Calculus

Motion and Manipulation

Linear Algebra: Vectors

- Directed line segment in n-dimensional space from the origin to a point $x=(x_1,\ldots,x_n)$:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

- Alternative notation

$$x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$$

- Zero vector is vector with all entries $x_i=0$

Vectors: Addition

- Sum x+y of two vectors of equal dimension n

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}: \qquad x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

- Head-to-tail method for graphical construction of vector sum

- Relevant to translation

Vectors: Scalar Multiplication

- Product cx of a scalar c and a vector of dimension n

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}: \qquad \boxed{cx = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}}$$

- Relevant to scaling

Vectors: Dot Product

- Dot product x•y of two vectors of equal dimension n

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}:$$

$$\boxed{x \bullet y = \sum_{i=1}^{n} x_i y_i}$$

- $x \bullet x = |x|^2$ where $|x|$ stands for the length or norm of vector x

- If θ is the angle between x and y then

$$x \bullet y = |x||y| \cos\theta$$

Vectors: Cross Product in 3D

- Cross product x × y of two vectors of dimension 3

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}: \qquad \boxed{x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}}$$

- Cross product x × y is perpendicular to x and y; the direction is given by the right hand rule

- If θ is again the angle between x and y then the magnitude of cross product x × y given by

$$|x \times y| = |x||y| \sin\theta$$

Vectors: Linear Independence

- A set V of vectors is linearly independent if no vector from V can be written as a linear combination of the other vectors from V
- A set V of vectors is a basis for a space S if the set V is linearly independent and every vector in S can be written as a linear combination of the vectors from V
- Basis is othogonal if the base vectors are mutually perpendicular
- Basis is orthonormal if it is orthogonal and the base vectors have unit length

Lines and Planes

- Vector equation of a line in 2D or 3D:

$$\mathbf{x} = \mathbf{p} + \lambda\mathbf{s}$$

where p is the 2D or 3D vector corresponding to a point on the line and s is a 2D or 3D direction vector of the line; λ is a parameter

- Vector equation of a plane in 3D:

$$\mathbf{x} = \mathbf{p} + \lambda\mathbf{s} + \mu\mathbf{t}$$

where p is the 3D vector corresponding to a point on the plane and s and t are (linearly independent) direction vectors for the plane; λ and μ are parameters

Linear Algebra: Matrices

- Rectangular array of entries, used to represent linear transformation from n-dimensional to m-dimensional space

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

- Zero matrix is matrix with all entries $a_{ij} = 0$

- For m=n, the identity matrix, usually referred to as I, has $a_{ii}=1$ and $a_{ij}=0$ for all $i \neq j$

- Vector is an (n x 1)-matrix

Matrices: Transpose

The transpose A^T of the m×n matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

is the n×m matrix with rows and columns of A exchanged so

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

Matrices: Addition

- Sum A+B of two m×n matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}:$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Matrices: Scalar Multiplication

- Product cA of a scalar c and an m×n matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}:$$

