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Rigid-Body Motion of Planar Mechanisms by Finite Element Method With Comparison to the Multibody Approach

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Abstract
In this paper the rigid-body motion of planar mechanisms (e.g. rigid rod, robotic arm, slider-crank mechanism) is considered by using two different approaches: FEM with modified truss elements named here as a stick model and the multibody method used in the engineering software (e.g. Adams, Modelica, etc.). The analysis shows that the stick model of mechanisms consisting of modified truss-type elements gives reliable results when compared to the solid model of mulibody approach by using the Adams software. The presented method of description of motion of rigid mechanism can be successfully used as a preliminary calculation of mechanism motion. The method is illustrated with planar examples of mechanisms, however with the use of proper elements the method can be extended to the 3D case.

Keywords: rigid-body motion, multibody dynamics, FEM, planar mechanism.

1 Introduction

Planar mechanisms are widely applied in the machine design. They are used both in a very simple structures such as foot-operated sewing devices, as well as in complex machines operated in severe conditions such as crankshaft-piston mechanism of high-speed internal combustion engines. Depending on the application, different requirements have to meet the individual parts of the mechanism, and therefore, the design of such mechanisms should be approached from different sites.

The calculation of mechanisms can be performed by using the environment of "multibody dynamics" (Frączek, 2002; Shabana, 2010). These calculations require significant computational power due to the large number of degrees of freedom of the model. A multibody dynamic system is one that consists of solid bodies, or links, that are connected to each other by joints that restrict their relative motion. The analysis

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of mechanism motion under the influence of forces is very important in machine evaluation. Product design frequently requires an understanding of how multiple moving parts interact with each other and their environment. Accurate modeling requires taking into account various physical phenomena like vibration and friction. Motion analysis enables the designer to evaluate and improve mechanisms for important characteristics like performance, durability and comfort.

At the stage of design process, since many different concepts have to be taken into account, it is very important for the designers not necessary the exact answer of the system, but the rough rating of mechanism motion in order to eliminate incorrect design solutions. Therefore there are growing in importance simple methods that are not required to be very precise but can give fast answer to the dynamic behavior of the considered model. The present paper uses the simple finite element stick model of mechanisms built of modified truss-type elements (Hać, 1996) for describing the rigid-body motion. The results for mechanisms considered are compared with those obtained in Adams environment.

2 Calculation of stick model of mechanism by FEM

Traditionally the method of determining the rigid body motion of mechanisms is based on the classical analysis which involves using one of the method of determining equations of motion (e.g. Lagrange's, Gibbs-Appel's or similar equations) and then applying it to the given example (Pars, 1965). Such method is very time-consuming since for each structure the equations of motion should be derived from the beginning. Moreover in the case of external torques applied to more than one link of the mechanism (e.g. retraction kinematic system of aircraft landing gear (Hać and From, 2008)) it is necessary to transform the external loadings to generalized coordinates. This procedure is sometimes more complicated than the derivation of the equations of motion itself.

The discrete methods such as finite element method are free of these disadvantages. The finite elements approach in order to obtain rigid-body motion has not been widely considered in research publications. The rigid finite element method (RFEM) was proposed by (Kruszewski et al., 1984; Adamiec-Wójcik and Wojciech, 1993). The RFEM method divides the mechanism link into rigid finite elements connected by spring-damping elements, and the rigid finite elements are characterized by one node with the appropriate mass and moments of inertia. In the method used in this paper the finite element assumed for rigid body motion consists of two nodes and the rigidity of this element is obtained by assuming appropriate shape function. The method used in the present paper is based on the finite element technique and obtained equations are solved by using the Newmark method. The shape function for rigid-body elements is taken from the earlier publications (Hać and Osiński, 1995; Hać, 1996) and the equations of motion for structures with rigid links are derived.

The method of obtaining rigid-body motion is presented in the closed form and the system equations can be solved by using the same procedures as for finite element analysis.

The shape function $[N_{szt}]$ for the rigid body motion taken in the analysis is of the following form (Hać and Osiński, 1995)

$$[N_{szt}] = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 1-\xi/L & 0 & \xi/L \end{bmatrix} \tag{1}$$

where $0 \leqslant \xi \leqslant L$, L is the length of a finite element. Since equation (1) represents the shape function for the rigid-body motion it is assumed that the longitudinal displacement of any point belonging to the finite element is equal to the arithmetic average of the displacements of its nodes and the transverse displacement of any element's point changes linearly with the length of the element.

