Reminder about Confidence Intervals

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Abstract

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Introduction: How to compute a confidence interval around $\mu_1 - \mu_2$. Considering the link between confidences intervals and NHST approach, we can think of confidence limits as the most extreme values of $\mu_1 - \mu_2$ that we could define as null hypothesis and that would not lead to rejecting the null hypothesis (**???**) (i.e that would be associated with a p-value that exactly equals $\frac{alpha}{2}$).

Under the assumption of iid normal distributions of residuals with equal variances across groups, in order to test the null hypothesis that $\mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$, we can compute the following quantity:

$$t_{Student} = \frac{(\bar{X_1} - \bar{X_2}) - (\mu_1 - \mu_2)_0}{SE} \tag{1}$$

With $SE = \sigma_{pooled} \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ and $\sigma_{pooled} = \sqrt{\frac{(n_1-1)*S_1^2+(n_2-1)*S_2^2}{n_1+n_2-2}}$.

Under the null hypothesis, this quantity will follow a central t- distribution with $n_1 + n_2 - 2$ degrees of freedom (see Figure 1) [1]. We can therefore easily define $(\mu_1 - \mu_2)_L$, the lower limit of the confidence interval, such as $\frac{(\bar{X_1}-\bar{X_2})-(\mu_1-\mu_2)_L}{SE}$ exactly equals the quantile (1-$\frac{\alpha}{2}$) of the central t-distribution of the null hypothesis $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_L$, and the upper limit $(\mu_1 - \mu_2)_U$ such as $\frac{(\bar{X_1}-\bar{X_2})-(\mu_1-\mu_2)_U}{SE}$ exactly equals the quantile $\frac{\alpha}{2}$ of the central t-distribution of the null hypothesis $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_U$:

$$Pr[t_{n_1+n_2-2} \geq \frac{(\bar{X_1} - \bar{X_2}) - (\mu_1 - \mu_2)_L}{SE}] = \frac{\alpha}{2} \tag{2}$$

[1] Distribution is central because under the null hypothesis, the quantity is a (supposed normal) centered variable, divided by SE, an independant variable closely related with the χ^2

