Chasles' theorem:

Any displacement of a rigid body is equivalent to the sum of a translation of any one point on that body and a rotation of the body about an axis through that point.

This describes complex motion as defined above and in Section 2.2 (p. 23). Note that equation 4.1c is an expression of Chasles' theorem.

4.4 GRAPHICAL POSITION ANALYSIS OF LINKAGES

For any one-*DOF* linkage, such as a fourbar, only one parameter is needed to completely define the positions of all the links. The parameter usually chosen is the angle of the input link. This is shown as θ_2 in Figure 4-4. We want to find θ_3 and θ_4. The link lengths are known. Note that we will consistently number the ground link as 1 and the driver link as 2 in these examples.

The graphical analysis of this problem is trivial and can be done using only high-school geometry. If we draw the linkage carefully to scale with rule, compass, and protractor in a particular position (given θ_2), then it is only necessary to measure the angles of links 3 and 4 with the protractor. Note that all link angles are measured from a positive X axis. In Figure 4-4, a *local xy* axis system, parallel to the *global XY* system, has been created at point A to measure θ_3. The accuracy of this graphical solution will be limited by our care and drafting ability and by the crudity of the protractor used. Nevertheless, a very rapid approximate solution can be found for any one position.

Figure 4-5 shows the construction of the graphical position solution. The four link lengths a, b, c, d and the angle θ_2 of the input link are given. First, the ground link (1) and the input link (2) are drawn to a convenient scale such that they intersect at the origin O_2 of the global XY coordinate system with link 2 placed at the input angle θ_2. Link 1 is drawn along the X axis for convenience. The compass is set to the scaled length of link 3, and an arc of that radius swung about the end of link 2 (point A). Then the compass is set to the scaled length of link 4, and a second arc swung about the end of link 1

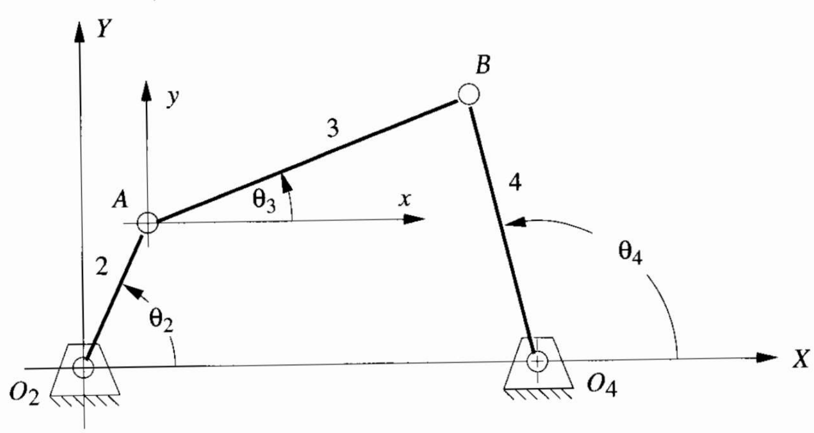


FIGURE 4-4
Measurement of angles in the fourbar linkage

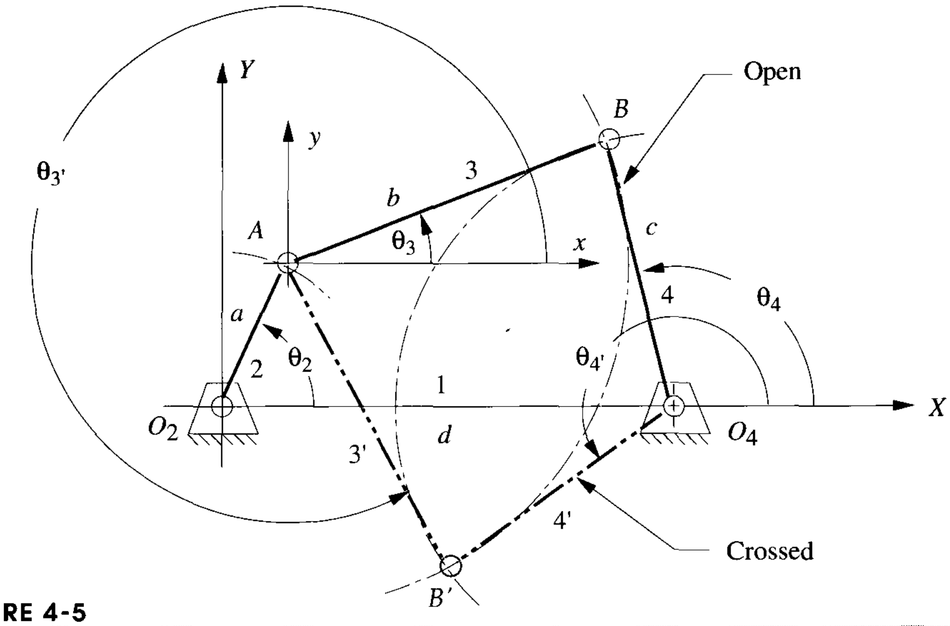


FIGURE 4-5

Graphical position solution to the open and crossed configurations of the fourbar linkage

(point O_4). These two arcs will have two intersections at B and B' that define the two solutions to the position problem for a fourbar linkage which can be assembled in two configurations, called circuits, labeled open and crossed in Figure 4-5. Circuits in linkages will be discussed in a later section.

