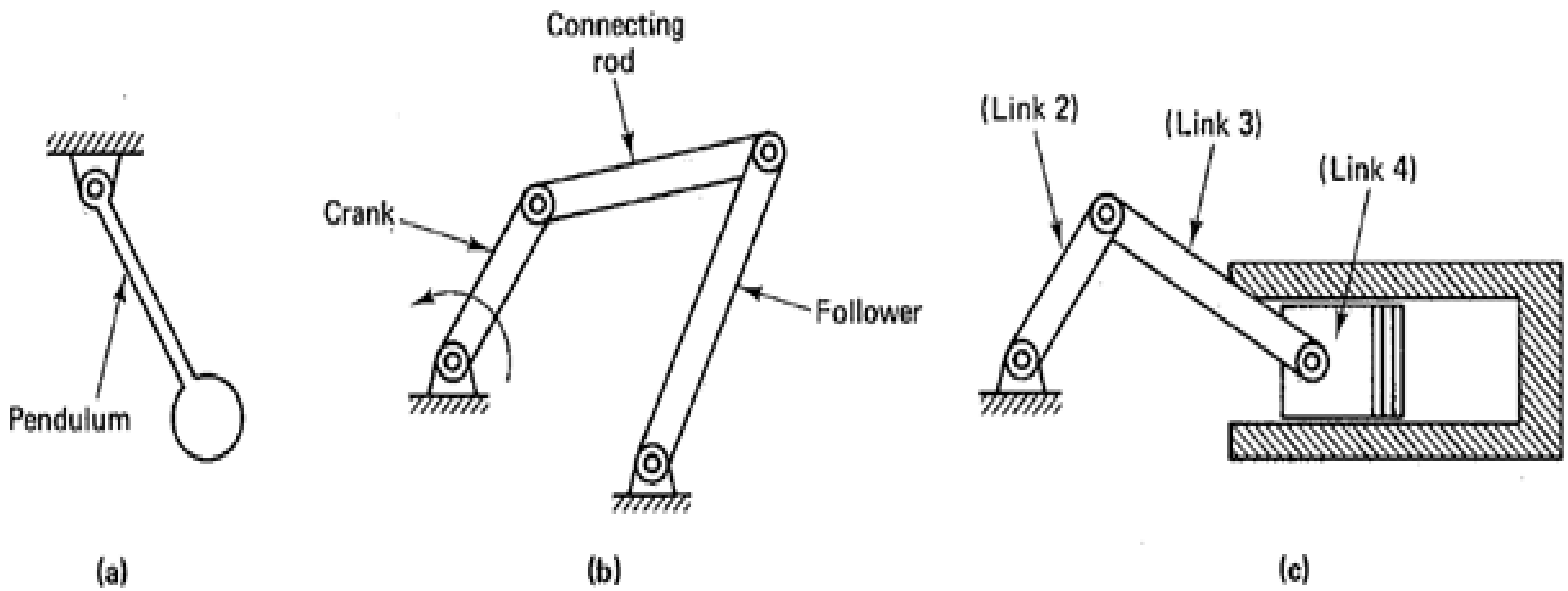
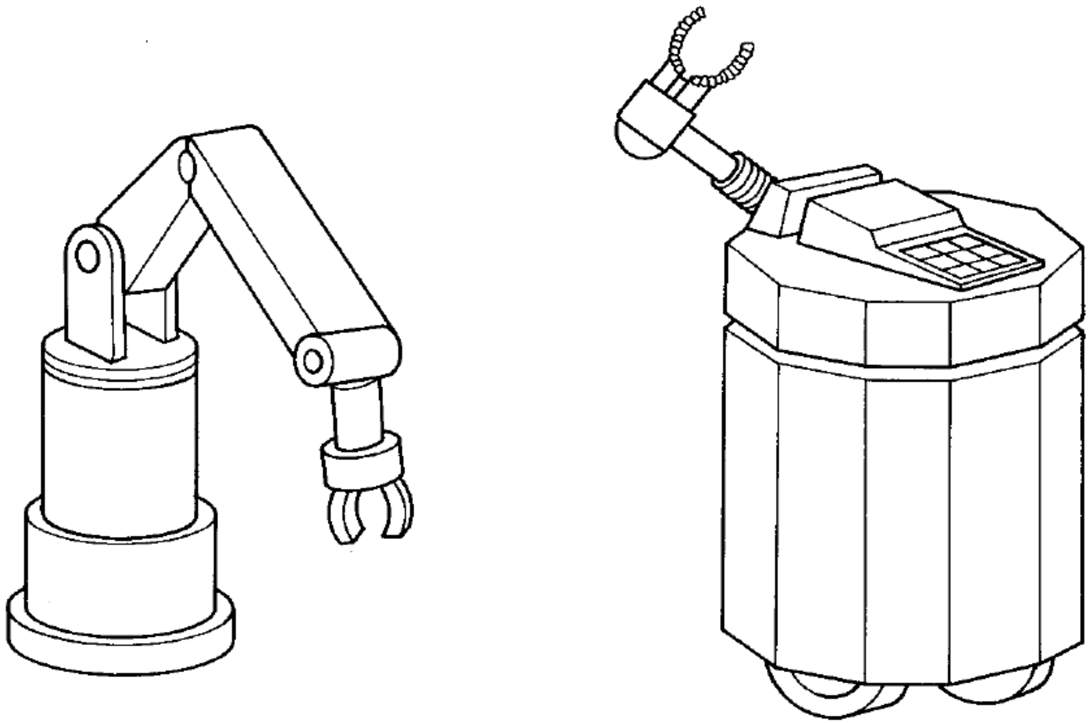
PART 3
Multibody Kinematics And Dynamics
Chapter 1 Multibody Mechanical Systems

2D Planar Mechanism



3D Spatial Mechanism



(a) (b)

1.2 Computer-Aided Design (CAD)

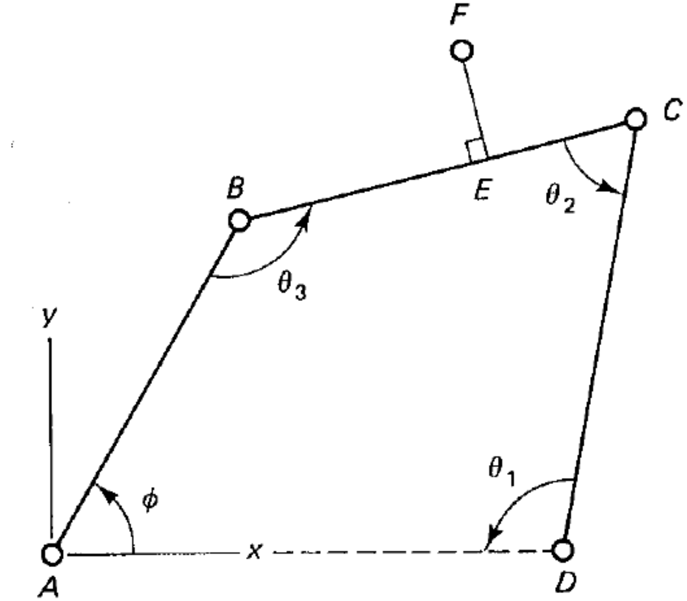
- Mechanics: Statics and dynamics.
- Dynamics : kinematics and kinetics.

- Kinematics is the study of motion, i.e., the study of displacement, velocity, and acceleration, regardless of the forces that produce the motion.

- Kinetics is the study of motion and its relationship with the forces that produce that motion.

1.3 Coordinate Systems

A four-bar mechanism with generalized coordinates.



4 Coordinate

$$\boldsymbol{q} = [\theta_1 \;\; \theta_2 \;\; \theta_3 \;\; \phi]^T$$

3 Constraints

$$(r^2 + l^2 + s^2 - d^2) - 2rl\cos\phi + 2ls\cos\theta_1 - 2rs\cos(\phi - \theta_1) = 0 \quad (1.1)$$

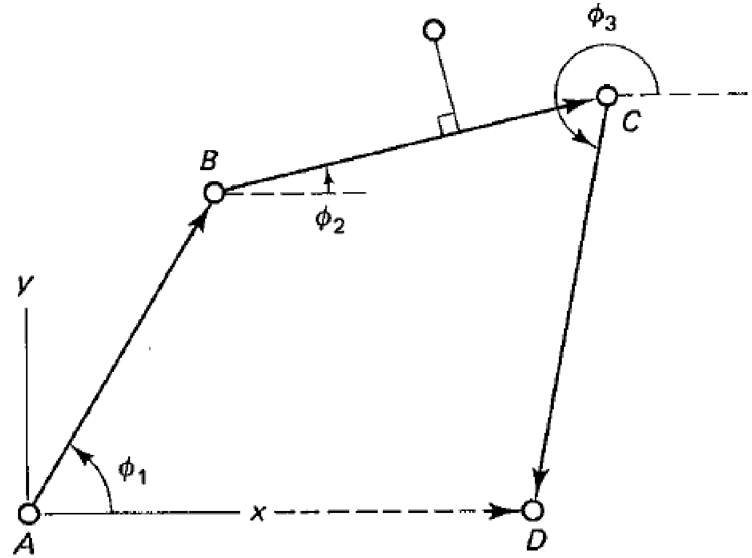
$$(r^2 + l^2 + s^2 - d^2) - 2rl\cos\phi + 2ds\cos\theta_2 = 0 \quad (1.2)$$

$$\phi + \theta_1 + \theta_2 + \theta_3 - 2\pi = 0 \quad (1.3)$$

degrees of freedom 4 – 3 = 1

3 coordinates

$$\boldsymbol{q} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$



2 constraints

$$r\cos\phi_1 + d\cos\phi_2 + s\cos\phi_3 - l = 0$$

$$r\sin\phi_1 + d\sin\phi_2 + s\sin\phi_3 = 0$$

$$\text{dof} = 3 - 2 = 1$$

Cartesian Coordinates

12 coordinates and 11 kinematic constraints

$$\boldsymbol{q} = \begin{bmatrix} x_1 & y_1 & \phi_1 & x_2 & y_2 & \phi_2 & x_3 & y_3 & \phi_3 \end{bmatrix}^T$$

