# Dimensional Synthesis of Six-Bar Linkage as a Constrained RPR Chain

**M. Plecnik and J.M. McCarthy**

**Abstract** In this paper, five positions of a planar RPR serial chain are specified and the synthesis equations for two RR constraints are solved to obtain a six-bar linkage. Analysis of the resulting linkage determines if it moves the end-effector smoothly through the five task positions without a branch defect. The design procedure presented randomly selects variations to the positions of the RPR chain in order to obtain new six-bar linkages. This dimensional synthesis algorithm yields a set of six-bar linkages that move the end-effector near the original task positions. This synthesis procedure is applied to the design of a linkage that generates a square pattern. The procedure yielded 122 defect-free linkages for one million iterations.

**Keywords** Linkage synthesis • Six-bar linkage synthesis • RPR chain • Defect-free synthesis

## 1 Introduction

This paper presents a dimensional synthesis procedure for a Watt I six-bar linkage that includes a prismatic joint. The designer specifies a planar RPR chain that reaches five specified task positions—R refers to a revolute joint and P to a prismatic joint. The task positions are defined in terms of the translation of the origin to a point **P** and the orientation $\phi$ of a moving reference frame $M$ relative to a fixed frame $F$. Two RR constraints are computed in order to guide the chain through the five task positions, however, a linkage can fail to move smoothly through the task positions due to what is called a branch defect. The procedure presented in this paper finds usable linkages that do not have this branch defect by searching within tolerance

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zones around the specified task positions. The design procedure is illustrated by the design of a RPR Watt I linkage that generates a square pattern for a screw insertion device.

## 2 Literature Review

The design of a six-bar linkage that includes an RPR chain was introduced by Bagci and Burke [1]. A strategy similar to our approach is found in Gatti and Mundo [4] who constrain a three degree-of-freedom planar six-bar chain using cams. Our approach follows Soh and McCarthy [11], who constrain a 3R chain.

Kinzel et al. [6] use computer-aided design based geometric constraint solvers to obtain a Stephenson III six-bar linkage, and Shiakolas et al. [10] use an evolutionary optimization algorithm. Our design equations are based on the synthesis of RR chains introduced by Burmester theory [3], also see McCarthy and Soh (2010), [7]. The design of four-bar linkages that do not branch is called solution rectification, Bawab et al. [2]. Plecnik and McCarthy [9] use a random search to find non-branching slider-crank function generators. Branching analyses for Watt I six-bars have been studied by Mirth and Chase [8] and more recently by Watanabe and Katoh [13] and by Ting et al. [12].

This paper introduces the combination of Burmester five-position synthesis to constrain an RPR chain with randomized search of tolerance zones in order to obtain usable six-bar linkages.

## 3 RPR Specification

The first step in the design procedure is to specify an RPR chain that reaches the five specified task positions

$$[T_i] = [T(\phi_i, x_i, y_i)] \quad i = 1, \ldots, 5, \tag{1}$$

where $[T]$ is a $3\times3$ planar homogeneous matrix capable of transforming coordinates to a fixed frame $F$ from a frame displaced by $\mathbf{P} = (x, y)$ and rotated at $\phi$. The specified parameters of the RPR chain are $\mathbf{O}$, $p$, and $\eta$ as shown in Fig. 1. Point $\mathbf{O} = (O_x, O_y)$ locates the fixed pivot, $p$ is the length from the slider to the end effector, and $\eta$ is the angular offset of the end effector frame from the **PA** frame.

The position $(x, y)$ and the orientation $\phi$ of the end effector is described by the following equations,

$$\begin{Bmatrix} s\cos\theta \\ s\sin\theta \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} - \begin{Bmatrix} O_x \\ O_y \end{Bmatrix} - \begin{Bmatrix} p\cos(\theta+\zeta) \\ p\sin(\theta+\zeta) \end{Bmatrix} \tag{2}$$