The angles of links 3 and 4 can be measured with a protractor. One circuit has angles θ_3 and θ_4, the other $\theta_{3'}$ and $\theta_{4'}$. A graphical solution is only valid for the particular value of input angle used. For each additional position analysis we must completely redraw the linkage. This can become burdensome if we need a complete analysis at every 1- or 2-degree increment of θ_2. In that case we will be better off to derive an analytical solution for θ_3 and θ_4 which can be solved by computer.

4.5 ALGEBRAIC POSITION ANALYSIS OF LINKAGES

The same procedure that was used in Figure 4-5 to solve geometrically for the intersections B and B' and angles of links 3 and 4 can be encoded into an algebraic algorithm. The coordinates of point A are found from

$$\begin{aligned} A_x &= a\cos\theta_2 \\ A_y &= a\sin\theta_2 \end{aligned} \tag{4.2a}$$

The coordinates of point B are found using the equations of circles about A and O_4.

$$b^2 = \left(B_x - A_x\right)^2 + \left(B_y - A_y\right)^2 \tag{4.2b}$$

$$c^2 = \left(B_x - d\right)^2 + B_y^2 \tag{4.2c}$$

which provide a pair of simultaneous equations in B_x and B_y.

Subtracting equation 4.2c from 4.2b gives an expression for B_x.

$$B_x = \frac{a^2 - b^2 + c^2 - d^2}{2(A_x - d)} - \frac{2A_y B_y}{2(A_x - d)} = S - \frac{2A_y B_y}{2(A_x - d)} \tag{4.2d}$$

Substituting equation 4.2d into 4.2c gives a quadratic equation in B_y which has two solutions corresponding to those in Figure 4-5.

$$B_y^2 + \left(S - \frac{A_y B_y}{A_x - d} - d \right)^2 - c^2 = 0 \tag{4.2e}$$

This can be solved with the familiar expression for the roots of a quadratic equation,

$$B_y = \frac{-Q \pm \sqrt{Q^2 - 4PR}}{2P} \tag{4.2f}$$

where :

$$P = \frac{A_y^2}{(A_x - d)^2} + 1 \qquad Q = \frac{2A_y(d - S)}{A_x - d}$$

$$R = (d - S)^2 - c^2 \qquad S = \frac{a^2 - b^2 + c^2 - d^2}{2(A_x - d)}$$

Note that the solutions to this equation set can be real or imaginary. If the latter, it indicates that the links cannot connect at the given input angle or at all. Once the two values of B_y are found (if real), they can be substituted into equation 4.2d to find their corresponding x components. The link angles for this position can then be found from

$$\begin{aligned} \theta_3 &= \tan^{-1}\left(\frac{B_y - A_y}{B_x - A_x} \right) \\ \theta_4 &= \tan^{-1}\left(\frac{B_y}{B_x - d} \right) \end{aligned} \tag{4.2g}$$

A two-argument arctangent function must be used to solve equations 4.2g since the angles can be in any quadrant. Equations 4.2 can be encoded in any computer language or equation solver, and the value of 82 varied over the linkage's usable range to find all corresponding values of the other two link angles.

Vector Loop Representation of linkages

An alternate approach to linkage position analysis creates a vector loop (or loops) around the linkage. This approach offers some advantages in the synthesis of linkages which will be addressed in Chapter 5. The links are represented as **position** vectors. Figure 4-6 shows the same fourbar linkage as in Figure 4-4 (p. 151) , but the links are now drawn as position vectors which form a vector loop. This loop closes on itself making the sum of the vectors around the loop zero. The lengths of the vectors are the link lengths which

