

- 1 Correlations between the sample mean difference and standardizers of all estimators, and
  - 2 implications on biases and variances of all estimators

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5 **Introduction**

The  $d$ -family effect sizes are commonly used with between-subject designs where individuals are randomly assigned into one of two independent groups and group means are compared. The population effect size is defined as

$$\delta = \frac{\mu_1 - \mu_2}{\sigma}$$

where both populations follow a normal distribution with mean  $\mu_j$  in the  $j^{th}$  population ( $j=1,2$ ) and common standard deviation  $\sigma$ . There exist different estimators of this population effect size, varying as a function of the chosen standardizer. When the equality of variances assumption is met,  $\sigma$  is estimated by pooling both sample standard deviations ( $S_1$  and  $S_2$ ):

$$S_{Cohen's\ d_s} = \sqrt{\frac{(n_1 - 1) \times S_1^2 + (n_2 - 1) \times S_2^2}{n_1 + n_2 - 2}}$$

- 6 When the equality of variances assumption is not met, we are considering three  
 7 alternative estimates:

- Using the standard deviation of the control group ( $S_c$ ) as standardizer:

$$S_{Glass's\ d_s} = S_c$$

- 8 - Using a standardizer that takes the sample sizes allocation ratio  $\left(\frac{n_1}{n_2}\right)$  into account:

$$S_{Shieh's\ d_s} = \sqrt{S_1^2/q_1 + S_2^2/q_2}; \quad q_j = \frac{n_j}{N} (j = 1, 2)$$

- Or using the square root of the non pooled average of both variance estimates ( $S_1^2$  and  $S_2^2$ ) as standardizer:

$$S_{Cohen's\ d_s^*} = \sqrt{\frac{(S_1^2 + S_2^2)}{2}}$$

As we previously mentioned, the use of these formulas requires to meet the assumption of normality. Using them when distributions are not normal will have consequences on both bias and variance of all estimators. More specifically, when samples are extracted from skewed distributions, correlations might occur between the sample mean difference ( $\bar{X}_1 - \bar{X}_2$ ) and standardizers ( $S$ ). Throughout this Supplemental Material, we will study when these correlations occur. To this end, we will distinguish 3 situations:

- when  $\sigma_1 = \sigma_2$  and  $n_1 = n_2$  (condition a);
  - when  $\sigma_1 = \sigma_2$  and  $n_1 \neq n_2$  (condition b);
  - when  $\sigma_1 \neq \sigma_2$  and  $n_1 = n_2$  (condition c).

Before studying conditions a, b and c, we will briefly introduce the impact of these correlations on the bias. Note that we will compute correlations using the coefficient of Spearman's  $\rho$ . We decided to use Spearman's  $\rho$  instead of Pearson's  $\rho$  because some plots revealed non-perfectly linear relations.

22 How correlations between the mean difference ( $\bar{X}_1 - \bar{X}_2$ ) and standardizers  
23 affect the bias of estimators.

When population distributions are right-skewed, there is a positive (negative) correlation between  $S_1$  ( $S_2$ ) and  $(\bar{X}_1 - \bar{X}_2)$ . When distributions are left-skewed, there is a negative (positive) correlation between  $S_1$  ( $S_2$ ) and  $(\bar{X}_1 - \bar{X}_2)$ . When the population mean difference ( $\mu_1 - \mu_2$ ) is positive (like in our simulations), all other parameters being equal, an estimator is always less biased and variable when choosing a standardizer that is positively correlated with  $\bar{X}_1 - \bar{X}_2$  than when choosing an estimator that is negatively correlated with  $\bar{X}_1 - \bar{X}_2$ . When the population mean difference is negative, the reverse is true.

<sup>31</sup> “All other parameters being equal” is mentioned because it is always possible that  
<sup>32</sup> other factors in action have an opposite effect on bias and variance in order that increasing  
<sup>33</sup> the magnitude of the correlation between  $S_i$  and  $\bar{X}_1 - \bar{X}_2$  does not necessarily reduce the

<sup>34</sup> bias and the variance. For example, when population variances are equal across groups and  
<sup>35</sup> sample sizes are unequal, we will see below that the lower  $n_j$ , the larger the magnitude of  
<sup>36</sup> the correlation between  $S_j$  and  $\bar{X}_1 - \bar{X}_2$ . When the correlation between  $S_j$  and  $\bar{X}_1 - \bar{X}_2$  is  
<sup>37</sup> positive, the smaller the sample size, the larger the positive correlation. At the same time,  
<sup>38</sup> we know that increasing the sample size decreases the bias. This is a nice example of  
<sup>39</sup> situations where two factors might have an opposite action on bias.

#### <sup>40</sup> Correlations between the mean difference ( $\bar{X}_1 - \bar{X}_2$ ) and all standardizers

##### <sup>41</sup> When equal population variances are estimated based on equal sample sizes <sup>42</sup> (condition a)

<sup>43</sup> While  $\bar{X}_j$  and  $S_j$  ( $j=1,2$ ) are uncorrelated when samples are extracted from  
<sup>44</sup> symmetric distributions (see Figure 1), there is a non-null correlation between  $\bar{X}_j$  and  $S_j$   
<sup>45</sup> when distributions are skewed (Zhang, 2007).

<sup>46</sup> More specifically, when distributions are right-skewed, there is a **positive** correlation  
<sup>47</sup> between  $\bar{X}_j$  and  $S_j$  (see the two top plots in Figure 2), resulting in a *positive* correlation  
<sup>48</sup> between  $S_1$  and  $\bar{X}_1 - \bar{X}_2$  and in a *negative* correlation between  $S_2$  and  $\bar{X}_1 - \bar{X}_2$  (see the  
<sup>49</sup> two bottom plots in Figure 2). This can be explained by the fact that  $\bar{X}_1$  and  $\bar{X}_1 - \bar{X}_2$  are  
<sup>50</sup> positively correlated while  $\bar{X}_2$  and  $\bar{X}_1 - \bar{X}_2$  are negatively correlated (of course,  
<sup>51</sup> correlations would be trivially reversed if we computed  $\bar{X}_2 - \bar{X}_1$  instead of  $\bar{X}_1 - \bar{X}_2$ ).

<sup>52</sup> One should also notice that both correlations between  $S_j$  and  $\bar{X}_1 - \bar{X}_2$  are equal, in  
<sup>53</sup> absolute terms (possible tiny differences might be observed due to sampling error in our  
<sup>54</sup> simulations). As a consequence, when computing a standardizer taking both  $S_1$  and  $S_2$  into  
<sup>55</sup> account, it results in a standardizer that is uncorrelated with  $\bar{X}_1 - \bar{X}_2$  (see Figure 3).

<sup>56</sup> On the other hand, when distributions are left-skewed, there is a **negative**  
<sup>57</sup> correlation between  $\bar{X}_j$  and  $S_j$  (see the two top plots in Figure 4), resulting in a *negative*

58 correlation between  $S_1$  and  $\bar{X}_1 - \bar{X}_2$  and in a *positive* correlation between  $S_2$  and  $\bar{X}_1 - \bar{X}_2$   
 59 (see the two bottom plots in Figure 4).

60 Again, because correlations between  $S_j$  and  $\bar{X}_1 - \bar{X}_2$  are similar in absolute terms,  
 61 any standardizers taking both  $S_1$  and  $S_2$  into account will be uncorrelated with  $\bar{X}_1 - \bar{X}_2$   
 62 (see Figure 5).

63 **When equal population variances are estimated based on unequal sample sizes  
 64 (condition b)**

65 When distributions are skewed, there are again non-null correlations between  $\bar{X}_j$  and  
 66  $S_j$ , however  $\text{cor}(S_1, \bar{X}_1) \neq \text{cor}(S_2, \bar{X}_2)$ , because of the different sample sizes.

67 When distributions are skewed, one observes that the larger the sample size, the  
 68 lower the correlation between  $S_j$  and  $\bar{X}_j$  (See Figures 6 and 7).

69 This might explain that the magnitude of the correlation between  $S_j$  and  $\bar{X}_1 - \bar{X}_2$  is  
 70 lower in the larger sample (see bottom plots in Figures 8 and 9). With no surprise, there is  
 71 a positive (negative) correlation between  $S_1$  and  $\bar{X}_1 - \bar{X}_2$  and a negative (positive)  
 72 correlation between  $S_2$  and  $\bar{X}_1 - \bar{X}_2$  when distributions are right-skewed (left-skewed), as  
 73 illustrated in the two bottom plots of Figures 8 and 9.