$$x_1 - \frac{r}{2}\cos\phi_1 = 0$$

$$y_1 - \frac{r}{2}\sin\phi_1 = 0$$

$$x_1 + \frac{r}{2}\cos\phi_1 - x_2 + \frac{d}{2}\cos\phi_2 = 0$$

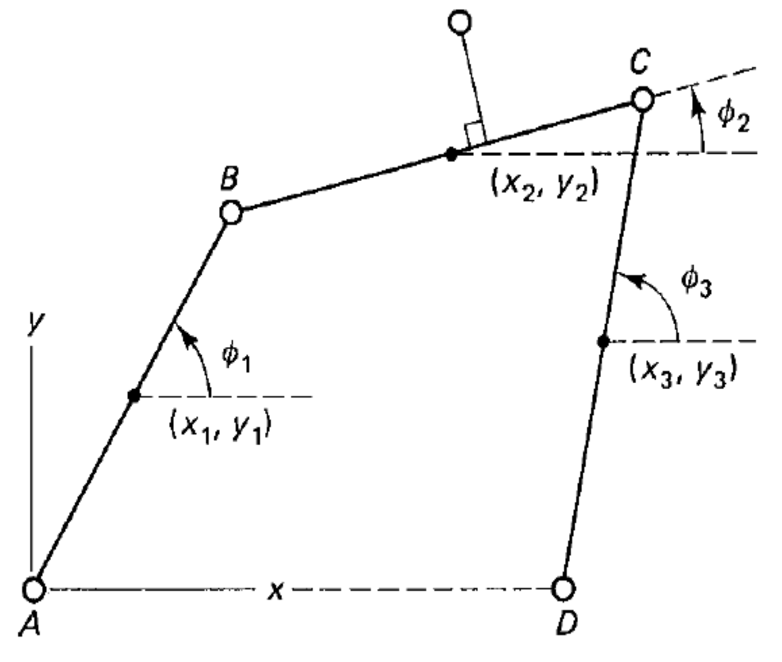
$$y_1 + \frac{r}{2}\sin\phi_1 - y_2 - \frac{d}{2}\sin\phi_2 = 0$$

$$x_2 + \frac{d}{2}\cos\phi_2 - x_3 - \frac{s}{2}\cos\phi_3 = 0$$

$$y_2 + \frac{d}{2}\sin\phi_2 - y_3 - \frac{s}{2}\sin\phi_3 = 0$$

$$x_3 - \frac{s}{2}\cos\phi_3 - l = 0$$

$$y_3 - \frac{s}{2}\sin\phi_3 = 0$$



$$\text{dof} = 9 - 8 = 1$$

Coordinates Systems

	Generalized coordinates	Relative coordinates	Cartesian coordinates
Number of coordinates	Minimum	Moderate	Large
Number of second-order differential equations	Minimum	Moderate	Large
Number of algebraic constraint equations	None	Moderate	Large
Order of nonlinearity	High	Moderate	Low
Derivation of the equations of motion	Hard	Moderate hard	Simple
Computational efficiency	Efficient	Efficient	Not as efficient
Development of a general-purpose computer program	Difficult	Relatively difficult	Easy

Chapter 2 Computation Kinematics

- A mechanism that is formed from a collection of links or bodies kinematically connected to one another.
- An open-loop mechanism may contain links with single joint.
- A closed-loop mechanism is formed from a closed chain, wherein each link is connected to at least two other links of the mechanism and it is possible to traverse a closed loop.

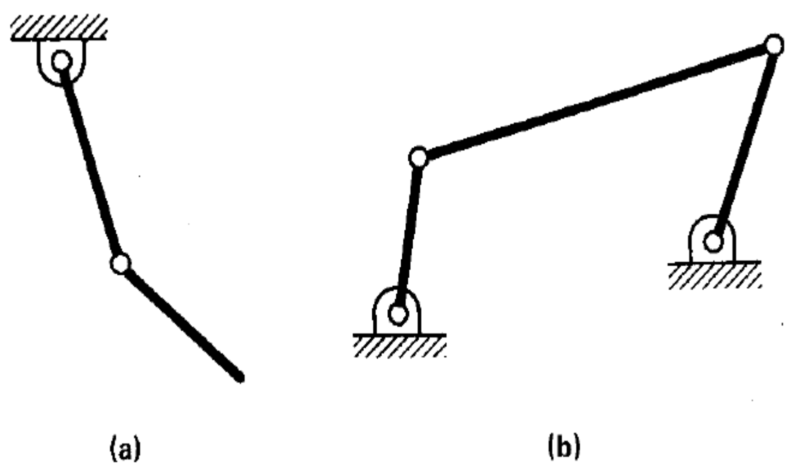


Fig. 2.1 (a) Open-loop mechanism—double pendulum and (b) closed-loop mechanism—four-bar linkage.

Single and Multi-Loop Mechanism

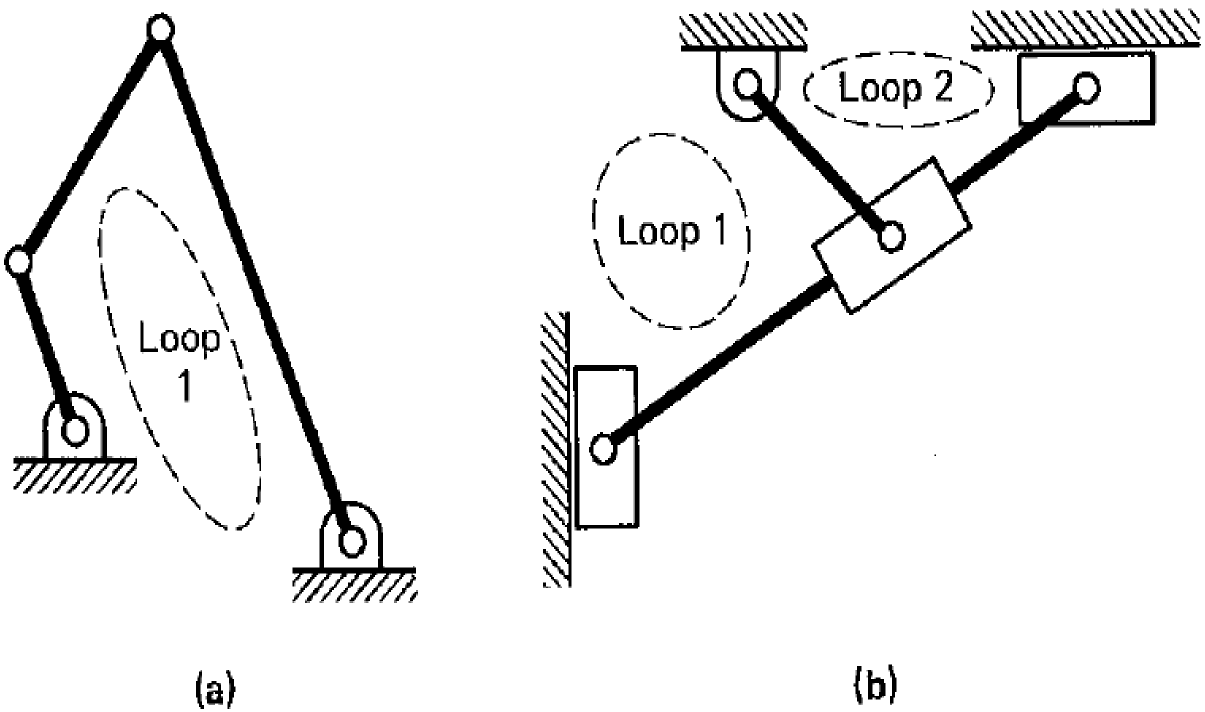


Figure 2.2 (a) Single-loop mechanism and (b) multi-loop mechanism.

