Supplemental Material

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Supplemental Material 1 :

Theoretical bias

For all "biased" estimators, when the population effect size is null so is the bias. We will therefore focus on configurations where there is a non-null population effect size. The sampling distribution of Cohen's d (and therefore its bias) is only known under the assumptions of normality and homoscedasticity. On the other hand, the biases of Glass's d, Cohen's d^* and Shieh's d are theoretically known for all configurations where the normality assumption is met. In order to simplify the analysis of their bias, it is convenient to subdivide all configurations into 3 conditions:

- when population variances are equal across groups;
- when population variances are unequal across groups, with equal sample sizes;
- when population variances are unequal across groups, with unequal sample sizes.

Preliminary note. For all previously mentioned estimators (Cohen's d, Glass's d, Cohen's d^* and Shieh's d), the theoretical expectation is computed by multiplying the population effect size (respectively δ_{Cohen}, δ_{Glass}, δ^*_{Cohen} and δ_{Shieh}) by the following multiplier coefficient:

$$\gamma = \frac{\sqrt{\frac{df}{2}} \times \Gamma\frac{df-1}{2}}{\Gamma\frac{df}{2}} \tag{1}$$

where df are the degrees of freedom (see the main article). γ is *always* positive, meaning that when the population effect size is not zero, all estimators will overestimate the population effect size. Moreover, its limit tends to 1 when the degrees of freedom (df) tend to infinity, meaning that the larger the degrees of freedom, the lower the bias.

While we focus on the theoretical bias of biased estimators when the normality assumption is met, it is interesting to notice that our main conclusions seem to generalize to :

- biased estimators when samples are extracted from symmetric distributions;

- unbiased estimators when samples are extracted from heavy-tailed symmetric distributions.

Cohen's *d* (see Table 2). Under the assumptions that independant residuals are normally distributed with equal variances, the **bias** of Cohen's d is a function of total sample size (N) and the population effect size (δ_{Cohen}):

- The larger the population effect size, the more Cohen's d will overestimate δ_{Cohen};

- The larger the total sample size, the lower the bias (see Figure 1);

- Of course, considering the degrees of freedom, the sample size ratio does not matter (i.e. the bias will decrease when increasing n_1, n_2 or both sample sizes).

Glass's *d* (see Table 3). Because degrees of freedom depend only on the control group size (neither on σ_1 nor on σ_2), there is no need to distinguish between cases where there is homoscedasticity or heteroscedasticity!

The **bias** of Glass's d is a function of the control group size (n_c) and the population effect size (δ_{Glass}):

- The larger the population effect size, the more Glass's d will overestimate δ_{Glass};

- The larger the size of the control group, the lower the bias (see the two top plots in Figure 2). On the other hand, increasing the experimental group size does not impact the bias (see the two bottom plots in Figure 2).

Cohen's *d (see Table 3).**

When variances are equal across populations.

When $\sigma_1 = \sigma_2 = \sigma$:

$$df_{Cohen's\ d^*} = \frac{(n_1-1)(n_2-1)(2\sigma^2)^2}{(n_2-1)\sigma^4 + (n_1-1)\sigma^4} = \frac{(n_1-1)(n_2-1)\times 4\sigma^4}{\sigma^4(n_1+n_2-2)} = \frac{4(n_1-1)(n_2-1)}{n_1+n_2-2}$$

One can see that degrees of freedom depend only on the total sample size (N) and the sample size allocation ratio $\left(\frac{n_2}{n_1}\right)$. As a consequence, the **bias** of Cohen's d^* is a function of the population effect size (δ^*_{Cohen}), the sample size allocation ratio $\left(\frac{n_2}{n_1}\right)$ and the total sample size (N).

- The larger the population effect size, the more Cohen's d^* will overestimate δ^*_{Cohen};
- The further the sample size allocation ratio is from 1, the larger the bias (see Figure 3);
- The larger the total sample size, the lower the bias (see Figure 4).

When variances are unequal across populations, with equal sample sizes.

When $n_1 = n_2 = n$:

$$df_{Cohen's\ d^*} = \frac{(n-1)^2(\sigma_1^2+\sigma_2^2)^2}{(n-1)(\sigma_1^4+\sigma_2^4)} = \frac{(n-1)(\sigma_1^4+\sigma_2^4+2\sigma_1^2\sigma_2^2)}{\sigma_1^4+\sigma_2^4}$$

One can see that degrees of freedom depend only on the total sample size (N) and the SD-ratio $\left(\frac{\sigma_2}{\sigma_1}\right)$. As a consequence, the **bias** of Cohen's d^* is a function of the population effect size (δ^*_{Cohen}), the SD-ratio $\left(\frac{\sigma_2}{\sigma_1}\right)$ and the total sample size (N):

- The larger the population effect size, the more Cohen's d^* will overestimate δ^*_{Cohen};
- The further the SD-ratio is from 1, the larger the bias (see Figure 5);
- The larger the total sample size, the lower the bias (see Figure 6).

Note: for a constant SD-ratio, σ_1 and σ_2 do not matter (see Figure 7).

When variances are unequal across populations, with unequal sample sizes.

The **bias** of Cohen's d^* is a function of the population effect size (δ^*_{Cohen}), the total sample size (N), and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- The larger the population effect size, the more Cohen's d^* will overestimate δ^*_{Cohen};

- The larger the total sample size, the lower the bias (see Figure 8);

- The smallest bias always occurs when there is a positive pairing between variances and sample sizes, because one gives more weight to the smallest variance, in the denominator of the *df* computation. Moreover, the further the SD-ratio is from 1, the further from 1 will also be the sample sizes ratio associated with the smallest bias (see Figure 9). This can be explained by splitting the numerator and the denominator in the *df* computation.

As illustrated in Figure 10, for any SD-ratio, the numerator of the degrees of freedom will be maximized when sample sizes are equal across groups (and is not impacted by the SD-ratio). On the other hand, the denominator will be minimized when there is a positive pairing between variances and sample sizes. For example, when $\sigma_1 > \sigma_2$, the smallest denominator occurs when $\frac{n_2}{n_1}$ reaches its minimum value and the further from 1 the SD-ratio, the larger the impact of the sample sizes ratio on the denominator.

Note: for a constant SD-ratio, the variance does not matter. (See Figure 11).

Shieh's *d* (see Table 3).

When variances are equal across populations.

When $\sigma_1 = \sigma_2 = \sigma$:

$$df_{Shieh's\ d} = \frac{\left(\frac{n_2\sigma^2 + n_1\sigma^2}{n_1 n_2}\right)^2}{\frac{(n_2-1)\left(\frac{\sigma^2}{n_1}\right)^2 + (n_1-1)\left(\frac{\sigma^2}{n_2}\right)^2}{(n_1-1)(n_2-1)}}$$

$$\leftrightarrow df_{Shieh's\ d} = \frac{[\sigma^2(n_1+n_2)]^2}{n_1^2 n_2^2} \times \frac{(n_1-1)(n_2-1)}{(n_2-1)\times\frac{\sigma^4}{n_1^2} + (n_1-1)\times\frac{\sigma^4}{n_2^2}}$$

$$\leftrightarrow df_{Shieh's\ d} = \frac{\sigma^4 N^2}{n_1^2 n_2^2} \times \frac{(n_1-1)(n_2-1)}{\sigma^4\left(\frac{n_2-1}{n_1^2} + \frac{n_1-1}{n_2^2}\right)}$$

$$\leftrightarrow df_{Shieh's\ d} = \frac{N^2(n_1-1)(n_2-1)}{n_1^2 n_2^2\left(\frac{n_2^2(n_2-1)+n_1^2(n_1-1)}{n_1^2 n_2^2}\right)}$$

$$\leftrightarrow df_{Shieh's\ d} = \frac{N^2(n_1-1)(n_2-1)}{n_2^2(n_2-1)+n_1^2(n_1-1)}$$

One can see that degrees of freedom depend only on the total sample size (N) and the sample size allocation ratio $\left(\frac{n_2}{n_1}\right)$. As a consequence, the **bias** of Shieh's d is a function of the population effect size (δ_{Shieh}), the sample size allocation ratio $\left(\frac{n_2}{n_1}\right)$ and the total sample size (N).

- The larger the population effect size, the more Shieh's d will overestimate δ_{Shieh};

- The further the sample size allocation ratio is from 1, the larger the bias (see Figure 12);

- For a constant sample sizes ratio, the larger the total sample size, the lower the bias (see Figure 13).

Note: when computing Cohen's d^*, degrees of freedom increased when adding subjects in either one or both groups, even when the sample size ratio increased. When computing Shieh's d, this is not true anymore: there is a larger impact of the sample sizes ratio such that moving the sample sizes ratio away from 1 when adding subjects in only one group can decrease the degrees of freedom and therefore, increase the bias (See Figure 14).

When variances are unequal across populations, with equal sample sizes.

When $n_1 = n_2 = n$:

$$df_{Shieh's\ d} = \frac{\left(\frac{\sigma_1^2+\sigma_2^2}{n}\right)^2}{\frac{(\sigma_1^2/n)^2+(\sigma_2^2/n)^2}{n-1}}$$

$$\leftrightarrow df_{Shieh's\ d} = \frac{(\sigma_1^2+\sigma_2^2)^2}{n^2} \times \frac{n-1}{\frac{\sigma_1^4+\sigma_2^4}{n^2}}$$

$$\leftrightarrow df_{Shieh's\ d} = \frac{(\sigma_1^2+\sigma_2^2)^2 \times (n-1)}{\sigma_1^4+\sigma_2^4}$$

One can see that degrees of freedom depend on the total sample size (N) and the SD-ratio $\left(\frac{\sigma_2}{\sigma_1}\right)$. As a consequence, the bias depends on the population effect size (δ_{Shieh}), the SD-ratio $\left(\frac{\sigma_2}{\sigma_1}\right)$ and the total sample size (N).

- The larger the population effect size, the more Shieh's d will overestimate δ_{Shieh};

- The further the SD-ratio is from 1, the larger the bias (see Figure 15);

- The larger the total sample size, the lower the bias (see Figure 16);

Note: for a constant SD-ratio, σ_1 and σ_2 do not matter (see Figure 17).

When variances are unequal across populations, with unequal sample sizes.

The **bias** of Shieh's d is a function of the population effect size (δ_{Shieh}), the sample sizes (n_1 and n_2), and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- The larger the population effect size, the more Shieh's d will overestimate δ_{Shieh};

- For a constant sample sizes ratio, the larger the sample sizes, the lower the bias (See Figure 18);

Note: When variances were equal across populations, adding subjects only in the first group had the same impact on degrees of freedom (and therefore on bias) than adding subjects only in the second group (see Figure 14). When variances are unequal across groups, this is not true anymore (see Figure 19).

- The smallest bias always occurs when there is a positive pairing between variances and sample sizes. Moreover, the further the SD-ratio is from 1, the further from 1 will also be the sample sizes ratio associated with the smallest bias (See Figure 20);

Moreover, for a constant SD-ratio, the variances do not matter (See Figure 21).

In summary. The **bias** of Cohen's d is a function of the population effect size δ_{Cohen} and the total sample size (N):

- When δ_{Cohen} is null, the bias is null. In all other configurations, the larger δ_{Cohen}, the more Cohen's d will overestimate δ_{Cohen};
- The bias decreases when the total sample size increases (it does not matter whether one adds subjects in only one group or in both).

The **bias** of Glass's d is a function of the population effect size (δ_{Glass}) and the size of the control group (n_e):

- When δ_{Glass} is null, the bias is null. In all other configurations, the larger δ_{Glass}, the more Glass's d will overestimate δ_{Glass};
- The bias decreases when the size of the control group increases. On the other hand, increasing the size of the experimental group does not impact the bias.

The **bias** of Cohen's d^* is a function of the population effect size (δ^*_{Cohen}), the total sample size, and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- When δ^*_{Cohen} is null, the bias is null. In all other configurations, the larger δ^*_{Cohen}, the more Cohen's d^* will overestimate δ^*_{Cohen};
- The bias decreases when the total sample size increases (it does not matter whether one adds subjects in only one group or in both);
- The smallest bias always occurs when there is a positive pairing between $\frac{\sigma_2}{\sigma_1}$ and $\frac{n_2}{n_1}$. Moreover, the larger the SD-ratio, the further from 1 is the sample sizes ratio associated with the smallest bias.

The **bias** of Shieh's d is a function of the population effect size (δ_{Shieh}), the total sample size, and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- When δ_{Shieh} is null, the bias is null. In all other configurations, the larger δ_{Shieh}, the more Shieh's d will overestimate δ_{Shieh};
- For a constant sample sizes ratio, the bias decreases when the total sample size increases;
- The smallest bias always occurs when there is a positive pairing between $\frac{\sigma_2}{\sigma_1}$ and $\frac{n_2}{n_1}$. Moreover, the larger the SD-ratio, the further from 1 is the sample sizes ratio associated with the smallest bias (for more details, see "Theoretical Bias, as a function of population parameters").

Theoretical variance

Note: while we focus on the theoretical variance of biased estimators (Cohen's d, Glass's d, Shieh's d and Cohen's d^*) when the normality assumption is met, it is interesting to notice that our main conclusions seem to generalize to biased estimators when samples are extracted from symmetric distributions. Moreover, unbiased estimators depend on the same factors as biased estimators, so our conclusions remain similar for unbiased estimators when samples are extracted from heavy-tailed symmetric distributions.

Cohen's *d*.

When variances are equal across populations.

When $\boldsymbol{\delta_{Cohen} = 0}$.

When the population effect size is zero, the variance of Cohen's d can be simplified as follows:

$$Var_{Cohen's\ d} = \frac{N(N-2)}{n_1 n_2 (N-4)}$$

The **variance** of Cohen's d is a function of total sample size (N) and the sample sizes allocation ratio ($\frac{n_2}{n_1}$):

- The larger the total sample size, the lower the variance. The variance tends to zero when the total sample size tends to infinity (see Figure 22);

- The further the sample sizes allocation ratio is from 1, the larger the variance (see Figure 23).

When $\boldsymbol{\delta_{Cohen} \neq 0}$.

While the variance of Cohen's d still depends on the total sample size and the sample sizes allocation ratio, it also depends on the population effect size (δ_{Cohen}). The larger the population effect size, the larger the variance. Note that the effect of the population effect size decreases when sample sizes increase since

$$\lim_{n_1 \to \infty} \left[\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)} \right)^2 \right] = 0$$

$$\lim_{n_2 \to \infty} \left[\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)} \right)^2 \right] = 0$$

$$\lim_{N \to \infty} \left[\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)} \right)^2 \right] = 0$$

This is illustrated in Figure 24.

In summary.

The variance of Cohen's d is a function of the population effect size (δ_{Cohen}), the total sample size (N) and the sample sizes ratio ($\frac{n_2}{n_1}$):

- The variance decreases when the total sample size increases;
- The variance also decreases when the sample sizes ratio gets closer to 1;
- Finally, the variance increases when δ_{Cohen} increases. Note that the effect of δ_{Cohen} is moderated by the total sample size (the larger N, the smaller the effect of δ_{Cohen} on the variance).

Glass's *d*.

When variances are equal across populations.

When $\boldsymbol{\delta_{Glass} = 0}$.

When the population effect size is zero, the variance of Glass's d can be simplified as follows:

$$Var_{Glass's\ d} = \frac{n_c - 1}{n_c - 3} \left(\frac{1}{n_c} + \frac{1}{n_e} \right)$$

In this configuration, the **variance** of Glass's d is a function of the sample sizes of both control (n_c) and experimental (n_e) groups as well as of the sample sizes allocation ratio $\left(\frac{n_c}{n_e}\right)$:

- The larger the sample sizes, the lower the variance (Figure 25);

The sample sizes ratio associated with the lowest variance is not exactly 1 (because of the term $\frac{df}{df-2}$, df depending only on n_c), but is very close to 1 (and the larger the total sample size, the closer to 1 is the sample sizes ratio associated with the lowest variance). The further from this sample size ratio, the larger the variance (see Figure 26).

When $\boldsymbol{\delta_{Glass}} \neq \mathbf{0}$.

While the variance of Glass's d still depends on the total sample size and the sample sizes allocation ratio, it also depends on the population effect size (δ_{Glass}). The larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the control group increases. On the other hand, the effect of the population effect size does *not* depend on the size of the experimental group since

$$\lim_{n_c \to \infty} \left[\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)} \right)^2 \right] = 0$$

$$\lim_{n_e \to \infty} \left[\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)} \right)^2 \right] \neq 0$$

These limits are illustrated in Figure 27.

Note: while the sample sizes ratio associated with the lowest variance was very close to 1 with a null population effect size, this is not true anymore when the population effect size is not zero. Indeed, because of the second term in the addition, when computing the variance, one gives much more weight to the effect size of the control group (see Figure 28), especially when the effect size gets larger. For example, when $\delta_{Glass}= 4$, the lowest variance will occur when n_c is approximately 3 times larger than n_e. When $\delta_{Glass}= 7$, the lowest variance will occur when n_c is approximately 5 times larger than n_e, etc.

When variances are unequal across populations, with equal sample sizes.

When $\boldsymbol{\delta_{Glass} = 0}$.

When the population effect size is zero, the variance of Glass's d can be simplified as follows:

$$Var_{Glass's\ d} = \frac{n-1}{n(n-3)}\left(1 + \frac{\sigma_e^2}{\sigma_c^2}\right)$$

where $n = N/2 =$ sample size of each group. The variance is therefore a function of the total sample size and the SD-ratio ($\frac{\sigma_c}{\sigma_e}$):

- The larger the total sample size, the lower the variance (See Figure 29);

- The larger the SD-ratio (i.e. the larger is σ_c in comparison with σ_e), the lower the variance (see Figure 30). However, the effect of the SD-ratio decreases when sample sizes increase, because $\lim_{n(=n_c=n_e)\to\infty}\left[\frac{df}{n(df-2)}\right] = 0$.

When $\boldsymbol{\delta_{Glass} \neq 0}$.

While the variance of Glass's d still depends on the total sample size and the SD-ratio, it also depends on the population effect size (δ_{Glass}). The larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the control group increases, as previously explained and illustrated in Figure 27.

When variances are unequal across populations, with unequal sample sizes.

When $\boldsymbol{\delta_{Glass} = 0}$.

When the population effect size is zero, the variance of Glass's d can be simplified as follows:

$$Var_{Glass's\ d} = \frac{n_c - 1}{n_c - 3}\left(\frac{1}{n_c} + \frac{\sigma_e^2}{n_e \sigma_c^2}\right)$$

The variance of Glass's d is therefore a function of the total sample size (N), the SD-ratio and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_c}{n_e} \times \frac{\sigma_c}{\sigma_e}\right)$:

- For any SD and sample sizes pairing, increasing n_c and/or n_e will decrease the variance (see Figure 31);
  - The effect of the sample sizes ratio depends on the SD-ratio:
    * We previously mentioned that when $\sigma_c = \sigma_e$, the variance is minimized when sample sizes of both groups are almost identical (see Figure 26), meaning that it is more efficient, in order to reduce variance, to add subjects uniformly in both groups;
    * When $\sigma_e > \sigma_c$, more weight is given to n_e, meaning that it is more efficient, in order to reduce variance, to add subjects in the experimental group (n_e; see bottom plots in Figure 31);
    * When $\sigma_c > \sigma_e$, less weight is given to n_e, meaning that it is more efficient, in order to reduce variance, to add sujects in the control group (n_c; see top plots in Figure 31).

- Finally, there is also a main effect of the SD-ratio: the larger is σ_c in comparison with σ_e, the lower the variance, as we can observe in Figure 32. We can also notice that in Figure 31, the maximum variance is much larger in the two bottom plots (where $\sigma_c < \sigma_e$) than in the two top plots (where $\sigma_c > \sigma_e$).

Note that the effect of the SD-ratio, and the interaction effect between SD-ratio and sample sizes ratio decreases when the sample size of the control group increases (because $\frac{n_c-1}{n_c-3}$ gets closer to 1).

When $\boldsymbol{\delta_{Glass} \neq 0}$.

While the variance of Glass's d still depends on the total sample size, the SD-ratio and the interaction between the SD-ratio and the sample sizes ratio, it also depends on the population effect size (δ_{Glass}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the sample size of the control group increases, as previously explained and illustrated in Figure 27.

Note: when the population effect size was null, when $\sigma_c < \sigma_e$, it was much more efficient to add subjects in the experimental group in order to reduce the variance (because much more weight was given to n_e). When $\delta_{Glass} \neq 0$, it is important to add subjects in both groups in order to reduce the variance (because $\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}\right)^2$ is only a function of the sample size of the control group). With huge population effect size, it is even always more important to add subjects in the control group (e.g. when $\delta_{Glass} = 30$).

In summary.

