# Position and Displacement Analysis

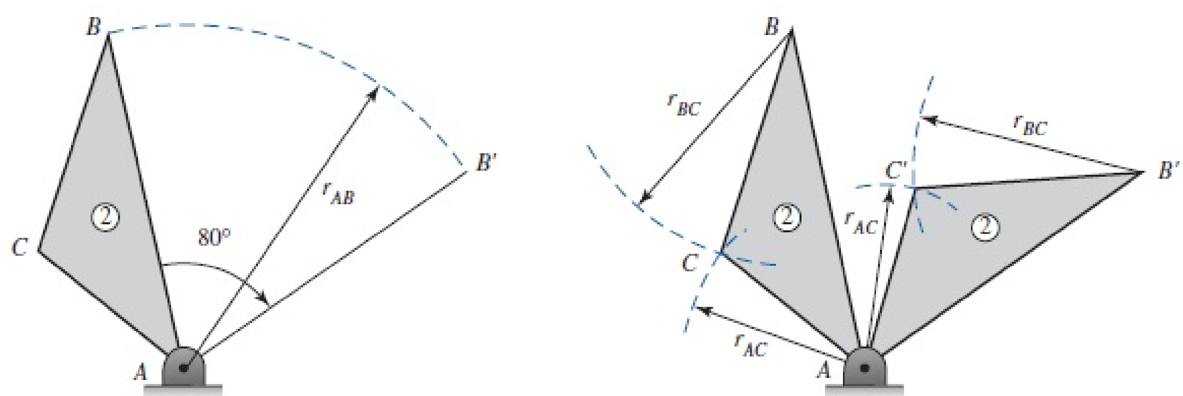
## Introduction:

In this chapter we introduce the tools to identifying the position of the different points and links in a given mechanism. Recall that for linkages with one degree of freedom, the position of one link or point can precisely determine the position of all other links or points. Likewise, for linkages with two degrees of freedom, the position of two links can precisely determine the position of all other links.

## Displacement Analysis

### Displacement of a single link

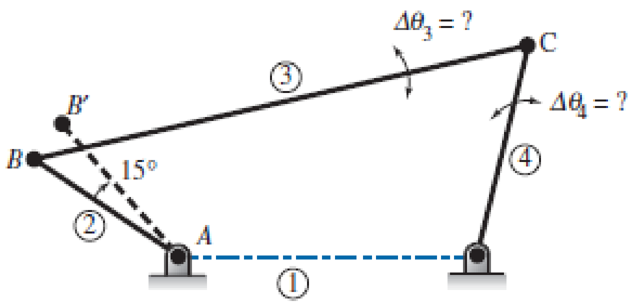
The displacement of one or more links (if M > 1), will identify the displacement of all other links (linear and/or angular). Whether this is a simple or complex link, the displacement of the link can be fully determined given the end positions of few points on the link. For instance, consider the kinematic diagram of a complex link as depicted in the figure below, the link is pin connected at A (full joint) and is rotated with an angular displacement of $80^o$ clockwise.  Point B is constrained to move along the circular arc of radium $r_{AB}$ to its end location B'. Because of the rigid character of the link, point C is constrained to move over the circular arc centered at A of radius $r_{AC}$ (constrained path), additionally for an observer moving with B, the displacement of point C is a circle of radius $r_{BC}$ (from B' the feasible path of C' is a circle centered at B' of radius $r_{BC}$), this is because the length $r_{BC}$ is unchanged. Given the constrained and feasible paths, the end location of point C (: C') is the point where these two paths meet.




### Displacement of combination of links

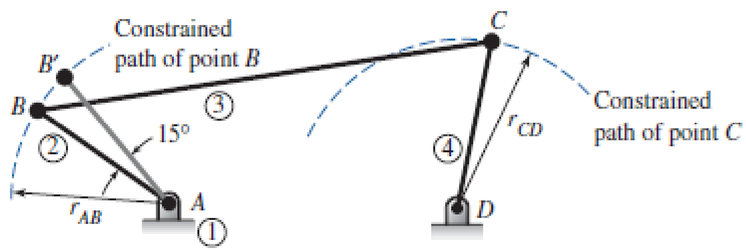
The procedure described in the previous section can be extended to a linkage of two or more links. Consider the linkage in the figure below. This linkage is made of four links (including the frame) and four full joints (pin joints). The mobility of the linkage is

