Proceedings of EuCoMeS, the first
European Conference on Mechanism Science
Obergurgl (Austria), February 21–26 2006
Manfred Husty and Hans-Peter Schröcker, editors

The Kinematic Synthesis of Mechanically Constrained Planar 3R Chains

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In this paper, we consider the problem of designing mechanical constraints for a planar serial chain formed with three revolute joints, denoted as the 3R chain. Our focus is on the various ways that two RR chains can be used to constrain the links of the 3R chain such that the system has one degree-of-freedom, yet passes through a set of five specified task positions.

The result of this synthesis process is a planar six-bar linkage with one degree-of-freedom, and we obtain designs for each of the known Watt and Stephenson six-bar topologies, except the Watt II. We demonstrate the synthesis process with an example.

Introduction

In this paper, we consider the planar robot formed by a serial chain constructed from three revolute joints, the planar 3R chain. Our goal is to mechanically constrain the relative movement of the joints so the end-effector reaches a specified set of task positions. This work is inspired by Krovi et al. (2002)(1), who derived synthesis equations for planar nR planar serial chains in which the n joints are constrained by a cable drive. They obtained a "single degree-of-freedom coupled serial chain" that they use to design an assistive device.

Rather than use cables to constrain the relative joint angles, we add two RR chains. The planar 3R robot consists of four bars, if we include its base. Therefore, the appropriate attachment of two RR chains results in a planar six-bar with seven joints forming a one degree-of-freedom system. The synthesis of these systems was first explored by Lin and Erdman (1987)(2), who used a complex vector formulation to obtain design equations for planar 3R and 4R chains, which they called triads and quadriads. Chase et al. (1987)(3) applied triad synthesis to the design of planar six-bar linkages, and Subbian and Flugrad (1994)(4) used it to design a planar eight-bar linkage.

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Our formulation of the synthesis problem follows Perez and McCarthy (2005)(5), who use a planar version of the dual quaternion kinematics equations of a serial chain as design equations. Also see Perez and McCarthy (2004)(6). In what follows, we present the design equations for planar RR and 3R chains. We then show the sequence of synthesis problems that constrain a 3R chain in a way that yields the various six-bar linkage topologies, Waldron and Kinzel (2004)(7), Tsai (2001)(8). Finally, we present an analysis routine that uses the Dixon determinant to solve the loop equations as presented by Wampler (2001)(9). An example illustrates our synthesis methodology.

The Relative Kinematics Equation of Planar 3R Robot

The kinematics equations of the planar 3R robot equate the 3×3 homogeneous transformation $[D]$ between the end-effector and the base frame to the sequence of local coordinate transformations around the joint axes and along the links of the chain (McCarthy 1990(10)),

$$[D] = [G][Z(\theta_1)][X(a_{12})][Z(\theta_2)][X(a_{23})][Z(\theta_3)][H]. \tag{1}$$

The parameters θ_i define the movement at each joint and $a_{i,j}$ define the length the links. The transformation $[G]$ defines the position of the base of the chain relative to the fixed frame, and $[H]$ locates the task frame relative to the end-effector frame. The matrix $[D]$ defines the coordinate transformation from the world frame F to the task frame M.

Lee and Mavroidis (2002)(11) show how to use the kinematics equations (1) as design equations for a spatial 3R chain. Rather than use these equations directly, we follow Perez and McCarthy (2005)(5) and construct the relative kinematics equations. To do this we select a reference configuration for the robot and obtain the transformation $[D_0]$, which locates the reference position of the task frame, denoted M_0.

Let $[D_i]$ be the transformation from the world frame to the the task frame M_i, then compute the relative transformation $[D_{0i}] = [D_i][D_0]^{-1}$, given by

$$[D_{0i}] = [D_i][D_0]^{-1} = [T(\Delta\theta_1, \mathbf{C}_1)][T(\Delta\theta_2, \mathbf{C}_2)][T(\Delta\theta_3, \mathbf{C}_3)], \tag{2}$$

where

$$\begin{aligned}
[T(\Delta\theta_1, \mathbf{C}_1)] =& [G][Z(\Delta\theta_1^i)][G^{-1}],\\
[T(\Delta\theta_2, \mathbf{C}_2)] =& ([G][Z(\theta_1^i)][X(a_{12})])[Z(\Delta\theta_2^i)]([G][Z(\theta_1^i)][X(a_{12})])^{-1}, \quad \text{and}\\
[T(\Delta\theta_3, \mathbf{C}_3)] =& ([G][Z(\theta_1^i)][X(a_{12})][Z(\theta_2^i)][X(a_{23})])[Z(\Delta\theta_3^i)]([G][Z(\theta_1^i)][X(a_{12})][Z(\theta_2^i)][X(a_{23})])^{-1}.
\end{aligned} \tag{3}$$

The relative joint angles are given by $\Delta\theta_j^i = \theta_j^i - \theta_j^0, j = 1, 2, 3$. The points $\mathbf{C}_j, j = 1, 2, 3$ are the poles of the displacements $[T(\Delta\theta_j, \mathbf{C}_j)]$, which means they are the coordinates of the joints of the 3R chain measured in the world frame, when the chain is in its reference configuration .

The Even Clifford Algebra $C^+(P^2)$

It is convenient at this point to introduce the complex numbers $e^{k\Delta\theta} = \cos\Delta\theta + k\sin\Delta\theta$ and $\mathbf{C} = c_x + kc_y$ to simplify the representation of the displacement $[T(\Delta\theta, \mathbf{C})]$—$k$ is the complex unit $k^2 = -1$. Let $\mathbf{X}_1 = x + ky$ be the coordinates of a point in the world frame in the first position and $\mathbf{X}_2 = X + kY$ be its coordinates in the second position, then this transformation becomes

$$\mathbf{X}_2 = e^{k\Delta\theta}\mathbf{X}_1 + (1 - e^{k\Delta\theta})\mathbf{C}. \tag{4}$$

The complex numbers $e^{k\Delta\theta}$ and $(1 - e^{k\Delta\theta})\mathbf{C}$ define the rotation and translation respectively, that form the planar displacement. The point $\mathbf{C}$ is the pole of the displacement.

The complex number form of a planar displacement can be expanded to define the even Clifford algebra of the projective plane P^2. Using the homogeneous coordinates of points in the projective plane as the vectors and a degenerate scalar product, we obtain an eight dimensional Clifford algebra, $C(P^2)$, McCarthy (1990)(10). This Clifford algebra has an even sub-algebra, $C^+(P^2)$, which is a set of four dimensional elements of the form

$$A = a_1 i\epsilon + a_2 j\epsilon + a_3 k + a_4. \tag{5}$$

The basis elements $i\epsilon$, $j\epsilon$, k and 1 satisfy the following multiplication table,

	$i\epsilon$	$j\epsilon$	k	1
$i\epsilon$	0	0	$-j\epsilon$	$i\epsilon$
$j\epsilon$	0	0	$i\epsilon$	$j\epsilon$
k	$j\epsilon$	$-i\epsilon$	-1	k
1	$i\epsilon$	$j\epsilon$	k	1

. (6)

Notice that the set of Clifford algebra elements $\mathbf{z} = x + ky$ formed using the basis element k is isomorphic to the usual set of complex numbers.