High and Low Pair of Kinematic Joint

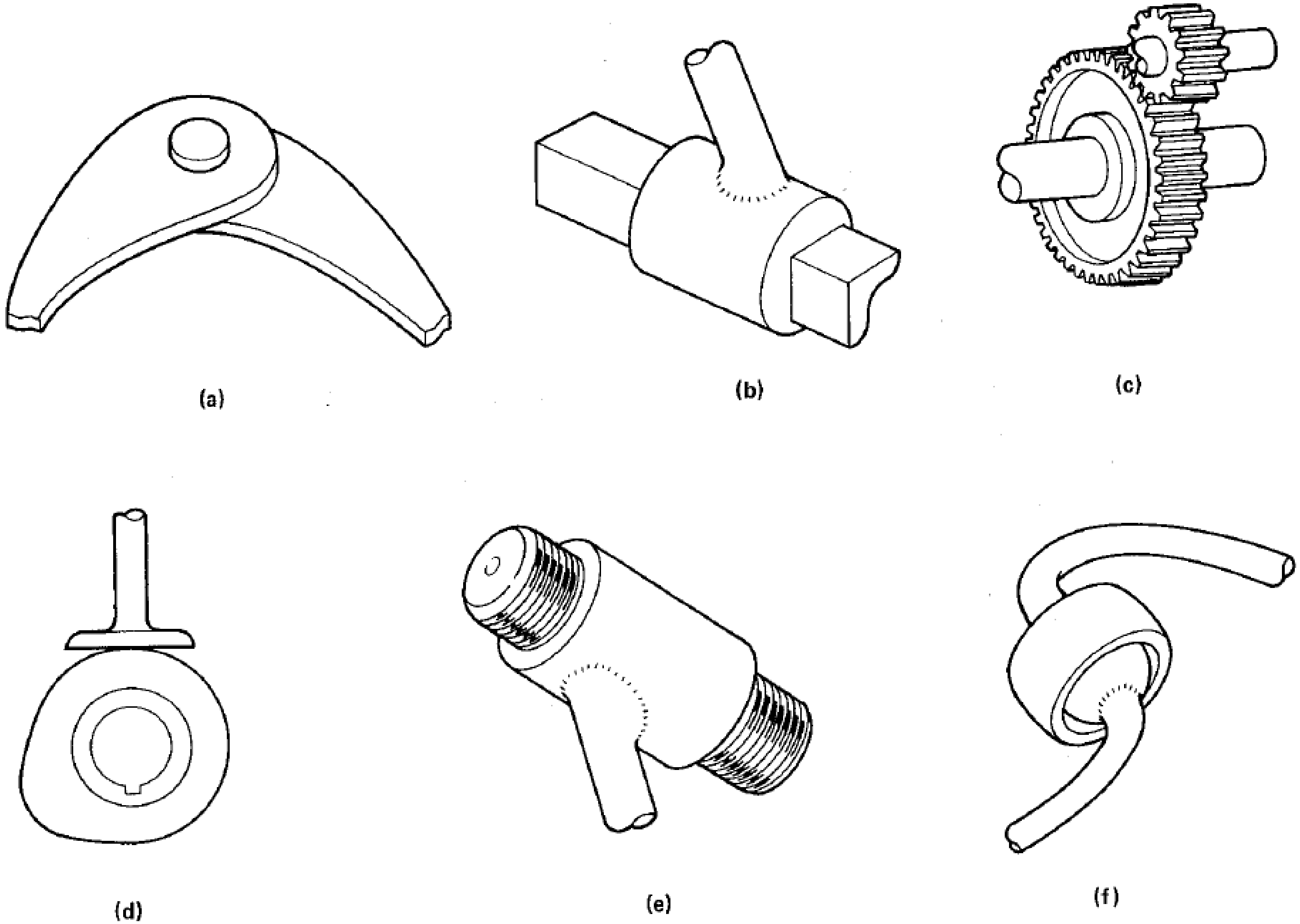


Figure 2.3 Example of kinematic pairs: (a) revolute joint, (b) translational joint, (c) gear set, (d) cam follower, (e) screw joint, and (f) spherical ball joint.

Generalized Coordinates

3 generalized coordinates,

2 algebraic constraint equations,

$$
\begin{aligned}
l_1 \cos\phi_1 + l_2 \cos\phi_2 - l_3 \cos\phi_3 - d_1 &= 0 \\
l_1 \sin\phi_1 + l_2 \sin\phi_2 - l_3 \sin\phi_3 - d_2 &= 0
\end{aligned}
\qquad (2.1)
$$

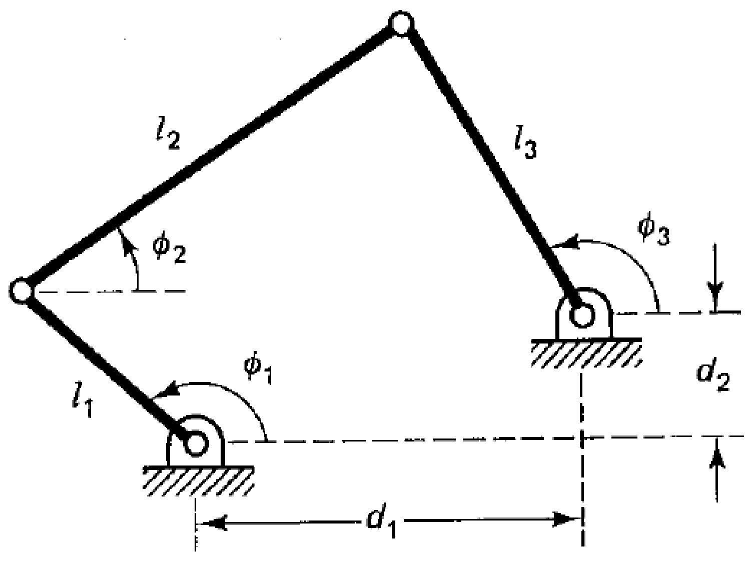
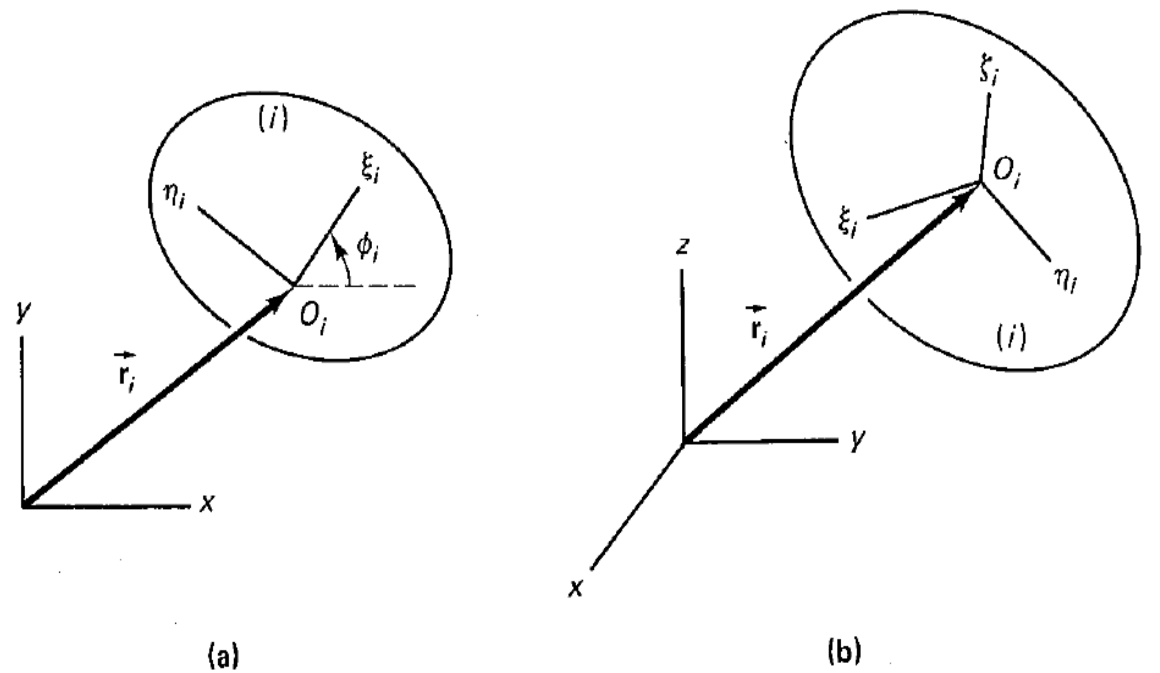


Fig. 2.5 four-bar mechanism.

Cartesian Coordinates

The column vector $\boldsymbol{q}_i \equiv [x, y, \phi]_i^T$ is the vector of coordinates for body i in a plane.



$\boldsymbol{q}_i \equiv \left[x, y, z, \phi_1, \phi_2, \phi_3\right]_i^T$ is the vector of coordinates for body i in three-dimensional space.

Constraint Equation

- A constraint equation describing a condition on the vector of coordinates of a system can be expressed as follows:

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\boldsymbol{q}) = 0 \tag{2.2}$$

- In some constraint equations, the variable time may appear explicitly:

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\boldsymbol{q}, t) = 0 \tag{2.3}$$

Redundant Constraint

- Kinematically equivalent.

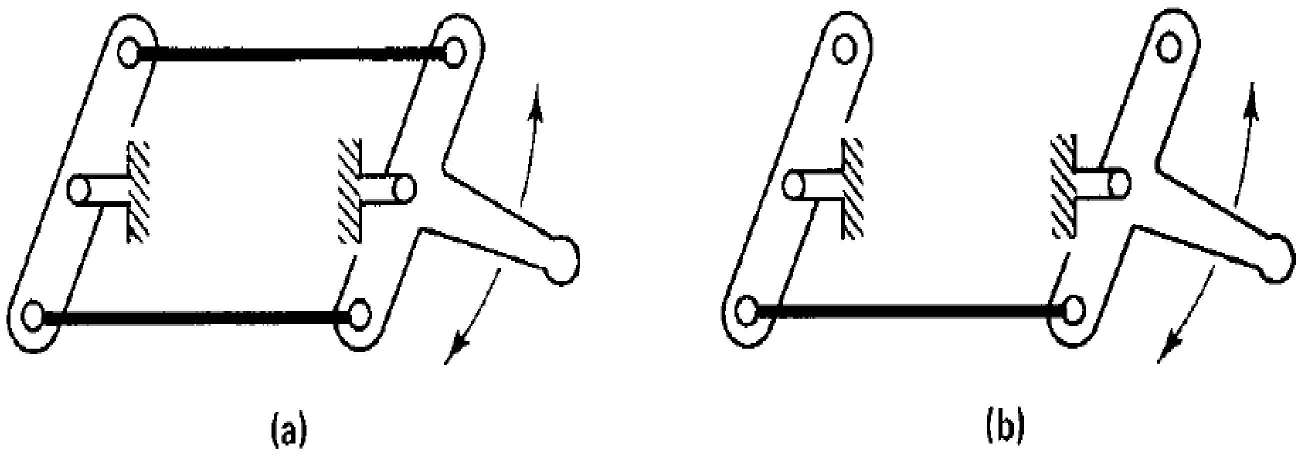
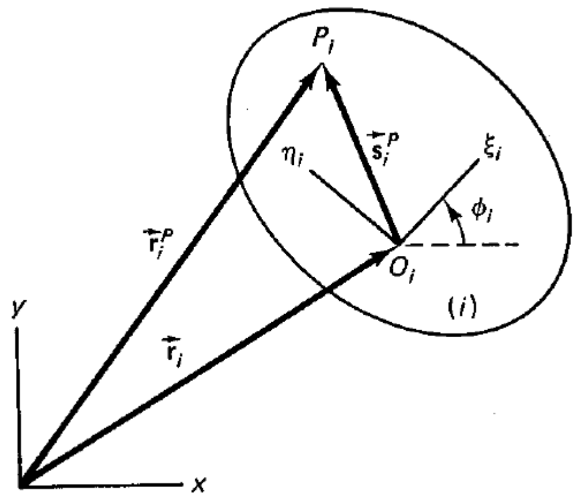


Figure 2.6 (a) A double parallel-crank mechanism and (b) its kinematically equivalent.