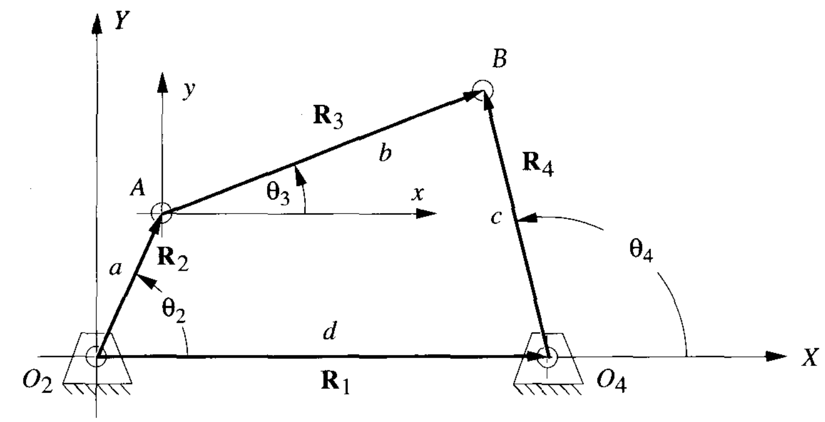


FIGURE 4-6

Position vector loop for a fourbar linkage

are known. The current linkage position is defined by the input angle θ_2 as it is a one-*DOF* mechanism. We want to solve for the unknown angles θ_3 and θ_4. To do so we need a convenient notation to represent the vectors.

Complex Numbers as Vectors

There are many ways to represent vectors. They may be defined in **polar coordinates,** by their *magnitude* and *angle*, or in **cartesian coordinates** as x and y components. These forms are of course easily convertible from one to the other using equations 4.0. The position vectors in Figure 4-6 can be represented as any of these expressions:

Polar form	Cartesian form	
$R\,@\angle\theta$	$r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$	(4.3a)
$re^{j\theta}$	$r\cos\theta + j\,r\sin\theta$	(4.3b)

Equation 4.3a uses **unit vectors** to represent the x and y vector component directions in the cartesian form. Figure 4-7 shows the unit vector notation for a position vector. Equation 4.3b uses **complex number notation** wherein the X direction component is called the *real portion* and the Y direction component is called the *imaginary portion*. This unfortunate term *imaginary* comes about because of the use of the notation j to represent the square root of minus one, which of course cannot be evaluated numerically. However, this *imaginary* number is used in a **complex number** as an **operator,** *not as a value*. Figure 4-8a shows the **complex plane** in which the *real* axis represents the X-directed component of the vector in the plane, and the *imaginary* axis represents the Y-directed component of the same vector. So, any term in a complex number which has no j operator is an x component, and a j indicates a y component.

Note in Figure 4-8b that each multiplication of the vector $\mathbf{R}_A$ by the operator j results in a *counterclockwise rotation* of the vector through 90 degrees. The vector $\mathbf{R}_B = j\mathbf{R}_A$ is directed along the *positive imaginary* or j axis. The vector $\mathbf{R}_C = j^2\,\mathbf{R}_A$ is directed along the *negative real* axis because $j^2 = -1$ and thus $\mathbf{R}_C = -\mathbf{R}_A$. In similar fashion, $\mathbf{R}_D = j^3\,\mathbf{R}_A = -j\mathbf{R}_A$ and this component is directed along the *negative j axis.*

