1   What measure of effect size when comparing two independent means?

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₁₆   What measure of effect size when comparing two independent means?

₁₇   Effect sizes are an important outcome of empirical research. Moving beyond decisions

₁₈ about statistical significance, there is a strong call for researchers to report and interpret

₁₉ effect sizes and associated confidence intervals. This practice is highly endorsed by the

₂₀ American Psychological Association (APA) and the American Educational Research

₂₁ Association (American Educational Research Association, 2006; American Psychological

₂₂ Association, 2010).

₂₃   In "between-subject" designs where individuals are randomly assigned into one of two

₂₄ independent groups and group scores are compared based on their means, the dominant

₂₅ estimator of effect size is Cohen's d_s, where the sample mean difference is divided by the

₂₆ pooled sample standard deviation (Peng, Chen, Chiang, & Chiang, 2013; Shieh, 2013). This

₂₇ estimator is available in many statistical software packages, such as SPSS and Stata.

₂₈ However, computing the pooled sample standard deviation assumes that both sample

₂₉ variances are estimates of a common population variance, which is known as the

₃₀ homogeneity of variance assumption. It has been widely argued that there are many fields in

₃₁ psychology where this assumption is ecologically unlikely (Delacre, Lakens, & Leys, 2017;

₃₂ Erceg-Hurn & Mirosevich, 2008; R. J. Grissom, 2000). The question how to deal with the

₃₃ assumption of equal variances has been widely explored in the context of hypothesis testing,

₃₄ and it is becoming increasingly common to by default report a t-test that does not assume

₃₅ equal variances, such as Welch's t-test.

₃₆   However, the question which effect size to report when equal variances are not assumed

₃₇ has received less attention. One possible reason is that researchers have not found consensus

₃₈ on which of the available options should be used (Shieh, 2013). Even within the very specific

₃₉ context of an estimate for the standardized sample mean difference there is little agreement

₄₀ about which estimator is the best choice. In this article, we will review the main candidates

₄₁ that have been proposed in the literature in the d family of effect sizes, namely Cohen's d_s,

42   Glass's d_s, Shieh's d_s and Cohen's d_s^*. We will compare these effect size measures through

43   simulations and suggest that Cohen's d_s^* is a better default than Cohen's d_s. We provide an

44   R package and Shiny app to compute relevant effect size measure and their confidence

45   intervals.

46   Before reviewing the most important effect size measures in the d-family, we will first

47   list the different purposes effect size measures serve, and discuss the relationship between

48   effect sizes, statistical, and practical significance. Based on a detailed description of the good

49   properties an effect size measure should possess, we will evaluate these properties in the

50   Monte Carlo simulations we performed to compare the different effect size estimators.

51   **Effect size estimators: definition and purposes**

52   The effect size is a measure of the magnitude of an effect. In the context of the

53   comparison of two groups based on their means, when the null hypothesis is the absence of

54   effect, d-family effect size estimators estimate the magnitude of the differences between

55   parameters of two populations groups are extracted from [e.g. the mean; Peng and Chen

56   (2014)]. Such a measure can be used for three different purposes.

57   First, effect size measures can be used for *interpretative* purposes. They allow

58   researchers to assess the practical significance of a result (i.e. statements about the relevance

59   of an effect in real life). In order to assess the meaningfulness of an effect, we should be able

60   to relate this effect size estimate with behaviors/meaningful consequences in the real world

61   (Andersen, McCullagh, & Wilson, 2007). This typically involves an analysis of the costs

62   (determined by a specific context) and the benefits (in part determined by the size of the

63   effect). It is important to remember an effect size is just a mathematical indicator of the

64   magnitude of a difference, which depends on the way a variable is converted into numerical

65   indicator. An effect size in itself is not a measure of the importance or the relevance of an

66   effect for real life [even if benchmarks for small, medium, or large effect sizes might have

67 contributed to such a misinterpretation; Stout and Ruble (1995)].

68   Second, effect size measures can be used for *comparative* purposes. They allow

69 researchers to assess the stability of results across designs, analyses, and sample sizes. This

70 includes statistically comparing and combining the results from two or more studies in a

71 meta-analysis. Third, effect size measures can be used for *inferential* purposes. The effect

72 sizes from previous studies can be used in an a-priori power analysis when planning a new

73 study (Lakens, 2013; Prentice & Miller, 1990; Stout & Ruble, 1995; Sullivan & Feinn, 2012;

74 Wilkinson & the Task Force on Statistical Inference, 1999). Moreover, hypothesis tests and

75 confidence intervals based on the same statistical quantity are directly related: if the area of

76 the null hypothesis is out of the $(1 - \alpha)$-confidence interval, then the hypothesis test would

77 also result in a *p*-value below the nominal alpha level. At the same time, the interval provide

78 extra information about the precision of the sample estimate for inferential purposes (Altman,

79 2005; Ellis, 2015), and which effect sizes are excluded. The narrower the interval, the higher

80 the precision, and the wider the confidence interval, the more the data lacks precision.

Properties of a good effect size estimator

82   The empirical value of an estimator (called the *estimate*) depends on the sample value.

83 Different samples extracted from the same population will lead to different sample estimates

84 of the population value. The *sampling distribution* of the estimator is the distribution of all

85 estimates, based on all possible samples of size *n* extracted from one population. Studying

86 the sampling distribution is useful, as it allows us to assess the qualities of an estimator.

87 More specifically, three desirable properties a good estimator should possess for inferential

88 purposes are: **unbiasedness**, **consistency** and **efficiency** (Wackerly, Mendenhall, &

89 Scheaffer, 2008).

90   An estimator is unbiased if the distribution of estimates is centered around the true

91 population parameter. On the other hand, an estimator is positively (or negatively) biased if

92 the distribution is centered around a value that is higher (or lower) than the true population

93 parameter (see Figure 1). In other words, examining the bias of an estimator tells us if

94 estimates are on average accurate. The *bias* of a point estimator $\hat{\delta}$ can be computed as

$$\delta_{bias} = E(\hat{\delta}) - \delta \tag{1}$$

95 where $E(\hat{\delta})$ is the expectation of the sampling distribution of the estimator and δ is the true
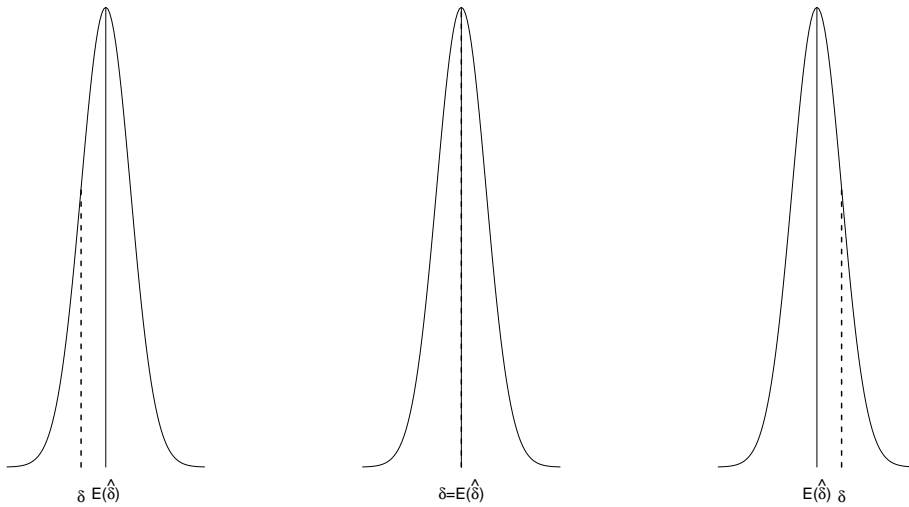
96 (population) parameter.

Figure 1. Sampling distribution for a positively biased (left), an unbiased (center) and a negatively biased estimator (right)

97   As we can see in Tables 1 and 2 the bias is directly related to the population effect size.

98 The larger the population effect size, the larger the bias. It is therefore also interesting to

99 examine the *relative bias*, defined as the ratio between the bias and the population effect size:

$$\delta_{relative\ bias} = \frac{E(\hat{\delta}) - \delta}{\delta} \tag{2}$$

100 While the bias informs us about the quality of estimates on average, in particular their

101 capacity of lying close to the true value, it says nothing about individual estimates. Imagine

102 a situation where the distribution of estimates is centered around the real parameter but

with such a large variance that some point estimates are very far from the center. This

would be problematic, since any single estimate might be very far from the true population

value. Therefore it is not only essential for an estimator to be unbiased, but it is also

desirable that the variability of its sampling distribution is small. Ideally, sample estimates

are close to the true population parameter. Among two unbiased estimators $\hat{\delta}_1$ and $\hat{\delta}_2$, we

therefore say that $\hat{\delta}_1$ is **more efficient** than $\hat{\delta}_2$ if

$$Var(\hat{\delta}_1) \leq Var(\hat{\delta}_2) \tag{3}$$

where $Var(\hat{\delta})$ is the variance of the sampling distribution of the estimator $\hat{\delta}$. Among all

unbiased estimators, the more efficient estimator will be the one with the smallest variance [1].