The variance of Glass's d is a function of the population effect size (δ_{Glass}), the SD-ratio, the total sample size and the interaction between sample sizes ratio and SD-ratio $\left(\frac{n_c}{n_e} \times \frac{\sigma_c}{\sigma_e}\right)$:

- The variance decreases when the SD-ratio increases (i.e. when $\sigma_e >> \sigma_c$);
- The variance always decreases when the control and/or the experimental group increases. The benefit of adding subjects rather in the control, in the experimental, or in both groups, in order to reduce the variance, varies as a function of the SD-ratio and the population effect size. The only situation where it is optimal to maximize the experimental group is when $\sigma_e > \sigma_c$ and $\delta_{Glass} \approx 0$. Most of the time, it is more efficient to maximize the control groups (e.g. anytime $\sigma_e < \sigma_c$, and when δ_{Glass} is very large) or to uniformly add subjects in both groups (e.g. when $\sigma_e > \sigma_c$ and δ_{Glass} is neither null nor huge);

- The variance increases when δ_{Glass} increases. Note that the effect of δ_{Glass} is moderated by the control group size (the larger n_e, the smaller the effect of δ_{Glass} on the variance).

Cohen's d^*.

When variances are equal across populations.

When $\boldsymbol{\delta^*_{Cohen} = 0}$.

When the population effect size is zero, the variance of Cohen's d^* is computed as follows :

$$Var_{Cohen's\ d^*} = \frac{df}{df-2} \times \frac{N}{n_1 n_2}$$

with

$$df = \frac{4(n_1-1)(n_2-1)}{n_1+n_2-2}$$

In this configuration, the degrees of freedom as well as the variance of Cohen's d^* depend on the total sample size (N) and the sample sizes allocation ratio ($\frac{n_2}{n_1}$):

- The further the sample sizes allocation ratio is from 1, the larger the variance (see Figure 33);

- The larger the total sample size, the lower the bias (see Figure 34).

When $\boldsymbol{\delta^*_{Cohen} \neq 0}$.

While the variance of Cohen's d^* still depends on the total sample size and the sample sizes ratio, it also depends on the population effect size (δ^*_{Cohen}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the degrees of freedom increase (i.e. when sample sizes increase and/or the sample sizes ratio get closer to 1), as illustrated in Figure 35 since

$$\lim_{df\to\infty}\left[\frac{df}{df-2} - \left(\frac{\sqrt{\frac{df}{2}} \times \Gamma\left(\frac{df-1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)}\right)^2\right] = 0$$

When variances are unequal across populations, with equal sample sizes.

When $\boldsymbol{\delta^*_{Cohen} = 0}$.

When the population effect size is zero, the variance of Cohen's d^* can be simplified as follows:

$$Var_{Cohen's\ d^*} = \frac{df}{df-2} \times \frac{2}{n}$$

where $n = N/2$=sample size of each group, and $df = \frac{(n-1)(\sigma_1^4+\sigma_2^4+2\sigma_1^2\sigma_2^2)}{\sigma_1^4+\sigma_2^4}$. In this configuration, the degrees of freedom as well as the variance of Cohen's d^* depend on the total sample size (N) and the SD-ratio $\left(\frac{\sigma_2}{\sigma_1}\right)$:

- The further the SD-ratio is from 1, the larger the variance (see Figure 36);

- The larger the total sample size, the lower the variance (see Figure 37).

Note: for a constant SD-ratio, the size of the variance does not matter (see Figure 38).

When $\boldsymbol{\delta^*_{Cohen} \neq 0}$.

While the variance of Cohen's d^* still depends on the total sample size and the SD-ratio, it also depends on the population effect size (δ^*_{Cohen}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the degrees of freedom increase (i.e. when the total sample size increases and/or the SD-ratio get closer to 1), as previously illustrated in Figure 35.

When variances are unequal across populations, with unequal sample sizes.

When $\boldsymbol{\delta^*_{Cohen} = 0}$.

When the population effect size is zero, the variance of Cohen's d^* can be simplified as follows:

$$Var_{Cohen's\ d^*} = \frac{df}{df-2} \times \frac{2\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}{\sigma_1^2 + \sigma_2^2}$$

with $df = \frac{(n_1-1)(n_2-1)(\sigma_1^2+\sigma_2^2)^2}{(n_2-1)\sigma_1^4+(n_1-1)\sigma_2^4}$. In this configuration, the degrees of freedom are a function of the total sample size (N) and the interaction between sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- The larger the total sample size, the lower the variance (illustration in Figure 39);

- The smallest variance always occurs when there is a positive pairing between variances and sample sizes, because one gives more weight to the smallest variance in the denominator of the *df* computation and in the numerator of the variance computation. Moreover, the further the SD-ratio is from 1, the further from 1 will also be the sample sizes ratio associated with the smallest variance (see Figure 40). This can be explained by splitting the numerator and the denominator of the *df* computation (see the file "Theoretical Bias, as a function of population parameters").

Note: for a constant SD-ratio, the variance does not matter. (See Figure 41).

When $\boldsymbol{\delta^*_{Cohen} \neq 0}$.

While the variance of Cohen's d^* still depends on the total sample size, the SD-ratio and the interaction between the sample sizes ratio and the SD-ratio, it also depends on the population effect size (δ^*_{Cohen}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the degrees of freedom increase (i.e. when the total sample size increases and/or when there is a positive pairing between the sample sizes ratio and the SD-ratio), as previously illustrated in Figure 35.

In summary.

The variance of Cohen's d^* is a function of the population effect size (δ^*_{Cohen}), the total sample size (N) and the interaction between sample sizes ratio and SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- The variance always decreases when the control and/or the experimental group increases. The benefit of adding subjects rather in the control or in the experimental group depends on the SD-ratio. Indeed, the smallest variance always occurs when there is a positive pairing between variances and sample sizes. Moreover, the further the SD-ratio is from 1, the further from 1 will also be the sample sizes ratio associated with the smallest variance;
- The variance increases when δ^*_{Cohen} increases. Note that the effect of δ^*_{Cohen} is moderated by the total sample size and the interaction between sample sizes ratio and SD-ratio.

Shieh's *d*.

When variances are equal across populations.

When $\boldsymbol{\delta_{Shieh} = 0}$.

When the population effect size is zero, the variance of Shieh's d can be simplified as follows:

$$Var_{Shieh's\ d} = \frac{df}{(df-2)N}$$

with $df = \frac{N^2(n_1-1)(n_2-1)}{n_2^2(n_2-1)+n_1^2(n_1-1)}$. In this configuration, the degrees of freedom as well as the variance of Shieh's d depend on the total sample size (N) and the sample sizes allocation ratio ($\frac{n_2}{n_1}$):

- The further the sample sizes allocation ratio is from 1, the larger the variance (see Figure 42);

- The larger the total sample size, the lower the variance. It does not matter whether the sample sizes ratio is constant (see Figure 43) or not (see Figure 44).

Note: in "Theoretical Bias, as a function of population parameters," we noticed that moving the sample sizes ratio away from 1 when adding subjects in only one group could decrease the degrees of freedom. However, due to the total sample size term (N) in the denominator of the variance computation, even when degrees of freedom decrease due to the fact that one adds subjects only in one group, the variance still decreases (because the denominator of the variance computation increases; see Figure 44).

When $\boldsymbol{\delta_{Shieh} \neq 0}$.

While the variance of Shieh's d still depends on the total sample size and the sample sizes ratio, it also depends on the population effect size (δ_{Shieh}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the degrees of freedom increase (i.e. when sample sizes increase, without increasing the sample sizes ratio, and/or the sample sizes ratio gets closer to 1), as previously illustrated in Figure 35.

Note: we previously noticed that when the effet size is zero, the variance of Shieh's d decreases, even when the sample sizes ratio increases. It is no longer true when there is a non-null effect size because the larger the sample sizes ratio, the more the variance will increase with increasing effect size.

When variances are unequal across populations, with equal sample sizes.

When $\boldsymbol{\delta_{Shieh} = 0}$.

When the population effect size is zero, the variance of Shieh's d can be simplified as follows:

$$Var_{Shieh's\ d} = \frac{df}{(df-2)N}$$

with $df = \frac{(\sigma_1^2+\sigma_2^2)^2 \times (n-1)}{\sigma_1^4+\sigma_2^4}$. In this configuration, the degrees of freedom as well as the variance of Shieh's d depend on the total sample size (N) and the SD-ratio ($\frac{\sigma_2}{\sigma_1}$).

- The further the SD-ratio is from 1, the larger the variance (see Figure 45);

- The larger the total sample size, the lower the variance (see Figure 46).

Note: for a constant SD-ratio, the size of the variance does not matter (see Figure 47).

When $\boldsymbol{\delta_{Shieh} \neq 0}$.

While the variance of Shieh's d still depends on the total sample size and the SD-ratio, it also depends on the population effect size (δ_{Shieh}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the degrees of freedom increase (i.e. when sample sizes increase and/or the SD-ratio gets closer to 1), as previously illustrated in Figure 35.

When variances are unequal across populations, with unequal sample sizes.

When $\boldsymbol{\delta_{Shieh} = 0}$.

When the population effect size is zero, the variance of Shieh's d can be simplified as follows:

$$Var_{Shieh's\ d} = \frac{df}{(df-2)N}$$

with $df = \frac{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}{\frac{(\sigma_1^2/n_1)^2}{n_1-1}+\frac{(\sigma_2^2/n_2)^2}{n_2-1}}$. In this configuration, the degrees of freedom as well as the variance of Shieh's d depend on the total sample size (N) and the interaction between the sample sizes ratio and the SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- The larger the total sample size, the lower the variance. It remains true when the sample sizes ratio is constant (see Figure 48) and when it is not (see Figure 49).

Note: When variances were equal across populations, adding subjects only in the first group had the same impact on the variance as adding subjects only in the second group (see Figure 44). When variances are unequal across groups, this is not true anymore (see Figure 49).

- The smallest variance always occurs when there is a positive pairing between variances and sample size. Moreover, the further the SD-ratio is from 1, the further from 1 will also be the sample sizes ratio associated with the smallest variance (see Figure 50).

Moreover, for a constant SD-ratio, the variances do not matter (See Figure 51).

When $\boldsymbol{\delta_{Shieh}} \neq \mathbf{0}$.

While the variance of Shieh's d still depends on the total sample size and the interaction between the sample sizes ratio and the SD-ratio, it also depends on the population effect size (δ_{Shieh}): the larger the population effect size, the larger the variance. However, the effect of the population effect size decreases when the degrees of freedom increase, as previously illustrated in Figure 35.

In summary.

The variance of Shieh's d is a function of the population effect size (δ_{Shieh}), the total sample size (N) and the interaction between sample sizes ratio and SD-ratio $\left(\frac{n_2}{n_1} \times \frac{\sigma_2}{\sigma_1}\right)$:

- The variance always decreases when the control and/or the experimental group increases. The benefit of adding subjects rather in the control or in the experimental group depends on the SD-ratio. Indeed, the smallest variance always occurs when there is a positive pairing between variances and sample sizes. Moreover, the further the SD-ratio is from 1, the further from 1 will also be the sample sizes ratio associated with the smallest variance;

- The variance increases when δ_{Shieh} increases. Note that the effect of δ_{Shieh} is moderated by the total sample size and the interaction between the sample sizes ratio and the SD-ratio.

Supplemental Material 2 :

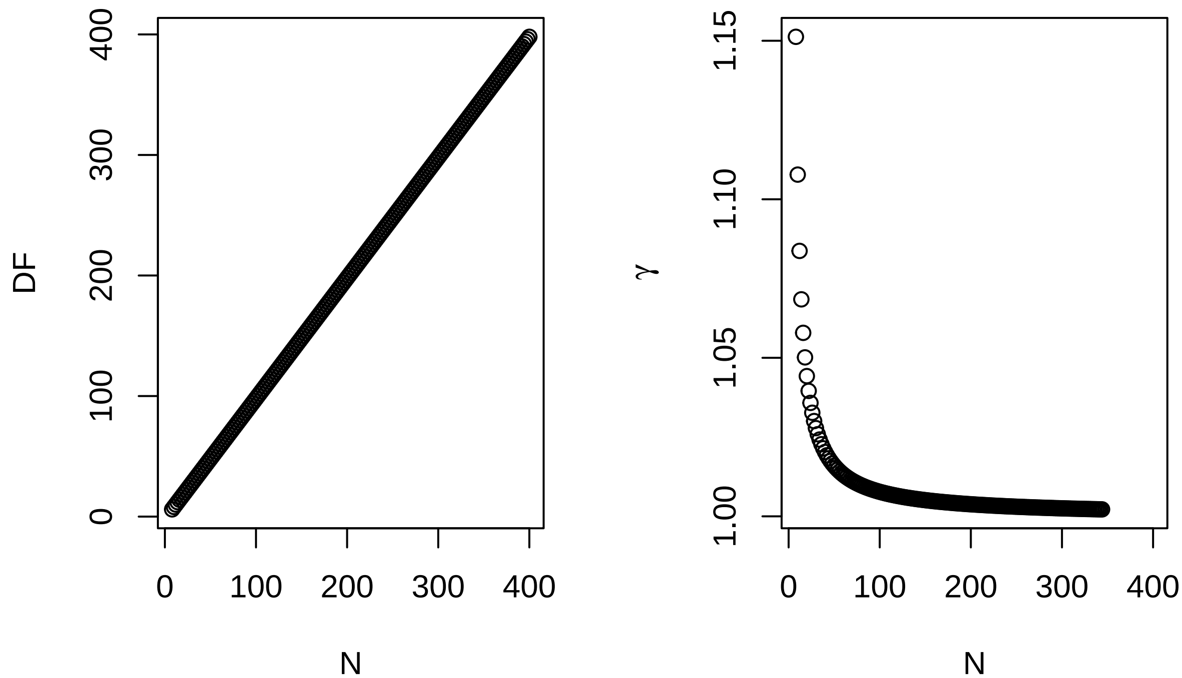


Figure 1. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d, when variances are equal across groups, as a function of the total sample size (N).

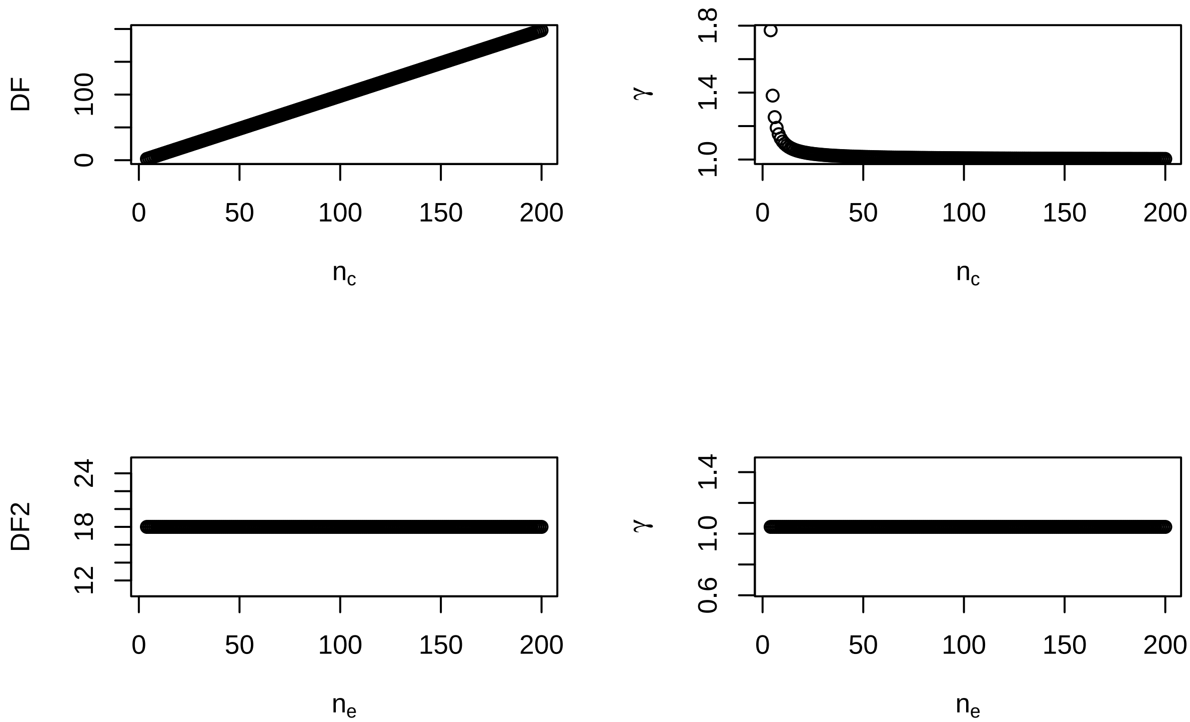


Figure 2. Degrees of freedom (DF) and γ, when computing the bias of Glass's d, when variances are equal across groups, as a function of n_c (top) and n_e (bottom).

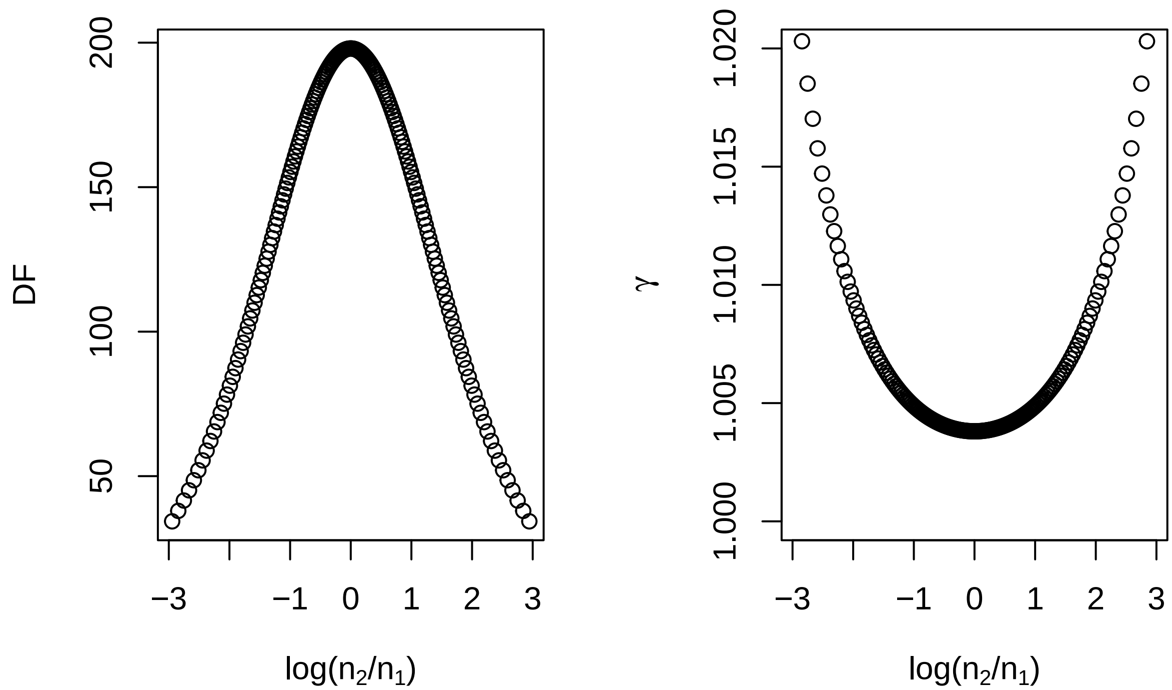


Figure 3. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances are equal across groups, as a function of the logarithm of the sample sizes ratio $log\left(\frac{n_2}{n_1}\right)$.

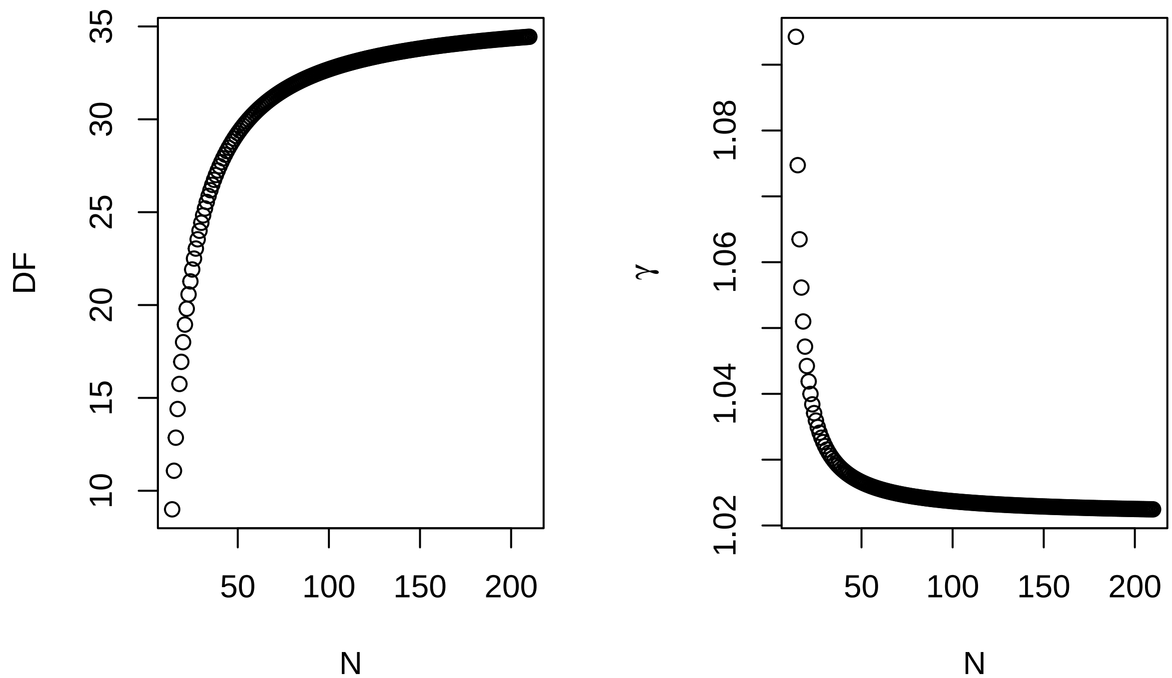


Figure 4. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances are equal across groups, as a function of the total sample size (N).

