

Argument to use the Hedges' g^* with the Welch's t -test : an investigation of Cohen's d and related effect size estimators in terms of interpretability, bias, precision and robustness

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Argument to use the Hedges' g^* with the Welch's t -test : an investigation of Cohen's d and related effect size estimators in terms of interpretability, bias, precision and robustness

Effect sizes are an important outcome of empirical research. Moving beyond decisions about statistical significance, there is a strong call for researchers to report and interpret effect sizes and associated confidence intervals. This practice is highly endorsed by the American Psychological Association (APA) and the American Educational Research Association (American Psychological Association, 2010; Duran et al., 2006).

In “between-subject” designs where individuals are randomly assigned into one of two independent groups and group scores are compared based on their means, the dominant estimator of effect size is Cohen's d , where the sample mean difference is divided by the pooled sample standard deviation (Peng et al., 2013; Shieh, 2013). This estimator is available in many statistical software packages, such as SPSS and Stata. However, computing the pooled sample standard deviation assumes that both sample variances are estimates of a common population variance, which is known as the homogeneity of variance assumption. It has been widely argued that there are many fields in psychology where this assumption is ecologically unlikely (Delacre et al., 2017; Erceg-Hurn & Miroseovich, 2008; Grissom, 2000). The question how to deal with the assumption of equal variances has been widely explored in the context of hypothesis testing, and it is becoming increasingly common to by default report a t -test that does not assume equal variances, such as Welch's t -test.

However, the question which effect size to report when equal variances are not assumed has received less attention. One possible reason is that researchers have not found consensus on which of the available options should be used (Shieh, 2013). Even within the very specific context of an estimate for the standardized sample mean difference there is little agreement about which estimator is the best choice. In this article, we will review the main candidates that have been proposed in the literature in the d family of effect sizes,

without (Cohen's d , Glass' d , Shieh's d and Cohen's d^*) and with correction for bias (Hedges' g , Glass' g , Shieh's g and Hedges' g^*). We provide an R package and Shiny app to compute relevant effect size measures and their confidence intervals.

Before reviewing the most important effect size measures in the d -family, we will first list the different purposes effect size measures serve, and discuss the relationship between effect sizes, statistical, and practical significance. Based on a detailed description of the good properties an effect size measure should possess, we will evaluate these properties in the Monte Carlo simulations we performed to compare the different effect size estimators with correction for bias.

Three purposes of effect size estimators

The effect size is a measure of the magnitude of an effect. In the context of the comparison of two groups based on their means, when the null hypothesis is the absence of effect, d -family effect size estimators estimate the magnitude of the differences between parameters of two populations groups are extracted from (e.g. the mean; Peng & Chen, 2014). Such a measure can be used for three different purposes.

First, effect size measures can be used for *interpretative* purposes. They allow researchers to assess the practical significance of a result (i.e. statements about the relevance of an effect in real life). In order to assess the meaningfulness of an effect, we should be able to relate this effect size estimate with behaviors/meaningful consequences in the real world (Andersen et al., 2007). This typically involves an analysis of the costs (determined by a specific context) and the benefits (in part determined by the size of the effect). It is important to remember an effect size is just a mathematical indicator of the magnitude of a difference, which depends on the way a variable is converted into numerical indicator. An effect size in itself is not a measure of the importance or the relevance of an effect for real life (even if benchmarks for small, medium, or large effect sizes might have

contributed to such a misinterpretation; Stout & Ruble, 1995).

Second, effect size measures can be used for *comparative* purposes. They allow researchers to assess the stability of results across designs, analyses, and sample sizes. This includes statistically comparing and combining the results from two or more studies in a meta-analysis.

Third, effect size measures can be used for *inferential* purposes. Hypothesis tests and confidence intervals based on the same statistical quantity are directly related : if the area of the null hypothesis is out of the $(1 - \alpha)$ -confidence interval, then the hypothesis test would also result in a p -value below the nominal alpha level. At the same time, the interval provides extra information about the precision of the sample estimate for inferential purposes (Altman, 2005; Ellis, 2010), and which effect sizes are excluded. The narrower the interval, the higher the precision, and the wider the confidence interval, the more the data lack precision. Effect size measures are also indirectly related to the hypothesis tests as effect sizes from previous studies can be used in an a-priori power analysis when planning a new study (Lakens, 2013; Prentice & Miller, 1992; Stout & Ruble, 1995; Sullivan & Feinn, 2012; Wilkinson, 1999).

Inferential properties of a good effect size estimator

The empirical value of an estimator (called the *estimate*) depends on the sample value. Different samples extracted from the same population will lead to different sample estimates of the population value. The *sampling distribution* of the estimator is the distribution of all estimates, based on all possible samples of size n extracted from one population. Studying the sampling distribution is useful, as it allows us to assess the qualities of an estimator. More specifically, three desirable properties a good estimator should possess for inferential purposes are : *unbiasedness*, *consistency* and *efficiency* (Wackerly et al., 2008).

An estimator is unbiased if the distribution of estimates is centered around the true population parameter. On the other hand, an estimator is positively (or negatively) biased if the distribution is centered around a value that is higher (or lower) than the true population parameter (see Figure 1). In other words, examining the bias of an estimator tells us if estimates are on average accurate. The *bias* of a point estimator $\hat{\delta}$ can be computed as

$$\delta_{bias} = E(\hat{\delta}) - \delta \quad (1)$$

where $E(\hat{\delta})$ is the expectation of the sampling distribution of the estimator and δ is the true (population) parameter.

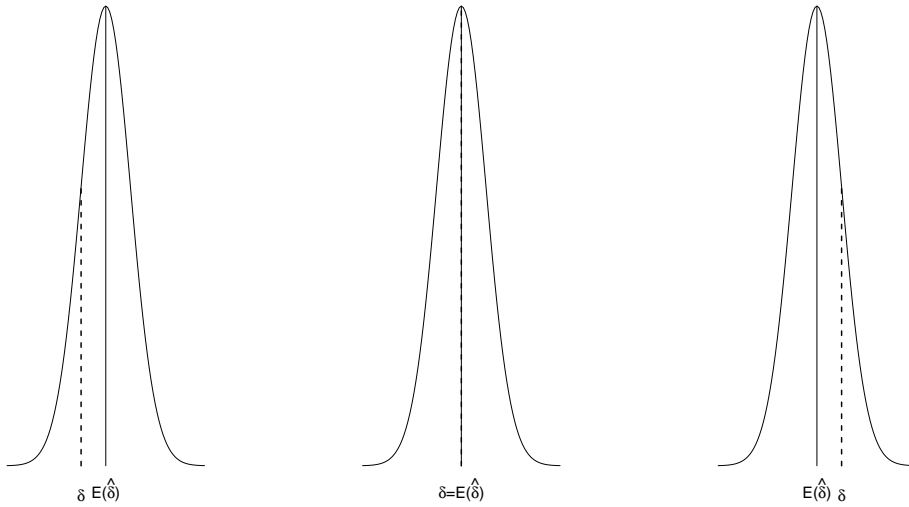


Figure 1. Sampling distribution for a positively biased (left), an unbiased (center) and a negatively biased estimator (right)

As we can see in Tables 1 and 2 the bias is directly related to the population effect size. The larger the population effect size, the larger the bias. It is therefore also interesting to examine the *relative bias*, defined as the ratio between the bias and the population effect size :

$$\delta_{relative\ bias} = \frac{E(\hat{\delta}) - \delta}{\delta}. \quad (2)$$

While the bias informs us about the quality of estimates on average, in particular their capacity of lying close to the true value, it says nothing about individual estimates. Imagine a situation where the distribution of estimates is centered around the real parameter but with such a large variance that some point estimates are very far from the center. This would be problematic, since any single estimate might be very far from the true population value. Therefore it is not only essential for an estimator to be unbiased, but it is also desirable that the variability of its sampling distribution is small. Ideally, sample estimates are close to the true population parameter. Among two unbiased estimators $\hat{\delta}_1$ and $\hat{\delta}_2$, we therefore say that $\hat{\delta}_1$ is *more efficient* than $\hat{\delta}_2$ if

$$Var(\hat{\delta}_1) \leq Var(\hat{\delta}_2) \quad (3)$$

where $Var(\hat{\delta})$ is the variance of the sampling distribution of the estimator $\hat{\delta}$. Among all unbiased estimators, the more efficient estimator will be the one with the smallest variance.¹ The variance of an estimator $\hat{\delta}$ is a function of its size (the larger the estimator, the larger the variance) and, therefore, we might be interested in evaluating the *relative variance* as the ratio between the variance and the square of the population estimator :

$$relative\ var(\hat{\delta}) = \frac{Var(\hat{\delta})}{\delta^2}. \quad (4)$$

Note that both unbiasedness and efficiency are very important when choosing an estimator. In some situations, it might be better to have a slightly biased estimator with low variance, (so that each estimate remains relatively close to the true parameter and one might be able to apply bias correction techniques) rather than an unbiased estimator with a large variance (Raviv, 2014).

Finally, the last property of a good point estimator is *consistency*. Consistency means that the bigger the sample size, the closer the estimate is to the population parameter. In other words, the estimates *converge* to the true population parameter.

¹ The Cramér-Rao inequality provides a theoretical lower bound for the variance of unbiased estimators. An estimator reaching this bound is therefore optimally efficient.

Different measures of effect sizes

The d -family effect sizes are commonly used for mean differences between groups or conditions. The population effect size is defined as

$$\delta = \frac{\mu_1 - \mu_2}{\sigma} \quad (5)$$

where both populations follow a normal distribution with mean μ_j in the j^{th} population ($j = 1, 2$) and standard deviation σ . There exist different estimators of this effect size measure. For all, the mean difference is estimated by the difference $\bar{X}_1 - \bar{X}_2$ of both sample means. When the equality of variances assumption is assumed, σ is estimated by pooling both sample standard deviations (S_1 and S_2). When the equality of variances assumption cannot be assumed, alternatives to the pooled standard deviation are available. In the next section we will present effect sizes that assume equal variances between groups (Cohen's d and Hedges' g), and effect sizes that do not assume equal variances (Glass' d , Shieh's d , Cohen's d^* , Glass' g , Shieh's g , and Hedges' g^*). For each effect size we will provide information about their theoretical bias, variance, and consistency.