The variance of an estimator $\hat{\delta}$ is a function of its size (the larger the estimator, the larger

the variance) and, therefore, we might be interested in evaluating the *relative variance* as the

ratio between the variance and the square of the population estimator:

$$relative\ var(\hat{\delta}_1) = \frac{Var(\hat{\delta})}{\delta^2} \tag{4}$$

Note that both unbiasedness and efficiency are very important when choosing an estimator.

In some situations, it might be better to have a slightly biased estimator with low variance,

(so that each estimate remains relatively close to the true parameter and one might be able

to apply bias correction techniques) rather than an unbiased estimator with a large variance

(Raviv, 2014).

Finally, the last property of a good point estimator is **consistency**. Consistency

means that the bigger the sample size, the closer the estimate is to the population parameter.

In other words, the estimates *converge* to the true population parameter.

Beyond these inferential properties, Cumming (2013) reminds us that it is desireable

for an estimator to have a constant value across designs, as this facilitates the

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[1] The Cramér-Rao inequality provides a theoretical lower bound for the variance of unbiased estimators. An estimator reaching this bound is therefore optimally efficient.

124 interpretability of effect sizes from different designs, and makes it easier to include effects in

125 a meta-analysis. In other words, effect sizes should be **generalizable**. To our concern, while

126 we focus on between-subject designs, we expect to highlight an estimate of the population

127 effect size with a constant value for any sample sizes ratio.

Different measures of effect sizes

128

The d-family effect sizes are commonly used for mean differences between groups or
conditions. The population effect size is defined as

$$\delta = \frac{\mu_1 - \mu_2}{\sigma}$$

129 where both populations follow a normal distribution with mean μ_j in the jth population

130 $(j = 1, 2)$ and common standard deviation σ. There exist different estimators of this effect

131 size measure. For all, the mean difference is estimated by the difference $\bar{X}_1 - \bar{X}_2$ of both

132 sample means. When the equality of variances assumption is assumed, σ is estimated by

133 pooling both sample standard deviations (S_1 and S_2). When the equality of variances

134 assumption cannot be assumed, alternatives to the common standard deviation are available.

135 In the next section we will present effect sizes that assume equal variances between groups

136 (Cohen's d_s and Hedges' g_s), and effect sizes that do not assume equal variances (Glass's d_s,

137 Shieh's d_s, Hedges' d_s^*, Glass's g_s, Shieh's g_s, and Hedges' g_s^*). For each effect size we will

138 provide information about their theoretical bias, variance, and consistency.

When variances are equal between groups

139

When we have good reasons to assume equality of variances between groups then the
most common estimator of δ is Cohen's d_s, where the sample mean difference is divided by a
pooled error term (Cohen, 1965):

$$Cohen's\ d_s = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{(n_1-1) \times S_1^2 + (n_2-1) \times S_2^2}{n_1+n_2-2}}}$$

140 where S_j is the standard deviation, and n_j the sample size of the jth sample ($j = 1, 2$). The

141 reasoning behind this measure is to make use of the fact that both samples share the same

142 population variance (Keselman, Algina, Lix, Deering, & Wilcox, 2008), which means a more

143 accurate estimation of the population variance can be achieved by pooling both estimates of

144 this parameter (i.e. S_1 and S_2). Since the larger the sample size, the more accurate the

145 estimate, we give more weight to the estimate based on the larger sample size. Cohen's d_s is

146 directly related to Student's t-statistic:

$$t_{Student} = \frac{Cohen's\ d_s}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \leftrightarrow Cohen's\ d_s = t_{Student} \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \tag{5}$$

147 Under the assumption of normality and equal variances between groups, Student's t-statistic

148 follows a t-distribution with known degrees of freedom

$$df_{Student} = n_1 + n_2 - 2 \tag{6}$$

149 and noncentrality parameter [2]

$$ncp_{Student} = \frac{\delta_{Cohen}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

150 where $\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sigma_{pooled}}$ and $\sigma_{pooled} = \sqrt{\frac{(n_1-1) \times \sigma_1^2 + (n_2-1) \times \sigma_2^2}{n_1+n_2-2}}$. The relationship described in

151 equation 5 and the theoretical distribution of Student's t-statistic allow us to theoretically

152 determine the sampling distribution of Cohen's d_s, and therefore, its theoretical expectation

153 and variance when the assumptions of normality and equal variances are met. All these

154 equations are provided in Table 1. For interested readers, Supplemental Material 1 provides

155 a detailed examination of the theoretical bias and variance of Cohen's d_s based on Table 1,

156 as well as the bias and variance of all other estimators described later, based on Tables 2 and

157 3)[3], with the goal to determine which parameters influence the bias and variance of different

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[2] Under the null hypothesis of no differences between sample means, Student's t-statistic will follow a central t-distribution with $n_1 + n_2 - 2$ degrees of freedom. However, when the null hypothesis is false, the distribution of this quantity will not be centered, and a noncentral t-distribution will arise.

[3] All supplemental Materials are available on Github: https://github.com/mdelacre/Effect-sizes/

158 estimators. The main results will be discussed in the section "Monte Carlo Simulations:

159 assessing the bias, efficiency and consistency of 5 estimators" below.

While Cohen's d_s is a consistent estimator, its bias and variance are substantial with

small sample sizes, even under the assumptions of normality and equal variances (Lakens,

2013). In order to compensate for Cohen's d_s bias with small sample sizes, Hedges and Olkin

(1985) defined a bias-corrected version:

$$Hedges'\ g_s = Cohen's\ d_s \times \frac{\Gamma(\frac{df_{Student}}{2})}{\sqrt{\frac{df_{Student}}{2}} \times \Gamma(\frac{df_{Student}-1}{2})}$$

where $df_{Student}$ has been defined in equation 6 and $\Gamma()$ is the gamma function [for integers,

$\Gamma(x)$ is the factorial of x minus 1: $\Gamma(x) = (x-1)!$; Goulet-Pelletier and Cousineau (2018)].

This equation can be approximated as follows:

$$Hedges'\ g_s = Cohen's\ d_s \times \left(1 - \frac{3}{4N - 9}\right)$$

160 where N is the total sample size. Hedges' g_s is theoretically unbiased when the assumptions

161 of normality and equal variances are met (see Table 1). Moreover, while the variance of both

162 Cohen's d_s and Hedges' g_s depend on the same parameters (i.e. the total sample size (N) and

163 the sample sizes ratio $\left(\frac{n_1}{n_2}\right)$), Hedges' g_s is less variable, especially with small sample sizes [4].

164 While the pooled error term is the best choice when variances are equal between

165 groups (R. J. Grissom & Kim, 2001), it may not be well advised for use with data that

166 violate this assumption (Cumming, 2013; R. J. Grissom & Kim, 2001; Grissom R. J. & Kim,

167 2005; Kelley, 2005; Shieh, 2013). When variances are unequal between groups, the expression

168 in equation A is no longer valid because both groups do not share a common population

169 variance. If we pool the estimates of two unequal population variances, the estimator of

170 effect size will be lower as it should be in case of positive pairing (i.e. the group with the

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[4] $.52 \leq \left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}\right]^2 < 1$ for $3 \leq df < \infty$. The larger the total sample size, the smaller the difference between the variance of Cohen's d_s and Hedges' g_s.

larger sample size is extracted from the population with the larger variance) and larger as it should be in case of negative pairing (i.e. the group with the larger sample size is extracted from the population with the smaller variance). Because the assumption of equal variances across populations is rarely realistic in practice (Cain, Zhang, & Yuan, 2017; Delacre, Lakens, & Leys, 2017; Delacre, Leys, Mora, & Lakens, 2019; Erceg-Hurn & Mirosevich, 2008; Glass, Peckham, & Sanders, 1972; R. J. Grissom, 2000; Micceri, 1989; Yuan, Bentler, & Chan, 2004), both Cohen's d_s and Hedges' g_s should be abandoned in favor of an alternative robust to unequal population variances.

Table 1

Expentency, bias and variance of Cohen's d_s and Hedges' g_s under the assumptions that independent residuals are normally distributed with equal variances across groups.

	df	Expectation	Variance
Cohen's d_s	$N-2$	$\delta_{Cohen} \times c_f$	$\frac{N \times df}{n_1 n_2 \times (df-2)} + \delta_{Cohen}^2 \left[\frac{df}{df-2} - c_f^2\right]$
		$\approx \frac{\delta_{Cohen}}{\left(1-\frac{3}{4N-9}\right)}$	$\approx \frac{N \times df}{n_1 n_2 \times (df-2)} + \delta_{Cohen}^2 \left[\frac{df}{df-2} - \left(\frac{1}{1-\frac{3}{4N-9}}\right)^2\right]$
Hedges' g_s	$N-2$	δ_{Cohen}	$Var(Cohen's\ d_s) \times \left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}\right]^2$
			$\approx Var(Cohen's\ d_s) \times \left[1 - \frac{3}{4N-9}\right]^2$

Note. $\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sigma_{pooled}}$ and $c_f = \frac{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}{\Gamma(\frac{df}{2})}$; Cohen's d_s is a biased estimator, because its expectation differs from the population effect size. Moreover, the larger the population estimator (δ_{Cohen}), the larger the bias. Indeed, the bias is the difference between the expectation and δ_{Cohen}: $\delta_{bias} = \delta_{Cohen} \times (c_f - 1)$. On the other hand, Hedges' g_s is an unbiased estimator, because its expectation equals δ_{Cohen}; equations in Table 1 require $df \geq 3$ (i.e. $N \geq 5$).