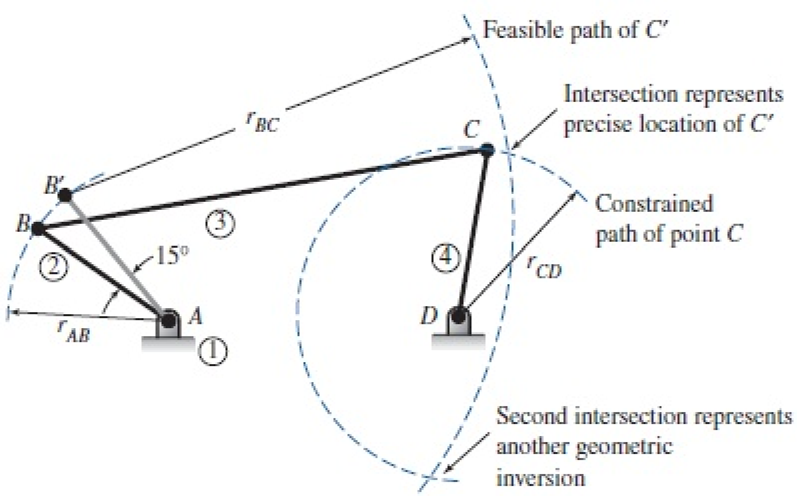
$$M = 3(4 - 1) - 4(3 - 1) = 1$$



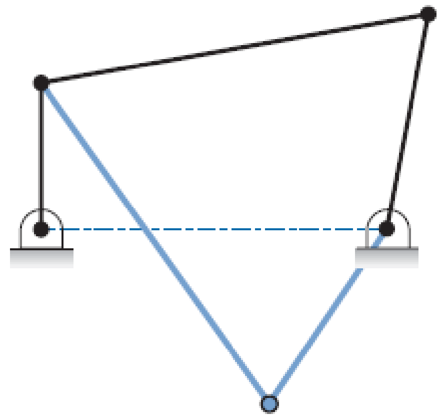

Link 2 is rotated clockwise with an angle of 15° about joint A. during this displacement; point B followed a circular path to B′. We would like to find the new location of point C as well as $\Delta\theta_3$ and $\Delta\theta_4$ (the angular displacements of the coupler and follower respectively).

Because all members of the linkage are rigid, the length of each link is unchanged and the relative displacement of two points on the link is a pure rotation about an axis perpendicular to the plane. Accordingly the new location of point C (: C′) will simultaneously be on the arc of circle centered at B′ with radius equal to the length of member BC (feasible path), and the arc of a circle centered at D with radius equal to the length of member CD (constrained path). Once C′ has been located, the position of links 3 and 4 can be drawn.



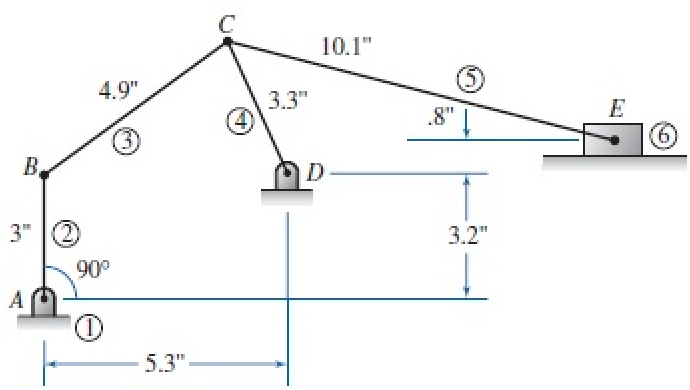
Notice that the constrained and feasible paths intersect at more than one location. The second intersection point corresponds to another possible configuration of the mechanism (a **geometric inversion**). In the current case, it should be discarded since that would require that the mechanism be disassembled and reassembled from its current initial position.



- The final configuration along with the angular changes $\Delta\theta_3$ and $\Delta\theta_4$ can be identified graphically or determined analytically using the rules of trigonometry.