When variances are equal between groups

When we have good reasons to assume equality of variances between groups then the most common estimator of δ is Cohen's d , where the sample mean difference is divided by a pooled error term (Cohen, 1965) :

$$Cohen's\ d = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{(n_1-1) \times S_1^2 + (n_2-1) \times S_2^2}{n_1 + n_2 - 2}}}$$

where S_j is the standard deviation, and n_j the sample size of the j^{th} sample ($j = 1, 2$). The reasoning behind this measure is to make use of the fact that both samples share the same population variance (Keselman et al., 2008), which means a more accurate estimation of the population variance can be achieved by pooling both estimates of this parameter

(i.e. S_1 and S_2). Since the larger the sample size, the more accurate the estimate, we give more weight to the estimate based on the larger sample size. Cohen's d is directly related to Student's t -statistic :

$$t_{Student} = \frac{Cohen's\ d}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \leftrightarrow Cohen's\ d = t_{Student} \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad (6)$$

Under the assumption of normality and equal variances between groups, Student's t -statistic follows a t -distribution with known degrees of freedom

$$df_{Student} = n_1 + n_2 - 2 \quad (7)$$

and noncentrality parameter²

$$ncp_{Student} = \frac{\delta_{Cohen}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where $\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sigma_{pooled}}$, $\sigma_{pooled} = \sqrt{\frac{(n_1 - 1) \times \sigma_1^2 + (n_2 - 1) \times \sigma_2^2}{n_1 + n_2 - 2}}$ and σ_j is the standard deviation of the j^{th} population ($j = 1, 2$). The relationship described in equation 6 and the theoretical distribution of Student's t -statistic allow us to determine the sampling distribution of Cohen's d , and therefore, its expectation and variance when the assumptions of normality and equal variances are met. All these equations are provided in Table 1. For interested readers, Supplemental Material 1 provides a detailed examination of the theoretical bias and variance of Cohen's d based on Table 1, as well as the bias and variance of all other estimators described later, based on Tables 2 and 3, with the goal to determine which parameters influence the bias and variance of different estimators. The main results will be discussed in the section "Monte Carlo Simulations : assessing the bias, efficiency and consistency of 5 estimators" below.

While Cohen's d is a consistent estimator, its bias and variance are substantial with small sample sizes, even under the assumptions of normality and equal variances (Lakens,

² Under the null hypothesis of no differences between sample means, Student's t -statistic will follow a central t -distribution with $n_1 + n_2 - 2$ degrees of freedom. However, when the null hypothesis is false, the distribution of this quantity will not be centered, and a noncentral t -distribution will arise.

2013). In order to compensate for Cohen's d bias with small sample sizes, Hedges and Olkin (1985) defined a bias-corrected version :

$$Hedges' g = Cohen's d \times \frac{\Gamma(\frac{df_{Student}}{2})}{\sqrt{\frac{df_{Student}}{2}} \times \Gamma(\frac{df_{Student}-1}{2})}$$

where $df_{Student}$ has been defined in equation 7 and $\Gamma()$ is the gamma function (for integers, $\Gamma(x)$ is the factorial of x minus 1 : $\Gamma(x) = (x - 1)!$; Goulet-Pelletier & Cousineau, 2018).

This equation can be approximated as follows :

$$Hedges' g = Cohen's d \times \left(1 - \frac{3}{4N - 9}\right)$$

163 where N is the total sample size. Hedges' g is theoretically unbiased when the assumptions
164 of normality and equal variances are met (see Table 1). Moreover, while the variance of
165 both Cohen's d and Hedges' g depend on the same parameters (i.e. the total sample size
166 (N) and the sample sizes ratio $\left(\frac{n_2}{n_1}\right)$, Hedges' g is less variable, especially with small
167 sample sizes.³

168 While the pooled error term is the best choice when variances are equal between
169 groups (Grissom & Kim, 2001), it may not be well advised for use with data that violate
170 this assumption (Cumming, 2013; Grissom & Kim, 2001; Grissom & Kim, 2005; Kelley,
171 2005; Shieh, 2013). When variances are unequal between groups, the expression in
172 equation 7 is no longer valid because both groups do not share a common population
173 variance. If we pool the estimates of two unequal population variances, the estimator of
174 effect size will be smaller as it should be in case of positive pairing (i.e. the group with the
175 larger sample size is extracted from the population with the larger variance) and larger as
176 it should be in case of negative pairing (i.e. the group with the larger sample size is

³ In Table 1, one can see that the variance of Hedges' g equals the variance of Cohen's d , multiplied by $\left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}\right]^2$. This term is always less than 1 and tends to 1 when the sample sizes tend to infinity ($.52 \leq \left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}\right]^2 < 1$ for $3 \leq df < \infty$). As a consequence, the larger the total sample size, the smaller the difference between the variance of Cohen's d and Hedges' g .

177 extracted from the population with the smaller variance). Because the assumption of equal
178 variances across populations is rarely realistic in practice (Cain et al., 2017; Delacre et al.,
179 2017; Delacre et al., 2019; Erceg-Hurn & Mirosevich, 2008; Glass et al., 1972; Grissom,
180 2000; Micceri, 1989; Yuan et al., 2004), both Cohen's d and Hedges' g should be abandoned
181 in favor of an alternative robust to unequal population variances.

Table 1

Expectation, bias and variance of Cohen's d and Hedges' g under the assumptions that independent residuals are normally distributed with equal variances across groups.

	df	Expectation	Variance
Cohen's d	$N - 2$	$\delta_{Cohen} \times c_f$	$\frac{N \times df}{n_1 n_2 \times (df - 2)} + \delta_{Cohen}^2 \left[\frac{df}{df - 2} - c_f^2 \right]$ $\approx \frac{N \times df}{n_1 n_2 \times (df - 2)} + \delta_{Cohen}^2 \left[\frac{df}{df - 2} - \left(\frac{1}{1 - \frac{3}{4N - 9}} \right)^2 \right]$
Hedges' g	$N - 2$	δ_{Cohen}	$Var(Cohen's\ d_s) \times \left[\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})} \right]^2$ $\approx Var(Cohen's\ d_s) \times \left[1 - \frac{3}{4N - 9} \right]^2$

Note. $\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sigma_{pooled}}$ and $c_f = \frac{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}{\Gamma(\frac{df}{2})}$; Cohen's d is a biased estimator, because its expectation differs from the population effect size. Moreover, the larger the population estimator (δ_{Cohen}), the larger the bias. Indeed, the bias is the difference between the expectation and δ_{Cohen} : $\delta_{bias} = \delta_{Cohen} \times (c_f - 1)$. On the other hand, Hedges' g is an unbiased estimator, because its expectation equals δ_{Cohen} ; equations in this table require $df \geq 3$ (i.e. $N \geq 5$).

When variances are unequal between populations

In his review, Shieh (2013) mentions three options available in the literature to deal with the case of unequal variances : (A) the Glass' d , (B) the Shieh's d and (C) the Cohen's d^* .

Glass' d . When comparing one control group with one experimental group, Glass et al. (1981) recommend using the standard deviation of the control group as standardizer. This yields

$$Glass' d = \frac{\bar{X}_e - \bar{X}_c}{S_c}$$

where \bar{X}_e and \bar{X}_c are the sample means of the experimental and control groups, and S_c is the sample SD of the control group. One argument in favour of using S_c as standardizer is the fact that it is not affected by the experimental treatment. When it is easy to identify which group is the “control” one, it is therefore convenient to compare the effect size estimation of different designs studying the same effect (Cumming, 2013). However, defining this group is not always obvious (Coe, 2002). This could induce large ambiguity because depending on the chosen SD as standardizer, measures could be substantially different (Shieh, 2013). As we shall see later through our simulations, this estimator also does not present good inferential properties. The distribution of Glass' d is defined as in Algina et al. (2006) :

$$Glass' d \sim \sqrt{\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \times \sigma_c^2}} \times t_{df,ncp} \quad (8)$$

where n_c and n_e are the sample sizes of the control and experimental groups, and df and ncp are defined as follows :

$$df = n_c - 1 \quad (9)$$

$$ncp = \frac{\delta_{Glass}}{\sqrt{\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \times \sigma_c^2}}}$$

where $\delta_{Glass} = \frac{\mu_c - \mu_e}{\sigma_c}$ and μ_c and μ_e are respectively the mean of the populations control and experimental groups are extracted from. Thanks to equation 8, we can compute its

204 theoretical expectation and variance when the assumption of normality is met (see Table
205 2), and therefore determine which factors influence bias and variance, and how they do so
206 (see Supplemental Material 1).

Table 2

Expectation, bias and variance of Glass' d and Cohen's d^* and Shieh's d under the assumption that independent residuals are normally distributed.

	df	Expectation	Variance
Glass' d	$n_c - 1$	$\delta_{Glass} \times c_f$	$\frac{df}{df-2} \times \left(\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \sigma_c^2} \right) + \delta_{Glass}^2 \left(\frac{df}{df-2} - c_f^2 \right)$
Cohen's d^*	$\frac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$	$\delta_{Cohen}^* \times c_f$	$\frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2+\sigma_2^2} + (\delta_{Cohen}^*)^2 \left(\frac{df}{df-2} - c_f^2 \right)$
		$\approx \delta_{Cohen}^* \times \frac{4df-1}{4(df-1)}$	$\approx \frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2+\sigma_2^2} + (\delta_{Cohen}^*)^2 \left[\frac{df}{df-2} - \left(\frac{4}{4(df-1)} \right)^2 \right]$
Shieh's d	$\frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}}$	$\delta_{Shieh} \times c_f$	$\frac{df}{(df-2)N} + \delta_{Shieh}^2 \left(\frac{df}{df-2} - c_f^2 \right)$

Note. $c_f = \frac{\sqrt{\frac{df}{2}} \times \Gamma(\frac{df-1}{2})}{\Gamma(\frac{df}{2})}$; $\delta_{Glass} = \frac{\mu_c - \mu_e}{\sigma_c}$, $\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}}$ and $\delta_{Cohen}^* = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$; all estimators are biased estimators, because their expectations differ from the population effect size δ . Moreover, the larger the population estimator (δ), the larger the bias. Indeed, the bias is the difference between the expectation and δ : $\delta_{bias} = \delta \times (c_f - 1)$; equations in this table require $df \geq 3$ and at least 2 subjects per group.