When variances are unequal between populations

In his review, Shieh (2013) mentions three options available in the literature to deal with the case of unequal variances: the sample mean difference divided by (A) the Glass's d_s, (B) the Shieh's d_s and (C) the non pooled average of both variance estimates.

Glass's d_s. When comparing one control group with one experimental group, Glass, McGav, and Smith (2005) recommend using the standard deviation of the control group as standardizer. This yields

$$Glass's\ d_s = \frac{\bar{X}_e - \bar{X}_c}{S_c}$$

where $\bar{X}_e$ and $\bar{X}_c$ are the sample means of the experimental and control groups, and S_c is the sample SD of the control group. One argument in favour of using S_c as standardizer is the fact that it is not affected by the experimental treatment. When it is easy to identify which group is the "control" one, it is therefore convenient to compare the effect size estimation of different designs studying the same effect (Cumming, 2013). However, defining this group is not always obvious (Coe, 2002). This could induce large ambiguity because depending on the chosen SD as standardizer, measures could be substantially different (Shieh, 2013). The distribution of Glass's d_s is defined as in Algina, Keselman, and Penfield (2006):

$$Glass's\ d_s \sim \sqrt{\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \times \sigma_c^2}} \times t_{df,ncp} \tag{7}$$

where n_c and n_e are the sample sizes of the control and experimental groups, and df and ncp are defined as follows:

$$df = n_c - 1 \tag{8}$$

$$ncp = \frac{\delta_{Glass}}{\sqrt{\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \times \sigma_c^2}}}$$

where $\delta_{Glass} = \frac{\mu_c - \mu_e}{\sigma_c}$ and μ_c and μ_e are respectively the mean of the populations control and experimental groups are extracted from. Thanks to equation 7, we can compute its theoretical expectation and variance when the assumption of normality is met (see Table 2),

₂₀₀ and therefore determine which factors influence bias and variance, and how they do so (see

₂₀₁ Supplemental Material 1).

Table 2

Expectency, bias and variance of Glass's d_s and Cohen's d_s^* and Shieh's d_s under the assumption that independent residuals are normally distributed.

	df	Expectation	Variance
Glass's d_s	$n_c - 1$	$\delta_{Glass} \times c_f$	$\frac{df}{df-2} \times \left(\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \sigma_c^2}\right) + \delta_{Glass}^2 \left(\frac{df}{df-2} - c_f^2\right)$
Cohen's d_s^*	$\frac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$	$\delta_{Cohen}^* \times c_f$	$\frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2+\sigma_2^2} + (\delta_{Cohen}^*)^2 \left(\frac{df}{df-2} - c_f^2\right)$
		$\approx \delta_{Cohen}^* \times \frac{4df-1}{4(df-1)}$	$\approx \frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2+\sigma_2^2} + (\delta_{Cohen}^*)^2 \left[\frac{df}{df-2} - \left(\frac{4df-1}{4(df-1)}\right)^2\right]$
Shieh's d_s	$\frac{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1}+\frac{(\sigma_2^2/n_2)^2}{n_2-1}}$	$\delta_{Shieh} \times c_f$	$\frac{df}{(df-2)N} + \delta_{Shieh}^2 \left(\frac{df}{df-2} - c_f^2\right)$

Note. $c_f = \frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}$, $\delta_{Glass} = \frac{\mu_c-\mu_e}{\sigma_c}$, $\delta_{Shieh} = \frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N}+\frac{\sigma_2^2}{n_2/N}}}$ and $\delta_{Cohen}^* = \frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}}$; all estimators are biased estimators, because their expectations differ from the population effect size δ. Moreover, the larger the population estimator (δ), the larger the bias. Indeed, the bias is the difference between the expectation and δ: $\delta_{bias} = \delta \times (c_f - 1)$. equations require $df \geq 3$ and at least 2 subjects per group.

Shieh's d$_s$.   Kulinskaya and Staudte (2007) were the first to recommend the use of a standardizer that takes the sample sizes allocation ratios into account, in addition to the variance of both samples. Shieh (2013), following Kulinskaya and Staudte (2007), proposed a modification of the exact *SD* of the sample mean difference:

$$Shieh's\ d_s = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{S_1^2/q_1 + S_2^2/q_2}}; \quad q_j = \frac{n_j}{N}(j = 1, 2)$$

where $N = n_1 + n_2$. Shieh's d_s is directly related with Welch's *t*-statistic:

$$Shieh's\ d_s = \frac{t_{Welch}}{\sqrt{N}} \leftrightarrow t_{welch} = Shieh's\ d_s \times \sqrt{N} \tag{9}$$

The exact distribution of Welch's *t*-statistic is more complicated than the exact distribution of Student's *t*-statistic, but it can be approximated, under the assumption of normality, by a *t*-distribution with degrees of freedom and noncentrality parameters (Shieh, 2013; Welch, 1938):

$$df_{Welch} = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}} \tag{10}$$

$$ncp_{Welch} = \delta_{Shieh} \times \sqrt{N} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n1} + \frac{\sigma_2^2}{n_2}}}$$

where $\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}}$. The relationship described in equation 9 and the theoretical distribution of Welch's *t*-statistic allow us to theoretically approximate the sampling distribution of Shieh's d_s. Based on the sampling distribution of Shieh's d_s, we can estimate its theoretical expectation and variance under the assumption of normality (see Table 2), and thereby determine which factors influence bias and variance, and how they do so (see Supplemental Material 1).

As demonstrated in the Appendix, when variances and sample sizes are equal across groups, the biases and variances of Cohen's d_s and Shieh's d_s are identical except for multiplication by a constant. The same is true for the estimators δ_{Cohen} and δ_{Shieh}:

$$\delta_{Cohen} = 2 \times \delta_{Shieh} \quad \text{(considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \tag{11}$$

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$$Bias_{Cohen's\ d_s} = 2 \times Bias_{Shieh's\ d_s} \quad \text{(considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \tag{12}$$

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$$Var_{Cohen's\ d_s} = 4 \times Var_{Shieh's\ d_s} \quad \text{(considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \tag{13}$$

222 We can deduce from equations 11, 12 and 13 that relative to their respective population

223 effect size, Cohen's d_s and Shieh's d_s are equally accurate. This is a good illustration of our

224 motivation to favor relative bias and variance (previously defined in equations 2 and 4) over

225 the most commonly used raw bias and variance (previously defined in equations 1 and 3).

226   When sample sizes are not equal, according to the statistical properties of Welch's

227 statistic under heteroscedasticity, Shieh's d_s accounts for the allocation ratio of sample sizes

228 to each condition. The lack of generality caused by taking this specificity of the design into

229 account has led Cumming (2013) to question its usefulness in terms of interpretability: when

230 the mean difference $(\bar{X}_1 - \bar{X}_2)$, S_1, and S_2 remain constant, Shieh's d_s will vary as a

231 function of the sample sizes allocation ratio (unlike Cohen's d_s^* that we will define below). At

232 the population level, δ_{Shieh} also depends on the sample sizes allocation ratio, as illustrated in

233 the following shiny application: https://mdelacre.shinyapps.io/shiehvsCohen/.

Cohen's d_s^*.   An effect size estimator based on the sample mean difference divided by

the square root of the non pooled average of both variance estimates was suggested by Welch

(1938). Here, we indicate the difference between Cohen's d_s (based on the pooled standard

deviations) and Cohen's d_s^* with an asterisk. This yields:

$$Cohen's\ d_s^* = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{(S_1^2 + S_2^2)}{2}}}$$

234 where $\bar{X}_j$ is the mean and S_j is the standard deviation of the jth sample (j = 1,2). We know

235 the distribution of Cohen's d_s^* (Huynh, 1989):

$$Cohen's\ d_s^* \sim \sqrt{\frac{2(n_2 \times \sigma_1^2 + n_1 \times \sigma_2^2)}{n_1 n_2 (\sigma_1^2 + \sigma_2^2)}} \times t_{df^*, ncp^*} \tag{14}$$

236 Where df^* and ncp^* are defined as follows:

$$df^* = \frac{(n_1 - 1)(n_2 - 1)(\sigma_1^2 + \sigma_2^2)^2}{(n_2 - 1)\sigma_1^4 + (n_1 - 1)\sigma_2^4} \tag{15}$$

$$ncp^* = \delta^*_{Cohen} \times \sqrt{\frac{n_1 n_2 (\sigma_1^2 + \sigma_2^2)}{2(n_2 \sigma_1^2 + n_1 \sigma_2^2)}} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

where $\delta^*_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}}$. Using equation 14 we can compute its theoretical expectation and variance when the assumption of normality is met (see Table 2), and therefore determine which factors influence bias and variance, and how they do so (see Supplemental Material 1). This estimator has been widely criticized, because it results in a variance term of an artificial population (i.e. since the variance term does not estimate the variance of one or the other group, the composite variance is an estimation of the variance of an artificial population) and is therefore very difficult to interpret (R. J. Grissom & Kim, 2001), and unless both sample sizes are equal, the variance term does not correspond to the variance of the mean difference (Shieh, 2013).

However, we will show throughout the simulation section that this estimator exhibits very good inferential properties. Moreover, it has a constant value across sample sizes ratios, as shown in the Shiny App at https://mdelacre.shinyapps.io/shiehvsCohen/.