$$cA = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$$

Matrix-Vector Multiplication

- Product Ax of an m×n matrix A and vector x of dimension n with

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}:$$

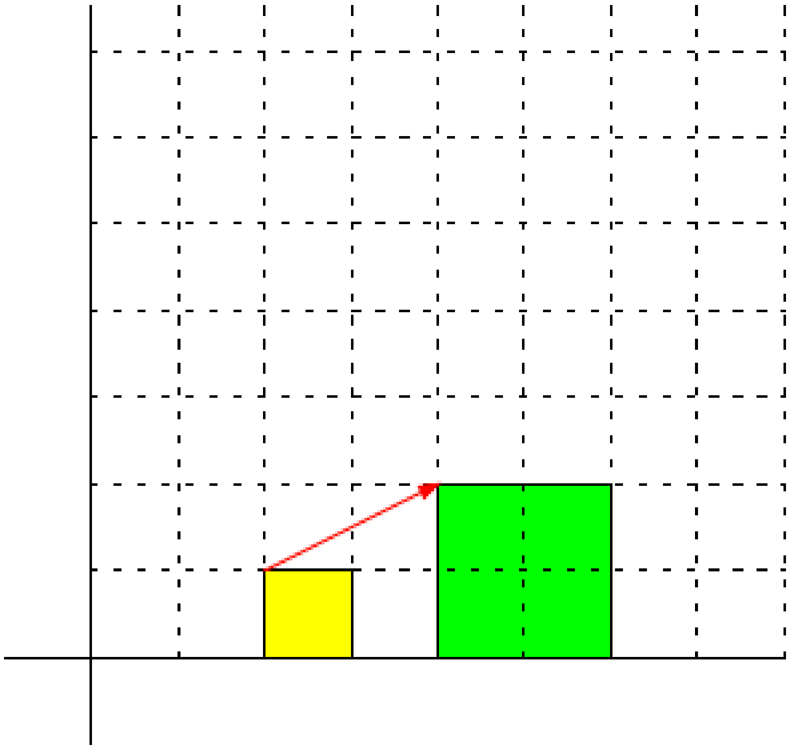
$$\boxed{Ax = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}} \quad \text{with} \quad \boxed{b_i = \sum_{j=1}^{n} a_{ij} x_j}$$

Scaling

- Uniform scaling in R^2

Example: with a factor 2 with respect to the origin:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

