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Solving a Planar Four-Bar Linkages Design Problem

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Abstract—**The problem of synthesizing a planar four-bar linkages whose curve passes through five prescribed precision points based on elimination methods has been a difficult problem in kinematics because it is of high polynomial degree and highly nonlinear. A new elimination method to solve the problem is presented. Based on the presented method, a 36th degree univariate polynomial of the problem is derived from the determinant of a Sylvester's matrix, which is relatively small in size, without factoring out or deriving the greatest common divisor. Conclusion shows that there are generically 36 non-degenerate solutions of the problem. To verify the obtained conclusion, the completely same result can be obtained using continuation method.**

I. Introduction

Path synthesis of planar four-bar linkages is to determine all lengths of links of the planar four-bar linkages whose curve passes through a given coupler curve or some given discrete points. The problem has always been a longstanding difficult and challenging task because it is f high polynomial degree and highly nonlinear. The solution methods to the problem can be broadly divided into two classes: direct method and indirect method. The direct method is to deduce the parameters of the mechanical linkages according to the kinematics principle. The direct method is the primary method in the early research on path synthesizing and can be divided into graphical methods [1] and elimination method. Planar four-bar linkages designed by the former method can only approximately satisfy the requirements of design, and the latter, although which is highly precise, progresses slowly because it appears to be infeasible.

However, the indirect method is to firstly extract some feature variables or similar coupler curves from the established atlas database, then determine the parameters of the unknown mechanical linkage according to the feature variables. This is the current focus of the research of path synthesis. The key technique of the method is to establish the electronic atlas database and effectively search the matching couple curve. As far as information goes, the main mathematical methods to extract features for the coupler curve are algebraic moment invariants [2], Fourier coefficients of Fourier descriptors [3-5], shape spectrum [6], approximating wavelet coefficients [7]. Rapid development of computer technology ensures the indirect method is a very effective and useful method used in the path synthesis of planar linkage mechanism.

With the development of sophisticated mechanisms, elimination method has made corresponding progress. Wang [8] provides the uniform model for synthesis optimization. Peng [9] accomplishes the problem of synthesizing a four-bar with given the specification of pivots of the crank whose coupler curve passes through five precision points. The problem is similar to the problem of rigid body guide and relatively simple. Wampler [10] shows that the problem of synthesizing a planar four-bar linkages with given pivots whose coupler curve passes through nine prescribed precision points has 1442 non-degenerate solutions after following 4326 solutions using the continuation method. But this method needs to utilize huge computer and cost nearly 332 machine hours of computing. Morgan [11] solves the problem of synthesizing a planar four-bar linkages with given pivots whose coupler curve passes through five prescribed precision points using the continuation method and proves the problem has 36 solutions. But He doesn't provide the numerical samples. Consequently, the problem is still quantitative analysis that obtaining the number of the solutions, rather than a qualitative that can deduce the equation with one variable. What's more, an analytical method to will contribute both in theory and practice to the problem.

In this paper, the problem of synthesizing a planar four-bar linkage whose coupler curve passes through five precision points is considered. A new elimination method to solve the problem is presented. Based on the presented method, a 36th degree univariate polynomial is derived from the determinant of a Sylvester's matrix, which is relatively small in size, without factoring out or deriving the greatest common divisor. Conclusions show that there are generically 36 non-degenerate solutions of the problem. The obtained conclusion is identical with the paper [11]. All computations are performed either symbolically or using rational arithmetic.

The rest of the paper is organized as follows. In Section Ⅱ, the mechanical linkage of the planar four-bar linkage is introduced. In Section Ⅲ, the synthesis problem is modeled as a system of polynomial equations. In Section Ⅳ, the reduction of the problem to a 36th degree univariate

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polynomial is presented, and all 36 non-degenerate solutions of the problem are obtained. In Section Ⅴ, the completely same result is obtained using continuation method. In Section Ⅵ, a numerical example demonstrates the presented method. In Section Ⅶ, conclusions are given.