McCarthy (1993)(12) shows that a displacement defined by a rotation by $\Delta\theta$ about the pole $\mathbf{C}$, given in Eq. 4, has the associated Clifford algebra element

$$C(\Delta\theta) = \big(1 + \frac{1}{2}(1 - e^{k\Delta\theta})\mathbf{C}i\epsilon\big)e^{k\Delta\theta/2}, \tag{7}$$

which is the Clifford algebra version of a relative displacement. Expand this equation to obtain the four dimensional vector

$$C(\Delta\theta) = -\sin\frac{\Delta\theta}{2}\mathbf{C}j\epsilon + e^{k\Delta\theta/2} = c_y \sin\frac{\Delta\theta}{2}i\epsilon - c_x \sin\frac{\Delta\theta}{2}j\epsilon + \sin\frac{\Delta\theta}{2}k + \cos\frac{\Delta\theta}{2}. \tag{8}$$

The components of $C(\Delta\theta)$ are the kinematic mapping used by Bottema and Roth (1979)(13) to study planar displacements—also see DeSa and Roth (1981)(14) and Ravani and Roth (1983)(15). Brunnthaler et al. 2005(17) and Schrocker et al. 2005(16) use this kinematic mapping to study the synthesis of planar four-bar linkages.

Clifford algebra kinematics equations

The Clifford algebra version of the relative kinematics equations are obtained as follows. Let the relative displacement of the frame M be defined by the rotation of $\Delta\rho$ about the pole $\mathbf{P}$, and let the coordinates in the reference position of the pivots of the 3R chain be given by $\mathbf{G} = g_x + kg_y$, $\mathbf{W} = w_x + kw_y$, and $\mathbf{H} = h_x + kh_y$, then we have

$$P(\Delta\rho) = G(\Delta\theta)W(\Delta\phi)H(\Delta\psi), \tag{9}$$

or

$$-\sin\frac{\Delta\rho}{2}\mathbf{P}j\epsilon + e^{k\Delta\rho/2} = \\ \big(-\sin\frac{\Delta\theta}{2}\mathbf{G}j\epsilon + e^{k\Delta\theta/2}\big)\big(-\sin\frac{\Delta\phi}{2}\mathbf{W}j\epsilon + e^{k\Delta\phi/2}\big)\big(-\sin\frac{\Delta\psi}{2}\mathbf{H}j\epsilon + e^{k\Delta\psi/2}\big), \tag{10}$$

where $\Delta\theta$, $\Delta\phi$ and $\Delta\psi$ are the rotations measured around $\mathbf{G}$, $\mathbf{W}$ and $\mathbf{H}$, respectively. We use this equation to design a planar 3R chain to reach five specified task positions.

A similar relative kinematics equation is obtained for planar RR chains, which we use to design constraints the 3R chain. As above let $\mathbf{G} = g_x + kg_y$ and $\mathbf{W} = w_x + kw_y$ be the fixed and moving pivots of the RR chain. Then, we have

$$P(\Delta\rho) = G(\Delta\theta)W(\Delta\phi), \tag{11}$$

that is

$$-\sin\frac{\Delta\rho}{2}\mathbf{P}j\epsilon + e^{k\Delta\rho/2} = \big(-\sin\frac{\Delta\theta}{2}\mathbf{G}j\epsilon + e^{k\Delta\theta/2}\big)\big(-\sin\frac{\Delta\phi}{2}\mathbf{W}j\epsilon + e^{k\Delta\phi/2}\big), \tag{12}$$

where $\Delta\theta$ and $\Delta\phi$ are the rotations measured around $\mathbf{G}$ and $\mathbf{W}$.

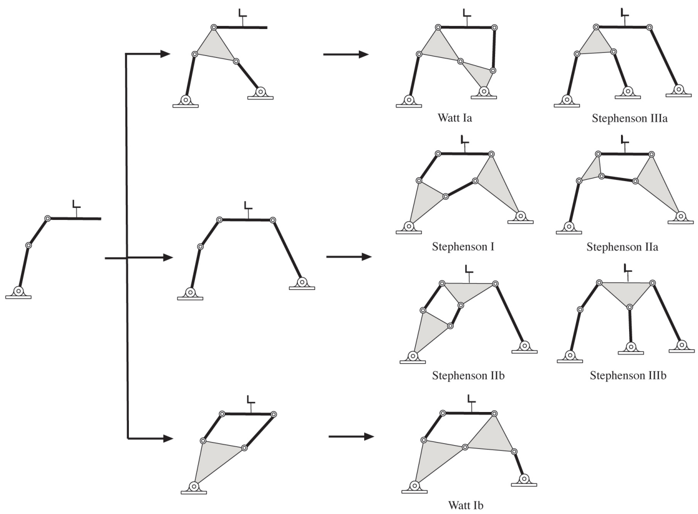

Figure 1: The various ways to add two RR constraints to a 3R chain.

Synthesis of Mechanically Constrained 3R Chains

In this section, we consider how two RR constrains are added to a 3R chain to mechanically constrain its movement to one degree-of-freedom. Given five task positions it is well-known that as many as four RR chains can be computed that guide the end-effector through the specified task positions. See for example Sandor and Erdman (1984)(18) and McCarthy (2000)(19).

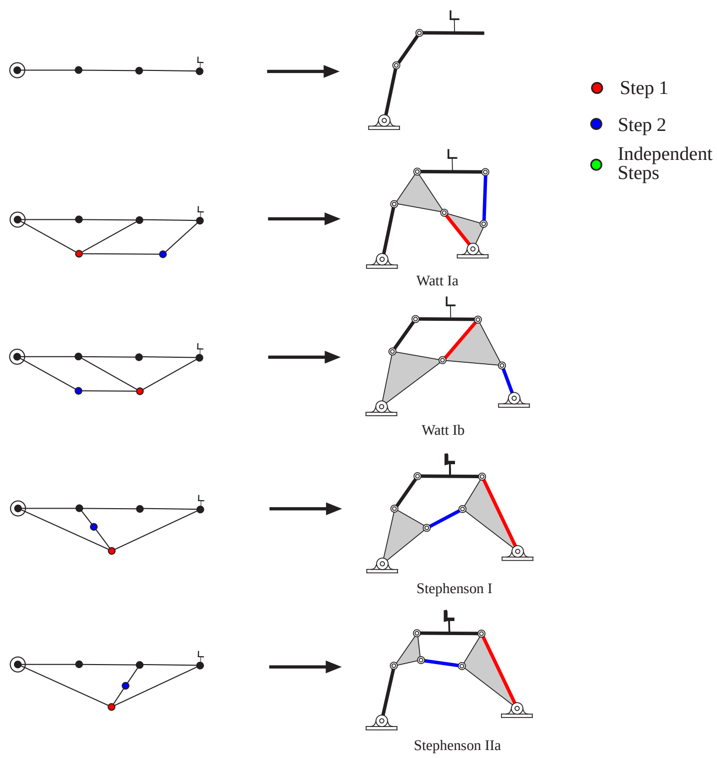

Figure 2: The linkage graphs show the synthesis sequence for the four constrained 3R chains in which the second RR chain connects to the first RR chain.

We present the RR chain design equations using Clifford algebra coordinates in the following sections.

Our synthesis of a mechanically constrained 3R chain proceeds in three steps. We first identify an 3R chain that reaches the five specified task positions. Inverse kinematics analysis of the 3R chain yields the configuration of the chain in each of the five positions, which allows us to determine the five relative positions of any pair of links in the chain.

The second step is to choose two links in the 3R chain and compute an RR chain that constrains their relative movement to that required by the five task positions. The solution of the design equations yield as many as four solutions. In order to identify the remaining connections, let the four links of the chain, including the ground, be labeled B_i, $i = 0, 1, 2, 3$. Clearly, we cannot constrain two consecutive links in the 3R chain, this leaves three cases: i) B_0B_2, ii) B_0B_3 and iii) B_1B_3. Figure 1 shows the introduction of this RR chain, which adds a link to the system that we denote at B_4. Analysis of this system determines the positions of B_4 relative to all of the remaining links in the chain.

