

MULTIBODY KINEMATICS AND DYNAMICS

Chapter 1 Multibody Mechanical Systems

1.1 Introduction

A mechanical system is defined as a collection of bodies (or links) in which some or all of the bodies can move relative to one another. An example of a simple mechanical system is the single pendulum as shown in Fig. 1.1(a), which contains two bodies--the pendulum and the ground. Other examples of more complex mechanical systems are the four-bar linkage and the slider-crank mechanism as shown in Fig. 1.1(b) and (c), respectively, which are commonly used in motion transmission such as internal-combustion engine. While the motion of the systems in Fig. 1.1 is planar (two-dimensional), other may experience spatial (three-dimensional) motion, such as the suspension and the steering system of an automobile.

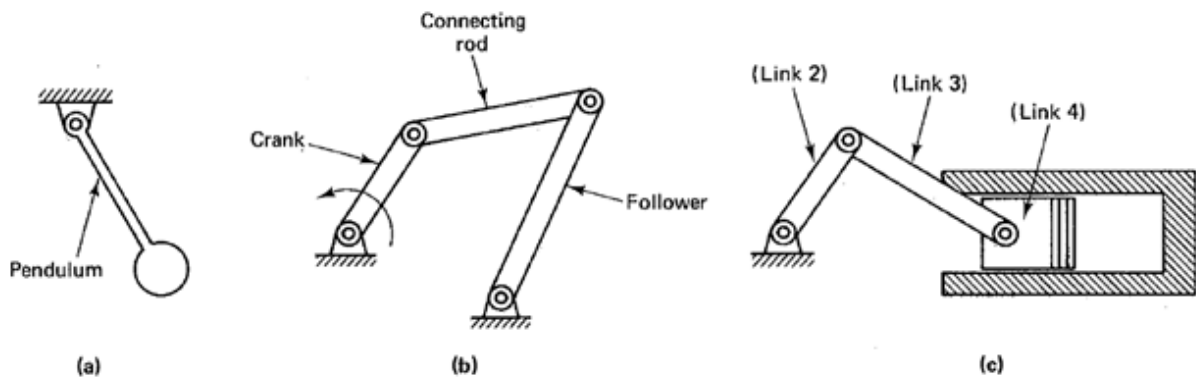


Figure 1.1 Examples of simple mechanical systems: (a) a single pendulum, (b) a four-bar mechanism, and (c) a slider-crank mechanism.

Another example of a mechanical system is a robot fixed to a stationary base or to a movable base as shown in Fig. 1.2. The motion of the end effector of a robot is controlled through force actuators located about each joint connecting the bodies that make up the robot.

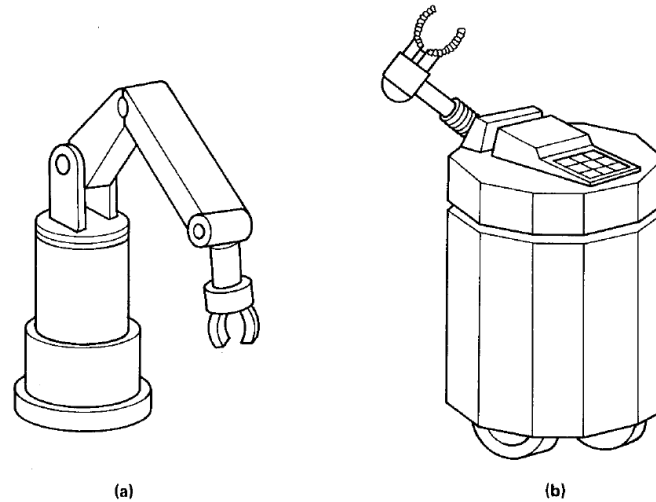


Figure 1.2 Examples of robot with (a) stationary base and (b) movable base.

Any mechanical system can be represented schematically as a multi-body system as shown in Fig. 1.3. The actual shape or outline of a body may not be of immediate concern in the process of analysis; of primary importance is the connectivity and the inertial characteristics of the bodies, the type and the location of the joints, and the physical characteristics of the springs, dampers in the system.

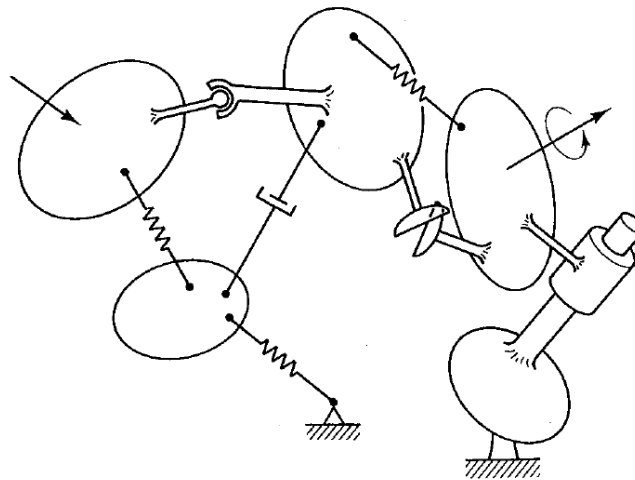


Figure 1.3 Schematic representation of a multi-body system.

1.2 Computer-Aided Design (CAD)

There are two different aspects to the study of a mechanical system: analysis and design. When a mechanical system is acted on by a given excitation, the system exhibits a certain response. The process which allows an engineer to study the response of a system to excitation is called analysis, which requires a complete knowledge of the physical characteristics such as material composition, shape, and arrangement of parts. The process of determining which physical characteristics are necessary for a mechanical system to perform a prescribed task is called design or synthesis, which requires the application of scientific techniques along with engineering judgment. Since the scientific aspect of a design process requires analysis techniques as a tool, it is important to learn about methods of analysis prior to design.

The branch of analysis which studies motion, time, and forces is called mechanics. It consists of two parts--statics and dynamics. Statics considers the analysis of stationary systems, in which time is not a factor, while dynamics deals with systems that are nonstationary, i.e., systems that change their response with respect to time. Dynamics is divided into two disciplines--kinematics and kinetics. Kinematics is the study of motion regardless of the forces that produce the motion; i.e., the study of displacement, velocity, and acceleration, while kinetics is the study of motion and its relationship with the forces that produce that motion.

The purpose of computer-aided analysis of mechanical systems is to develop computer formulation and solution of the equations of motion. This requires systematic techniques for formulating the equations and numerical methods for solving them. A computer program for the analysis of mechanical systems can be either a special-purpose program or a general-purpose program. A special-purpose program is a rigidly structured computer code with only one type of application. The equations of that particular application are derived a priori and then formulated into the program. As input to the program, the user can provide information such as the dimensions and physical characteristics of each part. Such a program can be made computationally efficient, but the drawback is its lack of flexibility for handling other types of applications.

A general-purpose program can analyze a variety of mechanical systems. For example, the planar motion of a four-bar linkage under applied loads and the spatial motion of a vehicle driven over a rough terrain can be simulated with the same general-purpose program. The input data to such a program are provided by the user and must completely describe the mechanical system under consideration. The input must contain such information as number of bodies, connectivity between the bodies, joint types, force elements, and geometric and physical characteristics. The program

then generates all of the governing equations of motion and solves them numerically. A general-purpose program, compared with a special-purpose program, is not computationally as efficient. The efficiency depends mainly upon the choice of coordinates and the method of numerical solution. The choice of coordinates directly influences both the number of the equations of motion and their order of nonlinearity.

1.3 Coordinate Systems

The governing equations of motion for a mechanical system can be derived and expressed in a variety of forms, dependent mainly upon the type of coordinates being employed. A set of coordinates \mathbf{q} selected for a system can describe the position of the elements in the system either with respect to each other or with respect to a common reference frame. In order to show how different sets of coordinates can lead to different formulations describing the same system, a four-bar linkage is considered for kinematic analysis.

The first formulation considers only one coordinate to describe the configuration of the system, since a four-bar linkage has only one degree of freedom. This is referred to as the generalized coordinate of the system. As shown in Fig. 1.4, the angle ϕ describing the orientation of the crank with respect to the ground can be selected as the generalized coordinate; i.e., $\mathbf{q} = [\phi]$

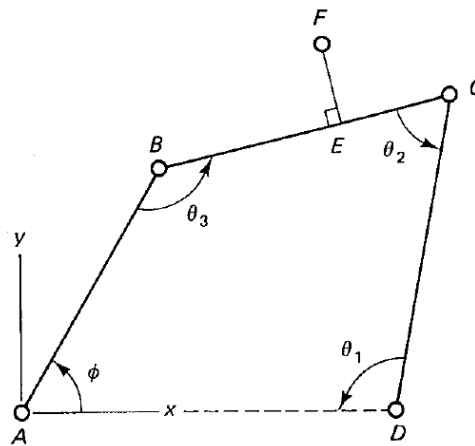


Figure 1.4 A four-bar mechanism with generalized coordinate ϕ .

For any given configuration; i.e., known ϕ , any other information on the position of any point in the system can be calculated. For example, the angles θ_1 , θ_2 , and θ_3 can be found by:

$$(r^2 + l^2 + s^2 - d^2) - 2rl \cos \phi + 2ls \cos \theta_1 - 2rs \cos(\phi - \theta_1) = 0 \quad (1.1)$$

$$(r^2 + l^2 + s^2 - d^2) - 2rl \cos \phi + 2ds \cos \theta_2 - 2rs \cos(\phi - \theta_2) = 0 \quad (1.2)$$

$$\phi + \theta_1 + \theta_2 + \theta_3 - 2\pi = 0 \quad (1.3)$$

where r , d , and s represent the lengths of the links, and l represents the distance between points A and D. These formulas are derived from simple geometric realizations. It is clear that for a given ϕ , Eq.(1.1) yields θ_1 , then Eq.(1.2) yields θ_2 , and finally Eq.(1.3) yields θ_3 . The solution of these equations requires direct substitution, and there is no need to solve a set of simultaneous algebraic equations. Now, it should be clear that the coordinates of a typical point attached to one of the links, say point F on link BC can be found easily.

The second way of formulating the kinematic equations for the four-bar linkage considers three coordinates. The selected coordinates may define the orientation of each moving body with respect to a nonmoving body or with respect to another moving body, referred to as relative coordinates. As shown in Fig. 1.5, angles ϕ_1 , ϕ_2 , and ϕ_3 , are selected as a set of generalized coordinates

$$\mathbf{q} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \quad (1.4)$$

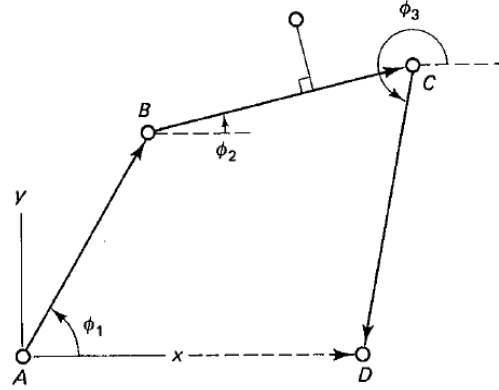


Figure 1.5 Relative coordinates describing the configuration of a four-bar system.

These angles are measured between the positive x -axis and the positive vectors representing the links. Since the four-bar linkage has only one degree of freedom, the three coordinates are not independent. Two loop equations relating these coordinates can be written as:

$$\begin{aligned} r \cos \hat{f}_1 + d \cos \hat{f}_2 + s \cos \hat{f}_3 - l &= 0 \\ r \sin \hat{f}_1 + d \sin \hat{f}_2 + s \sin \hat{f}_3 &= 0 \end{aligned} \quad (1.5)$$

For any given configuration; i.e., known ϕ_1 , the set of two simultaneous algebraic equations must be solved for ϕ_2 , and ϕ_3 . After Eq. (1.5) is solved, other information such as the coordinates of point F can be calculated.

The third formulation uses three Cartesian coordinates per link--the coordinates of the center point of each link and the angle of the link measured with respect to x -axis as shown in Fig. 1.6. Thus the set of coordinates describing the configuration of the four-bar linkage is

$$\mathbf{q} = [x_1 \ y_1 \ \phi_1 \ x_2 \ y_2 \ \phi_2 \ x_3 \ y_3 \ \phi_3]^T \quad (1.6)$$

These coordinates are dependent upon each other through eight equations:

$$\begin{aligned} x_1 - \frac{r}{2} \cos \phi_1 &= 0 \\ y_1 - \frac{r}{2} \sin \phi_1 &= 0 \\ x_1 + \frac{r}{2} \cos \phi_1 - x_2 + \frac{d}{2} \cos \phi_2 &= 0 \\ y_1 + \frac{r}{2} \sin \phi_1 - y_2 - \frac{d}{2} \sin \phi_2 &= 0 \\ x_2 + \frac{d}{2} \cos \phi_2 - x_3 - \frac{s}{2} \cos \phi_3 &= 0 \\ y_2 + \frac{d}{2} \sin \phi_2 - y_3 - \frac{s}{2} \sin \phi_3 &= 0 \\ x_3 - \frac{s}{2} \cos \phi_3 - l &= 0 \\ y_3 - \frac{s}{2} \sin \phi_3 &= 0 \end{aligned} \quad (1.7)$$

For any known configuration, any of the nine variables can be specified, and then the remaining eight variables can be found by solving the set of eight nonlinear algebraic equations in eight unknowns.

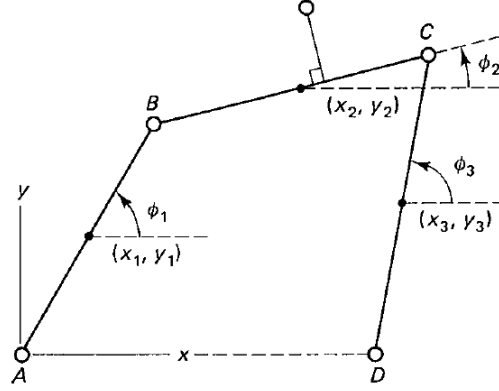


Figure 1.6 Cartesian coordinates describing the configuration of a four-bar linkage.