Shieh's d . Kulinskaya and Staudte (2007) were the first to recommend the use of a standardizer that takes the sample sizes allocation ratios into account, in addition to the variance of both samples. Shieh (2013), following Kulinskaya and Staudte, proposed a modification of the exact SD of the sample mean difference :

$$Shieh's\ d = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{S_1^2/q_1 + S_2^2/q_2}}; \quad q_j = \frac{n_j}{N} (j = 1, 2)$$

211 where $N = n_1 + n_2$. Shieh's d is directly related with Welch's t -statistic :

$$Shieh's\ d = \frac{t_{Welch}}{\sqrt{N}} \leftrightarrow t_{welch} = Shieh's\ d \times \sqrt{N}. \quad (10)$$

212 The exact distribution of Welch's t -statistic is more complicated than the exact distribution
213 of Student's t -statistic, but it can be approximated, under the assumption of normality, by
214 a t -distribution with degrees of freedom and noncentrality parameters (Welch, 1938)

$$df_{Welch} = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}} \quad (11)$$

$$ncp_{Welch} = \delta_{Shieh} \times \sqrt{N} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

215 where $\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}}$. The relationship described in equation 10 and the theoretical
216 distribution of Welch's t -statistic allow us to approximate the sampling distribution of
217 Shieh's d . Based on the sampling distribution of Shieh's d , we can estimate its theoretical
218 expectation and variance under the assumption of normality (see Table 2), and thereby
219 determine which factors influence bias and variance, and how they do so (see Supplemental
220 Material 1).

221 As demonstrated in Appendices 1 and 2, when variances and sample sizes are equal
222 across groups, the biases and variances of Cohen's d and Shieh's d are identical except for
223 multiplication by a constant. The same is true for the estimators δ_{Cohen} and δ_{Shieh} :

$$\delta_{Cohen} = 2 \times \delta_{Shieh} \quad (\text{considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \quad (12)$$

$$Bias_{Cohen's d} = 2 \times Bias_{Shieh's d} \quad (\text{considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \quad (13)$$

$$Var_{Cohen's d} = 4 \times Var_{Shieh's d} \quad (\text{considering } \sigma_1 = \sigma_2 \text{ and } n_1 = n_2) \quad (14)$$

We can deduce from equations 12, 13 and 14 that relative to their respective population effect size, Cohen's d and Shieh's d are equally accurate. In other words, their relative bias and variance are identical. This is a good illustration of our motivation to favor relative bias and variance (previously defined in equations 2 and 4) over the most commonly used raw bias and variance (previously defined in equations 1 and 3).

When sample sizes are not equal, according to the statistical properties of Welch's statistic under heteroscedasticity, Shieh's d accounts for the allocation ratio of sample sizes to each condition. The lack of generality caused by taking this specificity of the design into account has led Cumming (2013) to question its usefulness in terms of interpretability : when the mean difference ($\bar{X}_1 - \bar{X}_2$), S_1 , and S_2 remain constant, Shieh's d will vary as a function of the sample sizes allocation ratio (unlike Cohen's d^* that we will define below). At the population level, δ_{Shieh} also depends on the sample sizes allocation ratio, as illustrated in the following shiny application : <https://effectsize.shinyapps.io/ShiehvsCohen/>.

Cohen's d^* . An effect size estimator based on the sample mean difference divided by the square root of the non pooled average of both variance estimates was suggested by Welch (1938). Here, we indicate the difference between Cohen's d (based on the pooled standard deviations) and Cohen's d^* with an asterisk. This yields :

$$Cohen's d^* = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{(S_1^2 + S_2^2)}{2}}}$$

where \bar{X}_j is the mean and S_j is the standard deviation of the j^{th} sample ($j = 1, 2$). We know the distribution of Cohen's d^* (Huynh, 1989) :

$$Cohen's d^* \sim \sqrt{\frac{2(n_2 \times \sigma_1^2 + n_1 \times \sigma_2^2)}{n_1 n_2 (\sigma_1^2 + \sigma_2^2)}} \times t_{df^*, ncp^*} \quad (15)$$

where df^* and ncp^* are defined as follows :

$$df^* = \frac{(n_1 - 1)(n_2 - 1)(\sigma_1^2 + \sigma_2^2)^2}{(n_2 - 1)\sigma_1^4 + (n_1 - 1)\sigma_2^4} \quad (16)$$

$$ncp^* = \delta_{Cohen}^* \times \sqrt{\frac{n_1 n_2 (\sigma_1^2 + \sigma_2^2)}{2(n_2 \sigma_1^2 + n_1 \sigma_2^2)}} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

where $\delta_{Cohen}^* = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}}$. Using equation 15 we can compute its theoretical expectation and variance when the assumption of normality is met (see Table 2), and therefore determine which factors influence bias and variance, and how they do so (see Supplemental Material 1). This estimator has been widely criticized, because it results in a variance term of an artificial population (i.e. since the variance term does not estimate the variance of one or the other group, the composite variance is an estimation of the variance of an artificial population; Grissom & Kim, 2001), and unless both sample sizes are equal, the variance term does not correspond to the variance of the mean difference (Shieh, 2013). However, we will show throughout the simulation section that this estimator exhibits very good inferential properties. Moreover, it has a constant value across sample sizes ratios, as shown in the Shiny App at <https://effectsize.shinyapps.io/ShiehvsCohen/>.

Glass' g , Shieh's g and Hedges' g^* . As for Cohen's d , a Hedges' correction can be applied in order to compensate for the bias of Glass' d , Shieh's d and Cohen's d^* with small sample sizes (see Table 2). This correction has the following general form :

$$g = d \times \frac{\Gamma(\frac{\nu}{2})}{\sqrt{\frac{\nu}{2}} \times \Gamma(\frac{\nu-1}{2})}$$

where the distinct values of ν are provided in equation 9 for Glass' g , in equation 11 for Shieh's g and in equation 16 for Hedges' g^* . The three corrected estimators are theoretically unbiased when the assumption of normality is met. Their variance is a function of the same parameters as their biased equivalent. However, due to the correction they have a smaller variance, especially with small sample sizes, as shown in Table 3. In summary :

- The variances of Hedges' g^* and Shieh's g depend on the total sample size (N), their respective population effect size (δ), and the interaction between the sample sizes ratio and the SD -ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$.
- The variance of Glass' g also depends on N , δ and $\frac{n_c}{n_e} \times \frac{\sigma_c}{\sigma_e}$. In addition, there is also a main effect of the SD -ratio $\left(\frac{\sigma_c}{\sigma_e}\right)$ on its variance.

How these parameters influence the variance of the estimators will be summarized and illustrated in Monte Carlo simulations below.

Table 3

Expectation, bias and variance of Glass' g , Hedges' g^* and Shieh's g under the assumption that independent residuals are normally distributed.

	df	Expectation	Variance
Glass' g	$n_c - 1$	δ_{glass}	$Var(Glass' d) \times \left(\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2} \times \Gamma(\frac{df-1}{2})}} \right)^2$
Hedges' g^*	$\frac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$	δ_{Cohen}^*	$Var(Cohen's d_s^*) \times \left(\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2} \times \Gamma(\frac{df-1}{2})}} \right)^2$
Shieh's g	$\approx \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}}$	δ_{Shieh}	$Var(Shieh's d) \times \left(\frac{\Gamma(\frac{df}{2})}{\sqrt{\frac{df}{2} \times \Gamma(\frac{df-1}{2})}} \right)^2$

267 Note. $c_f = \frac{\sqrt{\frac{df}{2} \times \Gamma(\frac{df-1}{2})}}{\Gamma(\frac{df}{2})}$; $\delta_{Glass} = \frac{\mu_c - \mu_e}{\sigma_c}$, $\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}}$ and $\delta_{Cohen}^* = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}}$; all estimators are unbiased

268 estimators, because their expectations equal the population effect size δ ; equations in this table require $df \geq 3$ and at least 2

269 subjects per group.

Monte Carlo Simulations

Assessing the bias, efficiency and consistency of 5 estimators.

Method.

We performed Monte Carlo simulations using R (version 3.5.0) to assess the bias, efficiency and consistency of Hedges' g , Glass' g (using respectively the sample SD of the first or second group as a standardizer), Hedges' g^* and Shieh's g .

A set of 100,000 datasets was generated for 1,008 scenarios as a function of different criteria. In 252 scenarios, samples were extracted from a normally distributed population (in order to ensure the reliability of our calculation method) and in 756 scenarios, samples were extracted from non normal population distributions. In order to assess the quality of estimators under realistic deviations from the normality assumption, we referred to the review of Cain et al. (2017). Cain et al. (2017) investigated 1,567 univariate distributions from 194 studies published by authors in Psychological Science (from January 2013 to June 2014) and the American Education Research Journal (from January 2010 to June 2014). For each distribution, they computed Fisher's skewness

$$G_1 = \frac{\sqrt{n(n-1)}}{n-2} \frac{m_3}{\sqrt{(m_2)^3}}$$

and kurtosis

$$G_2 = \frac{n-1}{(n-2)(n-3)} \times \left[(n+1) \left(\frac{m_4}{(m_2)^2} - 3 \right) + 6 \right]$$

where n is the sample size and m_2 , m_3 and m_4 are respectively the second, third and fourth centered moments. They found values of kurtosis from $G_2 = -2.20$ to 1,093.48. According to their suggestions, throughout our simulations, we kept constant the population kurtosis value at the 99th percentile of their distribution of kurtosis, i.e. $G_2=95.75$. Regarding skewness, we simulated population parameter values which correspond to the 1st and 99th percentile of their distribution of skewness, i.e. respectively $G_1 = -2.08$ and $G_1 = 6.32$. We also simulated samples extracted from population where $G_1 = 0$, in order to assess the

293 main effect of high kurtosis on the quality of estimators. All possible combinations of
294 population skewness and kurtosis and the number of scenarios for each combination are
295 summarized in Table 4. To provide the reader with an intuition about the four
296 distributions we consider, we have plotted them in Figure 2.