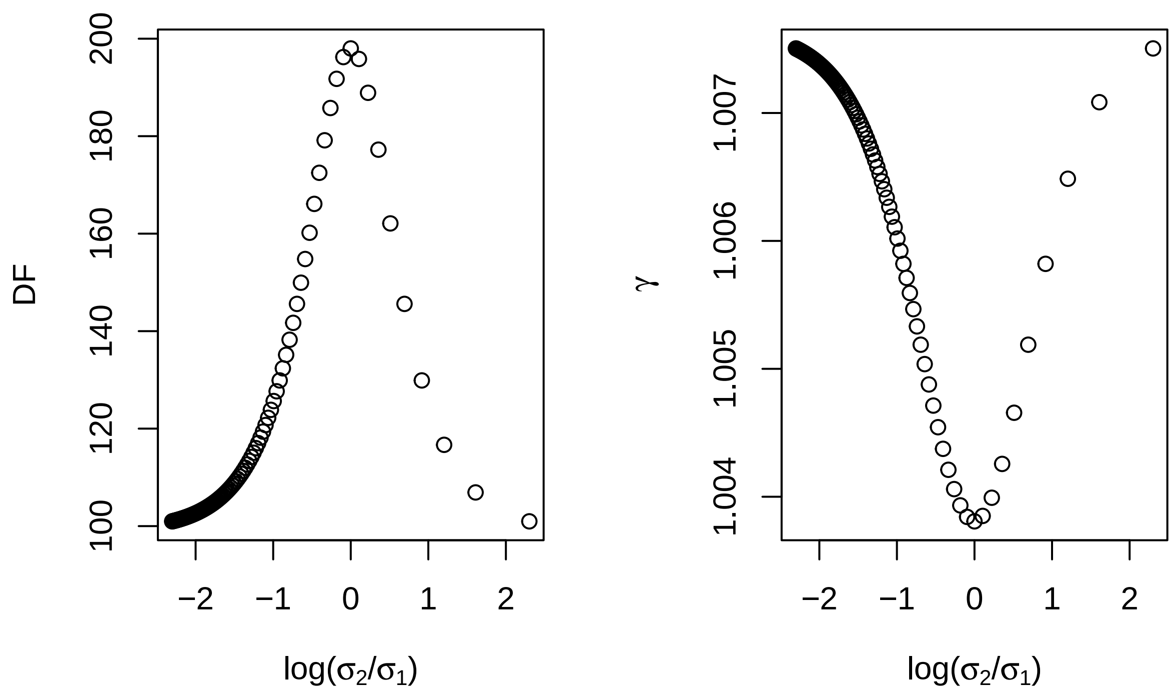


Figure 5. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances are unequal across groups and sample sizes are equal, as a function of the logarithm of the *SD*-ratio ($log\left(\frac{\sigma_2}{\sigma_1}\right)$).

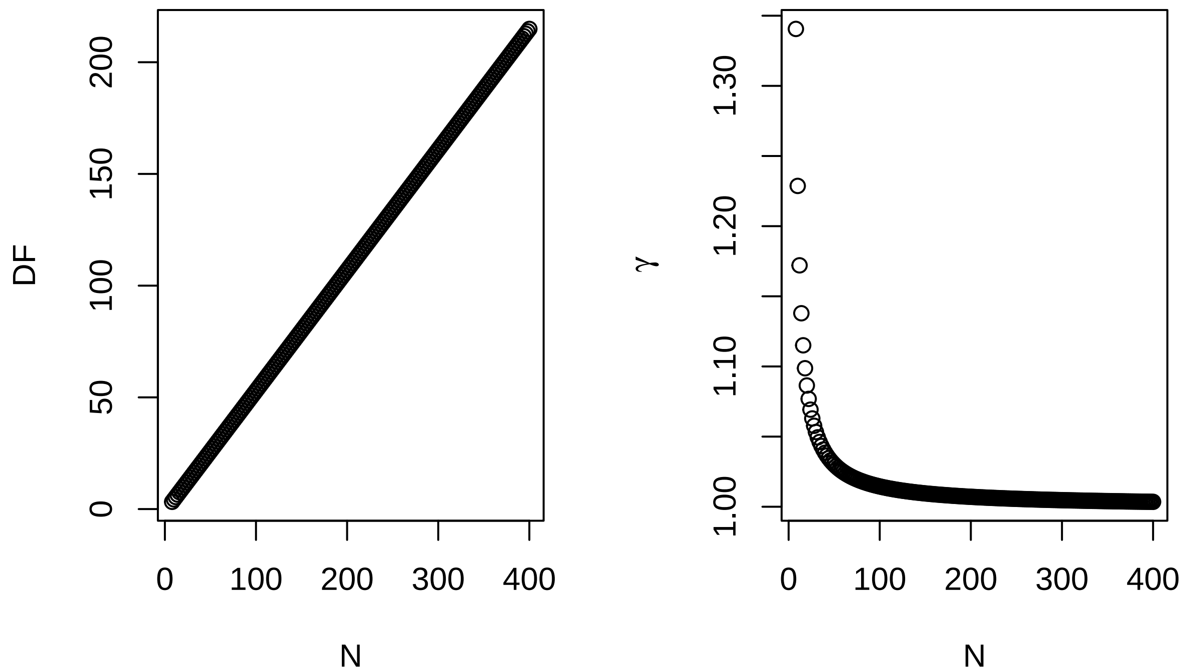


Figure 6. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances are unequal across groups and sample sizes are equal, as a function of the total sample size (N).

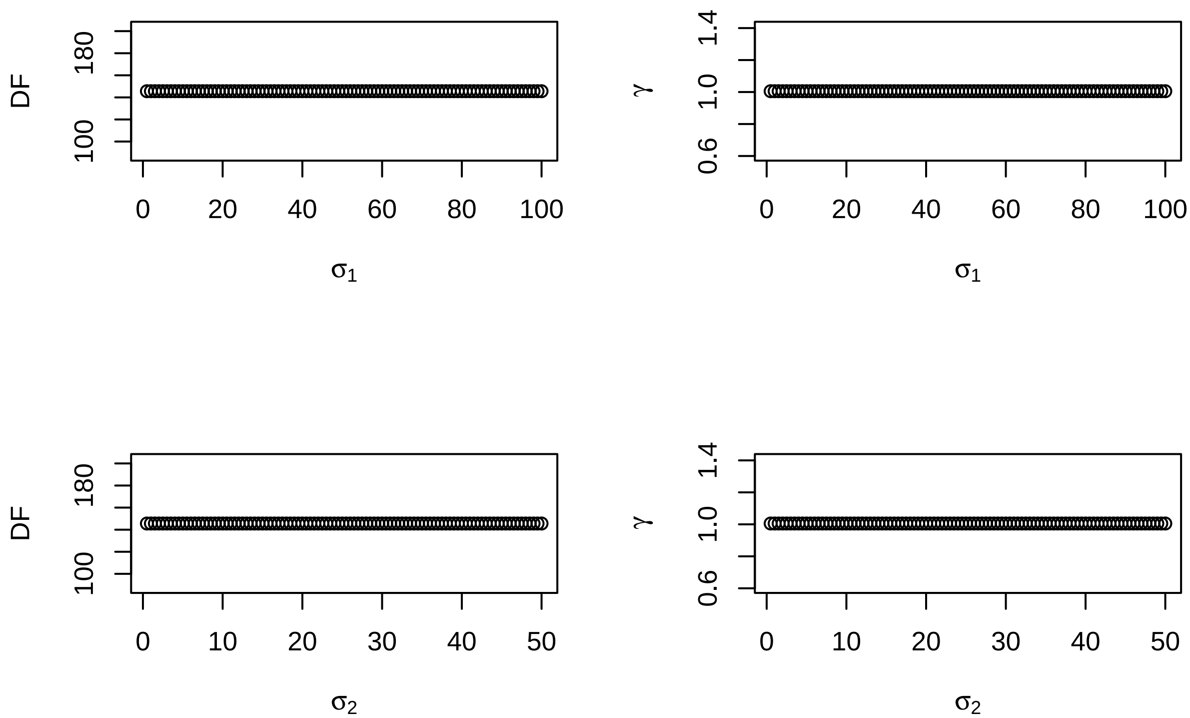


Figure 7. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances are unequal across groups and sample sizes are equal, as a function of σ_1 (top plots) and σ_2 (bottom plots), for a constant SD-ratio.

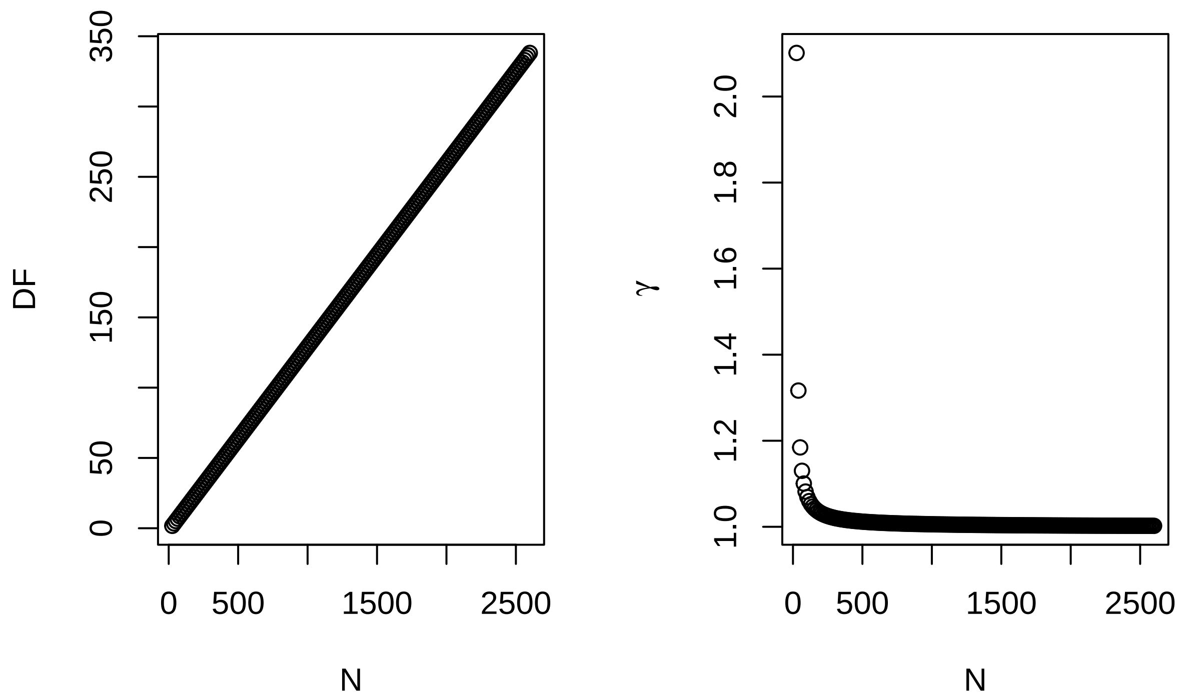


Figure 8. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances and sample sizes are unequal across groups, as a function of the total sample size (N).

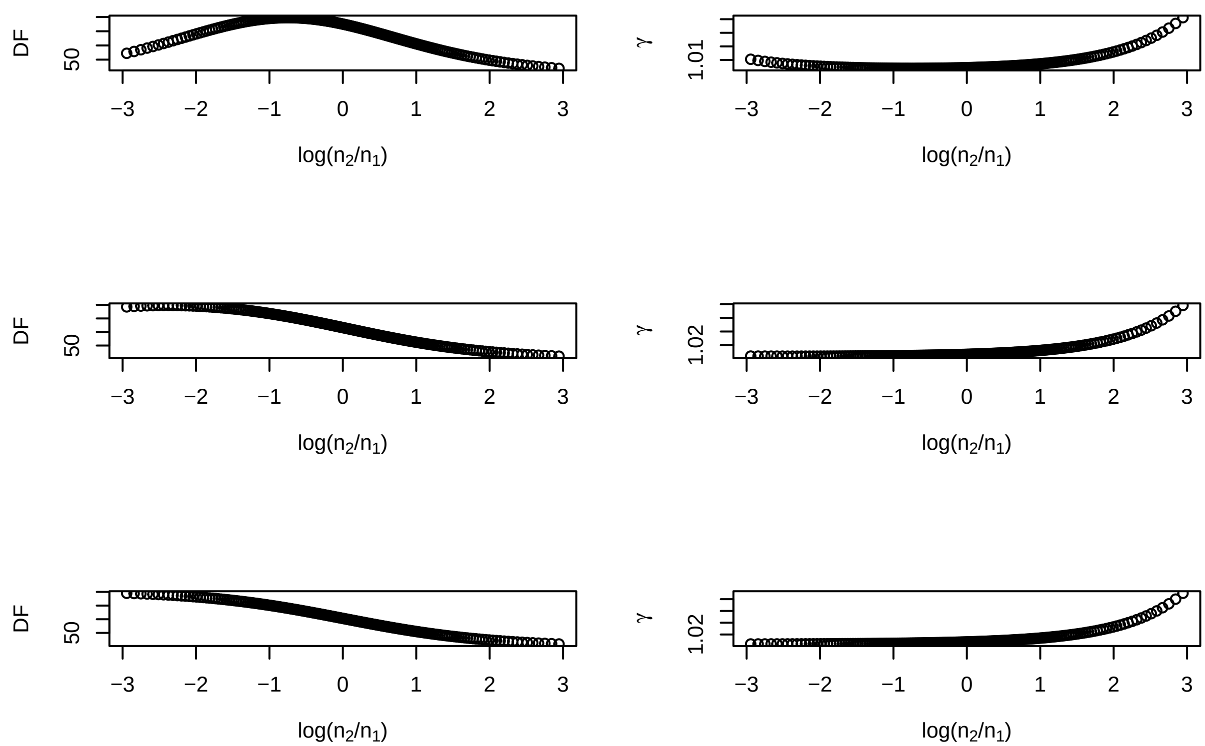


Figure 9. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^* when variances and sample sizes are unequal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$), when SD-ratio equals .68 (first row), .29 (second row) or .14 (third row).

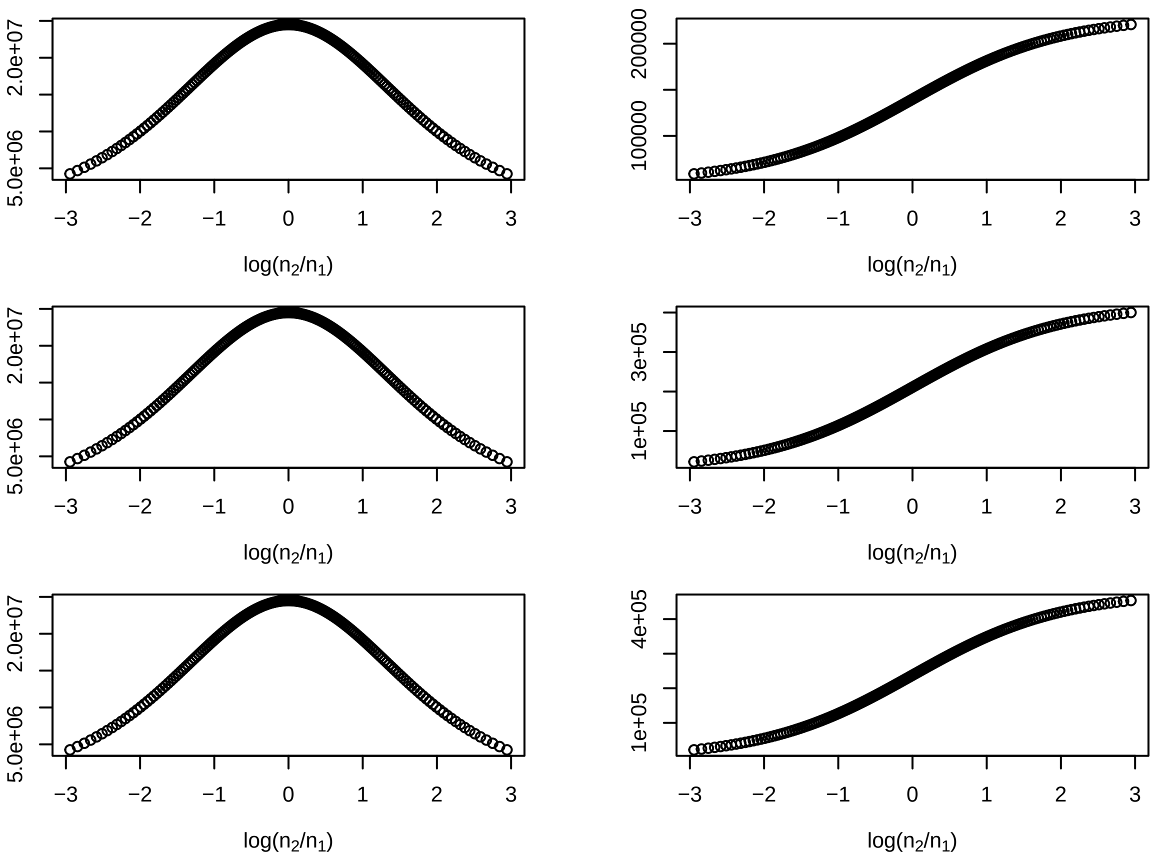


Figure 10. Numerator and denominator of the degrees of freedom (DF) computation, when computing the bias of Cohen's d^* when variances and sample sizes are unequal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$), when SD-ratio equals .68 (first row), .29 (second row) or .14 (third row).

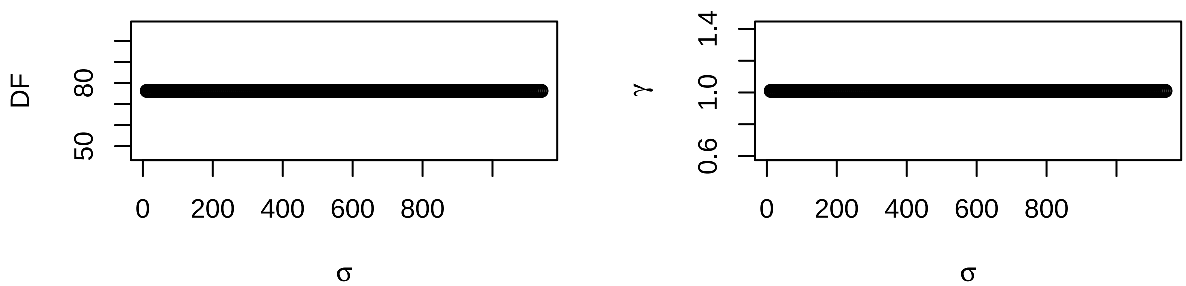


Figure 11. Degrees of freedom (DF) and γ, when computing the bias of Cohen's d^*, when variances and sample sizes are unequal across groups, as a function of $\sigma = \frac{(\sigma_1^2+\sigma_2^2)}{2}$, for a constant SD-ratio.

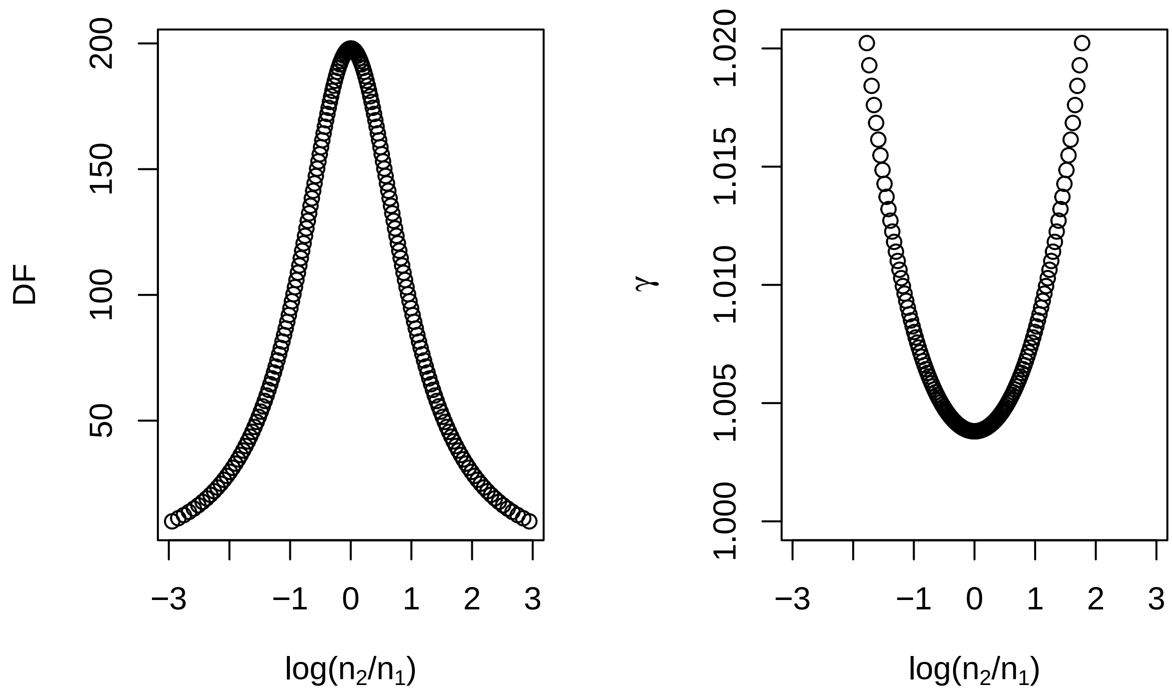


Figure 12. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances are equal across groups, as a function of the logarithm of the sample sizes ratio $(log\left(\frac{n_2}{n_1})\right)$.

