Laboratorio Economia e Finanza

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Arbitrage

Subject of this lecture

- Introduction to the *jargon* of quantitative finance
- Definitions
- Forward and Futures
- Using an arbitrage argument we determine the price of a forward contract
- We derive the put call parity
- We understand how to exploit arbitrage opportunities when they appear in the market
- The material developed in this lecture comes mainly from the Wilmott and Hull books, see Course Outline for exact references

Jargon

- **Equity:** the most basic financial instrument (also defined as stock, or share) it gives you the ownership of a small piece of a company
- The behavior of quoted prices of stocks is far from being predictable. If we could predict the behavior of stock prices in the future.. we could become very rich...
- However this does not mean that we cannot make predictions on the evolution of stock prices. We can but taking into account the probabilistic nature of their behavior..
- This reasoning extends to more complex financial products:
  - **Commodities**: raw products such as precious metals, oil, food products etc..notice that this kind of products usually shows seasonal effects
  - **Currencies**: exchange rate, the rate at which one currency can be exchanged for another... notice that if you can exchange dollar for pounds and then pounds for yen... this implies a relationship between dollar/pound, pound/yen and dollar yen... if the relationship moves out of the line..it would be possible to earn arbitrage profits
  - **Indices**: a weighted sum of a selection or basket of representative stocks, ex Standard and Poor 500...
  - **Fixed Income securities**: interest can be fixed or floating. Coupon-bearing bonds pay out a known amount (this is the coupon). At the end of your fixed term you normally receive the final coupon and the principal (the amount invested

Forward Contract

- **Forward Contract:** is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price. Money does not change hands until the delivery date or maturity of the contract. The asset traded could be a stock, a commodity or a currency.
- **Delivery Price**: The amount that is paid for the asset at the delivery date
- A **Futures contract** is very similar to a forward contract. Futures contracts are usually traded through an exchange, which standardizes the terms of the contracts. The profit or loss from the future position is calculated every day and the change in value is paid daily from one party to the other. Thus with futures contracts there is a gradual payment of funds from the beginning till maturity.
- Forwards and futures have two main uses, in **speculation and in hedging**.

Consider a forward contract

- This contract obliges us to hand over an amount $\$F$ at time T and to receive in exchange the underlying asset
- Let assume that the price today of the asset is $\$S(t)$. This is the spot price, the amount for which we could have immediately delivery of the asset.
- At maturity we will handover $\$F$ and receive the asset which is now worth the amount $\$S(T)$. Notice that we cannot know $\$S(T)$ until T.
- Notice that we know $F, S(t)$ and T..Is there any relationship among those quantities??....Notice that the forward contract entitles us to receive $\$S(T) - F$ but this quantity is unknown!
- However entering in a special contract we can eliminate all randomness.

No arbitrage (2)

- Enter into the forward contract!.. This cost us nothing, however it exposes us to the uncertain evolution of the asset!
- Simultaneously short the asset!!. What it means going short? Simply we are selling something that we don't own. It is possible to do this operation in many markets.
- What happens when we sell something that we do not own? At time t we receive an amount equal to $\$S(t)$ in cash due to the short sale of the asset. We are long on the forward contract and, at the same time, we have short asset position. Our net position is zero. We can put the money received from selling short to the asset in the bank. In exchange the bank pays an interest
- At maturity we hand over the amount F and receive the asset which we use to close our short position. Notice that the value $-F$ is guaranteed by the nature of the forward contract. What is left in our bank account?
- The amount $S(t)\, e^{r(T-t)}$
- Net position at maturity: $S(t)\, e^{r(T-t)} - F$.....Here comes the concept of no arbitrage

No Arbitrage (3)

- Arbitrage:
  - We began with a portfolio **worth zero** and we end up with a predictable amount. That predictable amount should also be equal to zero!!
- Consequently:

$$F = S\left(t\right)e^{r\left(T-t\right)}$$

- This equation tells us the relationship between the spot and the forward price. It is a linear **relationship** meaning that the forward price is proportional to the spot price.

Cashflows

This peculiar, hedged, portfolio generates the following cash-flows

Holding	Worth today (t)	Worth at Maturity(T)
Forward	0	$S(T) - F$
Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)\, e^{r(T-t)}$
Total	0	$S(t)\, e^{r(T-t)} - F$

Notice, that our analysis holds as long as we can assume that the interest rate is constant.

Arbitrage opportunity?