The three preceding formulations with generalized coordinates, relative coordinates, and Cartesian coordinates describe the kinematics of a four-bar mechanism. For dynamic analysis, the differential equations of motion for the four-bar linkage, can also be derived in terms of any of these sets of coordinates. For the four-bar linkage, formulation with generalized coordinates yields one second-order differential equation in terms of ϕ , $\dot{\phi}$, and $\ddot{\phi}$. This equation is highly nonlinear and complex in terms of ϕ and $\dot{\phi}$. The equations of motion for the four-bar linkage in terms of the relative coordinates consist of three second-order differential equations in terms of ϕ_i , $\dot{\phi}_i$, and $\ddot{\phi}_i$ for $i = 1, 2$, and 3 . The order of nonlinearity of these equations is not as high or as complex as in the first case. However, with these three differential equations, the two algebraic constraint equations of Eq.(1.5) must be considered. Therefore, the governing equations of motion for this system in terms of relative coordinates are a mixed set of algebraic-differential equations. Similarly, in terms of the Cartesian coordinates, nine second-order differential equations can be derived. Those, in conjunction with the eight algebraic constraint equations of Eq.(1.7), would define the governing equations of motion for the four-bar linkage. These algebraic-differential equations are loosely coupled and have a relatively low order of nonlinearity when compared with the previous sets.

A general comparison between these three sets of coordinates, with regard to several crucial and important aspects, is summarized in Table 1.1. A general conclusion that can be made from this table is that the smaller the number of coordinates and equations, the higher the order of nonlinearity and complexity of the governing equations of motion, and vice versa.

Table 1.1 Selection of Coordinate in Multibody Dynamics Modeling.

| | Generalized coordinates | Relative coordinates | Cartesian coordinates |
|---|----------------------------|-------------------------|--------------------------|
| Number of coordinates | Minimum | Moderate | Large |
| Number of second-order differential equations | Minimum | Moderate | Large |
| Number of algebraic constraint equations | None | Moderate | Large |
| Order of nonlinearity | High | Moderate | Low |
| Derivation of the equations of motion | Hard | Moderate hard | Simple |
| Computational efficiency | Efficient | Efficient | Not as efficient |
| Development of a general-purpose computer program | Difficult | Relatively difficult | Easy |

Chapter 2 Computation Kinematics

Kinematic analysis is necessary to generate, transmit, or control motion by the use of cams, gears, and linkages. An analysis of the displacement, velocity and acceleration is necessary to determine the design geometry of the mechanical parts. As a result of the generated motion, forces must be accounted for in the design of parts to determine the motion of a system of rigid bodies that results from applied forces. Kinematics analysis requires solution of nonlinear algebraic equations. For small problems with only a few variables and a few equations, it might be possible to write and solve these equations by hand. However, for large problems with many variables, numerical methods are necessary choice for fast and accurate solution.

2.1 Introduction

A rigid body is defined as a system of particles for which distances between particles remain unchanged. If a particle on such a body is located by a position vector fixed to the body, the vector never changes its position relative to the body, even when the body is in motion. In reality, all solid materials change shape to some extent when forces are applied to them. Nevertheless, if movement associated with the changes in shape is small compared with the overall movement of the body, then the concept of rigidity is acceptable. For example, displacements due to elastic vibration of the connecting rod of an engine may be of no consequence in the description of engine dynamics as a whole, so the rigid-body assumption is clearly in order. On the other hand, if the problem is the stress/slash strength of the connecting rod due to vibration, then its deformation becomes of prime importance. A mechanism is a set of rigid elements that are arranged to produce a specified motion. This definition of a mechanism includes classical linkages, as well as interconnected bodies that make up a vehicle, a vending machine, aircraft landing gear, an engine, and many other mechanical systems.

Kinematic synthesis is the process of finding the geometry of a mechanism that yields a desired set of motion characteristics. Kinematic analysis is a tool to support the kinematic synthesis process. The individual bodies that collectively form a mechanism are said to be links. The combination of two links in contact constitutes a kinematic pair, or joint. An assemblage of interconnected links is called a kinematic chain. A mechanism is formed when at least one of the links of the kinematic chain is held fixed and any of its other links can move. The fixed link is called the ground or frame. If some links undergo motion in three-dimensional space, the mechanism is called a spatial mechanism.

A mechanism that is formed from a collection of links or bodies kinematically connected to one another. An open-loop mechanism may contain links with single joints. An example of this kind of mechanism is the double pendulum shown in Fig.

2.1(a). A closed-loop mechanism is formed from a closed chain, where each link is connected to at least two other links of the mechanism. Figure 2.1(b) shows a four-bar linkage in a closed-loop mechanism. Kinematic analysis considers systems containing only closed loops.

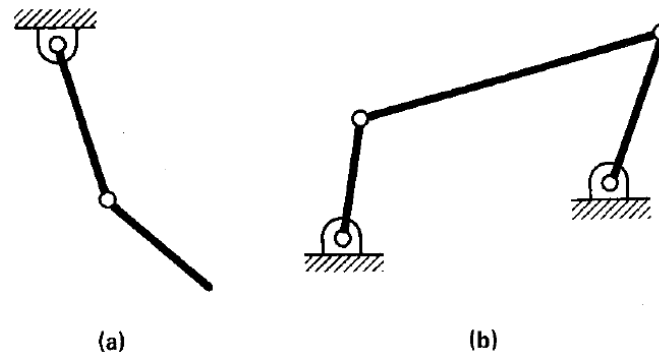


Fig. 2.1 (a) Open-loop mechanism—Double pendulum and (b) closed-loop mechanism—four-bar linkage.

A closed-loop mechanism may contain one or more loops (or closed paths) in its kinematic structure. If the number of loops in a closed-loop mechanism is 1, then the mechanism is called a single-loop mechanism. If the closed-loop mechanism contains more than one loop, then the mechanism is called a multi-loop mechanism. Figure 2.2(a) is an example of a single-loop mechanism, and Fig. 2.2(b) shows a multi-loop mechanism.

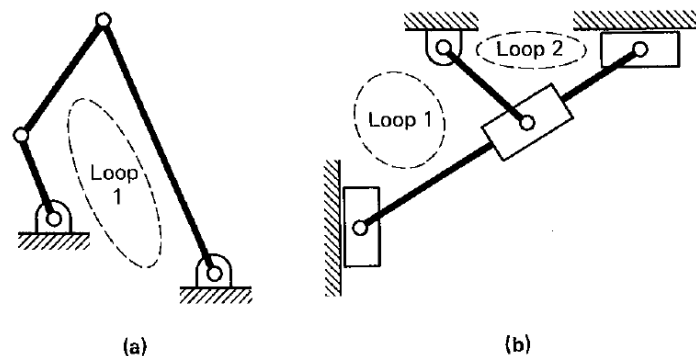


Figure 2.2 (a) Single-loop mechanism and (b) multi-loop mechanism.

Mechanisms and kinematic pairs can be classified in a number of different ways. One method is purely descriptive; e.g., gear pairs, cams and sliding pairs. However, kinematic pairs may be classified generally into two groups: joints with surface contact are referred to as lower pairs and those with point or line contact are referred to as higher pairs. Figure 2.3 gives a number of examples of kinematic pairs. The pairs (a), (b), (e), and (f) in Fig. 2.3 are examples of lower-pair joints, and pairs (c)

and (d) are examples of higher-pair joints. Relative motion between two bodies of a kinematic pair may be planar or spatial. For example, pairs (a), (b), (c), and (d) in Fig. 2.3 display relative motion between bodies in a manner that can be considered either for planar or spatial kinematic analysis. In contrast, pairs (e) and (f) can be studied only in a spatial kinematics.

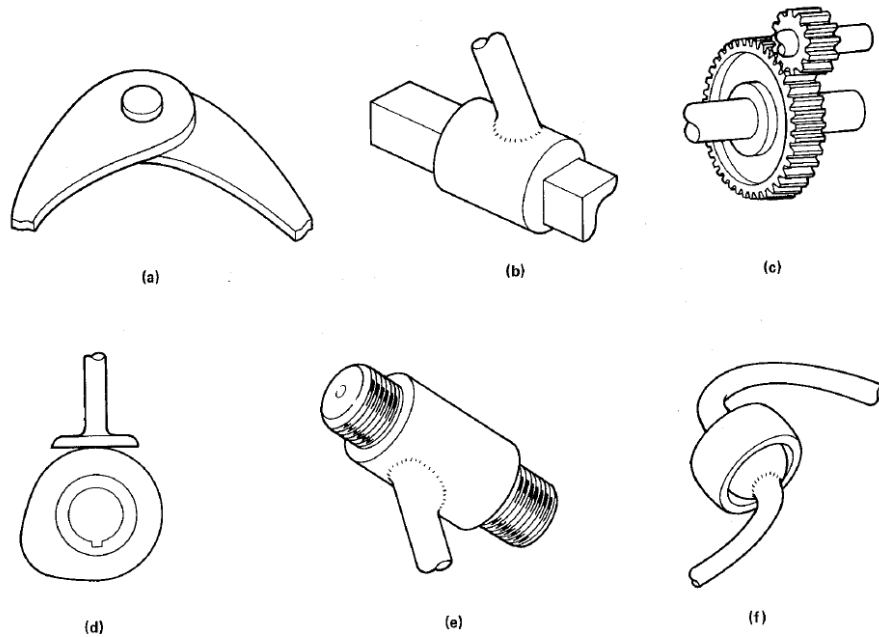


Figure 2.3 Example of kinematic pairs: (a) revolute joint, (b) translational joint, (c) gear set, (d) cam follower, (e) screw joint, and (f) spherical ball joint.

2.2 Coordinate System and Degree-of-freedom

Any set of parameters that uniquely specifies the position (configuration) of all bodies of a mechanism is called a set of coordinates. For systems in motion, these parameters vary with time. The term coordinates can refer to any of the commonly used coordinate systems, but it can also refer to any of an infinite variety of other sets of parameters that serve to specify the configuration of a system. Vectors of coordinates are designated by column vectors $\mathbf{q} \equiv [q_1, q_2, \dots, q_n]^T$, where n is the total number of generalized coordinates used in describing the system. Examples of commonly used coordinates are Lagrangian and Cartesian coordinates. The distinction between the two coordinates is that the former allows definition of the position of a body relative to a moving coordinate system, whereas the latter normally requires that the position of each body in space be defined relative to a fixed global coordinate system. Thus the Cartesian coordinate system requires that a large number of coordinates be defined to specify the position of each body of the system.

In order to specify the configuration of a planar system, a body-fixed $\xi\eta$ coordinate system is embedded in each body of the system as shown in Fig. 2.4(a). Body i can be located by specifying the global translational coordinates $\mathbf{r}_i = [x, y]_i^T$ of the origin of the body-fixed $\xi_i\eta_i$ reference system and the angle ϕ_i of rotation of this system relative to the global xy axes. The column vector $\mathbf{q}_i \equiv [x, y, \phi]_i^T$ is the vector of coordinates for body i in a plane.

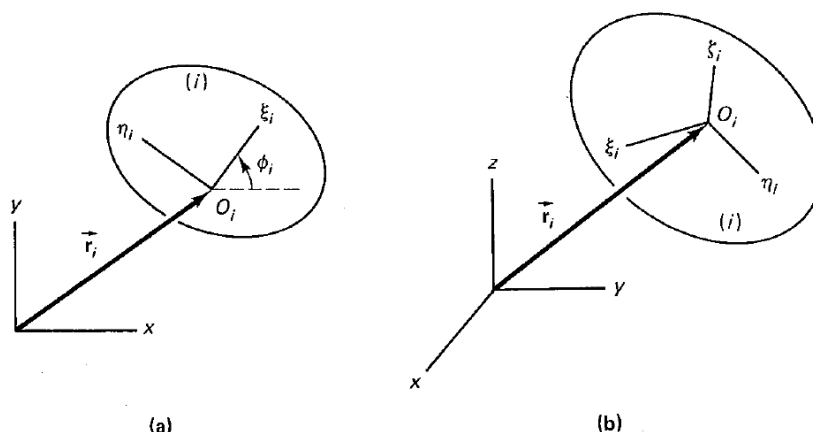


Figure 2.4 Global and body-fixed coordinate systems: (a) planar motion and (b) spatial motion.

For spatial systems, six coordinates are required to define the configuration of each body; e.g., body i shown in Fig. 2.4(b). The three components of the vector \mathbf{r}_i , the global translational coordinates $\mathbf{r}_i \equiv [x, y, z]_i^T$ to locate the origin of the body-fixed $\xi_i\eta_i\zeta_i$ reference system relative to the global xyz axes and three rotational coordinates ϕ_1 , ϕ_2 , and ϕ_3 specify the angular orientation of the body. Therefore, column vector $\mathbf{q}_i \equiv [x, y, z, \phi_1, \phi_2, \phi_3]_i^T$ is the vector of coordinates for body i in three-dimensional space. If a mechanism with b bodies is considered, the number of coordinates required is $n = 3 \times b$ if the system is planar, and $n = 6 \times b$ if the system is spatial. The overall vector of coordinates for the system is denoted by $\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \dots, \mathbf{q}_b^T]^T$. Since bodies making up a mechanism are interconnected by joints, all of the coordinates are not independent—there are equations of constraint relating the coordinates.