II. MECHANICAL LINKAGE

Mechanical linkages are collections of rigid bodies connected by joints that allow restricted relative movement. They have now been found a wide variety of applications, such as, the hinge for an automobile hood, an automobile suspension, aircraft landing gear, cranes, and bicycle derailleurs.

As the simplest planar linkage mechanism, planar four-bar linkage with one degree of freedom is shown in Fig. 1.

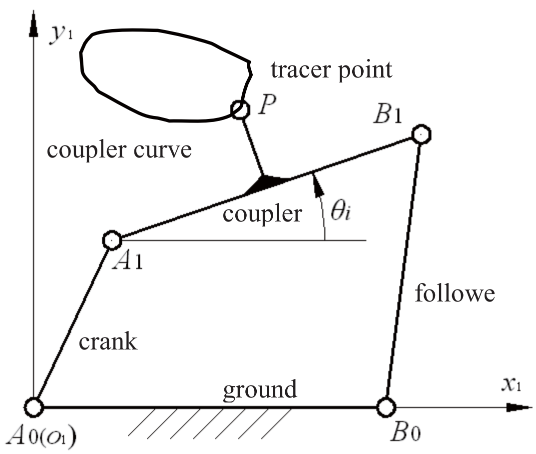

Fig. 1. Planar Four-bar Linkage

One link, called *ground*, serves as the reference frame for describing all relative movement. The other three links are called the *crank*, *coupler*, and *follower*. The crank and follower are attached to the ground link, and rotate around it on revolute joints. The other ends of the crank and follower are constrained to move relative to each other via the coupler link, which is the source of the nonlinearity in the four-bar linkage`s behavior. The *tracer point* is a point rigidly attached to the coupler. The path in space traced by the tracer point is called the *coupler curve*, which is a closed curve.

III. MATHEMATIC MODEL

As shown in the Figure 1, A_0 and B_0 are the given fixed pivots. A_1 and B_1 are revolute pairs. P is a tracer point rigidly attached to the coupler. The problem of synthesizing a planar four-bar linkage with given fixed pivots is to determine all lengths of links of the planar four-bar so that allow the tracer point P passes through five given precision points $P_i(p_{ix}, p_{iy})(i=1, 2, \ldots, 5)$. Without loss of generality, we may set A_0 and P_1 as the origin points of coordinates $O_1x_1y_1$, fixed with the rack and coordinates $O_2x_2y_2$, fixed with the linkage separately.

In the Figure 1, Let the coordinates of A_0 and B_0 in $O_1x_1y_1$ are (a_{0x}, a_{0y}) and(b_{0x}, b_{0y}), the coordinates of A_1 and B_1 in $O_2x_2y_2$ are (a_{1x}, a_{1y}) and (b_{1x}, b_{1y}), the relative displacement matrix of rigid body from P_i to P_1 is $\boldsymbol{D}_{1i}$. With given lengths of linkages, the constraint equations corresponding to the conditions of constraint length of each linkage are as follows

$$(\boldsymbol{D}_{1i}\boldsymbol{A}_1-\boldsymbol{A}_0)^{\mathrm{T}}(\boldsymbol{D}_{1i}\boldsymbol{A}_1-\boldsymbol{A}_0)-(\boldsymbol{A}_1-\boldsymbol{A}_0)^{\mathrm{T}}(\boldsymbol{A}_1-\boldsymbol{A}_0)=0 \tag{1}$$

$$(\boldsymbol{D}_{1i}\boldsymbol{B}_1-\boldsymbol{B}_0)^{\mathrm{T}}(\boldsymbol{D}_{1i}\boldsymbol{B}_1-\boldsymbol{B}_0)-(\boldsymbol{B}_1-\boldsymbol{B}_0)^{\mathrm{T}}(\boldsymbol{B}_1-\boldsymbol{B}_0)=0 \tag{2}$$

where, $\boldsymbol{A}_i=[a_{ix},\ a_{iy},\ 1]^{\mathrm{T}}$, $\boldsymbol{B}_i=[b_{ix},\ b_{iy},\ 1]^{\mathrm{T}}$,$(i$=0,1),

$$\boldsymbol{D}_{1i}=\begin{bmatrix}\cos\theta_{1i} & -\sin\theta_{1i} & p_{ix}-p_{1x}\cos\theta_{1i}+p_{1y}\sin\theta_{1i}\\ \sin\theta_{1i} & \cos\theta_{1i} & p_{iy}-p_{1x}\sin\theta_{1i}-p_{1y}\mathrm{con}\theta_{1i}\\ 0 & 0 & 1\end{bmatrix} \tag{3}$$