Chapter 3 Planar Kinematics in Cartesiam Coordinates

Inertial system X − Y

Body−fixed system ξ − η



$$\boldsymbol{r}_i^P = \boldsymbol{r}_i + \boldsymbol{A}_i \boldsymbol{s}_i'^P \qquad (3.1)$$

Coordinate transformation matrix

$$\boldsymbol{A}_i = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}_i \qquad (3.2)$$

Kinematics of Mechanism

Mainly composed of revolute joint and translation joint

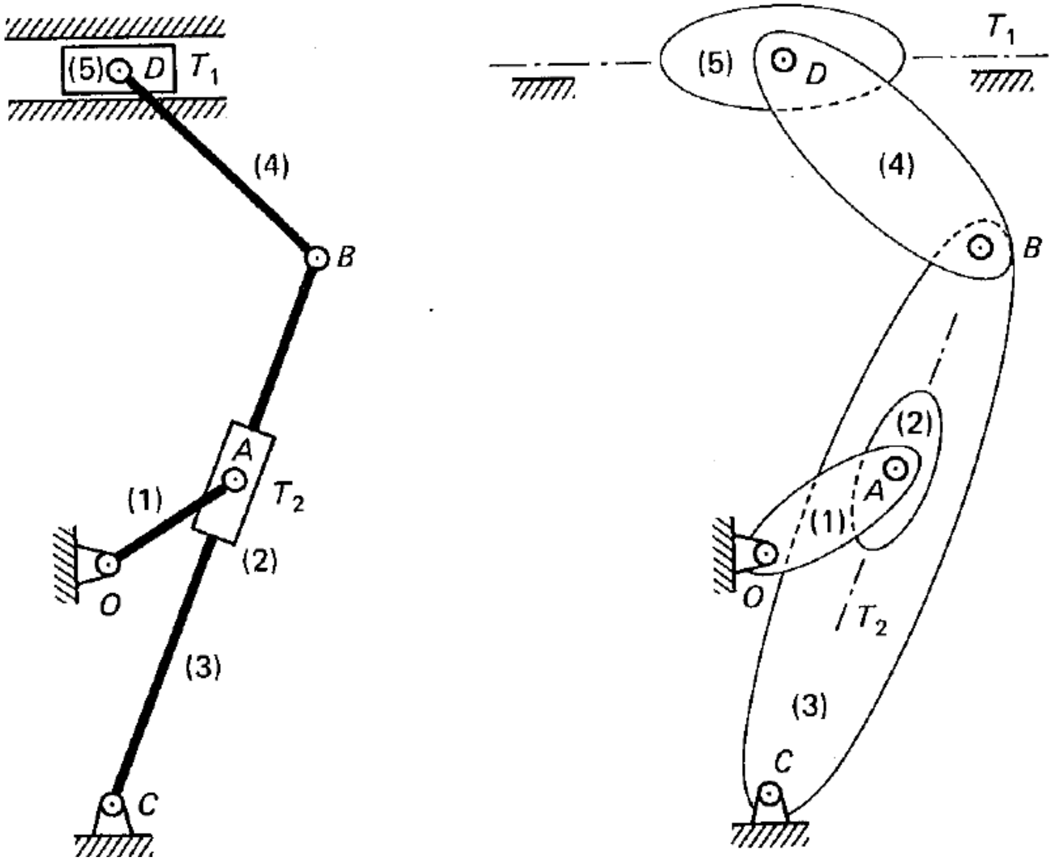


Figure 3.2 Quick-return mechanism: (a) schematic presentation and (b) its equivalent representation without showing the actual outlines.

Revolute Joint

$$\boldsymbol{r}_i + \boldsymbol{s}_i^P - \boldsymbol{r}_j - \boldsymbol{s}_j^P = 0 \tag{3.7}$$

$$\boldsymbol{\Phi}^{(r,2)} \equiv \boldsymbol{r}_i + \boldsymbol{A}_i \boldsymbol{s}_i'^P - \boldsymbol{r}_j - \boldsymbol{A}_j \boldsymbol{s}_j'^P = 0 \tag{3.8}$$

$$\Phi^{(r,2)} \equiv \begin{bmatrix} x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j \\ y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.9}$$

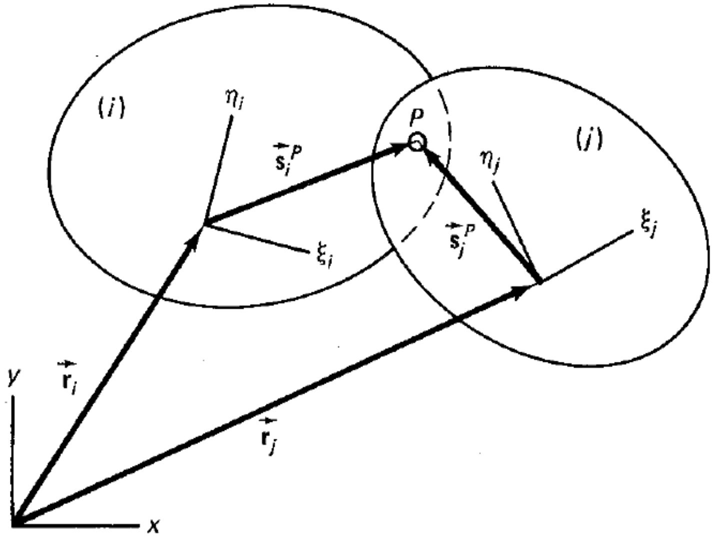


Figure 3.3 Revolute joint P connecting bodies i and j.

Constraint Jacobian Matrix

by differentiating the constraint equations

$$\mathbf{\Phi}(\boldsymbol{q}) = 0$$

$$\frac{\partial \mathbf{\Phi}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0, \mathbf{\Phi}_q \dot{\boldsymbol{q}} = 0$$

$$\mathbf{\Phi}_q \ddot{\boldsymbol{q}} + (\mathbf{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} = 0$$

$$\mathbf{\Phi}_q \ddot{\boldsymbol{q}} = -(\mathbf{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} \equiv \gamma$$

Time Derivative of Revolute Joint Constraint

$$\Phi^{(r,1st)} \equiv x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j = 0$$

$$\Phi^{(r,2nd)} \equiv y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j = 0$$

$$\dot{x}_i - (x_i^P \sin f_i + h_i^P \cos f_i)\dot{f}_i - \dot{x}_j + (x_j^P \sin f_j + h_j^P \cos f_j)\dot{f}_j = 0$$

$$\dot{y}_i + (x_i^P \cos f_i - h_i^P \sin f_i)\dot{f}_i - \dot{y}_j - (x_j^P \cos f_j - h_j^P \sin f_j)\dot{f}_j = 0$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

Time Derivative of Constraint

$$\Phi^{(r,1st)} \equiv x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j = 0$$

$$\Phi^{(r,2nd)} \equiv y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j = 0$$

$$\dot{x}_i - (x_i^P \sin f_i + h_i^P \cos f_i)\dot{f}_i - \dot{x}_j + (x_j^P \sin f_j + h_j^P \cos f_j)\dot{f}_j = 0$$

$$\dot{y}_i + (x_i^P \cos f_i - h_i^P \sin f_i)\dot{f}_i - \dot{y}_j - (x_j^P \cos f_j - h_j^P \sin f_j)\dot{f}_j = 0$$

$$\begin{aligned} &\ddot{x}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\ddot{\phi}_i - (\xi_i^P \cos\phi_i + \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - \ddot{x}_j \\ &\qquad + (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\ddot{\phi}_j + (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j^2 = 0 \end{aligned}$$

$$\begin{aligned} &\ddot{y}_i - (\xi_i^P \cos\phi_i + \eta_i^P \sin\phi_i)\ddot{\phi}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - \ddot{y}_j \\ &\qquad - (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\ddot{\phi}_j - (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j^2 = 0 \end{aligned}$$

or

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 5 & 6 & 0 & 7 & 8 \end{bmatrix} \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \\ \ddot{x}_j \\ \ddot{y}_j \\ \ddot{\phi}_j \end{bmatrix} = \gamma^{(r,2)}$$

$$\gamma^{(r,2)} = \begin{bmatrix} (\xi_i^P \cos\phi_i + \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - (\xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j)\dot{\phi}_j^2 \\ (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j^2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_i^P - x_i)\dot{\phi}_i^2 - (x_j^P - x_j)\dot{\phi}_j^2 \\ (y_i^P - y_i)\dot{\phi}_i^2 - (y_j^P - y_j)\dot{\phi}_j^2 \end{bmatrix}$$