The matrix differential equation of motion for a single finite element of the mechanism is developed by using Lagrange's equations of motion. The kinetic energy is determined and the element inertia matrix $[M_{e0}]$ is obtained of the standard form for the finite element analysis:

$$[M_{e0}] = \rho A \int_0^L [N_{szt}]^T [N_{szt}] \, d\xi = \frac{\rho A L}{12} \begin{bmatrix} 3 & 0 & 3 & 0 \\ 0 & 4 & 0 & 2 \\ 3 & 0 & 3 & 0 \\ 0 & 4 & 0 & 2 \end{bmatrix} \tag{2}$$

In the case of the rigid-body equations of motion it must be taken into account that displacements of both nodes in the longitudinal direction of elements are equal or the difference between them results from Hook's law. In the latter case it is necessary to build the element stiffness matrix based on one-dimensional stress analysis. Thus in spite that rigid mechanisms are considered, in the equation of motion has to appear also stiffness matrix. The reader can refer to (Hać, 2013) for more details regarding this problem.

The obtained equations of motion for the system are as follows

$$[M_0]\{\ddot{x}_0\} + [K_0]\{x_0\} = \{Q_0\} \tag{3}$$

where $[M_0]$, $[K_0]$ are the system matrices obtained from element matrices $[M_{e0}]$, $[K_{e0}]$; x_0 is the nodal displacement vector, and Q_0 is the external system force vector. The presence of stiffness matrix in equations (3) is only necessary due to proper modeling of displacements of the rigid element and has practically no influence on rigid body motion (the kinetic energy of an element and consequently element inertia matrix were taken for rigid elements). If the stiffness matrix was omitted, the nodes could displace in any direction - also in longitudinal direction of the element which is impossible due to the rigidity of elements.

3 Calculation of mechanisms by multibody dynamic method

The "multibody dynamics" calculations are possible for any mechanism. It is used e.g. in systems for obtaining rigid-body motion in Msc Adams software. Usage of such systems has a number of advantages, like the accuracy of calculation. The results obtaining for dynamic response of mechanisms by using "multibody dynamics" approach is very high, since theoretically it only depends on the accuracy of mapping of the members of mechanism to their solid models. However, this method has also some disadvantages. Apart from the fact that the cost of the license and the need of high experience and knowledge of designers, one needs computers with appropriate power in relation to the complexity of the model. In addition, the apparent ease of obtaining various calculation results (related to displacements, deformations, stress or parameters defined by the user functions) increases the uncertainty associated with the received results. Specifying the reference measurement directions or many other parameters can cause a number of problems and presentation of the results is not always intuitive. For these reasons, it is good to compare the obtained results with other methods of calculation, in order to verify the correctness of conducted analyzes.

Assuming that the bonds of multi-body system are holonomic and time-dependent, the equations of motion using multi-body approach in absolute coordinates can be determined by using the description of analytical mechanics. The kinematics and dynamics of rigid systems can be described by using the Lagrange equations of the form (4) - in a similar way there is analyzed mechanism motion by Adams software (Adams, 2011):

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{Q} \\ \mathbf{\Phi}(q,t) = 0 \end{cases} \tag{4}$$

where: **M**- positive define inertia matrix, $\boldsymbol{\lambda}$- Lagrange's multiplies, $\mathbf{\Phi}$- vector of generalized forces (potential and external forces), **Q**- external generalized forces $Q = Q(t,q,\dot{q})$, $\mathbf{\Phi}_q$- Jacobi matrix, q - generalized coordinates.

The equation (4) is a differential algebraic equation (DAE) of the index 3 (index 3 means that the equation bonds should be differentiated three times to get ODEs - ordinary differential equations). Integration of the DAE equations is very difficult and it is not easy to obtain the solution.

4 Examples

The discussed two methods of obtaining mechanism's rigid-body motion (i.e. the stick model and by using Adams software) are illustrated by some examples of 2D mechanisms. Gravity is not taken into account in the analysis.

4.1 Rotating rod

Rigid rod of length $l = 1.0$ [m], mass density $\rho = 7800$ [kg/m^3], cross-sectional area $A = 0.9295 \times 10^{-4}$ [m^2] , Young modulus $E = 2.1 \times 10^{11}$[N/m^2], pivotally mounted at one end, is set into rotational movement by constant torque 3 [Nm]. At the starting moment of the calculation rod is at rest. Calculation time is 0.4 [s]. In Table 1 there are presented results of values of the rotating angle for the theoretical solution and for the two cases considered, i.e.: 1) results obtained from classical theory of rigid particles; 2) results from stick model made of modified truss-type finite elements; and 3) results by using Adams software.

Table 1. Comparison of the results of calculations of the rotation angle of the rod [rad].