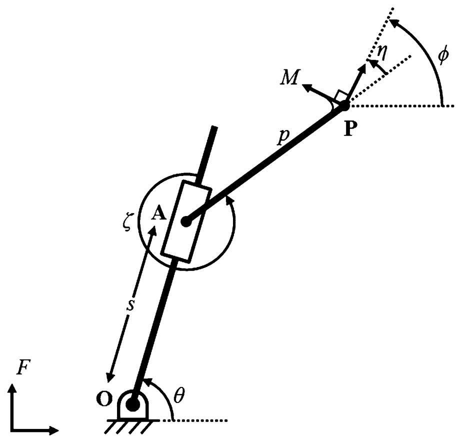


**Fig. 1** A three DOF open loop RPR chain

$$\phi = \theta + \zeta + \eta \mod 2\pi, \tag{3}$$

where joint parameters $s$, $\theta$, $\zeta$ are shown in Fig. 1. In order to find the inverse kinematics solution for $s$, $\theta$, and $\zeta$, Eq. (3) is substituted into Eq. (2) and the magnitude of the resulting vector equation is computed to obtain

$$s = \pm\sqrt{(x - O_x - p\cos(\theta - \eta))^2 + (y - O_y - p\sin(\theta - \eta))^2}. \tag{4}$$

The angle $\theta$ is computed by dividing the $y$-component of (2) by its $x$-component and applying an arctan function,

$$\theta = \arctan\left(\frac{y - O_y - p\sin(\theta - \eta)}{x - O_x - p\cos(\theta - \eta)}\right) \tag{5}$$

The computation of the final joint parameter $\zeta$ is straightforward from Eq. 3. Note that the positive and negative values of (4) correspond to two solutions of $\{s, \theta, \zeta\}$ which describe the same configuration of the RPR chain. Therefore only the positive value of $s$ will be considered.

## 4 Synthesis of the First RR Constraint

The inverse kinematics procedure (Sect. 3) is applied for all task positions to obtain $s_i$ and $\theta_i$ for $i = 1, \ldots, 5$. Next the slider link is connected to ground by a RR chain so that the resulting inverted slider-crank loop is capable of achieving each set of joint parameters $\{s_i, \theta_i\}$. This is accomplished by first defining the sliding point

$$\mathbf{A}_i = \begin{Bmatrix} O_x + s_i\cos\theta_i \\ O_y + s_i\sin\theta_i \end{Bmatrix} \tag{6}$$

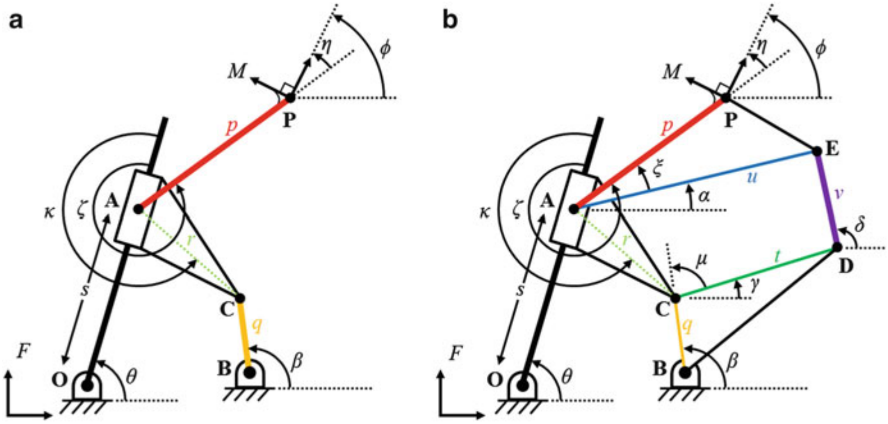


**Fig. 2** (**a**) First RR constraint is added, (**b**) Second RR constraint is added

which is used to create the transformations $[T(\theta_i, \mathbf{A}_i)]$ for point **C** (Fig. 2) from coordinates in link frame **CA** to coordinates in $F$. Then the relative transformations of **C** from its configuration in the first task position to the other four task positions can be defined as

$$[D_{1i}] = [T(\theta_i, \mathbf{A}_i)][T(\theta_1, \mathbf{A}_1)]^{-1} \quad i = 2, \ldots, 5. \tag{7}$$

Therefore the positions of **C** associated with the five task positions can be written as

$$\mathbf{C}_i = [D_{1i}]\mathbf{C}_1. \tag{8}$$

Next, the unknown ground point **B** is defined in $F$ as shown in Fig. 2. The constraint equations for an RR link that connects point **B** to **C** for all task positions is given by

$$([D_{1i}]\mathbf{C}_1 - \mathbf{B}) \cdot ([D_{1i}]\mathbf{C}_1 - \mathbf{B}) = q^2. \tag{9}$$

where $q$ is the constant length of link **CB**.