$$= \mathbf{s}_i^P \dot{\phi}_i^2 - \mathbf{s}_j^P \dot{\phi}_j^2$$

Translational Joint

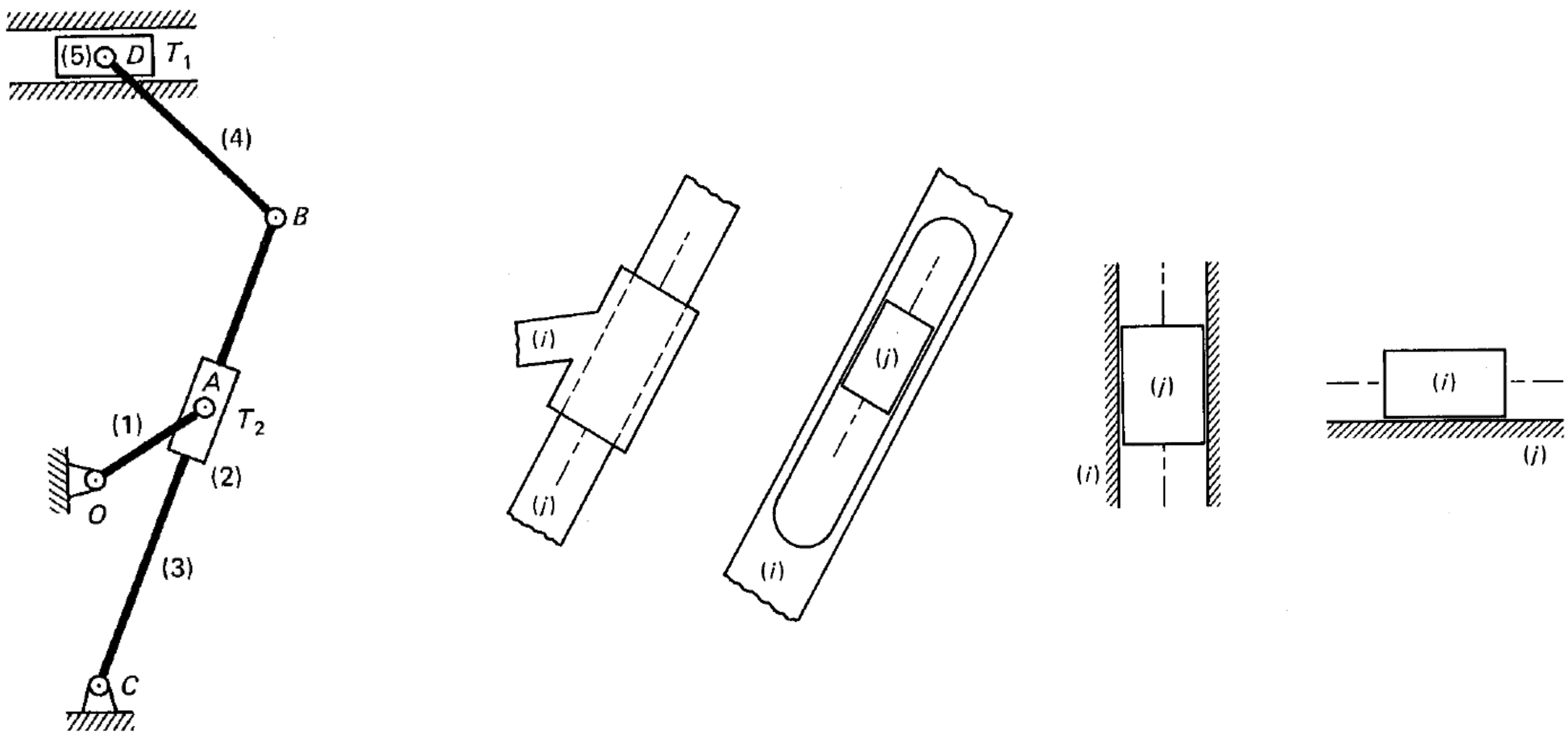
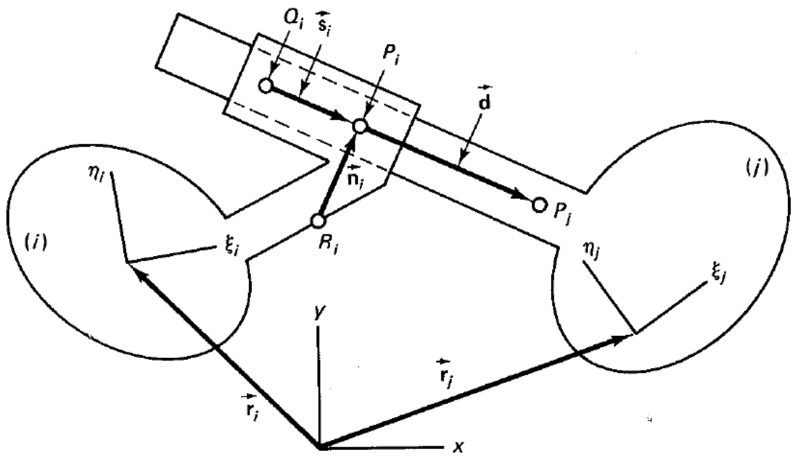


Figure 3.4 Different representations of a translational joint.

Translational Joint



$$\boldsymbol{n}_i^T \boldsymbol{d} = 0$$

$$\begin{bmatrix} x_i^P - x_j^R & y_i^P - y_j^R \end{bmatrix} \begin{bmatrix} x_i^P - x_j^R \\ y_i^P - y_j^R \end{bmatrix} = 0$$

$$\Phi^{(t,2)} = \begin{bmatrix} +(x_i^P - x_j^P)(y_i^P - y_i^Q) + (y_i^P - y_j^P)(x_i^P - x_j^Q) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.12)$$

Figure 3.5 A translational joint between bodies i and j .

Constraint Jacobian Matrix

$$\boldsymbol{\Phi}(\boldsymbol{q}) = 0$$

$$\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0, \boldsymbol{\Phi}_q \dot{\boldsymbol{q}} = 0$$

$$\boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} = 0$$

$$\boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} = -(\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} \equiv \gamma$$

Time Derivative of Translational Joint Constraint

$$
\boldsymbol{\Phi}^{(t,2)} = \begin{bmatrix} (x_i^P - x_i^Q)(y_j^P - y_i^P) - (y_i^P - y_i^Q)(x_j^P - x_i^P) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.12}
$$

$\Phi^{(t,2)}$	$(y_i^P - y_i^Q)$	$-(x_i^P - x_i^Q)$	$-(x_j^P - x_i)(x_i^P - x_i^Q)$ $-(y_j^P - y_i)(y_i^P - y_i^Q)$	$-(y_i^P - y_i^Q)$	$(x_i^P - x_i^Q)$	$(x_j^P - x_i)(x_i^P - x_i^Q)$ $+(y_j^P - y_i)(y_i^P - y_i^Q)$
	0	0	1	0	0	-1

$$
g = -2[(x_i^P - x_i^Q)(\dot{x}_i - \dot{x}_j) + (y_i^P - y_i^Q)(\dot{y}_i - \dot{y}_j)\dot{f}^{\dagger} - [(x_i^P - x_i^Q)(y_i - y_j) - (y_i^P - y_i^Q)(x_i - x_j)\dot{f}_i^2
$$

Driving Link

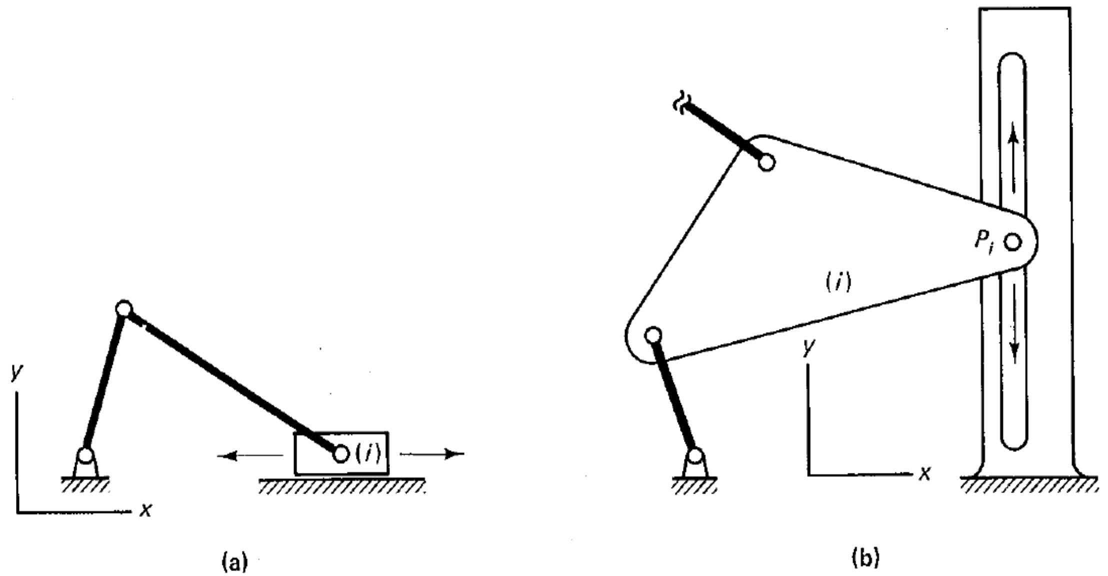
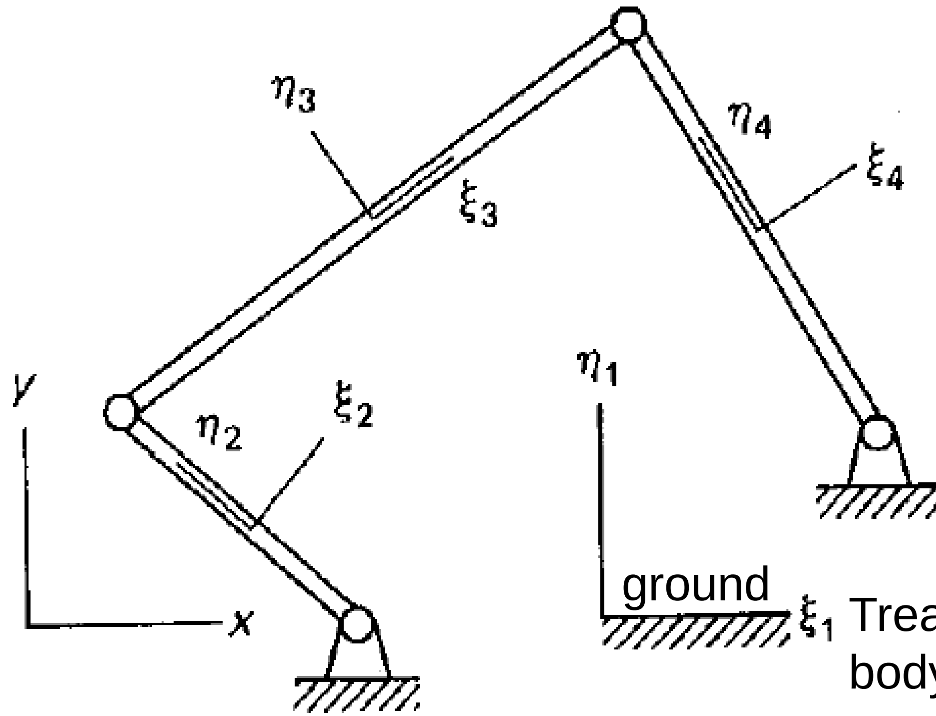


Figure 3.6 (a) The motion of the slider is controlled in the $x-$direction and (b) the motion of point P is controlled in the $y-$direction.