The minimum number of coordinates required to fully describe the configuration

of a system is called the number of degrees of freedom (DOF) of the system. Consider the triple pendulum shown in Fig. 2.5(a). Here, no fewer than three angles, ϕ_1 , ϕ_2 , and ϕ_3 , can uniquely determine the configuration of the system. Therefore, the triple pendulum has 3 degrees of freedom. Similarly, for the four-bar mechanism shown in Fig. 2.5(b), three variables ϕ_1 , ϕ_2 , and ϕ_3 , define the configuration of the system. However, the angles are not independent. There exist two algebraic constraint equations,

$$\begin{aligned} l_1 \cos \phi_1 + l_2 \cos \phi_2 - l_3 \cos \phi_3 - d_1 &= 0 \\ l_1 \sin \phi_1 + l_2 \sin \phi_2 - l_3 \sin \phi_3 - d_2 &= 0 \end{aligned} \quad (2.1)$$

which define loop closure of the mechanism. The two equations can be solved for ϕ_2 and ϕ_3 as a function of ϕ_1 . Therefore, ϕ_1 is the only variable needed to define the configuration of the system, and so there is only 1 degree of freedom for the four-bar mechanism.

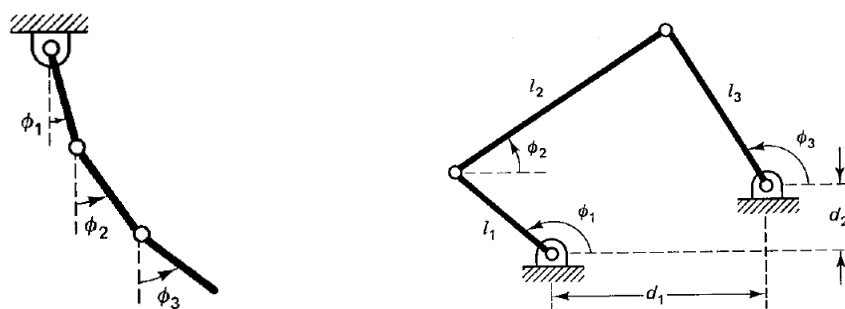


Fig. 2.5(a) Triple pendulum and (b) four-bar mechanism.

In a mechanical system, if k is the number of degrees of freedom of the system, then k independent coordinates are required to completely describe the system. These k quantities need not all have the dimensions of length. Depending on the system, it may be convenient to choose some coordinates with dimensions of length and some that are dimensionless, such as angles. Any set of coordinates that are independent and are equal in number to the number of degrees of freedom of the system is called a set of independent coordinates. Any remaining coordinates, which may be determined as a function of the independent coordinates, are called dependent coordinates.

2.3 Constraint Equations

A kinematic pair imposes certain conditions on the relative motion between the two bodies it comprises. When these conditions are expressed in analytical form, they are called equations of kinematic constraint. Since in a kinematic pair the

motion of one body fully or partially defines the motion of the other, it becomes clear that the number of degrees of freedom of a kinematic pair is less than the total number of degrees of freedom of two free rigid bodies. Therefore, a constraint is any condition that reduces the number of degrees of freedom in a system. A constraint equation describing a condition on the vector of coordinates of a system can be expressed as follows:

$$\Phi \equiv \Phi(\mathbf{q}) = 0 \quad (2.2)$$

In some constraint equations, the variable time may appear explicitly:

$$\Phi \equiv \Phi(\mathbf{q}, t) = 0 \quad (2.3)$$

For example, Eq. (2.1) describes two constraint equations for the four-bar mechanism, which has a vector of coordinates $\mathbf{q} = [\phi_1, \phi_2, \phi_3]^T$. Algebraic equality constraints in terms of the coordinates and time are said to be holonomic constraints. In general, if constraint equations contain inequalities or relations between velocity components that are not integrable in closed form, they are said to be nonholonomic constraints. The term constraint will refer to a holonomic constraint, unless specified otherwise.

A brief study of a mechanism is essential prior to actual kinematic or dynamic analysis. Knowledge of the number of degrees of freedom of the mechanism can be useful when constraint equations are being formulated. The pictorial description of a mechanism can often be misleading. Several joints may restrict the same degree of freedom and may therefore be equivalent or redundant. As an example, consider the double parallel-crank mechanism shown in Fig. 2.6(a). This system has 1 degree of freedom. If this system is modeled for kinematic analysis as four moving bodies and six revolute joints, the set of constraint equations will contain redundant equations. The reason for redundancy becomes clear when one of the coupler links is removed to obtain the mechanism shown in Fig. 2.6(b). The two mechanisms are kinematically equivalent.

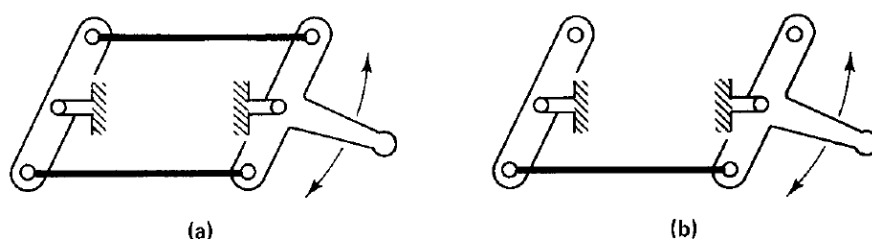


Figure 2.6 (a) A double parallel-crank mechanism and (b) its kinematically equivalent.

For a system having m independent constraint equations and n coordinates, the number of degrees of freedom is determined as follows:

$$k = n - m \quad (2.4)$$

In planar motion, a moving body can have three coordinates, and a revolute joint introduces two constraint equations. For the mechanism of Fig. 2.6(b), there are three moving bodies ($n = 3 \times 3 = 9$) and four revolute joints ($m = 4 \times 2 = 8$). Therefore, $k = 9 - 8 = 1$ DOF.

Chapter 3 Planar Kinematics

If all links of a mechanical system undergo motion in one plane or in parallel planes, the system is said to be experiencing planar motion. Kinematic analysis of mechanical systems using Cartesian coordinates is no different in principle from the method of analysis with Lagrangian coordinates. The use of Cartesian coordinates, however, results in a larger number of coordinates and constraint equations. The number of degrees of freedom of a system, however, is the same regardless of the type of coordinates used. Since the number of independent coordinates is equal to the number of degrees of freedom of a system, then the number of dependent Cartesian coordinates is generally greater than the number of dependent Lagrangian coordinates.

3.1 Cartesian Coordinates

In order to specify the configuration or state of a planar mechanical system, it is first necessary to define coordinates that specify the location of each body. Let xy the coordinate system be a global reference frame as shown in Fig. 3.1. Define a body-fixed $\xi_i\eta_i$ coordinate system and body i can be located in the plane by specifying the global coordinates $\mathbf{r}_i = [x, y]_i^T$ of the origin of the body-fixed coordinate system and the angle ϕ_i of rotation of this system relative to the global coordinate system. This angle is considered positive if the rotation from positive x axis to positive ξ_i axis is counterclockwise.

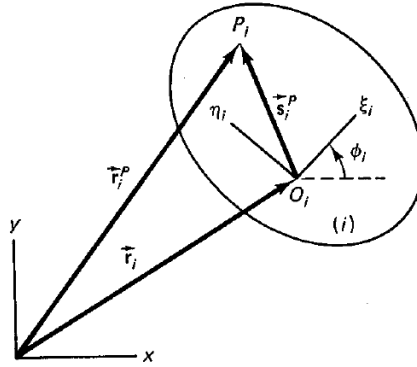


Figure 3.1 Locating point P relative to the body-fixed and global coordinates system.

A point P_i on body i can be located from the origin of the $\xi_i\eta_i$ axes by the vector \mathbf{s}_i^P . The coordinates of point P_i with respect to the $\xi_i\eta_i$ coordinate system are ξ_i^P and η_i^P . The local (body-fixed) components of vector \mathbf{s}_i^P are shown as

$\mathbf{s}_i'^P = [\xi^P, \eta^P]_i^T$. Since P_i is a fixed point on body i , ξ_i^P and η_i^P are constants, and therefore $\mathbf{s}_i'^P$ is a constant vector. The global xy components of vector \mathbf{s}_i^P are shown as \mathbf{s}_i^P . The elements of \mathbf{s}_i^P vary when body i rotates. Point P_i , may also be located by its global coordinates $\mathbf{r}_i^P = [x^P, y^P]_i^T$. It is clear that the components of \mathbf{r}_i^P are not necessarily constant, since body i may be in motion.

The relation between the local and global coordinates of point P_i is

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i'^P \quad (3.1)$$

where

$$\mathbf{A}_i = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}_i \quad (3.2)$$

is the rotational transformation matrix for body i . The transformation matrix \mathbf{A}_i is the simplified form of the following 3×3 matrix.

$$\mathbf{A}_i = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}_i$$

Equation (3.1) in expanded form can be written as

$$\begin{bmatrix} x^P \\ y^P \end{bmatrix}_i = \begin{bmatrix} x \\ y \end{bmatrix}_i + \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}_i \begin{bmatrix} \xi^P \\ \eta^P \end{bmatrix}_i$$

or

$$(3.3)$$

Now that

$$\mathbf{s}_i^P = \mathbf{A}_i \mathbf{s}_i'^P \quad (3.4)$$

is the relationship between the local and global components of vector \mathbf{s}_i^P .

The vector of coordinates for body i is denoted by the vector

$$\mathbf{q}_i = [\mathbf{r}^T, \phi]_i^T = [x, y, \phi]_i^T \quad (3.5)$$

For a mechanical system with b bodies, the coordinate vector is the $3 \times b$ vector.

$$\begin{aligned} \mathbf{q} &= [\mathbf{q}_1^T, \mathbf{q}_1^T, \dots, \mathbf{q}_b^T]^T \\ &= [x_1, y_1, \phi_1, x_2, y_2, \phi_2, \dots, x_b, y_b, \phi_b]^T \end{aligned} \quad (3.6)$$

where \mathbf{q} without a subscript denotes the vector of coordinates for the entire system.

3.2 Kinematic Constraints

In a mechanical system, the links and bodies may be interconnected by one or more kinematic joints. For example, the quick-return mechanism shown in Fig. 3.2(a) consists of five moving bodies interconnected by five revolute joints and two translational sliding joints. Since this mechanism undergoes planar motion, the motion of each moving body is described by three coordinates—two translational and one rotational. The kinematic joints in this system can be described as algebraic constraint equations.

In general formulation of lower-pair joints does not require any information on the shape (outline) of the connected bodies. For example, the quick-return mechanism of Fig. 3.2(a) can also be represented as shown in Fig. 3.2(b), where the body outlines are drawn arbitrarily. To derive constraint equations describing each joint, one need know only the position of the joint with respect to the bodies to which it is connected. In some higher-pair joints, either the entire shape or a partial shape of the body outline must be known. For example, in analyzing the motion of a cam-follower pair, the full or partial outlines of the cam and the follower must be described. In some other higher-pair joints, instead of the shape of the outline, the shape or curvature of a slot on one of the bodies must be known.

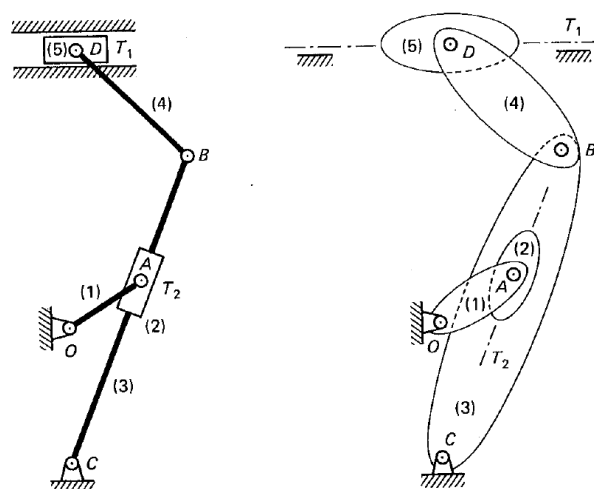


Figure 3.2 Quick-return mechanism: (a) schematic presentation and (b) its equivalent representation without showing the actual outlines.

The constraint equations are denoted by Φ with a superscript indicating the constraint type and the number of algebraic equations of that expression. For example, $\Phi^{(r,2)}$ denotes the revolute joint constraint which contains two equations, and $\Phi^{(t,2)}$ denotes the translational joint constraint which contains one equation.

(1) Revolute Joints (LP)

Revolute and translational joints are lower-pair kinematic joints. Examples of

revolute joints are joints A, B, C, D , and O in the quick-return mechanism of Fig. 3.2. Schematic representation of a revolute joint connecting to bodies i and j is shown in Fig. 3.3. The center of the joint is denoted by point P . This point can be considered to be two coincident points; point P_i on body i and point P_j on body j . Location of point P on body i and body j can be described by the two vectors s_i^P and s_j^P , where $s_i^P = [\xi_i^P, \eta_i^P]^T$ and $s_j^P = [\xi_j^P, \eta_j^P]^T$ are constants. The constraint equations for the revolute joint are obtained from the vector loop equation

$$\mathbf{r}_i + \mathbf{s}_i^P - \mathbf{r}_j - \mathbf{s}_j^P = \mathbf{0}$$

which is equivalent to

$$\Phi^{(r,2)} \equiv \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i'^P - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j'^P = \mathbf{0} \quad (3.7)$$

More explicitly, Eq.(3.7) can be written in the form

$$\Phi^{(r,2)} \equiv \begin{bmatrix} x_i^P - x_j^P \\ y_i^P - y_j^P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.8)$$

Equation (3.8) can be written in expanded form, using Eq. 3.3, as

$$\Phi^{(r,2)} \equiv \begin{bmatrix} x_i + \xi_i^P \cos \phi_i - \eta_i^P \sin \phi_i - x_j - \xi_j^P \cos \phi_j + \eta_j^P \sin \phi_j \\ y_i + \xi_i^P \sin \phi_i + \eta_i^P \cos \phi_i - y_j - \xi_j^P \sin \phi_j - \eta_j^P \cos \phi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.9)$$

The two constraints of Eq. (3.7) reduce the number of degrees of freedom of the system by 2. Therefore, if the two bodies of Fig. 3.3 are not connected to any other bodies, then they have 4 degrees of freedom.

