Problem Set 1

Let be the following sample expected return vector, μ, and the sample covariance matrix, Ω, obtained from 3 time series of yearly stock returns (denoted as **1**, **2**, and **3**):

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.09 \\ 0.05 \end{pmatrix}; \qquad \Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.1 & -0.06 \\ 0.1 & 0.5 & -0.1 \\ -0.06 & -0.1 & 0.1 \end{pmatrix}.$$

Answer the following questions:

a) Obtain the correlation matrix.

b) Obtain the percentage contribution to risk (PCTR) for each of the three stocks in both an equally-weighted portfolio and a portfolio containing 20% in stock 1, 60% in stock 2 and 20% in stock 3.

c) Obtain the mean-variance (MV) portfolio frontier by considering stocks 1 and 3 such that only long positions are allowed. What is the composition, mean and risk of the portfolio with minimum variance?

d) Obtain the graphic corresponding to the investment opportunity set for all the portfolios we can build from only long positions of the three stocks. **Help**: Make a Monte-Carlo simulation of 4,000 portfolios (use **Matlab**) and compute for each portfolio its mean and standard deviation.

e) Compute the efficient frontier (EF) by allowing short-selling. **Remember** that the portfolio weight composition for any frontier portfolio, w_p , is just given by

$$w_p = \mathrm{g} + \mathrm{h}\mu_p;$$

$$\mathrm{g} = \frac{1}{D}\left[B\left(\Omega^{-1}l\right) - A\left(\Omega^{-1}\mu\right)\right], \quad \mathrm{h} = \frac{1}{D}\left[C\left(\Omega^{-1}\mu\right) - A\left(\Omega^{-1}l\right)\right];$$

$$A = l'\Omega^{-1}\mu; \; B = \mu'\Omega^{-1}\mu > 0; \; C = l'\Omega^{-1}l > 0; \; D = BC - A^2 > 0.$$

where l denotes the column vector of ones and μ_p denotes the portfolio expected return. **Remember** that the EF contains all portfolios with expected returns higher or equal to the mean of the global minimum variance (**gmv**) portfolio. The **gmv** composition is $w_{gmv} = \frac{\Omega^{-1} l}{C}$ and the mean and variance are, respectively, A/C and 1/C.

Which stocks in the EF do increase when increasing μ_p? Make a graphic for each single stock weight in the EF by using a grid of values for μ_p (just start with the expected return of **gmv**).

f) Let **d** be the portfolio in the EF as the maximum expected portfolio return per unit of risk (=standard deviation) with portfolio composition given by

$$w_d = \frac{\Omega^{-1}\mu}{A},$$

Compute the portfolio composition, mean and risk. Show that both the expected return and variance are, respectively, B/A and B/A^2.

g) **Remember** that any portfolio **p** from the portfolio frontier (see section e) satisfy the following result:

$$R_p = \alpha R_{gmv} + (1-\alpha) R_d,$$

$$\mu_p = \alpha \mu_{gmv} + (1-\alpha)\mu_d = \mu_d + \alpha\left(\mu_{gmv} - \mu_d\right),$$

$$\Rightarrow \quad \alpha = \left(\mu_p - \mu_d\right) / \left(\mu_{gmv} - \mu_d\right),$$

$$\Rightarrow \quad w_p = \alpha w_{gmv} + (1-\alpha) w_d,$$

where R_k denotes the return of portfolio k.

Given the above result, obtain the portfolio composition, mean and risk for those portfolios in the frontier with expected returns of 5% and 15%. Are these portfolios efficient?

