Exercises: Particle Filtering and HMM Inference

Hidden Weather and Umbrellas

We model the daily weather as a Hidden Markov Model with *hidden* state

$$X_t \in \{\text{Rainy (R)}, \text{Sunny (S)}\}$$

and *observations*

$$E_t \in \{\text{Umbrella (U)}, \text{No Umbrella (N)}\},$$

indicating whether your colleague carries an umbrella on day t.

Model parameters

Initial distribution.

$$P(X_1 = R) = 0.5, \quad P(X_1 = S) = 0.5.$$

Transition model.

$$P(R \mid R) = 0.7, \quad P(S \mid R) = 0.3,$$
$$P(S \mid S) = 0.8, \quad P(R \mid S) = 0.2.$$

Observation model.

$$P(U \mid R) = 0.9, \quad P(N \mid R) = 0.1,$$
$$P(U \mid S) = 0.2, \quad P(N \mid S) = 0.8.$$

Observation sequence

Over four consecutive days you record:

$$e_{1:4} = (U, U, N, U).$$

Tasks

(A1) Forward algorithm (filtering).
Use the forward algorithm to compute the joint quantities

$$\alpha_t(x) = P(e_{1:t}, X_t = x)$$

for $t = 1, 2, 3, 4$ and for $x \in \{R, S\}$. Then compute the *filtered beliefs*

$$P(X_t = x \mid e_{1:t}) = \frac{\alpha_t(x)}{\sum_{x'} \alpha_t(x')}$$

for each t and x.

Present your results in a table with columns t, $\alpha_t(R)$, $\alpha_t(S)$, $P(X_t = R \mid e_{1:t})$, $P(X_t = S \mid e_{1:t})$.

(A2) Viterbi algorithm (most likely state sequence).
Use the Viterbi algorithm to compute the most likely sequence of hidden weather states given the observations:

$$x^*_{1:4} = \arg\max_{x_{1:4}} P(x_{1:4} \mid e_{1:4}).$$

Show the dynamic programming recurrence for the Viterbi scores $\delta_t(x)$ and backpointers, and give the final most likely sequence $x^*_{1:4}$.