The third step consists of adding the second RR, which can now connect any two of the five bodies $B_i, i = 0, 1, \ldots, 4$, again assuming the two are not consecutive. The five relative positions of the two bodies yields design equations that yield as many as four of these RR chains. Figure 1 shows that we obtain the following seven six-bar linkage topologies, which can be identified as i) (B_0B_2, B_3B_4) known as a Watt I linkage, ii) (B_0B_2, B_0B_3), the Stephenson III, iii) (B_0B_3, B_1B_4), the Stephenson I, iv) (B_0B_3, B_2B_4), the Stephenson II, v) (B_0B_3, B_1B_3),

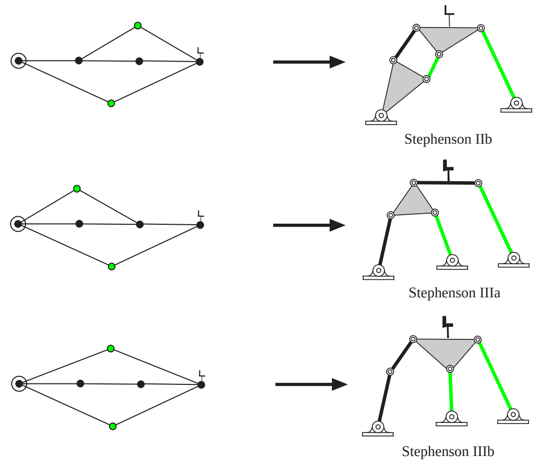

Figure 3: The linkage graphs show the synthesis sequence for the three constrained 3R chains in which the two RR chains are attached independently.

the Stephenson II, vi) (B_0B_3, B_0B_3), the Stephenson III, and vii) (B_1B_3, B_0B_4), the Watt I linkage.

The terms Watt I, II and Stephenson I, II, III are well-known names for six-bar linkage topologies, see for example Tsai (2001)(8). Our listing does not include the Watt II because, in this topology the end-effector is not a floating link. Notice that in this list the Watt I, Stephenson II and Stephenson III topologies are duplicated. In the first instance the synthesis sequence (B_0B_2, B_3B_4) yields the same Watt I topology as (B_1B_3, B_0B_4), however they result in a different form for the input link B_1. Similarly, synthesis sequences for the two Stephenson II linkages result in different links that act as the end-effector, or moving frame. This is true for the two Stephenson III topologies as well. Thus, our design procedure yields seven different constrained 3R chains.

Another illustration of the ways that a 3R serial chain can be constrained to obtain a one degree-of- freedom system are shown in Figure 2 and Figure 3. The linkage graph is constructed by identifying each link as a vertex, and each joint as an edge. The introduction of an RR chain adds a vertex and two edges to the graph.

Synthesis Equations for Planar RR and 3R Chains

In this section, we use the Clifford algebra formulation of the relative kinematics equations of RR and 3R chains to assemble design equations. This approach introduces the joint coordinates in the reference positions as design variables, as well as the relative joint angles that define the configuration of the chain in each of the specified task positions. We follow Perez and McCarthy (2005)(5) and eliminate the joint angles to obtain algebraic equations that can be solved for the joint coordinates.

The RR Chain

The relative kinematics equations givin in Eq 12 for the RR chain can be expanded to define the four-dimensional array

$$\begin{aligned} P(\Delta\rho) &= G(\Delta\theta)W(\Delta\phi) \\ &= \begin{Bmatrix} \sin(\frac{\Delta\theta}{2})\cos(\frac{\Delta\phi}{2})g_y + \cos(\frac{\Delta\theta}{2})\sin(\frac{\Delta\phi}{2})w_y - \sin(\frac{\Delta\theta}{2})\sin(\frac{\Delta\phi}{2})(g_x - w_x) \\ -\sin(\frac{\Delta\theta}{2})\cos(\frac{\Delta\phi}{2})g_x - \cos(\frac{\Delta\theta}{2})\sin(\frac{\Delta\phi}{2})w_x - \sin(\frac{\Delta\theta}{2})\sin(\frac{\Delta\phi}{2})(g_y - w_y) \\ \sin(\frac{\Delta\theta}{2})\cos(\frac{\Delta\phi}{2}) + \cos(\frac{\Delta\theta}{2})\sin(\frac{\Delta\phi}{2}) \\ \cos(\frac{\Delta\theta}{2})\cos(\frac{\Delta\phi}{2}) - \sin(\frac{\Delta\theta}{2})\sin(\frac{\Delta\phi}{2}) \end{Bmatrix}. \end{aligned} \tag{13}$$

Recall that $\mathbf{G} = (g_x, g_y)$ and $\mathbf{W} = (w_x, w_y)$ are the coordinates of the fixed and moving pivots of the chain.

In order to design an RR chain, we specify five task positions, $[T_i], i = 1, 2, \ldots, 5$ for the end-effector. We then choose the first task position $[T_1]$ as the reference, and compute the relative displacements $[T_{1i}] = [T_i][T_1^{-1}]$ for $i = 2, 3, 4, 5$. These relative displacements have the poles $\mathbf{P}_{1i}$ and the relative rotation angles $\Delta\rho^i = \rho_i - \rho_1$, which we use to assemble the Clifford algebra elements $P_{1i}(\Delta\rho^i)$. Equating these task positions to the relative kinematics equations of the RR chain, we obtain a set of four vector equations,

$$P_{1i}(\Delta\rho^i) = G(\Delta\theta^i)W(\Delta\phi^i), \quad i = 2, 3, 4, 5. \tag{14}$$

This set of equations can be expanded to yield,

$$\begin{Bmatrix} p_x^i \\ p_y^i \\ p_z^i \\ p_w^i \end{Bmatrix} = \begin{bmatrix} g_y & w_y & w_x - g_y & 0 \\ -g_x & -w_x & w_y - g_y & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \sin(\frac{\Delta\theta^i}{2})\cos(\frac{\Delta\phi^i}{2}) \\ \cos(\frac{\Delta\theta^i}{2})\sin(\frac{\Delta\phi^i}{2}) \\ \sin(\frac{\Delta\theta^i}{2})\sin(\frac{\Delta\phi^i}{2}) \\ \cos(\frac{\Delta\theta^i}{2})\cos(\frac{\Delta\phi^i}{2}) \end{Bmatrix}, \quad i = 2, 3, 4, 5, \tag{15}$$

which we write in the form

$$P_{1i}(\Delta\rho^i) = \big[M\big]\, V(\Delta\theta^i, \Delta\phi^i), \quad i = 2, 3, 4, 5. \tag{16}$$

The matrix $[M]$ above can be inverted symbolically, to yield

$$[M]^{-1} = \frac{1}{R^2} \begin{bmatrix} -(w_y - g_y) & w_x - g_x & \mathbf{W}\cdot(\mathbf{W} - \mathbf{G}) & 0 \\ w_y - g_y & -(w_x - g_x) & -\mathbf{G}\cdot(\mathbf{W} - \mathbf{G}) & 0 \\ w_x - g_x & w_y - g_y & \mathbf{G}\times\mathbf{W} & 0 \\ w_x - g_x & w_y - g_y & \mathbf{G}\times\mathbf{W} & R^2 \end{bmatrix}, \tag{17}$$

where the $\mathbf{G}\cdot\mathbf{W} = g_x w_x + g_y w_y$ and $\mathbf{G}\times\mathbf{W} = g_x w_y - g_y w_x$. The parameter R is the distance between the points $\mathbf{G}$ and $\mathbf{W}$, which is the length of the link connecting these joints. Thus, we can solve for the joint angle vectors $V(\Delta\theta^i, \Delta\phi^i)$ as

$$V(\Delta\theta^i, \Delta\phi^i) = [M]^{-1} P_{1i}(\Delta\rho^i), \quad i = 2, 3, 4, 5. \tag{18}$$

