1 Argument to use the Hedges' g^* with the Welch's t-test : an investigation of Cohen's d and

2 related effect size estimators in terms of interpretability, bias, precision and robustness

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18 Argument to use the Hedges' g^* with the Welch's t-test : an investigation of Cohen's d and

19   related effect size estimators in terms of interpretability, bias, precision and robustness

20   Effect sizes are an important outcome of empirical research. Moving beyond decisions

21 about statistical significance, there is a strong call for researchers to report and interpret

22 effect sizes and associated confidence intervals. This practice is highly endorsed by the

23 American Psychological Association (APA) and the American Educational Research

24 Association (American Psychological Association, 2010; Duran et al., 2006).

In "between-subject" designs where individuals are randomly assigned into one of two

independent groups and group scores are compared based on their means, the dominant

estimator of effect size is Cohen's d, where the sample mean difference is divided by the

pooled sample standard deviation (Peng et al., 2013; Shieh, 2013). This estimator is

available in many statistical software packages, such as SPSS and Stata. However,

computing the pooled sample standard deviation assumes that both sample variances are

estimates of a common population variance, which is known as the homogeneity of variance

assumption. It has been widely argued that there are many fields in psychology where this

assumption is ecologically unlikely (Delacre et al., 2017; Erceg-Hurn & Mirosevich, 2008;

Grissom, 2000). The question how to deal with the assumption of equal variances has been

widely explored in the context of hypothesis testing, and it is becoming increasingly

common to by default report a t-test that does not assume equal variances, such as Welch's

t-test.

38   However, the question which effect size to report when equal variances are not

39 assumed has received less attention. One possible reason is that researchers have not found

40 consensus on which of the available options should be used (Shieh, 2013). Even within the

41 very specific context of an estimate for the standardized sample mean difference there is

42 little agreement about which estimator is the best choice. In this article, we will review the

43 main candidates that have been proposed in the literature in the d family of effect sizes,

44 without (Cohen's d, Glass' d, Shieh's d and Cohen's d^*) and with correction for bias

45 (Hedges' g, Glass' g, Shieh's g and Hedges' g^*). We provide an R package and Shiny app

46 to compute relevant effect size measures and their confidence intervals.

47   Before reviewing the most important effect size measures in the d-family, we will first

48 list the different purposes effect size measures serve, and discuss the relationship between

49 effect sizes, statistical, and practical significance. Based on a detailed description of the

50 good properties an effect size measure should possess, we will evaluate these properties in

51 the Monte Carlo simulations we performed to compare the different effect size estimators

52 with correction for bias.

Three purposes of effect size estimators

The effect size is a measure of the magnitude of an effect. In the context of the

comparison of two groups based on their means, when the null hypothesis is the absence of

effect, d-family effect size estimators estimate the magnitude of the differences between

parameters of two populations groups are extracted from (e.g. the mean; Peng & Chen,

2014). Such a measure can be used for three different purposes.

59   First, effect size measures can be used for *interpretative* purposes. They allow

60 researchers to assess the practical significance of a result (i.e. statements about the

61 relevance of an effect in real life). In order to assess the meaningfulness of an effect, we

62 should be able to relate this effect size estimate with behaviors/meaningful consequences in

63 the real world (Andersen et al., 2007). This typically involves an analysis of the costs

64 (determined by a specific context) and the benefits (in part determined by the size of the

65 effect). It is important to remember an effect size is just a mathematical indicator of the

66 magnitude of a difference, which depends on the way a variable is converted into numerical

67 indicator. An effect size in itself is not a measure of the importance or the relevance of an

68 effect for real life (even if benchmarks for small, medium, or large effect sizes might have

⁶⁹ contributed to such a misinterpretation; Stout & Ruble, 1995).

⁷⁰   Second, effect size measures can be used for *comparative* purposes. They allow

⁷¹ researchers to assess the stability of results across designs, analyses, and sample sizes. This

⁷² includes statistically comparing and combining the results from two or more studies in a

⁷³ meta-analysis.

⁷⁴   Third, effect size measures can be used for *inferential* purposes. Hypothesis tests and

⁷⁵ confidence intervals based on the same statistical quantity are directly related : if the area

⁷⁶ of the null hypothesis is out of the $(1 - \alpha)$-confidence interval, then the hypothesis test

⁷⁷ would also result in a *p*-value below the nominal alpha level. At the same time, the interval

⁷⁸ provides extra information about the precision of the sample estimate for inferential

⁷⁹ purposes (Altman, 2005; Ellis, 2010), and which effect sizes are excluded. The narrower the

⁸⁰ interval, the higher the precision, and the wider the confidence interval, the more the data

⁸¹ lack precision. Effect size measures are also indirectly related to the hypothesis tests as

⁸² effect sizes from previous studies can be used in an a-priori power analysis when planning a

⁸³ new study (Lakens, 2013; Prentice & Miller, 1992; Stout & Ruble, 1995; Sullivan & Feinn,

⁸⁴ 2012; Wilkinson, 1999).

⁸⁵   **Inferential properties of a good effect size estimator**

The empirical value of an estimator (called the *estimate*) depends on the sample

value. Different samples extracted from the same population will lead to different sample

estimates of the population value. The *sampling distribution* of the estimator is the

distribution of all estimates, based on all possible samples of size n extracted from one

population. Studying the sampling distribution is useful, as it allows us to assess the

qualities of an estimator. More specifically, three desirable properties a good estimator

should possess for inferential purposes are : *unbiasedness*, *consistency* and *efficiency*

(Wackerly et al., 2008).

94   An estimator is unbiased if the distribution of estimates is centered around the true

95 population parameter. An estimator is positively (or negatively) biased if the distribution

96 is centered around a value that is higher (or lower) than the true population parameter

97 (see Figure 1). In other words, examining the bias of an estimator tells us if estimates are

98 on average accurate. The *bias* of a point estimator $\hat{\delta}$ can be computed as

$$\delta_{bias} = E(\hat{\delta}) - \delta \tag{1}$$

99 where $E(\hat{\delta})$ is the expectation of the sampling distribution of the estimator and δ is the

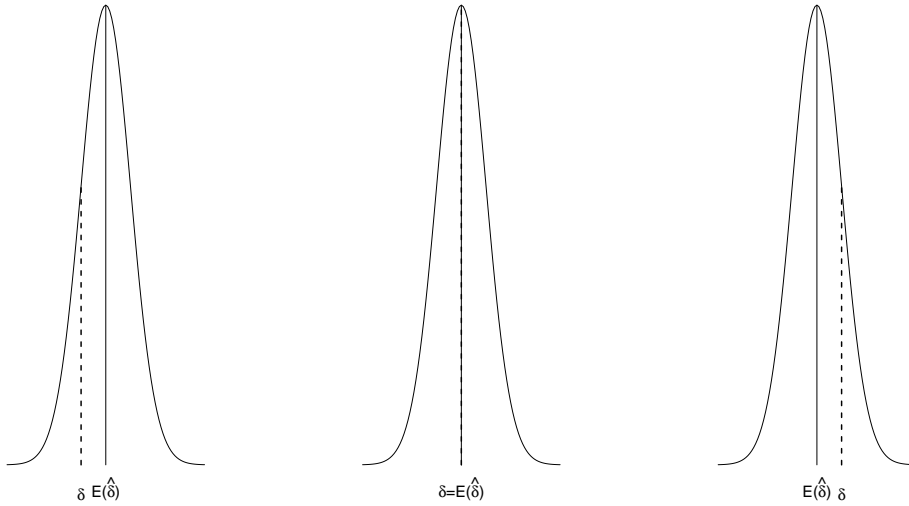100 true (population) parameter.

Figure 1. Sampling distribution for a positively biased (left), an unbiased (center) and a
negatively biased estimator (right)

101   As we can see in Tables 1 and 2 the bias is directly related to the population effect

102 size. The larger the population effect size, the larger the bias. As we shall see in the sequel,

103 each effect size estimator actually estimates a different expression of the population effect

104 size. It is therefore necessary to examine the *relative bias*, defined as the ratio between the

105 bias and the population effect size :

$$\delta_{relative\ bias} = \frac{E(\hat{\delta}) - \delta}{\delta}. \tag{2}$$

106 While the bias informs us about the quality of estimates on average, in particular their

107 capacity of lying close to the true value, it says nothing about individual estimates.

108 Imagine a situation where the distribution of estimates is centered around the real

109 parameter but with such a large variance that some point estimates are very far from the

110 center. This would be problematic, since any single estimate might be very far from the

111 true population value. Therefore it is not only essential for an estimator to be unbiased,

112 but it is also desirable that the variability of its sampling distribution is small. Ideally, all

113 sample estimates are close to the true population parameter. Among two unbiased

114 estimators $\hat{\delta}_1$ and $\hat{\delta}_2$, we therefore say that $\hat{\delta}_1$ is *more efficient* than $\hat{\delta}_2$ if

$$Var(\hat{\delta}_1) \leq Var(\hat{\delta}_2) \tag{3}$$

115 where $Var(\hat{\delta})$ is the variance of the sampling distribution of the estimator $\hat{\delta}$. This variance

116 inequality of course only makes sense if two estimators $\hat{\delta}_1$ and $\hat{\delta}_2$ estimate the same

117 population quantity, which is not the case here as explained before (each effect size

118 estimator corresponds to a distinct population effect size expression). Therefore, we rather

119 have to consider the *relative variance*, defined as the ratio between the variance of an

120 estimator and the square of the corresponding population effect size δ :

$$relative\ var(\hat{\delta}) = \frac{Var(\hat{\delta})}{\delta^2}. \tag{4}$$

121 Note that both unbiasedness and efficiency are very important when choosing an estimator.

122 In some situations, it might be better to have a slightly biased estimator with low variance,

123 (so that each estimate remains relatively close to the true parameter and one might be able

124 to apply bias correction techniques) rather than an unbiased estimator with a large

125 variance (Raviv, 2014).

126   Finally, the last property of a good point estimator is *consistency*. Consistency means

127 that the bigger the sample size, the closer the estimate is to the population parameter. In

128 other words, the estimates *converge* to the true population parameter.

Different measures of effect sizes

130   The *d*-family effect sizes are commonly used for mean differences between groups or

131 conditions. In all generality, the population effect size is defined as

$$\delta = \frac{\mu_1 - \mu_2}{\sigma} \tag{5}$$

132 where μ_1 and μ_2 respectively stand for the population means of population 1 and 2, and σ

133 represents the "variability" of the combined observations from both population. Typically

134 one assumes that both populations are independent and normally distributed (the

135 normality assumption however is often a too strong limitation; Delacre et al., 2017). In

136 some situations it is moreover assumed that the variances of both populations are equal, in

137 which case σ stands for the common standard deviation. There exist different estimators of

138 the effect size measure in expression 5. For all, the mean difference is estimated by the

139 difference $\bar{X}_1 - \bar{X}_2$ of both sample means. When the equality of variances assumption is

140 made, σ is estimated by pooling both sample standard deviations (S_1 and S_2). When the

141 equality of variances assumption cannot be made, the pooled standard deviation is a

142 possible but not recommendable choice as we explain later on, and alternatives are

143 available. In what follows, we shall present various effect size measures. For each effect

144 size, we will provide information about their theoretical bias, variance and consistency.