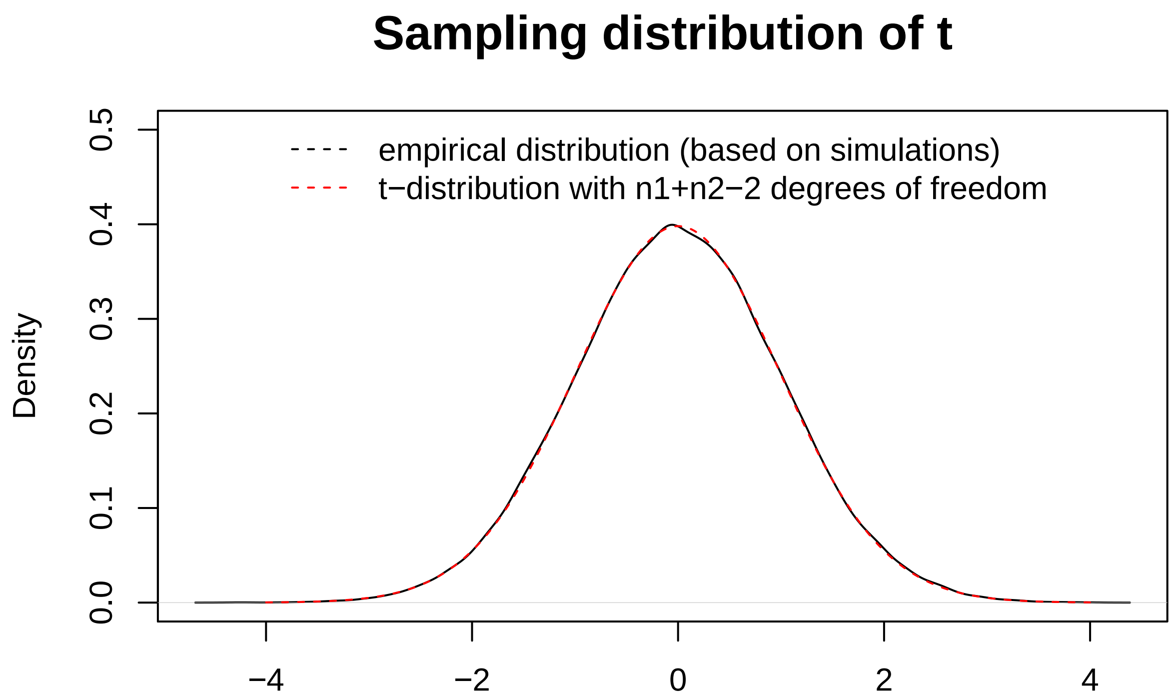

Figure 1. Sampling distribution of Student's t under the assumptions of normality and homoscedasticity

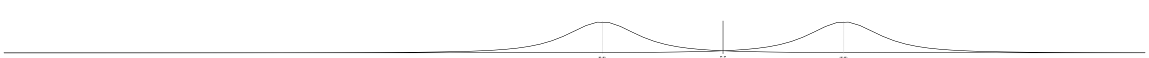

Figure 2. Sampling distribution. Both curves have the shape of a t distribution (df=n1+n2-2). Left (right) curve is placed so that the proportion of estimates larger (or lower) than the empirical mean difference exactly equals alpha/2

$$Pr[t_{n_1+n_2-2} \leq \frac{(\bar{X_1} - \bar{X_2}) - (\mu_1 - \mu_2)_U}{SE}] = \frac{\alpha}{2} \tag{3}$$

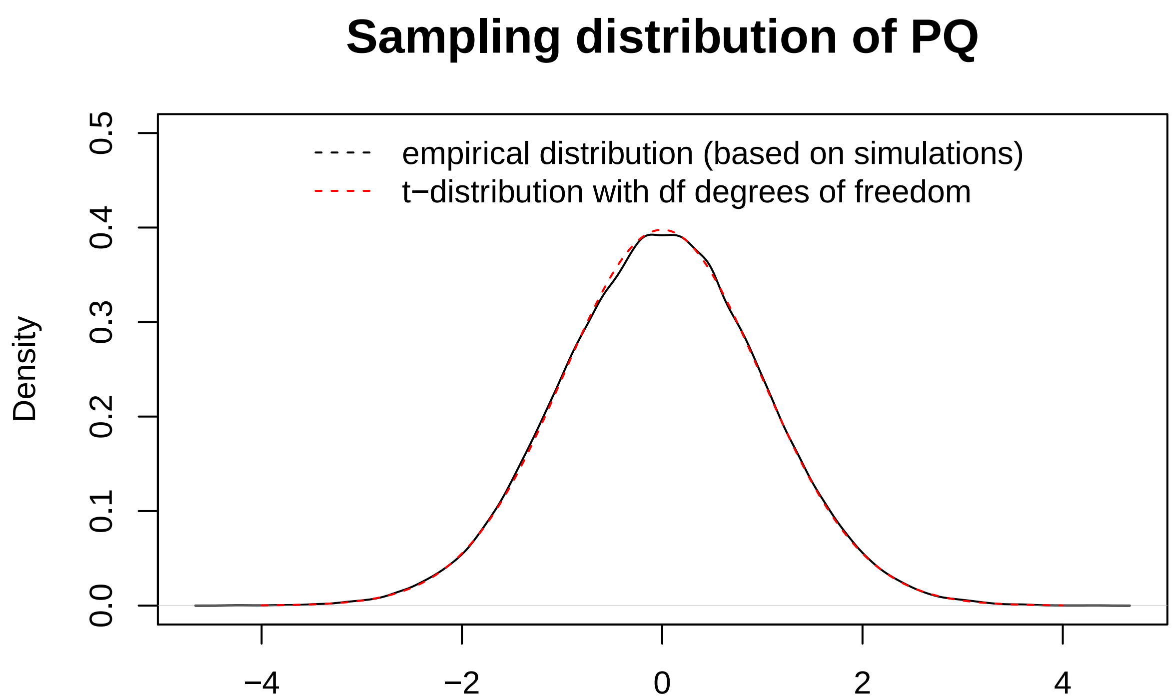

Figure 3. Sampling distribution of Welch's t under the assumptions of normality and heteroscedasticity