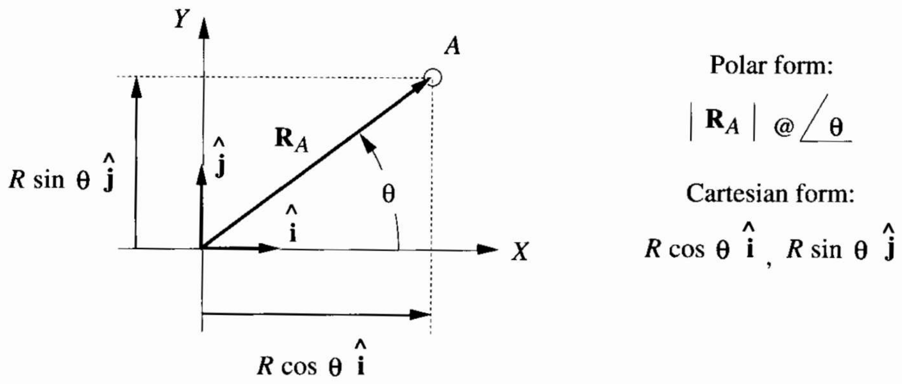


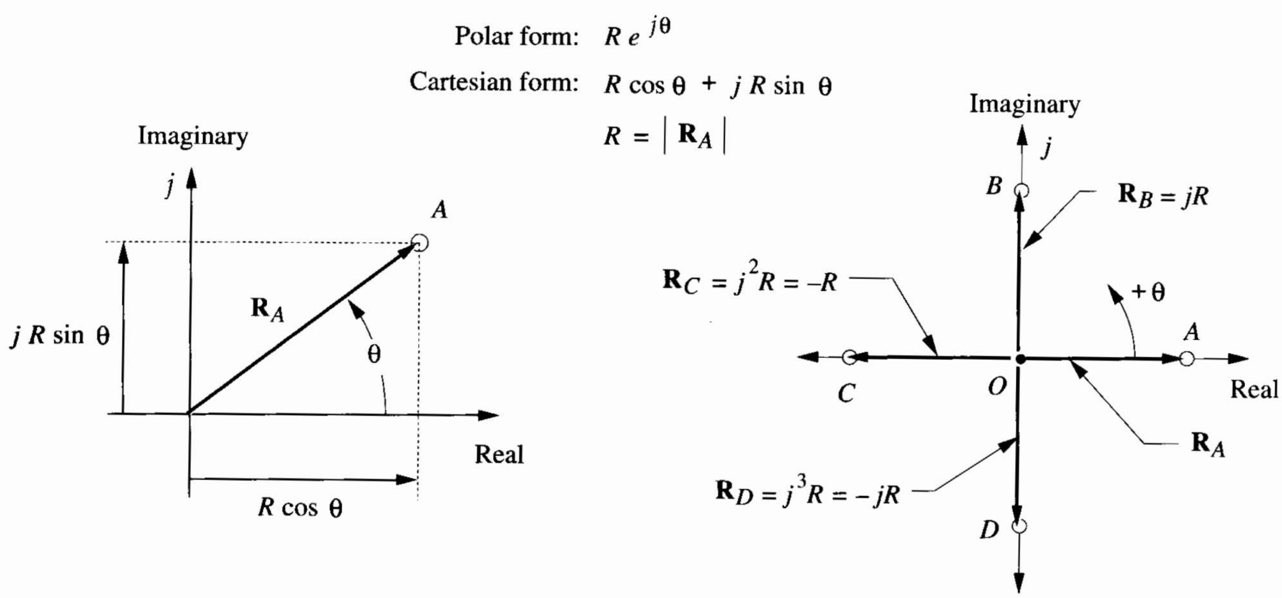
FIGURE 4-7

Unit vector notation for position vectors

One advantage of using this complex number notation to represent planar vectors comes from the **Euler identity**:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{4.4a}$$

Any two-dimensional vector can be represented by the compact polar notation on the left side of equation 4.4a. There is no easier function to differentiate or integrate, since it is its own derivative:



(a) Complex number representation of a position vector

(b) Vector rotations in the complex plane

FIGURE 4-8

Complex number representation of vectors in the plane

$$\frac{de^{j\theta}}{d\theta} = je^{j\theta} \tag{4.4b}$$

We will use this **complex number notation** for vectors to develop and derive the equations for position, velocity, and acceleration of linkages.

The Vector Loop Equation for a Fourbar Linkage

The directions of the position vectors in Figure 4-6 are chosen so as to define their angles where we desire them to be measured. By definition, *the angle of a vector is always measured at its root, **not at its head**.* We would like angle θ_4 to be measured at the fixed pivot O_4, so vector $\mathbf{R}_4$ is arranged to have its root at that point. We would like to measure angle θ_3 at the point where links 2 and 3 join, so vector $\mathbf{R}_3$ is rooted there. A similar logic dictates the arrangement of vectors $\mathbf{R}_1$ and $\mathbf{R}_2$. Note that the X (*real*) axis is taken for convenience along link 1 and the origin of the global coordinate system is taken at point O_2, the root of the input link vector, $\mathbf{R}_2$. These choices of vector directions and senses, as indicated by their arrowheads, lead to this vector loop equation:

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.5a}$$

An **alternate notation** for these position vectors is to use the labels of the points at the vector **tips** and **roots** (*in that order*) as subscripts. The second subscript is conventionally omitted if it is the origin of the global coordinate system (point O_2):

$$\mathbf{R}_A + \mathbf{R}_{BA} - \mathbf{R}_{BO_4} - \mathbf{R}_{O_4} = 0 \tag{4.5b}$$

Next, we substitute the complex number notation for each position vector. To simplify the notation and minimize the use of subscripts, we will denote the scalar lengths of the four links as a, b, c, and d. These are so labeled in Figure 4-6. The equation then becomes:

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0 \tag{4.5c}$$

These are three forms of the same vector equation, and as such can be solved for two unknowns. There are four variables in this equation, namely the four link angles. The link lengths are all constant in this particular linkage. Also, the value of the angle of link 1 is fixed (at zero) since this is the ground link. The *independent variable* is θ_2 which we will control with a motor or other driver device. That leaves the angles of link 3 and 4 to be found. We need algebraic expressions which define θ_3 and θ_4 as functions only of the constant link lengths and the one input angle, θ_2. These expressions will be of the form:

$$\begin{aligned} \theta_3 &= f\{a, b, c, d, \theta_2\} \\ \theta_4 &= g\{a, b, c, d, \theta_2\} \end{aligned} \tag{4.5d}$$

To solve the polar form, vector equation 4.5c, we must substitute the *Euler equivalents* (equation 4.4a) for the $e\,j\theta$ terms, and then separate the resulting cartesian form vector equation into two scalar equations which can be solved simultaneously for θ_3 and θ_4. Substituting equation 4.4a into equation 4.5c:

$$a(\cos\theta_2 + j\sin\theta_2) + b(\cos\theta_3 + j\sin\theta_3) - c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0 \tag{4.5e}$$