The components of the vectors $V(\Delta\theta^i, \Delta\phi^i) = (v_1^i, v_2^i, v_3^i, v_4^i)^T$ satisfy the relationship $v_1^i/v_4^i = v_3^i/{}^i v_2$, that is

$$\frac{\sin(\frac{\Delta\theta^i}{2})\cos(\frac{\Delta\phi^i}{2})}{\cos(\frac{\Delta\theta^i}{2})\cos(\frac{\Delta\phi^i}{2})} = \frac{\sin(\frac{\Delta\theta^i}{2})\sin(\frac{\Delta\phi^i}{2})}{\cos(\frac{\Delta\theta^i}{2})\sin(\frac{\Delta\phi^i}{2})}. \tag{19}$$

Expanding the expressions $\mathcal{R}_i : v_1^i v_2^i - v_3^i v_4^i = 0$, and factoring out the term $R^2 = (\mathbf{W}-\mathbf{G}).(\mathbf{W}-\mathbf{G})$, we obtain the four polynomial design equations,

$$\begin{aligned}\mathcal{R}_i :\ & (-p_w^i p_x^i - p_y^i p_z^i)w_x + (-p_w^i p_y^i + p_x^i p_z^i)w_y + g_x(p_w^i p_x^i - p_y^i p_z^i - (p_z^i)^2 w_x - p_w^i p_z^i w_y) + \\ & g_y(p_w^i p_y^i + p_x^i p_z^i + p_w^i p_z^i w_x - (p_z^i)^2 w_y) - (p_x^i)^2 - (p_y^i)^2 = 0, \quad i = 2,3,4,5. \end{aligned} \tag{20}$$

These equations can be solved to yield the four components of $\mathbf{G}$ and $\mathbf{W}$. The system can have as many as four roots yielding four different RR chains (Sandor and Erdman 1984(18), McCarthy 2000(19)).

The 3R Chain

The design of the planar 3R chain can be formulated in the same way as for the RR chain. Denote the coordinates of the three pivots in the reference configuration as $\mathbf{G} = (g_x, g_y)$, $\mathbf{W} = (w_x, w_y)$ and $\mathbf{H} = (h_x, h_y)$, then the relative kinematics equations of this chain can be expanded to yield

$$P(\Delta\rho) = G(\Delta\theta)W(\Delta\phi)H(\Delta\psi) = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

where

$$\begin{aligned}
q_1 =& g_y s\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} + w_y c\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} + (w_x - g_x)s\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}c\frac{\Delta\phi^i}{2} + \\
& h_y c\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2} + (h_x - g_x)s\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2} + (h_x - w_x)c\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2} - \\
& (g_y - w_y + h_y)s\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2}, \\
q_2 =& -g_x s\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} - w_x c\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} + (w_y - g_y)s\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}c\frac{\Delta\phi^i}{2} - \\
& h_x c\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2} + (h_y - g_y)s\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2} + (h_y - w_x)c\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2} \\
& + (g_x - w_x + h_x)s\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}s\frac{\Delta\phi^i}{2}, \\
q_3 =& s\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} + c\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} + c\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}s\frac{\Delta\psi^i}{2} - s\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}s\frac{\Delta\psi^i}{2}, \\
q_4 =& c\frac{\Delta\theta^i}{2}c\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} - s\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}c\frac{\Delta\psi^i}{2} - s\frac{\Delta\theta^i}{2}c\frac{\phi^i}{2}s\frac{\psi^i}{2} - c\frac{\Delta\theta^i}{2}s\frac{\Delta\phi^i}{2}s\frac{\Delta\psi^i}{2}.
\end{aligned} \tag{21}$$

The relative angles $\Delta\theta$, $\Delta\phi$ and $\Delta\psi$ define the rotations about the pivots $\mathbf{G}$, $\mathbf{W}$, and $\mathbf{H}$, respectively.

We formulate the design equations for this chain by specifying five task positions $[T_i], i = 1, \ldots, 5$. As above we choose the first as the reference positions can construct the relative kinematics equations for each of the four relative displacements,

$$P_{1i}(\Delta\rho^i) = G(\Delta\theta^i)W(\Delta\phi^i)H(\Delta\psi^i), \quad i = 2, \ldots, 5. \tag{22}$$

Expanding these equations and separating the joint variables, we obtain

$$P_{1i}(\Delta\rho^i) = \big[M(\Delta\theta^i)\big]\, V(\Delta\phi^i, \Delta\psi^i), \quad i = 2, \ldots, 5 \tag{23}$$

where

$$[M(\Delta\theta^i)] = [A_1]\cos(\frac{\Delta\theta^i}{2}) + [A_2]\sin(\frac{\Delta\theta^i}{2}) \tag{24}$$

and

$$[A_1] = \begin{bmatrix} h_y & w_y & h_x - w_x & 0 \\ -h_x & -w_x & h_y - w_y & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad [A_2] = \begin{bmatrix} h_x - g_x & w_x - g_x & w_y - g_y - h_y & g_y \\ h_y - g_y & w_y - g_y & -w_x + g_x + h_x & -g_x \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix}. \tag{25}$$

The matrix $[M(\Delta\theta^i)]$ can be inverted to yield

$$[M(\Delta\theta^i)]^{-1} = \frac{1}{R_1^2}([B_1]\cos(\frac{\Delta\theta^i}{2}) + [B_2]\sin(\frac{\Delta\theta^i}{2})) \tag{26}$$

where

$$\begin{aligned} [B_1] &= \begin{bmatrix} -(h_y - w_y) & h_x - w_x & \mathbf{H}.(\mathbf{H} - \mathbf{W}) & 0 \\ h_y - w_y & -(h_x - w_x) & -\mathbf{W}.(\mathbf{H} - \mathbf{W}) & 0 \\ h_x - w_x & h_y - w_y & \mathbf{W} \times \mathbf{H} & 0 \\ h_x - w_x & h_y - w_y & \mathbf{W} \times \mathbf{H} & R_1^2 \end{bmatrix}, \\ [B_2] &= \begin{bmatrix} -(h_x - w_x) & -(h_y - w_y) & (\mathbf{H} + \mathbf{G}) \times \mathbf{W} & \mathbf{H}.(\mathbf{G} + \mathbf{W} - \mathbf{H}) - \mathbf{G}.\mathbf{W}) \\ h_x - w_x & h_y - w_y & (\mathbf{W} - \mathbf{H}) \times \mathbf{G} & (\mathbf{W}. - \mathbf{H}).(\mathbf{G} - \mathbf{W}) \\ -(h_y - w_y) & h_x - w_x & \mathbf{H}.\mathbf{G} - \mathbf{G}.\mathbf{W} & \mathbf{H} \times (\mathbf{W} - \mathbf{G}) + \mathbf{W} \times \mathbf{G} \\ -(h_y - w_y) & h_x - w_x & (\mathbf{H} - \mathbf{G}).(\mathbf{G} - \mathbf{W} + \mathbf{H}) & \mathbf{H} \times (\mathbf{W} - \mathbf{G}) + \mathbf{W} \times \mathbf{G} \end{bmatrix}. \end{aligned} \tag{27}$$