$\theta_{1i}=\theta_i-\theta_1$, $i=2,3,4,5$, denotes the relative angle of the rigid body from P_i to P_1.

Substituting (3) and coordinates of points into (1), (2) and expanding, we obtain

$$c_i^{\ 2}+s_i^{\ 2}=1 \qquad i=2,3,4,5 \tag{4}$$

$$\begin{aligned}&2a_{1x}a_{0x}+2a_{1y}a_{0y}+2c_ih_{aci}-2a_{1x}p_{1x}+p_{1x}^{\ 2}\\&-2a_{1y}p_{1y}+p_{1y}^{\ 2}-2a_{0x}p_{ix}+p_{ix}^{\ 2}-2a_{oy}p_{iy}\\&+p_{iy}^{\ 2}+2h_{asi}s_i=0 \qquad i=2,3,4,5\end{aligned} \tag{5}$$

$$\begin{aligned}&2b_{1x}b_{0x}+2b_{1y}b_{0y}+2c_ih_{bci}-2b_{1x}p_{1x}+p_{1x}^{\ 2}\\&-2b_{1y}p_{1y}+p_{1y}^{\ 2}-2b_{0x}p_{ix}+p_{ix}^{\ 2}-2b_{oy}p_{iy}\\&+p_{iy}^{\ 2}+2h_{bsi}s_i=0 \qquad i=2,3,4,5\end{aligned} \tag{6}$$

In which, $c_i=\cos\theta_{1i}$, $s_i=\sin\theta_{1i}$,

$$\begin{aligned}h_{aci}=&-a_{1x}a_{0x}-a_{1y}a_{0y}+a_{0x}p_{1x}+a_{0y}p_{1y}\\&+a_{1x}p_{ix}-p_{1x}p_{ix}+a_{1y}p_{iy}-p_{1y}p_{iy}\end{aligned} \tag{7}$$

$$\begin{aligned}h_{asi}=&\,a_{1y}a_{0x}-a_{1x}a_{0y}+a_{0y}p_{1x}-a_{0x}p_{1y}\\&-a_{1y}p_{ix}+p_{1y}p_{ix}+a_{1x}p_{iy}-p_{1x}p_{iy}\end{aligned} \tag{8}$$

$$\begin{aligned}h_{bci}=&-b_{1x}b_{0x}-b_{1y}b_{0y}+b_{0x}p_{1x}+b_{0y}p_{1y}\\&+b_{1x}p_{ix}-p_{1x}p_{ix}+b_{1y}p_{iy}-p_{1y}p_{iy}\end{aligned} \tag{9}$$

$$\begin{aligned}h_{bsi}=&\,b_{1y}b_{0x}-b_{1x}b_{0y}+b_{0y}p_{1x}-b_{0x}p_{1y}\\&-b_{1y}p_{ix}+p_{1y}p_{ix}+b_{1x}p_{iy}-p_{1x}p_{iy}\end{aligned} \tag{10}$$

Equations (4)~(6) are consider as the mathematic model of the problem. There are totally 12 variants to be solved are a_{1x}, a_{1y}, b_{1x}, b_{1y}, c_i, s_i in these 12 equations and the remainders are the structural parameters of the planar four-bar linkage.