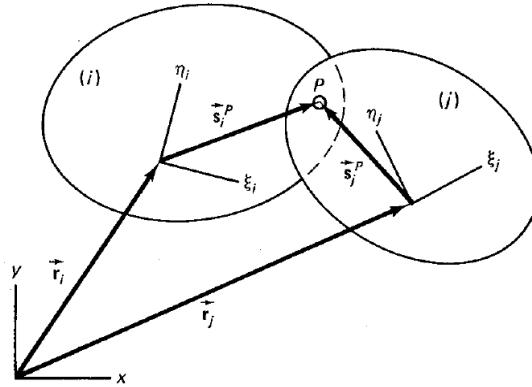


Figure 3.3 Revolute joint P connecting bodies i and j .

(2) Translational Joint

In the quick-return mechanism of Fig. 3.2, the two sliders (T_1 and T_2) are good examples of translational joints. This type of joint may appear in different shapes in a mechanism. Figure 3.4 illustrates several forms and presentations of translational

joints. In a translational joint, the two bodies translate with respect to each other parallel to an axis known as the line of translation; therefore, there is no relative rotation between the bodies. For a translational joint, there are an infinite number of parallel lines of translation. A constraint equation for eliminating the relative rotation between two bodies i and j is written as

$$\phi_i - \phi_j - (\phi_i^0 - \phi_j^0) = 0 \quad (3.10)$$

where ϕ_i^0 and ϕ_j^0 are the initial rotational angles. In order to eliminate the relative motion between the two bodies in a direction perpendicular to the line of translation, the two vectors s_i and d shown in Fig. 3.5 must remain parallel.

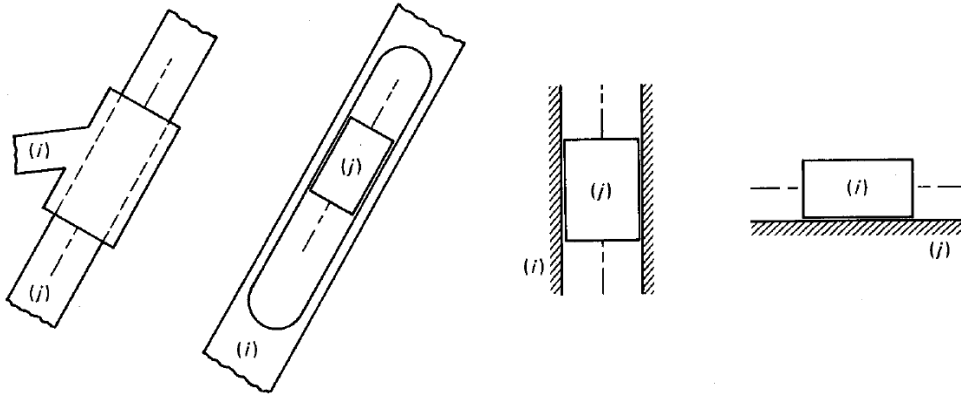


Figure 3.4 Different representations of a translational joint.

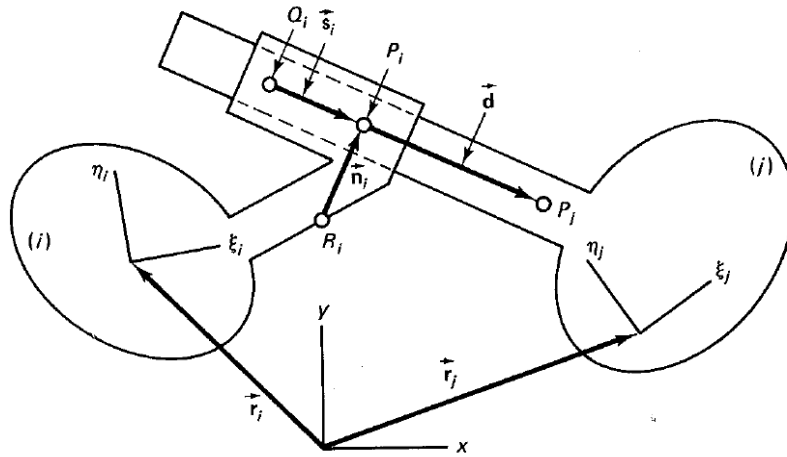


Figure 3.5 A translational joint between bodies i and j .

These vectors are defined by locating three points on the line of translation — two points on body i and one point on body j . This condition is enforced by letting the vector product of these two vectors be zero. A simple method would be to define

another vector \mathbf{n}_i perpendicular to the line of translation and to require that \mathbf{d} remain perpendicular to \mathbf{n}_i ; i.e.,

$$\mathbf{n}_i^T \mathbf{d} = 0$$

or

$$\begin{bmatrix} x_i^P - x_i^R & y_i^P - y_i^R \end{bmatrix} \begin{bmatrix} x_i^P - x_j^P \\ y_i^P - y_j^P \end{bmatrix} = 0$$

$$(x_i^P - x_i^R)(x_i^P - x_j^P) + (y_i^P - y_i^R)(y_i^P - y_j^P) = 0 \quad (3.11)$$

where

$$\mathbf{n}_i = \begin{bmatrix} x_i^P - x_i^R \\ y_i^P - y_i^R \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} x_i^P - x_j^P \\ y_i^P - y_j^P \end{bmatrix}$$

if $|\mathbf{n}_i| = |s_i|$, then

$$\mathbf{n}_i = \begin{bmatrix} x_i^P - x_i^R \\ y_i^P - y_i^R \end{bmatrix} = \begin{bmatrix} -(y_i^P - y_i^Q) \\ x_i^P - x_i^Q \end{bmatrix}$$

Therefore Eqs. (3.10) and (3.11) yield the two constraint equations for a translational joint as

$$\Phi_{(t,2)} = \begin{bmatrix} +(x_i^P - x_j^P)(y_i^P - y_i^Q) - (y_i^P - y_j^P)(x_i^P - x_j^Q) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.12)$$

Not that a translational joint reduces the number of degrees of freedom of a system by 2.

(3) Driving Link

In kinematically driven systems, the motion of one or more links (bodies) is usually defined. If kinematic analysis is to be performed then the motion of the driving link must be specified in the form of a driving constraint equation

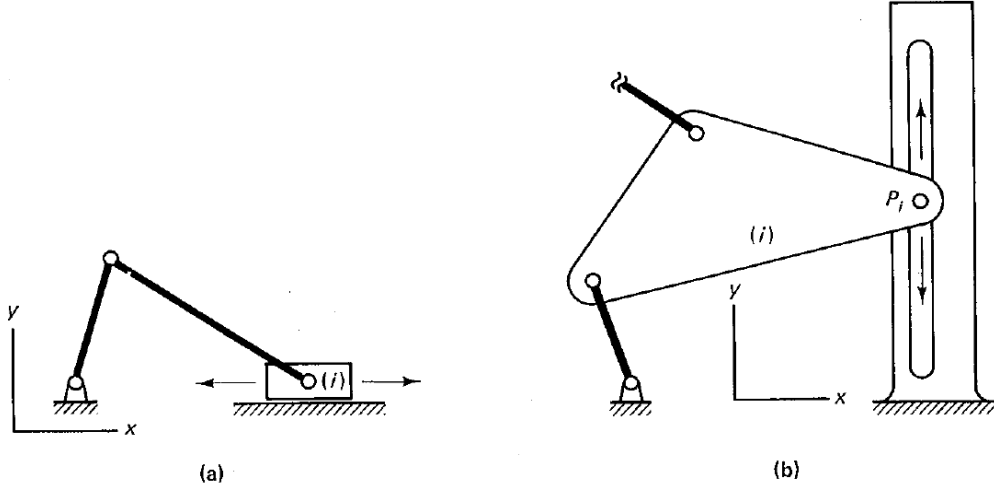


Figure 3.6 (a) The motion of the slider is controlled in the x -direction and (b) the motion of point P is controlled in the y -direction.

For example, if the motion of the slider shown in Fig. 3.6(a) is controlled as a function of time, then Eq.(3.13) can be used as the driving constraint. For the mechanism of Fig. 3.6(b), the motion of point P in the y -direction is controlled as a function of time, one moving constraint of the form

$$\Phi^{(d, 1)} \equiv y_i - d_i(t) = 0 \quad (3.13)$$

The method of appended driving constraints can now be stated in its most general form. If there are m kinematic constraints, then k driving constraints must be appended to the kinematic constraints to obtain $n = m + k$ equations:

$$\begin{aligned} \Phi &\equiv \Phi(q) = 0 \\ \Phi^{(d)} &\equiv \Phi(q, t) = 0 \end{aligned} \quad (3.14)$$

where superscript (d) denotes the driving constraints. Equation (3.14) represents n equations in n unknowns q which can be solved at any specified time t .

The velocity equations are obtained by taking the time derivative of Eq. (3.14):

$$\begin{aligned} \Phi_q \dot{q} &= 0 \\ \Phi_q^{(d)} \dot{q} + \Phi_t^{(d)} &= 0 \end{aligned} \quad (3.15)$$

or

$$\begin{bmatrix} \Phi_q \\ \Phi_q^{(d)} \end{bmatrix} \ddot{q} = \begin{bmatrix} 0 \\ -\Phi_t^{(d)} \end{bmatrix} \quad (3.16)$$

which represents n algebraic equations, linear in terms of \ddot{q} .

Similarly, the time derivative of Eq. (3.15) yields the acceleration equations:

$$\begin{aligned}\Phi_q \ddot{\mathbf{q}} + (\Phi_q \dot{\mathbf{q}})_q \dot{\mathbf{q}} &= 0 \\ \Phi_q^{(d)} \ddot{\mathbf{q}} + (\Phi_q^{(d)} \dot{\mathbf{q}})_q \dot{\mathbf{q}} + 2\Phi_{qt}^{(d)} \dot{\mathbf{q}} + \Phi_{tt}^{(d)} &= 0\end{aligned}\tag{3.17}$$

or

$$\begin{bmatrix} \Phi_q \\ \Phi_q^{(d)} \end{bmatrix} \ddot{\mathbf{q}} = \begin{bmatrix} -(\Phi_q \dot{\mathbf{q}})_q \dot{\mathbf{q}} \\ -(\Phi_q^{(d)} \dot{\mathbf{q}})_q \dot{\mathbf{q}} - 2\Phi_{qt}^{(d)} \dot{\mathbf{q}} - \Phi_{tt}^{(d)} \end{bmatrix}\tag{3.18}$$

which represents n algebraic equations linear in terms of $\ddot{\mathbf{q}}$. The term $-(\Phi_q \dot{\mathbf{q}})_q \dot{\mathbf{q}}$ in Eq. (3.18) is referred to as the right side of the kinematic acceleration equations and is represented as

$$\gamma = -(\Phi_q \dot{\mathbf{q}})_q \dot{\mathbf{q}}\tag{3.19}$$

3.3 Kinematic Analysis

The kinematic constraint equations Φ derived in the preceding sections for planar kinematic pairs are, in general, nonlinear in terms of the coordinates \mathbf{q} . If the number of coordinates describing the configuration of a mechanical system is n and the number of degrees of freedom of the system is k , then $m = n - k$ kinematic constraints can be defined as

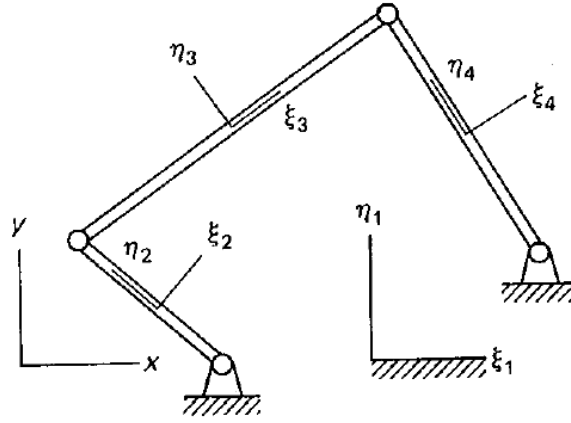
$$\Phi \equiv \Phi(\mathbf{q}) = 0\tag{3.20}$$

In addition, k driving constraint equations must be defined as

$$\Phi^{(d)} \equiv \Phi(\mathbf{q}, t) = 0\tag{3.21}$$

Equations (3.14) and (3.15) represent a set of n nonlinear algebraic equations which can be solved for n unknowns \mathbf{q} at any given time.

Consider a four-bar linkage in Cartesian coordinates. There are four bodies in the system—three moving links and one stationary link—the dimension of \mathbf{q} is $4 \times 3 = 12$. In the accompanying illustration, the body-fixed coordinates are attached to each link. Body 1 is assumed to be the frame, and bodies 2, 3, and 4 are the crank, the coupler, and the follower, respectively. Each revolute joint is represented by two algebraic equations, and therefore there are $4 \times 2 = 8$ algebraic equations representing all four revolute joints. In addition, since body 1 does not move with respect to the xy coordinate system, three simple constraints are needed. Therefore, the total number of kinematic constraints is $m = 3 + 8 = 11$. For the rotation of the crank, one driving constraint makes the total number of equations equal to 12.