145 Simulation-based results will be discussed in the section "Monte Carlo Simulations :

146 assessing the bias, efficiency and consistency of 5 estimators" below.

147 **Cohen's *d* and Hedges' *g***

When we have good reasons to assume equality of variances between groups then the

most common effect size is Cohen's *d*, where the sample mean difference is divided by a

pooled error term (Cohen, 1965) :

$$Cohen's\ d = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{(n_1-1)\times S_1^2+(n_2-1)\times S_2^2}{n_1+n_2-2}}}$$

148 where S_j is the standard deviation, and n_j the sample size of the jth sample ($j = 1, 2$). The

149 reasoning behind this measure is to make use of the fact that both samples share the same

150 population variance (Keselman et al., 2008), which means a more accurate estimation of

151 the population variance can be achieved by pooling both estimates of this parameter

152 (i.e. S_1 and S_2). Since the larger the sample size, the more accurate the estimate, we give

153 more weight to the estimate based on the larger sample size. Cohen's d is directly related

154 to Student's t-statistic :

$$t_{Student} = \frac{Cohen's\ d}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \leftrightarrow Cohen's\ d = t_{Student} \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \tag{6}$$

155 Under the assumption of normality and equal variances between groups, Student's

156 t-statistic follows a t-distribution with known degrees of freedom

$$df_{Student} = n_1 + n_2 - 2 \tag{7}$$

157 and noncentrality parameter[1]

$$ncp_{Student} = \frac{\delta_{Cohen}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

158 where $\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sigma_{pooled}}$, $\sigma_{pooled} = \sqrt{\frac{(n_1-1) \times \sigma_1^2 + (n_2-1) \times \sigma_2^2}{n_1 + n_2 - 2}}$ and σ_j is the standard deviation of the

159 j^{th} population ($j = 1, 2$). We attract the reader's attention to the act that δ_{Cohen} is the

160 expression of the effect size that the Cohen's d is estimating. The relationship described in

161 equation 6 and the theoretical distribution of Student's t-statistic allow us to determine the

162 sampling distribution of Cohen's d, and therefore, its expectation and variance when the

163 assumptions of normality and equal variances are met. All these equations are provided in

164 Table 1. For interested readers, Supplemental Material 1 provides a detailed examination

165 of the theoretical bias and variance of Cohen's d based on Table 1, as well as the bias and

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[1] Under the null hypothesis of no differences between sample means, Student's t-statistic will follow a

central t-distribution with $n_1 + n_2 - 2$ degrees of freedom. However, when the null hypothesis is false, the

distribution of this quantity will not be centered, and a noncentral t-distribution will arise.

166 variance of all other estimators described later, based on Tables 2 and 3, with the goal to

167 determine which parameters influence the bias and variance of different estimators.

While Cohen's d is a consistent estimator, its bias and variance are substantial with small sample sizes, even under the assumptions of normality and equal variances (Lakens, 2013). In order to compensate for Cohen's d bias with small sample sizes, Hedges and Olkin (1985) defined a bias-corrected version :

$$Hedges' \; g = Cohen's \; d \times \frac{\Gamma(\frac{df_{Student}}{2})}{\sqrt{\frac{df_{Student}}{2}} \times \Gamma(\frac{df_{Student}-1}{2})}$$

where $df_{Student}$ has been defined in equation 7 and $\Gamma()$ is the gamma function (for integers, $\Gamma(x)$ is the factorial of x minus 1 : $\Gamma(x) = (x-1)!$; Goulet-Pelletier & Cousineau, 2018). This equation can be approximated as follows :

$$Hedges' \; g = Cohen's \; d \times \left(1 - \frac{3}{4N-9}\right)$$

168 where N is the total sample size. Hedges' g is theoretically unbiased when the assumptions

169 of normality and equal variances are met (see Table 1). Moreover, while the variance of

170 both Cohen's d and Hedges' g depend on the same parameters (i.e. the total sample size

171 (N) and the sample sizes ratio $\left(\frac{n_2}{n_1}\right)$), Hedges' g is less variable, especially with small

172 sample sizes.[2]

173   While the pooled error term is the best choice when variances are equal between

174 groups (Grissom & Kim, 2001), it may not be well advised for use with data that violate

175 this assumption (Cumming, 2013; Grissom & Kim, 2001; Grissom & Kim, 2005; Kelley,

176 2005; Shieh, 2013). When variances are unequal between groups, the expression in

177 equation 5 is no longer valid because both groups do not share a common population

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[2] In Table 1, one can see that the variance of Hedges' g equals the variance of Cohen's d, multiplied by $\left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}\right]^2$. This term is always less than 1 and tends to 1 when the sample sizes tend to infinity ($.52 \leq \left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}\right]^2 < 1$ for $3 \leq df < \infty$). As a consequence, the larger the total sample size, the smaller the difference between the variance of Cohen's d and Hedges' g.

178   variance. If we pool the estimates of two unequal population variances, the estimator of

179   effect size will be smaller as it should be in case of positive pairing (i.e. the group with the

180   larger sample size is extracted from the population with the larger variance) and larger as

181   it should be in case of negative pairing (i.e. the group with the larger sample size is

182   extracted from the population with the smaller variance).

Table 1

Expectation, bias and variance of Cohen's d and Hedges' g under the assumptions that independent residuals are normally distributed with equal variances across groups.

	df	Expectation	Variance
Cohen's d	$N-2$	$\delta_{Cohen} \times c_f$	$\frac{N \times df}{n_1 n_2 \times (df-2)} + \delta_{Cohen}^2 \left[\frac{df}{df-2} - c_f^2\right]$
		$\approx \frac{\delta_{Cohen}}{\left(1 - \frac{3}{4N-9}\right)}$	$\approx \frac{N \times df}{n_1 n_2 \times (df-2)} + \delta_{Cohen}^2 \left[\frac{df}{df-2} - \left(\frac{1}{1-\frac{3}{4N-9}}\right)^2\right]$
Hedges' g	$N-2$	δ_{Cohen}	$Var(Cohen's\ d_s) \times \left[\frac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}\right]^2$
			$\approx Var(Cohen's\ d_s) \times \left[1 - \frac{3}{4N-9}\right]^2$

Note. $\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sigma_{pooled}}$ and $c_f = \frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}$; Cohen's d is a biased estimator, because its expectation differs from the population effect size. Moreover, the larger the population estimator (δ_{Cohen}), the larger the bias. Indeed, the bias is the difference between the expectation and δ_{Cohen}: $\delta_{bias} = \delta_{Cohen} \times (c_f - 1)$. On the other hand, Hedges' g is an unbiased estimator, because its expectation equals δ_{Cohen}; equations in this table require $df \geq 3$ (i.e. $N \geq 5$).

Three popular alternatives from the literature

In his review, Shieh (2013) mentions three options available in the literature to deal with the case of unequal variances : (A) the Glass' d, (B) the Shieh's d and (C) the Cohen's d^*.

Glass' d. When comparing one control group with one experimental group, Glass et al. (1981) recommend using the standard deviation of the control group as standardizer. This yields

$$Glass' \; d = \frac{\bar{X}_e - \bar{X}_c}{S_c}$$

where $\bar{X}_e$ and $\bar{X}_c$ are the sample means of the experimental and control groups, and S_c is the sample SD of the control group. One argument in favour of using S_c as standardizer is the fact that it is not affected by the experimental treatment. When it is easy to identify which group is the "control" one, it is therefore convenient to compare the effect size estimation of different designs studying the same effect (Cumming, 2013). However, defining this group is not always obvious (Coe, 2002). This could induce large ambiguity because depending on the chosen SD as standardizer, measures could be substantially different (Shieh, 2013). As we shall see later through our simulations, this estimator also does not present good inferential properties. The distribution of Glass' d is defined as in Algina et al. (2006) :

$$Glass' \; d \sim \sqrt{\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \times \sigma_c^2}} \times t_{df,ncp} \tag{8}$$

where n_c and n_e are the sample sizes of the control and experimental groups, σ_c^2 and σ_e^2 are the population variance of the control and experimental groups and and df and ncp are defined as follows :

$$df = n_c - 1 \tag{9}$$

$$ncp = \frac{\delta_{Glass}}{\sqrt{\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \times \sigma_c^2}}}$$

204   where $\delta_{Glass} = \frac{\mu_c - \mu_e}{\sigma_c}$ and μ_c and μ_e are respectively the mean of the populations control

205   and experimental groups are extracted from. Note that here the population effect size

206   δ_{Glass} is also based on the standard deviation σ_c of the control group. Thanks to equation

207   8, we can compute the theoretical expectation and variance of Glass's d when the

208   assumption of normality is met (see Table 2), and therefore determine which factors

209   influence bias and variance, and how they do so (see Supplemental Material 1).

Table 2

Expectation, bias and variance of Glass' d and Cohen's d* and Shieh's d under the assumption that independent residuals are normally distributed.

	df	Expectation	Variance
Glass' d	$n_c - 1$	$\delta_{Glass} \times c_f$	$\frac{df}{df-2} \times \left(\frac{1}{n_c} + \frac{\sigma_e^2}{n_e\sigma_c^2}\right) + \delta_{Glass}^2 \left(\frac{df}{df-2} - c_f^2\right)$
Cohen's d^*	$\frac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$	$\delta_{Cohen}^* \times c_f$	$\frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2+\sigma_2^2} + (\delta_{Cohen}^*)^2 \left(\frac{df}{df-2} - c_f^2\right)$
		$\approx \delta_{Cohen}^* \times \frac{4df-1}{4(df-1)}$	$\approx \frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2+\sigma_2^2} + (\delta_{Cohen}^*)^2 \left[\frac{df}{df-2} - \left(\frac{4df-1}{4(df-1)}\right)^2\right]$
Shieh's d	$\frac{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1}+\frac{(\sigma_2^2/n_2)^2}{n_2-1}}$	$\delta_{Shieh} \times c_f$	$\frac{df}{(df-2)N} + \delta_{Shieh}^2 \left(\frac{df}{df-2} - c_f^2\right)$

Note. $c_f = \frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}$; $\delta_{Glass} = \frac{\mu_c-\mu_e}{\sigma_c}$, $\delta_{Shieh} = \frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N}+\frac{\sigma_2^2}{n_2/N}}}$ and $\delta_{Cohen}^* = \frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}}$; all estimators are biased estimators, because their expectations differ from the population effect size δ. Moreover, the larger the population estimator (δ), the larger the bias. Indeed, the bias is the difference between the expectation and δ: $\delta_{bias} = \delta \times (c_f - 1)$; equations in this table require $df \geq 3$ and at least 2 subjects per group.

