Oplossingen Bayesian Networks

Mathis De Herdt

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Problem 1: Host's Console Crash

Let's define the binary variables based on the problem description. Note the interpretation of P:

- $P = 1$: Power is OK, $P = 0$: Power problem.
- $G = 1$: Game is buggy, $G = 0$: Game is not buggy.
- $M = 1$: Music app playing, $M = 0$: Music app not playing.
- $C = 1$: Console crash, $C = 0$: Console not crashed.

The given probabilities are:

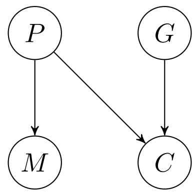
- $P(P = 1) = 0.97 \implies P(P = 0) = 0.03$
- $P(G = 1) = 0.35 \implies P(G = 0) = 0.65$
- $P(M = 1|P = 1) = 0.92 \implies P(M = 0|P = 1) = 0.08$
- $P(M = 1|P = 0) = 0.12 \implies P(M = 0|P = 0) = 0.88$
- $P(C = 1|P = 0, G = 1) = 0.98$
- $P(C = 1|P = 1, G = 1) = 0.45$
- $P(C = 1|P = 0, G = 0) = 0.88$
- $P(C = 1|P = 1, G = 0) = 0.08$

a) Bayesian Network and Factorization

The story implies:

- "P and G are a priori independent" $\implies$ P and G are roots.
- "Music... if power is fine" $\implies$ M depends on P.
- "Crashes... due to... power problem (P) or a buggy game (G)" $\implies$ C depends on both P and G.

The Bayesian Network diagram is:



The factorization of the joint probability distribution is:

$$P(P, G, M, C) = P(P) \cdot P(G) \cdot P(M|P) \cdot P(C|P, G)$$

b) Storage

- **Full Joint:** A full joint distribution over 4 binary variables has $2^4 = 16$ possible entries. To specify the distribution, we need $2^4 - 1 = 15$ independent parameters (the last one is constrained by the sum-to-one rule).
- **Factorization:** We count the independent parameters required for each Conditional Probability Table (CPT) in our factorization:
  - $P(P)$: 1 parameter ($P(P = 1)$).
  - $P(G)$: 1 parameter ($P(G = 1)$).
  - $P(M|P)$: 2 parameters ($P(M = 1|P = 1)$ and $P(M = 1|P = 0)$).
  - $P(C|P, G)$: 4 parameters (one for each of the $2^2 = 4$ combinations of parent states).

  The total number of parameters is $1 + 1 + 2 + 4 = 8$.

c) Inference

(i) $P(G = 1|C = 1)$

We use Bayes' rule: $P(G = 1|C = 1) = \frac{P(C=1|G=1)P(G=1)}{P(C=1)}$.

First, we find $P(C = 1|G = 1)$ by marginalizing over P:

$$P(C = 1|G = 1) = \sum_p P(C = 1|P = p, G = 1)P(P = p)$$

$$= P(C = 1|1, 1)P(P = 1) + P(C = 1|0, 1)P(P = 0)$$

$$= (0.45 \cdot 0.97) + (0.98 \cdot 0.03) = 0.4365 + 0.0294 = 0.4659$$

Next, we find the marginal $P(C = 1)$ using the law of total probability:

$$P(C = 1) = \sum_g P(C = 1|G = g)P(G = g)$$

$$= P(C = 1|G = 1)P(G = 1) + P(C = 1|G = 0)P(G = 0)$$

We need $P(C = 1|G = 0) = \sum_p P(C = 1|p, 0)P(p)$:

$$P(C = 1|G = 0) = P(C = 1|1, 0)P(P = 1) + P(C = 1|0, 0)P(P = 0)$$

$$= (0.08 \cdot 0.97) + (0.88 \cdot 0.03) = 0.0776 + 0.0264 = 0.104$$

Now, plug this back into the equation for $P(C = 1)$:

$$P(C = 1) = (0.4659 \cdot 0.35) + (0.104 \cdot 0.65) = 0.163065 + 0.0676 = 0.230665$$

Finally, we compute the desired probability:

$$P(G = 1|C = 1) = \frac{P(C = 1|G = 1)P(G = 1)}{P(C = 1)} = \frac{0.4659 \cdot 0.35}{0.230665} = \frac{0.163065}{0.230665} \approx 0.7069$$