Thus, we can solve for the joint angle vector $V(\Delta\phi^i, \Delta\psi^i)$ to obtain,

$$V(\Delta\phi^i, \Delta\psi^i) = [M(\Delta\theta^i)]^{-1} P(\Delta\rho^i). \tag{28}$$

As we saw above for the RR chain, the components of this vector satisfy the condition $\mathcal{R}_i$:

$v_1^i v_2^i - v_3^i v_4^i = 0$, which after factoring out $R_1^2 = (\mathbf{H} - \mathbf{W}).(\mathbf{H} - \mathbf{W})$ yield the design equations

$$\begin{aligned}
\mathcal{R}_i: \quad & \frac{1}{2}(2(p_x^i)^2 + 2(p_y^i)^2 - 2p_x^i p_z^i h_y - 2p_x^i p_z^i g_y + 2p_y^i p_z^i h_x + 2p_y^i p_z^i g_x + (p_z^i)^2 h_x g_x + (p_z^i)^2 g_x^2 + \\
& (p_z^i)^2 g_y h_y + (p_z^i)^2 g_y^2 + 2p_x^i p_w^i h_x - 2g_x p_x^i p_w^i + 2p_y^i p_w^i h_y - 2p_y^i p_w^i g_y + 2p_z^i p_w^i h_y g_x - \\
& 2p_z^i p_w^i h_x g_y - (p_w^i)^2 h_x g_x + (p_w^i)^2 g_x^2 - (p_w^i)^2 h_y g_y + (p_w^i)^2 g_y^2 + (p_z^i)^2 h_x w_x - (p_z^i)^2 g_x w_x + \\
& (p_w^i)^2 h_x w_x - (p_w^i)^2 g_x w_x + (p_z^i)^2 h_y w_y - (p_z^i)^2 g_y w_y + (p_w^i)^2 h_y w_y + (-g_x^2((p_z^i)^2 + (p_w^i)^2) - \\
& g_y^2((p_z^i)^2 + (p_w^i)^2) + 2p_y^i p_z^i w_x + (p_z^i)^2 h_x w_x - 2p_x^i p_w^i w_x + 2p_z^i p_w^i h_y w_x - (p_w^i)^2 h_x w_x + \\
& g_x(-2p_y^i p_z^i - (p_z^i)^2 h_x + 2p_x^i p_w^i - 2p_z^i p_w^i h_y + (p_w^i)^2 h_x + (p_z^i)^2 w_x + (p_w^i)^2 w_x) - 2p_x^i p_z^i w_y + \\
& (p_z^i)^2 h_y w_y - 2p_y^i p_w^i w_y - 2p_z^i p_w^i h_x w_y - (p_w^i)^2 h_y w_y + g_y(2p_x^i p_z^i - (p_z^i)^2 h_y + 2p_y^i p_w^i + \\
& 2p_z^i p_w^i h_x + (p_w^i)^2 h_y + (p_z^i)^2 w_y + (p_w^i)^2 w_y)) \cos(\Delta\theta^i) + (-2p_x^i p_z^i w_x + (p_z^i)^2 h_y w_x - \\
& 2p_y^i p_w^i w_x - 2p_z^i p_w^i h_x w_x - (p_w^i)^2 h_y w_x + g_y(2p_y^i p_z^i + (p_z^i)^2 h_x - 2p_x^i p_w^i + 2p_z^i p_w^i h_y - \\
& (p_w^i)^2 h_x + (p_z^i)^2 w_x + (p_w^i)^2 w_x) - 2p_y^i p_z^i w_y - (p_z^i)^2 h_x w_y + 2p_x^i p_w^i w_y - 2p_z^i p_w^i h_y w_y + \\
& (p_w^i)^2 h_x w_y + g_x(2p_x^i p_z^i - (p_z^i)^2 h_y + 2p_y^i p_w^i + 2p_z^i p_w^i h_x + (p_w^i)^2 h_y - (p_z^i)^2 w_y - \\
& (p_w^i)^2 w_y)) \sin(\Delta\theta^i)) = 0, \quad i = 2, \ldots, 5. \qquad (29)
\end{aligned}$$

These equations are easily derived using symbolic manipulation software such as Mathematica. We use these equations to design the 3R chain to reach five task positions.

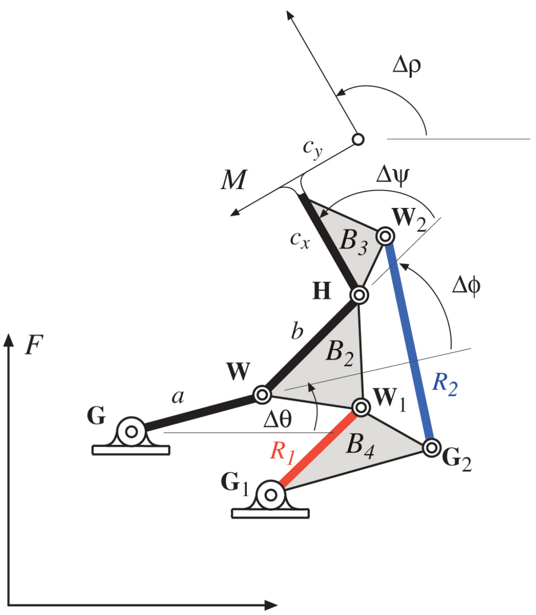

Figure 4: A 3R chain, **GWH**, constrained by two RR chains, $\mathbf{G}_1\mathbf{W}_1$ and $\mathbf{G}_2\mathbf{W}_2$, to form a Watt I linkage.

Table 1: Five task positions for the end-effector of the 3R chain.

Task	*Position Data* (ϕ, x, y)	$\Delta\theta_0$
1	$(70.09°, -460.72, -53.31)$	$0°$
2	$(11.38°, -357.22, 264.35)$	$-18°$
3	$(-53.41°, -65.30, 377.47)$	$-36°$
4	$(-110.19°, 137.99, 286.51)$	$-52°$
5	$(-174.91°, 220.27, 141.43)$	$-69°$

Example Synthesis of Constrained a 3R Chain: The Watt I

Our approach to the synthesis of a constrained 3R chain begins with the specification of a set of task positions for the end-effector, $[T_i], i = 1, \ldots, 5$. We use these positions to design the 3R chain which forms the base chain to which we attach two RR chains, see Figure 4. The Clifford algebra design equations (22) provide 12 equations in the 6 unknown coordinate $\mathbf{G}$, $\mathbf{W}$, and $\mathbf{H}$, and the 12 relative joint angles $\Delta\theta^i$, $\Delta\phi^i$, and $\Delta\psi^i$, $i = 2, 3, 4, 5$. Thus, these design equations have six parameters that the designer is free to specify. One choice is to specify $\mathbf{G}$, and the relative angles $\Delta\theta^i, i = 2, 3, 4, 5$. Using equation (28), we can eliminate the remaining joint angles, and obtain the equations (29), which can be solved for as many as four pairs of joints $\mathbf{W}$ and $\mathbf{H}$, to define the 3R chain.

Once the 3R chain is identified, the positions of its links B_1, B_2 and B_3 in each of the task positions can be determined by analysis of the chain. This means we can identify five positions $T_i^{B_2}, i = 1, \ldots, 5$ of the B_2 relative to the ground frame, which become the task positions for the design of an RR chain denoted $\mathbf{G}_1\mathbf{W}_1$ in Figure 4.

The addition of the RR chain $\mathbf{G}_1\mathbf{W}_1$ results in the addition of the link B_4, which takes the positions $[T_i^{B_4}], i = 2, \ldots, 5$, when the end-effector is in each of the specified task positions. We can now compute the relative positions $[S_i] = [T_i^{B_4}]^{-1}[T_i], i = 1, \ldots, 5$. The positions $[S_i]$ are now used as the task positions for the synthesis of the RR chain $\mathbf{G}_2\mathbf{W}_2$, which constrains the end-effector to the link B_4.