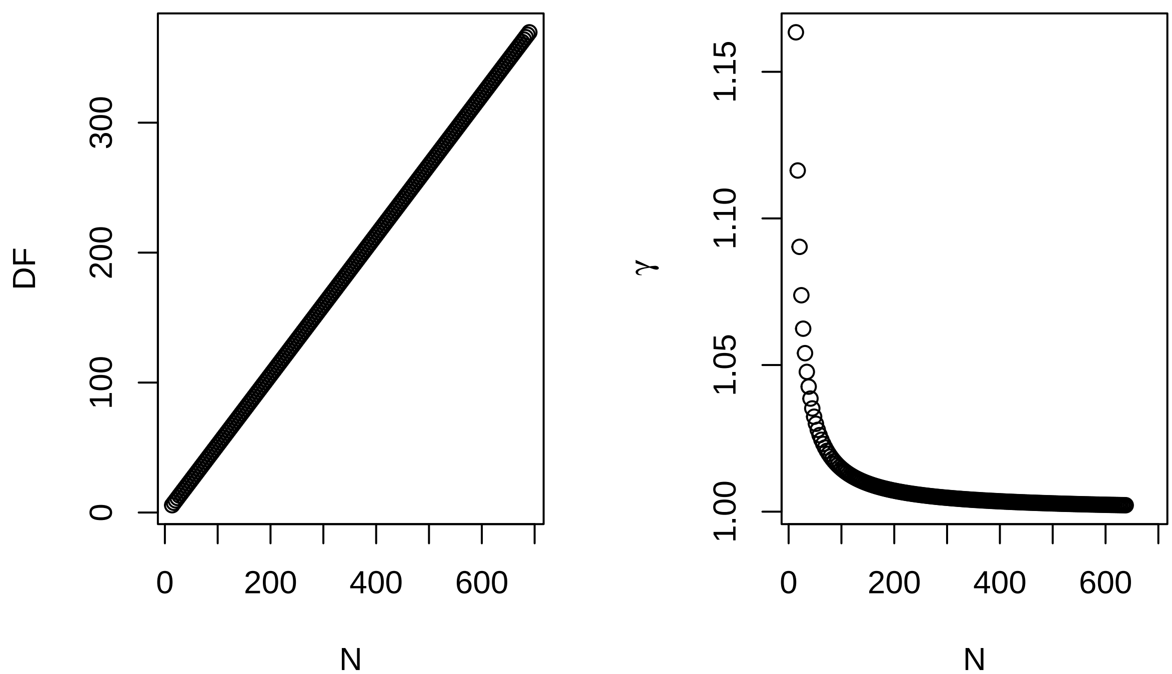


Figure 13. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances are equal across groups, as a function of the total sample size (N).

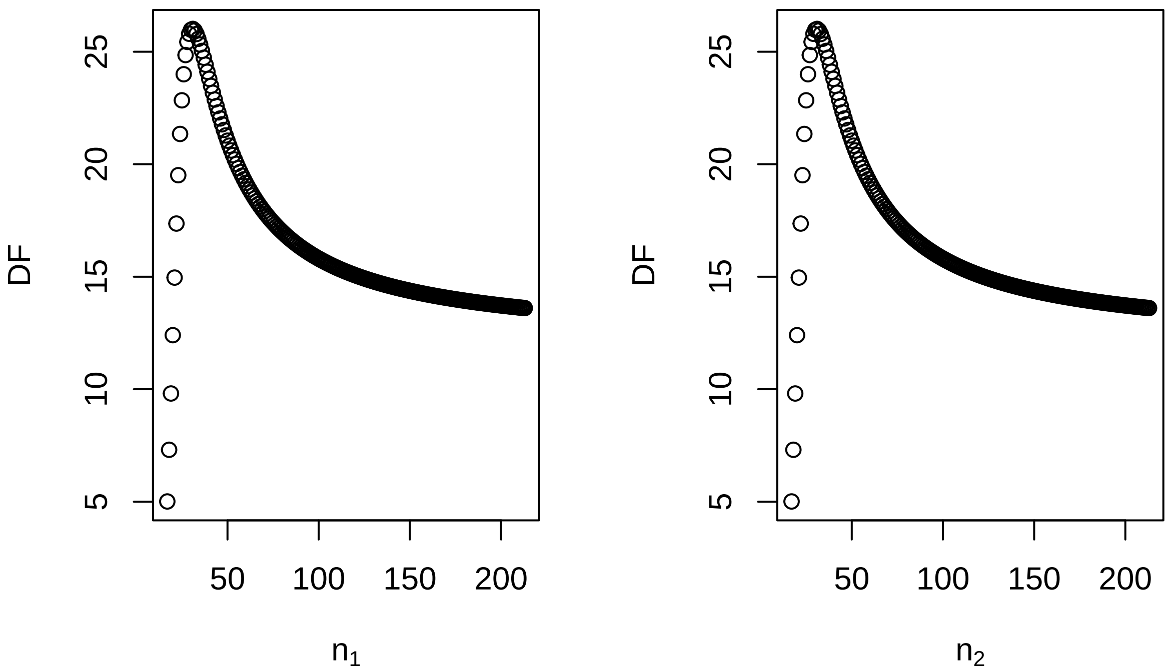


Figure 14. Degrees of freedom (DF), when computing the bias of Shieh's d, when variances are equal across groups, when adding subjects only in the first group (left) or in the second group (right).

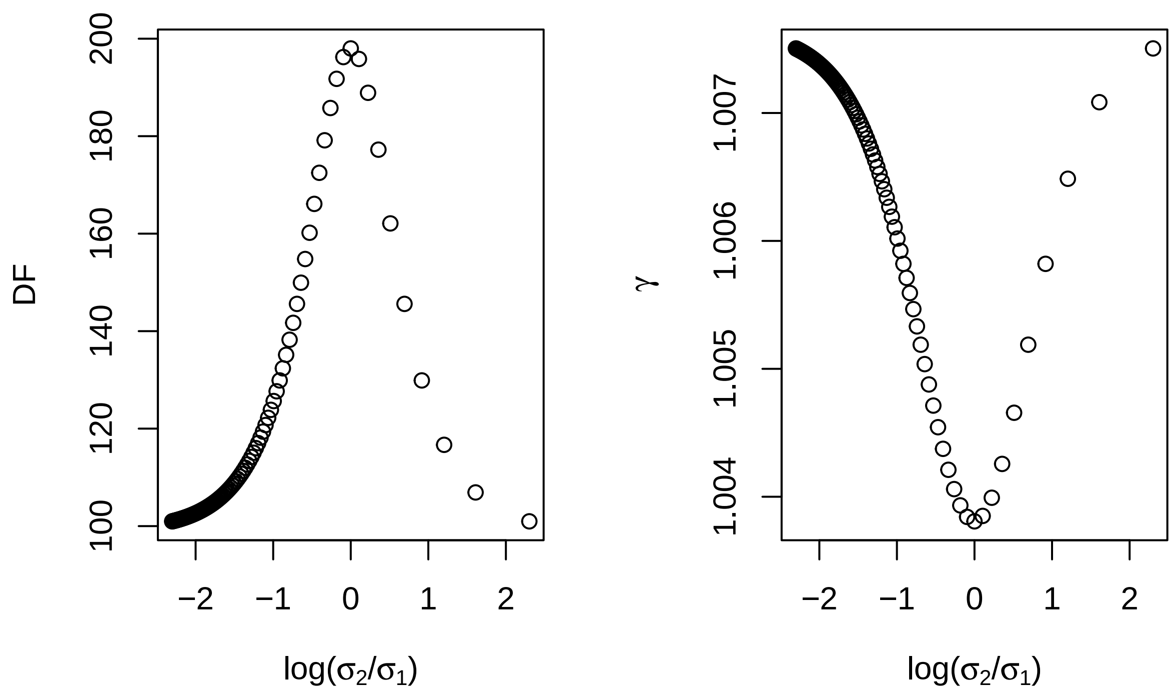


Figure 15. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances are unequal across groups and sample sizes are equal, as a function of the logarithm of the SD-ratio ($log\left(\frac{\sigma_2}{\sigma_1}\right)$).

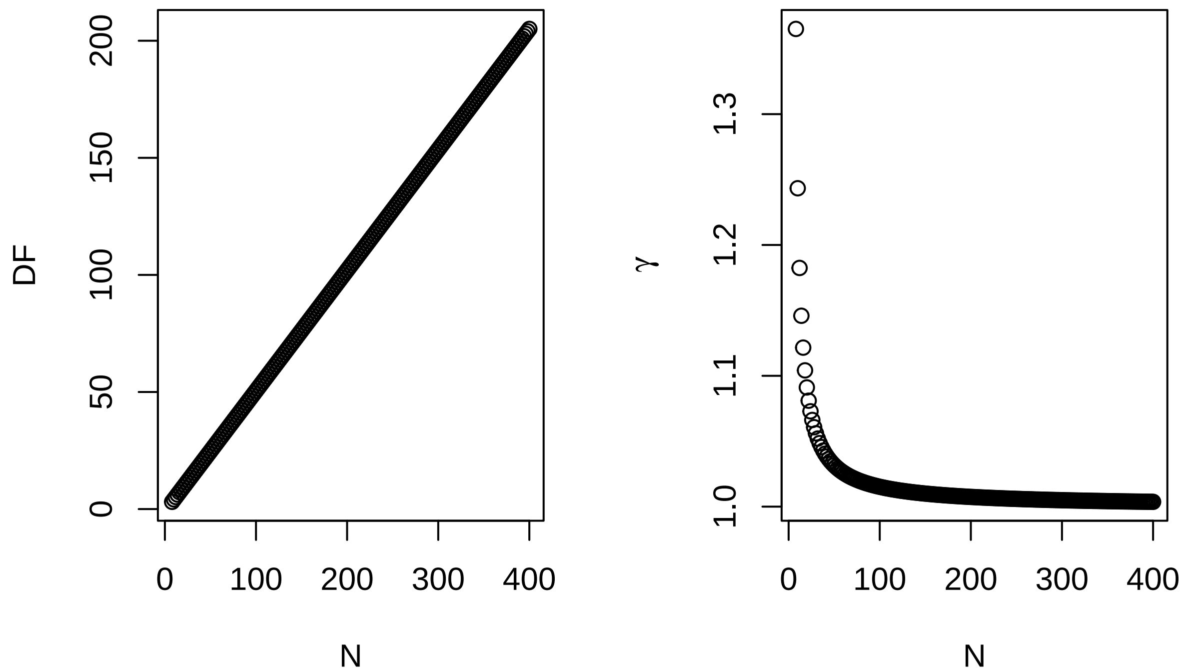


Figure 16. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances are unequal across groups and sample sizes are equal, as a function of the total sample size (N).

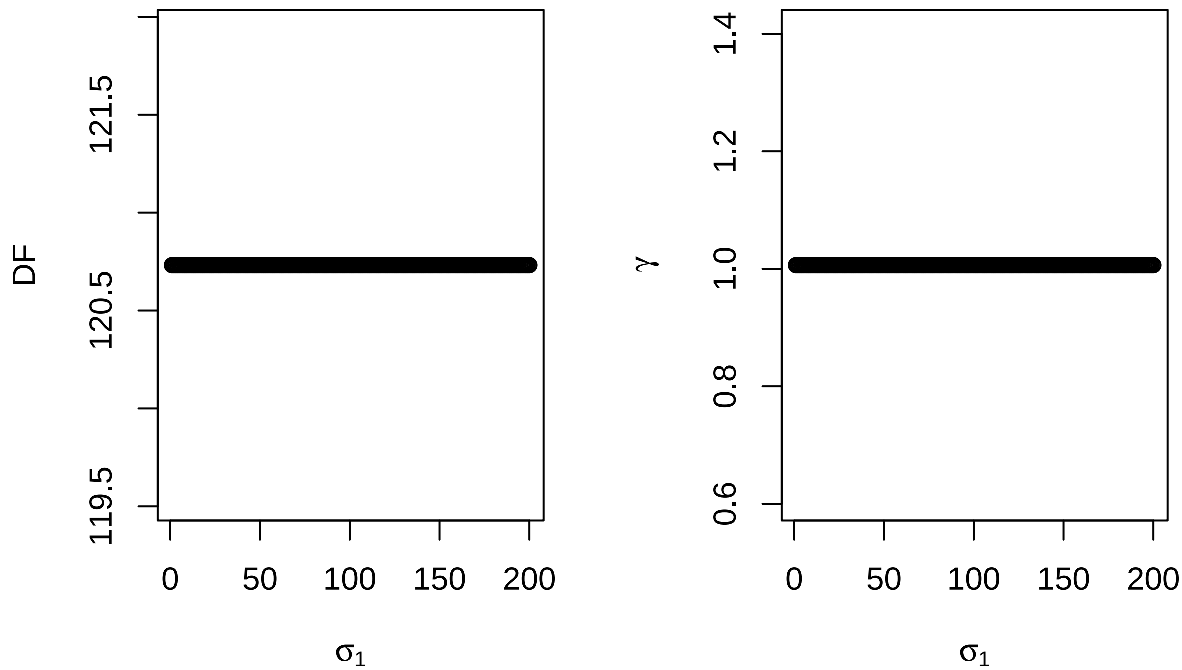


Figure 17. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances are unequal across groups and sample sizes are equal, as a function of σ_1, for a constant *SD*-ratio.

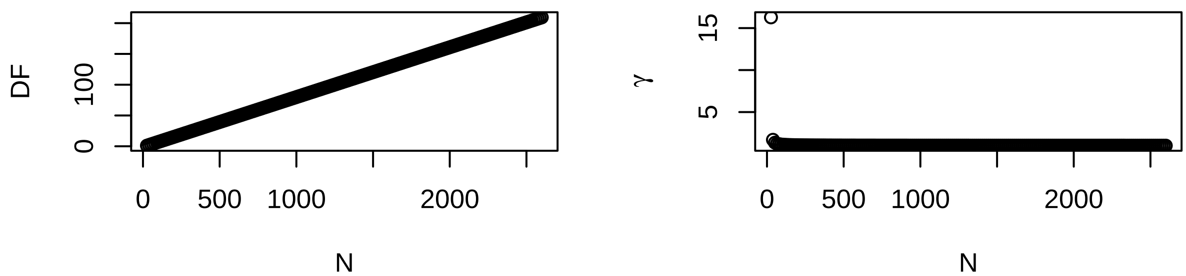


Figure 18. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances and sample sizes are unequal across groups, as a function of the total sample size (N).

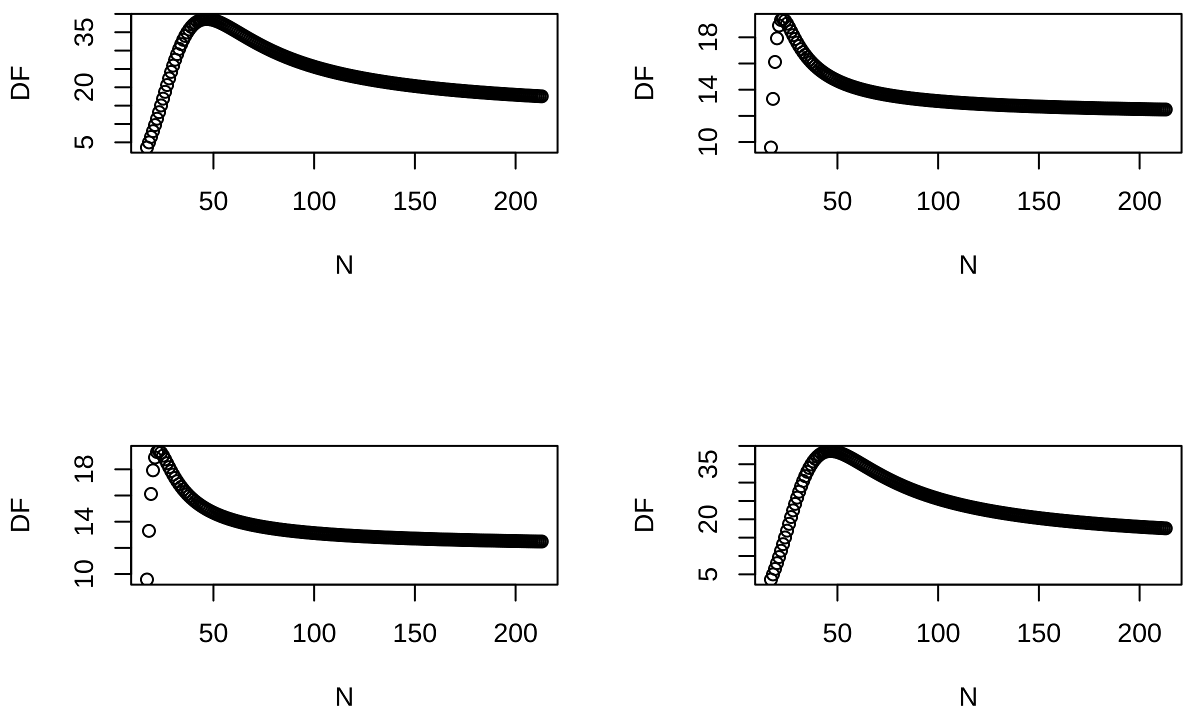


Figure 19. Degrees of freedom (DF), when computing the bias of Shieh's d, when variances and sample sizes are unequal across groups, as a function of the total sample size, when adding subjects only in one group (either in the first group; see top plots; or in the second group; see bottom plots), and $\sigma_1 > \sigma_2$ (left plots) or $\sigma_1 < \sigma_2$ (right plots).

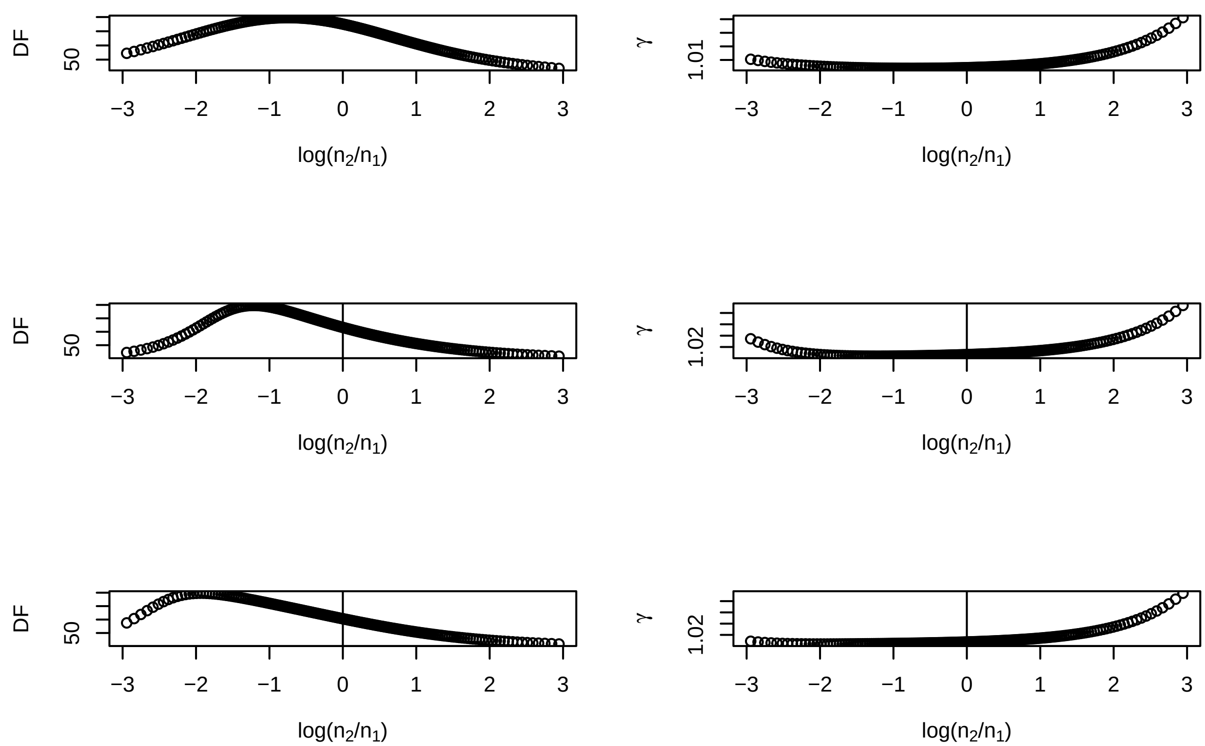


Figure 20. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances and sample sizes are unequal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$), when SD-ratio equals .68 (first row), .29 (second row) or .14 (third row).