(ii) $P(G=1|C=1,M=1)$

By definition of conditional probability: $P(G=1|C=1,M=1) = \frac{P(G=1,C=1,M=1)}{P(C=1,M=1)}$.
Numerator: $P(G=1,C=1,M=1) = \sum_p P(P=p,G=1,M=1,C=1)$

$$= \sum_p P(G=1)P(p)P(M=1|p)P(C=1|p,G=1)$$

$$= P(G=1)\cdot[P(P=1)P(M=1|1)P(C=1|1,1) + P(P=0)P(M=1|0)P(C=1|0,1)]$$

$$= 0.35\cdot[(0.97\cdot 0.92\cdot 0.45) + (0.03\cdot 0.12\cdot 0.98)]$$

$$= 0.35\cdot[0.40158 + 0.003528] = 0.35\cdot 0.405108 = 0.1417878$$

Denominator: $P(C=1,M=1) = \sum_g P(G=g,C=1,M=1)$

$$= P(G=1,C=1,M=1) + P(G=0,C=1,M=1)$$

We just calculated the $G=1$ term. Now for the $G=0$ term:

$$P(G=0,C=1,M=1) = P(G=0)\cdot[P(P=1)P(M=1|1)P(C=1|1,0)+P(P=0)P(M=1|0)P(C=1|0,0)]$$

$$= 0.65\cdot[(0.97\cdot 0.92\cdot 0.08) + (0.03\cdot 0.12\cdot 0.88)]$$

$$= 0.65\cdot[0.071392 + 0.003168] = 0.65\cdot 0.07456 = 0.048464$$

So, $P(C=1,M=1) = 0.1417878 + 0.048464 = 0.1902518$.
Finally:

$$P(G=1|C=1,M=1) = \frac{0.1417878}{0.1902518} \approx 0.74526$$

d) Independence

Based on the network structure and d-separation rules:

- **Independent pairs ($X \perp Y$):**
  - (G,P): **Independent**. They are two roots with no path between them.
  - (G,M): **Independent**. The only path is $G \to C \leftarrow P \to M$, which is blocked by the collider C.
- **Conditionally independent given exactly one variable ($X \perp Y|Z$):**
  - $(G,M)|P$: **Independent**. The path $G \to C \leftarrow P \to M$ is blocked by C (collider) and also by the given P (which blocks the $P \to M$ part).
  - $(C,M)|P$: **Independent**. The path $C \leftarrow P \to M$ is blocked by P. The path $C \leftarrow G$ is not relevant.
  - $(P,G)|M$: **Independent**. The path $P \to C \leftarrow G$ is blocked by the collider C. (M is not C or a descendant).

Problem 2: Party Noise Network

a) True/False using d-separation

(i) $h \perp s$: **True**. The path $h \to w \to e \leftarrow l \leftarrow s$ is blocked by the collider e. The path $h \to w \to e \to d \leftarrow b \leftarrow s$ is blocked by the collider d.

(ii) $h \perp l$: **True**. The path $h \to w \to e \leftarrow l$ is blocked by the collider e.

(iii) $h \perp x$: **False**. The path $h \to w \to e \to x$ is an active path (a chain).

(iv) $w \perp s$: **True**. The path $w \to e \leftarrow l \leftarrow s$ is blocked by collider e. The path $w \to e \to d \leftarrow b \leftarrow s$ is blocked by collider d.

(v) $x \perp b$: **True**. The path $x \leftarrow e \to d \leftarrow b$ is blocked by collider d. The path $x \leftarrow e \leftarrow l \leftarrow s \to b$ is blocked at e (collider) and s (common cause, but path from e is blocked).

b) Compute

We need to compute $P(d = 1)$, $P(d = 1|s = 1)$, and $P(d = 1|s = 0)$. We will use the structure: $P(d = 1|s) = \sum_{e,b} P(d = 1|e, b)P(e, b|s)$. From the graph, $e \perp b|s$. (The path $e \leftarrow l \leftarrow s \to b$ is blocked by s. The other path $e \leftarrow w \leftarrow h$ is independent of s and b). Therefore, $P(d = 1|s) = \sum_{e,b} P(d = 1|e, b)P(e|s)P(b|s)$.

First, let's find the components $P(e|s)$ and $P(b|s)$. $P(b|s)$ is given. For $P(e|s)$: $e = w \vee l$, so $e = 0 \iff w = 0$ and $l = 0$. $P(e = 0|s) = P(w = 0, l = 0|s) = P(w = 0|s)P(l = 0|s)$. w is d-separated from s, so $P(w = 0|s) = P(w = 0)$. $P(w = 1) = P(w|h = 1)P(h = 1) + P(w|h = 0)P(h = 0)$ $P(w = 1) = (0.5 \cdot 0.2) + (0.1 \cdot 0.8) = 0.1 + 0.08 = 0.18$. $P(w = 0) = 1 - 0.18 = 0.82$. So, $P(e = 0|s) = 0.82 \cdot P(l = 0|s)$.