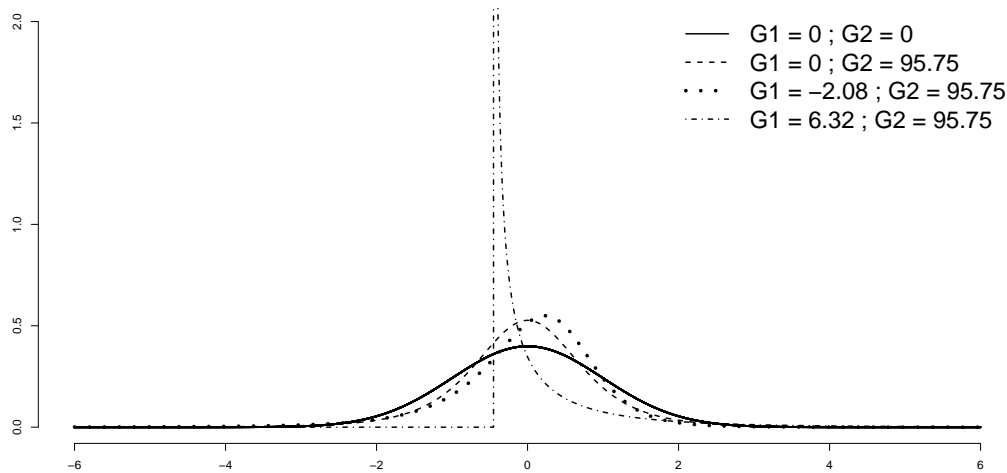


Figure 2. Plots of the four distributions considered in our simulations, with $\mu = 0$ and $\sigma = 1$. The undotted curve is the normal distribution.

Table 4

Number of combinations of skewness and kurtosis in our simulations.

		Kurtosis		
		0	95.75	TOTAL
Skewness	0	252	252	504
	-2.08	/	252	252
	6.32	/	252	252

Kurtosis			
TOTAL	252	756	1008

Note. Fisher's skewness (G1) and kurtosis (G2) are presented in Table 4. The combinations where both G1 and G2 equal 0 correspond to the normal case.

For the 4 resulting combinations of skewness and kurtosis (see Table 4), all other parameter values were chosen in order to illustrate the consequences of factors identified as playing a key role on the variance of unbiased estimators. We manipulated the population mean difference ($\mu_1 - \mu_2$), the sample sizes (n_1 and n_2), the sample size ratio (n -ratio = $\frac{n_2}{n_1}$), the population SD -ratio (i.e. $\frac{\sigma_2}{\sigma_1}$), and the sample size and population variance pairing ($\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}$). In our scenarios, μ_2 was always 0 and μ_1 varied from 1 to 4, in steps of 1 (so does $\mu_1 - \mu_2$). Thus, we only consider a positive population effect size, which will result in positive bias of our estimators for reasons explained in Supplemental Material 1.⁴ Moreover, σ_1 always equals 1, and σ_2 equals .1, .25, .5, 1, 2, 4 or 10, and therefore, the SD -ratio were 10, 4, 2, 1, .5, .25 or .1. The simulations for which both σ_1 and σ_2 equal 1 are the particular case of homoscedasticity, or equal population variances across groups. The sample sizes of both groups (n_1 and n_2) were 20, 50 or 100. When sample sizes of both groups are equal, the n -ratio equals 1 (this is known as a balanced design). All possible combinations of n -ratio and population SD -ratio were simulated in order to distinguish

⁴ In the original plan, we had added 252 simulations in which μ_1 and μ_2 were both null. We decided not to present the results of these simulations in the main article, because the relative bias and the relative variance appeared to us to be very useful to fully understand the comparison of the estimators, and computing them is impossible when the real mean difference is zero. Indeed, for these specific configurations, both relative bias and relative variance would have infinite values due to the presence of the population effect size term in their denominator. However, these extra simulations were included in the simulation checks, in Supplemental Material 2.

scenarios where both sample sizes and population variances are unequal across groups (with positive pairing when the group with the largest sample size is extracted from the population with the largest SD , and negative pairing when the group with the smallest sample size is extracted from the population with the smallest SD) and scenarios with no pairing between sample sizes and variances (sample sizes and/or population SD are equal across all groups). In sum, the simulations grouped over different sample sizes yield 4 conditions (a, b, c and d) based on the n -ratio, population SD -ratio, and sample size and population variance pairing, as summarized in Table 5. We chose to divide scenarios into these 4 conditions because analyses in Supplemental Material 1 revealed main and interaction effects of sample sizes ratio and SD -ratio on the bias and variance of some estimators.

Table 5

4 conditions based on the n -ratio and the SD -ratio.

		n -ratio		
		1	>1	<1
SD -ratio	1	a	b	b
	>1	c	d	d
	<1	c	d	d

Results.

Before presenting the comparison of the estimators for each condition, it is useful to make some general comments.

- 1) We previously discussed the fact that raw bias and variances are sometimes misleading. They can give the illusion of huge differences between two estimators, even if these differences only reflect a change of unit (i.e. different population effect sizes). To better understand this, imagine a sample of 15 people for whom we know the height (in meters) and we compute a sample variance of 0.06838. If we convert sizes to centimeters and compute the sample variance again, we find a measure of 683.8 (i.e. 10^4 larger). Both measures represent the same amount of variability, but they are expressed in different units. The same issue due to a change in scales occurs when comparing the estimates of different population measures. To avoid this possible confusion, we will only present the relative bias and relative variance in all Figures (and anytime we will mention the biases and variances in the results section, we will be referring to relative bias and variance). For interested readers, illustrations

of the raw bias and variance are available on Github at

<https://github.com/mdelacre/Effect-sizes/>.

2) For the sake of readability, the vertical axis differs across plots.

3) Throughout this section, we will *compare* the relative bias and variance of different estimators, but we do not present bias and variance in absolute terms. We chose very extreme (although realistic) conditions, and we know that none of the parametric measures of effect size will be robust against such extreme conditions. Our goal is therefore to study the robustness of the estimators against normality violations only in comparison with the robustness of other indicators, but not in absolute terms.

After these general remarks, we will analyze each condition separately. In all Figures presented below, for different sub-conditions, the averaged relative bias and relative variance of five estimators are presented. When describing the Glass' g estimators, we will systematically refer to the “control group” as the condition the standardizer is based on (i.e. the first group when using S_1 as standardizer, the second group when using S_2 as standardizer). The other condition will be referred to as the “experimental group.”

When variances are equal across groups

Figures 3 and 4 represent configurations where the equality of variances assumption is met. According to our expectations, one observes that the bias of all estimators is approximately zero as long as the normality assumption is met (first column in both Figures).⁵ However, the more the data generation process deviates from the normality assumption (i.e. when moving from left to right in the Figures), the larger the bias in the estimators.

We will observe that Glass' g should always be avoided when the equality of variance

⁵ When looking at relative bias for all estimators, the maximum departure from zero is 0.0064 when sample sizes are equal across groups, and 0.0065 with unequal sample sizes.

assumption is met. Hedges' g , Hedges' g^* and Shieh's g perform equally well as long as the sample size ratio is close to 1 (condition a; see Figure 3). However, when designs are highly unbalanced (condition b; see Figure 4), Shieh's g is not consistent anymore and while Hedges' g^* remains consistent, Hedges's g is a better estimator. For interested readers, these findings are detailed in the three paragraphs below.

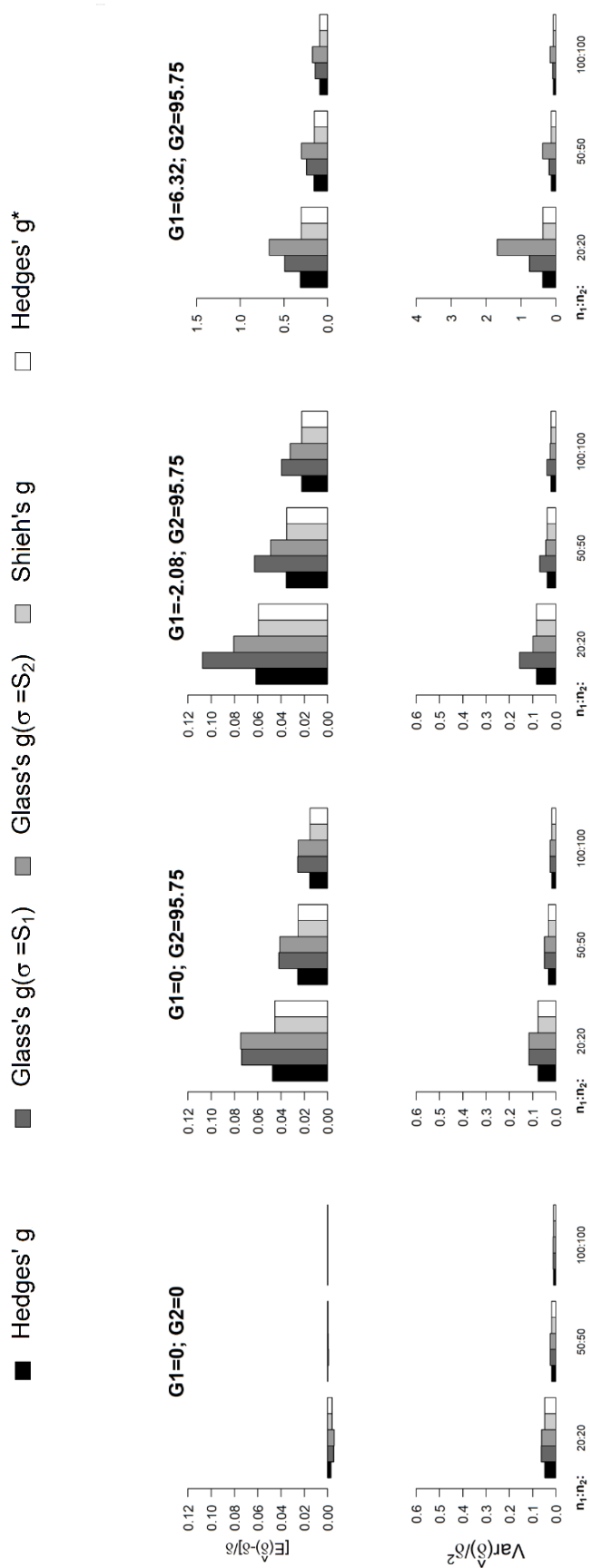


Figure 3. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are equal across groups (condition a)