Glass's g_s, Shieh's g_s and Hedges' g_s^*. As for Cohen's d_s, a Hedges' correction can be applied in order to compensate for the bias of Glass's d_s, Shieh's d_s and Cohen's d_s^* with small sample sizes (see Table 2). This correction has the following general form:

$$g_s = d_s \times \frac{\Gamma(\frac{\nu}{2})}{\sqrt{\frac{\nu}{2}} \times \Gamma(\frac{\nu-1}{2})}$$

where the distinct values of ν are provided in equation 8 for Glass's g_s, in equation 15 for Hedges' g_s^* and in equation 10 for Shieh's g_s. The three corrected estimators are theoretically unbiased when the assumption of normality is met. Their variance is a function of the same parameters as their biased equivalent. However, due to the correction they have a smaller variance, especially with small sample sizes, as shown in Table 3. In summary:

- The variances of Hedges' g_s^* and Shieh's g_s depend on the total sample size (N), their respective population effect size (δ), and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_1}{n_2} \times \frac{\sigma_1}{\sigma_2}\right)$.

257 • The variance of Glass's g_s also depends on N, δ and $\frac{n_c}{n_e} \times \frac{\sigma_c}{\sigma_e}$. In addition, there is also

258 a main effect of the SD-ratio $\left(\frac{\sigma_c}{\sigma_e}\right)$ on its variance.

259 How these parameters influence the variance of the estimators will be summarized and

260 illustrated in Monte Carlo simulations below.

Table 3

Expectancy, bias and variance of Glass's d_s and Cohen's d_s^* and Shieh's d_s under the assumption that independent residuals are normally distributed.

	df	Expectation	Variance
Glass's g_s	$n_c - 1$	δ_{glass}	$Var(Glass's\ d_s) \times \left(\dfrac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}}\times\Gamma\left(\frac{df-1}{2}\right)}\right)^2$
Cohen's g_s^*	$\dfrac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$	δ^*_{Cohen}	$Var(Cohen's\ d_s^*) \times \left(\dfrac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}}\times\Gamma\left(\frac{df-1}{2}\right)}\right)^2$
Shieh's g_s	$\approx \dfrac{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1}+\frac{(\sigma_2^2/n_2)^2}{n_2-1}}$	δ_{Shieh}	$Var(Shieh's\ d_s) \times \left(\dfrac{\Gamma\left(\frac{df}{2}\right)}{\sqrt{\frac{df}{2}}\times\Gamma\left(\frac{df-1}{2}\right)}\right)^2$

Note. $c_f = \dfrac{\sqrt{\frac{df}{2}}\times\Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}$; $\delta_{Glass} = \frac{\mu_c-\mu_e}{\sigma_c}$, $\delta_{Shieh} = \dfrac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N}+\frac{\sigma_2^2}{n_2/N}}}$ and $\delta^*_{Cohen} = \dfrac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}}$; all estimators are unbiased estimators, because their expectations equal the population effect size δ; equations require $df \geq 3$ and at least 2 subjects per group.

Monte Carlo Simulations

Assessing the bias, efficiency and consistency of 5 estimators.

Method.   We performed Monte Carlo simulations using R (version 3.5.0) to assess the bias, efficiency and consistency of Cohen's g_s, Glass's g_s (using respectively the sample SD of the first or second group as a standardizer), Hedges' g_s^* and Shieh's g_s.

A set of 100,000 datasets was generated for 1,008 scenarios as a function of different criteria. In 252 scenarios, samples were extracted from a normally distributed population (in order to ensure the reliability of our calculation method) and in 756 scenarios, samples were extracted from non normal population distributions. In order to assess the quality of estimators under realistic deviations from the normality assumption, we referred to the review of Cain, Zhang, and Yuan (2017). Cain et al. (2017) investigated 1,567 univariate distributions from 194 studies published by authors in Psychological Science (from January 2013 to June 2014) and the American Education Research Journal (from January 2010 to June 2014). For each distribution, they computed Fisher's skewness

$$G_1 = \frac{\sqrt{n(n-1)}}{n-2} \frac{m_3}{\sqrt{(m_2)^3}}$$

and kurtosis

$$G_2 = \frac{n-1}{(n-2)(n-3)} \times \left[(n+1) \left(\frac{m_4}{(m_2)^2} - 3 \right) + 6 \right]$$

where n is the sample size and m_2, m_3 and m_4 are respectively the second, third and fourth centered moments. They found values of kurtosis from G2 = -2.20 to 1,093.48. According to their suggestions, throughout our simulations, we kept constant the population kurtosis value at the 99th percentile of their distribution of kurtosis, i.e. G2=95.75. Regarding skewness, we simulated population parameter values which correspond to the 1st and 99th percentile of their distribution of skewness, i.e. respectively G1 = -2.08 and G1 = 6.32. We also simulated samples extracted from population where G1 = 0, in order to assess the main effect of high kurtosis on the quality of estimators. All possible combinations of population skewness and kurtosis and the number of scenarios for each combination are summarized in Table 4.

Table 4

Number of combinations of skewness and kurtosis in our simulations.

		Kurtosis		
		0	95.75	**TOTAL**
	0	252	252	**504**
Skewness	-2.08	/	252	**252**
	6.32	/	252	**252**
	TOTAL	**252**	**756**	**1008**

287   *Note.* Fisher's skewness (G1) and kurtosis (G2) are presented in Table 4. The 252

288   combinations where both G1 and G2 equal 0 correspond to the normal case.

289   For the 4 resulting combinations of skewness and kurtosis (see Table 4), all other

290   parameter values were chosen in order to illustrate the consequences of factors identified as

291   playing a key role on the variance of unbiased estimators. We manipulated the population

292   mean difference $(\mu_1 - \mu_2)$, the sample sizes (n), the sample size ratio $(n\text{-ratio} = \frac{n_1}{n_2})$, the

293   population *SD*-ratio (i.e. $\frac{\sigma_1}{\sigma_2}$), and the sample size and population variance pairing $\left(\frac{n_1}{n_2} \times \frac{\sigma_1}{\sigma_2}\right)$.

294   In our scenarios, μ_2 was always 0 and μ_1 varied from 1 to 4, in steps of 1 (so does $\mu_1 - \mu_2$)[5].

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[5] In the original plan, we had added 252 simulations in which μ_1 and μ_2 were both null. We decided not to

present the results of these simulations in the main article, because the relative bias and the relative variance

appeared to us to be very useful to fully understand the comparison of the estimators, and computing them

is impossible when the real mean difference is zero. Indeed, for these specific configurations, both relative

bias and relative variance would have infinite values due to the presence of the population effect size term in

295 Moreover, σ_1 always equals 1, and σ_2 equals .1, .25, .5, 1, 2, 4 or 10, and therefore, so were

296 the ratios of $\frac{\sigma_1}{\sigma_2}$. The simulations for which both σ_1 and σ_2 equal 1 are the particular case of

297 homoscedasticity, or equal population variances across groups. The sample sizes of both

298 groups (n_1 and n_2) were 20, 50 or 100. When sample sizes of both groups are equal, the

299 n-ratio equals 1 (this is known as a balanced design). All possible combinations of n-ratio

300 and population SD-ratio were simulated in order to distinguish scenarios where both sample

301 sizes and population variances are unequal across groups (with positive pairing when the

302 group with the largest sample size is extracted from the population with the largest SD, and

303 negative pairing when the group with the smallest sample size is extracted from the

304 population with the smallest SD) and scenarios with no pairing between sample sizes and

305 variances (sample sizes and/or population SD are equal across all groups). In sum, the

306 simulations grouped over different sample sizes yield 4 conditions (a, b, c and d) based on

307 the n-ratio, population SD-ratio, and sample size and population variance pairing, as

308 summarized in Table 5. We chose to divide scenarios into these 4 conditions because

309 analyses in Supplemental Material 1 revealed main and interaction effects of sample sizes

310 ratio and SD-ratio on the bias and variance of some estimators.

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their denominator. However, these extra simulations were included in the simulation checks, in Supplemental

Material 2.

Table 5

4 conditions based on the n-ratio and the SD-ratio.

		n-ratio		
		1	**>1**	**<1**
	1	a	b	b
SD-ratio	**>1**	c	d	d
	<1	c	d	d

Results.   Before presenting the comparison of the estimators for each condition, it is useful to make some general comments.

1) We previously discussed the fact that raw bias and variances are sometimes misleading. They can give the illusion of huge differences between two estimators, even if these differences only reflect a change of unit (i.e. different population effect sizes). To better understand this, imagine a sample of 15 people for whom we know the height (in meters) and we compute a sample variance of 0.06838. If we convert sizes to centimeters and compute the sample variance again, we find a measure of 683.8 (i.e. 10^4 larger). Both measures represent the same amount of variability, but they are expressed in different units. The same issue due to a change in scales occurs when comparing the estimates of different population measures. To avoid this possible confusion, we will only present the relative bias and relative variance in all Figures (and anytime we will mention the biases and variances in the results section, we will be referring to relative bias and variance). For interested readers, illustrations of the raw bias and variance are available on Github.