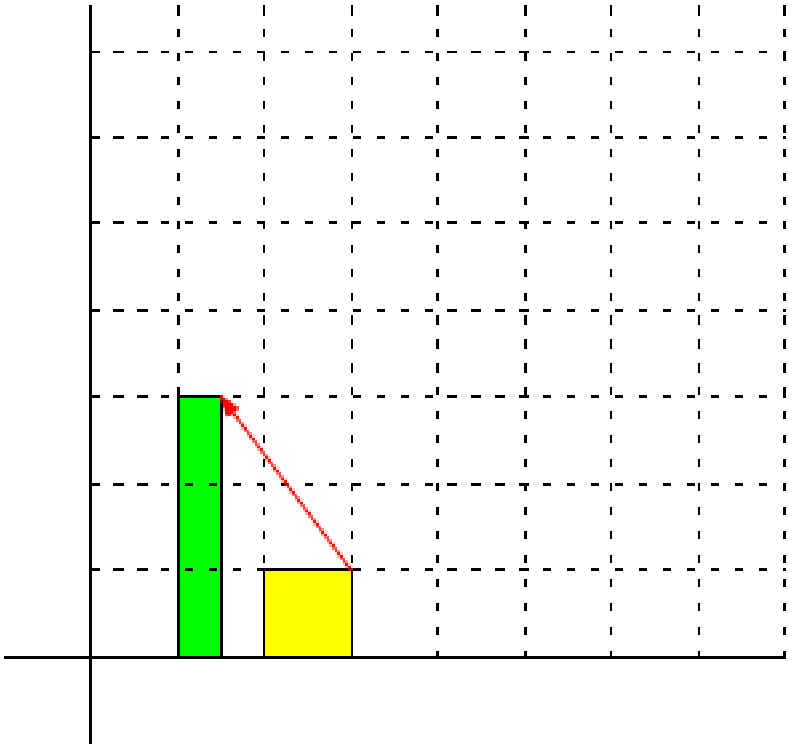


Scaling

- Non-uniform scaling in R^2

Example

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ 3y \end{pmatrix}$$

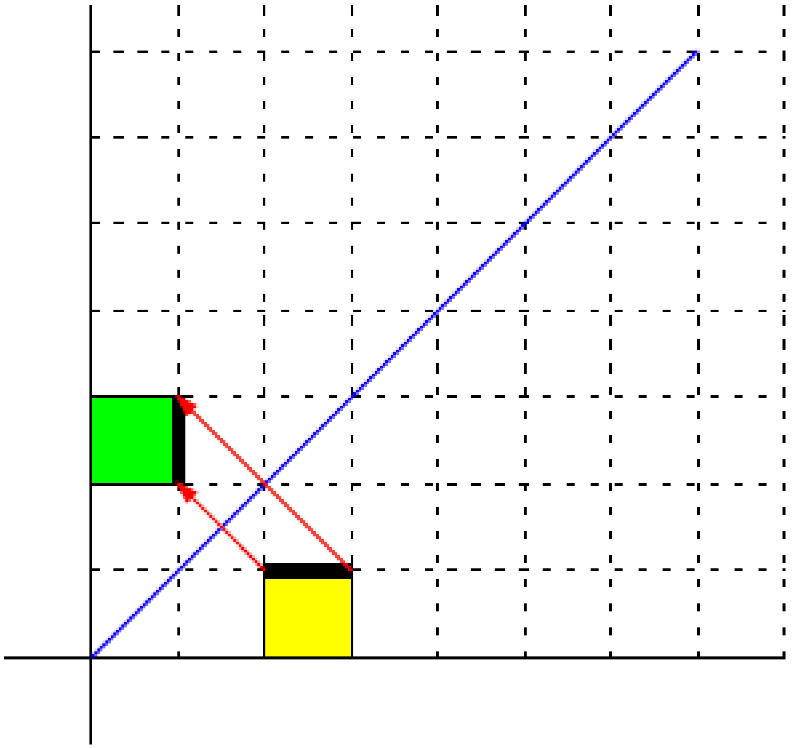


Reflection

- Reflection in R^2

Example: in the line y=x

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$



Rotation

- Rotation in R^2 by an angle θ about the origin

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Regular angles:

	0	π/6	π/4	π/3	π/2
sin	0	1/2	√2/2	√3/2	1
cos	1	√3/2	√2/2	1/2	0

Rules:

- sin (-x) = - sin x
- cos (-x) = cos x
- tan x = sin x / cos x

Rotation

- Rotation in R^3 by an angle θ about the x-axis (R_1), y-axis (R_2), and z-axis (R_3)

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation

- Translation by a vector t is not a linear transformation and can therefore not be formulated as a matrix-vector product involving a 2×2 matrix in R^2 or a 3×3 matrix in R^3

Solution: homogeneous coordinates, adding one dimension

Translation of point $x=(x_1,x_2,x_3)$ along a vector $t=(t_1,t_2,t_3)^T$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = \left(\begin{array}{c} x_1 + t_1 \\ x_2 + t_2 \\ x_3 + t_3 \\ \hline 1 \end{array}\right)$$

Homogeneous Coordinates

Let $R_i(\theta)$ be a (fundamental) rotation matrix, t be a translation vector, I be the identity matrix, and 0 be the zero vector

- Homogeneous translation matrix

$$\mathrm{Tran}(\mathbf{t}) = \begin{pmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

- Homogeneous rotation matrix

$$\mathrm{Rot}_i(\theta) = \begin{pmatrix} \mathbf{R}_i(\theta) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Matrix-Matrix Multiplication

- Product AB of m×n and n×p matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}:$$

$$AB = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}$$

with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrix-Matrix Multiplication

Example

$$\begin{pmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 2 \\ 37 & 5 & 16 \end{pmatrix}$$

Matrix-Matrix Multiplication

Properties

- Distributive over addition:
  - $A(B+C) = AB+AC$
  - $(A+B)C = AC + BC$
- Associative
  - $(AB)C=A(BC)$
- Not commutative, so in general
  - $AB \neq BA$

Composition

If A is the matrix of a linear transformation T_A and B is the matrix of a linear transformation T_B then

- $C=BA$ corresponds to the linear transformation that first performs T_A and then T_B and
- $C=AB$ corresponds to the linear transformation that first performs T_B and then T_A