Time [s]	Theory	Stick model	Adams	Ratio: stick model/Adams
0.1	0.057252	0.0573092	0.057252	1.000999
0.2	0.229008	0.2291253	0.229007	1.000512
0.3	0.515267	0.5154644	0.515262	1.000383
0.4	0.916031	0.9163897	0.916014	1.000087

When performing calculations in Adams one should check the weight, volume and mass moments of inertia elements. In defining the dimensions of the component type "link" (i.e. width, depth, length, density) it can appear that the values calculated by using the software are greater than it appears from the element volume (e.g. from the type of bounds used) and therefore also increases the weight and moment of inertia. One should select the "user input" and enter manually the mass and the moments of inertia of the element.

Theoretical calculations and the calculations obtained by Adams enviroment are consistent. The difference is very small and the results prove the correctness of the algorithm used for the stick model.

4.2 Robotic arm

The kinematics and dynamics of a single link planar manipulator is presented. The manipulator arm (Fig. 1) is modelled as one finite element and the mass at the end is treated as a lumped mass. The parameters used in the simulations are the same as in the example in section 4.1, but in addition, it is applied at the arm's end the nominal payload of mass $m = 0.3625$ [kg].

The input torque M_t is applied to the arm of the mechanism and is assumed to be

$$M_t = M_{t0} + M_{t1} \sin \alpha \tag{5}$$

where α is the angle of rotation and there are assumed the following data for torques:

$M_{t0} = 1$ [Nm], $M_{t1} = 0.5$ [Nm]. Time of calculation is equal to 1.0 [s], and zero initial conditions for rotation angle and for angular velocity are taken.

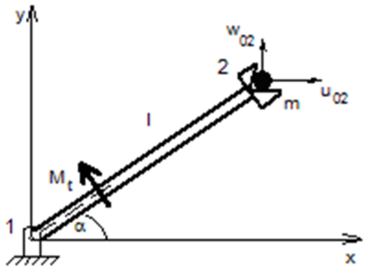

Fig. 1. Robotic arm.

For planar motion the nodal coordinates are elements of a four-dimensional vectorspace. Taking into account the boundary conditions, the vector of the unknown functions (nodal displacement vector x_0) consists of two elements: displacements u_0, and w_0 in the x, and y direction, respectively, of a moving node 2:

$$\{x_0\}^T = [u_{02}, w_{02}] \tag{6}$$

The coefficients matrices of equation (3) are calculated in every step of time discretization. The nodal displacement vector x_0 is obtained for successive periods of time. For the shape function (1) and taking into account the mass m at the end, the element inertia matrix is as follows

$$[M_e] = \frac{\rho AL}{12}\begin{bmatrix} 3 & & & Symm. \\ 0 & 4 & & \\ 3 & 0 & 3+m' & \\ 0 & 2 & 0 & 4+m' \end{bmatrix} \tag{7}$$

wherem' is obtained from the following formula: $m = (\rho AL/12)m'$. The global stiffness and inertia matrices for a single link manipulator arm are as follows:

$$[K] = \frac{EA}{L}\begin{bmatrix} a^2 & ab \\ ab & a^2 \end{bmatrix} \tag{8}$$

$$[M] = \frac{EA}{L}\begin{bmatrix} (3+m')a^2 + (4+m')b^2 & -ab \\ -ab & (3+m')b^2 + (4+m')a^2 \end{bmatrix} \tag{9}$$

where $a = \cos\alpha, b = \sin\alpha$. To estimate the accurateness of obtained results the kinematic and dynamic analysis of one link rigid manipulator arm is made by using Adams software. The comparison of results is presented in Table 2.

Table 2. Computer simulation results: robotic arm.

Time [s]	Stick model	Adams	Ratio: stick model/Adams
0.2	0.033203	0.033195	1.000241
0.4	0.133898	0.13388	1.000134
0.6	0.305413	0.305375	1.000124
0.8	0.553287	0.553190	1.000175
1.0	0.884964	0.884672	1.000330

From Table 2 it can be seen that the results obtained from the two methods considered are very close, so the algorithm based on truss-type elements for the stick model can be considered as correct.

4.3 Slider-crank mechanism

Figure 2 shows a slider-crank mechanism with the following characteristics: crank length and cross-sectional area $l_1 = 0.1$ [m], $A_1 = 2.564 \times 10^{-4}$ [m^2], connecting rod length and cross-sectional area $l_2 = 0.4$ [m], $A_2 = 1.282 \times 10^{-4}$ [m^2], mass at point B: $m = 0.2$ [kg], mass density $\rho = 7800$ [kg/m^3]. The input torque M_t is applied to the arm of the mechanism of the form presented in equation (3) (α is the crank angle), and for calculation there are assumed the following data:$M_{t0} = 0.1$ [Nm], $M_{t1} = 0.05$ [Nm], time of calculation is equal to 0.5 [s]. The zero initial conditions for the crank angle and its angular velocity were taken for numerical solution.