Calculations for $s = 1$:

- $P(l = 0|s = 1) = 1 - P(l = 1|s = 1) = 1 - 0.3 = 0.7$.
- $P(e = 0|s = 1) = P(w = 0)P(l = 0|s = 1) = 0.82 \cdot 0.7 = 0.574$.
- $P(e = 1|s = 1) = 1 - 0.574 = 0.426$.
- $P(b = 1|s = 1) = 0.6$, $P(b = 0|s = 1) = 0.4$.

Calculations for $s = 0$:

- $P(l = 0|s = 0) = 1 - P(l = 1|s = 0) = 1 - 0.05 = 0.95$.
- $P(e = 0|s = 0) = P(w = 0)P(l = 0|s = 0) = 0.82 \cdot 0.95 = 0.779$.
- $P(e = 1|s = 0) = 1 - 0.779 = 0.221$.
- $P(b = 1|s = 0) = 0.2$, $P(b = 0|s = 0) = 0.8$.

Now we can compute the probabilities for $d = 1$. Given $P(d = 1|e, b)$: $P(d|1, 1) = 0.9$, $P(d|1, 0) = 0.7$, $P(d|0, 1) = 0.8$, $P(d|0, 0) = 0.1$.

(ii) $P(d=1|s=1)$

$$P(d=1|s=1) = \sum_{e,b} P(d|e,b)P(e|s=1)P(b|s=1)$$

$$= P(d|1,1)P(e=1|1)P(b=1|1) + P(d|1,0)P(e=1|1)P(b=0|1)+$$

$$P(d|0,1)P(e=0|1)P(b=1|1) + P(d|0,0)P(e=0|1)P(b=0|1)$$

$$= (0.9 \cdot 0.426 \cdot 0.6) + (0.7 \cdot 0.426 \cdot 0.4) + (0.8 \cdot 0.574 \cdot 0.6) + (0.1 \cdot 0.574 \cdot 0.4)$$

$$= 0.23004 + 0.11928 + 0.27552 + 0.02296 = \mathbf{0.6478}$$

(iii) $P(d=1|s=0)$

$$P(d=1|s=0) = \sum_{e,b} P(d|e,b)P(e|s=0)P(b|s=0)$$

$$= P(d|1,1)P(e=1|0)P(b=1|0) + P(d|1,0)P(e=1|0)P(b=0|0)+$$

$$P(d|0,1)P(e=0|0)P(b=1|0) + P(d|0,0)P(e=0|0)P(b=0|0)$$

$$= (0.9 \cdot 0.221 \cdot 0.2) + (0.7 \cdot 0.221 \cdot 0.8) + (0.8 \cdot 0.779 \cdot 0.2) + (0.1 \cdot 0.779 \cdot 0.8)$$

$$= 0.03978 + 0.12376 + 0.12464 + 0.06232 = \mathbf{0.3505}$$

(i) $P(d=1)$

Using the law of total probability with $P(s=1) = 0.5$ and $P(s=0) = 0.5$:

$$P(d=1) = P(d=1|s=1)P(s=1) + P(d=1|s=0)P(s=0)$$

$$= (0.6478 \cdot 0.5) + (0.3505 \cdot 0.5)$$

$$= 0.5 \cdot (0.6478 + 0.3505) = 0.5 \cdot 0.9983 = \mathbf{0.49915}$$

Problem 3: D-Separation Quick Test

1. $X \to Z \to Y$ (Chain)
   - $X \perp Y$? **No**. The path $X \to Z \to Y$ is active.
   - $X \perp Y|Z$? **Yes**. The path $X \to Z \to Y$ is blocked by conditioning on Z.
2. $X \leftarrow Z \to Y$ (Common cause)
   - $X \perp Y$? **No**. The path $X \leftarrow Z \to Y$ is active.
   - $X \perp Y|Z$? **Yes**. The path $X \leftarrow Z \to Y$ is blocked by conditioning on Z.
3. $X \to Z \leftarrow Y$ (Collider / v-structure)
   - $X \perp Y$? **Yes**. The path $X \to Z \leftarrow Y$ is blocked by the collider Z (which is not conditioned on).
   - $X \perp Y|Z$? **No**. The path is opened by conditioning on the collider Z.
4. $X \leftarrow Z \leftarrow Y$ (Chain)
   - $X \perp Y$? **No**. The path $Y \to Z \to X$ is active.
   - $X \perp Y|Z$? **Yes**. The path $Y \to Z \to X$ is blocked by conditioning on Z.