Recall that $AB \neq BA$

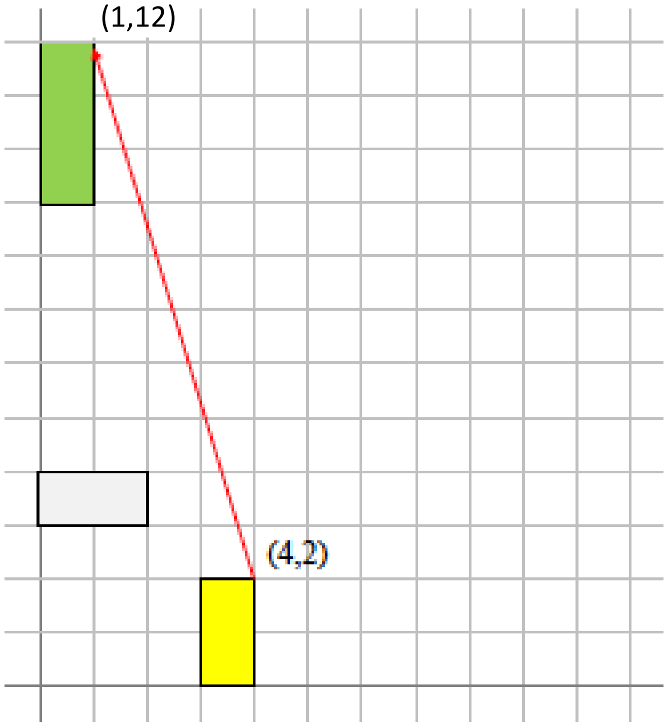
Example

Matrix for scaling after reflection

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 3 & 0 \end{pmatrix}$$

Image of (4,2)

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$



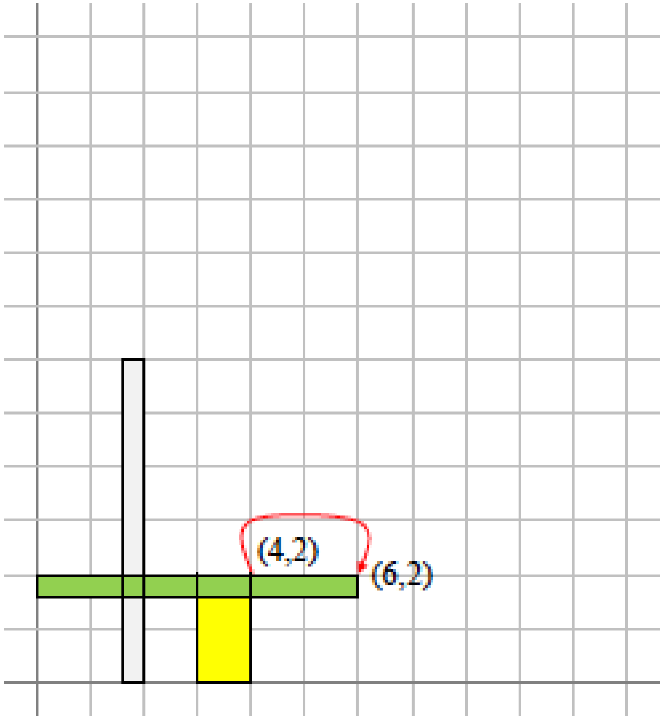
Example

Matrix for reflection after scaling

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ \frac{1}{2} & 0 \end{pmatrix}$$

Image of (4,2)

$$\begin{pmatrix} 0 & 3 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$



Another Example

- Rotation followed by translation

$$\mathrm{Tran}(t)\,\mathrm{Rot}_i(\theta) = \begin{pmatrix} I & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} R_i(\theta) & 0 \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} R_i(\theta) & t \\ 0^T & 1 \end{pmatrix}$$

- Translation followed by rotation

$$\mathrm{Rot}_i(\theta)\,\mathrm{Tran}(t) = \begin{pmatrix} R_i(\theta) & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & t \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} R_i(\theta) & R_i(\theta)t \\ 0^T & 1 \end{pmatrix}$$

Inverse

The inverse A^{-1} of a matrix A is a matrix that satisfies

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- A^{-1} exists if and only if
  - A is square (so if m=n) and
  - the determinant of A is nonzero
- If A is the matrix of a linear transformation T_A then A^{-1} is the matrix of the linear transformation that inverts T_A

Determinants in 2D and 3D

- Determinant of a 2×2 matrix

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Determinant of a 3×3 matrix

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Inverse

- Simple expression for inverse of a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{with } \det(A) \neq 0: \qquad \boxed{A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$

- Similar expressions can be obtained for larger square matrices but a common approach is to use Gaussian elimination