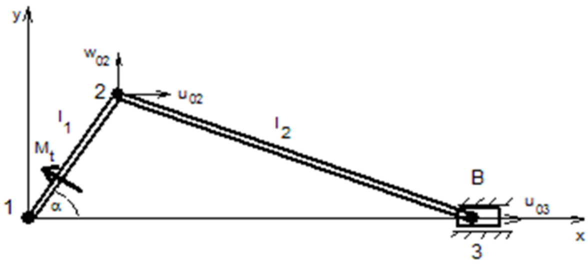

Fig. 2. Slider-crank mechanism.

The vector of unknown nodal displacements (i.e. a displacement vector after taking into account boundary conditions) x_0 contains three elements: displacements u_{02}, and w_{02} in the x, and y direction, respectively of a moving node 2 and displacement u_{03} of node 3:

$$\{x_0\}^T = [u_{02}, w_{02}, u_{03}] \tag{10}$$

The numerical simulation results are given in Table 3.

Table 3. Computer simulation results: slider-crank mechanism.

Time [s]	Stick model	Adams	Ratio: stick model/Adams
0.1	0.245477	0.245329	1.000603
0.2	0.802038	0.801768	1.000336
0.3	1.462893	1.463285	0.999732
0.4	2.448063	2.451681	0.998524
0.5	4.192739	4.210979	0.995668

The difference in the results obtained by the two methods is no more than 0.5%.

In order to comprehensively check the correctness of the results, except of the comparison of values of crank angle at given periods o time during motion, the comparison of acceleration at specific point of the mechanism is under investigation. Thus for the case of slider crank mechanism there are presented graphs of midpoint acceleration of the connecting rod for the two considered cases. In Fig. 3 the midpoint acceleration in X direction for the stick model of slider-crank mechanism is presented, and in Fig. 4 the same acceleration obtained by Adams software.

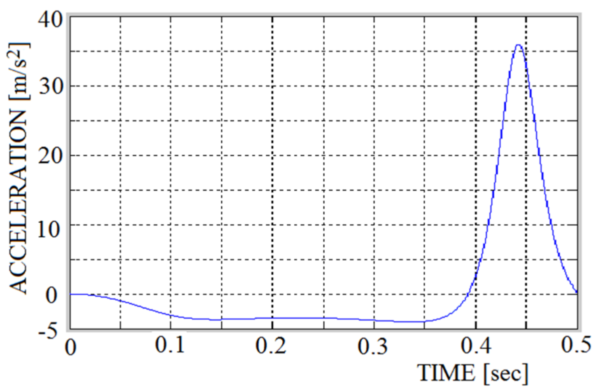

Fig. 3. Acceleration in X direction of connecting rod's midpoint of the slider-crank mechanism obtained for the stick model.

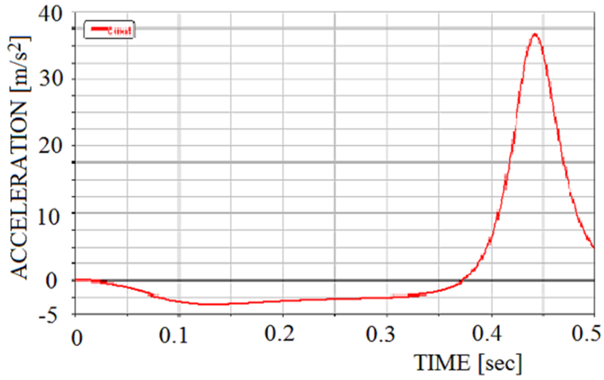

Fig. 4. Acceleration in X direction of connecting rod's midpoint of the slider-crank mechanism obtained by ADAMS program.

Analysis of plots received for acceleration of the slider-crank example shows that the computational algorithm used for the stick model is correct. Acceleration values as well as the graphs' shapes received for the specific point are comparable to those obtained in Adams environment.

5 Conclusions

A simple model (i.e. stick model consisting of modified truss-type finite elements) was presented for preliminary analysis of rigid-body motion of planar mechanisms. The validation of this model was conducted by comparison with the results obtained for solid model and by using multi-body approach used in Msc Adams environment. The examples of mechanisms considered were characterized by one degree of freedom (mechanisms with rigid links) and the comparison of the two methods presented concerned crank (or rod) angle or acceleration of specific points of mechanism (slider-crank mechanism). Based on the conducted analysis it can be concluded that the proposed stick model gives reliable results when compared to the solid model. By using the presented simple model based on modified truss-type elements, one can get correct results for displacement, velocity and acceleration of links of a planar rigid mechanism during motion.

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