IV. THE ELIMINATION PROCESS

Equations (4)~(6) form a polynomial system of 12 equations in the 12 unknowns. Since all the equations are second-degree, the total degree (product of the degrees of the polynomials) of the mathematic model equals 2^{12} = 4 096. It is currently difficult and formidable to obtain the solutions using the traditional elimination methods such as Dixon resultant method, Sylvester resultant method and Wu's method. Instead, we will use two-step elimination method to solve the problem. Firstly, we reduce the Groebner basis under lexicographic ordering of equations (4) ~ (6). Then, we analyze the variants of the basis and the terms of the different variants degrees, and then construct Sylvester resultant by hiding the proper variants such that the number of polynomials is equal to the number of monomials. At the same time, to reduce the time of symbol computing, we utilize the related principles in Linear Algebra to analysis the resultant， and minimize the degrees of the resultant which can also deduce a 36$^{\text{th}}$ univariate equation.

A. The Groebner Basis under Lexicographic Ordering

Groebner basis under lexicographic term ordering can be described as following. If $\boldsymbol{F}$ is a system of polynomial equations, the Groebner basis method reduces the problem of solving $\boldsymbol{F}$ to manipulating the monomials in $\boldsymbol{F}$. This transforms $\boldsymbol{F}$ into its Groebner basis representation $\boldsymbol{G}$ that generates the same ideal as $\boldsymbol{F}$ but is easier to solve. In brief, the Groebner basis for ideal $\boldsymbol{I}$ generated by $\boldsymbol{F}$□$k[x_1,\cdots,x_n]$ over field k, is a set of generators $\boldsymbol{G}$□$k[x_1,\cdots,x_n]$ that generate the same ideal $\boldsymbol{I}$, but are simpler inform and easier to solve. Groebner basis computations involve manipulation the monomials in $\boldsymbol{F}$ under term ordering, computing S-polynomials and their normal form reduction. The paper [12] contains tutorial material on the mathematical techniques. The Groebner package available in Methematica-5.2 will be utilized to yield a reduced Groebner basis of (4)~(10) under the degree lexicographic term ordering with $c_2 > s_2 > c_3 > s_3 > c_4 > s_4 > c_5 > s_5 > a_{1x} > a_{1y} > b_{1x} > b_{1y}$. We obtain a reduced Groebner basis with 89 polynomials in 125 unknown monomials, which are not reported herein doe to space limitations.

Suppressing the unknown a_{1y}, the reduced Groebner basis under degree lexicographic ordering can be viewed as a linear system of 89 equations in 89 unknown monomials and be expressed in the form of matrix

$$\boldsymbol{M}'_{89\times 89}\boldsymbol{t} = 0 \qquad (11)$$

Where, $\boldsymbol{M}'_{89\times 89}$ is the 89×89 coefficient matrix, $\boldsymbol{t}$ is the 89×1 column matrix of the unknown variables (the 89 monomials).

According to the Algebra, (4)~(6) has the solutions under the condition that the determinant of the matrix equate zero, i.e.

$$\det(\boldsymbol{M}'_{89\times 89}) = 0 \qquad (12)$$

Solving (12) will cost a large quantity of time and space of computer. So it is necessary to reduce the degree of the coefficient matrix to improve the efficiency.

B. Reducing the Degree of Resultant

For there is only one element in some columns of the matrix $\boldsymbol{M}'_{89\times 89}$ is the constant (e.g. j^{th} column) however others are zero, i.e. only the i^{th} equation has relation with j^{th} unknown variable, consequently, removing the i^{th} equation and decreasing an unknown variable at the same time will not affect solving the equations.

There remain 23 polynomial equations after similar removing above, given as g_i for i= 1, 2, 4, 7, 8, 9, 10, 14, 17, 20, 23, 27, 30, 35, 38, 44, 47, 54, 57, 65, 68, 77, 80. With the unknown a_{1y}, the 23 polynomial equations can be viewed as a linear system of 23 polynomials in 23 unknowns 1，a_{1x}，b_{1x}，b_{1y}，$a_{1x}b_{1y}$，$b_{1x}b_{1y}$，b_{1y}^2，c_2，$b_{1y}c_2$，c_3，$b_{1y}c_3$，c_4，$b_{1y}c_4$，c_5，$b_{1y}c_5$，s_2，$b_{1y}s_2$，s_3，$b_{1y}s_3$，s_4，$b_{1y}s_4$，s_5，and $b_{1y}s_5$, with their coefficients expressed in terms of a_{1y}.

$$\boldsymbol{M}_{23\times 23}\boldsymbol{T} = 0 \qquad (13)$$

Where, $\boldsymbol{M}_{23\times 23}$ is the 23×23 coefficient matrix of a_{1y}, and $\boldsymbol{T}$ is the 23×1 column matrix in 23 unknown variables (the 23 monomials).