These design equations can be simplified by cancelling $q^2$ to obtain four bilinear equations in the four unknowns $B_x$, $B_y$, $C_{x1}$, and $C_{y1}$. The solution of the design equations is known to yield as many as four real solutions [5]. One of these solutions will be the specified RP chain with fixed pivot $\mathbf{B} = \mathbf{O}$ and a moving pivot $\mathbf{C}_1$ at infinity. Therefore this design procedure will yield one or three additional real RR chains.

The pivots **O**, **A**, **C**, and **B** form an inverted slider-crank loop. The pivots **A** and **C** are attached to the slider and **O** and **B** are attached to ground. The vector $\mathbf{C} - \mathbf{A}$ has magnitude $r$ and the constant angle $\kappa$ measured from the vector $\mathbf{A} - \mathbf{O}$.

The input-output equation for an inverted slider-crank can be used to evaluate whether the synthesized RR constraint has a branch defect. The formulation of the input-output equation of an inverted slider-crank is similar to the RRRP linkage as found in [7]. For a given input $\theta$ this equation has two solutions, $\{s, \beta\}^+$ and $\{s, \beta\}^-$. These are assembled into the two sets

$$\sigma^+ = \{\{\theta_1, s_1^+, \beta_1^+\}, \dots, \{\theta_i, s_i^+, \beta_i^+\}, \dots, \{\theta_5, s_5^+, \beta_5^+\}\},$$
$$\sigma^- = \{\{\theta_1, s_1^-, \beta_1^-\}, \dots, \{\theta_i, s_i^-, \beta_i^-\}, \dots, \{\theta_5, s_5^-, \beta_5^-\}\}, \tag{10}$$

which are the two branches of the linkage. A linkage is usable if the configurations obtained for each of the task positions are on the same branch. A linkage that does not have the task positions on the same branch is not useful.

This procedure determines the values of $\{\theta_i, s_i, \beta_i\}$, $i = 1, \dots, 5$ and compares the results to $\sigma^+$ and $\sigma^-$. If all of the configurations are on one branch or the other, the linkage is considered to be usable. This procedure can fail due to the complexity of six-bar linkage coupler curves [8], therefore a final visual confirmation is necessary.

## 5 Synthesis of the Second RR Constraint

The synthesis and analysis procedures of Sect. 4 result in 0–3 partial six-bar assemblies passed on to the second RR constraint procedure. The second procedure parallels the first in that 0–3 complete six-bar mechanisms will result from each partial six-bar, allowing for a maximum synthesis of nine six-bar mechanisms.

The synthesis procedure for the second RR chain computes the points **D** in the link **CB** frame (Fig. 2) and **E** in the link **PA** frame. The relative transformations from the configuration of the first task position to all other task positions that define **D** and **E** are given by

$$[G_{1i}] = [T(\beta_i, \mathbf{B})][T(\beta_1, \mathbf{B})]^{-1} \tag{11}$$

$$[H_{1i}] = [T(\phi_i, x_i, y_i)][T(\phi_1, x_1, y_1)]^{-1} \tag{12}$$

respectively, so that locations $\mathbf{D}_i$ and $\mathbf{E}_i$ in $F$ are

$$\mathbf{D}_i = [G_{1i}]\mathbf{D}_1 \tag{13}$$

$$\mathbf{E}_i = [H_{1i}]\mathbf{E}_1 \quad i = 2, \dots, 5. \tag{14}$$

The constraint equation for an RR link that connects points **D** and **E** is given by

$$([G_{1i}]\mathbf{D}_1 - [H_{1i}]\mathbf{E}_1) \cdot ([G_{1i}]\mathbf{D}_1 - [H_{1i}]\mathbf{E}_1) = v^2, \tag{15}$$

where $v$ is the constant length of the link **ED**.