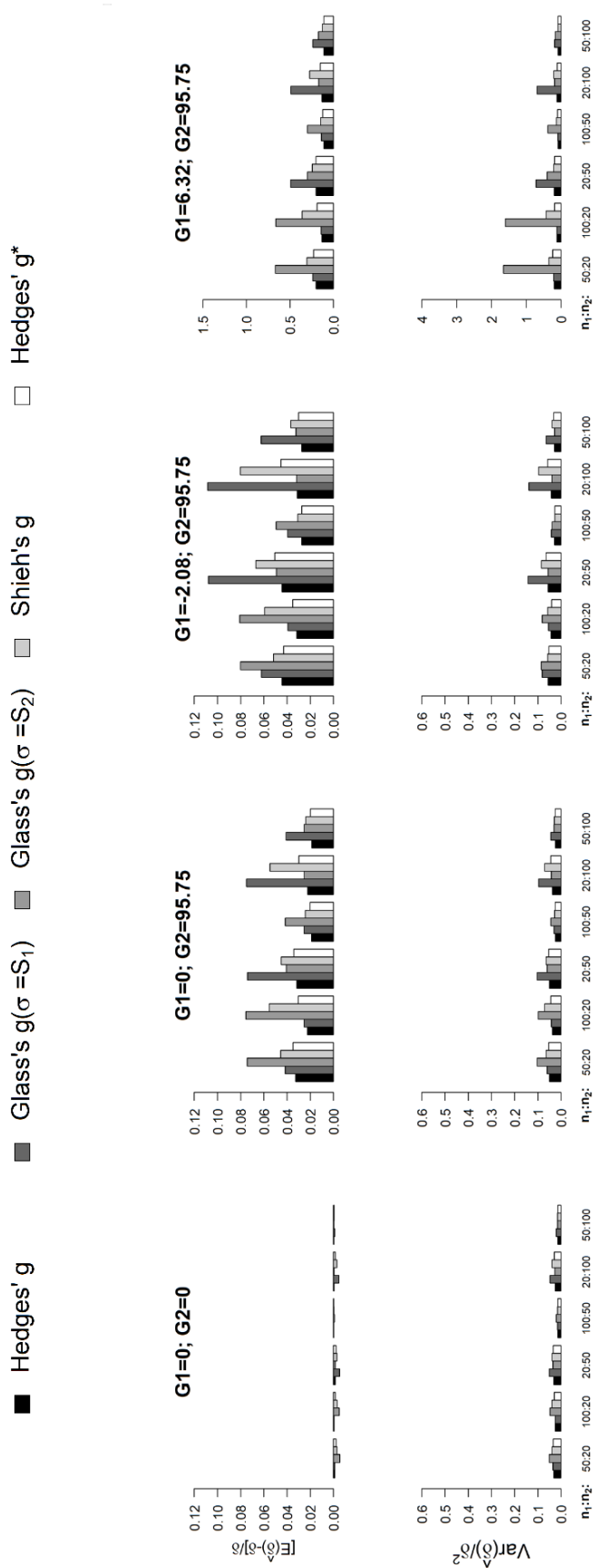


Figure 4. Bias and efficiency of estimators of standardized mean difference, when variances are equal across groups and sample sizes are unequal (condition b)

Figure 3 illustrates scenarios where both population variances and sample sizes are equal across groups (condition a). One can first notice that all estimators are consistent, as their bias and variance decrease when the total sample size increases. For any departure from the normality assumption, both bias and variance of Hedges' g , Shieh's g and Hedges' g^* are similar⁶ and smaller than the bias and variance of Glass' g estimates using either S_1 or S_2 as a standardizer. Moreover, when samples are extracted from skewed distributions, Glass' g will show different bias and variance as a function of the chosen standardizer (S_1 or S_2), even if both S_1 and S_2 are estimates of the same population variance, based on the same sample size. This is due to non-null correlations of opposite sign between the mean difference ($\bar{X}_1 - \bar{X}_2$) and respectively S_1 and S_2 . In Supplemental Material 3, we detailed in which situation a non-null correlation occurs between the sample mean difference ($\bar{X}_1 - \bar{X}_2$) and the standardizer of compared estimators as well as the way this correlation impacts the bias and variance of estimators.

Figure 4 illustrates scenarios where population variances are equal across groups, but sample sizes are unequal (condition b). For any departures from the normality assumptions, Hedges' g shows the smallest bias and variance. Hedges' g and Hedges' g^* are consistent estimators (i.e. the larger the sample sizes, the lower the bias and the variance), unlike Shieh's g and Glass' g . The bias of Glass' g does not depend either on the size of the experimental group or on the total sample size. The only way to decrease the bias of Glass' g is therefore to add subjects in the control group. On the other hand, the variance of Glass' g depends on both sample sizes, but not in an equivalent way : in order to reduce

⁶ While the bias and variance of Cohen's d , Cohen's d^* and Shieh's d are identical, the bias and variance of Hedges' g are marginally different from the bias and variance of Hedges' g^* and Shieh's g (these last two having identical bias and variance). Indeed, because of the sampling error, differences remain between sample variances, even when population variances are equal between groups. Since the Hedges' correction applied to Cohen's d does not contain the sample variances (unlike the correction applied on both other estimators), the bias and variance of Hedges' g are slightly different from the bias and variance of Hedges' g^* and Shieh's g .

the variance, it is much more efficient to add subjects in the control group and when the relative size of the experimental group decreases so does the variance, even when the total sample size is decreased. Regarding Shieh's g , for a given sample size ratio, the bias and variance will decrease when sample sizes increase. However, there is a large effect of the sample sizes ratio such that when the sample sizes ratio moves away from 1 by adding subjects, bias and variance might increase.⁷ On the other hand, when the sample sizes ratio moves closer to 1 by adding subjects, the bias will decrease.

When samples are extracted from skewed distributions and have unequal sizes (the two last columns in Figure 4), for a constant total sample size, Glass' g , Shieh's g and Hedges' g^* will show different bias and variance depending on which group is the largest one (e.g. when distributions are right-skewed, the bias and variance of all these estimators when n_1 and n_2 are respectively 50 and 20 are not the same as their bias and variance when n_1 and n_2 are respectively 20 and 50). This is due to a non-null correlation of opposite sign between the mean difference ($\bar{X}_1 - \bar{X}_2$) and their respective standardizers depending on which group is the largest one, as detailed in Supplemental Material 3. One observes that under these configurations, the bias and variance of Glass' g are sometimes a bit smaller and sometimes much larger than the bias and variance of Shieh's g and Cohen's d^* .⁸

⁷ Regarding variance, in Supplemental Material 1, we mentioned that when the population effect size is zero, the larger the total sample size, the lower the variance, whether the sample sizes ratio is constant or not. We also mentioned that this is no longer true when the population effect size is not zero. In our simulations the effect size is never zero. The effect size effect is partially visible in Figure 4 because we do not entirely remove the effect size effect when we divide the variance by δ^2 . This is due to the fact that one term, in the equation of the variance computation, does not depend on the effect size.

⁸ Supplemental Material 3 shows that when $\mu_1 - \mu_2 > 0$ (like in our simulations), all other parameters being equal, an estimator is always less biased and variable when choosing a standardizer that is positively correlated with $\bar{X}_1 - \bar{X}_2$. Supplemental Material 3 also shows that the smaller n_c , the larger the magnitude of correlation between S_c and $\bar{X}_1 - \bar{X}_2$. When $\text{cor}(S_c, \bar{X}_1 - \bar{X}_2)$ is positive, the positive effect of increasing the magnitude of the correlation is counterbalanced by the negative effect of reducing n_c . On the other

When variances are unequal across groups

Figures 5 to 10 represent configurations where the equality of variances assumption is not met. According to our expectations, one observes that the bias of all estimators is approximately zero as long as the normality assumption is met (first column in all Figures), and the further from the normality assumption (i.e. when moving from left to right in Figures), the larger the bias.⁹ It might be considered surprising that the bias of Hedges' g remains very small throughout these conditions. As discussed in the section "Different measures of effect size," Hedges' g should be avoided when population variances and sample sizes are unequal across groups, because of the pooled error term. When pooling the estimates of two unequal population variances, the resulting estimator will be smaller (in case of positive pairing) or larger (in case of negative pairing) than it should be. At the same time, when pooling two unequal population variances, the population effect size will also be smaller (in case of positive pairing) or larger (in case of negative pairing) as it should be. As a consequence, the distortion cannot be seen through the difference between the expected estimator and the population effect size measure. For this reason, the bias and variance of Hedges' g will not be taken into account in the following comparisons.

We will observe that when variances are unequal across populations, Glass' g sometimes performs better, but also sometimes performs much worse than Shieh's g and Hedges' g^* , both in terms of bias and variance. The performance of Glass' g highly depends on parameters that we cannot control (i.e. a triple interaction between the n -ratio, the SD -ratio and the correlation between the standardizer and the mean difference) and for this reason, we do not recommend using it. When the sample sizes ratio is close to 1,

hand, when $\text{cor}(S_c, \bar{X}_1 - \bar{X}_2)$ is negative, the negative effect of increasing the magnitude of the correlation is amplified by the negative effect of decreasing n_c . This explains why the difference between Glass' g and other estimators is larger when Glass' g is the least efficient estimator.

⁹ When looking at the relative bias for all estimators, the maximum departure from zero is 0.0173 when sample sizes are equal across groups, and 0.0274 when both sample sizes and variances differ across groups.