$\boldsymbol{T}$ = [1，a_{1x}，b_{1x}，b_{1y}，$a_{1x}b_{1y}$，$b_{1x}b_{1y}$，b_{1y}^2，c_2，$b_{1y}c_2$，c_3，$b_{1y}c_3$，c_4，$b_{1y}c_4$，c_5，$b_{1y}c_5$，s_2，$b_{1y}s_2$，s_3，$b_{1y}s_3$，s_4，$b_{1y}s_4$，s_5，$b_{1y}s_5$]$^{\text{T}}$

C. Derrving A Univariate Ploynomial

According to Algebra, the condition of the (13) having solutions is the determinant of the coefficient is zero, i.e.

$$\det(\boldsymbol{M}_{23\times 23}) = 0 \qquad (14)$$

Limited by the computer memory, it is impossible to expand the determent of matrix $\boldsymbol{M}_{23\times 23}$ symbolically. Numerical computation must be carried out. It is important to determine the degrees of the equation to obtain the number of solutions exactly. The maximum degree of the unknown variant of each column of matrix $\boldsymbol{M}_{23\times 23}$ are: 3, 2, 2, 2, 1, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1，with sum 36. Based on the calculation principle of the determent, the highest degree of the polynomials with only variant a_{1y} are no more than 36 after expanding the determinant.

Expanding directly the equation (14), we can obtain the following 36$^{\text{th}}$ degree univariate polynomial without factoring out or deriving any common divisor.

$$\sum_{i=0}^{36} s_i a_{1y}{}^i = 0 \qquad (15)$$

where, s_i (i=0,⋯,36) are real constants depending on input parameters only. 36 complex solutions will be obtained by solving equation (15).

D. Back substitution for other unknowns

Solving the linear system, which is obtained by removing any one row from the matrix $\boldsymbol{M}_{23\times 23}$ in (14) with a_{1y} replaced by a_{1yi} (i = 1, …, 36), solutions of a_{1x}, a_{1y}, b_{1x}, b_{1y}, c_i, and s_i can be easily computed in the complex domain. For one solution of a_{1y} there will be one solution of a_{1x}, a_{1y}, b_{1x}, b_{1y}, c_i, and s_i.

V. CONTINUATION METHOD

In the last decade, homotopy continuation method developed into a convenient, reliable tool for solving nonlinear polynomial systems. It is a numerical process which finds all isolated roots of a polynomial system. Starting at the roots of a suitable start system, whose roots are known, the method tracks the solution paths as the start system is continuously transformed into the target system. When the start system and the transformation procedure, called a homotopy, are chosen suitably, the endpoints of these solution paths are guaranteed to include all isolated solutions of the target system. Some publicly available software for polynomial continuation is available [13], [14]. In this section, we will solve the mathematic model of the problem of synthesizing a planar four-bar linkage by using homotopy continuation method.

Combining (4)~(6), we get the nonlinear algebraic system. On the ordinary personal computer of Intel Pentium III 2.93 GHz and RAM 256 M, we solve the nonlinear algebraic system by using PHCpack [14] which is a kind of Homotopy continuation method software exploited by Illinois University. We find that the result shows that totally 36 roots, which are same as the former results by using the presented algebraic method completely.

VI. Numerical Sample

The parameters of planar four-bar linkage are $a_{0x}= a_{0y}=0$, $b_{0x}=6$, $b_{0y}=0$, $p_{1x}=5$, $p_{1y}=6$, $p_{2x}=4$, $p_{2y}=7$, $p_{3x}=3$, $p_{3y}=5$, $p_{4x}=2$, $p_{4y}=3$, $p_{5x}=1$, $p_{5y}=2$. Following the steps above, the 36 non-degenerate solutions, including 10 real solutions, shown in table Ⅰ.