Shieh's *d*.   Kulinskaya and Staudte (2007) were the first to recommend the use of a standardizer that takes the sample sizes allocation ratios into account, in addition to the variance of both samples. Shieh (2013), following Kulinskaya and Staudte, proposed a modification of the exact *SD* of the sample mean difference :

$$Shieh's\ d = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{S_1^2/q_1 + S_2^2/q_2}}; \quad q_j = \frac{n_j}{N}(j = 1, 2)$$

214   where $N = n_1 + n_2$. Shieh's *d* is directly related with Welch's *t*-statistic :

$$Shieh's\ d = \frac{t_{Welch}}{\sqrt{N}} \leftrightarrow t_{welch} = Shieh's\ d \times \sqrt{N}. \tag{10}$$

215   The exact distribution of Welch's *t*-statistic is more complicated than the exact distribution

216   of Student's *t*-statistic, but it can be approximated, under the assumption of normality, by

217   a *t*-distribution with degrees of freedom and noncentrality parameters (Welch, 1938)

$$df_{Welch} = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}} \tag{11}$$

$$ncp_{Welch} = \delta_{Shieh} \times \sqrt{N} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n1} + \frac{\sigma_2^2}{n_2}}}$$

218   where $\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}}$ is the population effect size measure estimated by Shieh's *d*.

219   The relationship described in equation 10 and the theoretical distribution of Welch's

220   *t*-statistic allow us to approximate the sampling distribution of Shieh's *d*. Based on the

221   sampling distribution of Shieh's *d*, we can estimate its theoretical expectation and variance

222   under the assumption of normality (see Table 2), and thereby determine which factors

223   influence bias and variance, and how they do so (see Supplemental Material 1).

224   As demonstrated in Appendices 1 and 2, when population variances and sample sizes

225   are equal across groups, the biases and variances of Cohen's *d* and Shieh's *d* are identical

226   except for multiplication by a constant. The same is true for the population effect size

227   expressions δ_{Cohen} and δ_{Shieh} :

$$\delta_{Cohen} = 2 \times \delta_{Shieh} \quad (\text{considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \tag{12}$$

228

$$Bias_{Cohen's\ d} = 2 \times Bias_{Shieh's\ d} \quad (\text{considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \tag{13}$$

229

$$Var_{Cohen's\ d} = 4 \times Var_{Shieh's\ d} \quad (\text{considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \tag{14}$$

230 We can deduce from equations 12, 13 and 14 that relative to their respective population

231 effect size, Cohen's d and Shieh's d are equally accurate. In other words, their relative bias

232 and variance are identical.

233   When sample sizes are not equal, according to the statistical properties of Welch's

234 statistic under heteroscedasticity, Shieh's d accounts for the allocation ratio of sample sizes

235 to each condition. The lack of generality caused by taking this specificity of the design into

236 account has led Cumming (2013) to question its usefulness in terms of interpretability :

237 when the mean difference $(\bar{X}_1 - \bar{X}_2)$, S_1, and S_2 remain constant, Shieh's d will vary as a

238 function of the sample sizes allocation ratio (unlike Cohen's d^* that we will define below).

239 At the population level, δ_{Shieh} also depends on the sample sizes allocation ratio, as

240 illustrated in the following shiny application :

241 https://effectsize.shinyapps.io/ShiehvsCohen/.

Cohen's d^*.   An effect size estimator based on the sample mean difference divided

by the square root of the non pooled average of both variance estimates was suggested by

Welch (1938). Here, we indicate the difference between Cohen's d (based on the pooled

standard deviations) and Cohen's d^* with an asterisk. This yields :

$$Cohen's\ d^* = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{(S_1^2 + S_2^2)}{2}}}$$

242 where $\bar{X}_j$ is the mean and S_j is the standard deviation of the jth sample (j = 1,2). We

243 know the distribution of Cohen's d^* (Huynh, 1989) :

$$Cohen's\ d^* \sim \sqrt{\frac{2(n_2 \times \sigma_1^2 + n_1 \times \sigma_2^2)}{n_1 n_2 (\sigma_1^2 + \sigma_2^2)}} \times t_{df^*, ncp^*} \tag{15}$$

²⁴⁴ where df^* and ncp^* are defined as follows :

$$df^* = \frac{(n_1 - 1)(n_2 - 1)(\sigma_1^2 + \sigma_2^2)^2}{(n_2 - 1)\sigma_1^4 + (n_1 - 1)\sigma_2^4} \tag{16}$$

$$ncp^* = \delta_{Cohen}^* \times \sqrt{\frac{n_1 n_2(\sigma_1^2 + \sigma_2^2)}{2(n_2\sigma_1^2 + n_1\sigma_2^2)}} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

²⁴⁵ where $\delta_{Cohen}^* = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}}$ is the corresponding population effect size. Using equation 15 we

²⁴⁶ can compute its theoretical expectation and variance when the assumption of normality is

²⁴⁷ met (see Table 2), and therefore determine which factors influence bias and variance, and

²⁴⁸ how they do so (see Supplemental Material 1). This estimator has been widely criticized,

²⁴⁹ because it results in a variance term of an artificial population (i.e. since the variance term

²⁵⁰ does not estimate the variance of one or the other group, the composite variance is an

²⁵¹ estimation of the variance of an artificial population; Grissom & Kim, 2001). However, the

²⁵² same criticism would then apply to the celebrated Cohen's d because the pooled variance

²⁵³ also corresponds to the variance of an artificial population, unless when variances are equal.

²⁵⁴ But for equal variances the variance term of Cohen's d^* also yields σ^2, which is perfectly

²⁵⁵ interpretable. Moreover, we will show throughout the simulation section that this estimator

²⁵⁶ exhibits very good inferential properties. Moreover, it has a constant value across sample

²⁵⁷ sizes ratios, as shown in the Shiny App at https://effectsize.shinyapps.io/ShiehvsCohen/.

Glass' g, Shieh's g and Hedges' g^*.   As for Cohen's d, a Hedges' correction can

be applied in order to compensate for the bias of Glass' d, Shieh's d and Cohen's d^* with

small sample sizes (see Table 2). This correction has the following general form :

$$g = d \times \frac{\Gamma(\frac{\nu}{2})}{\sqrt{\frac{\nu}{2}} \times \Gamma(\frac{\nu-1}{2})}$$

²⁵⁸ where the distinct values of ν are provided in equation 9 for Glass' g, in equation 11 for

²⁵⁹ Shieh's g and in equation 16 for Hedges' g^*. The three corrected estimators are

²⁶⁰ theoretically unbiased when the assumption of normality is met. Their variance is a

²⁶¹ function of the same parameters as their biased equivalent. However, due to the correction

they have a smaller variance, especially with small sample sizes, as shown in Table 3. In summary :

- The variances of Hedges' g^* and Shieh's g depend on the total sample size (N), their respective population effect size (δ), and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$.

- The variance of Glass' g also depends on N, δ and $\frac{n_c}{n_e} \times \frac{\sigma_c}{\sigma_e}$. In addition, there is also a main effect of the SD-ratio $\left(\frac{\sigma_c}{\sigma_e}\right)$ on its variance.

How these parameters influence the variance of the estimators will be summarized and illustrated in Monte Carlo simulations below.

Table 3

Expectation, bias and variance of Glass' g, Hedges' g* and Shieh's g under the assumption that independent residuals are normally distributed.

	df	Expectation	Variance
Glass' g	$n_c - 1$	δ_{glass}	$Var(Glass'\ d) \times \left(\dfrac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}\right)^2$
Hedges' g*	$\dfrac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$	δ^*_{Cohen}	$Var(Cohen's\ d^*_s) \times \left(\dfrac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}\right)^2$
Shieh's g	$\approx \dfrac{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1}+\frac{(\sigma_2^2/n_2)^2}{n_2-1}}$	δ_{Shieh}	$Var(Shieh's\ d) \times \left(\dfrac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}\right)^2$

Note. $c_f = \dfrac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}$; $\delta_{Glass} = \frac{\mu_c-\mu_e}{\sigma_c}$, $\delta_{Shieh} = \dfrac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N}+\frac{\sigma_2^2}{n_2/N}}}$ and $\delta^*_{Cohen} = \dfrac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}}$; all estimators are unbiased estimators, because their expectations equal the population effect size δ; equations in this table require $df \geq 3$ and at least 2 subjects per group.

274 **Monte Carlo Simulations**

275   **Assessing the bias, efficiency and consistency of 5 estimators.**

276   *Method.*

277   We performed Monte Carlo simulations using R (version 3.5.0) to assess the bias,

278 efficiency and consistency of Hedges' g, Glass' g (using respectively the sample SD of the

279 first or second group as a standardizer), Hedges' g^* and Shieh's g.

280   A set of 100,000 datasets was generated for 1,008 scenarios as a function of different

281 criteria. In 252 scenarios, samples were extracted from a normally distributed population

282 (in order to ensure the reliability of our calculation method) and in 756 scenarios, samples

283 were extracted from non normal population distributions. In order to assess the quality of

284 estimators under realistic deviations from the normality assumption, we referred to the

285 review of Cain et al. (2017). Cain et al. (2017) investigated 1,567 univariate distributions

286 from 194 studies published by authors in Psychological Science (from January 2013 to June

287 2014) and the American Education Research Journal (from January 2010 to June 2014).

288 For each distribution, they computed Fisher's skewness

$$G_1 = \frac{\sqrt{n(n-1)}}{n-2} \frac{m_3}{\sqrt{(m_2)^3}}$$

289 and kurtosis

$$G_2 = \frac{n-1}{(n-2)(n-3)} \times \left[(n+1)\left(\frac{m_4}{(m_2)^2} - 3 \right) + 6 \right]$$

290 where n is the sample size and m_2, m_3 and m_4 are respectively the second, third and fourth

291 centered moments. They found values of kurtosis from $G_2 = $ -2.20 to 1,093.48. According

292 to their suggestions, throughout our simulations, we kept constant the population kurtosis

293 value at the 99th percentile of their distribution of kurtosis, i.e. $G_2 = 95.75$. Regarding

294 skewness, we simulated population parameter values which correspond to the 1st and 99th

295 percentile of their distribution of skewness, i.e. respectively $G_1 = $ -2.08 and $G_1 = 6.32$. We

296 also simulated samples extracted from population where $G_1 = 0$, in order to assess the

297 main effect of high kurtosis on the quality of estimators. All possible combinations of

298 population skewness and kurtosis and the number of scenarios for each combination are

299 summarized in Table 4. To provide the reader with an intuition about the four

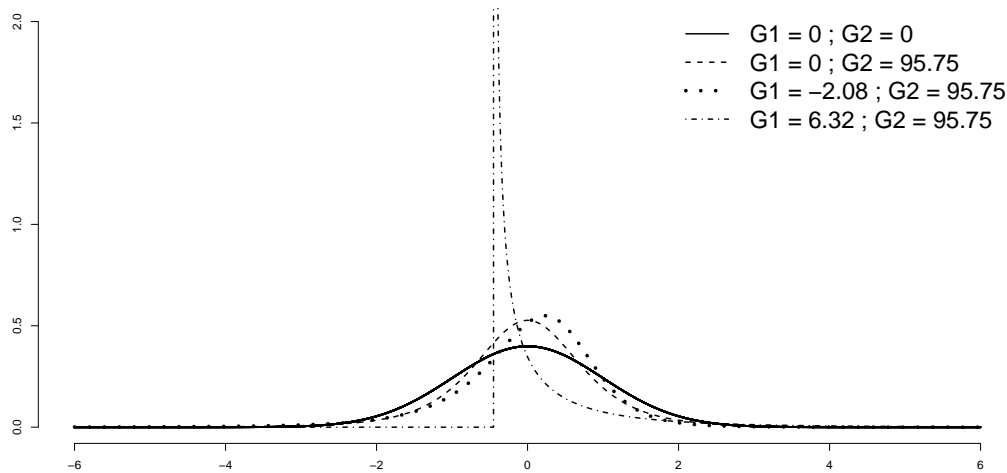300 distributions we consider, we have plotted them in Figure 2.