We used this synthesis methodology and the five task positions listed in Table 1 to compute the pivots of the Watt 1 six-bar linkage. We obtained four real solutions listed in Table 2. Of these design candidates, we found that only Design 1 passes through all five positions in the same assembly.

Analysis of a Constrained 3R Chain: The Watt I

In this section, we formulate the configuration analysis of a six-bar linkage to simulate its movement using complex number coordinates and the Dixon determinant as presented by Wampler (2001)(9). Our focus is on the Watt 1 six-bar, but the approach can be generalized to apply to all seven of the mechanically constrained 3R chains.

Consider the general Watt I linkage shown in Figure 5. Notice that we have introduced a coordinate frame for this analysis that the base pivot of the 3R chain, $\mathbf{G}$, as the origin and its x-axis is directed toward $\mathbf{G}_1$. We have renamed the pivots, the dimensions and angles to

Table 2: A Watt I linkage design.

Design	**G**	**W**	**H**	$\mathbf{G}_1$	$\mathbf{W}_1$
1−4	$(\mathbf{0},\mathbf{0})$	$(\mathbf{129.56},\mathbf{145.46})$	$(\mathbf{-235.36},\mathbf{-69.26})$	$(\mathbf{104.98},\mathbf{-65.52})$	$(\mathbf{45.73},\mathbf{37.46})$

Design	$\mathbf{G}_2$	$\mathbf{W}_2$
1	$(\mathbf{-36.52},\mathbf{5.08})$	$(\mathbf{-283.68},\mathbf{-56.47})$
2	$(-30.40, 106.48)$	$(-178.68, -161.06)$
3	$(45.73, 37.46)$	$(-235.36, -69.26)$
4	$(92.46, 38.29)$	$(-225.90, -58.15)$

facilitate the following derivation.

The complex vector loop equations

Using the notation in Figure 5, we formulate the vector equations of the loops formed by $\mathbf{C}_1\mathbf{C}_2\mathbf{C}_7\mathbf{C}_4$ and $\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3\mathbf{C}_6\mathbf{C}_5\mathbf{C}_4$, that is,

$$\begin{aligned}
\mathcal{F}_1 :&\quad l_1\cos\theta_1 + b_1\cos(\theta_2-\gamma) - b_2\cos(\theta_4+\eta) - l_0 = 0,\\
\mathcal{F}_2 :&\quad l_1\sin\theta_1 + b_1\sin(\theta_2-\gamma) - b_2\sin(\theta_4+\eta) = 0,\\
\mathcal{F}_3 :&\quad l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 - l_5\cos\theta_5 - l_0 = 0,\\
\mathcal{F}_4 :&\quad l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 - l_5\sin\theta_5 = 0.
\end{aligned} \tag{30}$$

The angle θ_1 is specified as the input to the six-bar linkage, thus these four equations $\mathcal{F}_i$ determine the joint angles $\theta_j, j = 2, 3, 4, 5$.

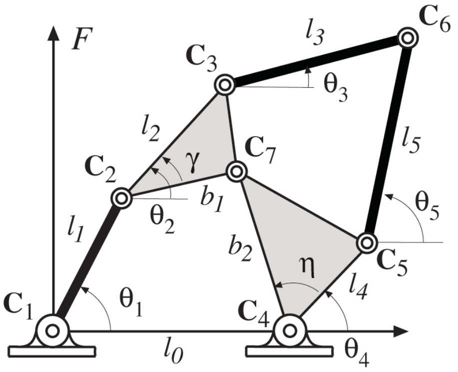

Figure 5: Six Bar Planar Linkage.

Now introduce the complex numbers $\Theta_j = e^{i\theta_j}$, where in this case $i^2 = -1$, so the four loop equations take the form of two complex loop equations,

$$\begin{aligned}
\mathcal{C}_1 :&\quad l_1\Theta_1 + b_1\Theta_2 e^{-i\gamma} - b_2\Theta_4 e^{i\eta} - l_0 = 0,\\
\mathcal{C}_2 :&\quad l_1\Theta_1 + l_2\Theta_2 + l_3\Theta_3 - l_4\Theta_4 - l_5\Theta_5 - l_0 = 0.
\end{aligned} \tag{31}$$

The complex conjugate of these two equations yields

$$\begin{aligned}
\mathcal{C}_1^\star: &\quad l_1\Theta_1^{-1} + b_1\Theta_2^{-1}e^{i\gamma_1} - b_2\Theta_4^{-1}e^{-i\eta_1} - l_0 = 0,\\
\mathcal{C}_2^\star: &\quad l_1\Theta_1^{-1} + l_2\Theta_2 + l_3\Theta_3^{-1} - l_4\Theta_4^{-1} - l_5\Theta_5^{-1} - l_0 = 0.
\end{aligned} \tag{32}$$

We solve these four equations for Θ_j, $j = 2, 3, 4$ using the Dixon determinant, (Wampler 2001(9)).

The Dixon determinant

We suppress Θ_3, so we have four complex equations in the three variables Θ_2, Θ_4 and Θ_5. We formulate the Dixon determinant by inserting each of the four functions $\mathcal{C}_1$, $\mathcal{C}_1^\star$, $\mathcal{C}_2$, $\mathcal{C}_2^\star$ as the first row, and then sequentially replacing the three variables by α_j in the remaining rows, to obtain,

$$\Delta(\mathcal{C}_1, \mathcal{C}_1^\star, \mathcal{C}_2, \mathcal{C}_2^\star) = \begin{vmatrix} \mathcal{C}_1(\Theta_2,\Theta_4,\Theta_5) & \mathcal{C}_1^\star(\Theta_2,\Theta_4,\Theta_5) & \mathcal{C}_2(\Theta_2,\Theta_4,\Theta_5) & \mathcal{C}_2^\star(\Theta_2,\Theta_4,\Theta_5)\\ \mathcal{C}_1(\alpha_2,\Theta_4,\Theta_5) & \mathcal{C}_1^\star(\alpha_2,\Theta_4,\Theta_5) & \mathcal{C}_2(\alpha_2,\Theta_4,\Theta_5) & \mathcal{C}_2^\star(\alpha_2,\Theta_4,\Theta_5)\\ \mathcal{C}_1(\alpha_2,\alpha_4,\Theta_5) & \mathcal{C}_1^\star(\alpha_2,\alpha_4,\Theta_5) & \mathcal{C}_2(\alpha_2,\alpha_4,\Theta_5) & \mathcal{C}_2^\star(\alpha_2,\alpha_4,\Theta_5)\\ \mathcal{C}_1(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_1^\star(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_2(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_2^\star(\alpha_2,\alpha_4,\alpha_5) \end{vmatrix}. \tag{33}$$

This determinant is zero when Θ_j, $j = 2, 4, 5$ satisfy the loop equations, because the elements of the first row become zero.