VII. Conclusion

A planar four-bar linkage design problem has been described, and a new elimination method to obtain complete closed-form solution has been given. Base on the presented method, we firstly derive the 36th degree univariate polynomial without factoring out or deriving the greatest common divisor. Simultaneously, the same result can be obtained with continuation method, which illustrating that the problem has 36 groups of solution in complex field exactly. The success in solving in this problem using the proposed method sheds the light on solving other similarly difficult mechanism synthesis problems.

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TABLE I
SOLUTIONS OF THE NUMERICAL SAMPLE

	a_{1x}	a_{1y}	b_{1x}	b_{1y}	c_2	s_2	c_3	s_3	c_4	s_4	c_5	s_5
1	4.1067	0.5418	22.0475	−0.4761	0.9903	0.1387	0.9336	0.3582	0.2943	−0.9557	0.5136	−0.858
2	4.7020	0.2907	8.3907	−0.6614	0.9896	0.1438	0.9387	0.3447	0.9095	0.4156	0.8735	0.4868
3	7.5888	−0.2655	8.8589	−0.7818	0.9887	0.15	0.9385	0.3454	0.8449	0.5349	0.5912	0.8065
4	4.2425	0.1647	12.5309	−4.5203	0.6014	−0.799	0.9459	0.3243	0.9428	0.3334	0.9628	0.2701
5	7.3704	0.0042	−16.0409	0.4475	0.989	0.1481	0.9333	0.3592	−0.8187	−0.5742	0.5378	0.8431
6	3.1257	2.188	8.2720	2.3869	0.997	−0.0772	0.6996	−0.7145	0.6437	0.7653	0.4527	0.8917
7	5.2669	1.6961	6.6653	5.5869	0.2813	−0.9596	0.8697	0.4935	−0.5798	−0.8148	0.1523	0.9883
8	4.3849	1.4406	13.9317	0.3020	0.9926	0.1213	0.2732	−0.962	0.7526	0.6584	0.5381	0.8429
9	3.0612	1.2296	7.4744	3.8540	0.9309	−0.3654	0.9032	0.4291	0.6277	−0.7785	0.8588	−0.5123
10	8.5476	−0.9057	−8.5379	11.1973	−0.5402	−0.8415	0.9465	0.3227	0.8683	0.4961	0.6347	0.7727
11	1.5678−2.6074*I	3.4352+2.819*I	6.0982−2.3443*I	3.4959−1.5719*I	1.859−6.4921*I	6.5631+1.8389*I	0.8307−5.1643*I	5.2579+0.8159*I	−0.3199−4.3793*I	4.4915−0.3119*I	−1.9840−4.3574*I	4.4521−1.9418*I
12	1.5678+2.6074*I	3.4352−2.819*I	6.0982+2.3443*I	3.4959+1.5719*I	1.859+6.4921*I	6.5631−1.8389*I	0.8307+5.1643*I	5.2579−0.8159*I	−0.3199+4.3793*I	4.4915+0.3119*I	−1.9840+4.3574*I	4.4521+1.9418*I
13	2.9448−2.2127*I	3.6766+1.6353*I	5.0878−0.4333*I	5.3753−0.2067*I	5.0635−4.9815*I	5.0311+5.0135*I	1.2209−0.0809*I	0.1392+0.7094*I	1.1466+1.8473*I	−2.0428+1.0369*I	−1.0748−0.1526*I	0.3329−0.4926*I