Equation 15 has the same structure as Eq. 9 forming four bilinear equations in four unknowns $D_{x1}$, $D_{y1}$, $E_{x1}$, and $E_{y1}$. Similar to the first RR case, four solutions are found, two of which may be imaginary and one that reproduces the link **CA**, that is $\mathbf{D}_1 = \mathbf{C}_1$ and $\mathbf{E}_1 = \mathbf{A}_1$. Therefore, either one or three links exist that connect link **CB** to the link **PA** for the given task positions. The moving pivot **D** is located on the link **CB** at a distance $t$ from the moving pivot **C** and at the angle $\mu$ relative to the vector $\mathbf{C} - \mathbf{B}$ . **E** is located on link **PA** by length $u$ from **A** and angular offset $\xi$ from $\mathbf{P} - \mathbf{A}$. Points **C**, **D**, **E**, and **A** form a 4R loop.

Using a process similar to the inverted slider-crank loop analysis presented in Sect. 4, the input-output equation of a 4R loop can be used to find branch defects. Formulation of the input-output equation for a 4R linkage is outlined in [7]. For a given input $\gamma$ this equation has two solutions, $\{\alpha, \delta\}^+$ and $\{\alpha, \delta\}^-$. These are assembled into the sets

$$\begin{aligned}\tau^+ &= \{\{\gamma_1, \alpha_1^+, \delta_1^+\}, \ldots, \{\gamma_i, \alpha_i^+, \delta_i^+\}, \ldots, \{\gamma_5, \alpha_5^+, \delta_5^+\}\},\\ \tau^- &= \{\{\gamma_1, \alpha_1^-, \delta_1^-\}, \ldots, \{\gamma_i, \alpha_i^-, \delta_i^-\}, \ldots, \{\gamma_5, \alpha_5^-, \delta_5^-\}\}.\end{aligned} \tag{16}$$

to form the two solution branches. The values of $\{\gamma_i, \alpha_i, \delta_i\}$, $i = 1, \ldots, 5$ are determined and compared to $\tau^+$ and $\tau^-$. If all configurations are on a single branch, then the linkage is considered usable. Note that this condition does not guarantee the mechanism will not travel through a singular configuration in between task positions.

## 6 Application of Design Methodology

This synthesis procedure begins with five task positions $[T_i]$ for $i = 1, \ldots, 5$ specified by the designer. Dimensional synthesis computes as many as nine design candidates. If the evaluation of branching for these candidate linkages fails, then the task positions are adjusted within specified tolerance zones. The adjustment process randomly selects values for $\phi_i$, $x_i$, and $y_i$, $i = 1, \ldots, 5$ that lie within regions defined by the designer. The synthesis procedure is repeated for the new task positions $[T_i]$.

This procedure was applied to the design of an RPR six-bar linkage that guides a screw insertion device. The goal is to position the end effector over four holes equally spaced on a 5.08 cm diameter bolt hole circle, stopping over each hole to insert a screw. The four holes are to be visited by the end effector in a star pattern, see Fig. 3.

The specified task positions and tolerance zones for this design problem are listed in Table 1. The dimensions of the RPR chain, **O**, $p$, $\eta$, are also listed in Table 1. Notice that tolerances were applied to the dimensions of the RPR linkage as well.

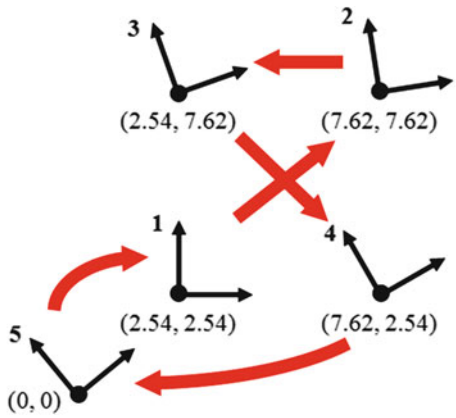


**Fig. 3** Original task positions with position in cm and orientation found in Table 1

**Table 1** Task position tolerance zones and RPR parameter tolerance zones

| Task positions, $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\phi_i$ | $0° \pm 10°$ | $10° \pm 10°$ | $20° \pm 10°$ | $30° \pm 10°$ | $40° \pm 10°$ |
| $x_i$ (cm) | $2.54 \pm 0$ | $7.62 \pm 0$ | $2.54 \pm 0$ | $7.62 \pm 0$ | $0 \pm 25.4$ |
| $y_i$ (cm) | $2.54 \pm 0$ | $7.62 \pm 0$ | $7.62 \pm 0$ | $2.54 \pm 0$ | $0 \pm 25.4$ |
| $O_x$ (cm) | $0 \pm 25.4$ | | | | |
| $O_y$ (cm) | $0 \pm 25.4$ | | | | |
| $p$ (cm) | $0 \pm 12.7$ | | | | |
| $\eta$ | $0 \pm 180°$ | | | | |