- Let's prove that this is should be the actual price through a contradiction argument. We can easily show that there exists an arbitrage opportunity
  - Let assume that $F < S(t)\, e^{r(T-t)}$. Enter in the contract that we define above.
  - At maturity you will have the money left in the bank, worth $S(t)\, e^{r(T-t)}$ a short asset and a long forward. Short asset and long forward cancel each other.
  - Then if $F < S(t)\, e^{r(T-t)}$ we make money at time T without investing any positive amount at time t.
- Let assume $F > S(t)\, e^{r(T-t)}$... then you enter in the opposite position, namely I agree to sell the asset in the future at the vale F. This contract has payoff $F - S(T)$. Namely you borrow $S(t)$ and buy the asset. At time T you hand over the asset and receive F. However, by assumption F is larger than the amount of money that you agreed to give back to the bank. Again, we are making a positive amount at time T without investing anything at time t
- Investor will act quickly to exploit the opportunity and in the process prices will adjust to eliminate it.

Numerical Example

- Let the Spot Price equal to 28.75, the one year forward price is 30.20. The one year interest rate is 4.92. Are these numbers consistent with no arbitrage?
- In order to check whether the price satisfy no arbitrage we need to compute
$$F - Se^{r(T-t)} = 30.2. - 28.75e^{0.0492*1} = 0.0001$$

This is effectively zero...
Namely if we know any three out of S, F, r and $T-t$ we can find the fourth assuming there are no arbitrage possibilities
Important: the forward price in no way depends on what the asset price is expected to do! Namely, our analysis shows that its price is not affected whether the asset is expected to increase or decrease in value

More about futures

- Delivery and Settlement:
  - The futures contract will specify when the asset is to be delivered. There may be some uncertainty in the precise delivery date. Most futures contracts are closed out before delivery, with the trader taking the opposite position before maturity. But if the position is not closed then delivery of the asset must be made made. When the asset is another financial contract the settlement is usually done by a cash exchange.
- Marking to market
  - To reduce the likelihood of one party defaulting, being unable or unwilling to pay up, the exchanges insist on traders depositing a sum of money to cover changes in the value of their positions. This money is deposited in a margin account. As the position is marked to market daily, money is deposited or withdrawn from this margin account. Margin comes in two forms, the **initial margin and the maintenance margin**. The initial margin is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks, etc.) must be deposited. The levels of these margins vary from market to market.

- Futures on commodities don't necessarily obey the no arbitrage law. This is because we shall take into account the storage cost of the commodity
  - Storage cost: holder must be compensated for keeping the commodity (and this increases the value of the future)
  - Convenience yield: there can be a benefit from holding the commodity, maybe you can use straight in the production (decrease the value of the future)
- Storage cost and convenience yield go in opposite direction

Put and Call Parity (1)

Let consider a put- and call- options on the same underlying. They share the same strike price and the same maturity. Is there any relationship among the price of the Call, c, and the price of the Put, p.
We'll see that through an arbitrage reasoning we can develop the mathematical relationship between these two prices.

Imagine that an investor holds the two portfolios defined below

1. Portfolio (A): one European call option plus a zero-coupon bond that provides a payoff of K at time T
2. Portfolio (C): one European put option plus one share of the stock.

Recap: Zero Coupon Bond

Simply a Zero Coupon Bond.. is a Bond that does not provide any coupons. A zero-coupon bond is a debt security that does not pay interest but instead trades at a deep discount, rendering a profit at maturity, when the bond is redeemed for its full face value. (Investopedia). The price:

$$Price = M/(1+r)^n$$

- $M = Maturity$ value or face value of the bond
- r = the rate of interest
- n = number of years until maturity

Normally in the market you know M, usually set to 1000 and $Price$. Hence you can determine r, the rate of interest

Cash flows, Portfolio A

The zero-coupon bond will be worth K at time T. If the stock price S_T at time T proves to be above K, then the call option in portfolio A will be exercised. Portfolio A is worth

$$S_T > K \rightarrow (S_T - K)) + K = S_T$$

1. exercise the option!
2. receive K from the investment in the zero coupon bond

If S_T proves to be less than K, then the call option in portfolio A will expire worthless; the portfolio is worth K at time T.

$$S_T < K \rightarrow (0) + K = K$$

1. don't exercise the option!
2. receive K from the investment in the zero coupon bond

Cash flows, Portfolio C

Share will be worth S_T at time T. If S_T proves to be below K, then the put option in portfolio C will be exercised. This means that

$$S_T < K \rightarrow (K - S_T) + S_T = S_T$$

1. Exercise the option and you get $K - S_T$
2. asset that you bought is worth S_T

If S_T proves to be greater than K, then the put option in portfolio C will expire worthless and the portfolio will be worth S_T at time T.

$$S_T > K \rightarrow 0 + S_T = S_T$$

1. don't exercise the option! (negative payoff otherwise)
2. asset that you bought is worth S_T

Table 10.2 Values of Portfolio A and Portfolio C at time T.

		$S_T > K$	$S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio C	Put Option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

Arbitrage (1)

So we have determined the value of both Portfolios in the two possible state of world.