14	2.9448+2.2127*I	3.6766−1.6353*I	5.0878+0.4333*I	5.3753+0.2067*I	5.0635+4.9815*I	5.0311−5.0135*I	1.2209+0.0809*I	0.1392−0.7094*I	1.1466−1.8473*I	−2.0428−1.0369*I	−1.0748+0.1526*I	0.3329+0.4926*I
15	6.1736−1.5521*I	3.7548+0.8552*I	5.4004−0.2680*I	6.4228−0.0287*I	−0.9427+0.4053*I	−0.7375−0.5180*I	0.7713−0.2489*I	0.7320+0.2622*I	0.3985−1.8676*I	2.1103+0.3527*I	−1.9684+1.3860*I	1.5142+1.8018*I
16	6.1736+1.5521*I	3.7548−0.8552*I	5.4004+0.2680*I	6.4228+0.0287*I	−0.9427−0.4053*I	−0.7375+0.5180*I	0.7713+0.2489*I	0.7320−0.2622*I	0.3985+1.8676*I	2.1103−0.3527*I	−1.9684−1.3860*I	1.5142−1.8018*I
17	0.7969−2.9296*I	3.4331+1.4748*I	5.6603−2.0625*I	3.4064−3.0814*I	0.4652−1.2543*I	1.5791+0.3696*I	0.3246−0.8955*I	1.3209+0.2200*I	0.3276−0.5507*I	1.10570+0.1632*I	0.3549−0.1633*I	0.9510+0.0609*I
18	0.7969+2.9296*I	3.4331−1.4748*I	5.6603+2.0625*I	3.4064+3.0814*I	0.4652+1.2543*I	1.5791−0.3696*I	0.3246+0.8955*I	1.3209−0.2200*I	0.3276+0.5507*I	1.10570−0.1632*I	0.3549+0.1633*I	0.9510−0.0609*I
19	1.4236−3.2816*I	2.7154+1.5159*I	6.7426−0.8876*I	−0.6243+0.3339*I	0.9856+0.0048*I	0.1716−0.0276*I	1.0084−0.0517*I	0.2133+0.2443*I	0.8541−0.2555*I	0.6657+0.3277*I	0.5903+0.0810*I	0.8134−0.0587*I
20	1.4236+3.2816*I	2.7154−1.5159*I	6.7426+0.8876*I	−0.6243−0.3339*I	0.9856−0.0048*I	0.1716+0.0276*I	1.0084+0.0517*I	0.2133−0.2443*I	0.8541+0.2555*I	0.6657−0.3277*I	0.5903−0.0810*I	0.8134+0.0587*I
21	1.6533−2.142*I	2.6386+1.6382*I	5.2148−0.3448*I	4.5613−0.5810*I	1.7565−0.7670*I	0.8903+1.5131*I	1.0545−0.0221*I	0.0685+0.3407*I	1.1343+0.1058*I	−0.2120+0.5661*I	1.0491−0.1536*I	0.3566+0.4518*I
22	1.6533+2.142*I	2.6386−1.6382*I	5.2148+0.3448*I	4.5613+0.5810*I	1.7565+0.7670*I	0.8903−1.5131*I	1.0545+0.0221*I	0.0685−0.3407*I	1.1343−0.1058*I	−0.2120−0.5661*I	1.0491+0.1536*I	0.3566−0.4518*I
23	2.5412−2.0311*I	2.5796+1.5116*I	5.9732−1.4572*I	−0.2307+0.315*I	0.9867+0.0074*I	0.1684−0.0432*I	1.0305−0.1721*I	0.4024+0.4407*I	0.8075−0.4139*I	0.8264+0.4045*I	0.5649−0.4492*I	0.9749+0.2603*I
24	2.5412+2.0311*I	2.5796−1.5116*I	5.9732+1.4572*I	−0.2307−0.315*I	0.9867−0.0074*I	0.1684+0.0432*I	1.0305+0.1721*I	0.4024−0.4407*I	0.8075+0.4139*I	0.8264−0.4045*I	0.5649+0.4492*I	0.9749−0.2603*I