The structure of the determinant Δ can be studied in detail by noting that the complex equations for each loop k have the general form

$$\mathcal{C}_k: \; c_{k0} + c_{k3}x + \sum_{j=2,4,5} c_{k,j}\Theta_j, \quad \text{and} \quad \mathcal{C}_k^\star: \; c_{k0}^\star + c_{k3}^\star x^{-1} + \sum_{j=2,4,5} c_{k,j}^\star\Theta_j^{-1}, \tag{34}$$

where x denotes the suppressed variable Θ_3. Clearly, the equations maintain this form when α_j replaces Θ_j. Now row reduce Δ by subtracting the second row from the first row, then the third from the second, and the fourth from the third, to obtain,

$$\begin{vmatrix} c_{12}(\Theta_2-\alpha_2) & c_{12}^*(\Theta_2^{-1}-\alpha_2^{-1}) & c_{22}(\Theta_2-\alpha_2) & c_{22}^*(\Theta_2^{-1}-\alpha_2^{-1})\\ c_{14}(\Theta_4-\alpha_4) & c_{14}^*(\Theta_4^{-1}-\alpha_4^{-1}) & c_{24}(\Theta_4-\alpha_4) & c_{24}^*(\Theta_4^{-1}-\alpha_4^{-1})\\ c_{15}(\Theta_5-\alpha_5) & c_{15}^*(\Theta_5^{-1}-\alpha_5^{-1}) & c_{25}(\Theta_5-\alpha_5) & c_{25}^*(\Theta_5^{-1}-\alpha_5^{-1})\\ \mathcal{C}_1(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_1^\star(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_2(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_2^\star(\alpha_2,\alpha_4,\alpha_5) \end{vmatrix}. \tag{35}$$

Notice that because $\Theta_j - \alpha_j = -\Theta_j\alpha_j(\Theta_j^{-1} - \alpha_j^{-1})$, we have that for each value $\Theta_j = \alpha_j$ this determinant is zero. Divide out these extraneous roots in the form of the terms $(\Theta_j^{-1} - \alpha_j^{-1})$ to define the determinant

$$\delta = \frac{\Delta(\mathcal{C}_1, \mathcal{C}_1^\star, \mathcal{C}_2, \mathcal{C}_2^\star)}{(\Theta_2^{-1}-\alpha_2^{-1})(\Theta_4^{-1}-\alpha_4^{-1})(\Theta_5^{-1}-\alpha_5^{-1})}, \tag{36}$$

that is

$$\delta = \begin{vmatrix} -c_{12}\Theta_2\alpha_2 & c_{12}^* & -c_{22}\Theta_2\alpha_2 & c_{22}^*\\ -c_{14}\Theta_4\alpha_4 & c_{14}^* & -c_{24}\Theta_4\alpha_4 & c_{24}^*\\ -c_{15}\Theta_5\alpha_5 & c_{15}^* & -c_{25}\Theta_5\alpha_5 & c_{25}^*\\ \mathcal{C}_1(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_1^\star(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_2(\alpha_2,\alpha_4,\alpha_5) & \mathcal{C}_2^\star(\alpha_2,\alpha_4,\alpha_5) \end{vmatrix}. \tag{37}$$

Wampler shows that this determinant expands to form the polynomial with the form

$$\delta = \mathbf{a}^T[W]\mathbf{t} = 0, \tag{38}$$

where $\mathbf{a}$ and $\mathbf{t}$ are the vectors of monomials,

$$\mathbf{a} = \begin{Bmatrix} \alpha_2 \\ \alpha_4 \\ \alpha_5 \\ \alpha_4\alpha_5 \\ \alpha_2\alpha_5 \\ \alpha_2\alpha_4 \end{Bmatrix}, \quad \text{and} \quad \mathbf{t} = \begin{Bmatrix} \Theta_2 \\ \Theta_4 \\ \Theta_5 \\ \Theta_4\Theta_5 \\ \Theta_2\Theta_5 \\ \Theta_2\Theta_4 \end{Bmatrix}. \tag{39}$$

The 6×6 matrix $[W]$ is given by

$$[W] = \begin{bmatrix} D_1 x + D_2 & A^T \\ A & -(D_1^* x^{-1} + D_2^*) \end{bmatrix}, \tag{40}$$

The matrices D_1 and D_2 are 3×3 diagonal matrices, given by

$$D_1 = \begin{bmatrix} b_1 b_3 l_3 l_5 e^{-i(\gamma+\eta)} & 0 & 0 \\ 0 & -b_1 b_3 l_3 l_5 e^{i(\gamma+\eta)} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

where

$$\begin{aligned} d_1 =& (l_1\Theta_1 - l_0)(b_1 b_3 l_5 e^{-i(\gamma_1 \eta_1)} - b_3 l_2 l_5 e^{-i\eta_1}) \\ d_2 =& (l_0 - l_1\Theta_1)(b_1 b_3 l_5 e^{i(\gamma_1+\eta_1)} - b_1 l_4 l_5 e^{i\gamma_1}) \\ d_3 =& (l_1\Theta_1 - l_0)(b_3 l_2 l_5 e^{-i\eta_1} - b_1 l_4 l_5 e^{i\gamma_1}). \end{aligned} \tag{41}$$

The 3×3 matrix $[A]$ is given by

$$[A] = \begin{bmatrix} 0 & b_1 b_3 l_5^2 e^{i(\gamma+\eta)} & -b_3^2 l_2 l_5 + b_1 b_3 l_4 l_5 e^{i(\gamma+\eta)} \\ -b_1 b_3 l_5^2 e^{-i(\gamma+\eta)} & 0 & b_1 b_3 l_4 l_5 e^{-i(\gamma+\eta)} - b_1^2 l_4 l_5 \\ b_3^2 l_2 l_5 - b_1 b_3 l_4 l_5 e^{-i(\gamma+\eta)} & -b_1 b_3 l_4 l_5 e^{i(\gamma+\eta)} + b_1^2 l_4 l_5 & 0 \end{bmatrix}. \tag{42}$$

Solving the loop equations

A set of values Θ_j that satisfy the loop equations (32) will also yield $\delta = 0$, which will be true for arbitrary values of the auxiliary variables α_j. Thus, solutions for these loop equations must also satisfy the matrix equation,

$$[W]\mathbf{t} = 0. \tag{43}$$

The matrix $[W]$ is a square, therefore this equation has solutions only if $\det[W] = 0$. Expanding this determinant we obtain a polynomial in $x = \Theta_3$.

The structure of $[W]$ yields,

$$[W]\mathbf{t} = \left[\begin{pmatrix} D_1 & 0 \\ A & -D_2^* \end{pmatrix} x - \begin{pmatrix} -D_2 & -A^T \\ 0 & D_1^* \end{pmatrix}\right]\mathbf{t} = [Mx - N]\mathbf{t} = 0 \tag{44}$$

Notice that the values of x that result in $\det[W] = 0$ are also the eigenvalues of the characteristic polynomial $p(x) = \det(Mx - N)$ of the generalized eigenvalue problem

$$N\mathbf{t} = xM\mathbf{t}. \tag{45}$$

Each value of $x = \Theta_3$ has an associated eigenvector $\mathbf{t}$ which yields the values of the remaining joint angles $\Theta_j, j = 2, 4, 5$.

It is useful to notice that each eigenvector $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5, t_6)^T$ is defined only up to a constant multiple, say μ. Therefore, it is convenient to determine the values Θ_j, by the computing the ratios,

$$\Theta_2 = \frac{t_5}{t_3} = \frac{\mu\Theta_2\Theta_5}{\mu\Theta_5}, \quad \Theta_4 = \frac{t_6}{t_1} = \frac{\mu\Theta_2\Theta_4}{\mu\Theta_2}, \quad \Theta_5 = \frac{t_4}{t_2} = \frac{\mu\Theta_4\Theta_5}{\mu\Theta_4}. \tag{46}$$

Conclusions

Our formulation of the synthesis of a mechanically constrained 3R chain designs two RR chains to reach five task positions, each of which can have as many as four solutions. If we assume the 3R chain has already been sized to reach the task positions, then we would expect at most four designs for each of the two RR chains, or 16 six-bar design candidates. However, in the case of the Watt I, Stephenson IIb, Stephenson IIIa, one RR synthesis problem results in an existing link, therefore these cases have at most 12 design candidates. For the Stephenson IIIb, both RR synthesis problems result in existing links, which means for this case there are six candidate designs.