This equation can now be separated into its real and imaginary parts and each set to zero.

real part (x component):

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so: (4.6a)

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$$

imaginary part (y component):

$$ja\sin\theta_2 + jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the j's divide out, so: (4.6b)

$$a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 = 0$$

The scalar equations 4.6a and 4.6b can now be solved simultaneously for θ_3 and θ_4. To solve this set of two simultaneous trigonometric equations is straightforward but tedious. Some substitution of trigonometric identities will simplify the expressions. The first step is to rewrite equations 4.6a and 4.6b so as to isolate one of the two unknowns on the left side. We will isolate θ_3 and solve for θ_4 in this example.

$$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d \tag{4.6c}$$

$$b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4 \tag{4.6d}$$

Now square both sides of equations 4.6c and 4.6d and add them:

$$b^2\left(\sin^2\theta_3 + \cos^2\theta_3\right) = \left(-a\sin\theta_2 + c\sin\theta_4\right)^2 + \left(-a\cos\theta_2 + c\cos\theta_4 + d\right)^2 \tag{4.7a}$$

Note that the quantity in parentheses on the left side is equal to 1, eliminating θ_3 from the equation, leaving only θ_4 which can now be solved for.

$$b^2 = \left(-a\sin\theta_2 + c\sin\theta_4\right)^2 + \left(-a\cos\theta_2 + c\cos\theta_4 + d\right)^2 \tag{4.7b}$$

The right side of this expression must now be expanded and terms collected.

$$b^2 = a^2 + c^2 + d^2 - 2ad\cos\theta_2 + 2cd\cos\theta_4 - 2ac\left(\sin\theta_2\sin\theta_4 + \cos\theta_2\cos\theta_4\right) \tag{4.7c}$$

To further simplify this expression, the constants K_1, K_2, and K_3 are defined in terms of the constant link lengths in equation 4.7c:

$$K_1 = \frac{d}{a} \qquad K_2 = \frac{d}{c} \qquad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac} \tag{4.8a}$$

and:

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4 \tag{4.8b}$$

If we substitute the identity $\cos(\theta_2 - \theta_4) = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4$, we get the form known as Freudenstein's equation.

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos\left(\theta_2 - \theta_4\right) \tag{4.8c}$$

In order to reduce equation 4.8b to a more tractable form for solution, it will be useful to substitute the *half angle identities* which will convert the sin θ_4 and cos θ_4 terms to tan θ_4 terms:

$$\sin\theta_4 = \frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)}; \qquad \cos\theta_4 = \frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} \tag{4.9}$$

This results in the following simplified form, where the link lengths and known input value (θ_2) terms have been collected as constants A, B, and C.

$$A\tan^2\left(\frac{\theta_4}{2}\right) + B\tan\left(\frac{\theta_4}{2}\right) + C = 0$$

$$\text{where:} \qquad \begin{aligned} A &= \cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3 \\ B &= -2\sin\theta_2 \\ C &= K_1 - (K_2+1)\cos\theta_2 + K_3 \end{aligned} \tag{4.10a}$$

Note that equation 4.10a is quadratic in form, and the solution is:

$$\tan\left(\frac{\theta_4}{2}\right) = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-B \pm \sqrt{B^2-4AC}}{2A}\right) \tag{4.10b}$$

Equation 4.10b has two solutions, obtained from the ± conditions on the radical. These two solutions, as with any quadratic equation, may be of three types: *real and equal, real and unequal, complex conjugate.* If the discriminant under the radical is negative, then the solution is complex conjugate, which simply means that the link lengths chosen are not capable of connection for the chosen value of the input angle θ_2. This can occur either when the link lengths are completely incapable of connection in any position or, in a non-Grashof linkage, when the input angle is beyond a toggle limit position. There is then no real solution for that value of input angle θ_2. Excepting this situation, the solution will usually be real and unequal, meaning there are two values of θ_4 corresponding to any one value of θ_2. These are referred to as the **crossed** and **open** configurations of the linkage and also as the two **circuits** of the linkage. In the fourbar linkage, the minus solution gives θ_4 for the open configuration and the positive solution gives θ_4 for the crossed configuration.

Figure 4-5 (p. 152) shows both crossed and open solutions for a Grashof crank-rocker linkage. The terms crossed and open are based on the assumption that the input link 2, for which θ_2 is defined, is placed in the first quadrant (i.e., $0 < \theta_2 < \pi/2$). A Grashof linkage is then defined as **crossed** if the two links adjacent to the shortest link cross one another, and as **open** if they do not cross one another in this position. Note that the configuration of the linkage, either crossed or open, is solely dependent upon the way that the links are assembled. You cannot predict, based on link lengths alone, which of the solutions will be the desired one. In other words, you can obtain either solution with the same linkage by simply taking apart the pin which connects links 3 and 4 in Figure 4-5, and moving those links to the only other positions at which the pin will again connect them. In so doing, you will have switched from one position solution, or **circuit**, to the other.