The first and second time derivatives of Eq. (3.14) yield velocity and acceleration equations. For position analysis using the Newton-Raphson algorithm, velocity analysis, and acceleration analysis, the Jacobian matrix of the kinematic constraints Φ_q is needed. Also for acceleration analysis, the right side of the kinematic acceleration equations, vector γ as given in Eq. (3.20), is needed. For each of the kinematic pairs discussed in the preceding sections closed-form expressions can be derived for the entries of the Jacobian matrix and the right-side vector of the acceleration equations. From these expressions plus the constraint equations, all of the necessary terms for kinematic analysis can be assembled systematically.

Derivation of the velocity and acceleration equations requires the time derivatives of the constraint equations. In turn, these require the derivatives of the coordinates of the points describing the kinematic joint. The xy coordinates of a point P attached to body i can be found from Eq. (3.1), where it is assumed that the position of the point on the body and the coordinates of the body are known. The velocity of the point in the xy coordinate system can be found by taking the time derivative of Eq. (3.1),

$$\dot{\mathbf{r}}_i^P = \dot{\mathbf{r}}_i + \dot{\mathbf{A}}_i \mathbf{s}_i^P = \dot{\mathbf{r}}_i + \mathbf{B}_i \mathbf{s}_i^P \dot{\phi}_i \quad (3.22)$$

where

$$\mathbf{B}_i = \begin{bmatrix} -\sin \phi & -\cos \phi \\ \cos \phi & -\sin \phi \end{bmatrix}_i \quad (3.23)$$

Similarly, the time derivative of Eq. (3.23) yield the acceleration of point P :

$$\begin{aligned} \ddot{\mathbf{r}}_i^P &= \ddot{\mathbf{r}}_i + \ddot{\mathbf{A}}_i \mathbf{s}_i^P \\ &= \ddot{\mathbf{r}}_i + \mathbf{B}_i \mathbf{s}_i^P \ddot{\phi}_i - \mathbf{A}_i \mathbf{s}_i^P \dot{\phi}_i^2 \end{aligned} \quad (3.24)$$

Knowing the velocity of body i , i.e., $\dot{\mathbf{q}}_i = [\dot{\mathbf{r}}^T, \dot{\mathbf{f}}^T]^T$ one can use Eq. (3.22) to find the velocity of P_i ; and similarly, knowing the velocity and the acceleration of body i ; i.e., $\dot{\mathbf{q}}_i$ and $\ddot{\mathbf{q}}_i$, we can find the acceleration of P_i , by using Eq. (3.24).

Systematic generation of the Jacobian matrix and the right side of the acceleration equations for some of the standard kinematic joints can best be illustrated by deriving these elements for a revolute joint. Consider the revolute-joint constraint equations:

$$\Phi^{(r,2)} \equiv \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i'^P - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j'^P = 0$$

and in expanded form they are

$$\Phi^{(r,1st)} \equiv x_i + \xi_i^P \cos \phi_i - \eta_i^P \sin \phi_i - x_j - \xi_j^P \cos \phi_j + \eta_j^P \sin \phi_j = 0$$

$$\Phi^{(r,2nd)} \equiv y_i + \xi_i^P \sin \phi_i + \eta_i^P \cos \phi_i - y_j - \xi_j^P \sin \phi_j - \eta_j^P \cos \phi_j = 0$$

The partial derivative of these equation with respect to \mathbf{q} , i.e., $\Phi_q^{(r,2)}$, provides two rows to the overall system Jacobin. Since $\Phi^{(r,2)}$ is a function of only \mathbf{q}_i and \mathbf{q}_j , $\Phi_q^{(r,2)}$ may have nonzero elements only in the columns associated with \mathbf{q}_i and \mathbf{q}_j . The entries of the Jacobian matrix can also be found by taking the time derivative of the constraint equations. The time derivative of the constraint equations for a revolute joint is

$$\dot{x}_i - (X_i^P \sin f_i + h_i^P \cos f_i) \dot{f}_i - \dot{x}_j + (X_j^P \sin f_j + h_j^P \cos f_j) \dot{f}_j = 0$$

$$\dot{y}_i + (X_i^P \cos f_i - h_i^P \sin f_i) \dot{f}_i - \dot{y}_j - (X_j^P \cos f_j - h_j^P \sin f_j) \dot{f}_j = 0$$

or

$$\begin{bmatrix} \textcircled{1} & 0 & \textcircled{2} & \textcircled{3} & 0 & \textcircled{4} \\ 0 & \textcircled{5} & \textcircled{6} & 0 & \textcircled{7} & \textcircled{8} \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\phi}_i \\ \dot{x}_j \\ \dot{y}_j \\ \dot{\phi}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To obtain the right side of the acceleration equation for a revolute joint, either Eq.

(3.19) can be used, or the velocity equations can be differentiated with respect to time to obtain the acceleration equation:

$$\begin{aligned} \ddot{x}_i - (x_i^P \sin f_i + h_i^P \cos f_i) \ddot{f}_i - (x_i^P \cos f_i - h_i^P \sin f_i) \dot{f}_i^2 - \ddot{x}_j \\ + (x_j^P \sin f_j + h_j^P \cos f_j) \ddot{f}_j + (x_j^P \cos f_j - h_j^P \sin f_j) \dot{f}_j^2 = 0 \\ \ddot{y}_i - (x_i^P \cos f_i - h_i^P \sin f_i) \ddot{f}_i - (x_i^P \sin f_i + h_i^P \cos f_i) \dot{f}_i^2 - \ddot{y}_j \\ - (x_j^P \cos f_j - h_j^P \sin f_j) \ddot{f}_j - (x_j^P \sin f_j + h_j^P \cos f_j) \dot{f}_j^2 = 0 \end{aligned}$$

or

$$\begin{bmatrix} \textcircled{1} & 0 & \textcircled{2} & \textcircled{3} & 0 & \textcircled{4} \\ 0 & \textcircled{5} & \textcircled{6} & 0 & \textcircled{7} & \textcircled{8} \end{bmatrix} \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \\ \ddot{x}_j \\ \ddot{y}_j \\ \ddot{\phi}_j \end{bmatrix} = \gamma^{(r,2)}$$

where

$$\begin{aligned} \gamma^{(r,2)} &= \begin{bmatrix} (\xi_i^P \cos \phi_i - \eta_i^P \sin \phi_i) \dot{\phi}_i^2 - (\xi_j^P \cos \phi_j - \eta_j^P \sin \phi_j) \dot{\phi}_j^2 \\ (\xi_i^P \sin \phi_i - \eta_i^P \cos \phi_i) \dot{\phi}_i^2 + (\xi_j^P \sin \phi_j + \eta_j^P \cos \phi_j) \dot{\phi}_j^2 \end{bmatrix} \\ &= \begin{bmatrix} (x_i^P - x_i) \dot{\phi}_i^2 - (x_j^P - x_j) \dot{\phi}_j^2 \\ (y_i^P - y_i) \dot{\phi}_i^2 - (y_j^P - y_j) \dot{\phi}_j^2 \end{bmatrix} \\ &= s_i^P \dot{\phi}_i^2 - s_j^P \dot{\phi}_j^2 \end{aligned}$$

The entries of the Jacobian matrix and the vector of the right side of the acceleration equations for a revolute joint can be generated systematically. Tables 3.1 and 3.2 summarize the elements of the Jacobian matrix and the vector γ , respectively, for some of the constraint equations of the basic joints. Similar elements can be derived for other kinematic pairs.

Table 3.1 Elements of the Jacobin matrix for some of the basic constraint equation.

| | $\partial\Phi/\partial x_i$ | $\partial\Phi/\partial y_i$ | $\partial\Phi/\partial \phi_i$ | $\partial\Phi/\partial x_j$ | $\partial\Phi/\partial y_j$ | $\partial\Phi/\partial \phi_j$ |
|----------------|-----------------------------|-----------------------------|--|-----------------------------|-----------------------------|---|
| $\Phi^{(r,2)}$ | 1 | 0 | $-(y_i^P - y_i)$ | -1 | 0 | $(y_j^P - y_j)$ |
| | 0 | 1 | $(x_i^P - x_i)$ | 0 | -1 | $-(x_j^P - x_j)$ |
| $\Phi^{(t,2)}$ | $(y_i^P - y_i^Q)$ | $-(x_i^P - x_i^Q)$ | $-(x_j^P - x_j)(x_i^P - x_i^Q)$ $-(y_j^P - y_j)(y_i^P - y_i^Q)$ | $-(y_i^P - y_i^Q)$ | $(x_i^P - x_i^Q)$ | $(x_j^P - x_j)(x_i^P - x_i^Q)$ $+(y_j^P - y_j)(y_i^P - y_i^Q)$ |
| | 0 | 0 | 1 | 0 | 0 | -1 |

Table 3.2 Vector γ for some of the basic constraint equation.

| γ | |
|----------------|---|
| $\Phi^{(r,2)}$ | $(x_i^P - x_i)\dot{f}_i^2 - (x_j^P - x_j)\dot{f}_j^2$ |
| | $(y_i^P - y_i)\dot{f}_i^2 - (y_j^P - y_j)\dot{f}_j^2$ |
| $\Phi^{(t,2)}$ | $-2[(x_i^P - x_i^Q)(\dot{x}_i - \dot{x}_j) + (y_i^P - y_i^Q)(\dot{y}_i - \dot{y}_j)]\dot{f}^*$ $-[(x_i^P - x_i^Q)(y_i - y_j) - (y_i^P - y_i^Q)(x_i - x_j)]\dot{f}_i^2$ |
| | 0 |

3.4 Kinematic Modeling of a Slider-Crank System

The slider-crank is one of the most widely used mechanisms in practice. This mechanism finds its greatest application in the internal-combustion engine. The mechanism is made of four links or bodies as illustrated in Fig. 3.7(a). The bodies are numbered from 1 to 4 as shown in Fig. 3.7(b). Body 1 is the fixed link (ground, chassis, or engine block), body 2 is the crank, body 3 is the connecting rod, and body 4 is the slider. Bodies 1 and 2, 2 and 3, and 3 and 4 are connected by revolute joints A, B, and O, respectively. Bodies 1 and 4 are connected by a translational joint T . The number of degrees of freedom for this mechanism is $k = 4 \times 3 - (3 \times 2 + 1 \times 2 + 3) - 1$, since there are $4 \times 3 = 12$ coordinates in the system, 3 revolute joints eliminate 6 DOF, 1 translational joint eliminates 2 DOF, and ground constraints on body 1 eliminate 3 DOF.

The body-fixed coordinates $\xi\eta$ are attached to each body, including the ground, as shown in Fig. 3.7(c). Positioning of these coordinate systems is quite arbitrary for kinematic analysis. However, it is good practice to locate the origin of the coordinate system at the center of gravity of the body. Furthermore, aligning at least one of the coordinate axes with the link axis or parallel to some line of certain geometric or kinematic importance may simplify the task of collecting data for the kinematic pairs in the system.

For the three revolute joints, the following data are obtained from Fig. 3.7(c):

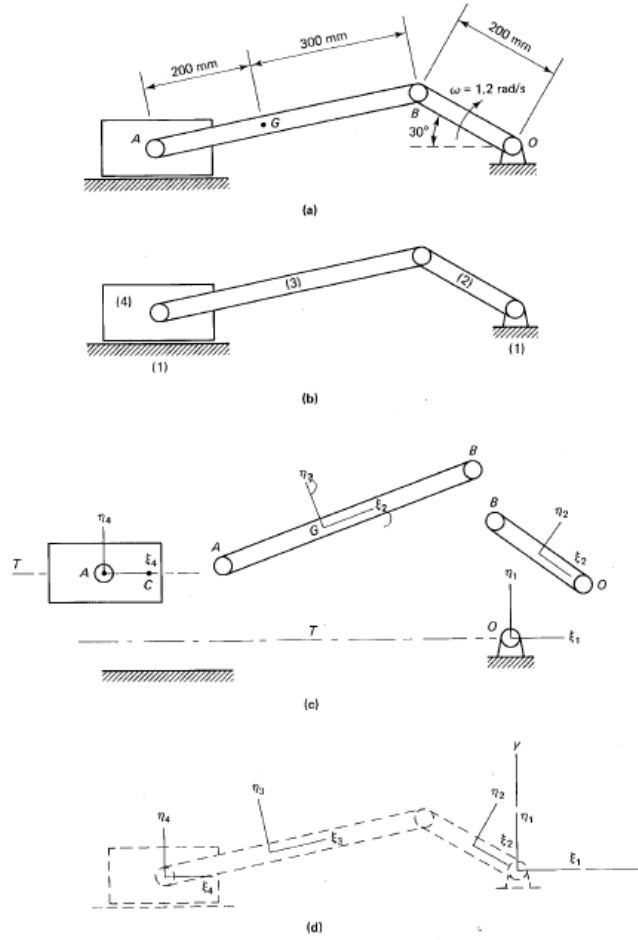


Figure 3.7 Kinematic modeling of a slider-crank mechanism.

$$\begin{aligned}
 x_4^A &= 0.0, & h_4^A &= 0.0, & x_3^A &= -200.0, & h_3^A &= 0.0 \\
 x_3^B &= 300.0, & h_3^B &= 0.0, & x_2^B &= -100.0, & h_2^B &= 0.0 \\
 x_2^O &= 100.0, & h_2^O &= 0.0, & x_1^O &= 0.0, & h_1^O &= 0.0
 \end{aligned} \tag{a}$$

For the translational joint, two points on body 4 and one point on body 1 are chosen. The location of these points is arbitrary, as long as they are on the same line of translation. These points are A, C, and O. The distance AC is taken here as 100 mm. The data that define the translational joint constraints are as follows:

$$\begin{aligned}
 \xi_4^A &= 0.0, & \eta_4^A &= 0.0, & \xi_4^C &= 100.0, & \eta_4^C &= 0.0, \\
 \xi_1^O &= 0.0, & \eta_1^O &= 0.0
 \end{aligned} \tag{b}$$

Additional constraints are needed to ensure that body 1 is the non-moving body. A global xy coordinate system is added to the configuration as shown in Fig. 3.7(d).