74 This might also explain that standardizers of Shieh's  $d_s$  and Cohen's  $d_s^*$  are  
 75 **correlated** with  $\bar{X}_1 - \bar{X}_2$  (see Figures 10 and 11):

76 - When computing  $S_{\text{Cohen's } d_s^*}$ , the same weight is given to both  $S_1$  and  $S_2$ . Therefore,  
 77 it does not seem surprising that the sign of the correlation between  $S_{\text{Cohen's } d_s^*}$  and  $\bar{X}_1 - \bar{X}_2$   
 78 is the same as the size of the correlation between  $\bar{X}_1 - \bar{X}_2$  and the  $SD$  of the smallest  
 79 sample;

80 - When computing  $S_{\text{Shieh's } d_s}$ , more weight is given to the  $SD$  of the smallest sample,  
 81 it is therefore not really surprising to observe that the correlation between  $S_{\text{Shieh's } d_s}$  and  
 82  $\bar{X}_1 - \bar{X}_2$  is closer of the correlation between the  $SD$  of the smallest group and  $\bar{X}_1 - \bar{X}_2$

83 (i.e.  $|cor(S_{Shieh's\ d_s}, \bar{X}_1 - \bar{X}_2)| > |cor(S_{Cohen's\ d_s^*}, \bar{X}_1 - \bar{X}_2)|$ );

84 - When computing  $S_{Cohen}$ , more weight is given to the  $SD$  of the largest sample,

85 which by compensation effect brings the correlation very close to 0.

86 The correlation between  $\bar{X}_1 - \bar{X}_2$  and respectively  $S_1, S_2$ , the standardizer of Cohen's

87  $d_s^*$ , the standardizer of Shieh's  $d_s$  and the standardizer of Cohen's  $d_s$  are summarized in

88 Table 1.

89 **When unequal population variances are estimated based on equal sample sizes**

90 **(condition c)**

91 When distributions are skewed, there are again non-null correlations between  $\bar{X}_j$  and

92  $S_j$ . As illustrated in Figures 12 and 13, the correlation remains the same for any

93 population  $SD$  ( $\sigma$ ). However, the magnitude of the correlation between  $S_j$  and  $\bar{X}_1 - \bar{X}_2$

94 differs: it is stronger in the sample extracted from the larger population variance (see

95 Figures 14 and 15).

96 This also explains that when computing a standardizer that takes both  $S_1$  and  $S_2$

97 into account, it results in a standardizer that is correlated with  $\bar{X}_1 - \bar{X}_2$  (see Figures 16

98 and 17). The correlation between the mean difference ( $\bar{X}_1 - \bar{X}_2$ ) and respectively the

99 standardizer of Shieh's  $d_s$ , Cohen's  $d_s^*$  and Cohen's  $d_s$  will have the same sign as the

100 correlation between ( $\bar{X}_1 - \bar{X}_2$ ) and the larger  $SD$ . Table 2 summarizes the sign of the

101 correlation between  $\bar{X}_1 - \bar{X}_2$  and respectively  $S_1, S_2$  and the three standardizers taking

102 both  $S_1$  and  $S_2$  into account (see "Others" in the Table).

Table 1

*Correlation between standardizers ( $S_1$ ,  $S_2$ ,  $S_{Cohen's\ d_s}$  and others) and  $\bar{X}_1 - \bar{X}_2$ , when samples are extracted from skewed distributions with equal variances, and  $n_1 = n_2$  (condition a) or  $n_1 \neq n_2$  (condition b)*

population distribution		
	right-skewed	left-skewed
When $n_1 = n_2$		
	$S_1$ : positive	$S_1$ : negative
	$S_2$ : negative	$S_2$ : positive
	$S_{Cohen's\ d_s}$ : null	$S_{Cohen's\ d_s}$ : null
	$S_{Shieh's\ d_s}$ : null	$S_{Shieh's\ d_s}$ : null
	$S_{Cohen's\ d_s^*}$ : null	$S_{Cohen's\ d_s^*}$ : null
When $n_1 > n_2$		
	$S_1$ : positive	$S_1$ : negative
	$S_2$ : negative	$S_2$ : positive
	$S_{Cohen's\ d_s}$ : null	$S_{Cohen's\ d_s}$ : null
	$S_{Shieh's\ d_s}$ : negative	$S_{Shieh's\ d_s}$ : positive
	$S_{Cohen's\ d_s^*}$ : positive (but very small)	$S_{Cohen's\ d_s^*}$ : negative (but very small)
When $n_1 < n_2$		
	$S_1$ : positive	$S_1$ : negative
	$S_2$ : negative	$S_2$ : positive
	$S_{Cohen's\ d_s}$ : negative (but very small)	$S_{Cohen's\ d_s}$ : positive (but very small)
	$S_{Shieh's\ d_s}$ : positive	$S_{Shieh's\ d_s}$ : negative
	$S_{Cohen's\ d_s^*}$ : positive	$S_{Cohen's\ d_s^*}$ : negative

Table 2

*Correlation between standardizers ( $S_1$ ,  $S_2$  and others) and  $\bar{X}_1 - \bar{X}_2$ , when samples are extracted from skewed distributions with equal sample sizes, as a function of the SD-ratio.*

<b>population distribution</b>		
	<i>right-skewed</i>	<i>left-skewed</i>
When $\sigma_1 = \sigma_2$	$S_1$ : <i>positive</i> $S_2$ : <i>negative</i> Others: <i>null</i>	$S_1$ : <i>negative</i> $S_2$ : <i>positive</i> Others: <i>null</i>
When $\sigma_1 > \sigma_2$	$S_1$ : <i>positive</i> $S_2$ : <i>negative</i> Others: <i>positive</i>	$S_1$ : <i>negative</i> $S_2$ : <i>positive</i> Others: <i>negative</i>
When $\sigma_1 < \sigma_2$	$S_1$ : <i>positive</i> $S_2$ : <i>negative</i> Others: <i>negative</i>	$S_1$ : <i>negative</i> $S_2$ : <i>positive</i> Others: <i>positive</i>

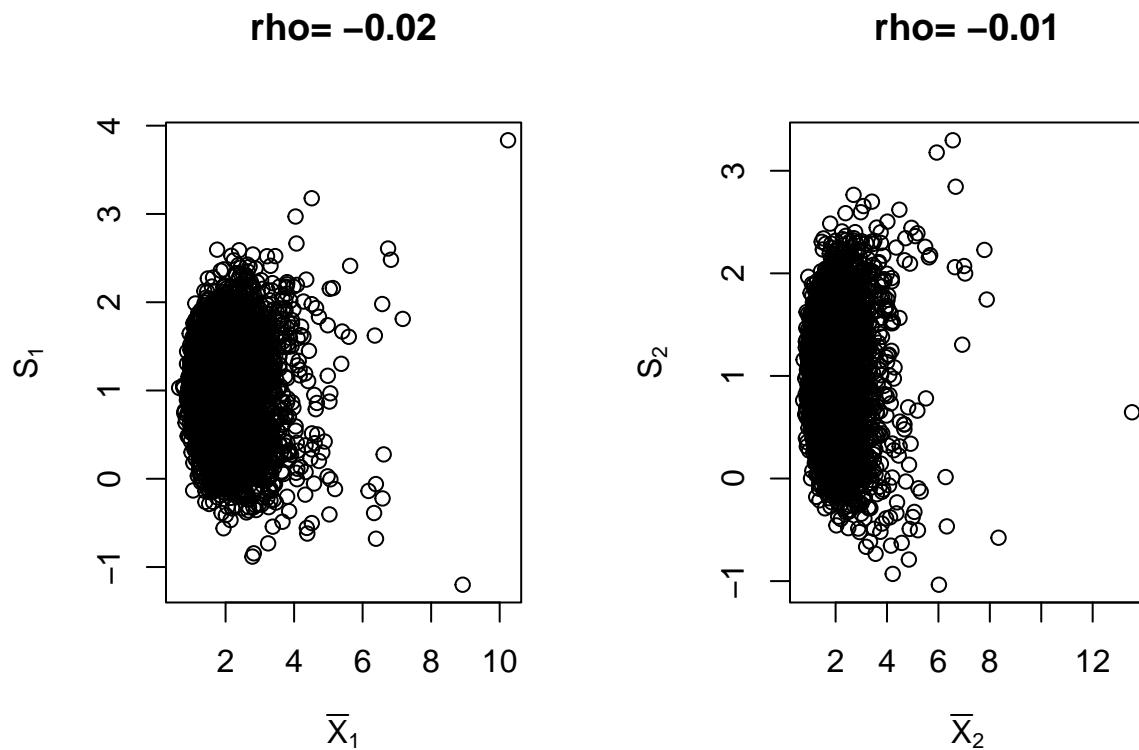


Figure 1.  $S_j$  as a function of  $\bar{X}_j$  ( $j=1,2$ ), when samples are extracted from symmetric distributions ( $\gamma_1 = 0$ )