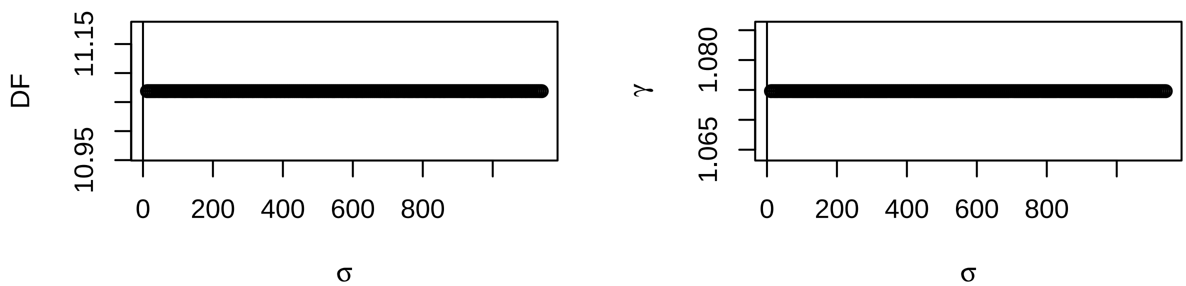


Figure 21. Degrees of freedom (DF) and γ, when computing the bias of Shieh's d, when variances and sample sizes are unequal across groups, as a function of σ_1 and σ_2, for a constant SD-ratio.

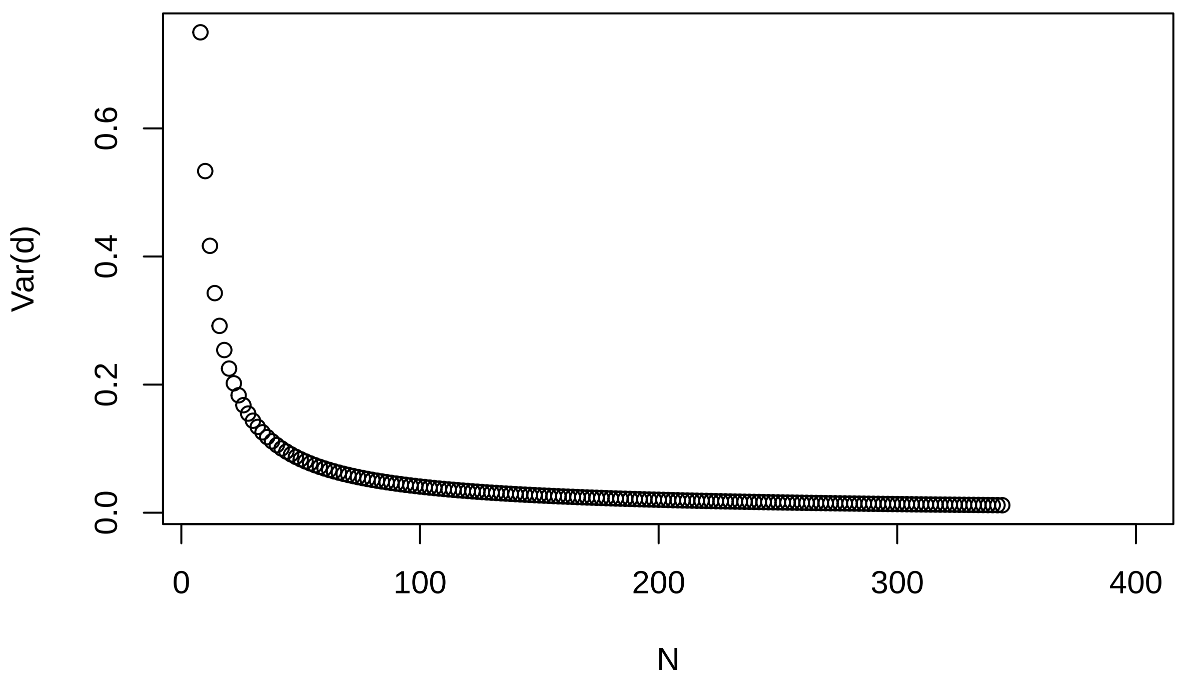


Figure 22. Variance of Cohen's d, when variances are equal across groups, as a function of the total sample size (N).

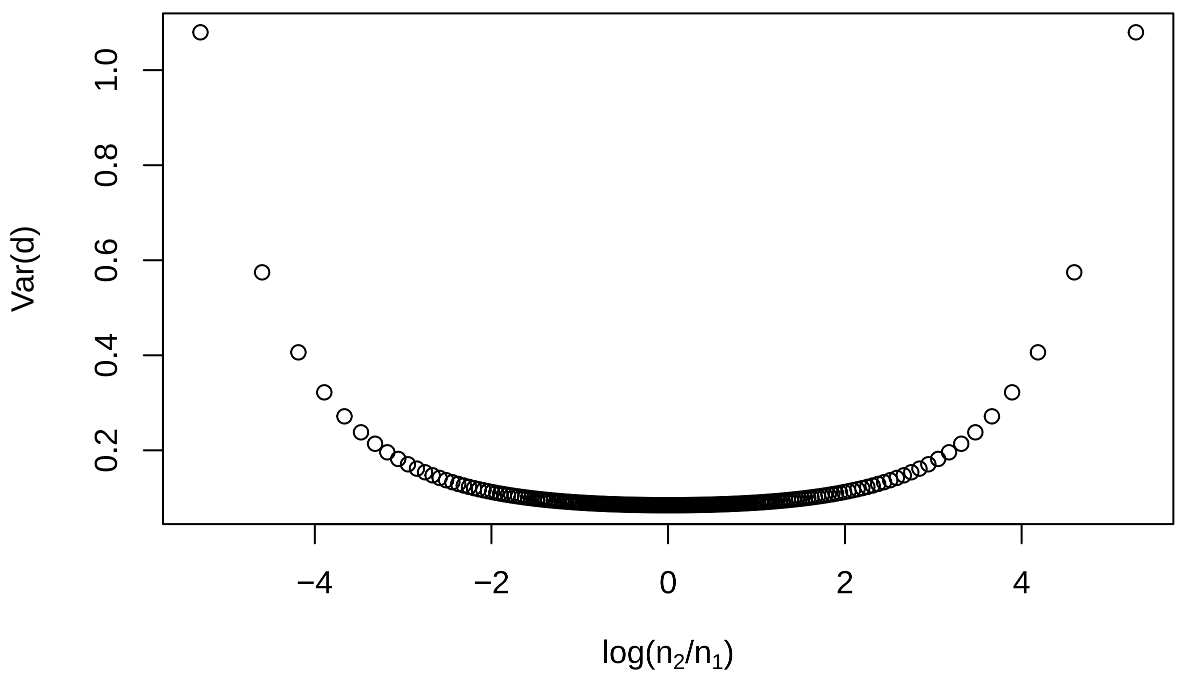


Figure 23. Variance of Cohen's d, when variances are equal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$).

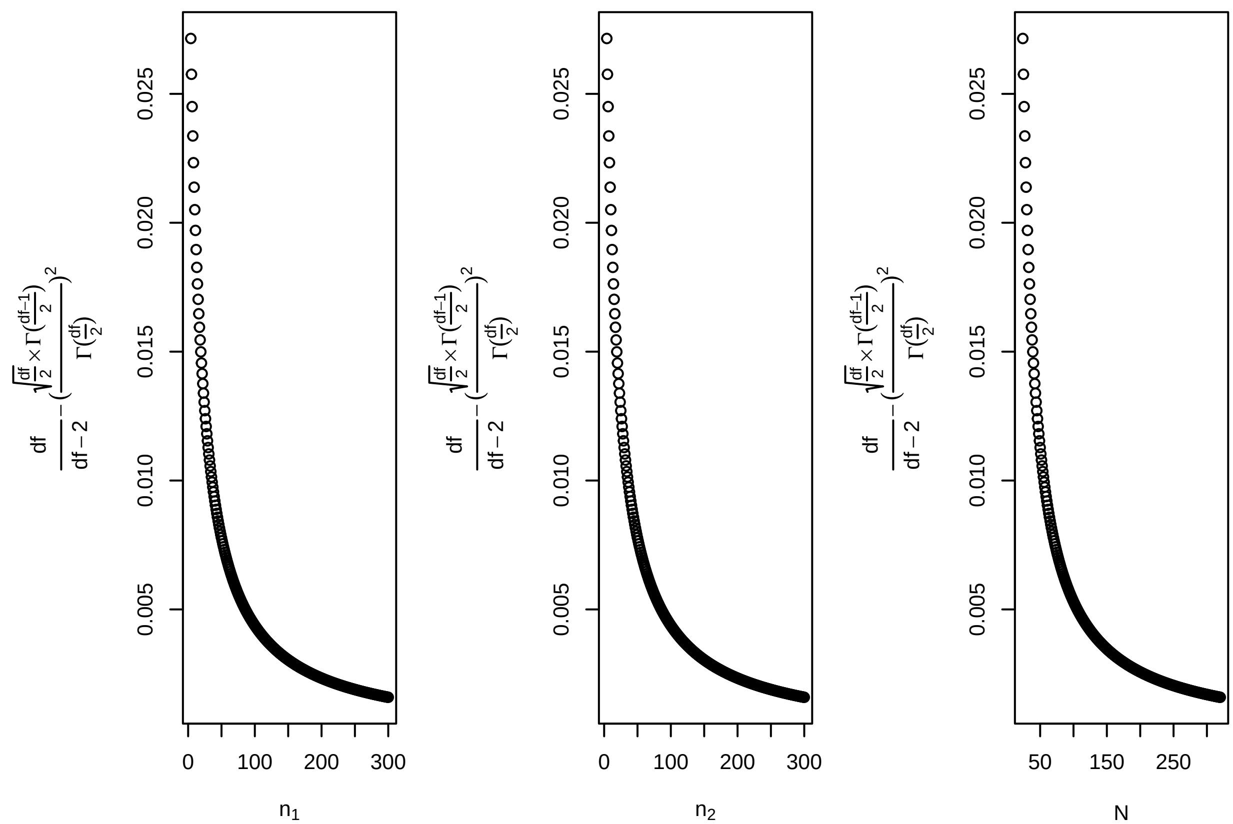


Figure 24. Effect size moderator, when computing the variance of Cohen's d, as a function of n_1 (left), n_2 (center) and $N = n_1 + n_2$ (right).

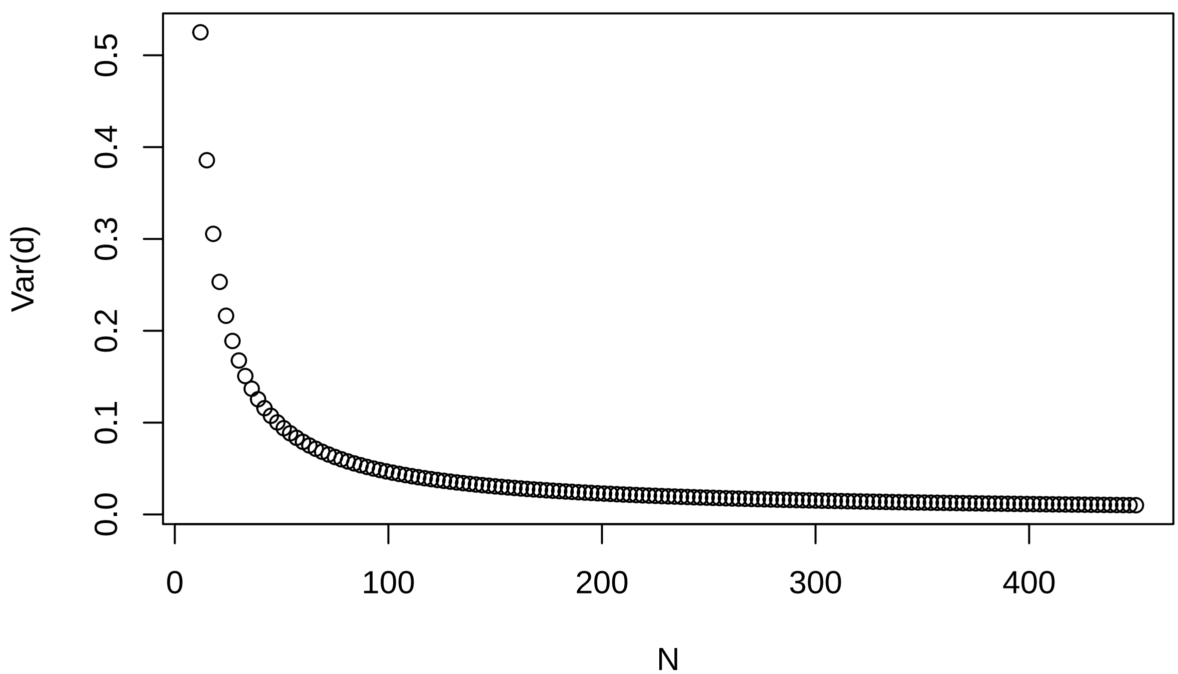


Figure 25. Variance of Glass's d, when variances are equal across groups, as a function of the total sample size (N).

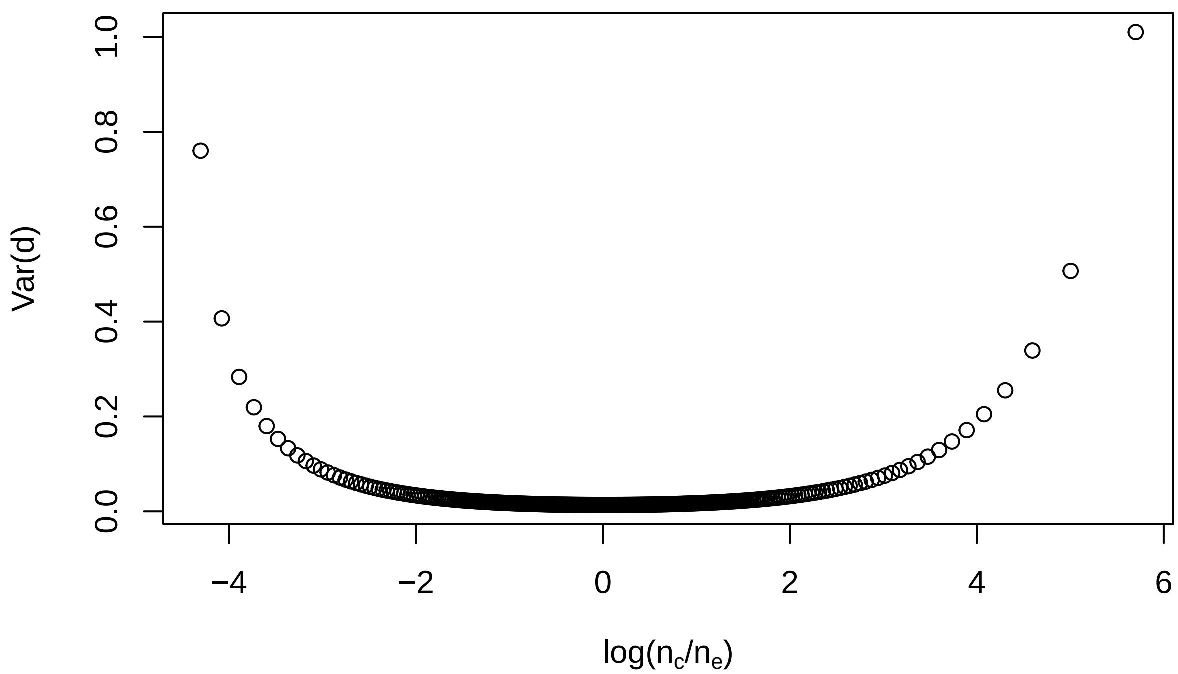


Figure 26. Variance of Glass's d, when variances are equal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_c}{n_e}\right)$).

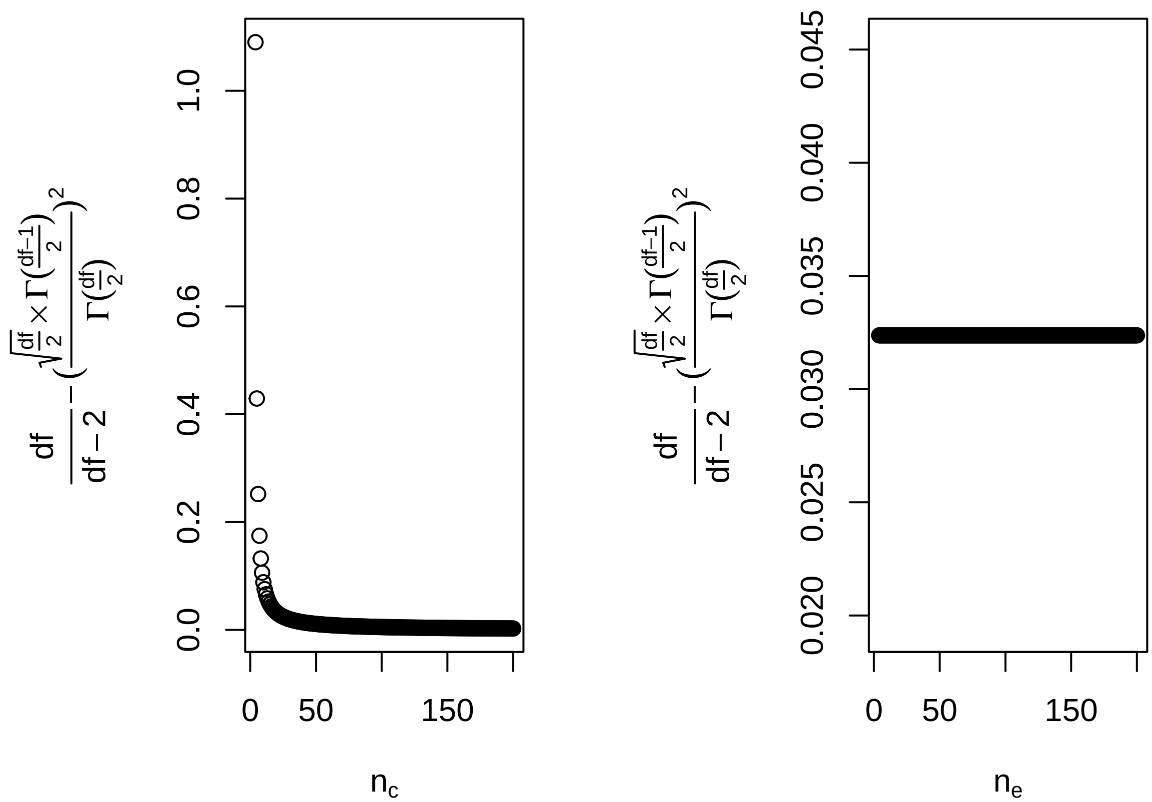


Figure 27. Effect size moderator, when computing the variance of Glass's *d*, as a function of the size of the control group (left) and experimental group (right).

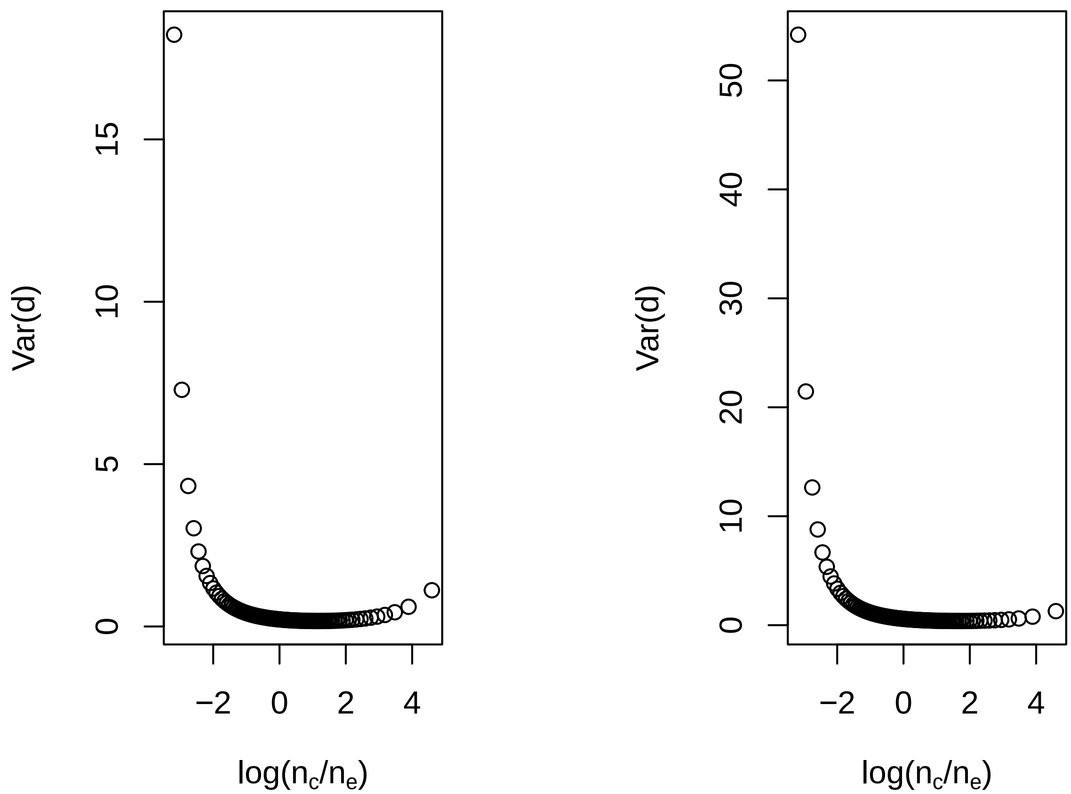


Figure 28. Variance of Glass's d, when variances are equal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_c}{n_e}\right)$) when δ_{Glass} equals 4 (left) or 7 (right).