The solution for angle θ_3 is essentially similar to that for θ_4. Returning to equations 4.6, we can rearrange them to isolate θ_4 on the left side.

$$c\cos\theta_4 = a\cos\theta_2 + b\cos\theta_3 - d \tag{4.6e}$$

$$c\sin\theta_4 = a\sin\theta_2 + b\sin\theta_3 \tag{4.6f}$$

Squaring and adding these equations will eliminate θ_4. The resulting equation can be solved for θ_3 as was done above for θ_4, yielding this expression:

$$K_1\cos\theta_3 + K_4\cos\theta_2 + K_5 = \cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3 \tag{4.11a}$$

The constant K_1 is the same as defined in equation 4.8b. K_4 and K_5 are:

$$K_4 = \frac{d}{b}; \qquad K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab} \tag{4.11b}$$

This also reduces to a quadratic form:

$$D\tan^2\left(\frac{\theta_3}{2}\right) + E\tan\left(\frac{\theta_3}{2}\right) + F = 0$$

$$\begin{aligned} \text{where:} \quad D &= \cos\theta_2 - K_1 + K_4\cos\theta_2 + K_5 \\ E &= -2\sin\theta_2 \\ F &= K_1 + (K_4 - 1)\cos\theta_2 + K_5 \end{aligned} \tag{4.12}$$

and the solution is:

$$\theta_{3_{1,2}} = 2\arctan\left(\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right) \tag{4.13}$$

As with the angle θ_4, this also has two solutions, corresponding to the crossed and open circuits of the linkage, as shown in Figure 4-5.

4.6 THE FOURBAR SLIDER-CRANK POSITION SOLUTION

The same vector loop approach as used above can be applied to a linkage containing sliders. Figure 4-9 shows an offset fourbar slider-crank linkage, inversion #1. The term **offset** means that *the slider axis extended does not pass through the crank pivot.* This is the general case. (The nonoffset slider-crank linkages shown in Figure 2-13 (p. 44) are the special cases.) This linkage could be represented by only three position vectors, $\mathbf{R}_2$, $\mathbf{R}_3$, and $\mathbf{R}_s$, but one of them ($\mathbf{R}_s$) will be a vector of varying magnitude and angle. It will be easier to use four vectors, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, and $\mathbf{R}_4$ with $\mathbf{R}_1$ arranged parallel to the axis of slid-

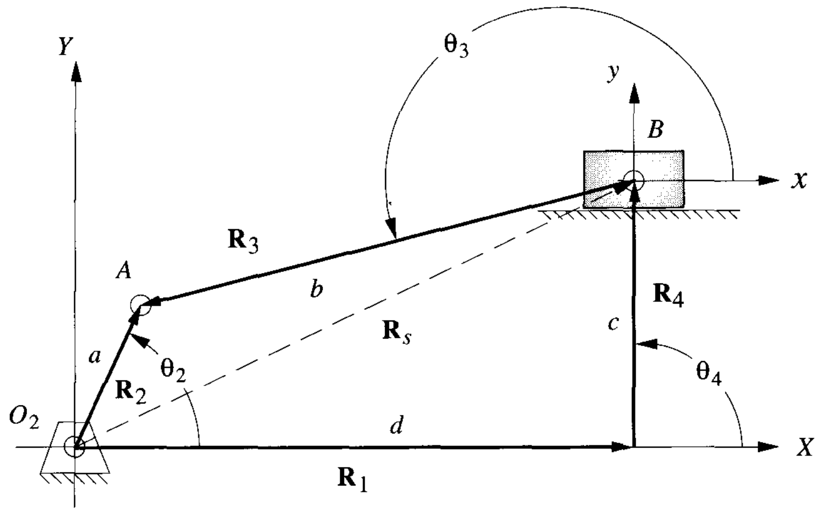


FIGURE 4-9

Position vector loop for a fourbar slider-crank linkage

ing and $\mathbf{R}_4$ perpendicular. In effect the pair of vectors $\mathbf{R}_1$ and $\mathbf{R}_4$ are orthogonal components of the position vector $\mathbf{R}_s$ from the origin to the slider.