For convenience, the xy coordinate system is positioned to coincide with the x_1h_1 coordinates. Therefore, the conditions

$$x_1 = 0.0, \quad y_1 = 0.0, \quad \phi_1 = 0.0 \quad (c)$$

must be satisfied. The driving constraint on ϕ_2 is written as

$$\phi_2 - 5.76 + 1.2t = 0.0 \quad (d)$$

where 5.76 rad is equal to $360^\circ - 30^\circ$. Since the crank rotates clockwise, a negative sign is selected for ω .

Since there are 12 coordinates in this problem, the computer program must generate 12 kinematic constraint equations. These equations can be obtained by substituting the data of Eqs. (a) to (d) in the constraint equations derived in the previous sections. Equation (c) yields three constraints:

$$\begin{aligned} \Phi_1 &\equiv x_1 = 0.0 \\ \Phi_2 &\equiv y_1 = 0.0 \\ \Phi_3 &\equiv \phi_1 = 0.0 \end{aligned} \quad (f)$$

Substitution of Eq. (a) into the revolute-joint constraints yields six constraint equations for the three revolute joints:

$$\begin{aligned} \Phi_4 &\equiv x_4 - x_3 + 200 \cos \phi_3 = 0 \\ \Phi_5 &\equiv y_4 - y_3 - 200 \sin \phi_3 = 0 \\ \Phi_6 &\equiv x_3 + 300 \cos \phi_3 - x_2 + 100 \cos \phi_2 = 0 \\ \Phi_7 &\equiv y_3 + 300 \sin \phi_3 - y_2 + 100 \sin \phi_2 = 0 \\ \Phi_8 &\equiv x_2 + 100 \cos \phi_2 - x_1 = 0 \\ \Phi_9 &\equiv y_2 + 100 \sin \phi_2 - y_1 = 0 \end{aligned}$$

Two constraint equations for the translational joint of Eq. (b)

$$\begin{aligned} \Phi_{10} &\equiv (+100 \sin \phi_4)(y_1 + 100 \cos \phi_1 - y_4) + (x_1 - 100 \sin \phi_1 - x_4)(+100 \cos \phi_4) = 0 \\ \Phi_{11} &\equiv \phi_4 - \phi_1 = 0 \end{aligned}$$

Equation (d) is the driving constraint:

$$\Phi_{12} \equiv \phi_2 - 5.76 + 1.2t = 0$$

The nonzero entries of the Jacobian matrix for these 12 constraints are found from Table 3.2. The positions of these entries are shown in Fig. 3.8 entries in the 12 x 12 matrix. The nonzero entries are:

$$\begin{aligned}
&\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{6}, \textcircled{10}, \textcircled{14}, \textcircled{18}, \textcircled{22}, \textcircled{26}, \textcircled{34}, \textcircled{36} = 1 \\
&\textcircled{4}, \textcircled{8}, \textcircled{12}, \textcircled{16}, \textcircled{20}, \textcircled{24}, \textcircled{35} = -1 \\
&\textcircled{7}, \textcircled{11}, \textcircled{21}, \textcircled{25}, = 0 \\
&\textcircled{5} = -200 \sin \phi_3 \\
&\textcircled{9} = 200 \cos \phi_3 \\
&\textcircled{13}, \textcircled{23} = 100 \sin \phi_2 \\
&\textcircled{15} = -300 \sin \phi_3 \\
&\textcircled{17}, \textcircled{27} = 100 \cos \phi_2 \\
&\textcircled{19} = 300 \cos \phi_3 \\
&\textcircled{28} = 100 \cos \phi_4 \\
&\textcircled{29} = -100 \sin \phi_4 \\
&\textcircled{30} = -10,000(\cos \phi_1 \cos \phi_4 + \sin \phi_1 \sin \phi_4) \\
&\textcircled{31} = -\textcircled{28} \\
&\textcircled{32} = -\textcircled{29} \\
&\textcircled{33} = 100[\cos \phi_4(y_1 + 100 \cos \phi_1 - y_4) - \sin \phi_4(x_1 - 100 \sin \phi_1 - x_4)]
\end{aligned}$$

Not that some of the entries at any given instant of time, depending on the value of the coordinates, may become zero. This example illustrates that about 83 percent of the entries of this Jacobian matrix are exactly zero. Therefore, the matrix is said to be sparse.

| | x_1 | y_1 | ϕ_1 | x_2 | y_2 | ϕ_2 | x_3 | y_3 | ϕ_3 | x_4 | y_4 | ϕ_4 |
|---------------------------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| $\partial \Phi_1 / \partial \dots$ | $\textcircled{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\partial \Phi_2 / \partial \dots$ | 0 | $\textcircled{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\partial \Phi_3 / \partial \dots$ | 0 | 0 | $\textcircled{3}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\partial \Phi_4 / \partial \dots$ | 0 | 0 | 0 | 0 | 0 | 0 | $\textcircled{4}$ | 0 | $\textcircled{5}$ | $\textcircled{6}$ | 0 | $\textcircled{7}$ |
| $\partial \Phi_5 / \partial \dots$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\textcircled{8}$ | $\textcircled{9}$ | 0 | $\textcircled{10}$ | $\textcircled{11}$ |
| $\partial \Phi_6 / \partial \dots$ | 0 | 0 | 0 | $\textcircled{12}$ | 0 | $\textcircled{13}$ | $\textcircled{14}$ | 0 | $\textcircled{15}$ | 0 | 0 | 0 |
| $\partial \Phi_7 / \partial \dots$ | 0 | 0 | 0 | 0 | $\textcircled{16}$ | $\textcircled{17}$ | 0 | $\textcircled{18}$ | $\textcircled{19}$ | 0 | 0 | 0 |
| $\partial \Phi_8 / \partial \dots$ | $\textcircled{20}$ | 0 | $\textcircled{21}$ | $\textcircled{22}$ | 0 | $\textcircled{23}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\partial \Phi_9 / \partial \dots$ | 0 | $\textcircled{24}$ | $\textcircled{25}$ | 0 | $\textcircled{26}$ | $\textcircled{27}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\partial \Phi_{10} / \partial \dots$ | $\textcircled{28}$ | $\textcircled{29}$ | $\textcircled{30}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\textcircled{31}$ | $\textcircled{32}$ | $\textcircled{33}$ |
| $\partial \Phi_{11} / \partial \dots$ | 0 | 0 | $\textcircled{34}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\textcircled{35}$ |
| $\partial \Phi_{12} / \partial \dots$ | 0 | 0 | 0 | 0 | 0 | $\textcircled{36}$ | 0 | 0 | 0 | 0 | 0 | 0 |

Chapter 4 Planar Dynamics

4.1 Rigid Body Dynamics

A body can be regarded as a collection of a very large number of particles and the location of the particles in a body relative to one another remains unchanged. In the discussion of the dynamics of a system of particles, the translational equation of motion was derived as

$$\mathbf{f} = m\ddot{\mathbf{r}} \quad (4.1)$$

The definition of the center of mass or centroid of a body is found from Eq. (4.2). The summation over the particles is replaced by an integral over the body volume, and the mass of the individual particle is replaced by the infinitesimal mass dm :

$$\mathbf{r} = \frac{1}{m} \int \mathbf{r}^P dm \quad (4.2)$$

where \mathbf{r} is the mass center and \mathbf{r}^P locates an infinitesimal mass. Vector \mathbf{r}^P is the sum of two vectors:

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s} \quad (4.3)$$

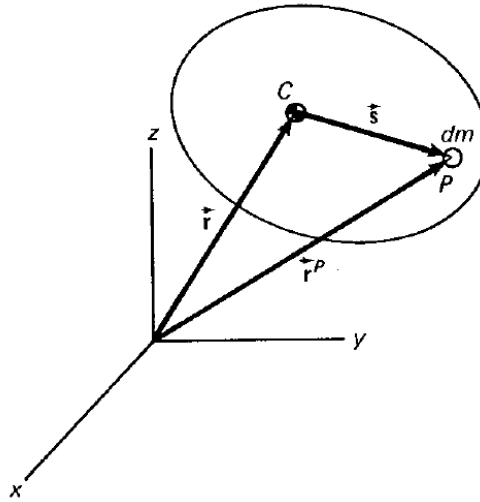


Figure 4.1 A body as a collection of infinitesimal masses.

Translational and rotational equations of motion for an unconstrained body are written as follows:

$$m_i \ddot{x}_i = f_{(x)i}$$

$$m_i \ddot{y}_i = f_{(y)i}$$

$$m_i \ddot{\tilde{r}}_i = n_i$$

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i \quad (4.4)$$

where the polar moment of inertia is denoted by μ . Equation (4.4) may also be expressed as

$$\mathbf{M}_i \ddot{\mathbf{q}}_i = \mathbf{g}_i \quad (4.5)$$

where

$$\mathbf{M}_i = \text{diag}[m, m, \mu]_i$$

$$\mathbf{q}_i = [x, y, \phi]_i^T$$

$$\mathbf{g}_i = [f_{(x)}, f_{(y)}, n]_i^T$$

For a system b unconstrained bodies, Eq. (4.5) is repeated b times as

$$\mathbf{M} \ddot{\mathbf{q}} = \mathbf{g}$$

where

$$\mathbf{M} = \text{diag}[\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_b]$$

$$\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \dots, \mathbf{q}_b^T]$$

$$\mathbf{g} = [\mathbf{g}_1^T, \mathbf{g}_2^T, \dots, \mathbf{g}_b^T]$$

The system mass matrix \mathbf{M} is a $3b \times 3b$ constant diagonal matrix, and vector $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ and \mathbf{g} are $3b$ -vectors.

For a system of b constrained bodies, the equations of motion can be written as

$$\mathbf{M} \ddot{\mathbf{q}} = \mathbf{g} + \mathbf{g}^{(c)} \quad (4.6)$$

where $\mathbf{g}^{(c)}$ is the vector of constraint reaction forces. Since Eq. (4.6), and hence $\mathbf{g}^{(c)}$ is describe in the same coordinate system as \mathbf{q} , then from it is found that

$$\mathbf{g}^{(c)} = \Phi_q^T \lambda \quad (4.7)$$

where $\Phi = 0$ represent m independent constraint equations. Substitution of Eq. (4.7) in Eq. (4.6) yields

$$\mathbf{M} \ddot{\mathbf{q}} + \Phi_q^T \lambda = \mathbf{g} \quad (4.8)$$

Equation (4.8) and the constraint equations $\Phi = 0$ together represent the equations of motion for a system of constrained bodies.

In kinematic analysis, the number of degrees of freedom of a system must be equal to the number of driver constraint equations. This means that m kinematic constraint equations and k driver equations provide n equations in n unknowns and so will yield a unique solution. However, in dynamic analysis, in general, there are no driving equations to be specified. Since $n > m$, there are more unknowns

in the constraint equations than there are equations, and so there is no unique solution to these equations. In dynamic analysis, a unique solution is obtained when the constraint equations are considered simultaneously with the differential equations of motion and a proper set of initial conditions is specified. These algebraic-differential equations are solved by numerical methods.

4.2 Constraint Force

The joint reaction forces can be expressed in terms of the Jacobian matrix of the constraint equations and a vector of Lagrange multipliers as

$$\mathbf{g}^{(c)} = \Phi_q^T \boldsymbol{\lambda} \quad (4.9)$$

Consider two bodies i and j connected by a revolute joint, as shown in Fig. 4.2 (a). The equations of motion for bodies i and j are

$$\mathbf{M}_i \ddot{\mathbf{q}}_i + \Phi_{qi}^T \boldsymbol{\lambda} = \mathbf{g}_i \quad (a)$$

and

$$\mathbf{M}_j \ddot{\mathbf{q}}_j + \Phi_{qj}^T \boldsymbol{\lambda} = \mathbf{g}_j \quad (a)$$

Using the entries of the Jacobian matrix for a revolute joint from Table 3.2, we can write Eq. (a) in the expanded form

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y_i^P - y_i) & (x_i^P - x_i) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i \quad (c)$$

Since there are two algebraic equations in the constraint equations for a revolute joint, Equation c can be written as the set of equations

$$m_i \ddot{x}_i = f_{(x)i} - l_1 \quad (4.10)$$

$$m_i \ddot{y}_i = f_{(y)i} - l_2 \quad (4.11)$$

$$m_i \ddot{\phi}_i = n - (y_i^P - y_i) l_1 + (x_i^P - x_i) l_2 \quad (4.12)$$

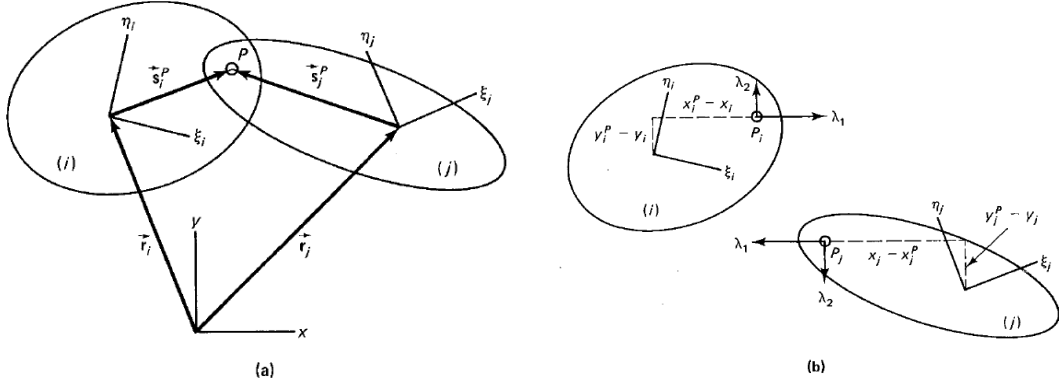


Figure 4.2 (a) Two bodies connected by a revolute joint and (b) free-body diagrams for the bodies.