**Example:**

Consider the following mechanism

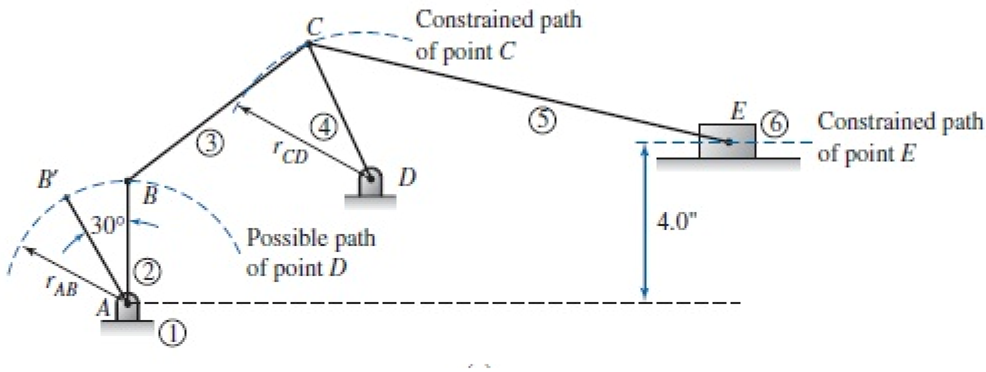



The mobility of this mechanism is

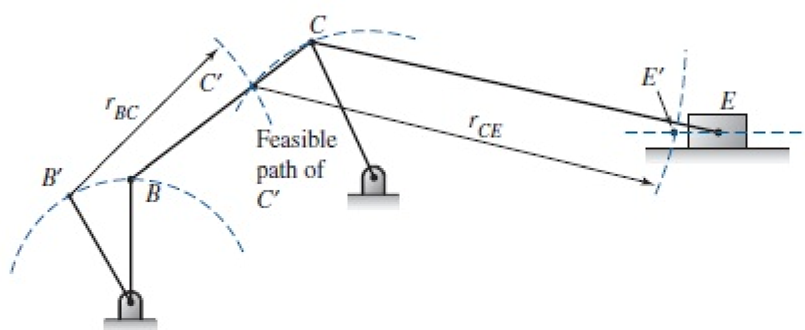
$$M = 3(5-1) - 5(3-1) - (3-2) = 1$$

Reposition the links of the mechanism as link 2 is displaced counterclockwise with an angle of 30$^{o}$ from its current position.

- The constrained paths of C and E are shown in the figure below




- The feasible paths of all of C′ and E′ are shown in the figure below




- The intersection of the constrained and feasible paths are the updated locations of C′, and E′
- **Graphically,** once the kinematic diagram is drawn to scale, the user can identify the new configuration with direct measurement using a ruler, protractor and a compass.
- The final configuration can also be determined **analytically**, this however may involve drawing convenient lines through the mechanism breaking it into triangles, the laws of general and right angle triangles are then employed to determine the length of the triangle sides and the magnitude of the interior angles. We will come back to this example later after establishing the necessary relations.

## Position Analysis: Analytical Method

As was previously stated, the final positions of the links can be identified analytically using the rules of trigonometry, in this section we emphasize the direction that is generally used using an series of applications.

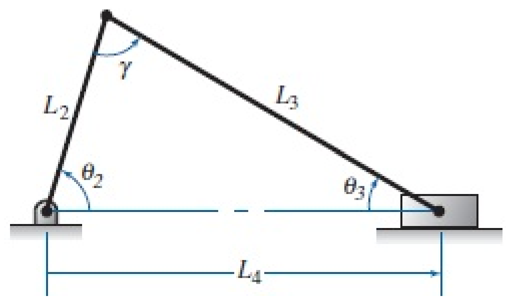
### In-line slider-crank mechanism

Consider the slider-crank mechanism shown in the figure below. It is termed an in-line slider-crank because the constrained path of the pin on the slider extends through the center of the crank rotation. Given the lengths L2 and L3 and the crank angle $\theta_2$ , the position of the slider and the interior joint angle can be determined from

$$\theta_3 = \sin^{-1}\left(\frac{L_2}{L_3} \sin\theta_2\right)$$

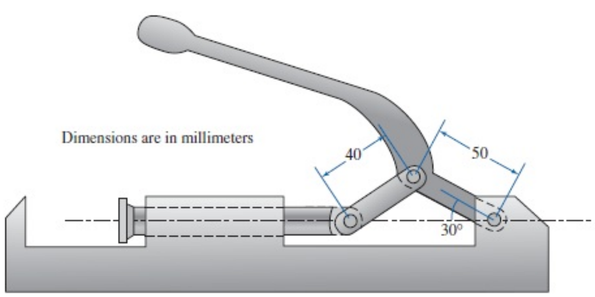
$$\gamma = 180 - (\theta_2 + \theta_3)$$