- $(AB)^{-1} = B^{-1}A^{-1}$

Moore-Penrose Pseudoinverse

- If the $m \times n$ real matrix A has linearly independent columns (and so $m > n$) then the $n \times m$ matrix

$$A^+ = (A^T A)^{-1} A^T$$

satisfies $A^+A = I$ and is referred to as a left inverse

- If the $m \times n$ real matrix A has linearly independent rows (and so $m < n$) then the $n \times m$ matrix

$$A^+ = A^T (A A^T)^{-1}$$

satisfies $A A^+ = I$ and is referred to as a right inverse

Systems of Linear Equations

The system of m linear equations in n variables

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

can be also be written as a matrix equation Ax=b or

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

Systems of Linear Equations

For a given matrix A and vector b solve x from

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Focus on the case where $m=n$. Similar approaches apply when $m<n$ or $m>n$

Gaussian Elimination

Augment matrix with righthand side of the equation, then transform matrix into the identity matrix by repeatedly

- interchanging two rows
- multiplying a single row by a constant
- adding a multiple of one row to another row

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

Augmented matrix is nothing more than a compact representation of the original system

Example

Solve

$$\begin{aligned} x_1 &+ x_2 &+ 2x_3 &= 17 \\ 2x_1 &+ x_2 &+ x_3 &= 15 \\ x_1 &+ 2x_2 &+ 3x_3 &= 26 \end{aligned}$$

Gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right)$$

Example

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 1 & 9 \\ 0 & -1 & -3 & -19 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & -2 & -10 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Example

Corresponds to system

$$\begin{aligned} x_1 \quad \quad \quad \quad &= 3 \\ x_2 \quad \quad &= 4 \\ x_3 &= 5 \end{aligned}$$

which is equivalent to the original system

Gaussian Elimination

Same approach works for matrix inversion: now place the identity matrix right of the vertical bar

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array} \right)$$

and then transform using the same three types of actions to get

$$\left(\begin{array}{c|c} \mathbf{I} & \mathbf{C} \end{array} \right)$$

Then $C = A^{-1}$

If A^{-1} is given then solving $Ax=b$ for x can be accomplished by $x=A^{-1}b$

Moore-Penrose Pseudoinverse

- If the $m \times n$ real matrix A has linearly independent rows (and so $m<n$) then the system $Ax=b$ is underdetermined and has infinitely many solutions. The matrix-vector product A^+b, where A^+ is the right inverse, gives the minimum-norm solution

- If the $m \times n$ real matrix A has linearly independent columns (and so $m>n$) then the system $Ax=b$ is overdetermined and has no solutions. The matrix-vector product A^+b, where A^+ is the left inverse, gives a least-squares approximation

Functions

Functions $f : R \to R$

- Polynomial functions

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0$$

- Exponential functions

$$f(x) = a^x$$

special case $f(x) = e^x$

Functions

- Logarithmic functions

$$f(x) = {}^{a}\log x$$

special case $f(x) = \ln x = {}^{e}\log x$

- Trigonometric functions

$$f(x) = \sin x$$

$$f(x) = \cos x$$

$$f(x) = \tan x$$

Useful rules:

$$\boxed{\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \end{aligned}}$$

Derivatives

Formal definition for a function $f : R \to R$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative describes the growth rate

Second (derivative of the derivative), third, i-th derivatives:

$$f''(x), f'''(x), f^{(i)}(x)$$

Alternative notation for first, second, ... derivatives if y=f(x):

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \quad \text{or} \frac{df}{dx}, \frac{d^2f}{dx^2}, \ldots$$

Derivatives of Common Functions

- $f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0$

$$f'(x) = k a_k x^{k-1} + (k-1) a_{k-1} x^{k-2} + \ldots + a_2 x + a_1$$

$f(x) = a$ $\qquad$ $f'(x) = 0$

- $f(x) = a^x$ $\qquad$ $f'(x) = a^x \ln a$

$f(x) = e^x$ $\qquad$ $f'(x) = e^x$

Derivatives of Common Functions

- $f(x) = {}^{a}\log x \qquad f'(x) = \dfrac{1}{x \ln a}, \quad x > 0$

  $f(x) = \ln x \qquad f'(x) = \dfrac{1}{x}, \quad x > 0$

- $f(x) = \sin x \qquad f'(x) = \cos x$

  $f(x) = \cos x \qquad f'(x) = -\sin x$

  $f(x) = \tan x \qquad f'(x) = \dfrac{1}{\cos^2 x}$

and more, see one of the many lists that are online

Rules for Derivatives

If

- $f(x) = cg(x)$
- $f(x) = g(x) + h(x)$
- $f(x) = g(x) - h(x)$
- $f(x) = g(x)h(x)$
- $f(x) = \dfrac{g(x)}{h(x)}$