In our numerical example we obtain four real designs for the Watt I system. The analysis of this six-bar linkage shows that it can have as many as four assemblies, one for each eigenvalue. We found that only one of the design candidates had all five task positions on the same assembly, and therefore moves smoothly through the task.

The design of mechanically constrained planar 3R chains is more complex than the design of planar four-bar linkages. However, this design process introduces the freedom to specify the 3R chain and its movement, as well as a number of ways in which to attach the two RR chains. In our experience, this provides the designer an opportunity to find successful designs, when a four-bar linkage is not satisfactory.

References

[1] Krovi, V., Ananthasuresh, G. K., and Kumar, V., 2002, "Kinematic and Kinetostatic Synthesis of Planar Coupled Serial Chain Mechanisms," *ASME Journal of Mechanical Design*, 124(2):301-312.

[2] Lin, C-S, and Erdman, A. G., 1987, "Dimensional Synthesis of Planar Triads for Six Positions," *Mechanism and Machine Theory*, 22:411-419.

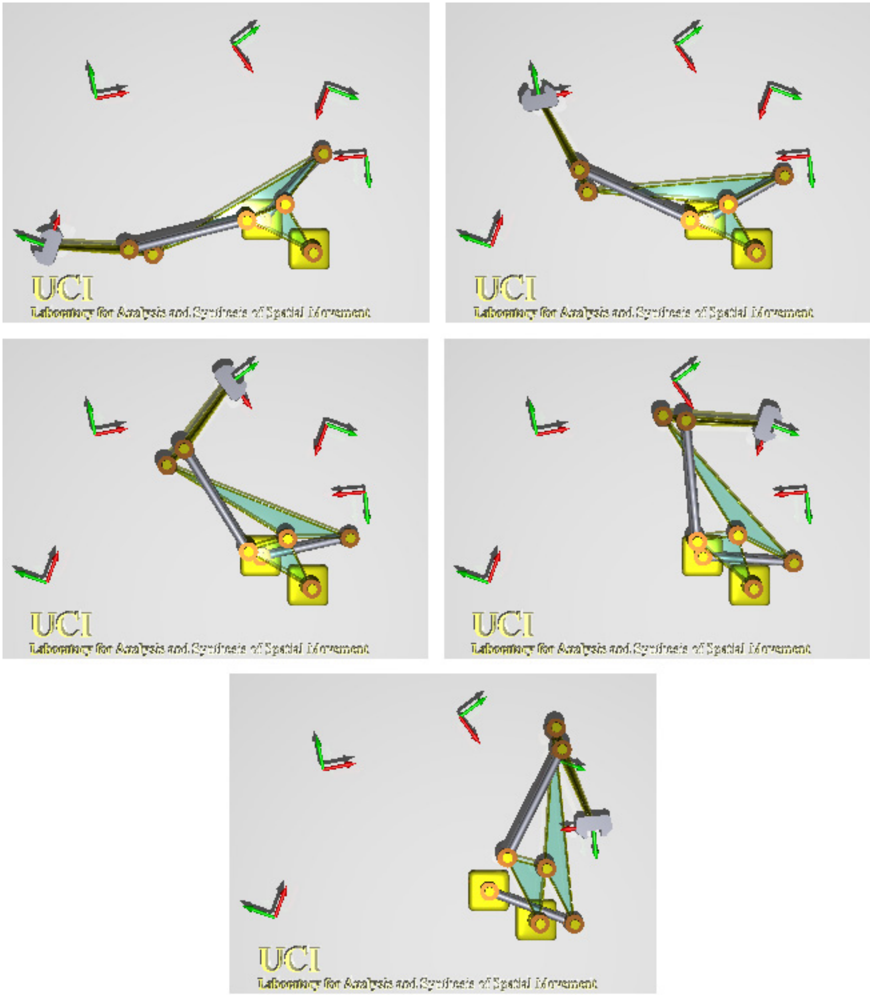

Figure 6: The example solution for a Watt I linkage reaching each of the five task positions.

[3] Chase, T. R., Erdman, A. G., and Riley, D. R., 1987, "Triad Synthesis for up to 5 Design Positions with Applications to the Design of Arbitrary Planar Mechanisms," *ASME Journal of Mechanisms, Transmissions and Automation in Design*, 109(4):426-434.

[4] Subbian, T., and Flugrad, D. R., 1994, "6 and 7 Position Triad Synthesis using Continuation Methods," *Journal of Mechanical Design*, 116(2):660-665.

[5] Perez, A., and McCarthy, J. M., 2005, "Clifford Algebra Exponentials and Planar Linkage Synthesis Equations," *ASME Journal of Mechanical Design*, 127(5):931-940, September.

[6] A. Perez and J. M. McCarthy 2004, Dual Quaternion Synthesis of Constrained Robotic Systems, *ASME Journal of Mechanical Design*, 126: 425-435

[7] K.J. Waldron, and G.L. Kinzel, *Kinematics, Dynamics and Design of Machinery-Second Edition*, John & Wiley Inc, 2004

[8] L.W. Tsai, *Enumeration of Kinematic Structures According to Function*, CRC Press, 2001

[9] C. W. Wampler, "Solving the Kinematics of Planar Mechanisms by Dixon Determinant and a Complex Plane Formulation", *ASME Journal of Mechanical Design*, 123(3), pp. 382-387

[10] J. M. McCarthy, *An Introduction to Theoretical Kinematics*, MIT Press, Cambridge, MA, 1990.

[11] Lee, E., and Mavroidis, D.,2002, "Solving the Geometric Design Problem of Spatial 3R Robot Manipulators Using Polynomial Homotopy Continuation," *ASME Journal of Mechanical Design*, 124(4), pp.652-661

[12] McCarthy, J.M., 1993, "Dual Quaternions and the Pole Triangle", *Modern Kinematics. Developments in the last 40 years.*, Arthur G. Erdman, editor. John Wiley and Sons, New York.

[13] Bottema, O., and Roth, B., 1979, *Theoretical Kinematics*, North Holland Press, NY.

[14] DeSa, S., and Roth, B., 1981, "Kinematic Mappings. Part2: Rational Algebraic Motions in the Plane", *ASME Journal of Mechanical Design*, **103**:712-717.

[15] Ravani, B., and Roth, B., 1983, "Motion Synthesis Using Kinematic Mapping," *ASME J. of Mechanisms, Transmissions, and Automation in Design*, 105(3):460–467.

[16] H. P. Schrocker, M. L. Husty and J. M. McCarthy, 2005, "Kinematic Mapping-based Evaluation of Assembly Modes for Planar Four-bar Synthesis," *Proceedings of the ASME International Design Engineering Technical Conferences*, Long Beach, CA, September 25-28.

[17] K. Brunnthaler, M. Pfurner, and M. Husty, 2005, "Synthesis of planar four-bar mechanisms," *Proceedings of MUSME 2005, the International Symposium on Multibody Systems and Mechatronics*, Uberlandia, Brazil, March 6-9.

[18] Sandor, G. N., and Erdman, A. G., 1984, *Advanced Mechanism Design: Analysis and Synthesis, Vol. 2.* Prentice-Hall, Englewood Cliffs, NJ.

[19] McCarthy, J. M., 2000, *Geometric Design of Linkages*, Springer-Verlag, New York.