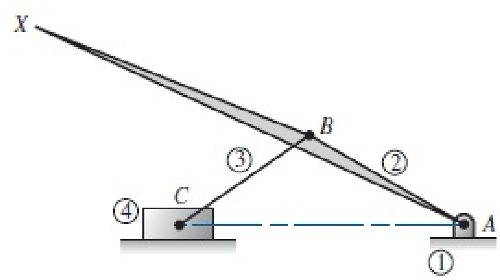
$$L_4 = \sqrt{L_2^2 + L_3^2 - 2\, L_2\, L_3\, \cos\gamma}$$




**Example**

Consider the toggle clamp shown in the figure below, the kinematic diagram of the mechanism is also included. Member AB is displaced counterclockwise about Joint A with an angular displacement of 15°. Determine the linear displacement of the clamp surface?



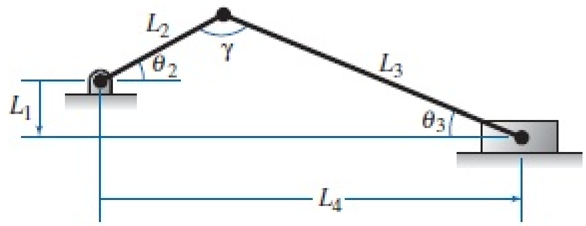

## Offset slider crank mechanism

In the offset slider crank mechanism shown in the figure below the constrained path of the pin on the slider does not extend through the center of rotation of the crank. Given the lengths L1, L2 and L3 and the crank angle $\theta_2$ , the position of the slider and the interior joint angle can be determined from

$$\theta_3 = \sin^{-1}\left(\frac{L_1 + L_2\, \sin\theta_2}{L_3}\right)$$

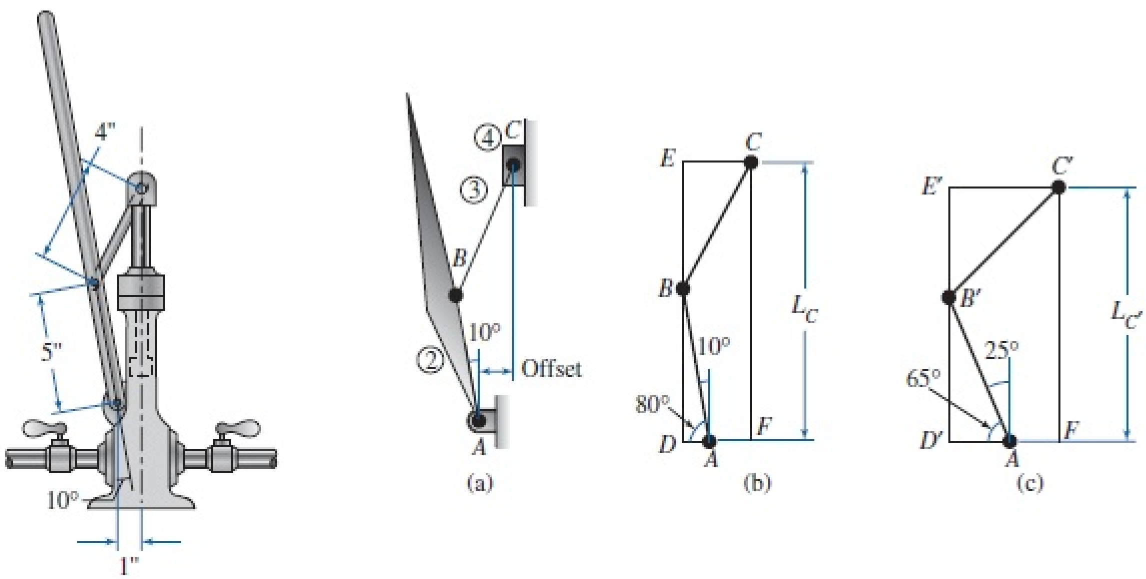
$$\gamma = 180 - (\theta_2 + \theta_3)$$

$$L_4 = L_2\, \cos\theta_2 + L_3\, \cos\theta_3$$




**Example:**

The figure below shows a concept for a hand pump used for increasing oil pressure in a hydraulic line. Analytically determine the displacement of the piston as the handle rotates 15° counterclockwise.