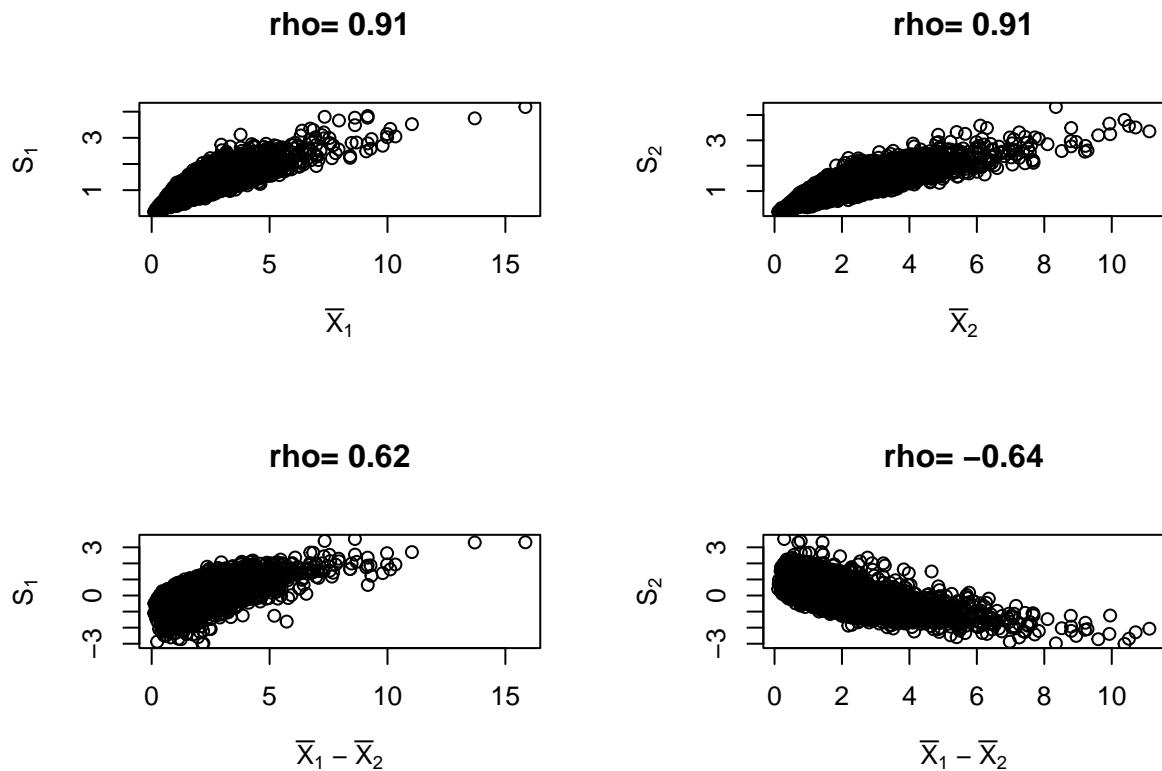


Figure 2.  $S_j$  ( $j=1,2$ ) as a function of  $\bar{X}_j$  (top plots) or  $\bar{X}_1 - \bar{X}_2$  (bottom plots), when samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ )

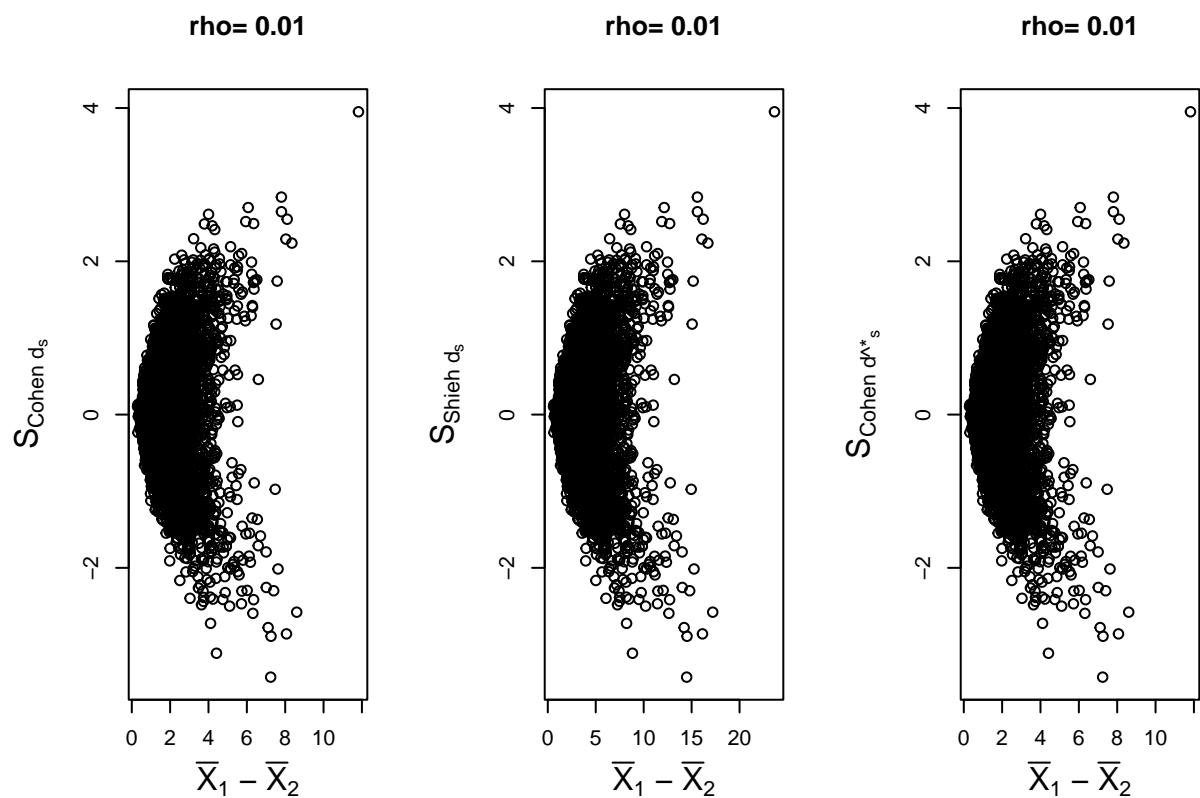


Figure 3.  $S_{Cohen's} d_s$ ,  $S_{Shieh's} d_s$  and  $S_{Cohen's} d_s^*$  as a function of the mean difference ( $\bar{X}_1 - \bar{X}_2$ ), when samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ )

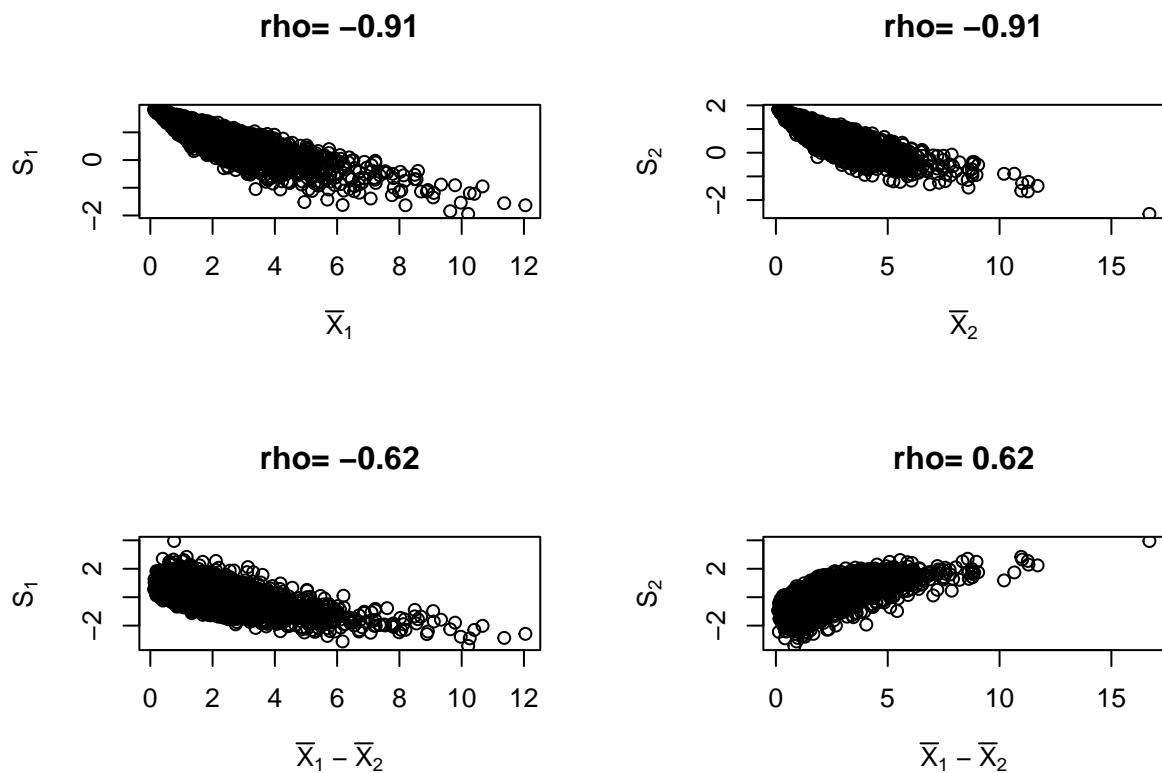


Figure 4.  $S_j$  ( $j=1,2$ ) as a function of  $\bar{X}_j$  (top plots) or  $\bar{X}_1 - \bar{X}_2$  (bottom plots), when samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ )

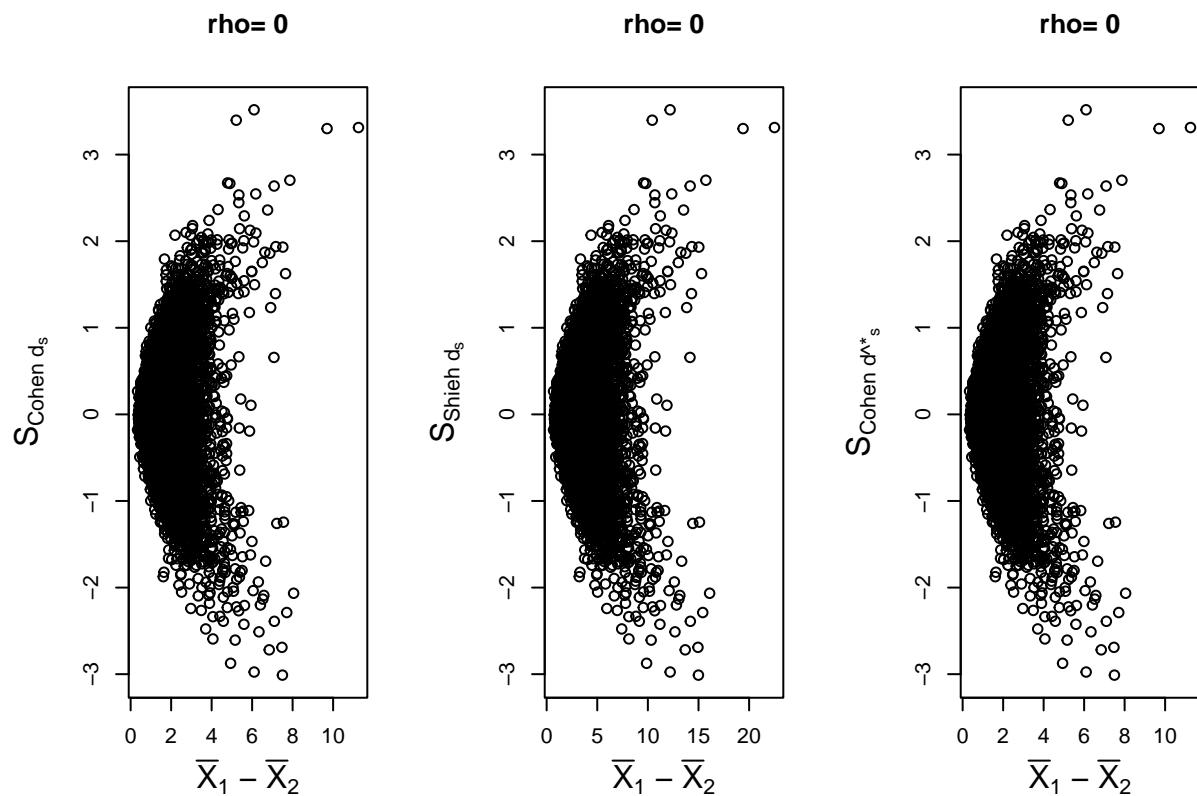


Figure 5.  $S_{Cohen's} d_s$ ,  $S_{Shieh's} d_s$  and  $S_{Cohen's} d_s^*$  as a function of the mean difference ( $\bar{X}_1 - \bar{X}_2$ ), when samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ )

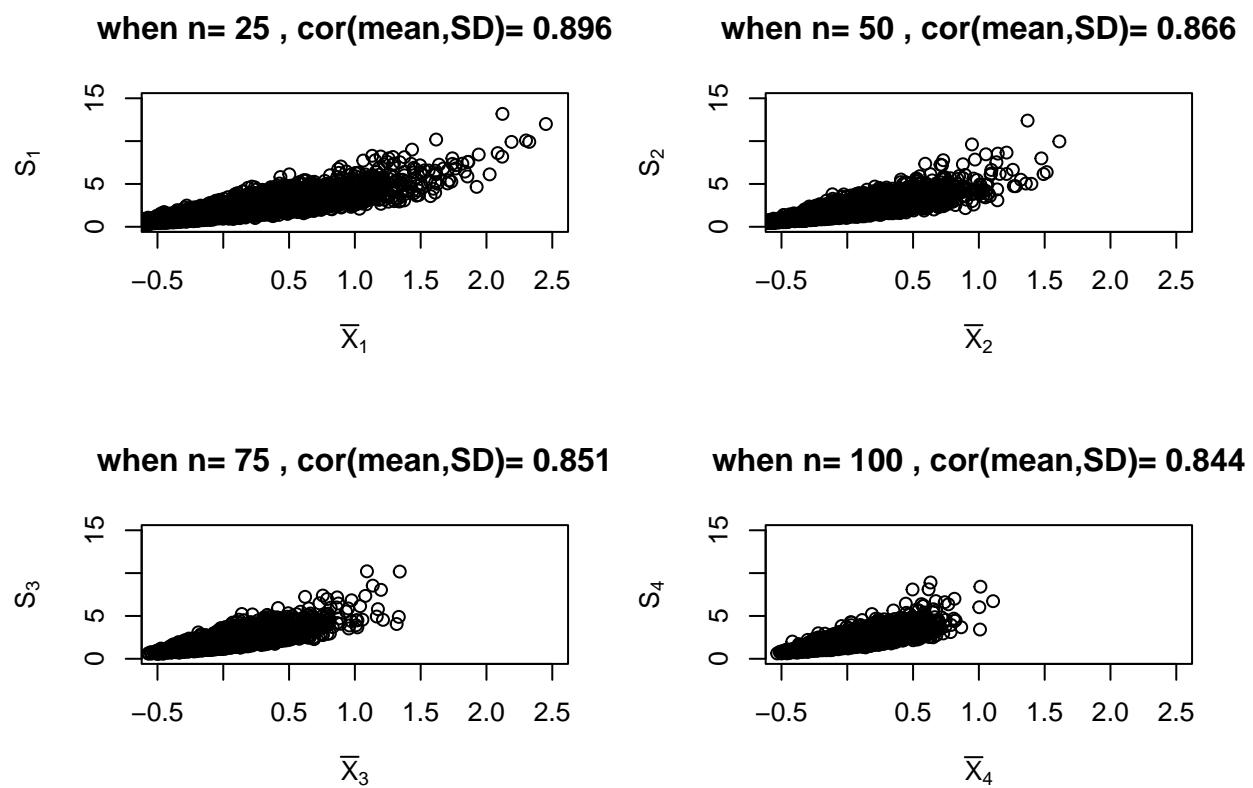


Figure 6. Correlation between  $S_j$  and  $\bar{X}_j$  when  $n = 25, 50, 75$  or  $100$  and samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ )

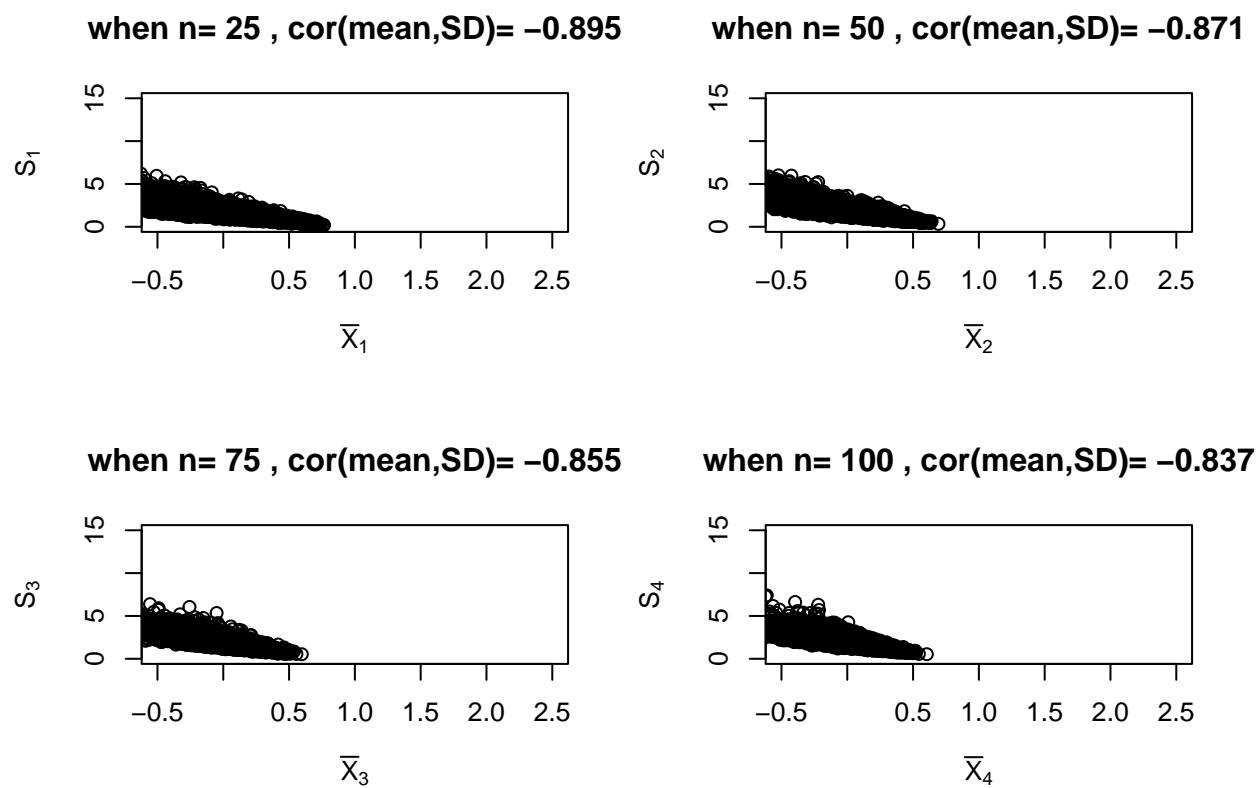


Figure 7. Correlation between  $S_j$  and  $\bar{X}_j$  when  $n = 25, 50, 75$  or  $100$  and samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ )

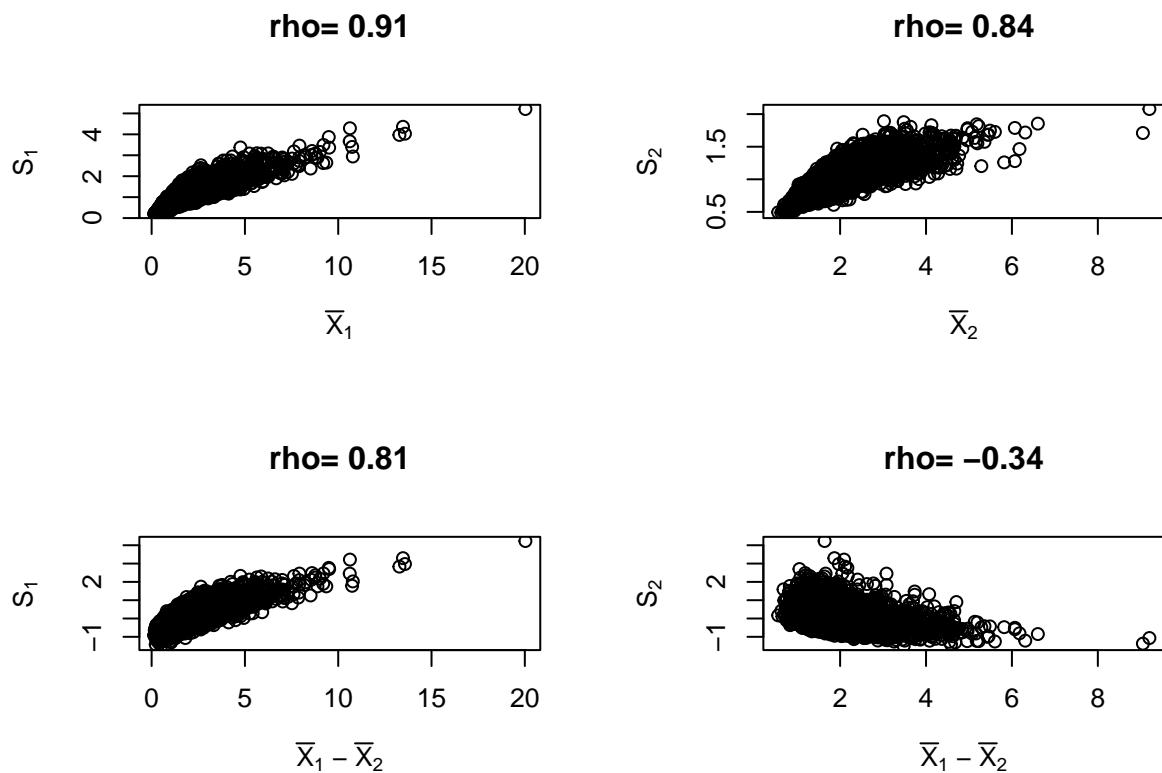


Figure 8.  $S_j$  ( $j=1,2$ ) as a function of  $\bar{X}_j$  (top plots) or  $\bar{X}_1 - \bar{X}_2$  (bottom plots), when samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ ), with  $n_1=20$  and  $n_2=100$

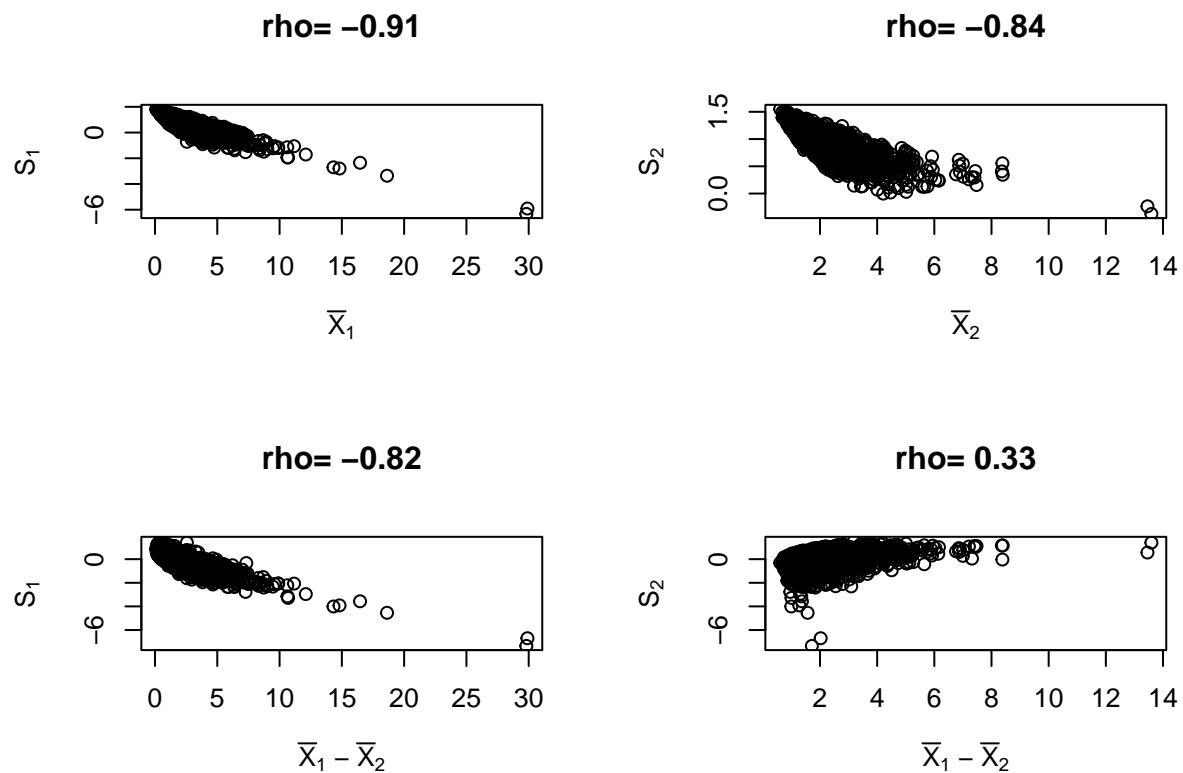


Figure 9.  $S_j$  ( $j=1,2$ ) as a function of  $\bar{X}_j$  (top plots) or  $\bar{X}_1 - \bar{X}_2$  (bottom plots), when samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ ), with  $n_1=20$  and  $n_2=100$