Figure 2. Plots of the four distributions considered in our simulations, with $\mu = 0$ and $\sigma = 1$. The undotted curve is the normal distribution.

Table 4

Number of combinations of skewness and kurtosis in our simulations.

		Kurtosis		
		0	95.75	**TOTAL**
	0	252	252	**504**
Skewness	-2.08	/	252	**252**
	6.32	/	252	**252**

	Kurtosis		
TOTAL	**252**	**756**	**1008**

301 *Note.* Fisher's skewness (G1) and kurtosis (G2) are presented in Table 4. The 252

302 combinations where both G1 and G2 equal 0 correspond to the normal case.

303 For the 4 resulting combinations of skewness and kurtosis (see Table 4), all other

304 parameter values were chosen in order to illustrate the consequences of factors identified as

305 playing a key role on the variance of unbiased estimators. We manipulated the population

306 mean difference ($\mu_1 - \mu_2$), the sample sizes (n_1 and n_2), the sample size ratio (n-ratio $=$

307 $\frac{n_2}{n_1}$), the population SD-ratio (i.e. $\frac{\sigma_2}{\sigma_1}$), and the sample size and population variance pairing

308 $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$. In our scenarios, μ_2 was always 0 and μ_1 varied from 1 to 4, in steps of 1 (so

309 does $\mu_1 - \mu_2$). Thus, we only consider a positive population effect size, which will result in

310 positive bias of our estimators for reasons explained in Supplemental Material 1. [3]

311 Moreover, σ_1 always equals 1, and σ_2 equals .1, .25, .5, 1, 2, 4 or 10, and therefore, the

312 SD-ratio were 10, 4, 2, 1, .5, .25 or .1. The simulations for which both σ_1 and σ_2 equal 1

313 are the particular case of homoscedasticity, or equal population variances across groups.

314 The sample sizes of both groups (n_1 and n_2) were 20, 50 or 100. When sample sizes of both

315 groups are equal, the n-ratio equals 1 (this is known as a balanced design). All possible

316 combinations of n-ratio and population SD-ratio were simulated in order to distinguish

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[3] In the original plan, we had added 252 simulations in which μ_1 and μ_2 were both null. We decided not to

present the results of these simulations in the main article, because the relative bias and the relative

variance appeared to us to be very useful to fully understand the comparison of the estimators, and

computing them is impossible when the real mean difference is zero. Indeed, for these specific

configurations, both relative bias and relative variance would have infinite values due to the presence of the

population effect size term in their denominator. However, these extra simulations were included in the

simulation checks, in Supplemental Material 2.

scenarios where both sample sizes and population variances are unequal across groups

(with positive pairing when the group with the largest sample size is extracted from the

population with the largest SD, and negative pairing when the group with the smallest

sample size is extracted from the population with the smallest SD) and scenarios with no

pairing between sample sizes and variances (sample sizes and/or population SD are equal

across all groups). In sum, the simulations grouped over different sample sizes yield 4

conditions (a, b, c and d) based on the n-ratio, population SD-ratio, and sample size and

population variance pairing, as summarized in Table 5. We chose to divide scenarios into

these 4 conditions because analyses in Supplemental Material 1 revealed main and

interaction effects of sample sizes ratio and SD-ratio on the bias and variance of some

estimators.

Table 5

4 conditions based on the n-ratio and the SD-ratio.

		n-ratio		
		1	**>1**	**<1**
	1	a	b	b
SD-ratio	**>1**	c	d	d
	<1	c	d	d

Results.

Before presenting the comparison of the estimators for each condition, it is useful to make some general comments.

1) We previously explained why we need to consider here relative bias and relative variance instead of raw bias and variance. Thus, anytime we will mention the biases and variances in the results section, we will be referring to relative bias and variance.

2) For the sake of readability, the vertical axis differs across plots.

3) Throughout this section, we will *compare* the relative bias and variance of different estimators. We chose very extreme (although realistic) conditions, and we know that none of the parametric measures of effect size will be robust against such extreme conditions. Our goal is therefore to study the robustness of the estimators against normality violations only in comparison with the robustness of other indicators, but not in absolute terms.

341   4) Because Cohen's δ is not an appropriate expression of the effect size when population

342   variances and sample sizes are unequal across groups, we will not plot the bias and

343   variance of Hedges' g under these specific conditions.

344   After these general remarks, we will analyze each condition separately. In all Figures

345 presented below, for different sub-conditions, the averaged relative bias and relative

346 variance of different estimators are presented. When describing the Glass' g estimators, we

347 will systematically refer to the "control group" as the condition the standardizer is based

348 on (i.e. the first group when using S_1 as standardizer, the second group when using S_2 as

349 standardizer). The other condition will be referred to as the "experimental group."

350   *When variances are equal across groups*

351   Figures 3 and 4 represent configurations where the equality of variances assumption is

352 met. According to our expectations, one observes that the bias of all estimators is

353 approximately zero as long as the normality assumption is met (first column in both

354 Figures).[4] However, the more the data generation process deviations from the normality

355 assumption (i.e. when moving from left to right in the Figures), the larger the bias in the

356 estimators.

357   We will observe that Glass' g should always be avoided when the equality of variance

358 assumption is met. Hedges' g, Hedges' g^* and Shieh's g perform equally well as long as the

359 sample size ratio is close to 1 (condition a; see Figure 3). However, when designs are highly

360 unbalanced (condition b; see Figure 4), Shieh's g is not consistent anymore and while

361 Hedges' g^* remains consistent, Hedges's g is a better estimator. For interested readers,

362 these findings are detailed in the three paragraphs below.

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[4] When looking at relative bias for all estimators, the maximum departure from zero is 0.0064 when sample

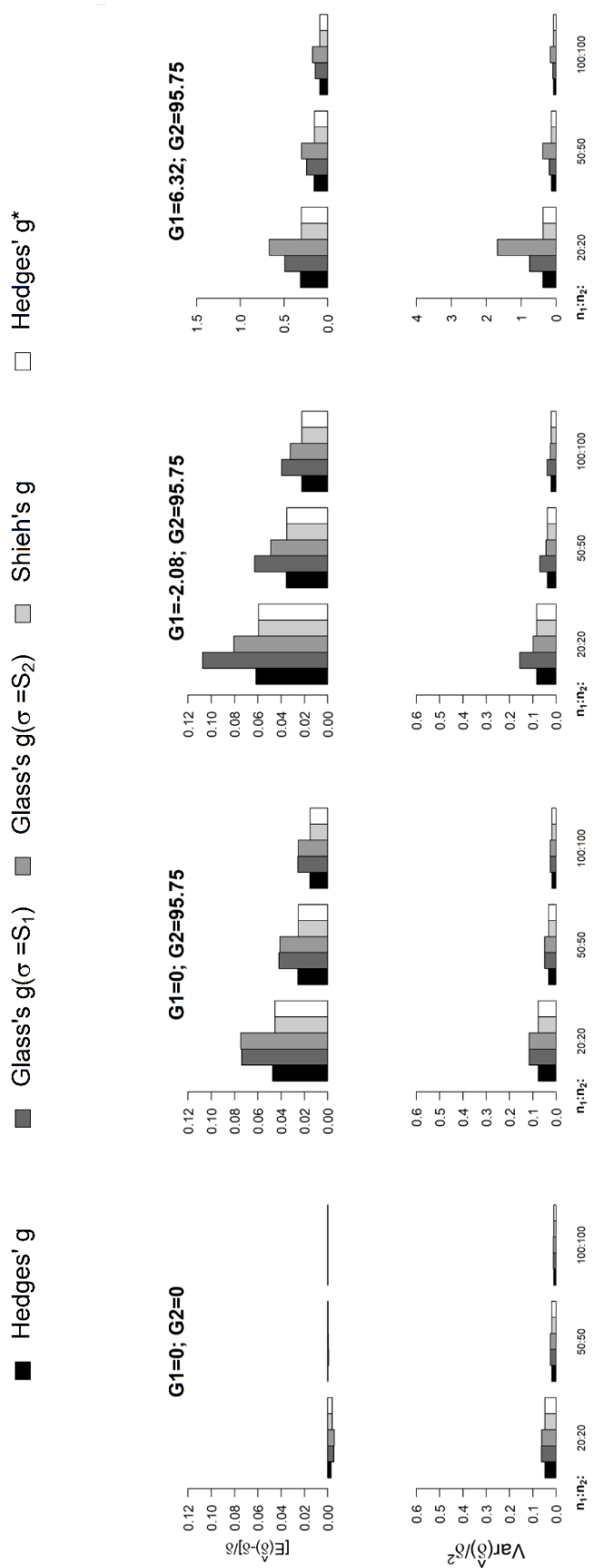sizes are equal across groups, and 0.0065 with unequal sample sizes.

Figure 3. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are equal across groups (condition a)
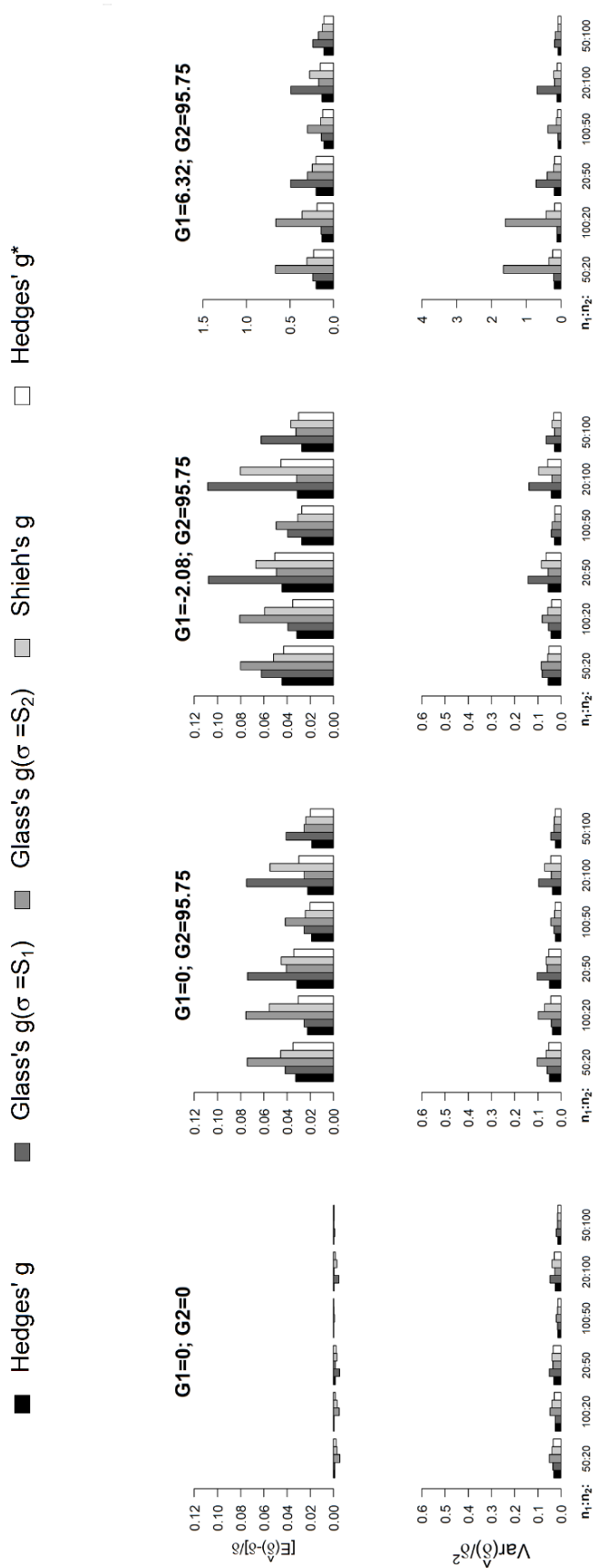
Figure 4. Bias and efficiency of estimators of standardized mean difference, when variances are equal across groups and sample sizes are unequal (condition b)