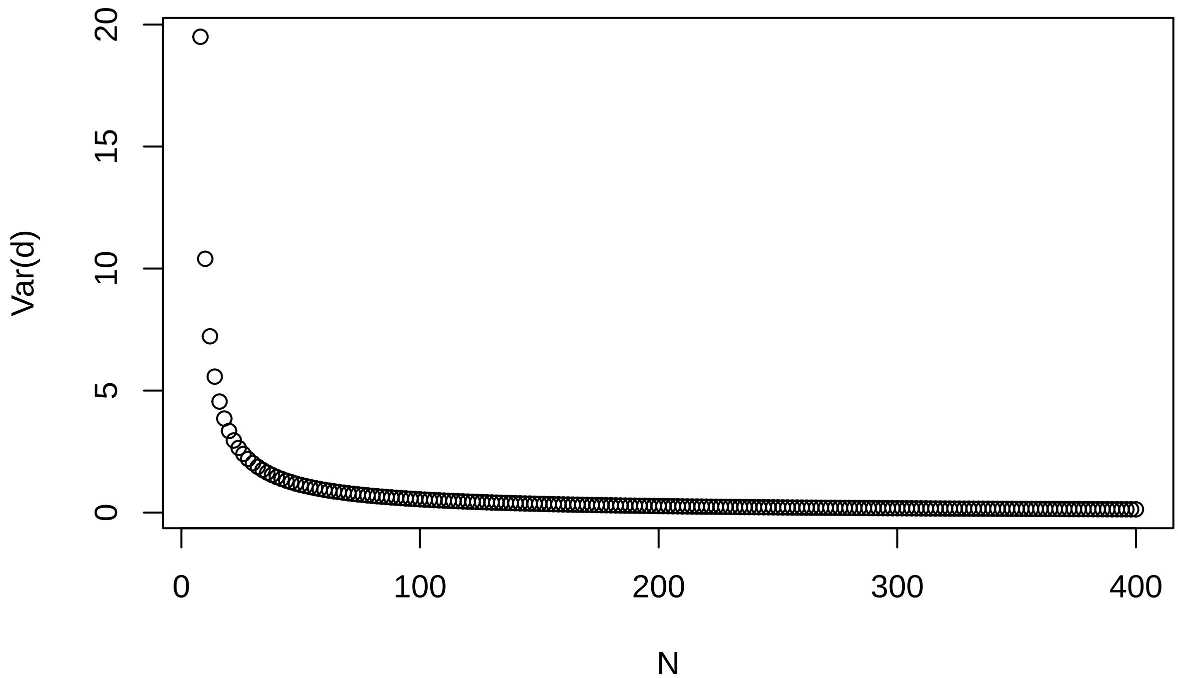


Figure 29. Variance of Glass's d, when variances are unequal across groups and sample sizes are equal, as a function of the total sample sizes (N).

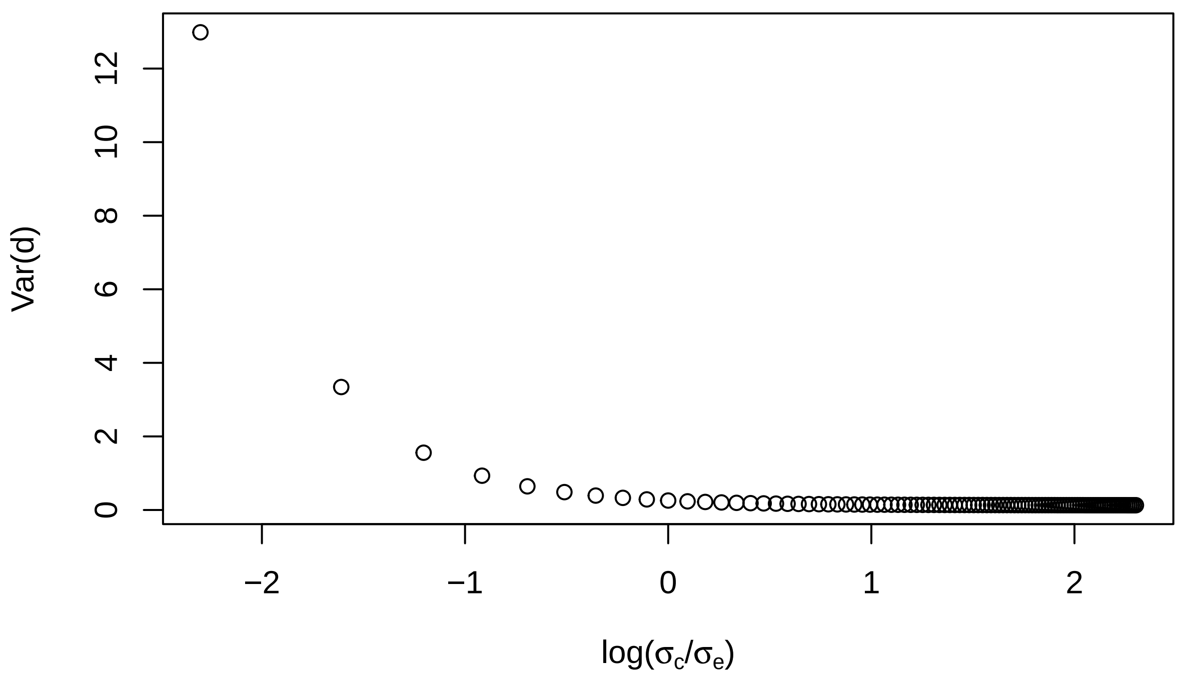


Figure 30. Variance of Glass's d, when variances are unequal across groups and sample sizes are equal, as a function of the logarithm of the SD-ratio ($log\left(\frac{\sigma_c}{\sigma_e}\right)$).

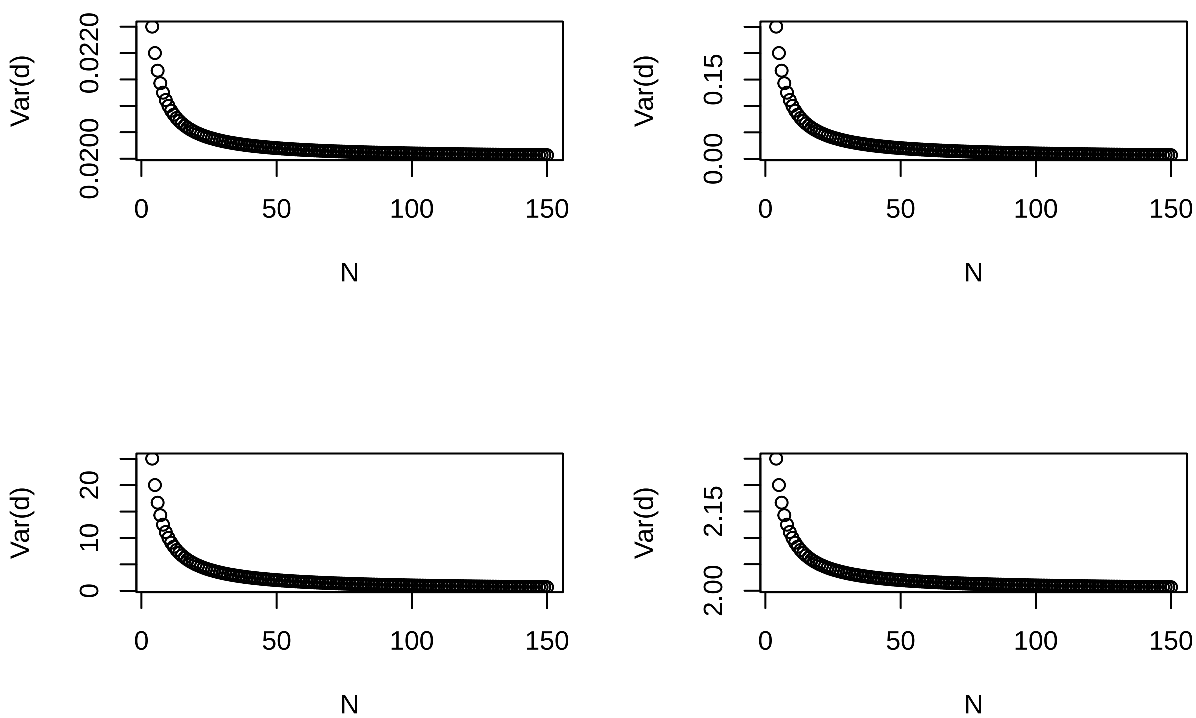


Figure 31. Variance of Glass's d, when variances and sample sizes are unequal across groups, as a function of the total sample sizes, when increasing only the control (right) or the experimental (left) group, when $\sigma_c > \sigma_e$ (top plots) or $\sigma_c < \sigma_e$ (bottom plots).

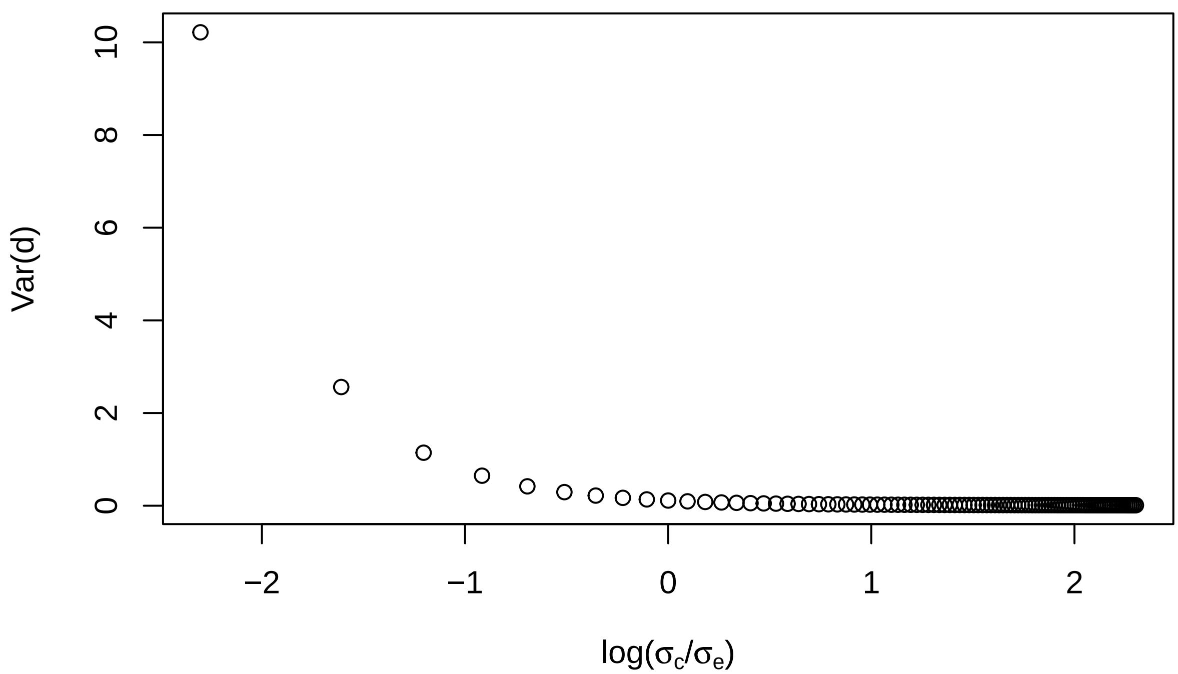


Figure 32. Variance of Glass's d, when sample sizes and variances are unequal across groups, as a function of the logarithm of the SD-ratio ($log\left(\frac{\sigma_c}{\sigma_e}\right)$).

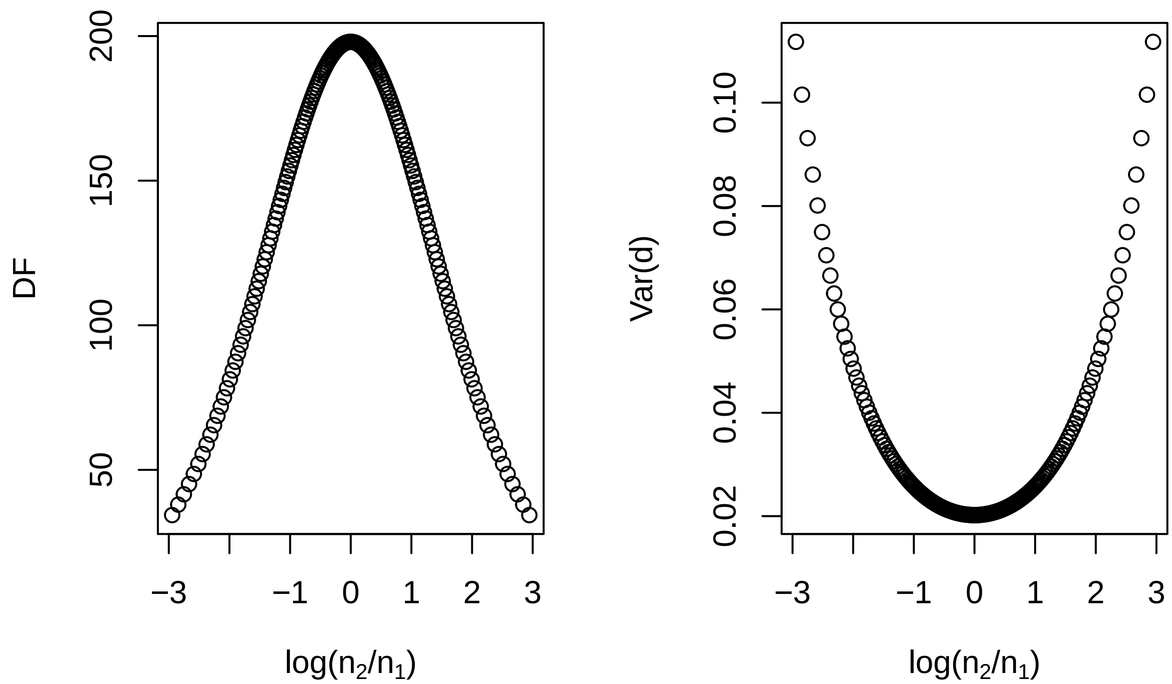


Figure 33. Variance of Cohen's d^* when variances are equal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$).

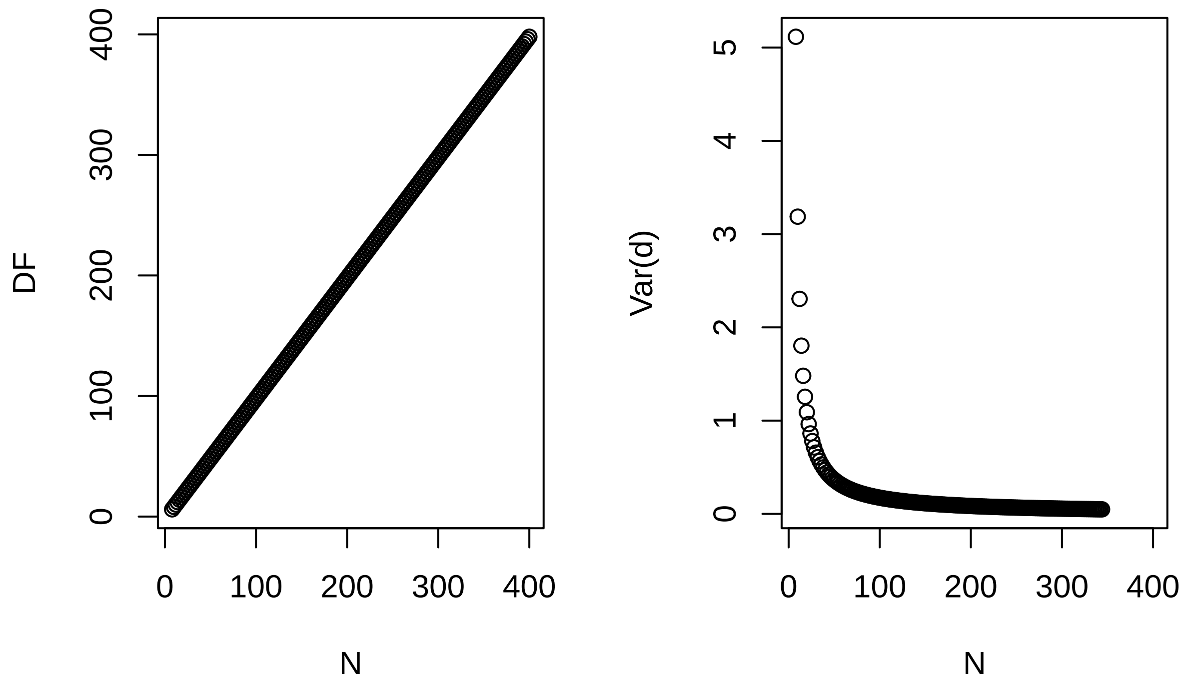


Figure 34. Variance of Cohen's d^* when variances are equal across groups, as a function of the total sample size (N).

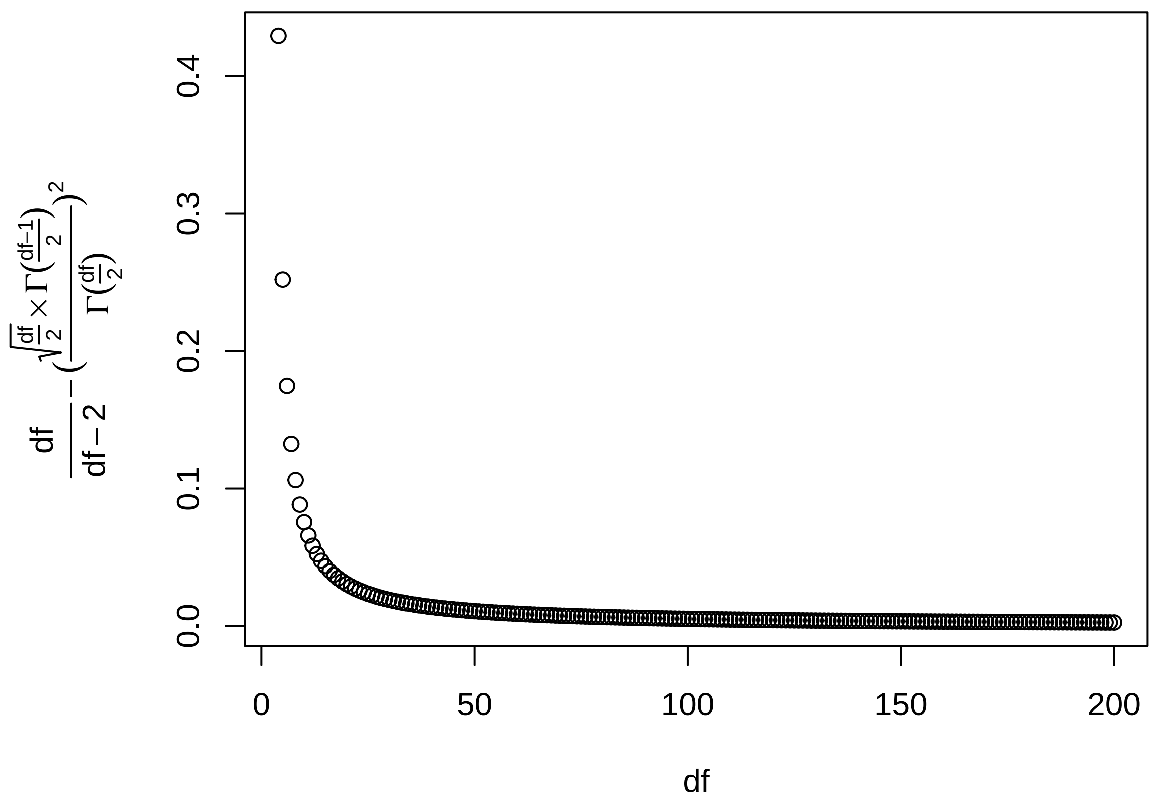


Figure 35. Effect size moderator (for all estimators), as a function of the degrees of freedom.