$$\Phi^{(d-2,1)} \equiv x_i - d_2(t) = 0 \tag{3.13}$$

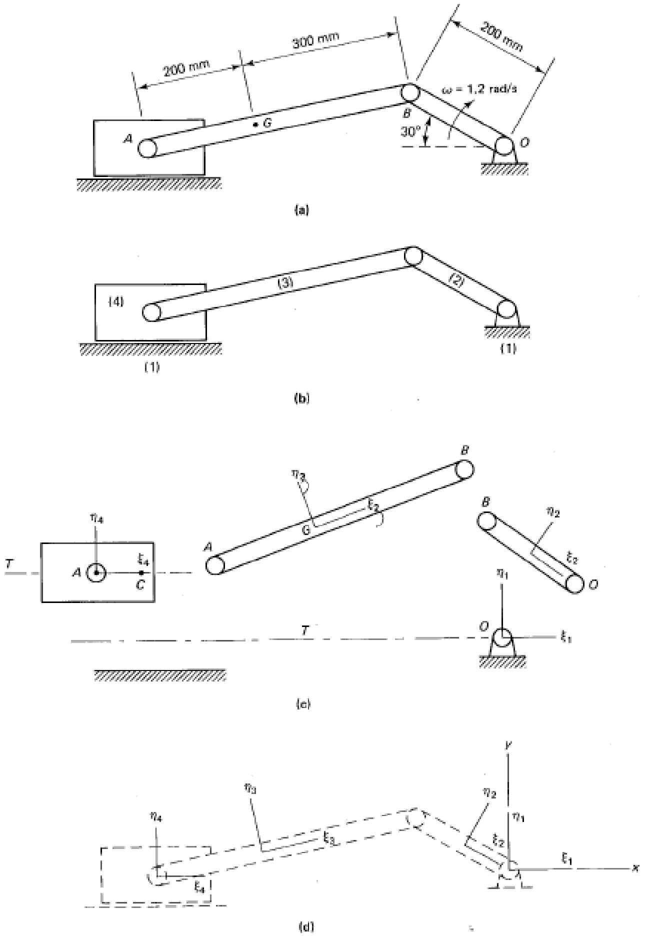
$$\Phi^{(d-5,1)} \equiv y_i^P - d_5(t) = 0 \tag{3.14}$$

3.3 Kinematic Analysis



Treat ground as a rigid body ξ_1 - h_1

Kinematic Modeling



Ground

$$x_1 = 0.0, \quad y_1 = 0.0, \quad \phi_1 = 0.0$$

Revolute joint

$$\xi_4^A = 0.0, \quad \eta_4^A = 0.0, \quad \xi_3^A = -200.0, \quad \eta_3^A = 0.0$$

$$\xi_3^B = 300.0, \quad \eta_3^B = 0.0, \quad \xi_2^B = -100.0, \quad \eta_2^B = 0.0$$

$$\xi_2^0 = 0.0, \quad \eta_2^0 = 0.0, \quad \xi_1^0 = 0.0, \quad \eta_1^0 = 0.0$$

translational joint

$$\xi_4^A = 0.0, \quad \eta_4^A = 0.0, \quad \xi_4^C = 100.0, \quad \eta_4^C = 0.0,$$

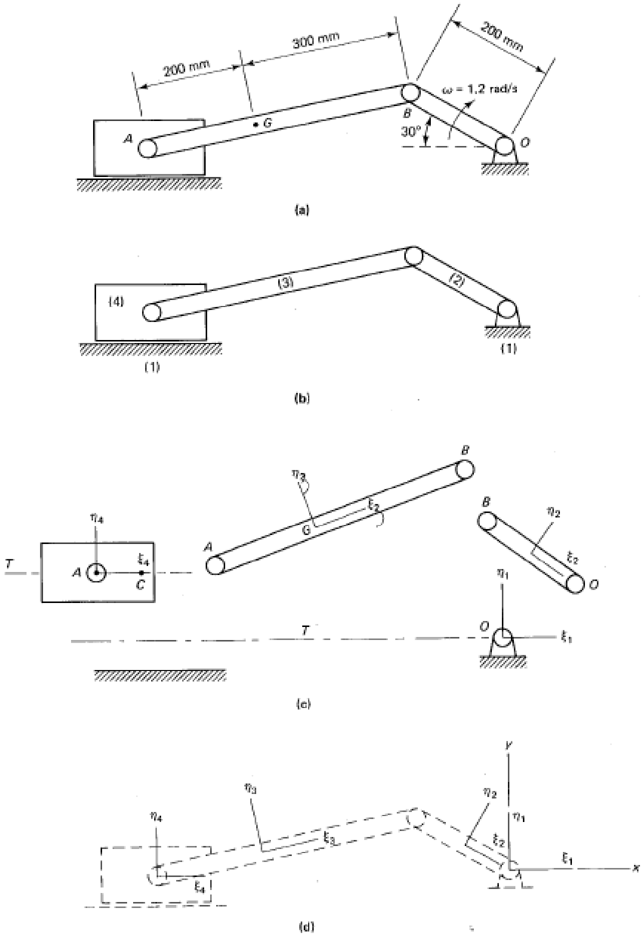
$$\xi_1^0 = 0.0, \quad \eta_1^0 = 0.0$$

driving constraint

$$\phi_2 - 5.76 + 1.2t = 0.0$$

Figure 3.7 Kinematic modeling of a slider-crank mechanism.

Kinematic Modeling



Ground

$$\Phi_1 \equiv x_1 = 0.0$$

$$\Phi_2 \equiv y_1 = 0.0$$

$$\Phi_3 \equiv \phi_1 = 0.0$$

revolute joints

$$\Phi_4 \equiv x_4 - x_3 + 200\cos\phi_3 = 0$$

$$\Phi_5 \equiv y_4 - y_3 + 200\sin\phi_3 = 0$$

$$\Phi_6 \equiv x_3 + 300\cos\phi_3 - x_2 + 100\cos\phi_2 = 0$$

$$\Phi_7 \equiv y_3 + 300\sin\phi_3 - y_2 + 100\sin\phi_2 = 0$$

$$\Phi_8 \equiv x_2 + 100\cos\phi_2 - x_1 = 0$$

$$\Phi_9 \equiv y_2 + 100\sin\phi_2 - y_1 = 0$$

translational joint

$$\Phi_{10} \equiv (-100\cos\phi_4)(y_1 - 100\sin\phi_1 - y_4) - (x_1 - 100\cos\phi_1 - x_4)(-100\sin\phi_4) = 0$$

$$\Phi_{11} \equiv \phi_4 - \phi_1 = 0$$

driving constraint

$$\Phi_{12} \equiv \phi_2 - 5.76 + 1.2t = 0$$

Figure 3.7 Kinematic modeling of a slider-crank mechanism.

Jacobian Matrix for Revolute Joint

$$
\begin{array}{r c}
 & \begin{array}{cccccccccccc} x_1 & y_1 & \phi_1 & x_2 & y_2 & \phi_2 & x_3 & y_3 & \phi_3 & x_4 & y_4 & \phi_4 \end{array} \\
\begin{array}{r}
\partial\Phi_1/\partial\ldots \\
\partial\Phi_2/\partial\ldots \\
\partial\Phi_3/\partial\ldots \\
\partial\Phi_4/\partial\ldots \\
\partial\Phi_5/\partial\ldots \\
\partial\Phi_6/\partial\ldots \\
\partial\Phi_7/\partial\ldots \\
\partial\Phi_8/\partial\ldots \\
\partial\Phi_9/\partial\ldots \\
\partial\Phi_{10}/\partial\ldots \\
\partial\Phi_{11}/\partial\ldots \\
\partial\Phi_{12}/\partial\ldots
\end{array}
&
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 5 & 6 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 9 & 0 & 10 & 11 \\
0 & 0 & 0 & 12 & 0 & 13 & 14 & 0 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 16 & 17 & 0 & 18 & 19 & 0 & 0 & 0 \\
20 & 0 & 21 & 22 & 0 & 23 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24 & 25 & 0 & 26 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\
28 & 29 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 31 & 32 & 33 \\
0 & 0 & 34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 \\
0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Figure 3.8 The Jacobian matrix.

Cont’d

$$
\begin{array}{cc}
 & \begin{matrix} x_1 & y_1 & \phi_1 & x_2 & y_2 & \phi_2 & x_3 & y_3 & \phi_3 & x_4 & y_4 & \phi_4 \end{matrix} \\
\begin{matrix} \partial\Phi_1/\partial\ldots \\ \partial\Phi_2/\partial\ldots \\ \partial\Phi_3/\partial\ldots \\ \partial\Phi_4/\partial\ldots \\ \partial\Phi_5/\partial\ldots \\ \partial\Phi_6/\partial\ldots \\ \partial\Phi_7/\partial\ldots \\ \partial\Phi_8/\partial\ldots \\ \partial\Phi_9/\partial\ldots \\ \partial\Phi_{10}/\partial\ldots \\ \partial\Phi_{11}/\partial\ldots \\ \partial\Phi_{12}/\partial\ldots \end{matrix} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 5 & 6 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 9 & 0 & 10 & 11 \\
0 & 0 & 0 & 12 & 0 & 13 & 14 & 0 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 16 & 17 & 0 & 18 & 19 & 0 & 0 & 0 \\
20 & 0 & 21 & 22 & 0 & 23 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24 & 25 & 0 & 26 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\
28 & 29 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 31 & 32 & 33 \\
0 & 0 & 34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 \\
0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$1,\ 2,\ 3,\ 6,\ 10,\ 14,\ 18,\ 22,\ 26\ \ 34,\ 36 = 1$$

$$4,\ 8,\ 12,\ 16,\ 20,\ 24,\ 35 = -1$$

$$7,\ 11,\ 21,\ 25,\ = 0$$

$$5 = -200 \sin \phi_3$$

$$9 = 200 \cos \phi_3$$

$$13,\quad 23 = 100 \sin \phi_2$$

$$15 = -300 \sin \phi_3$$

$$17,\quad 27 = 100 \cos \phi_2$$

$$19 = 300 \cos \phi_3$$

$$28 = 100 \cos \phi_4$$

$$29 = -100 \sin \phi_4$$

$$30 = -10,000(\cos \phi_1 \cos \phi_4 + \sin \phi_1 \sin \phi_4)$$

$$31 = -28$$

$$32 = -29$$

$$33 = 100[\cos \phi_4 (y_1 + 100 \cos \phi_1 - y_4) - \sin \phi_4 (x_1 - 100 \sin \phi_1 - x_4)]$$