Under the assumption of iid normal distributions of residuals with unequal variances across groups, in order to test the null hypothesis that $\mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$, we can compute the following quantity:

$$t_{Welch} = \frac{(\bar{X_1} - \bar{X_2}) - (\mu_1 - \mu_2)_0}{SE} \tag{4}$$

With $SE = \sqrt{\frac{S_1^2}{n1} + \frac{S_2^2}{n2}}$. Again, under the null hypothesis, we know that this quantity will follow a central t- distribution with $DF = \frac{(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2})^2}{\frac{(\frac{S_1^2}{n_1})^2}{n_1-1}+\frac{(\frac{S_2^2}{n_2})^2}{n_2-1}}$ degrees of freedom (see Figure 3). We can therefore easily define $(\mu_1 - \mu_2)_L$ such as $\frac{(\bar{X_1}-\bar{X_2})-(\mu_1-\mu_2)_L}{SE}$ exactly equals the quantile (1-$\frac{\alpha}{2}$) of the central t-distribution of the null hypothesis $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_L$, and the upper limit $(\mu_1 - \mu_2)_U$ such as $\frac{(\bar{X_1}-\bar{X_2})-(\mu_1-\mu_2)_U}{SE}$ exactly equals the quantile $\frac{\alpha}{2}$ of the central t-distribution of the null hypothesis $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_U$:

$$Pr[t_{DF} \geq \frac{(\bar{X_1} - \bar{X_2}) - (\mu_1 - \mu_2)_L}{SE}] = \frac{\alpha}{2} \tag{5}$$

$$Pr[t_{DF} \leq \frac{(\bar{X_1} - \bar{X_2}) - (\mu_1 - \mu_2)_U}{SE}] = \frac{\alpha}{2} \tag{6}$$

It is not the most conventional way of computing confidences limits around any mean differences, however this approach is interesting as it helps to understand how to compute confidence limits around a measure of effect size.

How to compute a confidence interval around Cohen's δ. We previously mentioned that if the null hypothesis is true,$t_{Student}$ (see equation (1)) will follow a central t-distribution. However, if the null hypothesis is false, the distribution of this quantity will not be centered, and noncentral t-distribution will arise (**???**), as illustrated in Figure 4.

Noncentral t-distributions are described by two parameters: degrees of freedom (df) and noncentrality parameter (that we will call Δ; **???**), the last being a function of δ and sample sizes n_1 and n_2:

$$\Delta = \frac{\mu_1 - \mu_2}{\sigma_{pooled}} \times \sqrt{\frac{n_1 \times n_2}{n_1 + n_2}} \tag{7}$$

Considering the link between Δ and δ, it is possible to compute confidence limits for Δ, and divide them by $\sqrt{\frac{n_1 \times n_2}{n_1+n_2}}$ in order to have confidence limits for δ. In other word, we first