427 Shieh's g and Hedges' g^* are both appropriate but the further the sample sizes ratio is from
428 1, the larger the bias of Shieh's g in order that in the end, the measure that we believe
429 performs best across scenarios is Hedges' g^* .

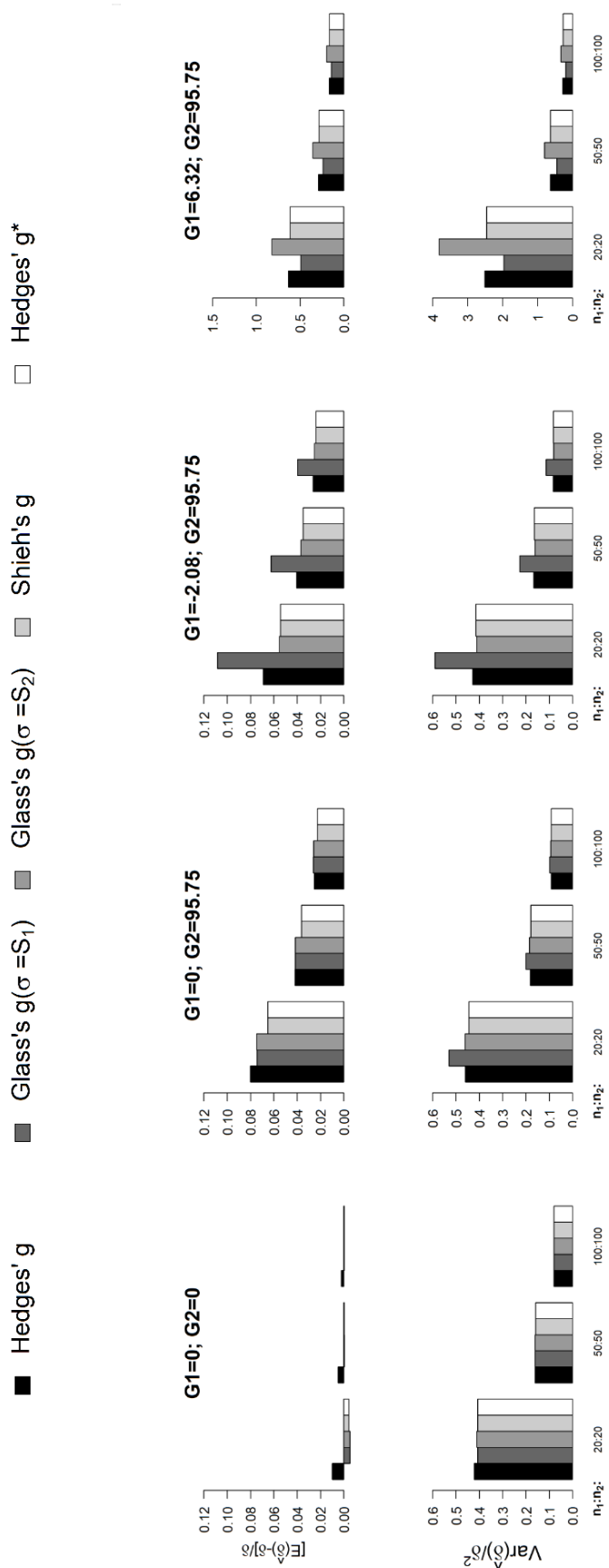


Figure 5. Bias and efficiency of estimators of standardized mean difference, when variances are unequal across groups and sample sizes are equal (condition c), as a function of sample sizes

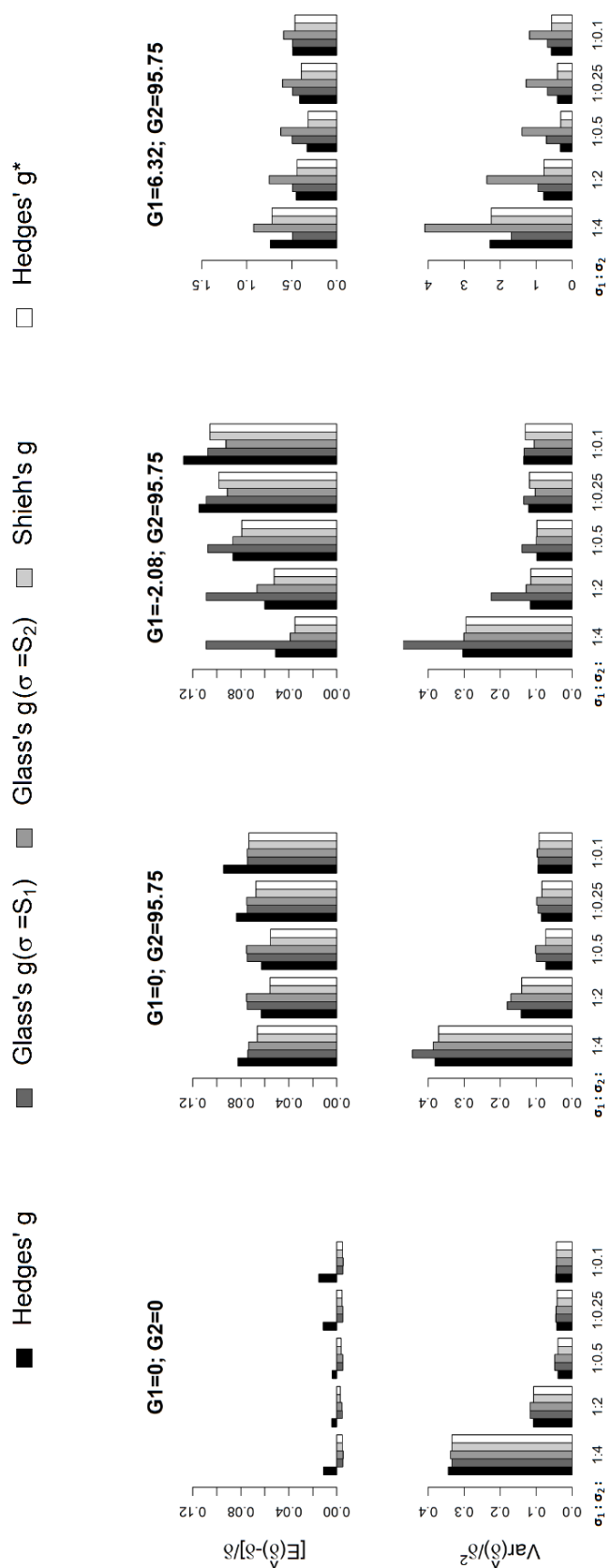


Figure 6. Bias and efficiency of estimators of standardized mean difference, when variances are unequal across groups and sample sizes are equal (condition c) as a function of the SD -ratio (when $n_1 = n_2 = 20$)

Figures 5 and 6 are dedicated to scenarios where population variances are unequal between groups and sample sizes are equal (condition c). In Figure 5, scenarios are subdivided as a function of the sample sizes and one can notice that all estimators are consistent, as their bias and variance decrease when the total sample size increases. In Figure 6, scenarios are subdivided as a function of the SD -ratio. Because the comparison pattern remains very similar for all sample sizes, we present only scenarios when sample sizes equal 20. One should first notice that for all estimators in Figure 6, the relative variance seems to be much larger when $S_2 > S_1$.¹⁰ This information should not be taken into account because it is only an artefact of our simulation conditions combined with the way we computed the relative variance.¹¹

When samples are extracted from skewed distributions, the bias and variance of Glass' g are sometimes smaller and sometimes larger than the bias of Shieh's g and Hedges' g^* . This is mainly due to the fact that when two samples of same sizes are extracted from two skewed distributions with unequal variances (the two last columns in Figure 6), there will be non-null correlations of opposite sign between the mean difference ($\bar{X}_1 - \bar{X}_2$) and the standardizer of *all* estimators, depending on which population variance is larger.¹²

Figures 7 to 10 are dedicated to scenarios where both sample sizes and population

¹⁰ The difference between the variance of estimators when the second group is 10 times larger than the first group was so large that we decided to not present it, for the sake of readability of the Figures.

¹¹ We previously mentioned that when dividing the variance by δ^2 , we do not entirely remove the effect size effect. Actually, we introduce δ^2 in the denominator of the first term, in the equation of the variance computation. Because we performed our simulations in order that σ_1 always equals 1, the smaller S_2 , the larger the population effect size and therefore, the smaller the relative variance.

¹² When population variances are unequal, a non-null correlation occurs between standardizer estimates and $\bar{X}_1 - \bar{X}_2$. For standardizers computed based on both S_1 and S_2 , the sign of the correlation between the standardizer and the mean difference will be the same as the sign of the correlation between the mean difference and the estimate of the larger population variance. For interested readers, this is detailed in Supplemental Material 3.

variances differ across groups. Due to a high number of combinations between the sample sizes ratio and the SD -ratio in our simulations, we decided to present only some conditions. Because equations in Table 3 revealed an interaction effect between the sample sizes ratio and the SD -ratio on the bias and variance of Hedges' g^* and Shieh's g (see Supplemental Material 1), we chose to present all configurations where the larger SD is 10 times larger than the smaller SD (Figures 7 and 8), and configurations where the larger SD is twice larger than the smaller SD (Figures 9 and 10), in order to compare the effect of the sample sizes ratio on the bias and variance of all estimators when the SD -ratio is large ($\frac{\sigma_2}{\sigma_1} = 10$ or $.1$) or medium ($\frac{\sigma_2}{\sigma_1} = 2$ or $.5$).