A free-body diagram for body i is shown in Fig. 4.2(b). Equation (4.10) indicates that besides $f_{(x)i}$, another force λ_1 acts in the x direction on body i . Similarly, a force λ_2 acts in the y direction on the same body. However, in order for Eq. 9.36 to be satisfied, forces λ_1 and λ_2 must act at point P_i . The moment arm of λ_1 is $y_i^P - y_i$, and hence a moment $(y_i^P - y_i)\lambda_1$ acts in the negative rotational direction. The moment arm of λ_2 is $x_i^P - x_i$, and so a moment $(x_i^P - x_i)\lambda_2$ acts in the positive rotational direction.

Equations of motion for body j , in the same form as Eq. (c) are written as follows:

$$\begin{bmatrix} m \\ m \\ \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_j + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ (y_j^P - y_j) & -(x_j^P - x_j) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_j \quad (d)$$

or

$$m_j \ddot{x}_j = f_{(x)j} + \lambda_1 \quad (4.13)$$

$$m_j \ddot{y}_j = f_{(y)j} + \lambda_2 \quad (4.14)$$

$$m_j \ddot{\phi}_j = n_j - (y_j^P - y_j)\lambda_1 + (x_j^P - x_j)\lambda_2 \quad (4.15)$$

It is shown in Fig. 4.2(b) that λ_1 and λ_2 are two forces acting at point P_j , in the negative x and y directions, respectively. The moment arm for λ_1 is $y_j^P - y_j$ which yields a positive moment $(y_j^P - y_j)\lambda_1$, and the moment arm for λ_2 , is $x_j^P - x_j$ which yields a moment $(x_j - x_j^P)\lambda_2$ or $-(x_j^P - x_j)\lambda_2$. The multipliers λ_1 and λ_2 can be positive or negative quantities. In any case, the reaction forces acting at the revolute joint on the connecting bodies are always equal in magnitude and opposite in direction.

Consider a system of two bodies connected by a revolute joint as shown in Fig. 4.2(a). The external forces acting on the system are gravity, a constant force of

10 N acting on body i in the negative x direction, and a constant force of 10 N acting on body j in the positive x direction. The constant quantities for this system are $m_i = 1.2$, $m_j = 2$, $\mu_i = 2.5$, $\mu_j = 4$, $s_i'^P = [0.9, 0.7]^T$, and $s_j'^P = [-1.3, 1]^T$

The constraint equations for this revolute joint are

$$\begin{aligned} x_i + 0.9 \cos f_i - 0.7 \sin f_i - x_j + 1.3 \cos f_j + \sin f_j &= 0 \\ y_i + 0.9 \sin f_i + 0.7 \cos f_i - y_j + 1.3 \sin f_j - \cos f_j &= 0 \end{aligned}$$

The Jacobian matrix for these constraints is

$$\Phi_q = \begin{bmatrix} 1 & 0 & -1.09 & -1 & 0 & 0.72 \\ 0 & 1 & 0.35 & 0 & -1 & 1.47 \end{bmatrix}$$

From Eqs. (4.10) through (4.12), the equations of motion for body i are

$$\begin{aligned} 1.2\ddot{x}_i - f_1 &= -10 \\ 1.2\ddot{y}_i - f_2 &= -11.77 \\ 2.5\ddot{f}_i + 1.09f_1 - 0.35f_2 &= 0 \end{aligned} \quad (4.16)$$

Similarly, Eqs. (4.13) through (4.15) provide equations of motion for body j :

$$\begin{aligned} 2\ddot{x}_j + f_1 &= 10 \\ 2\ddot{y}_j + f_2 &= -19.62 \\ 4\ddot{f}_j - 0.72f_1 - 1.47f_2 &= 0 \end{aligned} \quad (4.17)$$

Equations (4.16-7) are six equations in eight unknowns, and therefore two more equations are needed. These two additional equations are the kinematic acceleration equations. The second-time derivative of the constraint equations (refer to Table 3.2) can be used to obtain the acceleration equations for the revolute joint as follows:

$$\begin{aligned} \ddot{x}_i - 1.09\ddot{f}_i - \ddot{x}_j + 0.72\ddot{f}_j &= 0 \\ \ddot{y}_i + 0.35\ddot{f}_i - \ddot{y}_j + 1.47\ddot{f}_j &= 0 \end{aligned} \quad (4.18)$$

The right side of the acceleration equations is $\gamma = [0.0017, -0.0002]^T$. These equations can be solved to find

$$\ddot{f}_i = [-2.571, -10.154, -3.061]^T, \quad \ddot{f}_j = [1.534, -9.604, 1.096]^T, \text{ and}$$

$$\lambda = [6.915, -0.413]^T. \quad \text{Hence, } f_i^{(c)} = [6.915, -0.413]^T \text{ and}$$

$$f_j^{(c)} = [-6.915, 0.413]^T.$$

If a translational joint is considered between bodies i and j as shown in Fig. 4.3(a), the equations of motion for body i can be written as

$$m_i \ddot{x}_i = f_{(x)i} + (y_i^P - y_i^Q) / l_1 \quad (4.19)$$

$$m_i \ddot{y}_i = f_{(y)i} + (x_i^P - x_i^Q) / l_1 \quad (4.20)$$

$$m_i \ddot{\theta}_i = n_i - [(x_j^P - x_i)(x_i^P - x_i^Q) + (y_j^P - y_i)(y_i^P - y_i^Q)] / l_1 + l_2 \quad (4.21)$$

The free-body diagram for body i is shown in Fig. 4.3(b). In this diagram the force associated with i is the reaction force caused by the first constraint equation. It is a simple matter to show that this force, \mathbf{k}_I , is perpendicular to the line of translation. The contribution of the second constraint equation is a couple acting on body i . Note that λ_2 may be a positive or negative quantity.

In order to find a simpler physical description of the reaction force \mathbf{k}_I , one should not locate the points P_i , Q_i , and P_j , arbitrarily on the line of translation. These points can be selected to coincide with the edges of the slider, as shown in Fig. 4.4(a). If P_j is allowed to slide with the slider, then the reaction force \mathbf{k}_I , always acts at the edge of the slider as shown in Fig. 4.4(b).

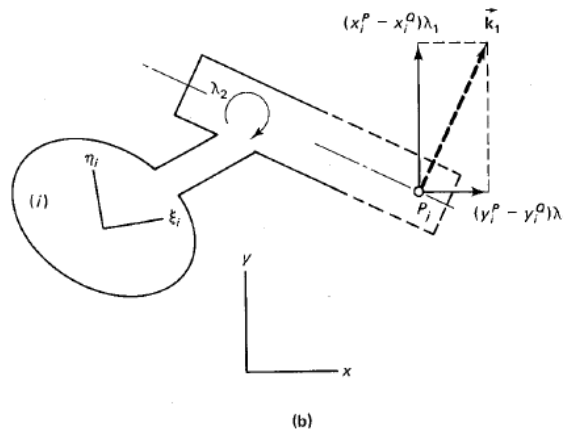
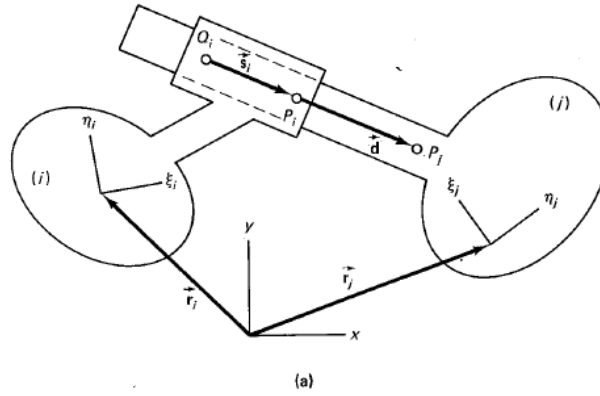


Figure 4.3 (a) Two bodies connected by a translational joint and (b) the reaction forces acting on body i associated with a translational joint.

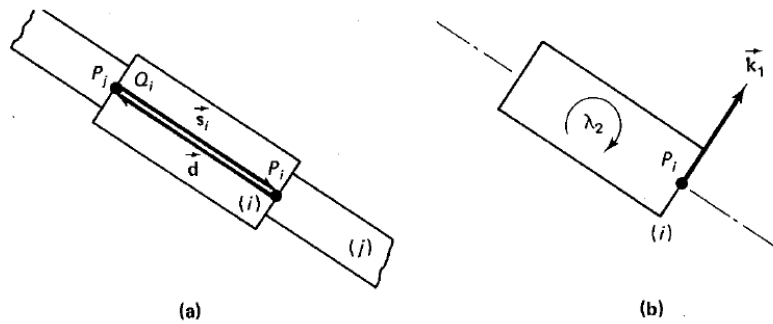


Figure 4.4 (a) A typical translational joint and (b) Forces acting on body i by the sliding body j .

4.3 Equations of Motion

For an unconstrained mechanical system, the equations of motion are

$$M\ddot{q} = g \quad (4.22)$$

For a constrained mechanical system with m independent constraints

$$\Phi = 0 \quad (4.23)$$

the velocity and acceleration equations are

$$\Phi_q \dot{\mathbf{q}} = 0 \quad (4.24)$$

and

$$\Phi_q \ddot{\mathbf{q}} - \gamma = 0 \quad (4.25)$$

The equations of motion for this constrained system are

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_q^T \lambda = \mathbf{g} \quad (4.26)$$

Equation (4.25) can be appended to Eq. (4.26) and the result can be written as

$$\begin{bmatrix} \mathbf{M} & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \gamma \end{bmatrix} \quad (4.27)$$

The Jacobian matrix Φ_q , is a function of \mathbf{q} , and vectors \mathbf{g} and γ are functions of \mathbf{q} , $\dot{\mathbf{q}}$, and t . Therefore at any given instant, if \mathbf{q} and $\dot{\mathbf{q}}$ are known, Eq. (4.27) provides $n+m$ linear algebraic equations in $n+m$ unknowns that can be solved for $\ddot{\mathbf{q}}$ and λ .

If the position and velocity vectors are appended together as the vector then the velocity and acceleration vectors are represented in the vector

$$\mathbf{y} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \quad (4.28)$$

At time $t = t^i$, vector $\dot{\mathbf{y}}^{(i)}$ can be integrated numerically to obtain $\mathbf{y}^{(i+1)}$, where $t^i + \Delta t$; i.e.,

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} \quad (4.29)$$

Initially, at $i=0$, the initial conditions on \mathbf{q} and $\dot{\mathbf{q}}$ are required to start the integration process.

$$\dot{\mathbf{y}}^{(i)} \xrightarrow{\text{(integrate)}} \mathbf{y}^{(i+1)} \quad (4.30)$$

For a constrained mechanical system, the equations of motion are, from Eq. 9.55,

$$\begin{bmatrix} \mathbf{M} & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \gamma \end{bmatrix} \quad (4.31)$$

with initial conditions \mathbf{q}^0 and $\dot{\mathbf{q}}^0$. The Jacobian Φ_q is a function of \mathbf{q} and \mathbf{g} and γ are functions of \mathbf{q} and $\dot{\mathbf{q}}$ that can be evaluated at the initial time. Hence, Eq. (4.31) can be solved for the unknowns at the initial time, i.e., $\ddot{\mathbf{q}}^0$ and λ^0 . The initial conditions on \mathbf{q} and $\dot{\mathbf{q}}$ for a constrained system cannot be specified arbitrarily. The initial conditions \mathbf{q}^0 and $\dot{\mathbf{q}}^0$ must satisfy the constraint equations; i.e.,

$$\Phi = 0 \quad (\text{for } \mathbf{q} = \mathbf{q}^0) \quad (4.32)$$

and

$$\Phi_q \dot{\mathbf{q}} = 0 \quad (\text{for } \mathbf{q} = \mathbf{q}^0 \text{ and } \dot{\mathbf{q}} = \dot{\mathbf{q}}^0) \quad (4.33)$$

For the constrained equations of motion, vectors \mathbf{y} and $\dot{\mathbf{y}}$ are as defined, and a numerical integration algorithm is applied solving the constrained equations of motion.

Chapter 5 Euler Angles and Bryant Angles for Spatial Motion

Among the most common parameters used to describe the angular orientation of a body in space are Euler angles. The angular orientation of a given body-fixed coordinate system $\xi\eta\zeta$ can be envisioned to be the result of three successive rotations. The three angles of rotation corresponding to the three successive rotations are defined as Euler angles. The sequence of rotations used to define the final orientation of the coordinate system is to some extent arbitrary. A total of twelve conventions is possible in a right-hand coordinate system. For the Euler angles described here, a particular sequence of rotations known as the x convention is considered. Another convention, known as the xyz convention, is also discussed here; the parameters associated with this convention are often referred to as Bryant angles.

5.1 Euler Angles

Euler angles provide a set of three coordinates without any constraint equations. The sequence of rotations employed in the x convention starts by rotating the initial system of xyz axes counterclockwise about the z axis by an angle ψ , as shown in Fig. 5.1. The resulting coordinate system is labeled $\xi''\eta''\zeta''$. In the second step the intermediate $\xi''\eta''\zeta''$ axes are rotated about ξ'' counterclockwise by an angle θ to produce another intermediate set, the $\xi'\eta'\zeta'$ axes. Finally, the $\xi'\eta'\zeta'$ axes are rotated counterclockwise about ζ' by an angle ϕ to produce the desired $\xi\eta\zeta$ system of axes. The angles ψ , θ , and ϕ , which are the Euler angles, completely specify the orientation of the $\xi\eta\zeta$ system relative to the xyz system and can therefore act as a set of three independent coordinates.