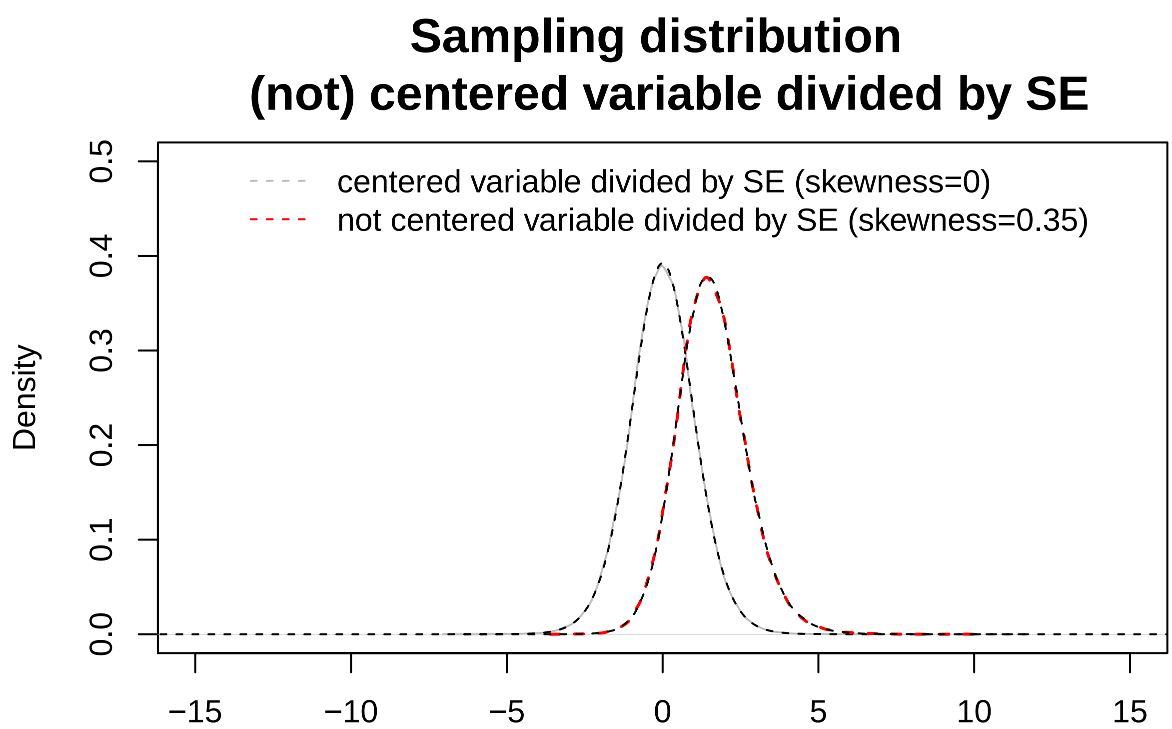

Figure 4. Sampling distribution of centered mean difference divided by SE (in grey, i.e. pivotal quantity) and not centered mean difference divided by SE (in red), assuming normality and homoscedasticity.

need to determine the noncentrality parameters of the t-distributions for which $t_{Student}$ corresponds respectively to the $1 - \frac{\alpha}{2}$ and to the $\frac{\alpha}{2}$ th. quantile:

$$P[t_{df,\Delta_L} \geq t_{Student}] = \frac{\alpha}{2}$$

$$P[t_{df,\Delta_U} \leq t_{Student}] = \frac{\alpha}{2}$$

With $df = n_1 + n_2 - 2$. Second, we divide Δ_L and Δ_U by $\sqrt{\frac{n_1 \times n_2}{n_1+n_2}}$ in order to define δ_L and δ_U:

$$\delta_L = \frac{\Delta_L}{\sqrt{\frac{n_1 \times n_2}{n_1 + n_2}}}$$

$$\delta_U = \frac{\Delta_U}{\sqrt{\frac{n_1 \times n_2}{n_1 + n_2}}}$$

How to determine the confidence interval around Shieh's $\delta*$

Like $t_{Student}$, t_{Welch} (see equation (4)) will follow a central t-distribution only if the null hypothesis is true. If the null hypothesis is false, it will follow a noncentral t-distribution, as illustrated in Figure 5.