## Four-bar linkage

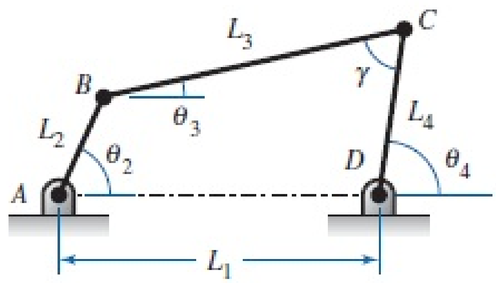
Consider the four-bar mechanism shown in the figure below. Given the lengths L1, L2, L3 and L4 and the crank angle $\theta_2$ , the position of the slider and the interior joint angles can be determined from the following equations

$$BD = \sqrt{L_1^2 + L_2^2 - 2\, L_1\, L_2\, \cos\theta_2}$$

$$\gamma = \cos^{-1}\left[\frac{L_3^2 + L_4^2 - (BD)^2}{2\, L_3 L_4}\right]$$

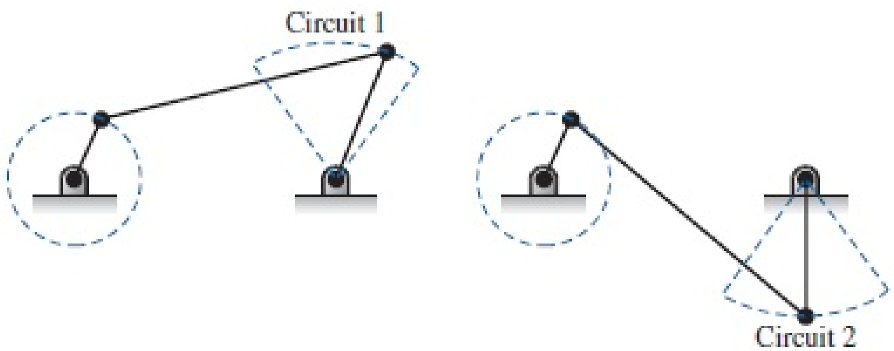
$$\theta_3 = 2\, \tan^{-1}\left[\frac{-L_2\, \sin\theta_2 + L_4 \sin\gamma}{L_1 + L_3 - L_2\, \cos\theta_2 - L_4\, \cos\gamma}\right]$$

$$\theta_4 = 2\, \tan^{-1}\left[\frac{L_2\, \sin\theta_2 - L_3 \sin\gamma}{L_4 - L_1 + L_2\, \cos\theta_2 - L_3\, \cos\gamma}\right]$$




**Note:**

The equations presented above are applicable to a Four-Bar mechanism assembled as depicted above. The figure below shows another possible configuration of the four-bar mechanism (circuit.2), a geometrical inversion of the first configuration realizable by dismantling the mechanism at joint C, inverting member DC then reassembling.



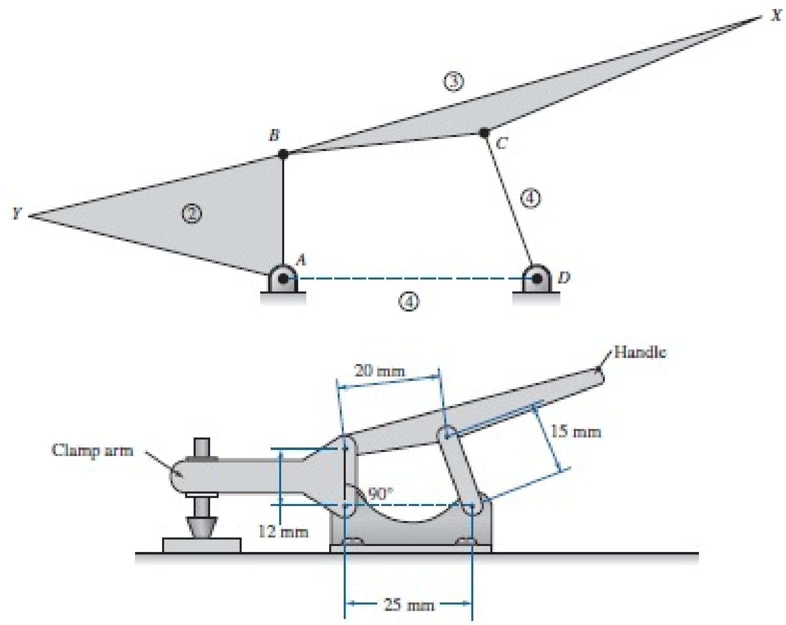

For this inversion the first two equations from the first configuration are still applicable, last two equations however must slightly be modified according to