- If $S_T > K$, both portfolios are worth S_T at time T;
- if $S_T < K$, both portfolios are worth K at time T.

Hence both portfolios are worth:

$$max\left(S_T, K\right)$$

when the options expire at time T. We are considering European options, which can be exercised only at time T.

Arbitrage (2)

Portfolios A and C are worth te same at time T... If we assume that in the market there are no arbitrage opportunities they must have identical values today! Let assume otherwise...In this case, an arbitrageur could buy the less expensive portfolio and sell the more expensive one... Because the portfolios are guaranteed to cancel each other out at time T, the arbitrage profit equals to the difference in the values of the two portfolios!

Portfolios at time t

Consider Portfolio A at period 0. It is made by two components which are worth at $t = 0$

- Call is worth c
- zero coupon bond is worth Ke^{-rT}

Consider Portfolio C at period 0

- the put option is worth p
- the stock is worth S_0

Exploiting the assumption of the absence of arbitrage of opportunities we can set equal the value of portfolio A, $c + Ke^{-rT}$, to the value of portfolio B, $p + S_0$. Namely the equation below must hold:

$$c + Ke^{-rT} = p + S_0$$

Put Call Parity

$$c + Ke^{-rT} = p + S_0$$

The equation above defines the relationship known as **put call parity**

It shows that the value of a European call with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and exercise date, and vice versa.

Let assume that the equation defining the put call parity does not hold.. Can we earn money? Yes Let assume that S_0=31; $K = 30$, $r = 0.1$; $c = 3$; $p = 2.25$, and T is equal to 3 months.

- $c + Ke^{-rT} = 3 + 30e^{-0.1*(3/12)} = \$32.26$
- $p + S_0 = 2.25 + 31 = \$33.25$

With this data Portfolio C is overpriced to Portfolio A. What can we do? Buy the call, short the put and the stock. We get some cash flows:

$$-3 + 2.25 + 31 = \$30.25$$

that we can invest at the risk free rate and get at time T, after 3 months the amount

$$30.25e^{0.1*0.25} = \$31.02$$

Arbitrage Opportunities (1)

Notice that the stock price at time T can be either below or above the price of 30.
When the price is less than \$30 the put will be exercised. It means that I buy the stock for \$30 and close my short position in the stock as well. When the price of the stock is greater than 30 I exercise the call and buy the stock at for this amount to close my short position in the stock. In both cases I get:

$$\$31.02 - 30 = 1.02$$

A net profit!

Let assume that $c = 3$ and $p = 1$. In this case Portfolio A is overpriced relative to portfolio C.

- $c + Ke^{-rT} = 3 + 30e^{-0.1*(3/12)} = \$32.26$
- $p + S_0 = 1 + 31 = \$32$

How can we make money? At the beginning short the assets of Portfolio A and buy the assets of Portfolio C. Namely we short the call and buy the put and the stock. This requires an initial investment equal to:

$$31 + 1 - 3 = 29$$

Arbitrage Opportunities (3)

We finance this investment borrowing at the risk-free interest rate. Meaning that we repay at T, after 3 months in our example, the amount:

$$29e^{0.1*0.25} = 29.73$$

At T either the put or the call will be exercised. In both cases we will be able to sold the stock at a price equal to 30. Then we make:

$$30 - 29.73 = 0.27$$

Table 10.3 Arbitrage opportunities when put–call parity does not hold. Stock price = \$31; interest rate = 10%; call price = \$3. Both put and call have strike price of \$30 and three months to maturity.

Three-month put price = \$2.25	***Three-month put price = \$1***
Action now:	*Action now*:
Buy call for \$3	Borrow \$29 for 3 months
Short put to realize \$2.25	Short call to realize \$3
Short the stock to realize \$31	Buy put for \$1
Invest \$30.25 for 3 months	Buy the stock for \$31
Action in 3 months if $S_T > 30$:	*Action in 3 months if* $S_T > 30$:
Receive \$31.02 from investment	Call exercised: sell stock for \$30
Exercise call to buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27
Action in 3 months if $S_T < 30$:	*Action in 3 months if* $S_T < 30$:
Receive \$31.02 from investment	Exercise put to sell stock for \$30
Put exercised: buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27

Recap

- We learnt some key technical terms
- Focus on the concept arbitrage
- determine the price of a forward contract through an arbitrage argument
- Again with an arbitrage argument we derived the put call parity condition for European Options
- We explore how to profit arbitrage opportunities when they appear into the market. Unfortunately banks are very fast and face much lower transaction costs..

1. Define a VBA function that given $S(t)$, r and T determines F
2. Define a VBA function that given p, then price of the Put, r, K and T determines c. Exploit the put call parity
3. Define a VBA function that given c, then price of the Put, r, K and T determines p. Exploit the put call parity