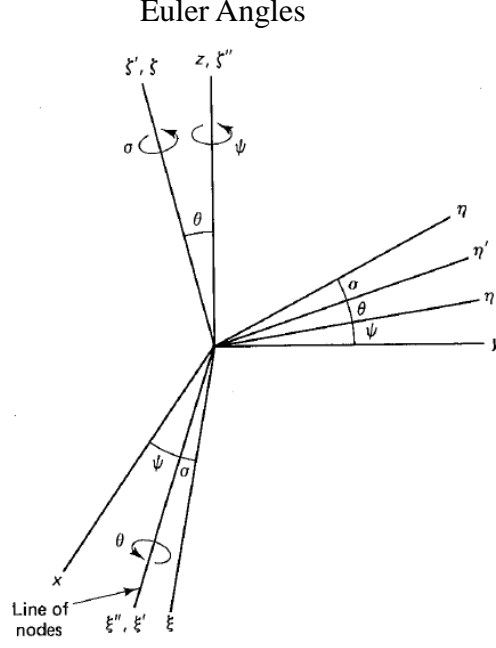


Figure 5.1 The rotations defining the Euler Angles.

The elements of the complete transformation matrix \mathbf{A} can be obtained as the triple product of the matrices that define the separate rotations, i.e., the matrices

$$\mathbf{D} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $c \equiv \text{cosine}$ and $s \equiv \text{sine}$. Hence, $\mathbf{A} \equiv \mathbf{DCB}$ is found to be

$$\mathbf{A} = \begin{bmatrix} c\psi c\phi - s\psi c\theta s\phi & -c\psi s\phi - s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi + c\psi c\theta s\phi & -s\psi s\phi + c\psi c\theta c\phi & -c\psi s\theta \\ s\theta s\phi & s\theta c\phi & c\theta \end{bmatrix} \quad (5.1)$$

It can be verified that matrix \mathbf{A} is orthonormal, i.e., that $\mathbf{A}^T = \mathbf{A}^{-1}$.

The advantage of having three independent rotational coordinates, instead of nine dependent direction cosines, is offset by the disadvantage that the elements of \mathbf{A} in terms of the Euler angles are complicated trigonometric functions. Figure 5.2 shows that if $\theta = n\pi$, $n = \pm 1, \pm 2, \dots$, the axes of the first and third rotations coincide, so that ψ and ϕ cannot be distinguished. This fact is illustrated by setting $\theta = 0$ in \mathbf{A} to obtain

$$\mathbf{A} = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & -c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \theta = 0$$

where $\alpha = \psi + \phi$.

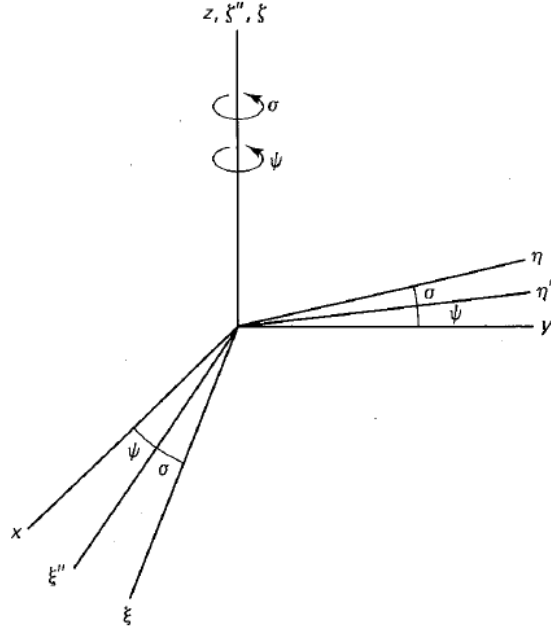


Figure 5.2 Euler angles for the case $\theta = n\pi$ ($n = \pm 1, \pm 2, \dots$).

It may be necessary to calculate Euler angles that correspond to a known transformation matrix. For this purpose, the following formulas are deduced from Eq. 5.1:

$$\begin{aligned} \cos \theta &= a_{33} & \sin \theta &= \pm \sqrt{1 - \cos^2 \theta} \\ \cos \psi &= \frac{-a_{23}}{\sin \theta} & \sin \psi &= \frac{a_{13}}{\sin \theta} \\ \cos \phi &= \frac{a_{32}}{\sin \theta} & \sin \phi &= \frac{a_{31}}{\sin \theta} \end{aligned} \quad (5.2)$$

These formulas show that numerical difficulties are to be expected for values of θ that are close to the critical values $n\pi$, $n = \pm 1, \pm 2, \dots$.

5.2 Time Derivatives of Euler Angles

The general rotation associated with ω can be considered equivalent to three successive rotations with angular velocities $\omega_{(\psi)} = \dot{\psi}$, $\omega_{(\theta)} = \dot{\theta}$, and $\omega_{(\phi)} = \dot{\phi}$. Hence,

This vector sum can be obtained as the sum of three separate angular velocity vectors.

This vector sum cannot be obtained easily, since the directions $\omega_{(\psi)}$, $\omega_{(\theta)}$, and $\omega_{(\phi)}$

$\omega_{(\psi)} = \dot{\psi}$, $\omega_{(\theta)} = \dot{\theta}$, and $\omega_{(\phi)} = \dot{\phi}$ are not orthogonally placed: $\omega_{(\psi)}$ is along the global

z axis and, $\omega_{(\theta)}$ is along the line of nodes, while $\omega_{(\phi)}$ is along the body ζ axis.

However, the orthonormal transformation matrices **B**, **C**, and **D** may be used to determine the components of these vectors along any desired set of axes.

Figure 5.3 can be used to obtain the components of the velocity vector $\boldsymbol{\omega}$ in the $\xi\eta\zeta$ axes in terms of Euler angles and rates. Since $\dot{\psi}$ is parallel to the z axis, its components along the body axes are given by applying the orthonormal transformation $\mathbf{B}^T \mathbf{C}^T$.

$$\begin{aligned}\dot{y}_{(x)} &= \sin q \sin f \\ \dot{y}_{(h)} &= \sin q \cos f \\ \dot{y}_{(z)} &= \cos q\end{aligned}$$

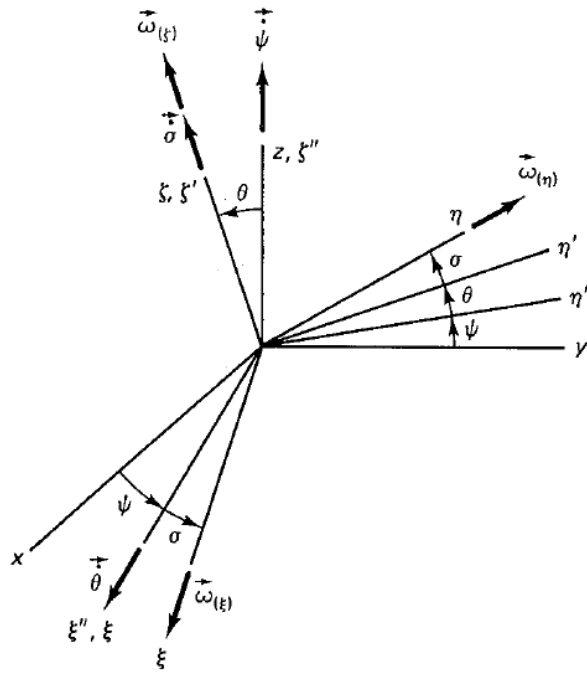


Figure 5.3 Euler angle rates.

The line of nodes, which is the direction of $\dot{\theta}$, coincides with the ξ' axis, and so the components of $\dot{\theta}$ with respect to the body axes are furnished by applying only the final orthonormal transformation \mathbf{B}^T :

$$\begin{aligned}\dot{q}_{(x)} &= \dot{q} \cos f \\ \dot{q}_{(h)} &= -\dot{q} \sin f \\ \dot{q}_{(z)} &= 0\end{aligned}$$

No transformation is necessary for the component of \dot{f} , which lies along the ζ axis. When these components of the separate angular velocities are added, the components of $\boldsymbol{\omega}$ with respect to the body axes are

$$\begin{aligned}
W_{(x)} &= \dot{y} \sin q \sin f + \dot{q} \cos f \\
W_{(y)} &= \dot{y} \sin q \cos f - \dot{q} \sin f \\
W_{(z)} &= \dot{y} \cos q + \dot{f}
\end{aligned}$$

or, in matrix form,

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \sin \theta \sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & -\sin \phi & 0 \\ \cos \theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \quad (5.3)$$

In addition, the Euler angle rates can be expressed in terms of $\omega_{(\xi)}$, $\omega_{(\eta)}$ and $\omega_{(\zeta)}$.

Since Euler angle rates are not orthogonal, the inverse of the matrix of Eq. (5.3) yields

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\sin \theta} \begin{bmatrix} \sin \phi & \cos \phi & 0 \\ \cos \phi \sin \theta & -\sin \phi \sin \theta & 0 \\ -\sin \phi \cos \theta & -\cos \phi \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} \quad (5.4)$$

Similar techniques may be applied to express the components of $\dot{\omega}$ along the xyz axes, in terms of Euler angles and rates. Equation (5.4) shows, again, that numerical problems will arise if θ is close to the critical values $n\pi$, $n=0, \pm 1, \dots$

5.3 Bryant Angles

The Bryant angle convention considers rotations about axes other than those for the Euler angles. The first rotation may be carried out counterclockwise about the x axis through an angle ϕ_1 ; the resultant coordinate system will be labeled $\xi''\eta''\zeta''$, as shown in Fig. 5.4. The second rotation, through an angle ϕ_2 counterclockwise about the η'' axis, produces the coordinate system $\xi'\eta'\zeta'$. Finally, the third rotation, counterclockwise about the ζ' axis through an angle ϕ_3 , results in the $\xi\eta\zeta$ coordinate system. The transformation matrices for the individual rotations are

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & -s\phi_1 \\ 0 & s\phi_1 & c\phi_1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c\phi_2 & 0 & s\phi_2 \\ 0 & 1 & 0 \\ -s\phi_2 & 0 & c\phi_2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the matrix of the complete transformation, $\mathbf{A} = \mathbf{DCB}$, is

$$\mathbf{A} = \begin{bmatrix} c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \\ c\phi_1 s\phi_3 + s\phi_1 s\phi_2 c\phi_3 & c\phi_1 c\phi_3 + s\phi_1 s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 + c\phi_1 s\phi_2 c\phi_3 & s\phi_1 c\phi_3 + c\phi_1 s\phi_2 s\phi_3 & c\phi_1 c\phi_2 \end{bmatrix} \quad (5.5)$$

Again, it may be necessary to calculate Bryant angles that correspond to a known transformation matrix. This can be done, with the help of formulas derived from

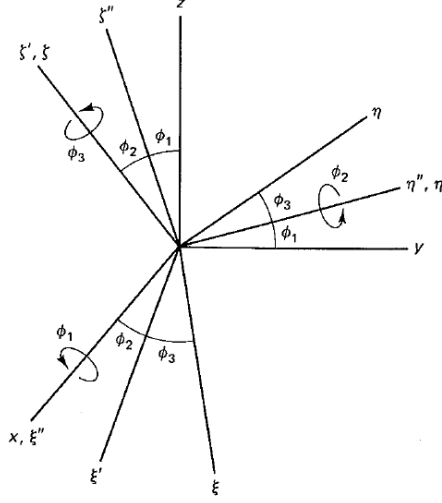


Figure 5.4 Rotations defining Bryant angles.

Eq. (5.5), to obtain

$$\begin{aligned} \sin \phi_2 &= a_{13} & \cos \phi_2 &= \pm \sqrt{1 - \sin^2 \phi_2} \\ \sin \phi_1 &= \frac{-a_{23}}{\cos \phi_2} & \cos \phi_1 &= \frac{a_{33}}{\cos \phi_2} \\ \sin \phi_3 &= -\frac{a_{12}}{\cos \phi_2} & \cos \phi_3 &= \frac{a_{11}}{\cos \phi_2} \end{aligned} \quad (5.6)$$

It can be observed again that there exists a critical case, namely, when $\phi_2 = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \dots$, in which the axes of the first and third rotations coincide, so that the rotation angles ϕ_1 and ϕ_3 become indistinguishable.

5.4 Time Derivative of Bryant Angles

The relationship between angular velocity ω and Bryant angles and rates can be found in a similar fashion to that for the Euler rates. The transformation matrix for the velocity components is

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \cos \phi_1 \cos \phi_3 & \sin \phi_3 & 0 \\ -\cos \phi_2 \sin \phi_3 & \cos \phi_3 & 0 \\ \sin \phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} \quad (5.7)$$

The inverse transformation can be found to be

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} = \frac{1}{\cos \phi_2} \begin{bmatrix} \cos \phi_3 & -\sin \phi_3 & 0 \\ \sin \phi_3 \cos \phi_2 & \cos \phi_3 \cos \phi_2 & 0 \\ -\cos \phi_3 \sin \phi_2 & \sin \phi_3 \sin \phi_2 & \cos \phi_2 \end{bmatrix} \begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} \quad (5.8)$$

It can be seen that Eq. (5.8) fails numerically in the vicinity of the critical values

$$\phi_2 = \pi/2 + n\pi, n = 0, +1, \dots$$