$$\theta_3 = 2\ \tan^{-1}\left[\frac{-L_2\ \sin\theta_2 - L_4 \sin\gamma}{L_1 + L_3 - L_2\ \cos\theta_2 - L_4\ \cos\gamma}\right]$$

$$\theta_4 = 2\ \tan^{-1}\left[\frac{L_2\ \sin\theta_2 + L_3 \sin\gamma}{L_4 - L_1 + L_2\ \cos\theta_2 - L_3\ \cos\gamma}\right]$$

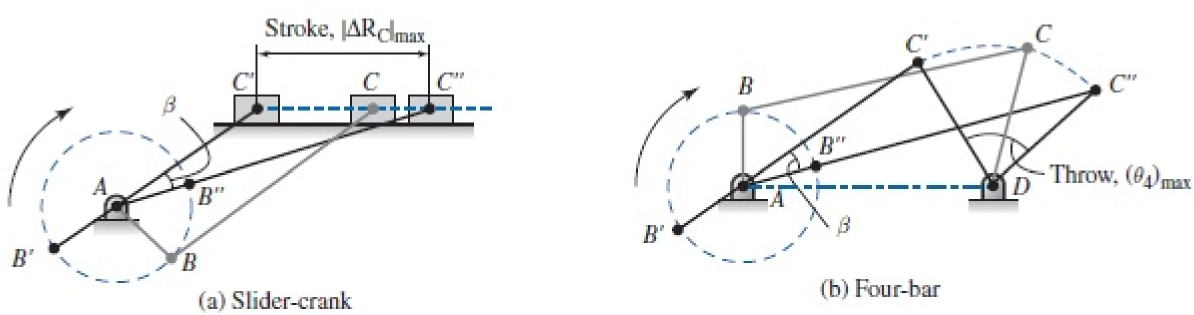
**Example:**

The figure below shows a toggle clamp used for securing a workpiece during a machining operation. Analytically determine the angle that the handle must be displaced in order to lift the clamp arm 30° clockwise.




## Limiting position in a mechanism

In some mechanisms, the follower is set to move between two limiting positions, such as in the case of a slider crank or a four-bar rocker mechanism as shown in the figure below where the extreme positions occur when the crank and coupler are perfectly aligned. The term stoke is used to measure the position of the follower between these extreme positions (linear displacement in the case of the slider crank and an angular displacement in the case of the four-bar rocker mechanism). The preceding analysis can be used to identify the extreme locations along with the imbalance angle ($\beta$) defined as the angle between the coupler configurations at the two limiting positions.



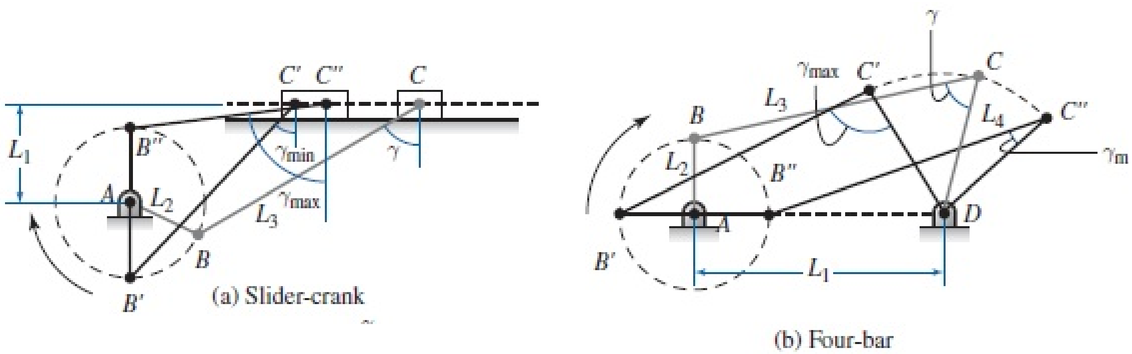

**Notes:**

1. One of the advantages of using an offset in the slider-crank mechanism is the ability to control time ratio of the forward and return stroke. For these mechanisms the motor rotates with a constant angular velocity, so in the case of an in-line slider crank mechanism, the angular

displacement of the crank in the forward stroke is identical to that of the backward stroke and the time ratio is 1. With the offset slider crank the angular displacement of the crank is different between the forward and backward motion of the slider and consequently the time ratio is different than 1. For an estimate of the time ratio we use the following equation,