Figure 3 illustrates scenarios where both population variances and sample sizes are equal across groups (condition a). One can first notice that all estimators are consistent, as their bias and variance decrease when the total sample size increases. For any departure from the normality assumption, both bias and variance of Hedges' g, Shieh's g and Hedges' g^* are similar[5] and smaller than the bias and variance of Glass' g estimates using either S_1 or S_2 as a standardizer. Moreover, when samples are extracted from skewed distributions, Glass' g will show different bias and variance as a function of the chosen standardizer (S_1 or S_2), even if both S_1 and S_2 are estimates of the same population variance, based on the same sample size. This is due to non-null correlations of opposite sign between the mean difference ($\bar{X}_1 - \bar{X}_2$) and respectively S_1 and S_2. In Supplemental Material 3, we detailed in which situation a non-null correlation occurs between the sample mean difference ($\bar{X}_1 - \bar{X}_2$) and the standardizer of compared estimators as well as the way this correlation impacts the bias and variance of estimators.

Figure 4 illustrates scenarios where population variances are equal across groups, but sample sizes are unequal (condition b). For any departures from the normality assumptions, Hedges' g shows the smallest bias and variance. Hedges' g and Hedges' g^* are consistent estimators (i.e. the larger the sample sizes, the lower the bias and the variance), unlike Shieh's g and Glass' g. The bias of Glass' g does not depend either on the size of the experimental group or on the total sample size. The only way to decrease the bias of Glass' g is therefore to add subjects in the control group. On the other hand, the variance of Glass' g depends on both sample sizes, but not in an equivalent way : in order to reduce

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[5] While the bias and variance of Cohen's d, Cohen's d^* and Shieh's d are identical, the bias and variance of Hedges' g are marginally different from the bias and variance of Hedges' g^* and Shieh's g (these last two having identical bias and variance). Indeed, because of the sampling error, differences remain between sample variances, even when population variances are equal between groups. Since the Hedges' correction applied to Cohen's d does not contain the sample variances (unlike the correction applied on both other estimators), the bias and variance of Hedges' g are slighly different from the bias and variance of Hedges' g^* and Shieh's g.

384 the variance, it is much more efficient to add subjects in the control group and when the

385 relative size of the experimental group decreases so does the variance, even when the total

386 sample size is decreased. Regarding Shieh's g, for a given sample size ratio, the bias and

387 variance will decrease when sample sizes increase. However, there is a large effect of the

388 sample sizes ratio such that when the sample sizes ratio moves away from 1 by adding

389 subjects, bias and variance might increase.[6] On the other hand, when the sample sizes

390 ratio moves closer to 1 by adding subjects, the bias will decrease.

391   When samples are extracted from skewed distributions and have unequal sizes (the

392 two last columns in Figure 4), for a constant total sample size, Glass' g, Shieh's g and

393 Hedges' g^* will show different bias and variance depending on which group is the largest

394 one (e.g. when distributions are right-skewed, the bias and variance of all these estimators

395 when n_1 and n_2 are respectively 50 and 20 are not the same as their bias and variance when

396 n_1 and n_2 are respectively 20 and 50). This is due to a non-null correlation of opposite sign

397 between the mean difference $(\bar{X}_1 - \bar{X}_2)$ and their respective standardizers depending on

398 which group is the largest one, as detailed in Supplemental Material 3. One observes that

399 under these configurations, the bias and variance of Glass' g are sometimes a bit smaller

400 and sometimes much larger than the bias and variance of Shieh's g and Cohen's d^*.[7]

———

[6] Regarding variance, in Supplemental Material 1, we mentioned that when the population effect size is zero, the larger the total sample size, the lower the variance, whether the sample sizes ratio is constant or not. We also mentioned that this is no longer true when the population effect size is not zero. In our simulations the effect size is never zero. The effect size effect is partially visible in Figure 4 because we do not entirely remove the effect size effect when we divide the variance by δ^2. This is due to the fact that one term, in the equation of the variance computation, does not depend on the effect size.

[7] Supplemental Material 3 shows that when $\mu_1 - \mu_2 > 0$ (like in our simulations), all other parameters being equal, an estimator is always less biased and variable when choosing a standardizer that is positively correlated with $\bar{X}_1 - \bar{X}_2$. Supplemental Material 3 also shows that the smaller n_c, the larger the magnitude of correlation between S_c and $\bar{X}_1 - \bar{X}_2$. When $cor(S_c, \bar{X}_1 - \bar{X}_2)$ is positive, the positive effect of increasing the magnitude of the correlation is counterbalanced by the negative effect of reducing n_c. On the other

When variances are unequal across groups

Figures 5 to 10 represent configurations where the equality of variances assumption is

not met. According to our expectations, one observes that the bias of all estimators is

approximately zero as long as the normality assumption is met (first column in all

Figures), and the further from the normality assumption (i.e. when moving from left to

right in Figures), the larger the bias.[8]

We observe that when variances are unequal across populations, Glass' g sometimes

performs better, but also sometimes performs much worse than Shieh's g and Hedges' g^*,

both in terms of bias and variance. The performance of Glass' g highly depends on

parameters that we cannot control (i.e. a triple interaction between the n-ratio, the

SD-ratio and the correlation between the standardizer and the mean difference) and for

this reason, we do not recommend using it. When the sample sizes ratio is close to 1,

Shieh's g and Hedges' g^* are both appropriate but the further the sample sizes ratio is from

1, the larger the bias of Shieh's g such that, in the end, the measure that we believe

performs best across scenarios is Hedges' g^*.

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hand, when $cor(S_c, \bar{X}_1 - \bar{X}_2)$ is negative, the negative effect of increasing the magnitude of the correlation is amplified by the negative effect of decreasing n_c. This explains why the difference between Glass' g and other estimators is larger when Glass' g is the least efficient estimator.

[8] When looking at the relative bias for all estimators, the maximum departure from zero is 0.0173 when sample sizes are equal across groups, and 0.0274 when both sample sizes and variances differ across groups.
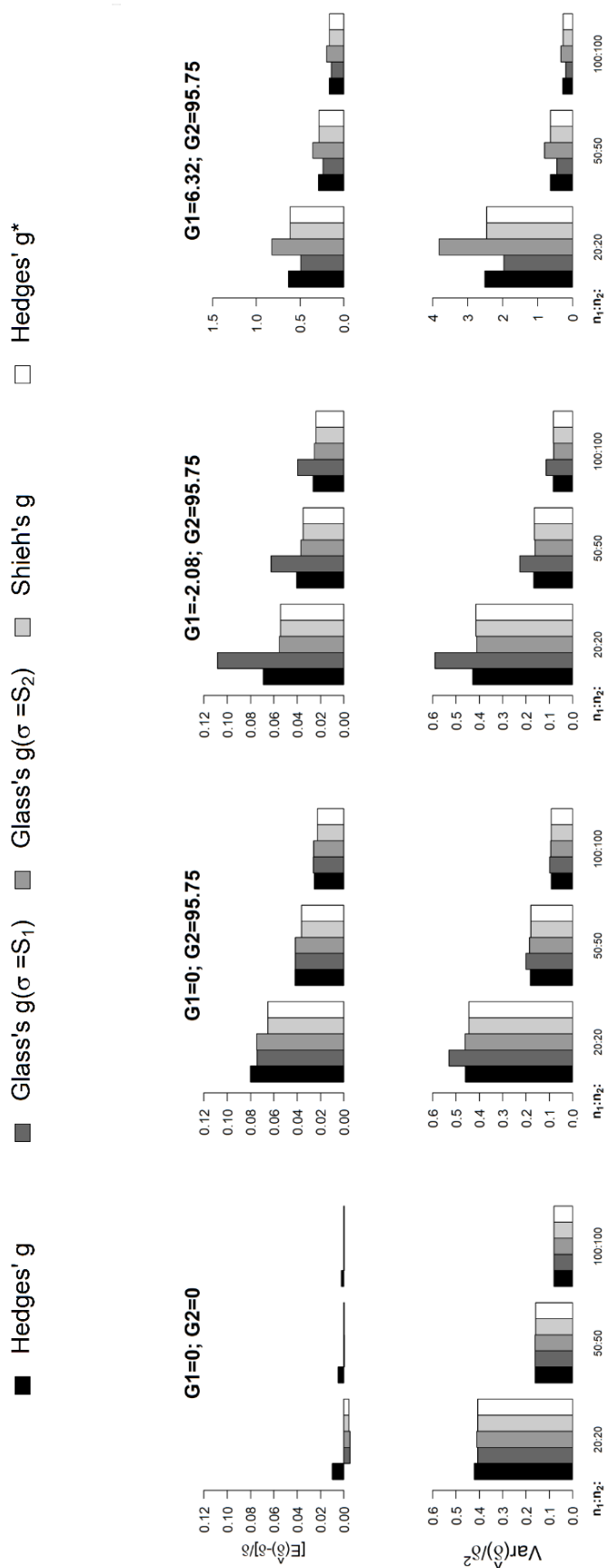
Figure 5. Bias and efficiency of estimators of standardized mean difference, when variances are unequal across groups and sample sizes are equal (condition c), as a function of sample sizes
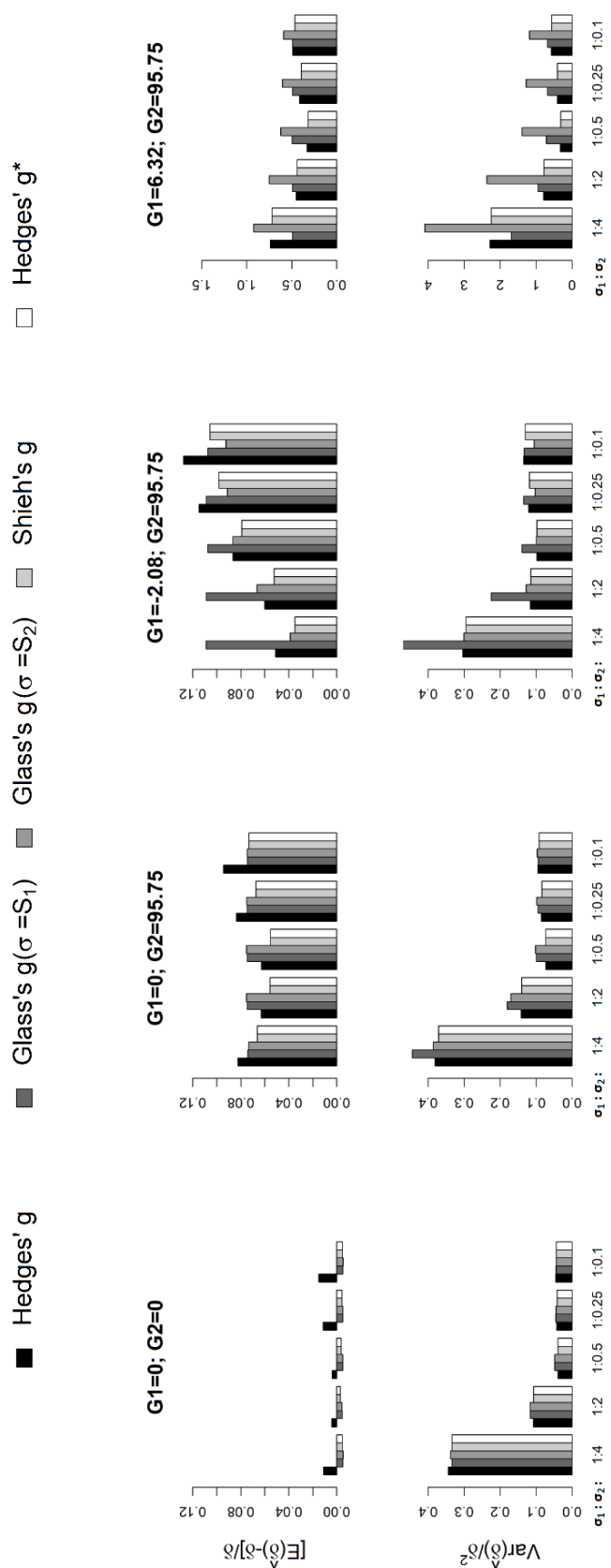
Figure 6. Bias and efficiency of estimators of standardized mean difference, when variances are unequal across groups and sample sizes are equal (condition c) as a function of the SD-ratio (when $n_1 = n_2 = 20$)