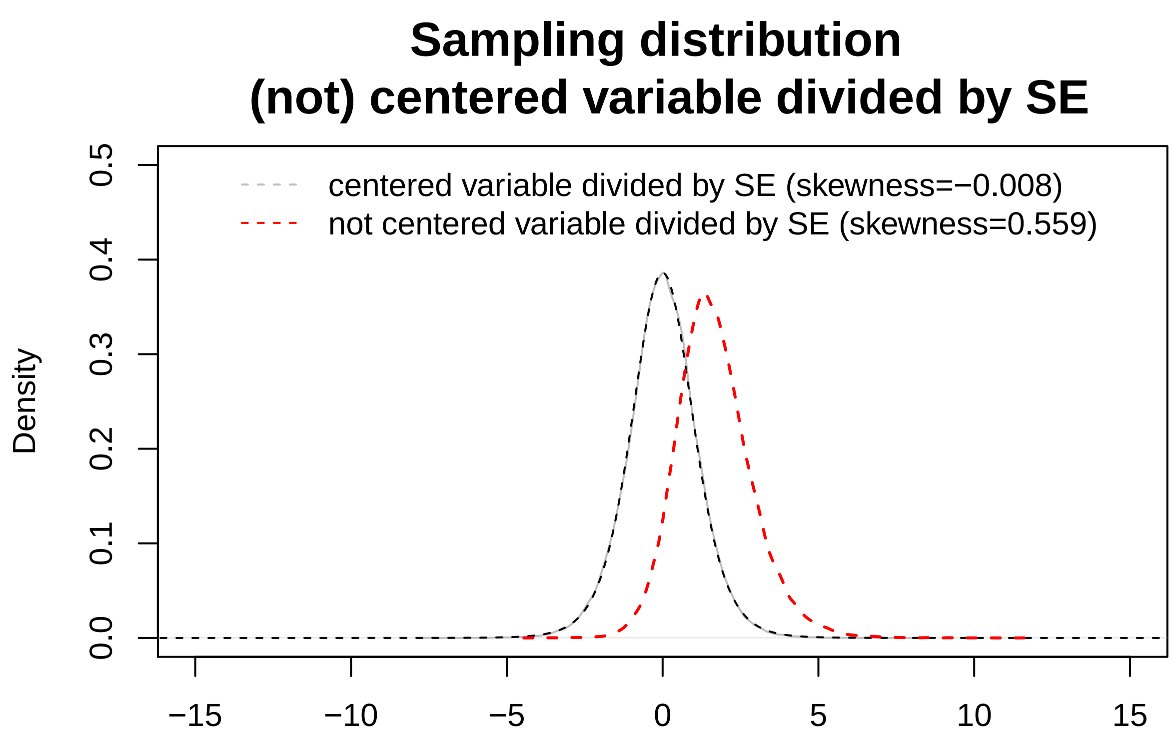

Figure 5. Sampling distribution of centered mean difference divided by SE (in grey, i.e. pivotal quantity) and not centered mean difference divided by SE (in red), assuming normality and homoscedasticity.

The noncentrality parameter $\Delta*$ is a function of $\delta*$ and total sample size $N = n_1 + n_2$ (**???**)

$$\Delta* = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}} \times \sqrt{N} \tag{8}$$

Considering the link between Δ and δ, we can compute confidence limits for $\Delta*$, and divide them by $\sqrt{N}$ in order to have confidence limits for $\delta*$. We first need to determine the noncentrality parameters of the distributions for which t_{Welch} corresponds respectively to the $1 - \frac{\alpha}{2}$ and to the $\frac{\alpha}{2}$ th. quantile.

$$P[t_{v,\Delta*_L} \geq t_{Welch}] = \frac{\alpha}{2}$$

and

$$P[t_{v,\Delta*_U} \leq t_{Welch}] = \frac{\alpha}{2}$$

.

With v approximated by $\hat{v} = \frac{(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2})^2}{\frac{(\frac{S_1^2}{n_1})^2}{n_1 - 1} + \frac{(\frac{S_2^2}{n_2})^2}{n_2 - 1}}$ (**???**)

Second, we divide $\Delta*_L$ and $\Delta*_U$ by $\sqrt{N}$ in order to have $\delta*_L$ and $\delta*_U$ (i.e. confidences limits for Shieh's $\delta*$).