h) Consider an investor with a quadratic expected utility function. Then, its optimization program is just given by

$$E\left[U\left(R_p\right)\right] = w'\mu - \frac{b}{2}\left(w'\Omega w\right),$$

$$R_p = w'R = w_1R_1 + w_2R_2 + w_3R_3; \quad w'l = 1,$$

where b>0 is the parameter of the investor's risk-aversion level, R_P the portfolio return containing the three stocks. **Remember** that the optimal portfolio composition (short-selling is allowed) can be expressed as

$$w^* = w_{gmv} + \frac{A}{b}\left(w_d - w_{gmv}\right),$$

where A is given in section c. Assume that you invest all your wealth into three different portfolios such that each follows a certain target. One portfolio is for 'retirement' with b=0.5, another is for 'education' of children with b=0.3 and finally, the last portfolio is for 'bequest' with b=0.1. The total investor's wealth is shared across the different portfolios in the following way: 60% in the portfolio for retirement, 25% in the portfolio for education, and finally the remaining 15% in the portfolio for bequest. Compute the optimal portfolio composition, mean and risk for each target portfolio and the mixed portfolio (i.e., the one containing the three target portfolios). Are the three target portfolios efficient? Does the 'mixed portfolio' belong to the PF? In case of being the 'mixed portfolio' in the PF, then try to obtain this portfolio as a combination of portfolios **gmv** and **d**.

i) Suppose that the return for the risk-free asset with return r=4%. Obtain the tangency portfolio, denoted as **e**, in the PF (corresponding to the three risky assets allowing short-selling, see section e), its mean and risk. **Remember** that the portfolio composition of **e** is just given by

$$w_e = \frac{\Omega^{-1}(\mu - rl)}{l'\Omega^{-1}(\mu - rl)}.$$

Finally, obtain and represent the capital allocation line (CAL). Does a portfolio containing 50% in the risk-free asset, 15% in stock 1, 20% in stock 2 and 15% in stock 3 belong to the CAL?

j) Consider the following utility function:

Al ser cúbica, tiene Sk, no solo media y varianza

$$\mathrm{U}(\mathrm{W}) = \mathrm{W} - 0.0114\,\mathrm{W}^2 + 0.0002\,\mathrm{W}^3,$$

where W denotes the wealth. Suppose an investor with the above utility function aims to maximize his expected utility function for wealth at time 1:

$$\mathrm{W}_1 = \mathrm{W}_0\left(1 + \mathrm{R}_p\right),$$

where W_0 is his initial wealth and R_P is the return of portfolio p from the CAL obtained in section i. Does either the skewness or kurtosis level of R_P, or both, become relevant for the investor's optimal portfolio selection? Try to explain the answer.

k) Consider now the following utility function:

$$U(W) = \frac{W^{1-a}}{1-a}; \qquad a \neq 1,$$

where **a** denotes the constant relative risk aversion (CRRA) parameter for this utility function. Try to answer the same question made in section j. (**Help**: Use the Taylor expansion for the utility function until the fourth order and then, obtain the expected utility).

l) Make the same analysis as in section k but now changing the above utility for the negative exponential utility function, i.e.

$$U(W) = -e^{-\lambda W},$$

where $\lambda > 0$ denotes the constant absolute risk aversion (CARA) parameter.

m) By using **Matlab**, try to compute the EF already obtained in section e but now only long positions in stocks are allowed.

n) Assume that the vector containing the three single stock returns, denoted as R, is Normal multivariate distributed. Thus,

$$R \sim N(\mu, \Omega), \qquad [1]$$

where μ and Ω are given at the beginning of this exercise. Let Γ be the **Cholesky decomposition** (lower triangular matrix) from the correlation matrix, already obtained in section a. Then, the above expression can be rewritten in the following way:

$$R = \mu + \Lambda \Gamma z \qquad z = (z_1, z_2, z_3)' \sim N(0, I), \qquad [2]$$

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \dfrac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{1-\rho_{13}^2-\dfrac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix},$$

such that ρ_{ij} is the correlation between the stock returns i and j, while σ_i is the standard deviation of the stock return i. **Remember** that the Cholesky decomposition will be very useful for Monte Carlo simulation under a multivariate framework.

Obtain the theoretical mean, variance, skewness and kurtosis of the **gmv** portfolio (see section e) by assuming the Normal multivariate distribution for the three stock returns in [1]. Run a total of 100,000 simulated returns (using **Matlab**) for the **gmv** portfolio according to [2] and compute the sample mean, standard deviation, skewness and kurtosis. Compare both theoretical and simulated moments.