⁴¹⁶   Figures 5 and 6 are dedicated to scenarios where population variances are unequal

⁴¹⁷ between groups and sample sizes are equal (condition c). In Figure 5, scenarios are

⁴¹⁸ subdivided as a function of the sample sizes and one can notice that all estimators are

⁴¹⁹ consistent, as their bias and variance decrease when the total sample size increases. In

⁴²⁰ Figure 6, scenarios are subdivided as a function of the SD-ratio. Because the comparison

⁴²¹ pattern remains very similar for all sample sizes, we present only scenarios when sample

⁴²² sizes equal 20. One should first notice that for all estimators in Figure 6, the relative

⁴²³ variance seems to be much larger when $S_2 > S_1$.[9] This information should not be taken

⁴²⁴ into account because it is only an artefact of our simulation conditions combined with the

⁴²⁵ way we computed the relative variance.[10]

⁴²⁶   When samples are extracted from skewed distributions, the bias and variance of

⁴²⁷ Glass' g are sometimes smaller and sometimes larger than the bias of Shieh's g and Hedges'

⁴²⁸ g^*. This is mainly due to the fact that when two samples of same sizes are extracted from

⁴²⁹ two skewed distributions with unequal variances (the two last columns in Figure 6), there

⁴³⁰ will be non-null correlations of opposite sign between the mean difference $(\bar{X}_1 - \bar{X}_2)$ and

⁴³¹ the standardizer of *all* estimators, depending on which population variance is larger.[11]

⁴³²   Figures 7 to 10 are dedicated to scenarios where both sample sizes and population

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[9] The difference between the variance of estimators when the second group is 10 times larger than the first group was so large that we decided to not present it, for the sake of readability of the Figures.

[10] We previously mentioned that when dividing the variance by δ^2, we do not entirely remove the effect size effect. Actually, we introduce δ^2 in the denominator of the first term, in the equation of the variance computation. Because we performed our simulations in order that σ_1 always equals 1, the smaller S_2, the larger the population effect size and therefore, the smaller the relative variance.

[11] When population variances are unequal, a non-null correlation occurs between standardizer estimates and $\bar{X}_1 - \bar{X}_2$. For standardizers computed based on both S_1 and S_2, the sign of the correlation between the standardizer and the mean difference will be the same as the sign of the correlation between the mean difference and the estimate of the larger population variance. For interested readers, this is detailed in Supplemental Material 3.

433   variances differ across groups. Due to a high number of combinations between the sample

434   sizes ratio and the SD-ratio in our simulations, we decided to present only some

435   conditions. Because equations in Table 3 revealed an interaction effect between the sample

436   sizes ratio and the SD-ratio on the bias and variance of Hedges' g^* and Shieh's g (see

437   Supplemental Material 1), we chose to present all configurations where the larger SD is 10

438   times larger than the smaller SD (Figures 7 and 8), and configurations where the larger

439   SD is twice larger than the smaller SD (Figures 9 and 10), in order to compare the effect

440   of the sample sizes ratio on the bias and variance of all estimators when the SD-ratio is

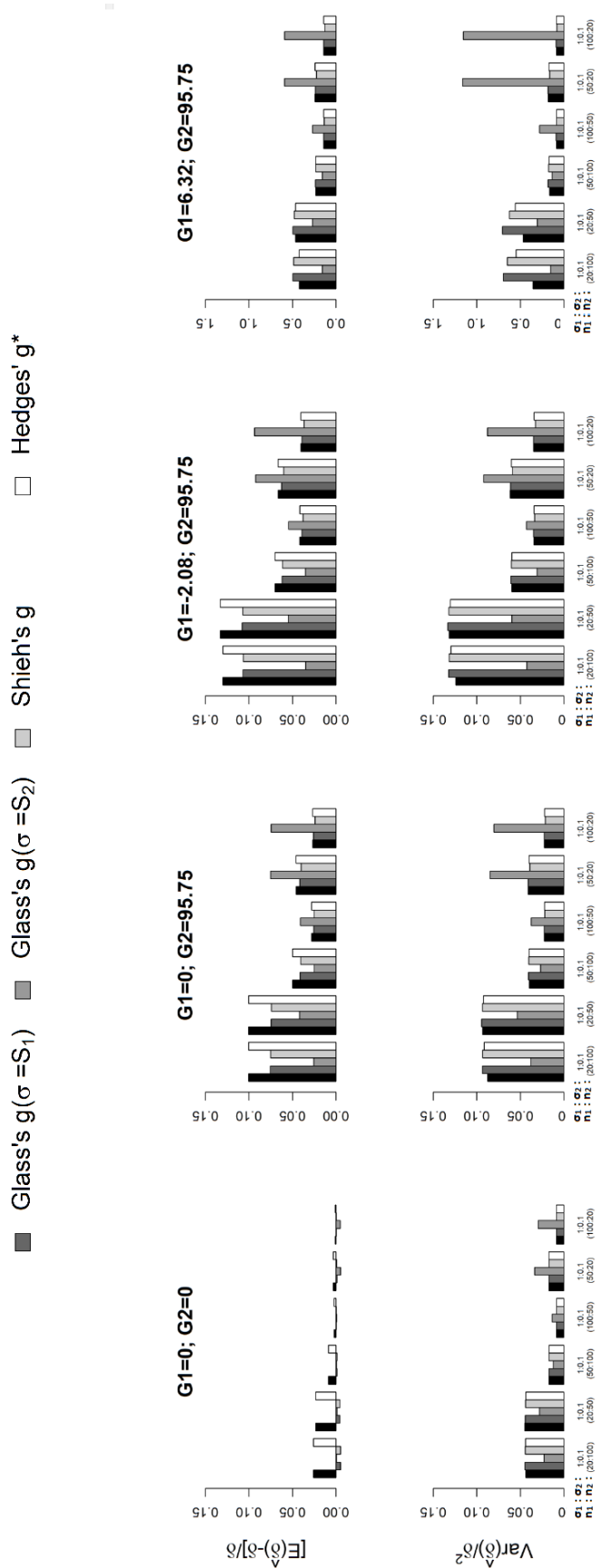441   large ($\frac{\sigma_2}{\sigma_1} = 10$ or .1) or medium ($\frac{\sigma_2}{\sigma_1} = 2$ or .5).