25	2.6461−0.8375*I	2.8213+0.6514*I	4.8601−0.7619*I	0.1284−0.3865*I	0.9919+0.0154*I	0.1596−0.0958*I	0.7389−0.1734*I	0.7182+0.1784*I	0.4698−0.2226*I	0.9175+0.114*I	0.1715−0.2366*I	1.0140+0.0400*I
26	2.6461+0.8375*I	2.8213−0.6514*I	4.8601+0.7619*I	0.1284+0.3865*I	0.9919−0.0154*I	0.1596+0.0958*I	0.7389+0.1734*I	0.7182−0.1784*I	0.4698+0.2226*I	0.9175−0.114*I	0.1715+0.2366*I	1.0140−0.0400*I
27	2.3025−0.6337*I	2.523+0.3460*I	4.9577−0.4260*I	0.0271−0.4575*I	0.9940+0.0177*I	0.1573−0.1117*I	0.7530−0.0842*I	0.6701+0.0946*I	0.6044−0.0753*I	0.8022+0.0567*I	0.4875−0.0337*I	0.8740+0.0188*I
28	2.3025+0.6337*I	2.523−0.3460*I	4.9577+0.4260*I	0.0271+0.4575*I	0.9940−0.0177*I	0.1573+0.1117*I	0.7530+0.0842*I	0.6701−0.0946*I	0.6044+0.0753*I	0.8022−0.0567*I	0.4875+0.0337*I	0.8740−0.0188*I
29	2.0086−0.5425*I	1.9985+0.4596*I	8.4019−1.6860*I	0.1576+1.5684*I	0.9923+0.0144*I	0.1550−0.0920*I	0.9369+0.0594*I	−0.3833+0.1453*I	0.8167−0.0789*I	0.5925+0.1088*I	0.9575−0.0332*I	0.3082+0.1030*I
30	2.0086+0.5425*I	1.9985−0.4596*I	8.4019+1.6862*I	0.1576−1.5684*I	0.9923−0.0144*I	0.1550+0.0920*I	0.9369−0.0594*I	−0.3833−0.1453*I	0.8167+0.0789*I	0.5925−0.1088*I	0.9575+0.0332*I	0.3082−0.1030*I
31	−0.9849+0.5949*I	1.9802+0.2347*I	2.9988−0.5233*I	7.7556+2.3543*I	0.7434+0.0329*I	0.6706−0.0364*I	0.9901+0.0161*I	0.16970−0.0942*I	1.1346+0.1739*I	0.3271−0.6034*I	1.3128−0.7328*I	0.9345+1.0294*I
32	−0.9849−0.5949*I	1.9802−0.2347*I	2.9988+0.5233*I	7.7556−2.3543*I	0.7434−0.0329*I	0.6706+0.0364*I	0.9901−0.0161*I	0.1697+0.0942*I	1.1346−0.1739*I	0.3271+0.6034*I	1.3128+0.7328*I	0.9345−1.0294*I
33	0.6167−1.0437*I	0.8536+0.0118*I	5.2104+2.1443*I	3.5133−1.0945*I	1.0094−0.0659*I	0.2442+0.2724*I	0.9481+0.0256*I	0.3275−0.0740*I	1.0981+0.1425*I	0.2986−0.5241*I	1.4236+0.4360*I	0.5747−1.0801*I
34	0.6167+1.0437*I	0.8536−0.0118*I	5.2104−2.1443*I	3.5133+1.0945*I	1.0094+0.0659*I	0.2442−0.2724*I	0.9481−0.0256*I	0.3275+0.0740*I	1.0981−0.1425*I	0.2986+0.5241*I	1.4236−0.4360*I	0.5747+1.0801*I
35	12.7770−0.7936*I	−0.7800+0.0057*I	3.0212−1.9856*I	−0.4094+0.1394*I	0.9884	0.1516+0.0001*I	0.9314+0.0018*I	0.3641−0.0046*I	0.7192+0.0266*I	0.6959−0.0275*I	−0.4940−0.5912*I	1.0853−0.2691*I
36	12.7770+0.7936*I	−0.7800−0.0057*I	3.0212+1.9856*I	−0.4094−0.1394*I	0.9884	0.1516−0.0001*I	0.9314−0.0018*I	0.3641+0.0046*I	0.7192−0.0266*I	0.6959+0.0275*I	−0.4940+0.5912*I	1.0853+0.2691*I