2) For the sake of readability, for each condition, the scale we used for the ordinate axis of our plots when comparing the relative bias of all estimators and $G_1 = 6.32$ and $G_2 = 95.75$ is different from the scale we used for all other combinations G_1/G_2. In the same way of thinking, we also used different scales for the ordinate axis of our plots that compare the relative variance of all estimators, as a function of the condition.

3) Throughout this section, we will *compare* the relative bias and variance of different estimators, but we do not present bias and variance in absolute terms. We chose very extreme (although realistic) conditions, and we know that none of the parametric measures of effect size will be robust against such extreme conditions. Our goal is therefore to study the robustness of the estimators against normality violations only in comparison with the robustness of other indicators, but not in absolute terms.

After these general remarks, we will analyze each condition separately. In all Figures presented below, for different sub-conditions, the averaged relative bias and relative variance of five estimators are presented. When describing the Glass's g_s estimators, we will systematically refer to the "control group" as the condition the standardizer is based on (i.e. the first group when using S_1 as standardizer, the second group when using S_2 as standardizer). The other condition will be referred to as the "experimental group."

When variances are equal across groups.   Figures 2 and 3 represent configurations where the equality of variances assumption is met. According to our expectations, one observes that the bias of all estimators is approximately zero as long as the normality assumption is met (first column in both Figures)[6]. However, the more the data generation process deviations from the normality assumption (i.e. when moving from left to right in the Figures), the larger the bias in the estimators.

We will observe that Glass's g_s should always be avoided when the equality of variance

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[6] When looking at relative bias for all estimators, the maximum departure from zero is 0.0064 when sample sizes are equal across groups, and 0.0065 with unequal sample sizes.

350 assumption is met. Hedge's g_s, Hedges' g_s^* and Shieh's g_s perform equally well as long as the

351 sample size ratio is close to 1 (condition a; see Figure 2). However, when designs are highly

352 unbalanced (condition b; see Figure 3), Shieh's g_s is not consistent anymore, while Hedges'

353 g_s^* remain consistent, Hedges's g_s is a better estimator. For interested readers, these findings

354 are detailed in the three paragraphs below.

355   Figure 2 illustrates scenarios where both population variances and sample sizes are

356 equal across groups (condition a). One can first notice that all estimators are consistent, as

357 their bias and variance decrease when the total sample size increases. For any departure

358 from the normality assumption, both bias and variance of Hedges' g_s, Shieh's g_s and Hedges'

359 g_s^* are similar[7] and smaller than the bias and variance of Glass's g_s estimates using either S_1

360 or S_2 as a standardizer. Moreover, when samples are extracted from skewed distributions,

361 Glass's g_s will show different bias and variance as a function of the chosen standardizer (S_1

362 or S_2), even if both S_1 and S_2 are estimates of the same population variance, based on the

363 same sample size. This is due to non-null correlations of opposite sign between the mean

364 difference ($\bar{X}_1 - \bar{X}_2$) and respectively S_1 and S_2. In Supplemental Material 3, we detailed in

365 which situation a non-null correlation occurs between the sample mean difference ($\bar{X}_1 - \bar{X}_2$)

366 and the standardizer of compared estimators as well as the way this correlation impacts the

367 bias and variance of estimators.

368   Figure 3 illustrates scenarios where population variances are equal across groups, but

369 sample sizes are unequal (condition b). For any departures from the normality assumptions,

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[7] While the bias and variance of Cohen's d_s, Cohen's d_s^* and Shieh's d_s are identical, the bias and variance of

Hedges' g_s are marginally different from the bias and variance of Hedges' g_s^* and Shieh's g_s (these last two

having identical bias and variance). Indeed, because of the sampling error, differences remain between sample

variances, even when population variances are equal between groups. Since the Hedges' correction applied to

Cohen's d_s does not contain the sample variances (unlike the correction applied on both other estimators),

the bias and variance of Hedges' g_s are slighly different from the bias and variance of Hedges' g_s^* and Shieh's

g_s.

Hedges' g_s shows the smallest bias and variance. Hedges' g_s and Hedges' g_s^* are consistent

estimators (i.e. the larger the sample sizes, the lower the bias and the variance), unlike

Shieh's g_s and Glass's g_s. The bias of Glass's g_s does not depend either on the size of the

experimental group or on the total sample size. The only way to decrease the bias of Glass's

g_s is therefore to add subjects in the control group. On the other hand, the variance of

Glass's g_s depends on both sample sizes, but not in an equivalent way: in order to reduce the

variance, it is much more efficient to add subjects in the control group and when the size of

the experimental group decreases so does the variance, even when the total sample size is

increased. Regarding Shieh's g_s, for a given sample size ratio, the bias and variance will

decrease when sample sizes increase. However, there is a large effect of the sample sizes ratio

such that when the sample sizes ratio moves away from 1 by adding subjects, bias and

variance might increase.[8] On the other hand, when the sample sizes ratio moves closer to 1

by adding subjects, the bias will decrease.

When samples are extracted from skewed distributions and have unequal sizes (the two

last columns in Figure 3), for a constant total sample size, Glass's g_s, Shieh's g_s and Hedges'

g_s^* will show different bias and variance depending on which group is the largest one

(e.g. when distributions are right-skewed, the bias and variance of all these estimators when

n_1 and n_2 are respectively 50 and 20 are not the same as their bias and variance when n_1

and n_2 are respectively 20 and 50). This is due to a non-null correlation of opposite sign

between the mean difference $(\bar{X}_1 - \bar{X}_2)$ and their respective standardizers depending on

which group is the largest one, as detailed in Supplemental Material 3. One observes that

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[8] Regarding variance, in Supplemental Material 1, we mentioned that when the population effect size is zero,

the larger the total sample size, the lower the variance, whether the sample sizes ratio is constant or not. We

also mentioned that this is no longer true when the population effect size is not zero. In our simulations the

effect size is never zero. The effect size effect is partially visible in Figure 3 because we do not entirely

remove the effect size effect when we divide the variance by δ^2. This is due to the fact that one term, in the

equation of the variance computation, does not depend on the effect size.

under these configurations, the bias and variance of Glass's g_s are sometimes a bit smaller

and sometimes much larger than the bias and variance of Shieh's g_s and Cohen's d_s^*. [9]

When variances are unequal across groups.   Figures 4 to 9 represent configurations

where the equality of variances assumption is not met. According to our expectations, one

observes that the bias of all estimators is approximately zero as long as the normality

assumption is met (first column in the three Figures), and the further from the normality

assumption (i.e. when moving from left to right in Figures), the larger the bias[10]. It might

be considered surprising that the bias of Hedges' g_s remains very small throughout these

conditions. As discussed in the section "Different measures of effect size," Hedges' g_s should

be avoided when population variances and sample sizes are unequal across groups, because of

the pooled error term. When pooling the estimates of two unequal population variances, the

resulting estimator will be lower (in case of positive pairing) or larger (in case of negative

pairing) than it should be. At the same time, when pooling two unequal population

variances, the population effect size will also be lower (in case of positive pairing) or larger

(in case of negative pairing) as it should be. As a consequence, the distortion cannot be seen

through the difference between the expected estimator and the population effect size

measure. For this reason, the bias and variance of Hedges' g_s will not be taken into account

in the following comparisons.

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[9] Supplemental Material 3 shows that when the $\mu_1 - \mu_2 > 0$ (like in our simulations), all other parameters

being equal, an estimator is always less biased and variable when choosing a standardizer that is positively

correlated with $\bar{X}_1 - \bar{X}_2$. Supplemental Material 3 also shows that the smaller n_c, the larger the magnitude

of correlation between S_c and $\bar{X}_1 - \bar{X}_2$. When $cor(S_c, \bar{X}_1 - \bar{X}_2)$ is positive, the positive effect of increasing

the magnitude of the correlation is counterbalanced by the negative effect of reducing n_c. On the other hand,

when $cor(S_c, \bar{X}_1 - \bar{X}_2)$ is negative, the negative effect of increasing the magnitude of the correlation is

amplified by the negative effect of decreasing n_c. This explains why the difference between Glass's g_s and

other estimators is larger when Glass's g_s is the least efficient estimator.

[10] When looking at the relative bias for all estimators, the maximum departure from zero is 0.0173 when

sample sizes are equal across groups, and 0.0274 when both sample sizes and variances differ across groups.

We will observe that when variances are unequal across populations, Glass's g_s sometimes performs better, but also sometimes performs much worst than Shieh's g_s and Hedges' g_s^*, both in terms of bias and variance. The performance of Glass's g_s highly depends on parameters that we cannot control (i.e. a triple interaction between the n-ratio, the SD-ratio and the correlation between the standardizer and the mean difference) and for this reason, we do not recommend using it. When the sample sizes ratio is close to 1, Shieh's g_s and Hedges' g_s^* are both appropriate but the further the sample sizes ratio is from 1, the larger the bias of Shieh's g_s in order that in the end, the measure that we believe performs best across scenarios is Hedges' g_s^*.

Figures 4 and 5 are dedicated to scenarios where population variances are unequal between groups and sample sizes are equal (condition c). In Figure 4, scenarios are subdivided as a function of the sample sizes and one can notice that all estimators are consistent, as their bias and variance decrease when the total sample size increases. In Figure 5, scenarios are subdivided as a function of the SD-ratio. Because the comparison pattern remains very similar for all sample sizes, we present only scenarios when sample sizes equal 20. One should first notice that for all estimators in Figure 5, the relative variance seems to be much larger when $S_2 > S_1$. [11] This information should not be taken into account because it is only an artefact of our simulation conditions combined with the way we computed the relative variance. [12]

When samples are extracted from skewed distributions, the bias and variance of Glass's g_s are sometimes smaller and sometimes larger than the bias of Shieh's g_s and Hedges' g_s.