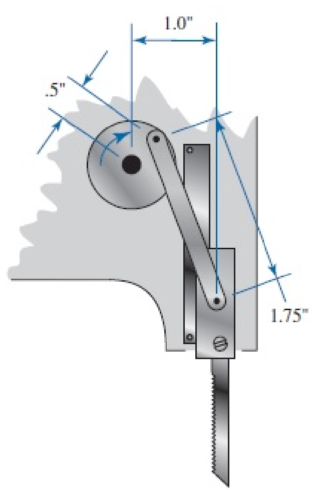
$$Time\ ratio(Q) = \frac{Time\ of\ slower\ stroke}{Time\ of\ faster\ strole} \approx \frac{\pi + \beta}{\pi - \beta}$$

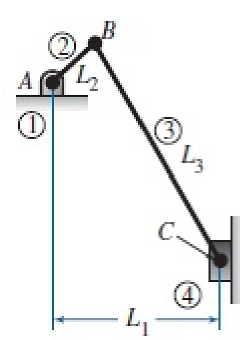
2. In mechanisms such as the slider-crank and four-bar mechanisms the transmission angle $(\gamma)$quantifies the force transmission through the linkage. For the four-bar mechanism $(\gamma)$ is the angle between the coupler and the follower links, for the slider-crank mechanism it is the angle between the coupler and the line normal to the sliding direction as indicated for the limiting positions in the figure below. Ideally we want this angle to be 90$^o$ since then the action of the two force member (coupler) will be fully transmitted to the follower. An angle less than 90$^o$ will affect the mechanical efficiency of the mechanism, a rule of thumb is to keep this angle larger than 45o at all times.




**Example**

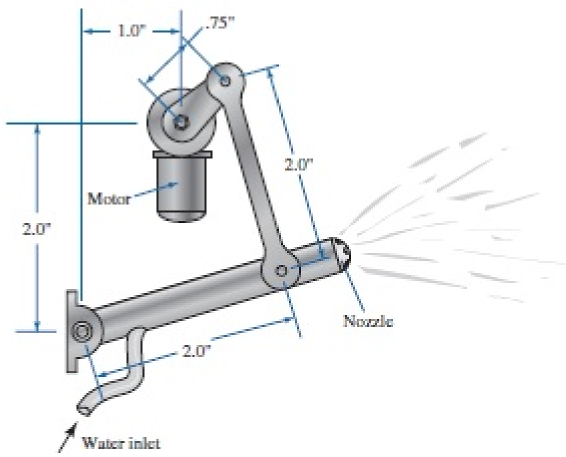
The mechanism shown in the figure below is the driving linkage for a reciprocating saber saw. Determine analytically the value of the stroke between the limiting positions of the saw blade.

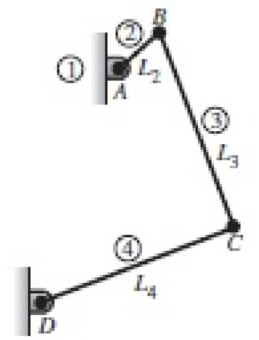


**Example**

The figure below illustrates a linkage that operates a water nozzle at an automatic car wash. Determine the value of the angular stroke between the limiting positions of the nozzle.




**Note:**

The configuration of a mechanism upon incremental changes in the input motion is called a phase, in the analysis of a mechanism it may be required that you identify the phase of the mechanism at different increments of the input motion, even sometimes tracing the path of one or more points, the methodology so far introduced, can be applied to produce these results.

**Example**

The mechanism shown in the figure below is designed to push parts from one conveyor to another. During the transfer, the parts must be rotated as shown. Analytically determine the position of the pusher rod at several phases of its motion.