3 positions

7.1 Algorithm SLID-FG2-1

Purpose: Design a slider-crank for prescribed range of motion of the slider and of the driving crank. (see Figure 12 for nomenclature)

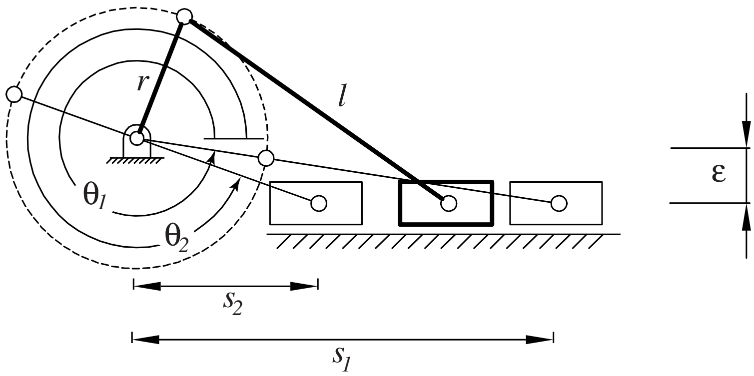

Figure 12: Function generator slider-crank: Nomenclature

Data: $\Delta\theta = \theta_2 - \theta_1 + \pi$, range of rotation of the crank, $\Delta s = s_1 - s_2$, slider displacement between design positions.
Unknowns: ε, slider-crank offset, r, crank length and l, coupler length.
Source:[2]
Computational procedure

1. Assume ε

2. Compute

$$s_1 = \frac{1}{2}\Delta s + \frac{1}{2}\sqrt{\Delta s^2 - 4\,(\varepsilon\ \Delta s\ \cot\Delta\theta + \varepsilon^2)}$$
$$s_2 = s_1 - \Delta s$$

3. Compute

$$\theta_1 = \arctan\left(-\frac{\varepsilon}{s_1}\right)$$
$$\theta_2 = \theta_1 + \Delta\theta + \pi$$

4. Compute

$$r = \frac{1}{2}\left(\frac{s_1}{\cos\theta_1} - \frac{s_2}{\cos\theta_2}\right)$$
$$l = \frac{1}{2}\left(\frac{s_1}{\cos\theta_1} + \frac{s_2}{\cos\theta_2}\right)$$

The choice of ε could be tied to an optimum value of the transmission angle μ.

For this purpose the values of μ and $\mu_{\min}$ can be respectively computed as follows:

$$\mu = \arctan\left[\frac{\varepsilon + r\sin\theta}{\sqrt{l^2 - (\varepsilon + r\sin\theta)^2}}\right] - \frac{\pi}{2}$$
$$\mu_{\min} = \arctan\sqrt{\left(\frac{l}{r+\varepsilon}\right)^2 - 1}$$

Numerical example
Design a slider crank mechanism with $\Delta\theta$ =145 ° and Δs=3.30.
If ε = 0.9 is assumed, then r = 1.501426, l = 2.726228, θ_1 = -12.29°, and θ_2 = 312.71°, $\mu_{\min}$ = 28.25°.
Program: `SLIDFG1.ch`

Graphical synthesis procedures are reported by J. Volmer [17][7] and D.C. Tao [14][8]

7.2 Algorithm SLID-FG2-2

Purpose: Design a slider-crank for prescribed dead-center positions. (see Figure 12 for nomenclature). The formulas are those reported in the previous subsection, but used with a different sequence.

[7]see p.480-483
[8]see p.36-43

Data: θ_1, θ_2, crank angular positions at dead-center configurations, s_1,s_2, slider position at dead-center configurations.
Unknowns: ε, slider-crank offset, r, crank length and l, coupler length, $\mu_{\min}$, minimum value of transmission angle.
Source:[2]
Computational procedure

1. Compute

$$r = \frac{1}{2}\left(\frac{s_1}{\cos\theta_1} - \frac{s_2}{\cos\theta_2}\right)$$
$$l = \frac{1}{2}\left(\frac{s_1}{\cos\theta_1} + \frac{s_2}{\cos\theta_2}\right)$$

2. Compute

$$\varepsilon = -s_1 \tan\theta_1$$

3. Compute

$$\mu_{\min} = \arctan\sqrt{\left(\frac{l}{r+\varepsilon}\right)^2 - 1}$$

Numerical example
Design a slider crank mechanism if the following dead-center positions are prescribed $\theta_1 =$ -16°, $\theta_2 =$ -54°, $s_1 = 4.000000$, $s_2 = 1.000000$. The computed slider-crank dimensions are: $r = 1.229948$, $l = 2.931250$, $\varepsilon = 1.146982$, $\mu_{\min} = 35.81°$.
Program: `SLIDFG2.ch`

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