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[11] The difference between the variance of estimators when the second group is 10 times larger than the first group was so large that we decided to not present it, for the sake of readability of the Figures.

[12] We previously mentioned that when dividing the variance by δ^2, we do not entirely remove the effect size effect. Actually, we introduce δ^2 in the denominator of the first term, in the equation of the variance computation. Because we performed our simulations in order that σ_1 always equals 1, the smaller S_2, the larger the population effect size and therefore, the lower the relative variance.

430   This is mainly due to the fact that when two samples of same sizes are extracted from two

431   skewed distributions with unequal variances (the two last columns in Figure 5), there will be

432   non-null correlations of opposite sign between the mean difference $(\bar{X}_1 - \bar{X}_2)$ and the

433   standardizer of *all* estimators, depending on which population variance is larger [13].

434   Figures 6 to 9 are dedicated to scenarios where both sample sizes and population

435   variances differ across groups. Due to a high number of combinations between the sample

436   sizes-ratio and the SD-ratio in our simulations, we decided to present only some conditions.

437   Because equations in Table 3 revealed an interaction effect between the sample sizes ratio

438   and the SD-ratio on the bias and variance of Hedges' g_s^* and Shieh's g_s (see Supplemental

439   Material 1), we chose to present all configurations where the larger SD is 10 times larger

440   than the smaller SD (Figures 6 and 7), and configurations where the larger SD is twice

441   larger than the smaller SD (Figures 8 and 9), in order to compare the effect of the sample

442   sizes ratio on the bias and variance of all estimators when the SD-ratio is large

443   $\left(\frac{\sigma_1}{\sigma_2} = 10 \; or \; .1 \right)$ or medium $\left(\frac{\sigma_1}{\sigma_2} = 2 \; or \; .5\right)$.

444   When distributions are symmetric, the bias of Glass's g_s only depends on the size of

445   the control group and is therefore not impacted by either the sample sizes ratio or the total

446   sample size. When comparing Figures 6 to 9, one can also notice that the bias of Glass's g_s

447   does not depend on the SD-ratio either. Unlike the bias of Glass's g_s, its variance depends

448   on both sample sizes, but not in an equivalent way. In most scenarios it is more efficient, in

449   order to reduce the variance of Glass's g_s, to add subjects in the control group. Regarding

450   Hedges' g_s^* and Shieh's g_s, their respective biases and variances depend on an interaction

451   effect between the sample sizes ratio and the SD-ratio $\left(\frac{n_1}{n_2} \times \frac{\sigma_1}{\sigma_2}\right)$: the sample sizes ratio

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[13] When population variances are unequal, a non-null correlation occurs between standardizer estimates and $\bar{X}_1 - \bar{X}_2$. For standardizers computed based on both S_1 and S_2, the sign of the correlation between the standardizer and the mean difference will be the same as the sign of the correlation between the mean difference and the estimate of the larger population variance. For interested readers, this is detailed in Supplemental Material 3.

₄₅₂ associated with the smallest bias and variance is not the same when the more variable group

₄₅₃ is 10 times more variable than the other group (Figures 6 and 7) than when it is only twice

₄₅₄ more variable (Figures 8 and 9). However, the respective biases and variances of Hedges' g_s^*

₄₅₅ and Shieh's g_s are always smaller when there is a positive pairing between sample sizes and

₄₅₆ variances. When samples are extracted from skewed distributions, the bias and variance of

₄₅₇ Glass's g_s are sometimes smaller and sometimes larger than the bias of Shieh's g_s and

₄₅₈ Hedges' g_s^*, due to a combination of three factors: (1) which group is larger, (2) which group

₄₅₉ has the smallest standard deviation and (3) what is the correlation between the standardizer

₄₆₀ and the mean difference.

Recommendations
₄₆₁

₄₆₂   We recommend using Hedges' g_s^* in order to assess the magnitude of the effect when

₄₆₃ comparing two independent means, because a) it does not rely on the equality of population

₄₆₄ variances assumption (unlike Hedges' g_s), b) it is always consistent (unlike Shieh's g_s), c) it

₄₆₅ is easy to interpret (Hedges' g_s^* can be interpreted in the same way as Hedges' g_s) and d) it

₄₆₆ remains constant for any sample sizes ratio, even when population variances are unequal

₄₆₇ across groups, as shown in the Shiny App.

₄₆₈   Effect sizes estimates such as Hedges' g_s^* should always be reported with a confidence

₄₆₉ interval. To help researchers compute Hedges' g_s^* and it's confidence interval we created the

₄₇₀ R package *deffectsize* (see https://github.com/mdelacre/deffectsize). The *datacohen_CI*

₄₇₁ function was built in order to compute point estimators and confidence intervals based on

₄₇₂ raw data and the *cohen_CI* function was built in order to compute point estimators and

₄₇₃ confidence intervals based on descriptive statistics (sample means, sample variances and

₄₇₄ sample sizes). By default, unbiased Cohen's g_s^* is computed but it is also possible to compute

₄₇₅ biased estimators (e.g. Cohen's d_s^*) and/or to use a pooled error term as standardizer by

₄₇₆ assuming that the equality of population variances is met (e.g. Hedges' g_s or Cohen's d_s,

₄₇₇ depending on whether we choose to compute unbiased or biased estimator). Other functions

478 (*datashieh_CI*, *shieh_CI*, *dataglass_CI* and *glass_CI*) are available in order to compute

479 Shieh's g_s (or Shieh's d_s) and Glass's g_s (or Glass's d_s) as well as their respective confidence

480 intervals, even though we don't recommend to use these effect sizes by default. Researchers

481 who do not use R can use a Shiny app to compute point estimators and confidence intervals

482 based on descriptive statistics: https://mdelacre.shinyapps.io/deffectsize-Shinyapp/.