Figure 7. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_1 is 10 times larger than σ_2
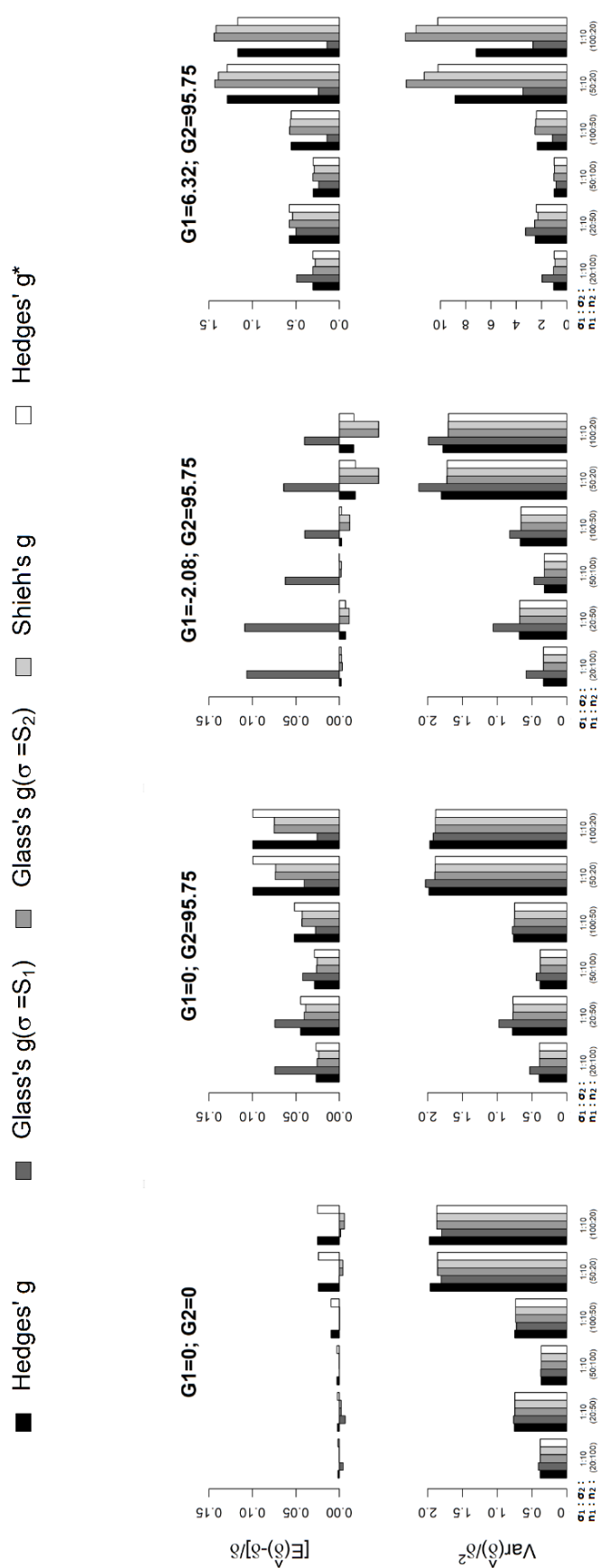
Figure 8. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_2 is 10 times larger than σ_1
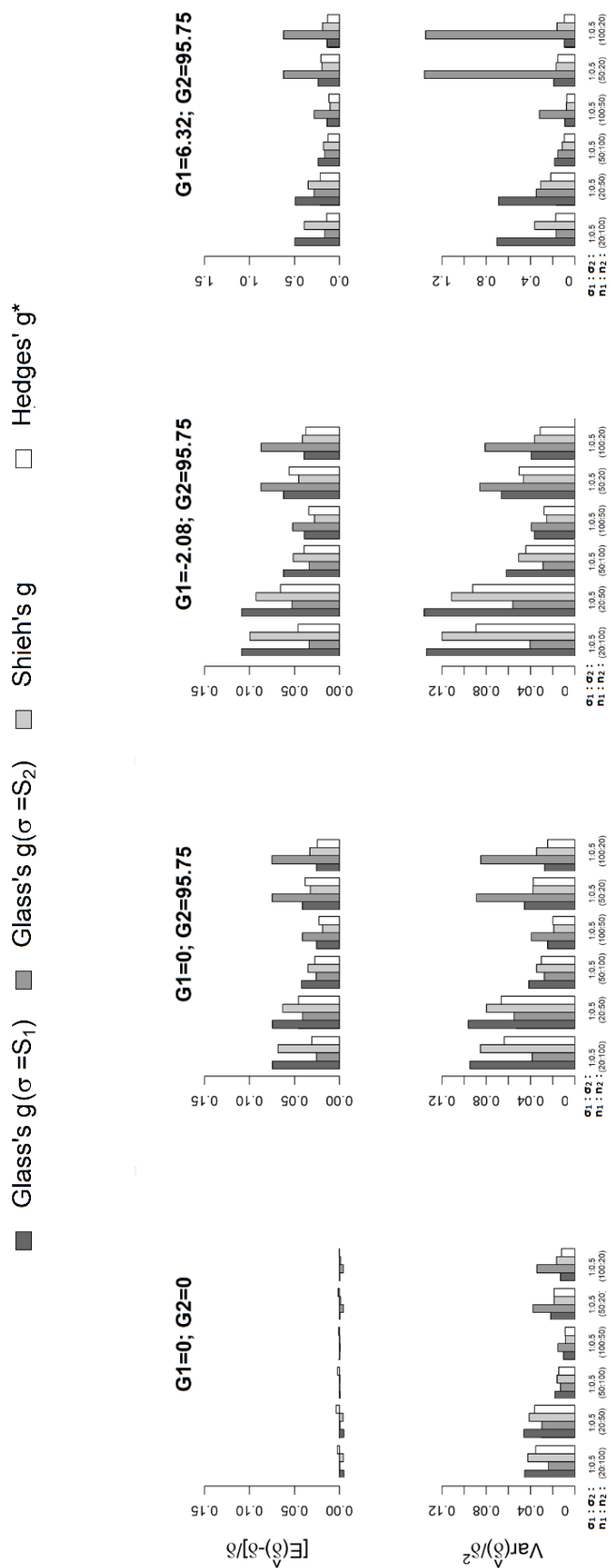
Figure 9. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_1 is twice larger than σ_2
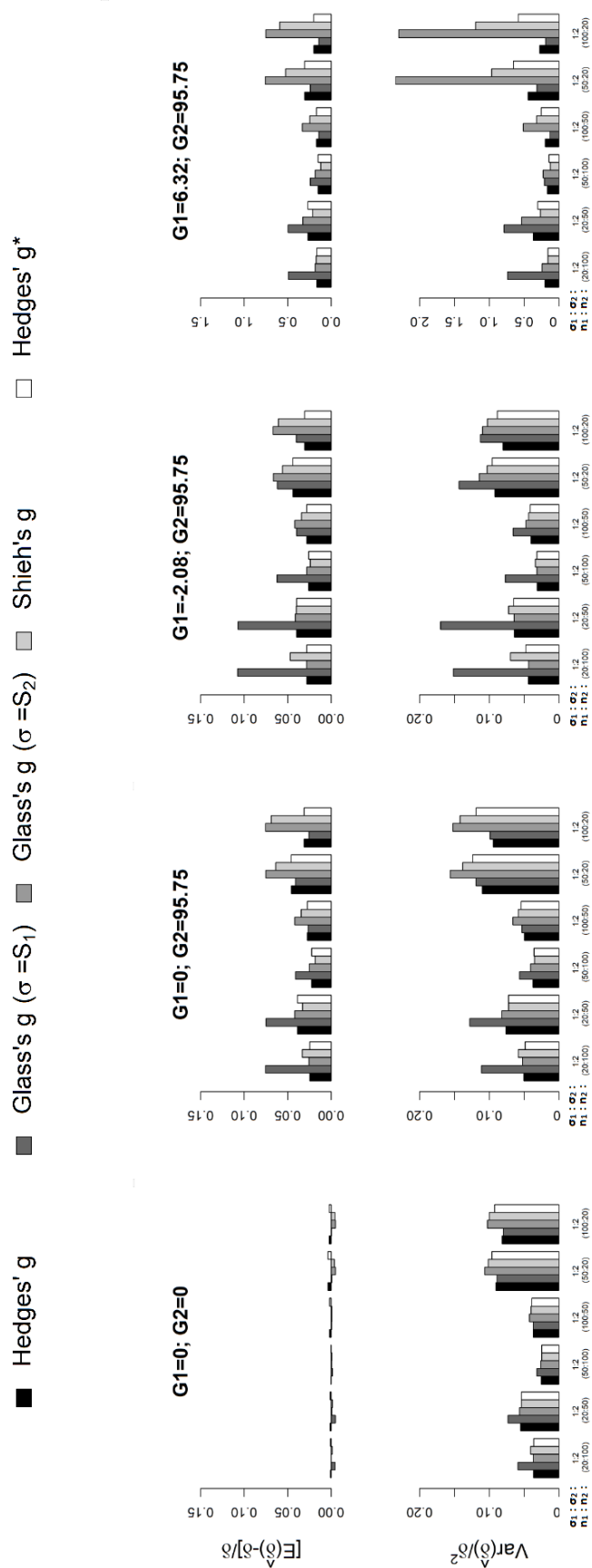
Figure 10. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_2 is twice larger than σ_1

When distributions are symmetric, the bias of Glass' g only depends on the size of
the control group and is therefore not impacted by either the sample sizes ratio or the total
sample size. When comparing Figures 7 to 10, one can also notice that the bias of Glass' g
does not depend on the SD-ratio either. Unlike the bias of Glass' g, its variance depends
on both sample sizes, but not in an equivalent way. In most scenarios it is more efficient, in
order to reduce the variance of Glass' g, to add subjects in the control group. Regarding
Hedges' g^* and Shieh's g, their respective biases and variances depend on an interaction
effect between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$: the sample sizes ratio
associated with the smallest bias and variance is not the same when the more variable
group is 10 times more variable than the other group (Figures 7 and 8) than when it is
only twice more variable (Figures 9 and 10). However, the respective biases and variances
of Hedges' g^* and Shieh's g are always smaller when there is a positive pairing between
sample sizes and variances. When samples are extracted from skewed distributions, the
bias and variance of Glass' g are sometimes smaller and sometimes larger than the bias of
Shieh's g and Hedges' g^*, due to a combination of three factors : (1) which group is larger,
(2) which group has the smallest standard deviation and (3) what is the correlation
between the standardizer and the mean difference.

Recommendations

We recommend using Hedges' g^* in order to assess the magnitude of the effect when
comparing two independent means, because a) it does not rely on the equality of
population variances assumption \footnote{The assumption of equal variances across
populations is rarely realistic in practice (Cain et al., 2017; Delacre et al., 2017; Delacre et
al. , 2019; Erceg-Hurn & Mirosevich, 2008; Glass et al., 1972; Grissom, 2000; Micceri, 1989;
Yuan et al., 2004) (unlike Hedges' g), b) it is always consistent (unlike Shieh's g), c) it is
easy to interpret (Hedges' g^* can be interpreted in the same way as Hedges' g) and d) it
remains constant for any sample sizes ratio, even when population variances are unequal

across groups, as shown in the Shiny App at https://effectsize.shinyapps.io/ShiehvsCohen/.

Effect size estimates such as Hedges' g^* should always be reported with a confidence interval. To help researchers compute Hedges' g^* and its confidence interval we created the R package *deffectsize* (see https://github.com/mdelacre/deffectsize). The *datacohen_CI* function was built in order to compute point estimators and confidence intervals based on raw data and the *cohen_CI* function was built in order to compute point estimators and confidence intervals based on descriptive statistics (sample means, sample variances and sample sizes). By default, unbiased Hedges' g^* is computed but it is also possible to compute biased estimators (e.g. Cohen's d^*) and/or to use a pooled error term as standardizer by assuming that the equality of population variances is met (e.g. Hedges' g or Cohen's d, depending on whether we choose to compute unbiased or biased estimator). Other functions (*datashieh_CI*, *shieh_CI*, *dataglass_CI* and *glass_CI*) are available in order to compute Shieh's g (or Shieh's d) and Glass' g (or Glass' d) as well as their respective confidence intervals, even though we do not recommend to use these effect sizes by default. Researchers who do not use R can use a Shiny app to compute point estimators and confidence intervals based on descriptive statistics :

https://effectsize.shinyapps.io/deffsize/.

References

Algina, J., Keselman, H. J., & Penfield, R. D. (2006). Confidence intervals for an effect size when variances are not equal. *Journal of Modern Applied Statistical Methods*, *5*(1), 2–13. https://doi.org/10.22237/jmasm/1146456060

Altman, D. G. (2005). Why we need confidence intervals. *World Journal of Surgery*, *29*(5), 554–556. https://doi.org/10.1007/s00268-005-7911-0

American Psychological Association. (2010). *Publication manual of the american psychological association [APA]* (6th ed.). Washington, DC: American Psychological Association.