It simplifies the analysis to arrange one coordinate axis parallel to the axis of sliding. The variable-length, constant-direction vector $\mathbf{R}_1$ then represents the slider position with magnitude d. The vector $\mathbf{R}_4$ is orthogonal to $\mathbf{R}_1$ and defines the constant magnitude **offset** of the linkage. Note that for the special-case, nonoffset version, the vector $\mathbf{R}_4$ will be zero and $\mathbf{R}_1 = \mathbf{R}_s$. The vectors $\mathbf{R}_2$ and $\mathbf{R}_3$ complete the vector loop. The coupler's position vector $\mathbf{R}_3$ is placed with its root at the slider which then defines its angle θ_3 at point B. This particular arrangement of position vectors leads to a vector loop equation similar to the pin-jointed fourbar example:

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.14a}$$

Compare equation 4.14a to equation 4.5a (p. 156) and note that the only difference is the sign of $\mathbf{R}_3$. This is due solely to the somewhat arbitrary choice of the sense of the position vector $\mathbf{R}_3$ in each case. The angle θ_3 must always be measured at the root of vector $\mathbf{R}_3$, and in this example it will be convenient to have that angle θ_3 at the joint labeled B. Once these arbitrary choices are made it is crucial that the resulting algebraic signs be carefully observed in the equations, or the results will be completely erroneous. Letting the vector magnitudes (link lengths) be represented by a, b, c, d as shown, we can substitute the complex number equivalents for the position vectors.

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \tag{4.14b}$$

Substitute the Euler equivalents:

$$\begin{aligned} &a(\cos\theta_2 + j\sin\theta_2) - b(\cos\theta_3 + j\sin\theta_3) \\ &\quad - c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0 \end{aligned} \tag{4.14c}$$

Separate the real and imaginary components:

real part (x component):

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so: (4.15a)

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0$$

imaginary part (y component):

$$ja\sin\theta_2 - jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the j's divide out, so: (4.15b)

$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0$$

We want to solve equations 4.15 simultaneously for the two unknowns, link length d and link angle θ_3. The independent variable is crank angle θ_2. Link lengths a and b, the offset c, and angle θ_4 are known. But note that since we set up the coordinate system to be parallel and perpendicular to the axis of the slider block, the angle θ_1 is zero and θ_4 is 90°. Equation 4.15b can be solved for θ_3 and the result substituted into equation 4.15a to solve for d. The solution is:

$$\theta_{3_1} = \arcsin\left(\frac{a\sin\theta_2 - c}{b}\right) \tag{4.16a}$$

$$d = a\cos\theta_2 - b\cos\theta_3 \tag{4.16b}$$

Note that there are again two valid solutions corresponding to the two circuits of the linkage. The arcsine function is multivalued. Its evaluation will give a value between ±90° representing only one circuit of the linkage. The value of d is dependent on the calculated value of θ_3. The value of θ_3 for the second circuit of the linkage can be found from:

$$\theta_{3_2} = \arcsin\left(-\frac{a\sin\theta_2 - c}{b}\right) + \pi \tag{4.17}$$

4.7 AN INVERTED SLIDER-CRANK POSITION SOLUTION

Figure 4-l0a shows inversion #3 of the common fourbar slider-crank linkage in which the sliding joint is between links 3 and 4 at point *B*. This is shown as an offset slider-crank mechanism. The slider block has pure rotation with its center offset from the slide axis. (Figure 2-l3c, p. 44, shows the nonoffset version of this linkage in which the vector R.t is zero.)

The global coordinate system is again taken with its origin at input crank pivot 02 and the positive X axis along link 1, the ground link. A local axis system has been placed at point *B* in order to define 83. Note that there is a fixed angle 'Ywithin link 4 which defines the slot angle with respect to that link.

In Figure 4-l0b the links have been represented as position vectors having senses consistent with the coordinate systems that were chosen for convenience in defining the