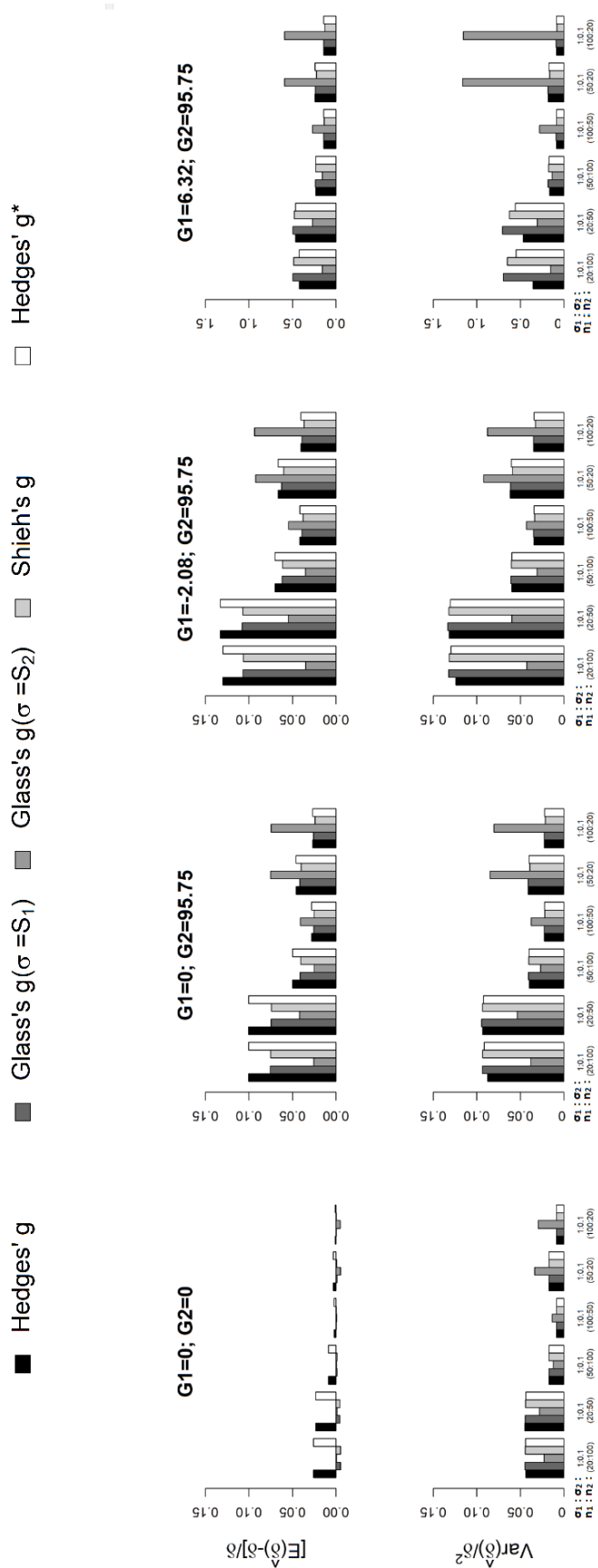


Figure 7. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_1 is 10 times larger than σ_2

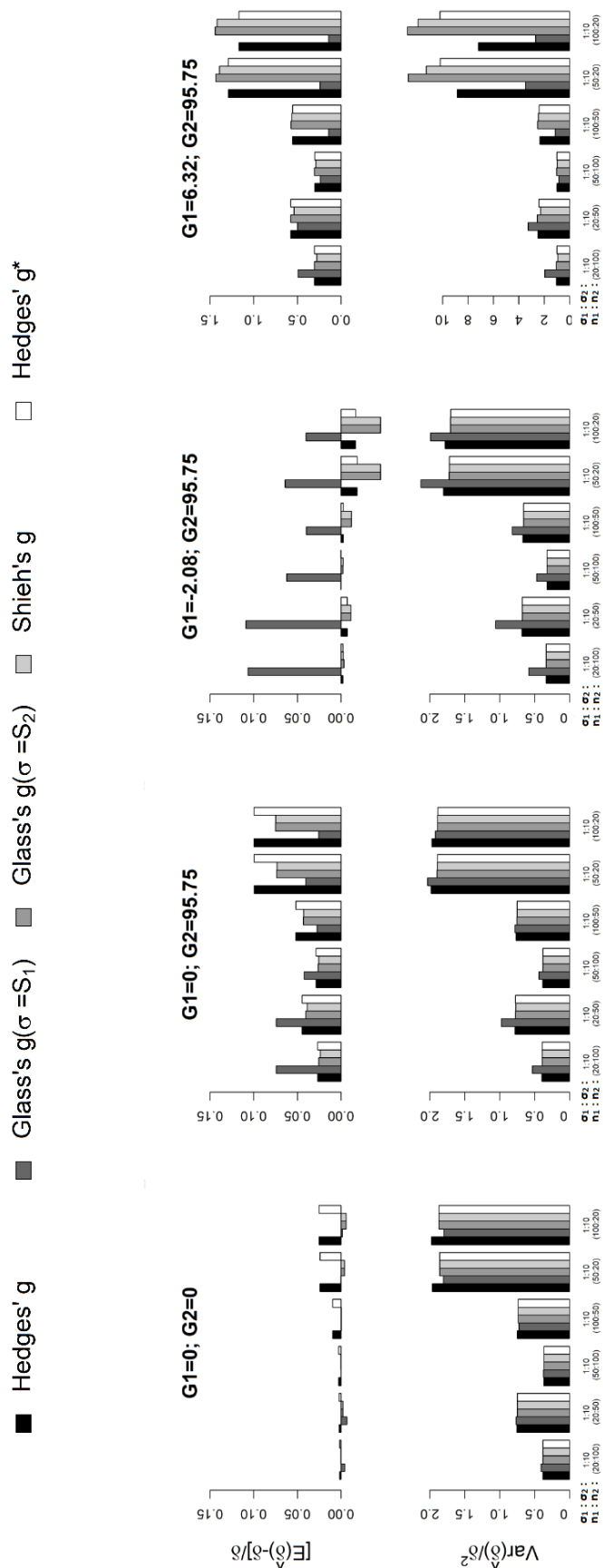


Figure 8. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_2 is 10 times larger than σ_1

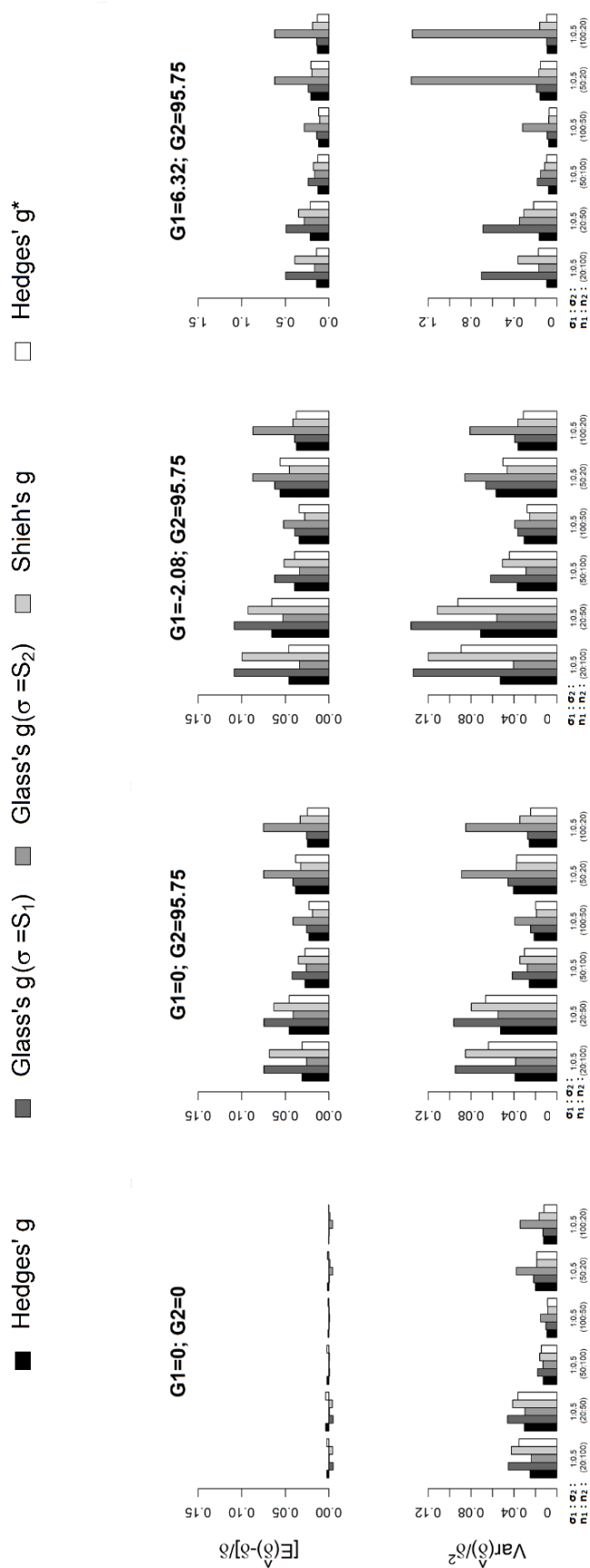


Figure 9. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_1 is twice larger than σ_2