Andersen, M. B., McCullagh, P., & Wilson, G. J. (2007). But what do the numbers really tell us?: Arbitrary metrics and effect size reporting in sport psychology research. *Journal of Sport and Exercise Psychology, 29*(5), 664–672. https://doi.org/10.1123/jsep.29.5.664

Cain, M. K., Zhang, Z., & Yuan, K.-H. (2017). Univariate and multivariate skewness and kurtosis for measuring nonnormality: Prevalence, influence and estimation. *Behavior Research Methods, 49*(5), 1716–1735. https://doi.org/10.3758/s13428-016-0814-1

Coe, R. (2002). It's the effect size, stupid: What effect size is and why it is important. Paper presented at Annual Conference of the British Educational Research Association, University of Exeter, Exeter, United-Kingdom.

Cohen, J. (1965). Some statistical issues in psychological research. In B. B. Wolmann (Ed.), *Handbook of clinical psychology* (pp. 95–121). New York, NY: McGraw-Hill.

Cumming, G. (2013). Cohen's *d* needs to be readily interpretable: Comment on shieh (2013). *Behavior Research Methods, 45*(4), 968–971. https://doi.org/10.3758/s13428-013-0392-4

Delacre, M., Lakens, D., & Leys, C. (2017). Why psychologists should by default use welch's *t*-test instead of student's *t*-test. *International Review of Social Psychology, 30*(1), 92–101. https://doi.org/10.5334/irsp.82

Duran, R. P., Eisenhart, M. A., Erickson, F. D., Grant, C. A., Green, J. L., Hedges, L. V., & Schneider, B. L. (2006). Standards for reporting on empirical social science research in AERA publications: American Educational Research Association. *Educational Researcher, 35*(6), 33–40.

Ellis, P. D. (2010). *The essential guide to effect sizes: Statistical power, meta-analysis, and the interpretation of research results.* Cambridge, United-Kingdom: Cambridge University Press.

Erceg-Hurn, D. M., & Mirosevich, V. M. (2008). Modern robust statistical methods: An easy way to maximize the accuracy and power of your research. *American Psychologist*, *63*(7), 591–601. https://doi.org/10.1037/0003-066X.63.7.591

Glass, G. V., McGaw, B., & Smith, M. L. (1981). *Meta-analysis in social research.* Beverly Hills, CA: Sage.

Goulet-Pelletier, J.-C., & Cousineau, D. (2018). A review of effect sizes and their confidence intervals, part i: The cohen's *d* family. *The Quantitative Methods for Psychology*, *14*(4), 242–265. https://doi.org/10.20982/tqmp.14.4.p242

Grissom, R. J. (2000). Heterogeneity of variance in clinical data. *Journal of Consulting and Clinical Psychology*, *68*(1), 155–165. https://doi.org/10.1037/0022-006X.68.1.155

Grissom, R. J., & Kim, J. J. (2001). Review of assumptions and problems in the appropriate conceptualization of effect size. *Psychological Methods*, *6*(2), 135–146. https://doi.org/10.1037/1082-989X.6.2.135

Grissom, R. J., & Kim, J. J. (2005). *Effect sizes for research: A broad practical approach.* Mahwah, NJ: Lawrence Erlbaum Associates.

Hedges, L. V., & Olkin, I. (1985). *Statistical methods for meta-analysis.* Orlando, FL: Academic Press.

Huynh, C.-L. (1989). A unified approach to the estimation of effect size in meta-analysis. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Kelley, K. (2005). The effects of nonnormal distributions on confidence intervals around the standardized mean difference: Bootstrap and parametric confidence intervals. *Educational and Psychological Measurement*, *65*(1), 51–69. https://doi.org/10.1177/0013164404264850

Keselman, H. J., Algina, J., Lix, L. M., Wilcox, R. R., & Deering, K. N. (2008). A generally robust approach for testing hypotheses and setting confidence intervals for effect sizes. *Psychological Methods*, *13*(2), 110–129. https://doi.org/10.1037/1082-989X.13.2.110

Kulinskaya, E., & Staudte, R. G. (2007). Confidence intervals for the standardized effect arising in the comparison of two normal populations. *Statistics in Medicine*, *26*(14), 2853–2871. https://doi.org/10.1002/sim.2751

Lakens, D. (2013). Calculating and reporting effect sizes to facilitate cumulative science: A practical primer for *t*-tests and ANOVAs. *Frontiers in Psychology*, *4*(863), 1–12. https://doi.org/10.3389/fpsyg.2013.00863

Peng, C.-Y. J., Chen, L.-T., Chiang, H.-M., & Chiang, Y.-C. (2013). The impact of APA and AERA guidelines on effect size reporting. *Educational Psychology Review*, *25*(2), 157–209. https://doi.org/10.1007/s10648-013-9218-2

Prentice, D. A., & Miller, D. T. (1992). When small effects are impressive. *Psychological Bulletin*, *112*(1), 160–164. https://doi.org/10.1037/0033-2909.112.1.160

Raviv, E. (2014, June 2). Bias vs. consistency. Retrieved July 12, 2021, from https://eranraviv.com/bias-vs-consistency/

Shieh, G. (2013). Confidence intervals and sample size calculations for the standardized mean difference effect size between two normal populations under heteroscedasticity. *Behavior Research Methods*, *45*(4), 955–967. https://doi.org/10.3758/s13428-013-0320-7

Stout, D. E., & Ruble, T. L. (1995). Assessing the practical significance of empirical results in accounting education research: The use of effect size information. *Journal of Accounting Education*, *13*(3), 281–298. https://doi.org/10.1016/0748-5751(95)00010-J

Sullivan, G. M., & Feinn, R. (2012). Using effect size—or why the *p* value is not enough. *Journal of Graduate Medical Education*, *4*(3), 279–282. https://doi.org/10.4300/JGME-D-12-00156.1

Welch, B. L. (1938). The significance of the difference between two means when the population variances are unequal. *Biometrika*, *29*(3), 350–362. https://doi.org/10.2307/2332010

572   Wilkinson, L. (1999). Statistical methods in psychology journals: Guidelines and

573   explanations. *American Psychologist*, *54*(8), 594–604.

574   https://doi.org/10.1037/0003-066X.54.8.594

Appendix

575 **The bias of Cohen's d is twice as large as the bias of Shieh's d when population**

576 **variances and sample sizes are equal across groups: mathematical**

577 **demonstration.**

578   As mentioned in Table 1, the bias of Cohen's d is defined as

$$Bias_{Cohen's\ d} = \delta_{Cohen} \times \left(\frac{\sqrt{\frac{df_{Student}}{2}} \times \Gamma\left(\frac{df_{Student}-1}{2}\right)}{\Gamma\left(\frac{df_{Student}}{2}\right)} - 1 \right) \qquad (17)$$

with

$$\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{(n_1-1)\times\sigma_1^2+(n_2-1)\times\sigma_2^2}{n_1+n_2-2}}}$$

and

$$df_{Student} = n_1 + n_2 - 2$$

579   As mentioned in Table 2, the bias of Shieh's d is defined as

$$Bias_{Shieh's\ d} = \delta_{Shieh} \times \left(\frac{\sqrt{\frac{df_{Welch}}{2}} \times \Gamma\left(\frac{df_{Welch}-1}{2}\right)}{\Gamma\left(\frac{df_{Welch}}{2}\right)} - 1 \right) \qquad (18)$$

with

$$\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}} \qquad (N = n_1 + n_2)$$

and

$$df_{Welch} = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}}$$

580   When $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$, δ_{Cohen} is twice larger than δ_{Shieh}, as shown

581 below in equations 19 and 20:

$$\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{2(n-1)\sigma^2}{2(n-1)}}} = \frac{\boldsymbol{\mu_1 - \mu_2}}{\boldsymbol{\sigma}} \qquad (19)$$

582

$$\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{2\left(\frac{\sigma^2}{n/(2n)}\right)}} = \frac{\boldsymbol{\mu_1 - \mu_2}}{\boldsymbol{2\sigma}} \tag{20}$$

583 Moreover, degrees of freedom associated with Student's t-test and Welch's t-test are

584 identical, as shown below in equations 21 and 22:

$$df_{Student} = \boldsymbol{2(n - 1)} \tag{21}$$

585

$$df_{Welch} = \frac{[2(\sigma^2/n)]^2}{\frac{2(\sigma^2/n)^2}{n-1}} = \boldsymbol{2(n - 1)} \tag{22}$$

586 Equations 17 and 18 can therefore be redefined as follows:

$$Bias_{Cohen's\,d} = \frac{\mu_1 - \mu_2}{\sigma} \times \left(\frac{\sqrt{n - 1} \times \Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n - 1)} - 1\right) \tag{23}$$

587

$$Bias_{Shieh's\,d} = \frac{\mu_1 - \mu_2}{\boldsymbol{2\sigma}} \times \left(\frac{\sqrt{n - 1} \times \Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n - 1)} - 1\right) \tag{24}$$

588 We can therefore conclude that the bias of Cohen's d is twice larger than the bias of

589 Shieh's d.

590 **The variance of Cohen's d is four times larger than the bias of Shieh's d when**

591 **population variances and sample sizes are equal across groups: mathematical**

592 **demonstration.**

593   The variance of Cohen's d is defined in Table 1 as

$$Var_{Cohen's\ d} = \frac{N \times df_{Student}}{n_1 n_2 \times (df_{Student} - 2)} + \delta^2_{Cohen} \left[\frac{df_{Student}}{df_{Student} - 2} - \left(\frac{\sqrt{\frac{df_{Student}}{2}} \times \Gamma\left(\frac{df_{Student} - 1}{2}\right)}{\Gamma\left(\frac{df_{Student}}{2}\right)} \right)^2 \right]$$

(25)

594 and the variance of Shieh's d is defined in Table 2 as

$$Var_{Shieh's\ d} = \frac{df_{Welch}}{(df_{Welch} - 2)N} + \delta^2_{Shieh} \left[\frac{df_{Welch}}{df_{Welch} - 2} - \left(\frac{\sqrt{\frac{df_{Welch}}{2}} \times \Gamma\left(\frac{df_{Welch} - 1}{2}\right)}{\Gamma\left(\frac{df_{Welch}}{2}\right)} \right)^2 \right]$$

(26)

595   We have previously shown in equations 21 and 22 that degrees of freedom associated

596 with Student's t-test and Welch's t-test equal $2(n-1)$, when $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$.

597 As a consequence, the first term of the addition in equation 25 is 4 times larger than the

598 first term of the addition in equation 26:

$$\frac{N \times df_{Student}}{n_1 n_2 \times (df_{Student} - 2)} = \frac{2n \times 2(n-1)}{n^2 \times (2n - 4)} = \frac{4(n-1)}{n(2n-4)}$$

599

$$\frac{df_{Welch}}{(df_{Welch} - 2)N} = \frac{2(n-1)}{2n(2n-4)} = \frac{n-1}{n(2n-4)}$$

600 We have also previously shown in equations 19 and 20 that δ_{Cohen} is twice larger than

601 δ_{Shieh} when $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$ and, therefore, δ^2_{Cohen} is four times larger than

602 δ^2_{Shieh}. As a consequence, the second term of the addition in equation 25 is also 4 times

603 larger than the second term of the addition in equation 26. Because both terms of the

604 addition in equation 25 are four times larger than those in equation 26, we can conclude

605 that the variance of Cohen's d is four times larger than the variance of Shieh's d.