then

- $f'(x) = cg'(x)$
- $f'(x) = g'(x) + h'(x)$
- $f'(x) = g'(x) - h'(x)$
- $f'(x) = g'(x)h(x) + g(x)h'(x)$
- $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$

Chain Rule for Derivatives

If then

- $f(x) = g(h(x))$
- $f'(x) = g'(h(x))h'(x)$

Example:

$$f(x) = \sin e^{x^2}$$

$$f'(x) = \cos e^{x^2} \cdot e^{x^2} \cdot 2x = 2xe^{x^2} \cos e^{x^2}$$

Multivariate Functions

Functions $f : R^n \rightarrow R$

$$x = (x_1, \ldots, x_n)^T$$

$$y = f(x_1, \ldots, x_n)$$

- Partial derivative with respect to x_i, denoted by

$$\frac{\partial y}{\partial x_i}, \frac{\partial f}{\partial x_i}, \text{or} \quad f_{x_i}(x_1, \ldots, x_n)$$

treats x_i as a variable and all other x_j with $j \neq i$ as constants

Example: $y = f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^3$

$$\frac{\partial y}{\partial x_1} = 2x_1 + x_2$$

Multivariate Functions

- The **gradient**

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{pmatrix}$$

is the vector of all partial derivatives

Example: $y = f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^3$

$$\nabla f = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 3x_2^2 \end{pmatrix}$$

Multivariate Functions

- The **Hessian**

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

is the $n \times n$ matrix of all second partial derivatives

Vector-Valued Functions

Functions $f : R^n \to R^m$

$$x = (x_1, \ldots, x_n)^T, f = (f_1, \ldots, f_m)^T$$

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ y_2 &= f_2(x_1, \ldots, x_n) \\ &\vdots \\ y_m &= f_m(x_1, \ldots, x_n) \end{aligned}$$

- Partial derivative for f_j with respect to x_i, denoted by $\dfrac{\partial f_j}{\partial x_i}$

Vector-Valued Functions

- The **Jacobian**

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

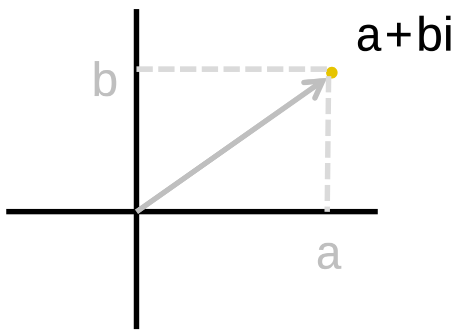
is the $m \times n$ matrix of the partial derivatives of all functions

Complex Numbers

- Two-dimensional extension of real numbers

$$z = a + bi$$

where a is referred to as the real part of z and b is referred to as the imaginary part of z



- Convention: $i^2=-1$

- Complex conjugate $\bar{z} = a - bi$

Complex Numbers

Rules

- $(a+bi) + (c+di) = (a+c) + (b+d)i$
- $(a+bi) - (c+di) = (a-c) + (b-d)i$
- $(a+bi)(c+di) = ac + bic + adi + bidi = (ac-bd) + (ad+bc)i$
- $$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Complex Numbers

- Euler's formula

$$e^{ix} = \cos x + i \sin x$$

- Rotation of a point (a,b) in the plane by an angle θ about the origin can be accomplished by multiplying $e^{i\theta}$ and a+bi:

$$e^{i\theta} \cdot (x + yi) = (\cos\theta + i\sin\theta)(x + yi) = (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$$

Compare with:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$