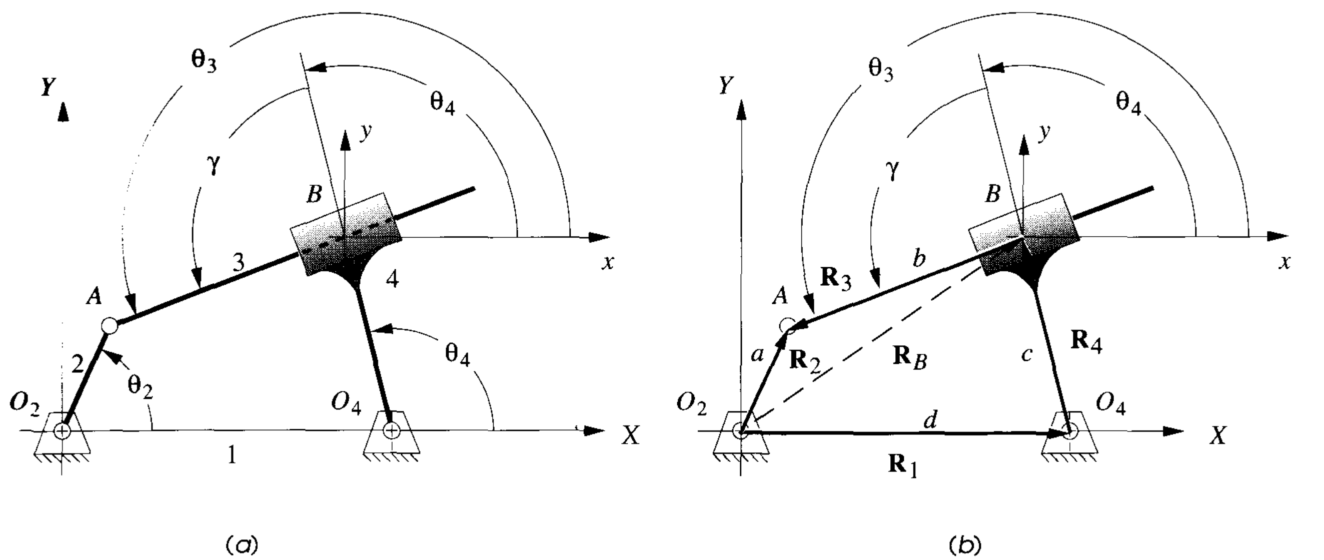


FIGURE 4-10
Inversion #3 of the slider-crank fourbar linkage

link angles. This particular arrangement of position vectors leads to the same vector loop equation as the previous slider-crank example. Equations 4.14 and 4.15 apply to this inversion as well. Note that the absolute position of point B is defined by vector $\mathbf{R}_B$ which varies in both magnitude and direction as the linkage moves. We choose to represent $\mathbf{R}_B$ as the vector difference $\mathbf{R}_2 - \mathbf{R}_3$ in order to use the actual links as the position vectors in the loop equation.

All slider linkages will have at least one link whose effective length between joints will vary as the linkage moves. In this example the length of link 3 between points A and B, designated as b, will change as it passes through the slider block on link 4. Thus the value of b will be one of the variables to be solved for in this inversion. Another variable will be θ_4, the angle of link 4. Note however, that we also have an unknown in θ_3, the angle of link 3. This is a total of three unknowns. Equations 4.15 can only be solved for two unknowns. Thus we require another equation to solve the system. There is a fixed relationship between angles θ_3 and θ_4, shown as γ in Figure 4-10, which gives the equation:

$$\theta_3 = \theta_4 \pm \gamma \tag{4.18}$$

where the + sign is used for the open and the – sign for the crossed configuration.

Substituting equation 4.18 into equations 4.15 yields:

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0 \tag{4.19a}$$

$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0 \tag{4.19b}$$

These have only two unknowns and can be solved simultaneously for θ_4 and b. Equation 4.19b can be solved for link length b and substituted into equation 4.19a.

$$b = \frac{a\sin\theta_2 - c\sin\theta_4}{\sin\theta_3} \tag{4.20a}$$

$$a\cos\theta_2 - \frac{a\sin\theta_2 - c\sin\theta_4}{\sin\theta_3}\cos\theta_3 - c\cos\theta_4 - d = 0 \tag{4.20b}$$

Substitute equation 4.18 and after some algebraic manipulation, equation 4.20 can be reduced to:

$$P\sin\theta_4 + Q\cos\theta_4 + R = 0 \tag{4.21}$$

where:

$$P = a\sin\theta_2 \sin\gamma + (a\cos\theta_2 - d)\cos\gamma$$
$$Q = -a\sin\theta_2 \cos\gamma + (a\cos\theta_2 - d)\sin\gamma$$
$$R = -c\sin\gamma$$

Note that the factors P, Q, R are constant for any input value of θ_2. To solve this for θ_4, it is convenient to substitute the tangent half angle identities (equation 4.9, p.158) for the sin θ_4 and cos θ_4 terms. This will result in a quadratic equation in tan $(\theta_4 / 2)$ which can be solved for the two values of θ_4.

$$P\frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} + Q\frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} + R = 0 \tag{4.22a}$$

This reduces to:

$$(R-Q)\tan^2\left(\frac{\theta_4}{2}\right) + 2P\tan\left(\frac{\theta_4}{2}\right) + (Q+R) = 0$$

let:

$$S = R - Q; \qquad T = 2P; \qquad U = Q + R$$

then:

$$S\tan^2\left(\frac{\theta_4}{2}\right) + T\tan\left(\frac{\theta_4}{2}\right) + U = 0 \tag{4.22b}$$

and the solution is:

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-T \pm \sqrt{T^2 - 4SU}}{2S}\right) \tag{4.22c}$$

As was the case with the previous examples, this also has a crossed and an open solution represented by the plus and minus signs on the radical. Note that we must also calculate the values of link length b for each θ_4 by using equation 4.20a. The coupler angle θ_3 is found from equation 4.18.