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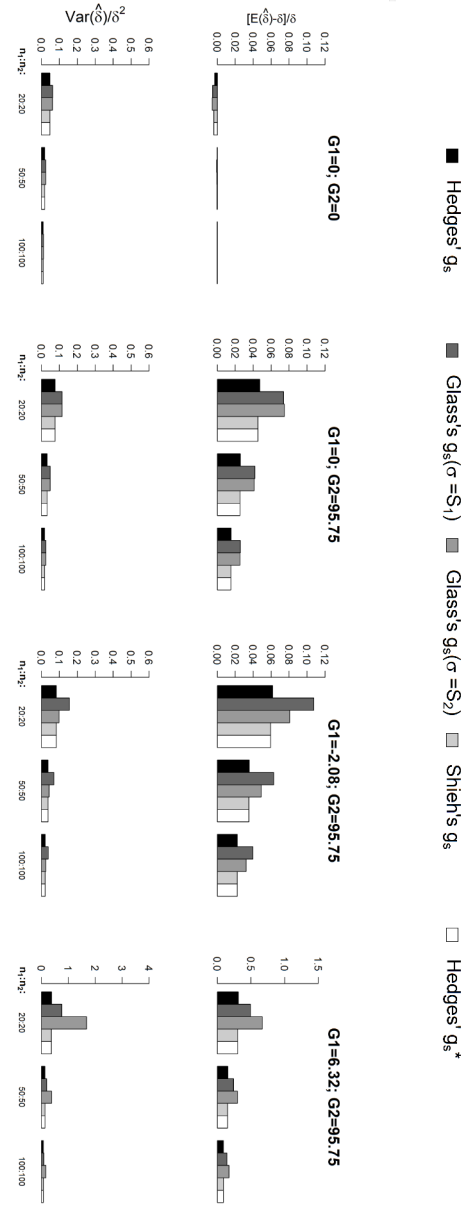
Figure 2. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are equal across groups (condition a)
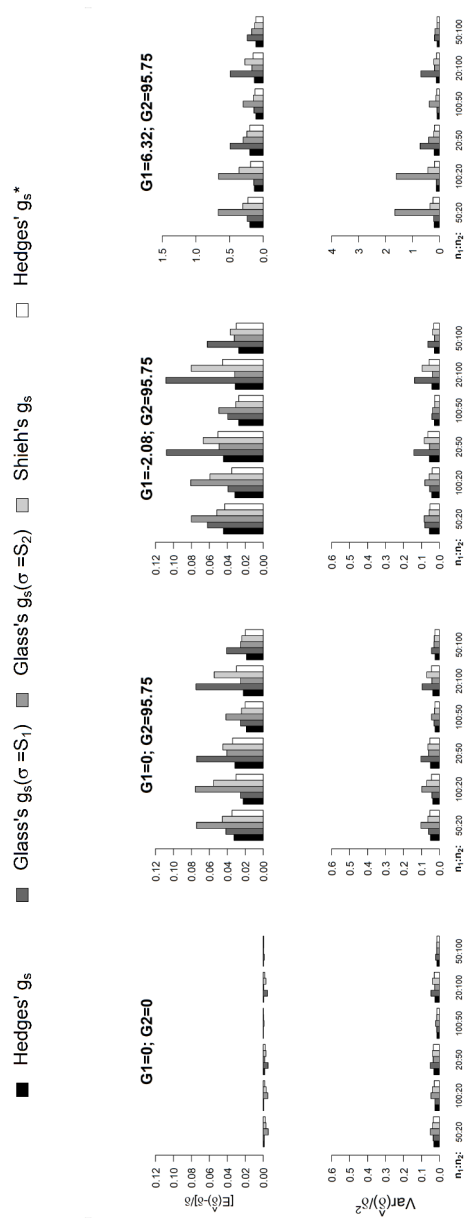
Figure 3. Bias and efficiency of estimators of standardized mean difference, when variances are equal across groups and sample sizes are unequal (condition b)
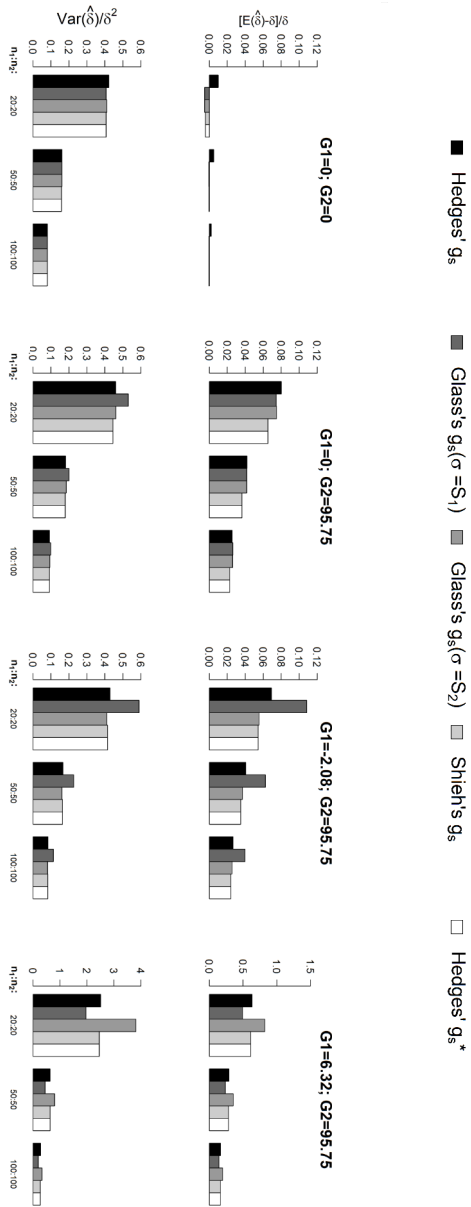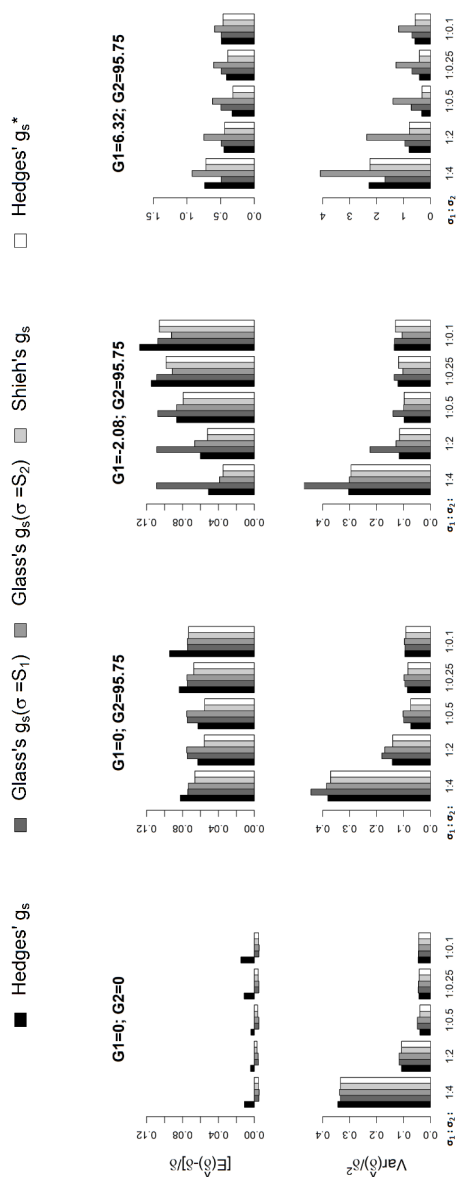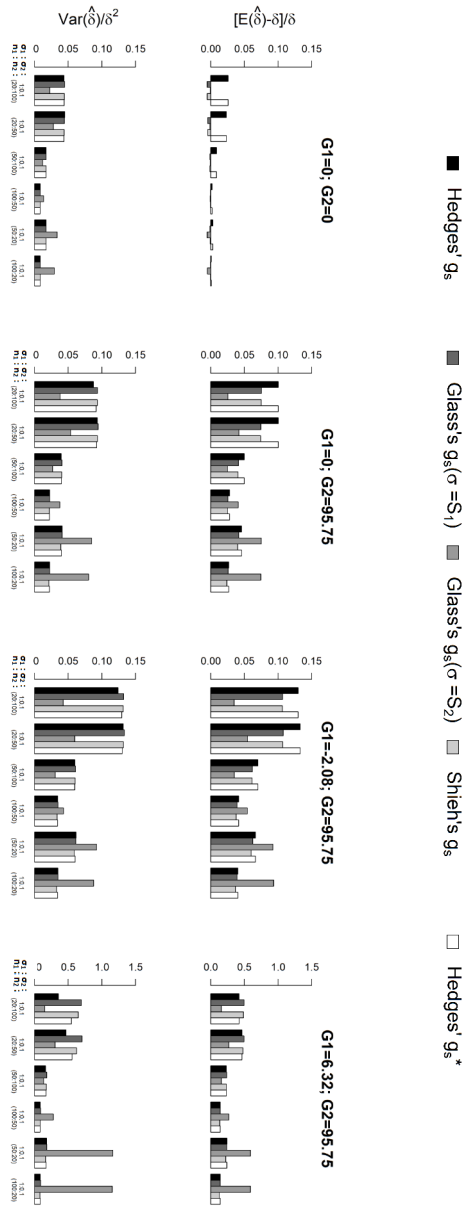
Figure 4. Bias and efficiency of estimators of standardized mean difference, when variances are unequal across groups and sample sizes are equal (condition c), as a function of n-ratio

Figure 5. Bias and efficiency of estimators of standardized mean difference, when variances are unequal across groups and sample sizes are equal (condition c) as a function of the *SD*-ratio (when $n_1 = n_2 = 20$)

Figure 6. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), $\frac{\sigma_1}{\sigma_2} = 10$ and $\sigma_1 > \sigma_2$
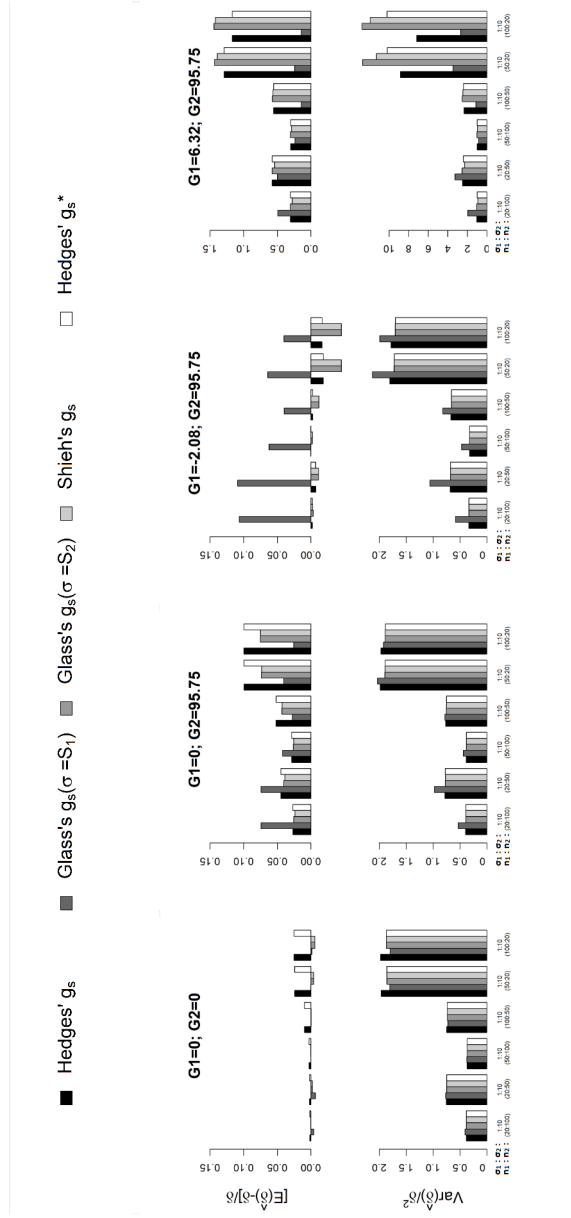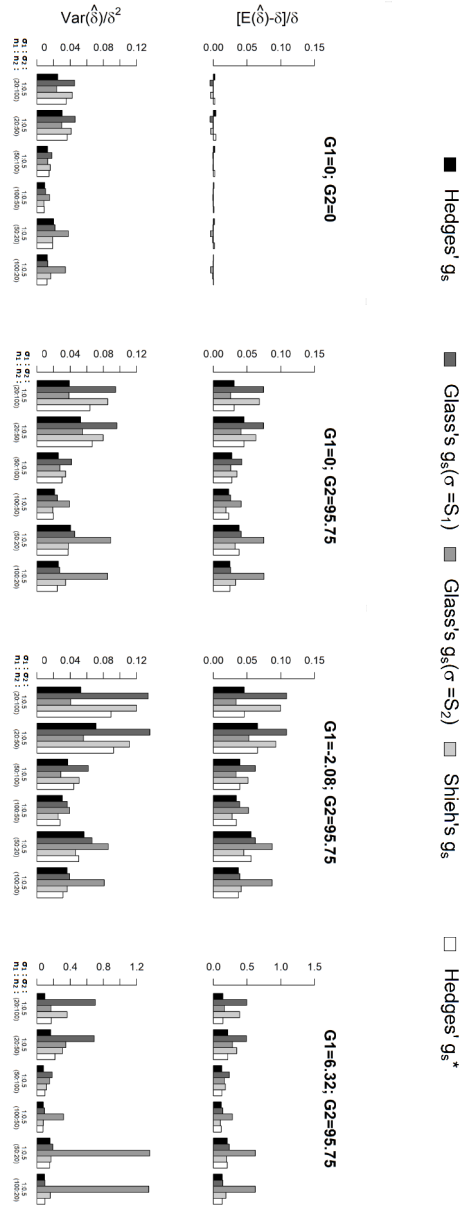
Figure 7. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), $\frac{\sigma_1}{\sigma_2} = 10$ and $\sigma_1 < \sigma_2$