Solution Technique

kinematic constraints $\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\boldsymbol{q}) = 0$

$$\boldsymbol{\Phi}^{(d)} \equiv \boldsymbol{\Phi}(\boldsymbol{q}, t) = 0 \qquad (3.14)$$

velocity equations $\boldsymbol{\Phi}_q \dot{\boldsymbol{q}} = 0$

$$\boldsymbol{\Phi}_q^{(d)} \dot{\boldsymbol{q}} + \boldsymbol{\Phi}_t^{(d)} = 0 \qquad (3.15)$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{q}} \\ \boldsymbol{\Phi}_{\boldsymbol{q}}^{(d)} \end{bmatrix} \ddot{\boldsymbol{q}} = \begin{bmatrix} 0 \\ -\boldsymbol{\Phi}_{\boldsymbol{t}}^{(d)} \end{bmatrix} \qquad (3.16)$$

acceleration $\boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} = 0$

$$\boldsymbol{\Phi}_q^{(d)} \ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_q^{(d)} \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} + 2\boldsymbol{\Phi}_{qt}^{(d)} \dot{\boldsymbol{q}} + \boldsymbol{\Phi}_{tt}^{(d)} = 0 \qquad (3.17)$$

$$\begin{bmatrix} \boldsymbol{\Phi}_q \\ \boldsymbol{\Phi}_q^{(d)} \end{bmatrix} \ddot{\boldsymbol{q}} = \begin{bmatrix} -(\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} \\ -(\boldsymbol{\Phi}_q^{(d)} \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} - 2\boldsymbol{\Phi}_{qt}^{(d)} \dot{\boldsymbol{q}} - \boldsymbol{\Phi}_{tt}^{(d)} \end{bmatrix} = \begin{bmatrix} \gamma \\ -(\boldsymbol{\Phi}_q^{(d)} \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} - 2\boldsymbol{\Phi}_{qt}^{(d)} \dot{\boldsymbol{q}} - \boldsymbol{\Phi}_{tt}^{(d)} \end{bmatrix} \qquad (3.18)$$

CHAPTER 4 Planar Dynamics
4.1 Rigid Body Dynamics

$$m_i \ddot{x}_i = f_{(x)i}$$

$$m_i \ddot{y}_i = f_{(y)i}$$

$$m_i \ddot{f}_i = n_i$$

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i \quad (4.4)$$

$$\boldsymbol{M}_i \ddot{\boldsymbol{q}}_i = \boldsymbol{g}_i$$

Constraint Force

$\mathbf{\Phi}(\boldsymbol{q}, t) = 0, \ \boldsymbol{\phi} \in \mathbb{R}^{n\times 1}$

$\mathbf{\Phi}_q \delta \boldsymbol{q} = 0, \quad \boldsymbol{\phi}_q \in \mathbb{R}^{m\times n}$

There exists Lagrange Multiplier $\boldsymbol{\lambda}, \ \boldsymbol{\lambda} \in \mathbb{R}^{m\times 1}$

such that $\boldsymbol{\lambda}^T \mathbf{\Phi}_q$ is that constraint force.

The equations of motion can be written as

$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{g} + \boldsymbol{g}^{(c)}$

$\boldsymbol{g}^{(c)} = -\boldsymbol{\phi}_q^T \boldsymbol{\lambda}$

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \boldsymbol{g} \qquad (4.8)$$

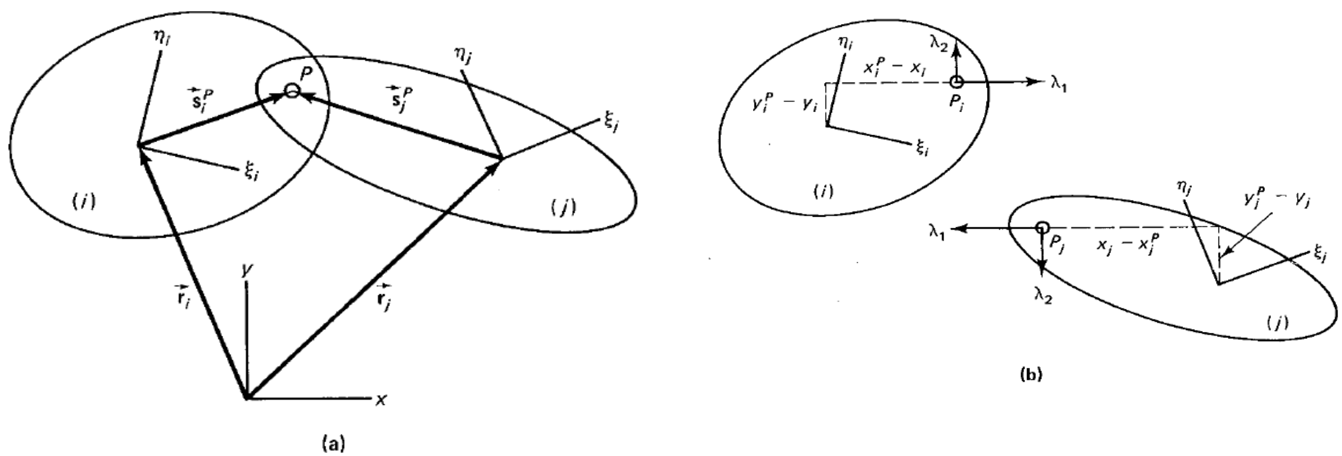
Dynamic System with Revolute Joint

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y_i^P - y_i) & (x_i^P - x_i) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i \tag{4.9}$$

$$m_i \ddot{x}_i = f_{(x)i} - \lambda_1 \tag{4.10}$$

$$m_i \ddot{y}_i = f_{(y)i} - \lambda_2 \tag{4.11}$$

$$\mu_i \ddot{\phi}_i = n_i - (y_i^P - y_i)\lambda_1 + (x_i^P - x_i)\lambda_2 \tag{4.12}$$



Physical meaning of constraint force

Dynamic System with Translational Joint

For a translational joint between *i* and *j*,

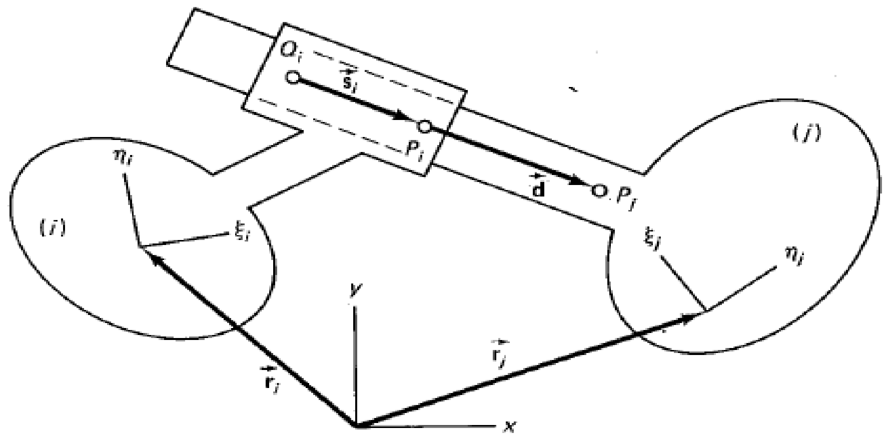
the equation of motion for body *i* can be written as

$$m_i \ddot{x}_i = f_{(x)i} + (y_i^P - y_i^Q)\lambda_1 \quad (4.19)$$

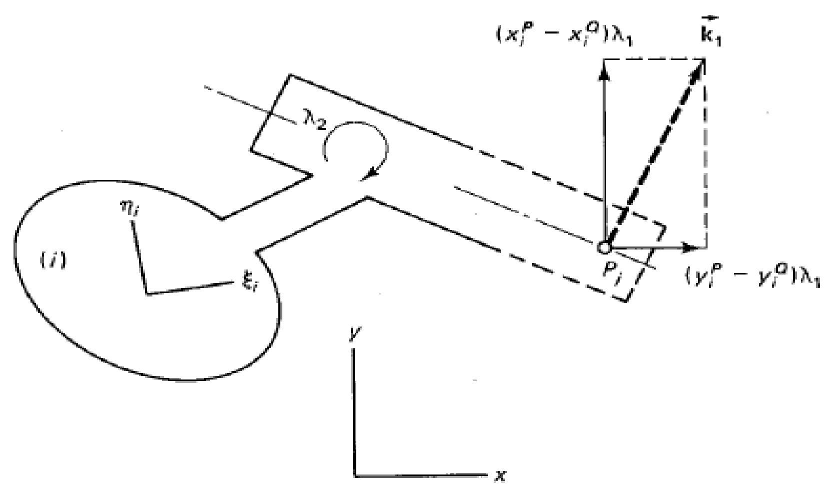
$$m_i \ddot{y}_i = f_{(y)i} + (x_i^P - x_i^Q)\lambda_1 \quad (4.20)$$

$$\mu_i \ddot{\phi}_i = n_i - [(x_j^P - x_i)(x_i^P - x_i^Q) + (y_j^P - y_i)(y_i^P - y_i^Q)]\lambda_1 + \lambda_2 \quad (4.21)$$

Physical Meaning of Constraint Force



(a)



(b)

Formulation of Multi-body Dynamic Systems

$$\boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} - \gamma = 0 \tag{4.25}$$

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_q^T \boldsymbol{\lambda} = \boldsymbol{g} \tag{4.26}$$

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{\Phi}_q^T \\ \boldsymbol{\Phi}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g} \\ \gamma \end{bmatrix} \tag{4.27}$$

$n+m$ linear algebraic equations in $n+m$ unknowns for $\ddot{q}$ and λ.