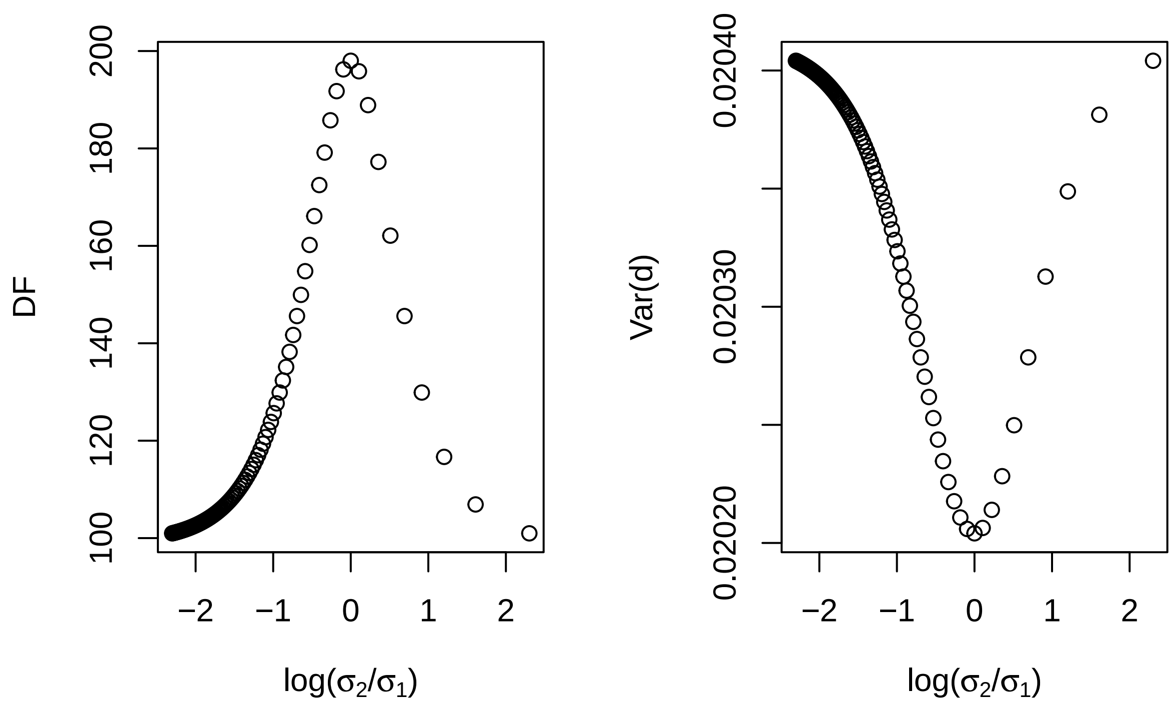


Figure 36. Variance of Cohen's d^* when variances are unequal across groups and sample sizes are equal, as a function of the logarithm of the SD-ratio ($log\left(\frac{\sigma_2}{\sigma_1}\right)$).

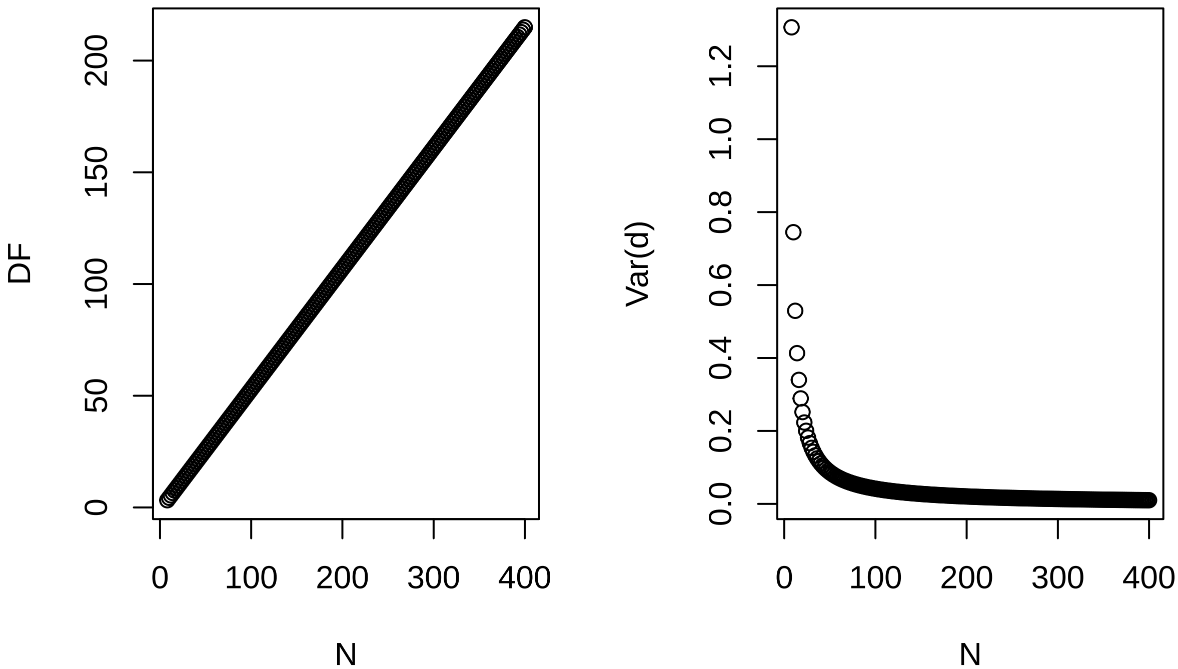


Figure 37. Variance of Cohen's d^* when variances are unequal across groups and sample sizes are equal, as a function of the total sample size (N).

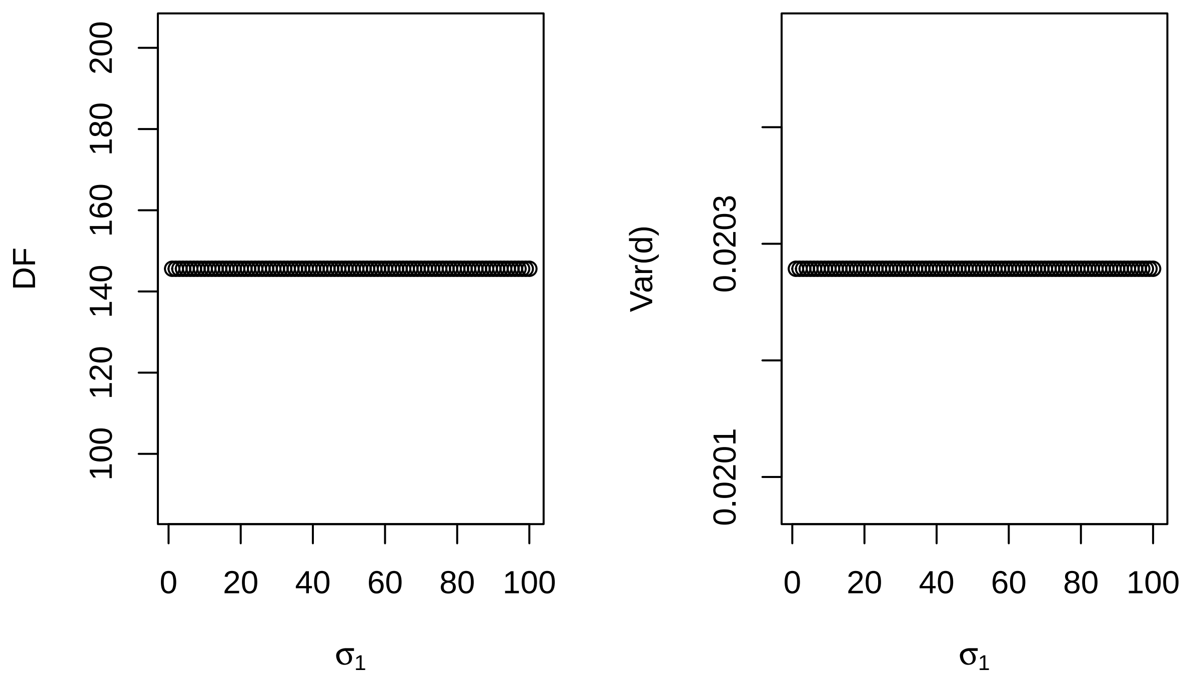


Figure 38. Variance of Cohen's d^*, when variances are unequal across groups and sample sizes are equal, as a function of σ_1 and σ_2, for a constant *SD*-ratio.

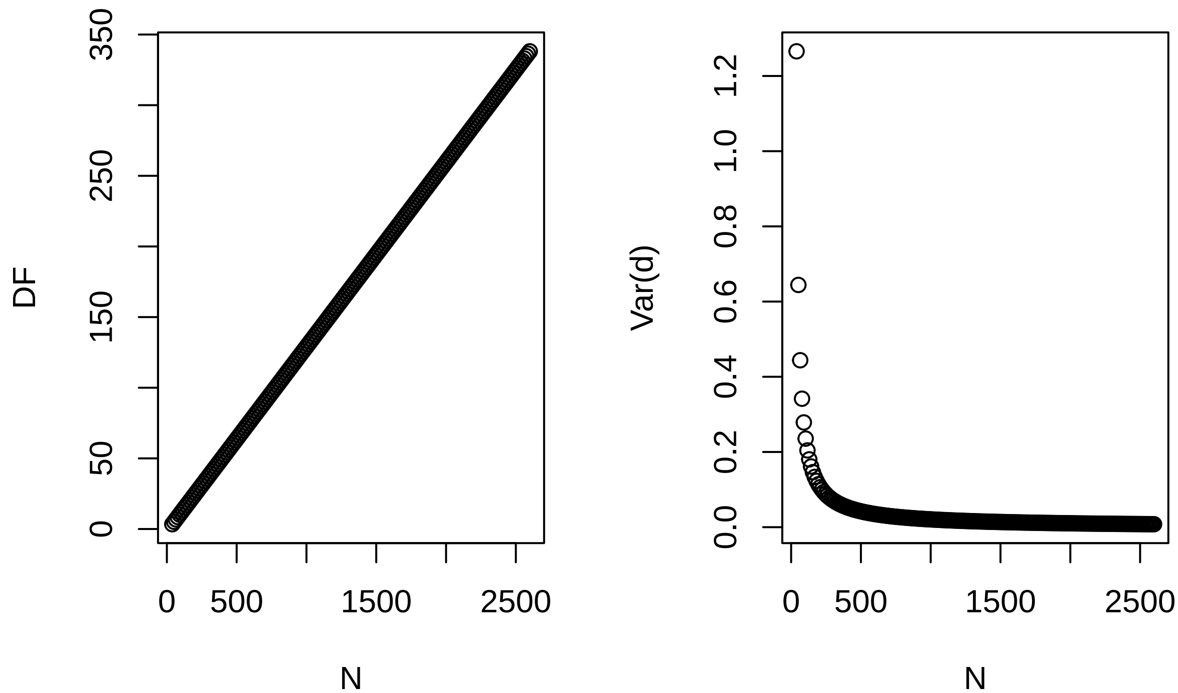


Figure 39. Variance of Cohen's d^* when variances and sample sizes are unequal across groups, as a function of the total sample size (N).

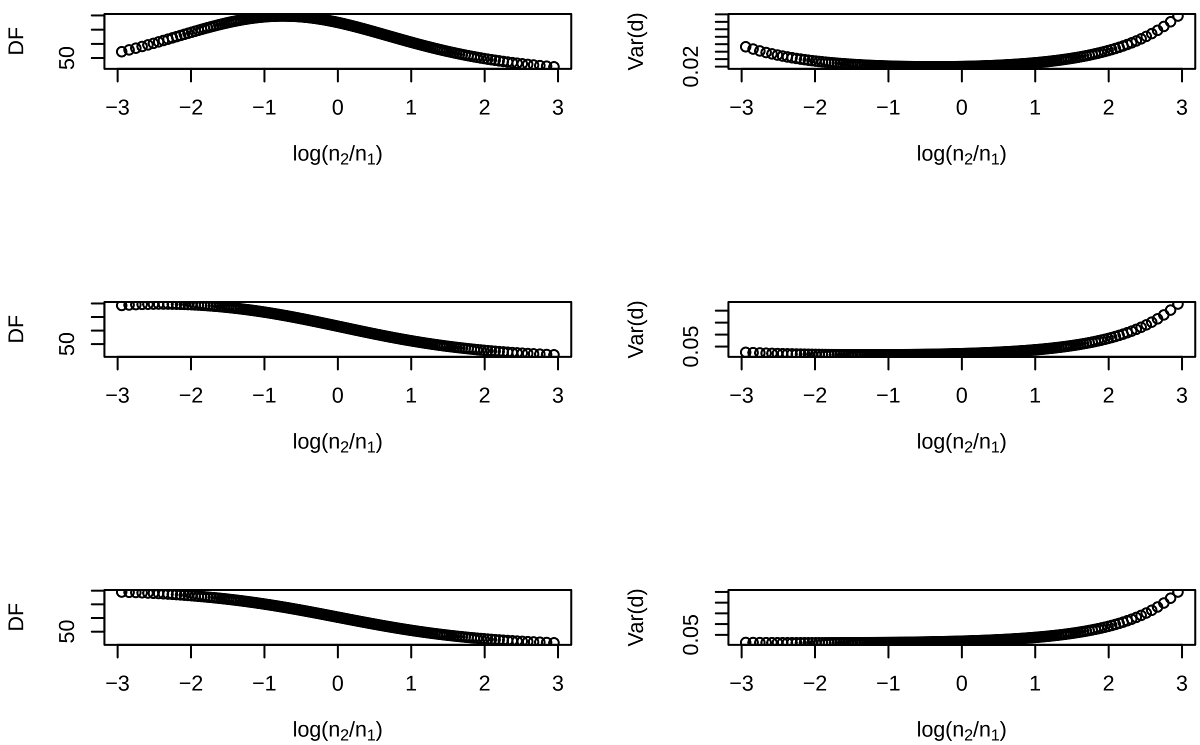


Figure 40. The variance of Cohen's d^*, when variances and sample sizes are unequal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$), when SD-ratio equals .68 (first row), .29 (second row) or .14 (third row).

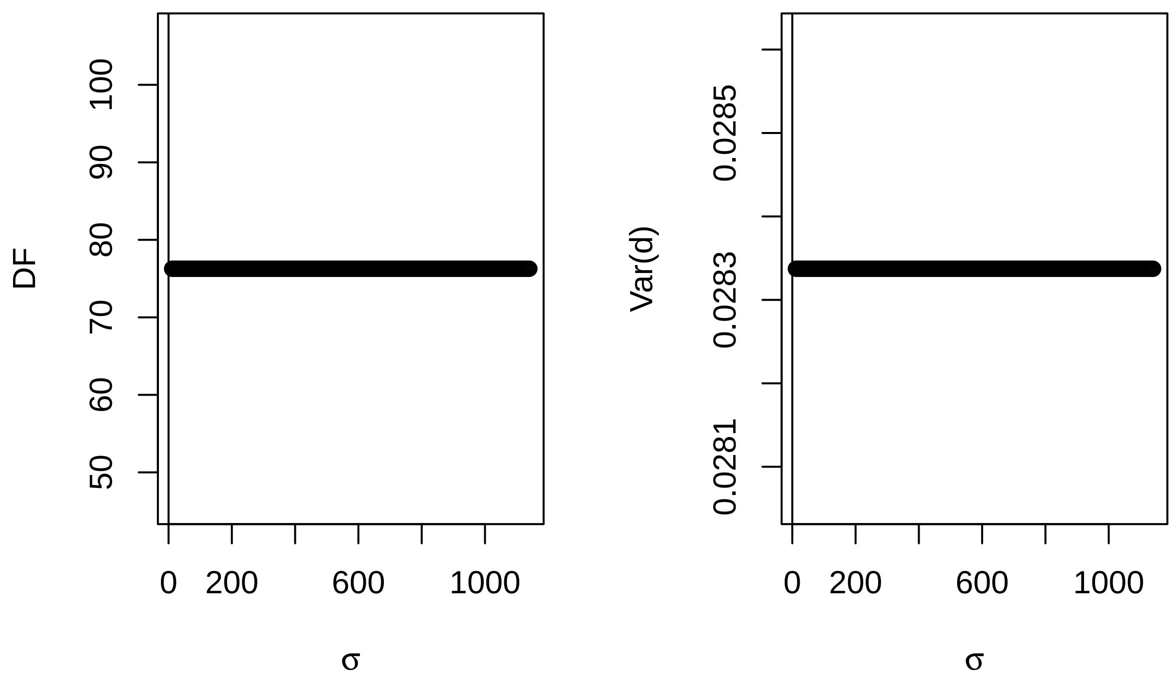


Figure 41. Variance of Cohen's d^*, when variances and sample sizes are unequal across groups, as a function of σ_1 and σ_2, for a constant SD-ratio.

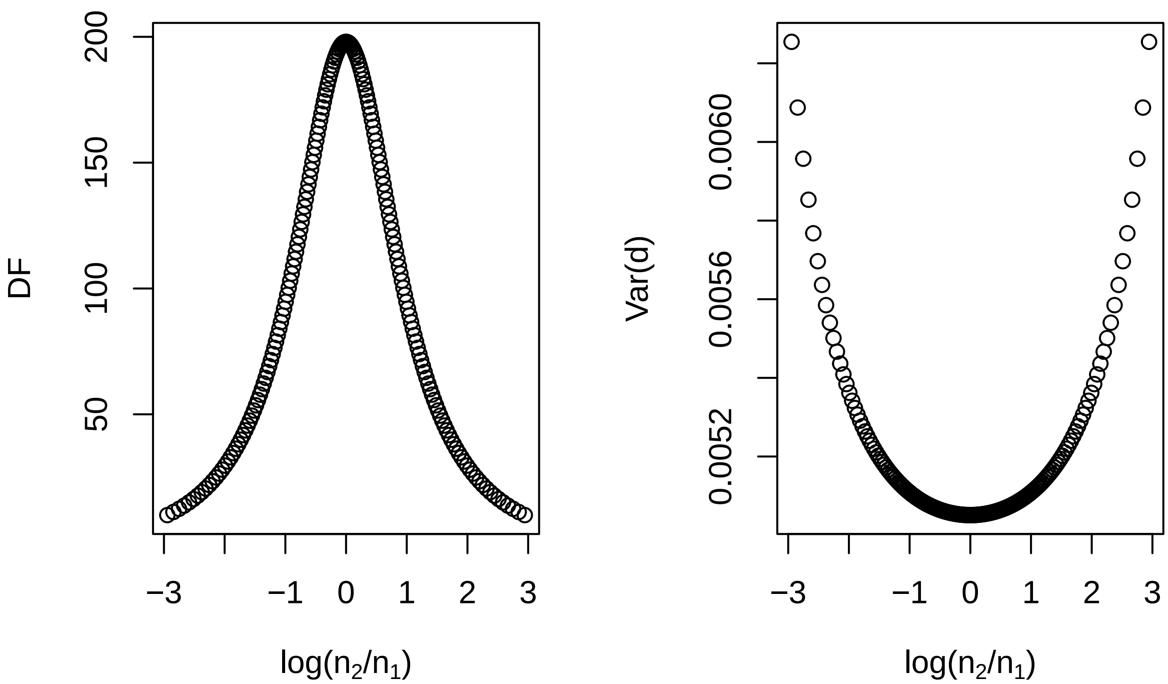


Figure 42. Variance of Shieh's d when variances are equal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$).

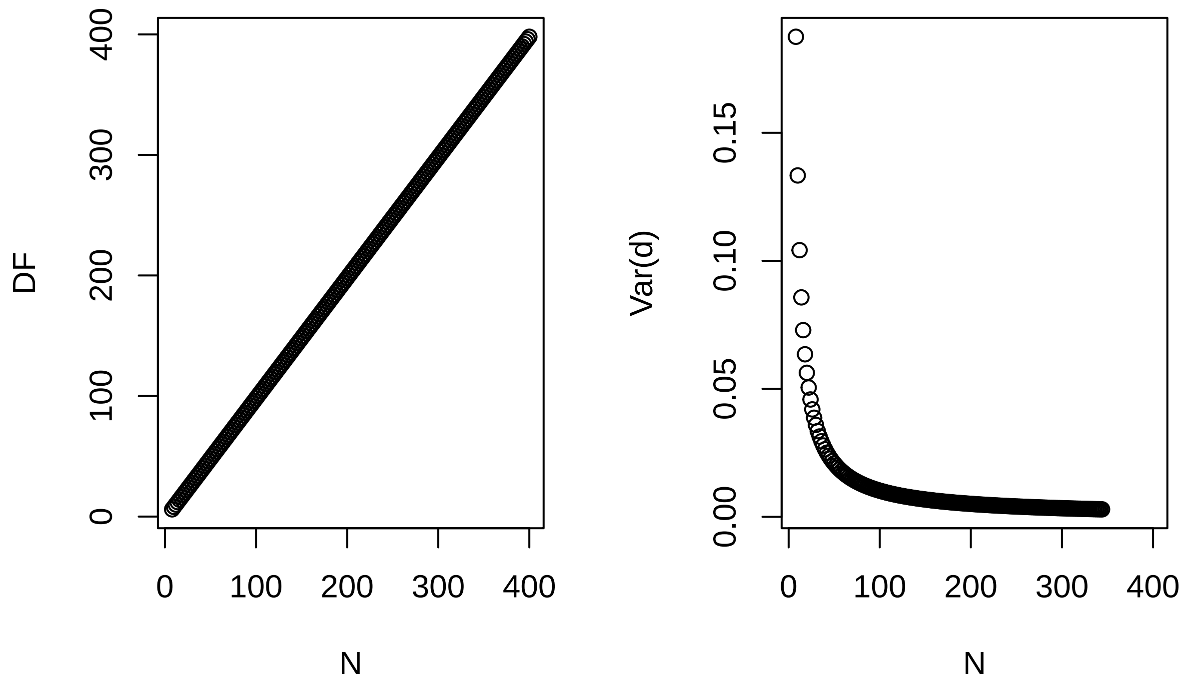


Figure 43. Variance of Shieh's d when variances are equal across groups, as a function of the total sample size (N), for a constant sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$).