Figure 8. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), $\frac{\sigma_1}{\sigma_2} = 2$ and $\sigma_1 > \sigma_2$

Figure 9. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), $\frac{\sigma_1}{\sigma_2} = 2$ and $\sigma_1 < \sigma_2$
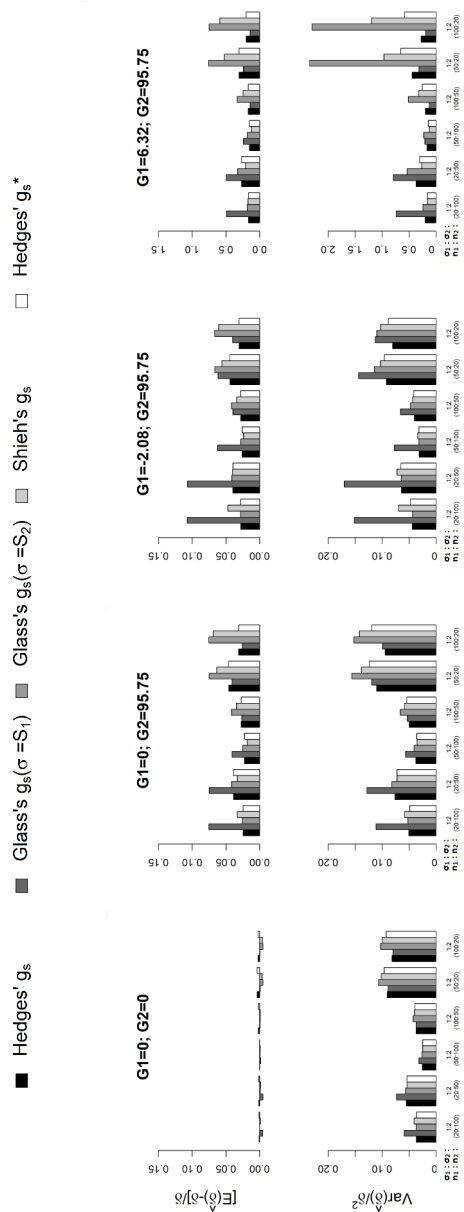
Appendix

583 **The bias of Cohen's d_s is twice as large as the bias of Shieh's d_s when**

584 **population variances and sample sizes are equal across groups: mathematical**

585 **demonstration.**

586   As mentioned in Table 1, the bias of Cohen's d_s is defined as

$$Bias_{Cohen's\ d_s} = \delta_{Cohen} \times \left(\frac{\sqrt{\frac{df_{Student}}{2}} \times \Gamma\left(\frac{df_{Student}-1}{2}\right)}{\Gamma\left(\frac{df_{Student}}{2}\right)} - 1 \right) \tag{16}$$

with

$$\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{(n_1-1)\times\sigma_1^2+(n_2-1)\times\sigma_2^2}{n_1+n_2-2}}}$$

and

$$df_{Student} = n_1 + n_2 - 2$$

587   As mentioned in Table 2, the bias of Shieh's d_s is defined as

$$Bias_{Shieh's\ d_s} = \delta_{Shieh} \times \left(\frac{\sqrt{\frac{df_{Welch}}{2}} \times \Gamma\left(\frac{df_{Welch}-1}{2}\right)}{\Gamma\left(\frac{df_{Welch}}{2}\right)} - 1 \right) \tag{17}$$

with

$$\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}} \quad (N = n_1 + n_2)$$

and

$$df_{Welch} = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}}$$

588   When $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$, δ_{Cohen} is twice larger than δ_{Shieh}, as shown below

589 in equations 18 and 19:

$$\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{2(n-1)\sigma^2}{2(n-1)}}} = \frac{\boldsymbol{\mu_1 - \mu_2}}{\boldsymbol{\sigma}} \tag{18}$$

590

$$\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{2\left(\frac{\sigma^2}{n/(2n)}\right)}} = \frac{\boldsymbol{\mu_1 - \mu_2}}{\boldsymbol{2\sigma}} \tag{19}$$

591 Moreover, degrees of freedom associated with Student's t-test and Welch's t-test are

592 identical, as shown below in equations 20 and 21:

$$df_{Student} = \mathbf{2(n-1)} \tag{20}$$

593

$$df_{Welch} = \frac{[2(\sigma^2/n)]^2}{\frac{2(\sigma^2/n)^2}{n-1}} = \mathbf{2(n-1)} \tag{21}$$

594 Equations 16 and 17 can therefore be redefined as follows:

$$Bias_{Cohen's\,d_s} = \frac{\mu_1 - \mu_2}{\sigma} \times \left(\frac{\sqrt{n-1} \times \Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n-1)} - 1\right) \tag{22}$$

595

$$Bias_{Shieh's\,d_s} = \frac{\mu_1 - \mu_2}{\mathbf{2\sigma}} \times \left(\frac{\sqrt{n-1} \times \Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n-1)} - 1\right) \tag{23}$$

596 We can therefore conclude that the bias of Cohen's d_s is twice larger than the bias of

597 Shieh's d_s.

598 **The variance of Cohen's d_s is four times larger than the bias of Shieh's d_s when**

599 **population variances and sample sizes are equal across groups: mathematical**

600 **demonstration.**

601   The variance of Cohen's d_s is defined in Table 1 as

$$Var_{Cohen's\ d_s} = \frac{N \times df_{Student}}{n_1 n_2 \times (df_{Student} - 2)} + \delta^2_{Cohen} \left[\frac{df_{Student}}{df_{Student} - 2} - \left(\frac{\sqrt{\frac{df_{Student}}{2}} \times \Gamma\left(\frac{df_{Student} - 1}{2}\right)}{\Gamma\left(\frac{df_{Student}}{2}\right)} \right)^2 \right]$$
$$(24)$$

602 and the variance of Shieh's d_s is defined in Table 2 as

$$Var_{Shieh's\ d_s} = \frac{df_{Welch}}{(df_{Welch} - 2)N} + \delta^2_{Shieh} \left[\frac{df_{Welch}}{df_{Welch} - 2} - \left(\frac{\sqrt{\frac{df_{Welch}}{2}} \times \Gamma\left(\frac{df_{Welch} - 1}{2}\right)}{\Gamma\left(\frac{df_{Welch}}{2}\right)} \right)^2 \right] \quad (25)$$

603   We have previously shown in equations 20 and 21 that degrees of freedom associated

604 with Student's t-test and Welch's t-test equal $2(n-1)$, when $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$.

605 As a consequence, the first term of the addition in equation 24 is 4 times larger than the first

606 term of the addition in equation 25:

$$\frac{N \times df_{Student}}{n_1 n_2 \times (df_{Student} - 2)} = \frac{2n \times 2(n-1)}{n^2 \times (2n-4)} = \frac{\mathbf{4(n-1)}}{\mathbf{n(2n-4)}}$$

607

$$\frac{df_{Welch}}{(df_{Welch} - 2)N} = \frac{2(n-1)}{2n(2n-4)} = \frac{\mathbf{n-1}}{\mathbf{n(2n-4)}}$$

608 We have also previously shown in equations 18 and 19 that δ_{Cohen} is twice larger than δ_{Shieh}

609 when $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$ and, therefore, δ^2_{Cohen} is four times larger than δ^2_{Shieh}. As

610 a consequence, the second term of the addition in equation 24 is also 4 times larger than the

611 second term of the addition in equation 25. Because both terms of the addition in equation

612 24 are four times larger than those in equation 25, we can conclude that the variance of

613 Cohen's d_s is four times larger than the variance of Shieh's d_s.