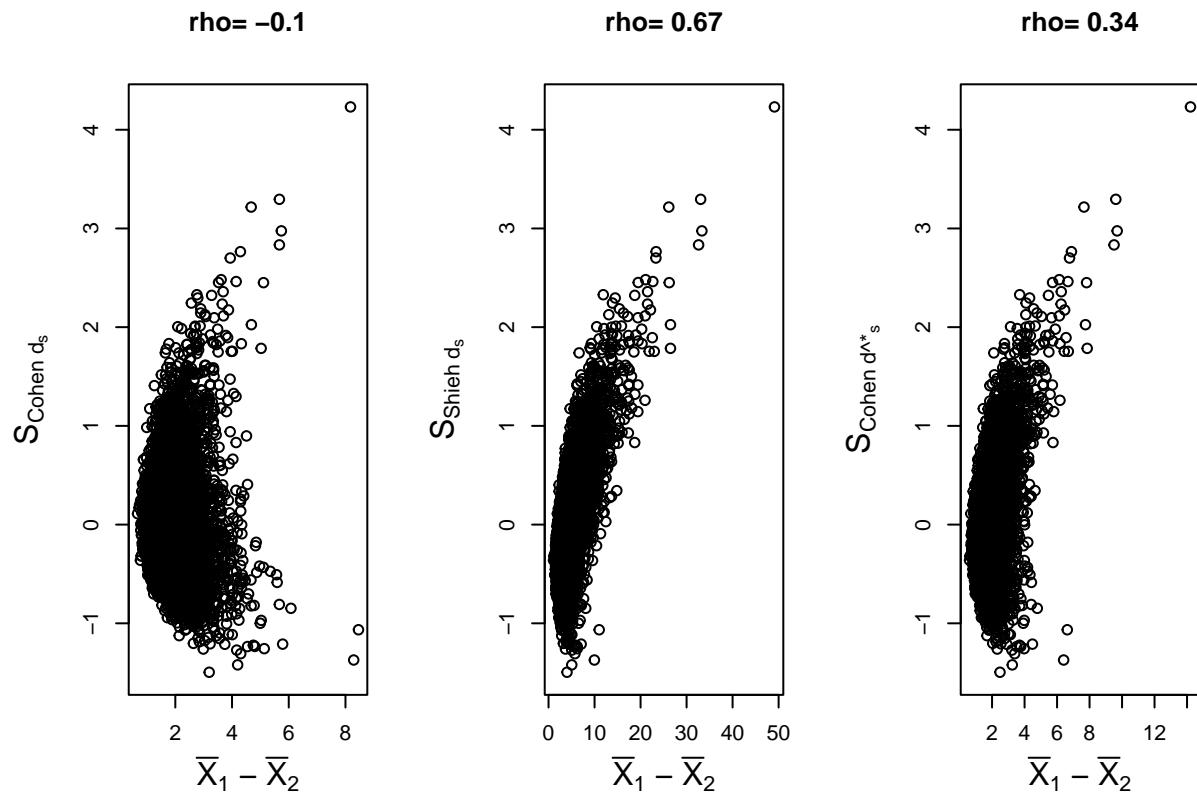


Figure 10.  $S_{Cohen's\ ds}$ ,  $S_{Shieh's\ ds}$  and  $S_{Cohen\ d^*s}$  as a function of the mean difference ( $\bar{X}_1 - \bar{X}_2$ ), when samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ , with  $n_1=20$  and  $n_2=100$ )

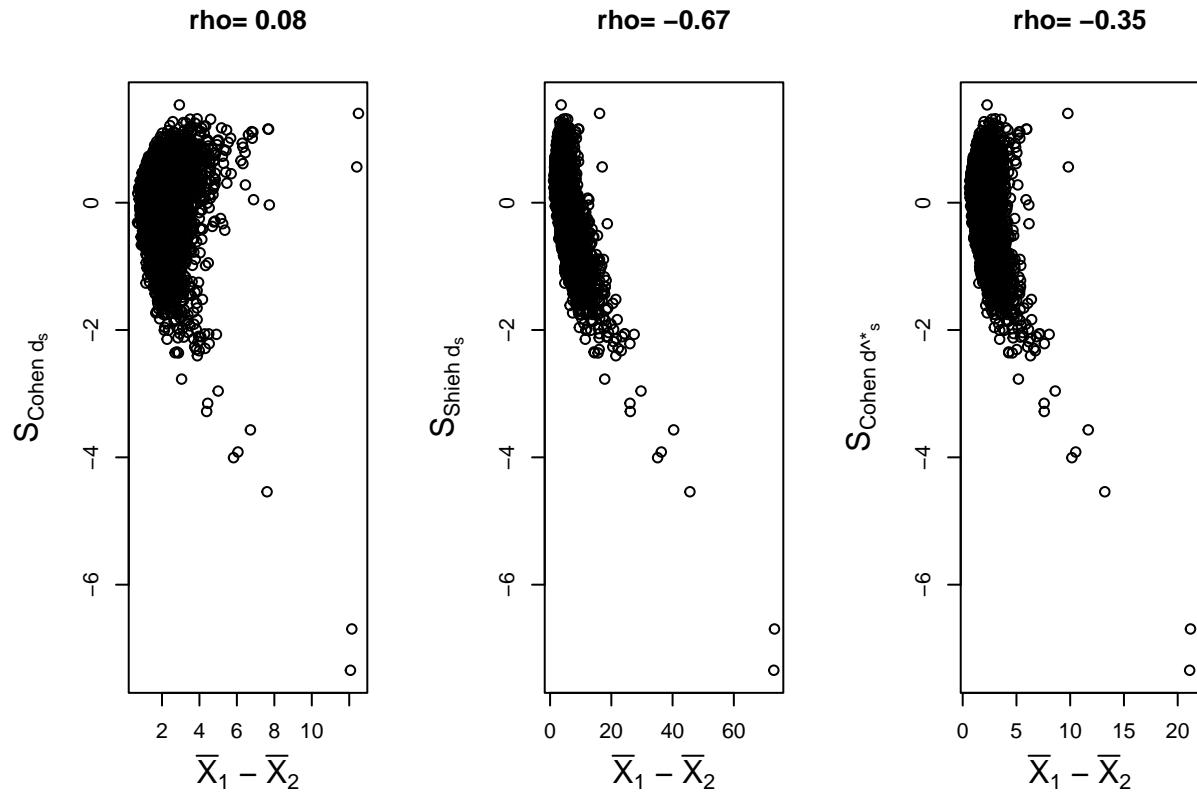


Figure 11.  $S_{Cohen's\ ds}$ ,  $S_{Shieh's\ ds}$  and  $S_{Cohen's\ ds^*}$  as a function of the mean difference ( $\bar{X}_1 - \bar{X}_2$ ), when samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ ), with  $n_1=20$  and  $n_2=100$

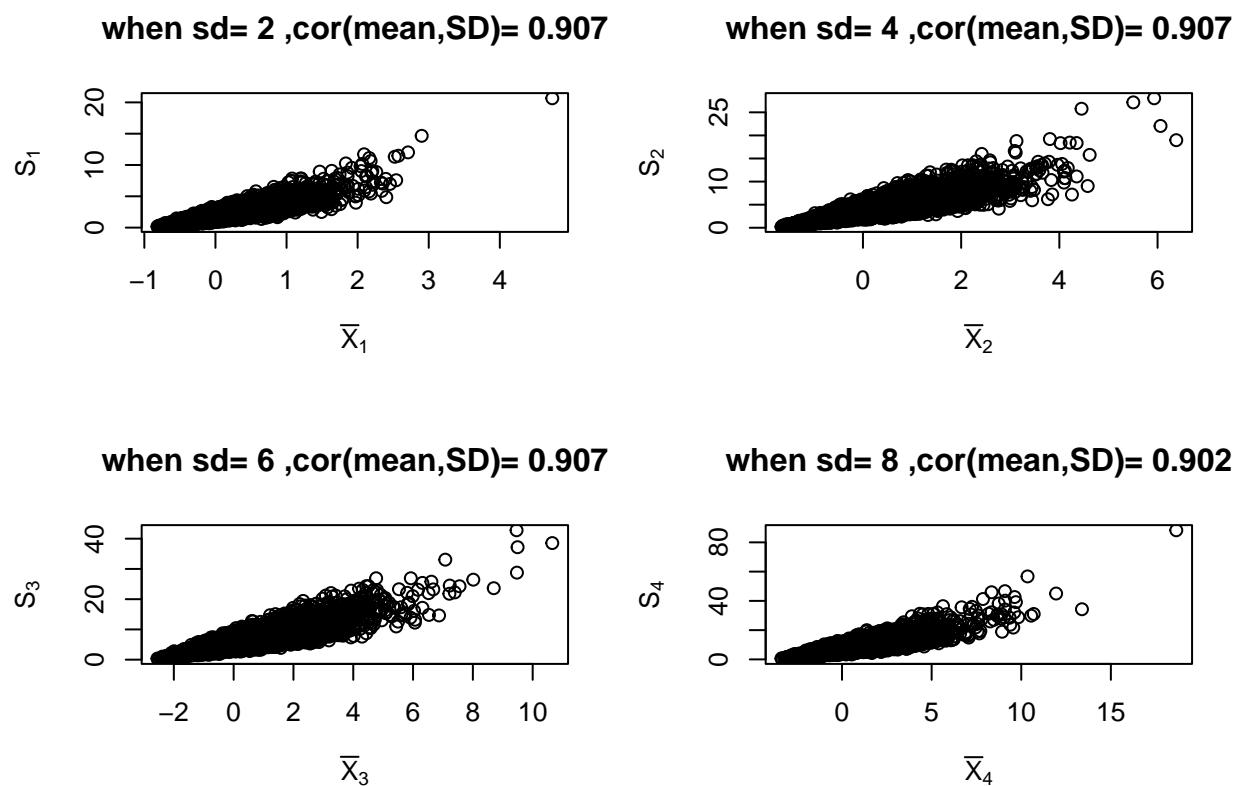


Figure 12. Correlation between  $S_j$  and  $\bar{X}_j$  when  $SD = 2, 4, 6$  or  $8$  and samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ )