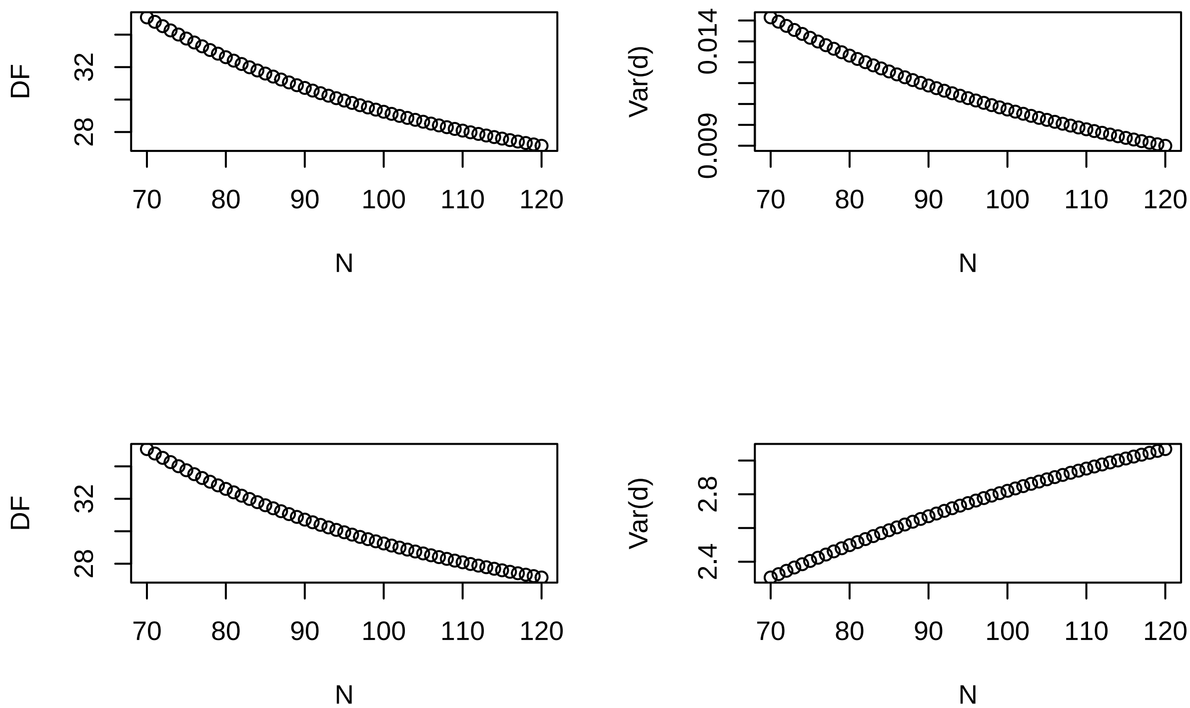


Figure 44. Variance of Shieh's d when variances are equal across groups, as a function of the total sample size (N), when adding subjects only in one group (either in the first group; see top plots; or in the second group; see bottom plots).

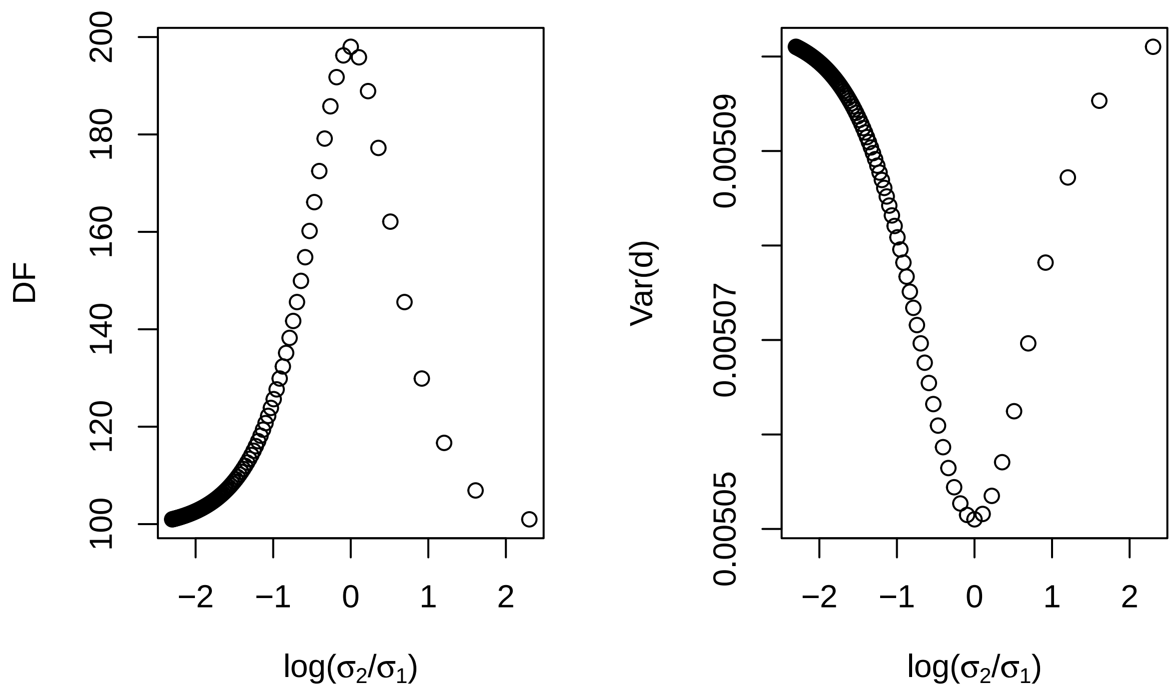


Figure 45. Variance of Shieh's d when variances are unequal across groups and sample sizes are equal, as a function of the logarithm of the SD-ratio ($log\left(\frac{\sigma_2}{\sigma_1}\right)$).

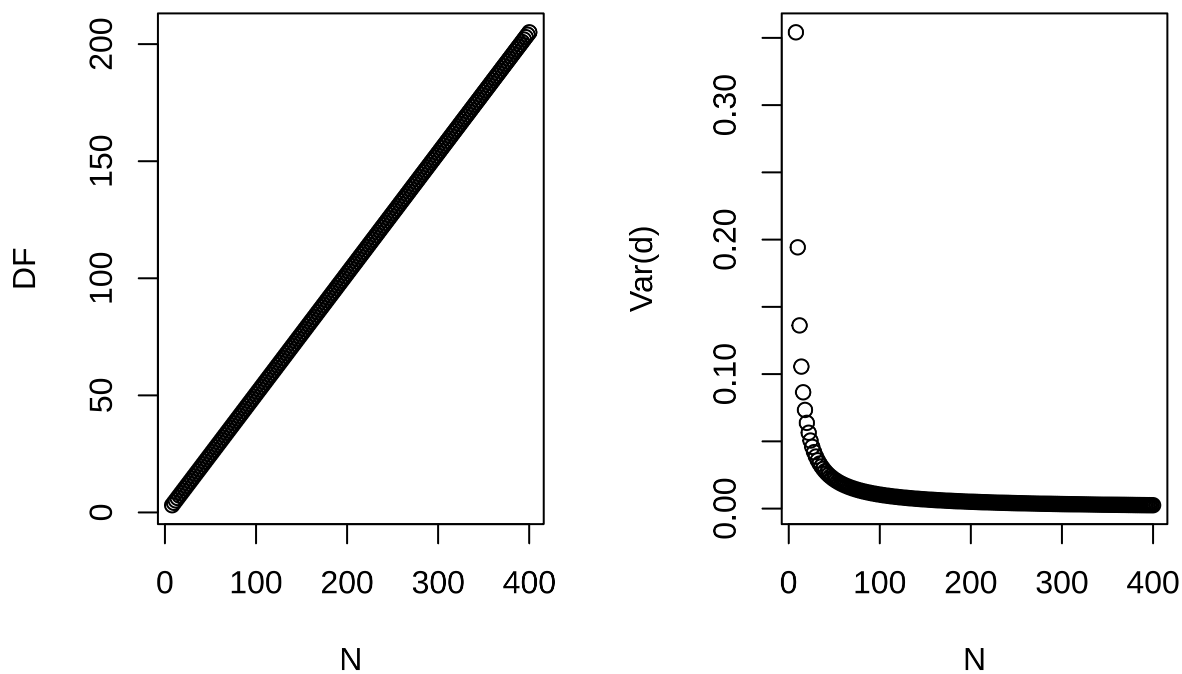


Figure 46. Variance of Shieh's d when variances are unequal across groups and sample sizes are equal, as a function of the total sample size (N).

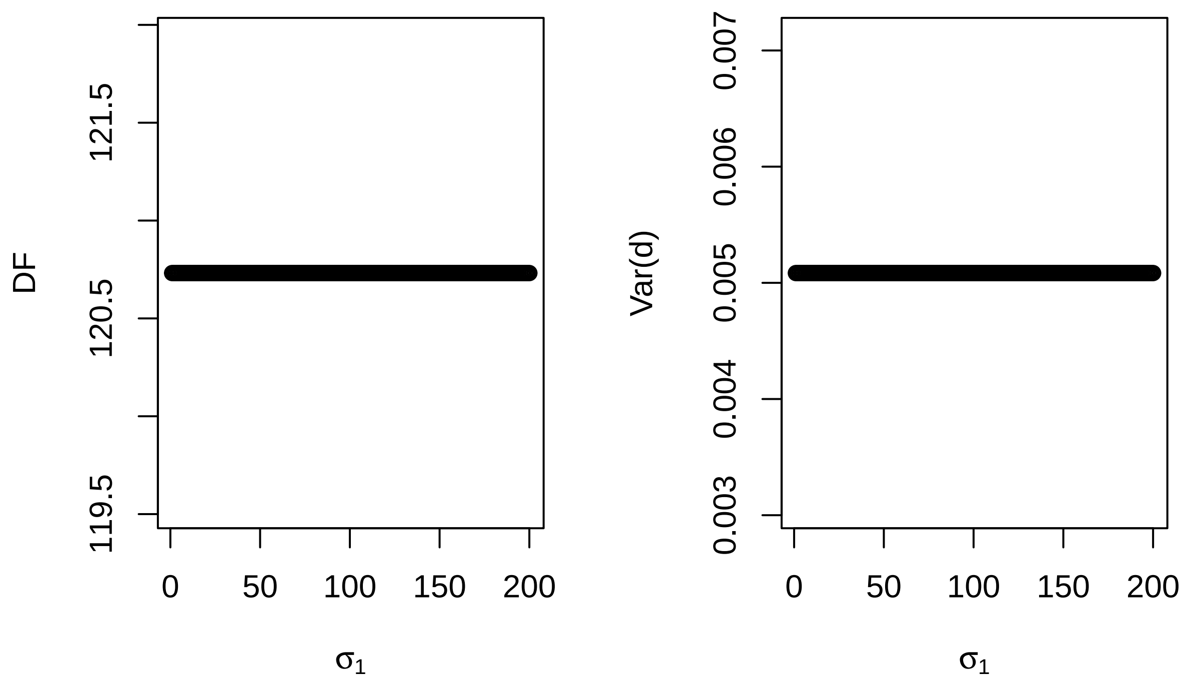


Figure 47. Variance of Shieh's d, when variances are unequal across groups and sample sizes are equal, as a function of σ_1 and σ_2, for a constant SD-ratio.

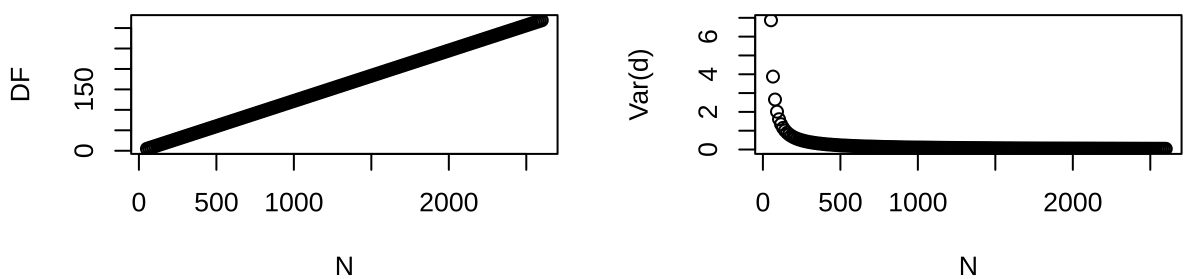


Figure 48. Variance of Shieh's d when variances and sample sizes are unequal across groups, as a function of the total sample size (N), for a constant sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$).

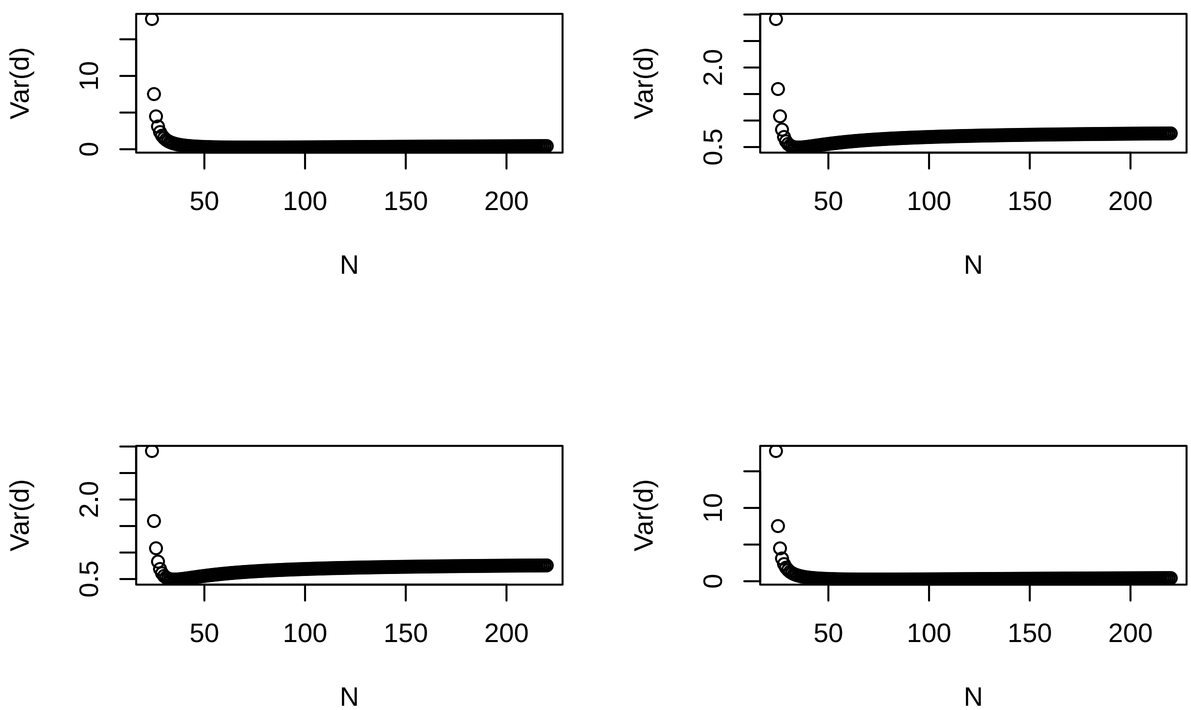


Figure 49. Variance of Shieh's d when variances and sample sizes are unequal across groups, as a function of the total sample size (N), when adding subjects only in one group (either in the first group; see left plots; or in the second group; see right plots), and $\sigma_1 > \sigma_2$ (top plots) or $\sigma_1 < \sigma_2$ (bottom plots).

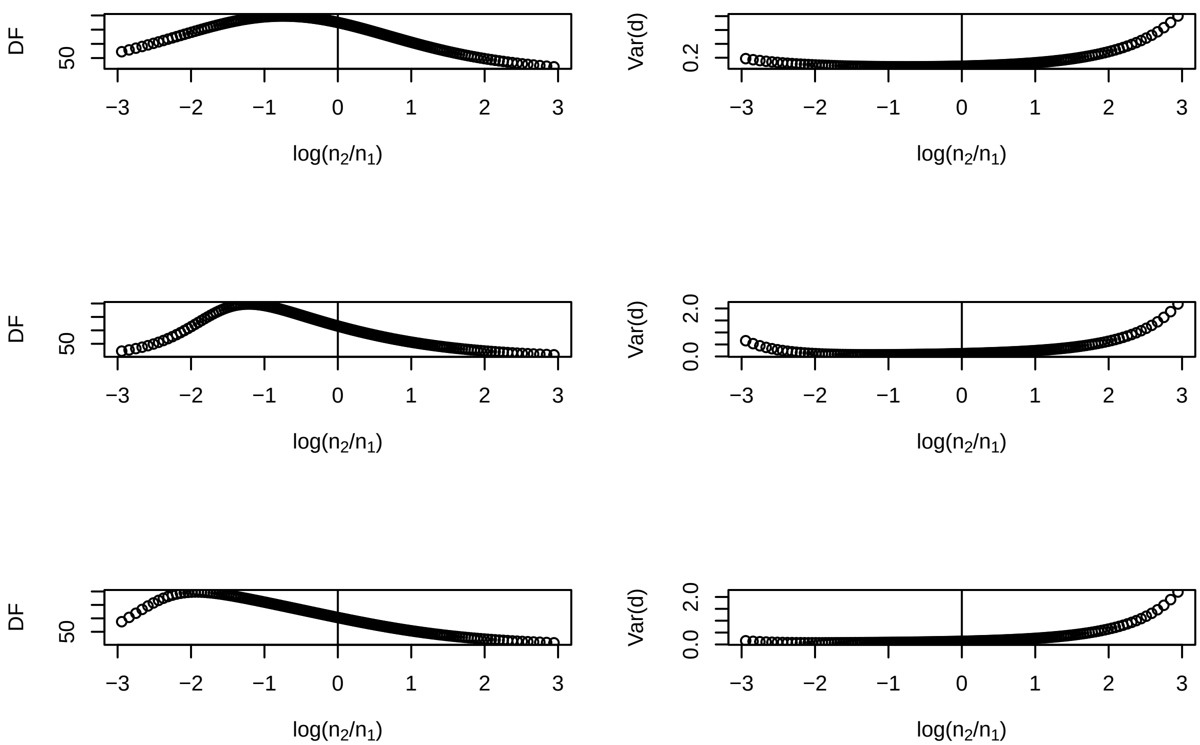


Figure 50. The variance of Shieh's d, when variances and sample sizes are unequal across groups, as a function of the logarithm of the sample sizes ratio ($log\left(\frac{n_2}{n_1}\right)$), when SD-ratio equals .68 (first row), .29 (second row) or .14 (third row).

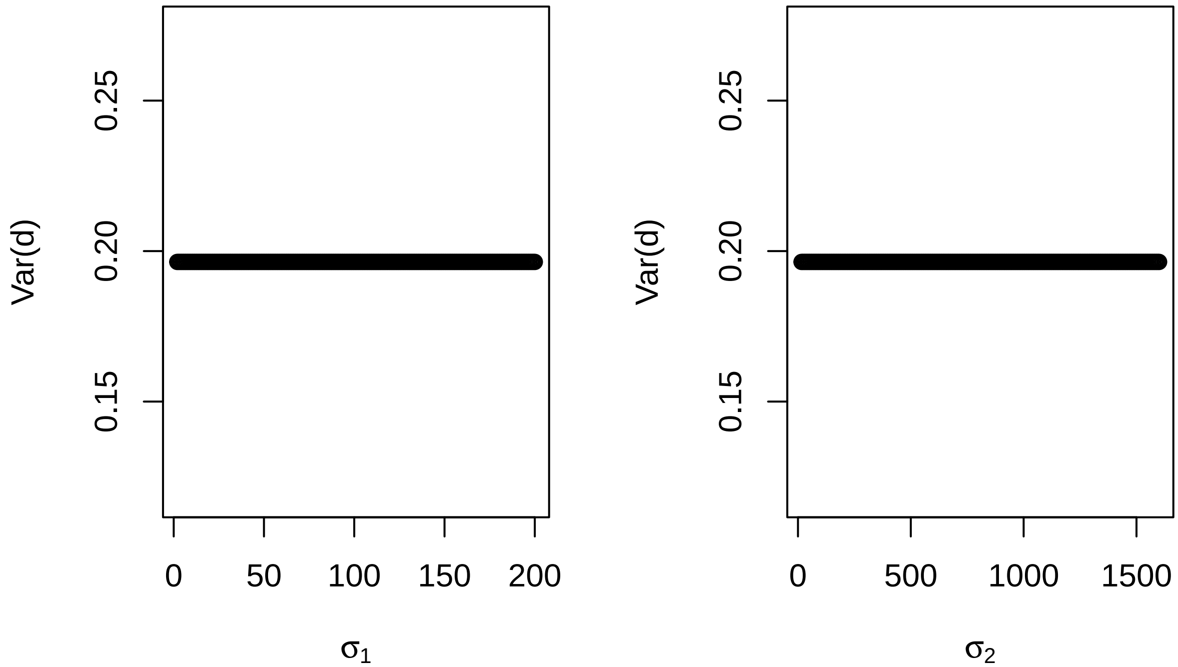
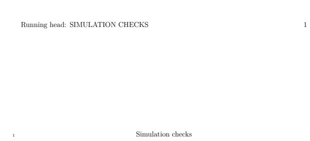


Figure 51. Variance of Shieh's d, when variances and sample sizes are unequal across groups, as a function of σ_1 (left) or σ_2 (right), for a constant SD-ratio.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Simulation checks

Figure 52. your caption