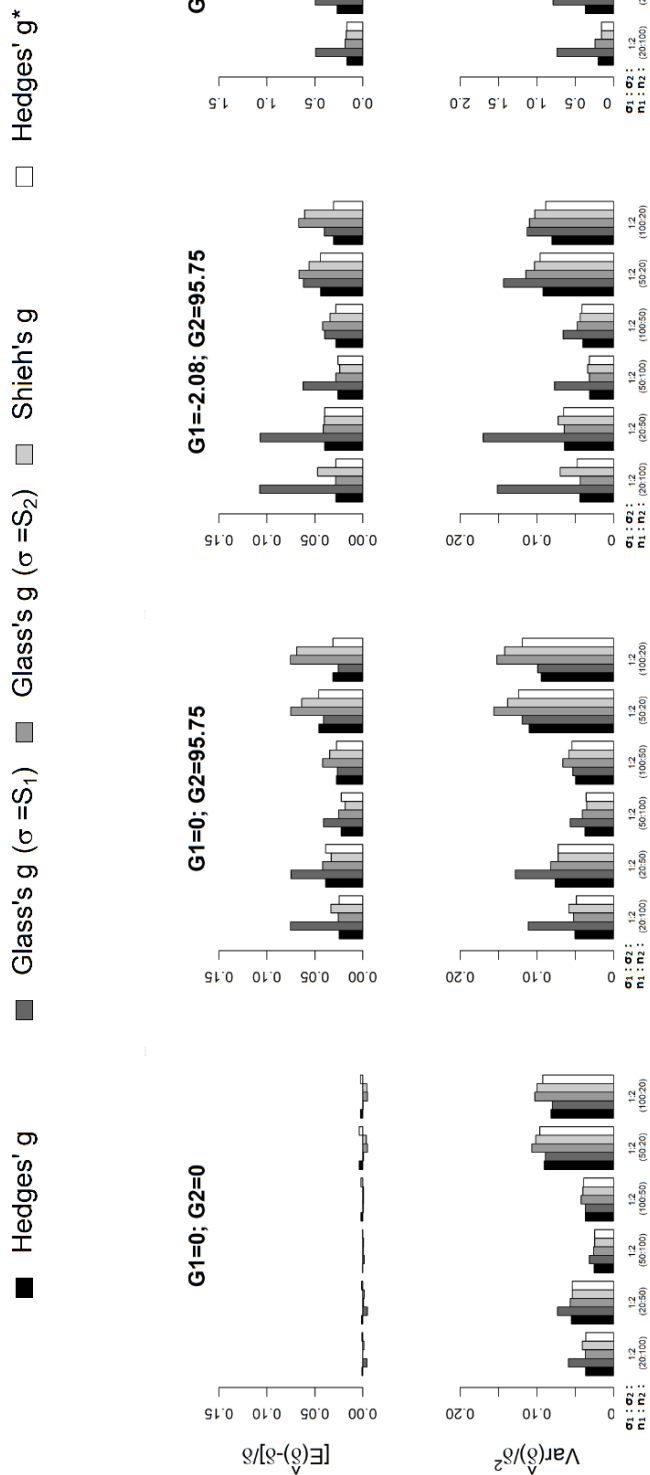


Figure 10. Bias and efficiency of estimators of standardized mean difference, when variances and sample sizes are unequal across groups (condition d), and σ_2 is twice larger than σ_1

When distributions are symmetric, the bias of Glass' g only depends on the size of the control group and is therefore not impacted by either the sample sizes ratio or the total sample size. When comparing Figures 7 to 10, one can also notice that the bias of Glass' g does not depend on the SD -ratio either. Unlike the bias of Glass' g , its variance depends on both sample sizes, but not in an equivalent way. In most scenarios it is more efficient, in order to reduce the variance of Glass' g , to add subjects in the control group. Regarding Hedges' g^* and Shieh's g , their respective biases and variances depend on an interaction effect between the sample sizes ratio and the SD -ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$: the sample sizes ratio associated with the smallest bias and variance is not the same when the more variable group is 10 times more variable than the other group (Figures 7 and 8) than when it is only twice more variable (Figures 9 and 10). However, the respective biases and variances of Hedges' g^* and Shieh's g are always smaller when there is a positive pairing between sample sizes and variances. When samples are extracted from skewed distributions, the bias and variance of Glass' g are sometimes smaller and sometimes larger than the bias of Shieh's g and Hedges' g^* , due to a combination of three factors: (1) which group is larger, (2) which group has the smallest standard deviation and (3) what is the correlation between the standardizer and the mean difference.

Recommendations

We recommend using Hedges' g^* in order to assess the magnitude of the effect when comparing two independent means, because a) it does not rely on the equality of population variances assumption (unlike Hedges' g), b) it is always consistent (unlike Shieh's g), c) it is easy to interpret (Hedges' g^* can be interpreted in the same way as Hedges' g) and d) it remains constant for any sample sizes ratio, even when population variances are unequal across groups, as shown in the Shiny App at <https://effectsize.shinyapps.io/ShiehvsCohen/>.

Effect sizes estimates such as Hedges' g^* should always be reported with a confidence interval. To help researchers compute Hedges' g^* and its confidence interval we created the

R package *deffectsize* (see <https://github.com/mdelacre/deffectsize>). The *datacohen_CI* function was built in order to compute point estimators and confidence intervals based on raw data and the *cohen_CI* function was built in order to compute point estimators and confidence intervals based on descriptive statistics (sample means, sample variances and sample sizes). By default, unbiased Hedges' g^* is computed but it is also possible to compute biased estimators (e.g. Cohen's d^*) and/or to use a pooled error term as standardizer by assuming that the equality of population variances is met (e.g. Hedges' g or Cohen's d , depending on whether we choose to compute unbiased or biased estimator). Other functions (*datashieh_CI*, *shieh_CI*, *dataglass_CI* and *glass_CI*) are available in order to compute Shieh's g (or Shieh's d) and Glass' g (or Glass' d) as well as their respective confidence intervals, even though we do not recommend to use these effect sizes by default. Researchers who do not use R can use a Shiny app to compute point estimators and confidence intervals based on descriptive statistics : <https://effectsize.shinyapps.io/deffsize/>.

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Appendix

546 The bias of Cohen's d is twice as large as the bias of Shieh's d when population
 547 variances and sample sizes are equal across groups: mathematical
 548 demonstration.

549 As mentioned in Table 1, the bias of Cohen's d is defined as

$$Bias_{Cohen's d} = \delta_{Cohen} \times \left(\frac{\sqrt{\frac{df_{Student}}{2}} \times \Gamma\left(\frac{df_{Student}-1}{2}\right)}{\Gamma\left(\frac{df_{Student}}{2}\right)} - 1 \right) \quad (17)$$

with

$$\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{(n_1-1)\times\sigma_1^2 + (n_2-1)\times\sigma_2^2}{n_1+n_2-2}}}$$

and

$$df_{Student} = n_1 + n_2 - 2$$

550 As mentioned in Table 2, the bias of Shieh's d is defined as

$$Bias_{Shieh's d} = \delta_{Shieh} \times \left(\frac{\sqrt{\frac{df_{Welch}}{2}} \times \Gamma\left(\frac{df_{Welch}-1}{2}\right)}{\Gamma\left(\frac{df_{Welch}}{2}\right)} - 1 \right) \quad (18)$$

with

$$\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1/N} + \frac{\sigma_2^2}{n_2/N}}} \quad (N = n_1 + n_2)$$

and

$$df_{Welch} = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1} + \frac{(\sigma_2^2/n_2)^2}{n_2-1}}$$

551 When $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$, δ_{Cohen} is twice larger than δ_{Shieh} , as shown
 552 below in equations 19 and 20:

$$\delta_{Cohen} = \frac{\mu_1 - \mu_2}{\sqrt{\frac{2(n-1)\sigma^2}{2(n-1)}}} = \frac{\mu_1 - \mu_2}{\sigma} \quad (19)$$

$$\delta_{Shieh} = \frac{\mu_1 - \mu_2}{\sqrt{2\left(\frac{\sigma^2}{n/(2n)}\right)}} = \frac{\mu_1 - \mu_2}{2\sigma} \quad (20)$$

Moreover, degrees of freedom associated with Student's t -test and Welch's t -test are identical, as shown below in equations 21 and 22:

$$df_{Student} = 2(n - 1) \quad (21)$$

$$df_{Welch} = \frac{[2(\sigma^2/n)]^2}{\frac{2(\sigma^2/n)^2}{n-1}} = 2(n - 1) \quad (22)$$

Equations 17 and 18 can therefore be redefined as follows:

$$Bias_{Cohen's\ d} = \frac{\mu_1 - \mu_2}{\sigma} \times \left(\frac{\sqrt{n-1} \times \Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n-1)} - 1 \right) \quad (23)$$

$$Bias_{Shieh's\ d} = \frac{\mu_1 - \mu_2}{2\sigma} \times \left(\frac{\sqrt{n-1} \times \Gamma\left(\frac{2n-3}{2}\right)}{\Gamma(n-1)} - 1 \right) \quad (24)$$

We can therefore conclude that the bias of Cohen's d is twice larger than the bias of Shieh's d .

The variance of Cohen's d is four times larger than the bias of Shieh's d when population variances and sample sizes are equal across groups: mathematical demonstration.

The variance of Cohen's d is defined in Table 1 as

$$Var_{Cohen's d} = \frac{N \times df_{Student}}{n_1 n_2 \times (df_{Student} - 2)} + \delta_{Cohen}^2 \left[\frac{df_{Student}}{df_{Student} - 2} - \left(\frac{\sqrt{\frac{df_{Student}}{2}} \times \Gamma\left(\frac{df_{Student}-1}{2}\right)}{\Gamma\left(\frac{df_{Student}}{2}\right)} \right)^2 \right] \quad (25)$$

and the variance of Shieh's d is defined in Table 2 as

$$Var_{Shieh's d} = \frac{df_{Welch}}{(df_{Welch} - 2)N} + \delta_{Shieh}^2 \left[\frac{df_{Welch}}{df_{Welch} - 2} - \left(\frac{\sqrt{\frac{df_{Welch}}{2}} \times \Gamma\left(\frac{df_{Welch}-1}{2}\right)}{\Gamma\left(\frac{df_{Welch}}{2}\right)} \right)^2 \right] \quad (26)$$

We have previously shown in equations 21 and 22 that degrees of freedom associated with Student's t -test and Welch's t -test equal $2(n-1)$, when $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$. As a consequence, the first term of the addition in equation 25 is 4 times larger than the first term of the addition in equation 26:

$$\frac{N \times df_{Student}}{n_1 n_2 \times (df_{Student} - 2)} = \frac{2n \times 2(n-1)}{n^2 \times (2n-4)} = \frac{4(n-1)}{n(2n-4)}$$

$$\frac{df_{Welch}}{(df_{Welch} - 2)N} = \frac{2(n-1)}{2n(2n-4)} = \frac{n-1}{n(2n-4)}$$

We have also previously shown in equations 19 and 20 that δ_{Cohen} is twice larger than δ_{Shieh} when $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$ and, therefore, δ_{Cohen}^2 is four times larger than δ_{Shieh}^2 . As a consequence, the second term of the addition in equation 25 is also 4 times larger than the second term of the addition in equation 26. Because both terms of the addition in equation 25 are four times larger than those in equation 26, we can conclude that the variance of Cohen's d is four times larger than the variance of Shieh's d .