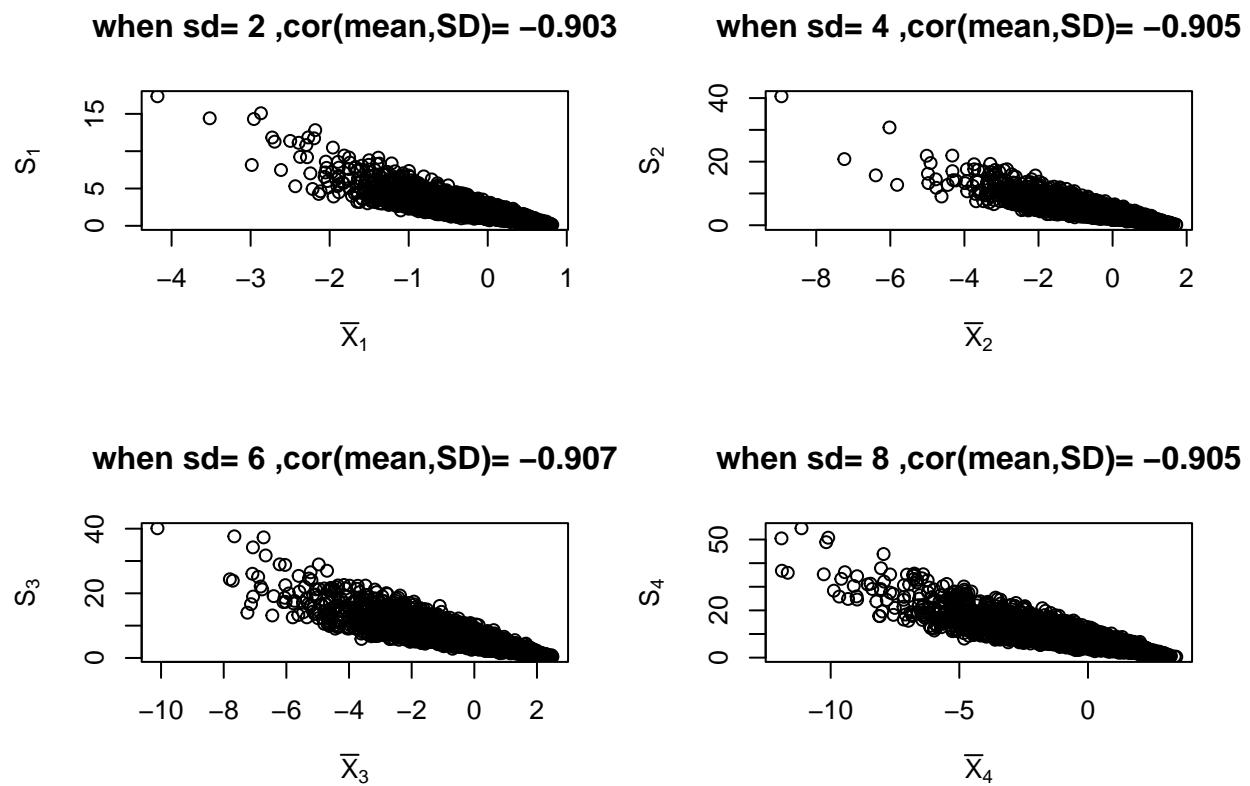


Figure 13. Correlation between  $S_j$  and  $\bar{X}_j$  when  $SD = 2, 4, 6$  or  $8$  and samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ )

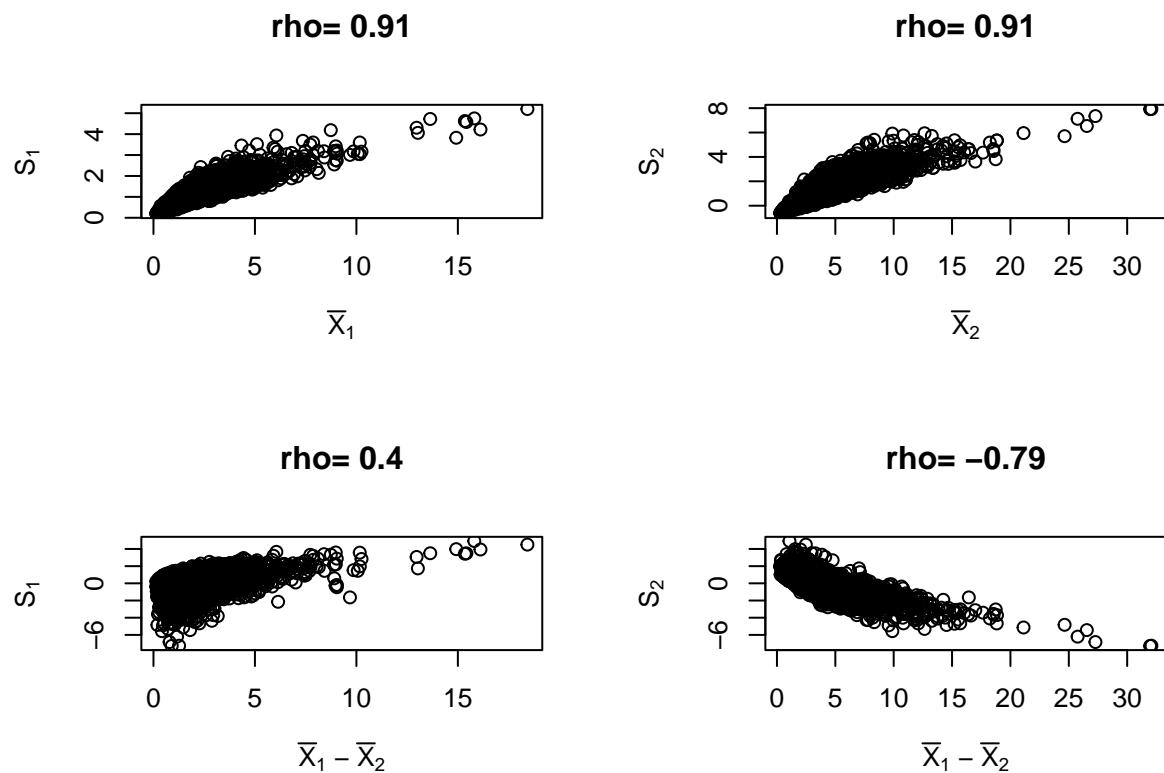


Figure 14.  $S_j$  ( $j=1,2$ ) as a function of  $\bar{X}_j$  (top plots) or  $\bar{X}_1 - \bar{X}_2$  (bottom plots), when samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ ), with  $S_1=2$  and  $S_2=4$