Solution Technique

At any given instant t

(1) Solve $\begin{cases} \Phi_q(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0} \\ \Phi_q^{(d)}(\boldsymbol{q},t)\dot{\boldsymbol{q}} = \text{(the right hand side)} \end{cases}$

n equations for n unknowns $\boldsymbol{q}(t)$

(2) Solve $\begin{cases} \Phi_q(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0} \\ \Phi_q^{(d)}(\boldsymbol{q},t)\dot{\boldsymbol{q}} = \text{(the right hand side)} \end{cases}$

n equations for n unknowns $\dot{\boldsymbol{q}}(t)$

(3) Solve $\begin{cases} \Phi_q(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{\gamma} \\ \Phi_q^{(d)}(\boldsymbol{q})\boldsymbol{q} = \text{(the right hand side)} \end{cases}$

n equations for n unknowns $\ddot{\boldsymbol{q}}(t)$

Euler Angles

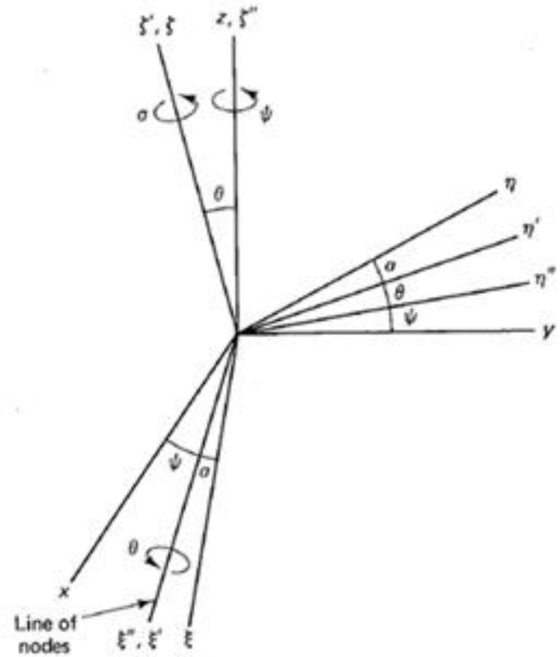


Figure 5.1 The rotations defining the Euler Angles.

$$\mathbf{A} \equiv \mathbf{DCB} \qquad \mathbf{D} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

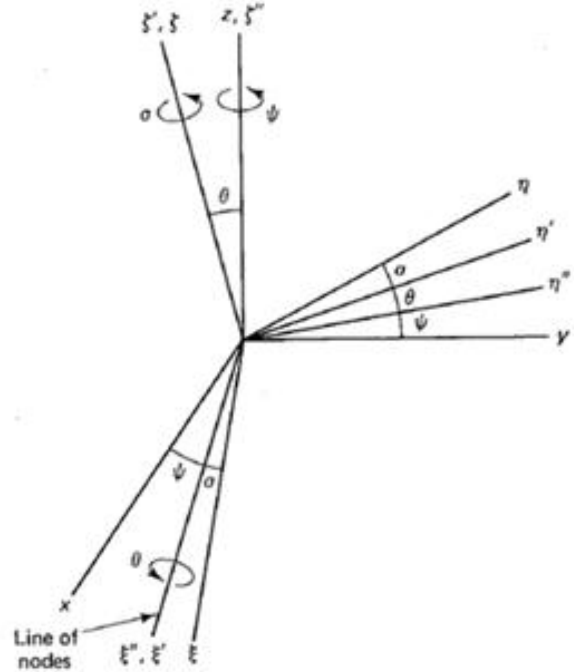


Figure 5.1 The rotations defining the Euler Angles.

$$\mathbf{A} = \begin{bmatrix} c\psi c\phi - s\psi c\theta s\phi & -c\psi s\phi - s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi + c\psi c\theta s\phi & -s\psi s\phi + c\psi c\theta c\phi & -c\psi s\theta \\ s\theta s\phi & s\theta c\phi & c\theta \end{bmatrix} \quad (5.1)$$

Time Derivatives of Euler Angles

$$w_{(x)} = \dot{y}\sin q \sin f + \dot{q}\cos f$$

$$w_{(h)} = \dot{y}\sin q \cos f - \dot{q}\sin f$$

$$w_{(z)} = \dot{y}\cos q + \dot{f}$$

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Bryant Angles

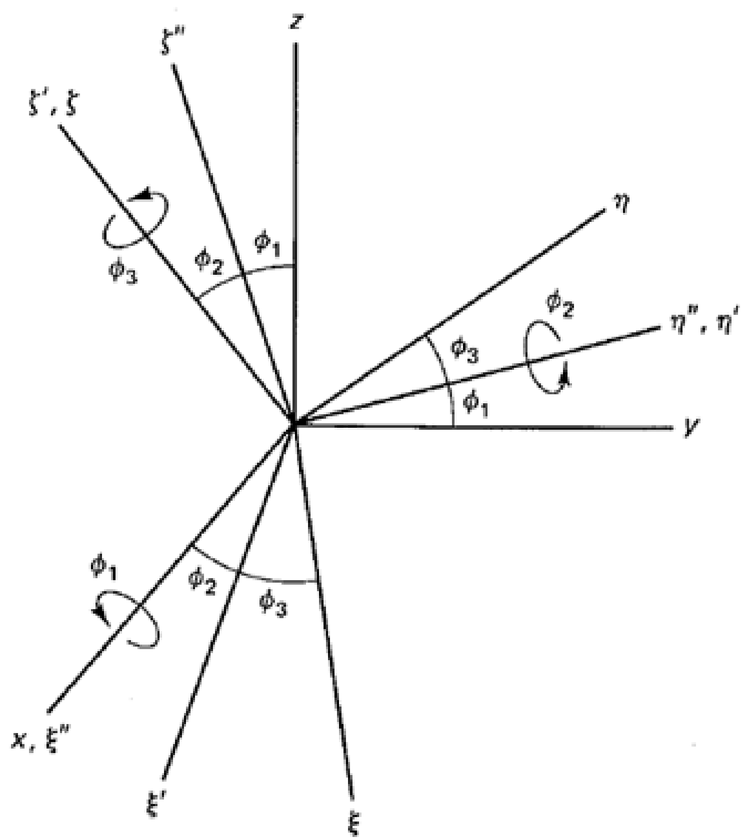


Figure 5.4 Rotations defining Bryant angles.

Bryant Angles

$$\mathbf{A} = \mathbf{DCB} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & -s\phi_1 \\ 0 & s\phi_1 & c\phi_1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c\phi_2 & 0 & s\phi_2 \\ 0 & 1 & 0 \\ -s\phi_2 & 0 & c\phi_2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

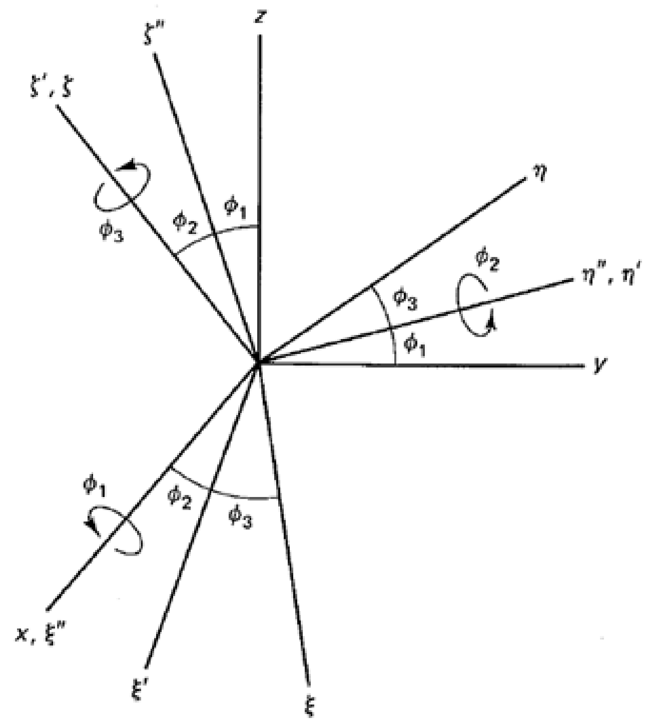


Figure 5.4 Rotations defining Bryant angles.

Bryant Angles

$$\mathbf{A}=\begin{bmatrix} c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \\ c\phi_1 s\phi_3 + s\phi_1 s\phi_2 c\phi_3 & c\phi_1 c\phi_3 + s\phi_1 s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 + c\phi_1 s\phi_2 c\phi_3 & s\phi_1 c\phi_3 + c\phi_1 s\phi_2 s\phi_3 & c\phi_1 c\phi_2 \end{bmatrix}$$

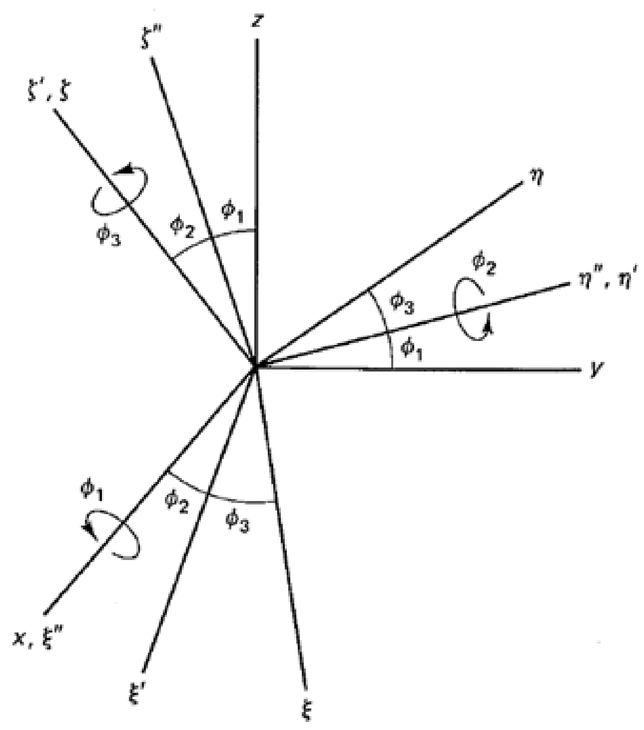


Figure 5.4 Rotations defining Bryant angles.

Time Derivative of Bryant Angles

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \cos\phi_1 \cos\phi_3 & \sin\phi_3 & 0 \\ -\cos\phi_2 \sin\phi_3 & \cos\phi_3 & 0 \\ \sin\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}$$