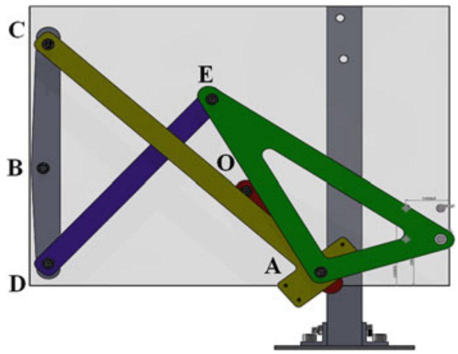


**Fig. 4** Useful screw insertion linkage design

This design procedure was executed for one million iterations and yielded 122 useful linkages including the design shown in Fig. 4. The computation was performed on a 3.01 GHz AMD Phenom(tm) II X4 945 processor. The run-time for this Mathematica program was approximately 0.11 s/iteration, or 30 h of computation.

## 7 Conclusions

This paper presents a synthesis procedure for six-bar mechanisms as constrained RPR chains. Analysis of the six-bar linkage identifies branch defects. By searching random variations within tolerances around a nominal task, a number of usable designs are obtained. This procedure does not guarantee the elimination of linkage designs that branch, but the authors' experience are that they rarely appear in the results of the algorithm. An example synthesis of an RPR six-bar linkage to guide a screw insertion device through a square pattern yielded 122 usable six-bar linkage designs in a search of one million tasks.

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## References

1. Bagci, C., Burke, D.: Optimum synthesis of coupler curve and uniform rotary motion driven multiloop mechanisms generating complex output motions. J. Mech. Des. **115**, 967–977 (1993)
2. Bawab, S., Kinzel, G.L., Waldron, K.J.: Rectified synthesis of six-bar mechanisms with well-defined transmission angles for four-position motion generation. J. Mech. Des. **118**, 377–384 (1996)
3. Burmester, L.: Lehrbuch der Kinematik. Felix, Leipzig (1886)
4. Gatti, G., Mundo, D.: Optimal synthesis of six-bar cammed-linkages for exact rigid-body guidance. Mech. Mach. Theory **42**, 1069–1081 (2007)
5. Hartenberg, R.S., Denavit, J.: Kinematic Synthesis of Linkages. McGraw-Hill, New York (1964)
6. Kinzel, E.C., Schmiedeler, J.P., Pennock, G.R.: Function generation with finitely separated precision points using geometric constraint programming. J. Mech. Des. **129**, 1185 (2007)
7. McCarthy, J.M., Soh, G.S.: Geometric Design of Linkages. Springer, New York (2010)
8. Mirth, J.A., Chase, T.R.: Circuit analysis of watt chain six-bar mechanisms. J. Mech. Des. **115**, 214–223 (1993)
9. Plecnik, M.M., McCarthy, J.M.: Five position synthesis of a slider-crank function generator. In: Proceedings of the 2011 International Design Engineering Technical Conferences DETC2011-47581, Washington, DC (2011)
10. Shiakolas, P.S., Koladiya, D., Kebrle, J.: On the optimum synthesis of six-bar linkages using differential evolution and the geometric centroid of precision positions technique. Mech. Mach. Theory **40**, 319–335 (2005)
11. Soh, G.S., McCarthy, J.M.: The synthesis of six-bar linkages as constrained planar 3R chains. Mech. Mach. Theory **43**, 160–170 (2008)
12. Ting, K., Xue, C., Wang, J., Currie, K.R.: Stretch rotation and complete mobility identification of Watt six-bar chains. Mech. Mach. Theory **44**, 1877–1886 (2009)
13. Watanabe, K., Katoh, H.: Identification of motion domains of planar six-link mechanisms of the Stephenson-type. Mech. Mach. Theory **39**, 1081–1099 (2004)