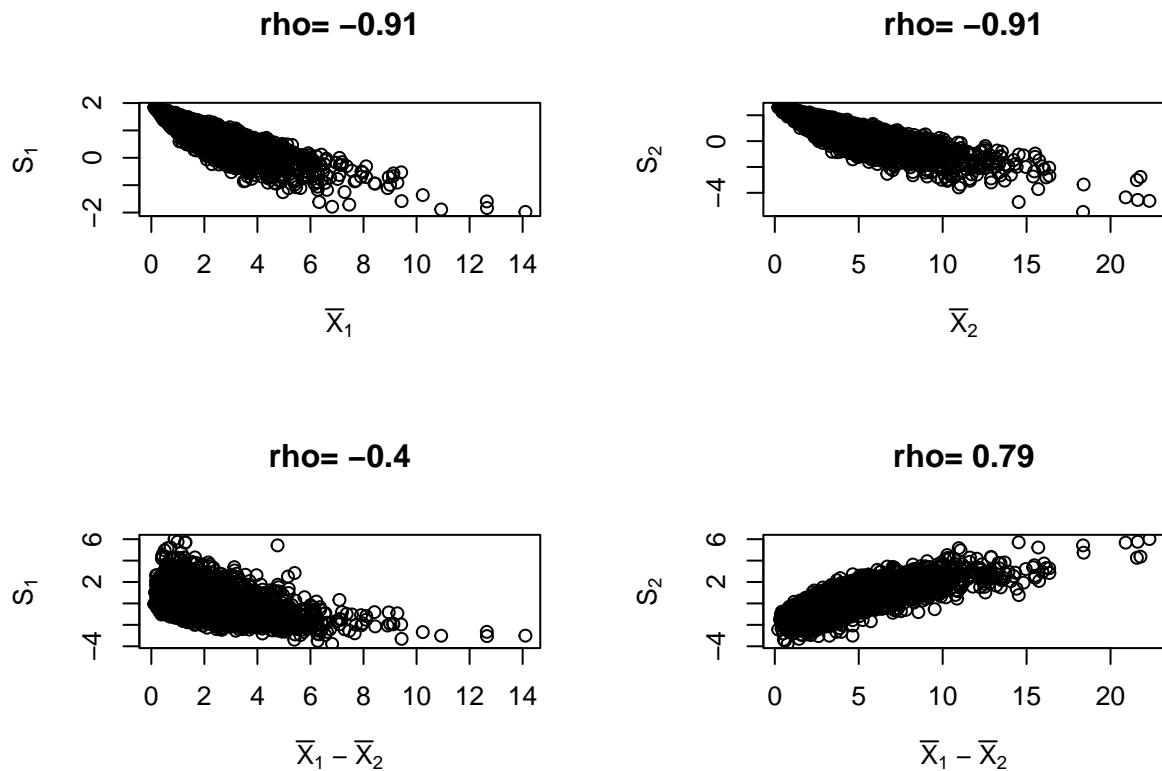


Figure 15.  $S_j$  ( $j=1,2$ ) as a function of  $\bar{X}_j$  (top plots) or  $\bar{X}_1 - \bar{X}_2$  (bottom plots), when samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ ), with  $S_1=2$  and  $S_2=4$

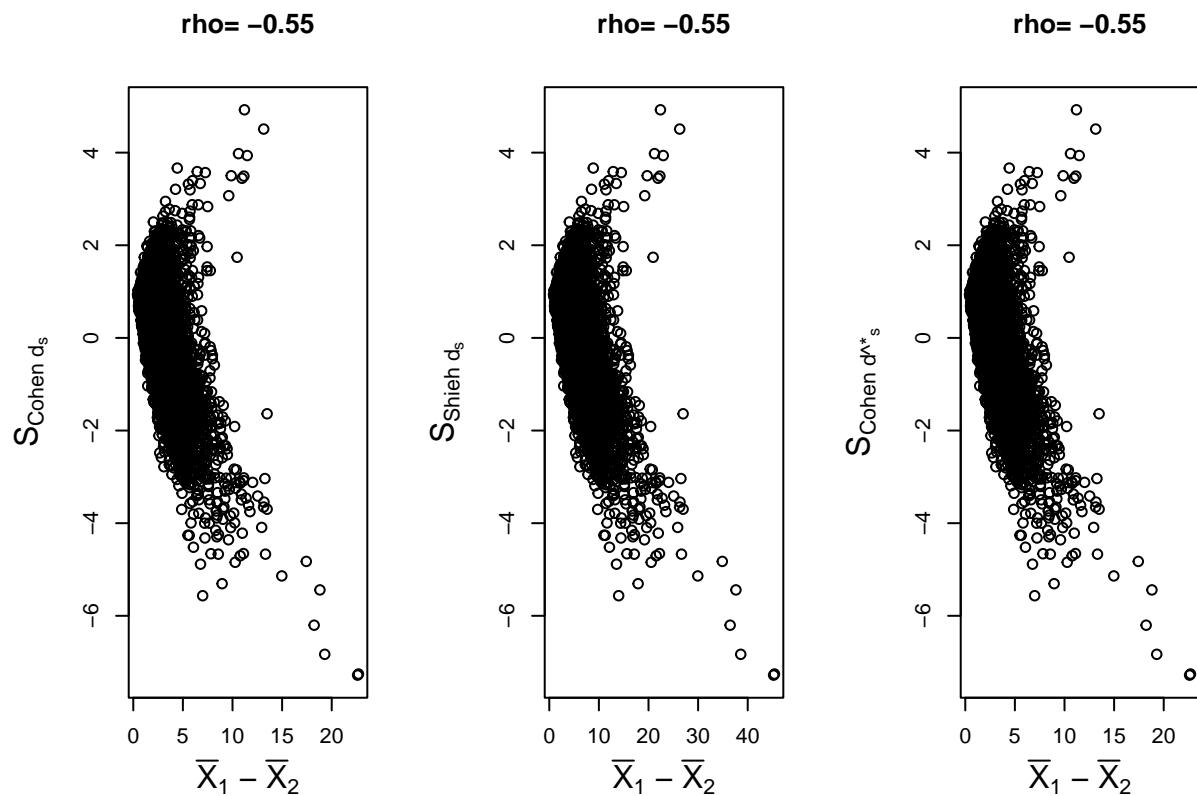


Figure 16.  $S_{Cohen's\,d_s}$ ,  $S_{Shieh's\,d_s}$  and  $S_{Cohen's\,d_s^*}$  as a function of the mean difference  $(\bar{X}_1 - \bar{X}_2)$ , when samples are extracted from right skewed distributions ( $\gamma_1 = 6.32$ ), with  $S_1=2$  and  $S_2=4$

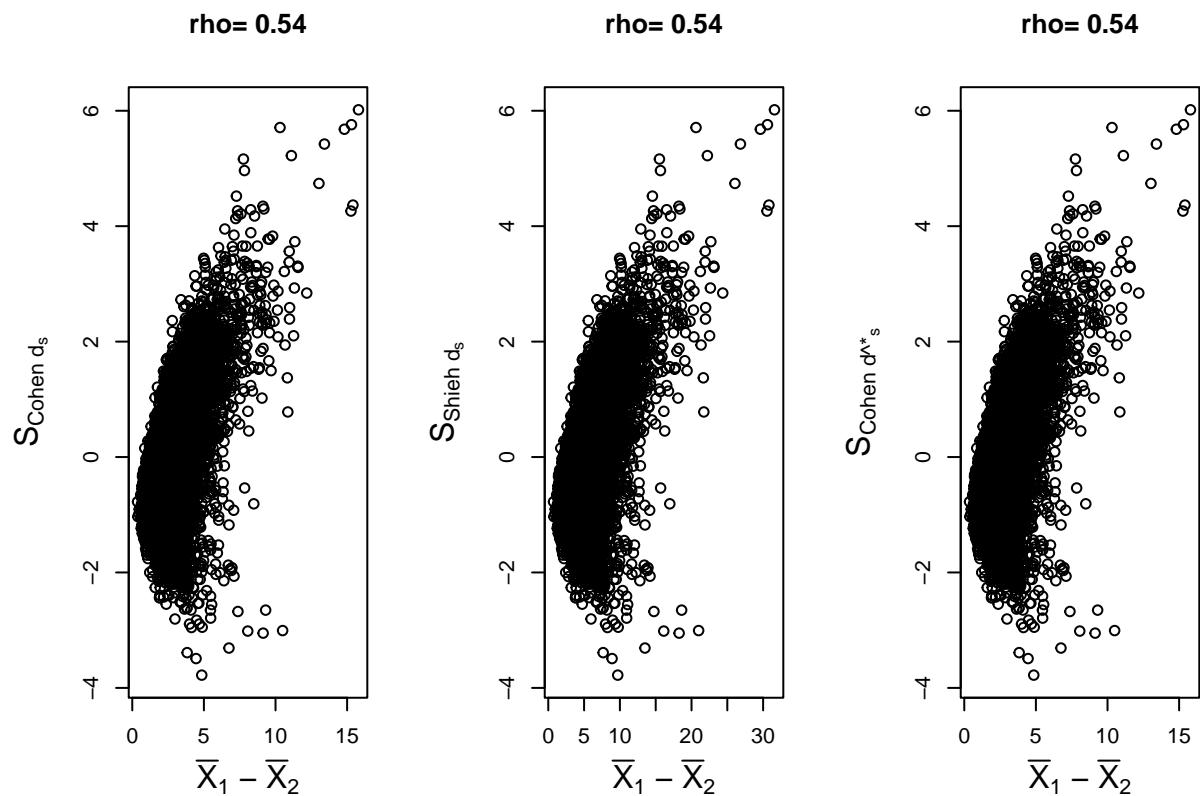


Figure 17.  $S_{Cohen\text{'s } d_s}$ ,  $S_{Shieh\text{'s } d_s}$  and  $S_{Cohen\text{'s } d_s^*}$  as a function of the mean difference ( $\bar{X}_1 - \bar{X}_2$ ), when samples are extracted from left skewed distributions ( $\gamma_1 = -6.32$ ), with  $S_1=2$  and  $S_2=4$