Laboratorio Economia e Finanza

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Pricing with Arbitrage. Introduction to Quantitative Finance

- Introduction to Option Pricing in discrete time. Then we introduce how to model stock prices in continous time
- First we introduce the role of the Jensen's inequality.
- Afterwards we set up a two period model that describe the random evolution of a stock.
- We employ our model to determine the price of an European Call Option, taking advantage of the no arbitrage assumption
- Finally we briefly introduce how to model stock prices in continuous time.
- References: Hull book, Wilmott book and Shreve book

- Consider a stock that is worth 100 dollars at period 0. At period 1 is worth either 150 with probability $\frac{1}{2}$ and 50 with probability $\frac{1}{2}$. Hence a European call option, with strike price 100 is worth 50 in the good state and 0 in the bad state
- It is easy to see that the expected value of the stock price in one year time is 100. What is the expected value of the European call option?.. Notice that it's not zero!
- If the stocks falls to 50 the payoff is zero. If it rises to 150 the payoff is 50.
- Therefore the average payoff is 25, which gives us some idea of the option's value. This is not the exact one!!! We'll see that when determining the option value the probabilities do not matter

Notice

$$E\,(Stock))=0$$

$$E\,[Payoff\,(Stock)]=25$$

Let consider a convex function $f\,(s)$. By Jensen's Inequality

$$E\,[f\,(s)]\geq fE\,[S]....$$

How much greater is the LHS??
Let define $S=\bar{S}+\varepsilon$ with $E\,[\varepsilon]=0$, and $\bar{S}=E\,[S]$
Let's take a Taylor Expansion around $\bar{S}$..

$$E\,[f\,(S)]=E\left[f\left(\bar{S}+\varepsilon\right)\right]=E\left[f\left(\bar{S}\right)+\varepsilon f'\left(\bar{S}\right)+\tfrac{1}{2}\varepsilon^2 f''\left(\bar{S}\right).....\right]$$
$$\approx f\left(\bar{S}\right)++\tfrac{1}{2}f''\left(\bar{S}\right)E\left[\varepsilon^2\right]=fE\,[S]+\tfrac{1}{2}f''\left(\bar{S}\right)E\left[\varepsilon^2\right]$$

- Hence $E\left[f\left(S\right)\right]$ is greater by a quantity $\frac{1}{2}f''\left(\bar{S}\right)E\left[\varepsilon^2\right]$
- Convexity gives value to the option: (Remind the definition: $f\left(\lambda x_1+\left(1-\lambda\right)x_2\right)\leq\lambda f\left(x_1\right)+\left(1-\lambda\right)f\left(x_2\right)$
- Modelling the random component $E\left(\varepsilon^2\right)$, the randomness of the underlying, its volatility, is central to price options!
- Modelling randomness requires knowledge of Probability theory!

There are three form of analysis common in the financial world

1. Fundamental analysis
   - is all about trying to determine the 'correct' worth of a company by an in-depth study of balance sheets, management teams, patent applications, competitors, lawsuits, etc. In other words, getting to the heart of the firm, doing lots of accounting and projections and what-not. You need a degree in accounting and lot of time
2. Technical Analysis
   - is when you don't care anything about the company other than the information contained within its stock price history. You draw trendlines, look for specific patterns in the share price and make predictions accordingly. Academic evidence suggests that this method does not work
3. **Quantitative Analysis**
   - most successful form analysis over the last 50 years, forming a solid foundation for portfolio theory, derivatives pricing and risk management. Quantitative analysis is all about treating financial quantities such as stock prices or interest rates as random, and then choosing the best models for that randomness.

Two period model (1)

In this Section we develop a simple two period model that describe the evolution of a Stock price in our ”imaginary” world. We determine, exploiting a no arbitrage assumption, the price of an European Call. **Assumptions**:

- S_0 is the price of the Stock at time zero. Let assume that $S_0 = 4$
- At period 1 the price of the Stock can either go up $S_{1,up} = 8$, with probability p or go down with probability $1 - p$ and it will be worth $S_{1,down} = 2$
- we set p=1/2
- Let assume that the risk free interest rate measured with continuous compounding is equal to 0.223144. You should be able to determine the equivalent r with 1 year compounding, we employ the last one: $r = 0.25$

Let's consider a European Call Option with Strike price equal to $K = 5$. At period 1, depending on the state of the world, the European Option is worth:

- In the *up* state of the world the owner of the call option gets a profit equal to $S_{1,up} = max(8 - 5, 0) = 3$
- In the *down* state of the world the owner of the call option gets a profit equal to $max(2 - 5, 0) = 0$. In this state the stock is worth only 2

Question: What should be the price of the call option at period 0?

We answer this question using a no-arbitrage argument. To do so we build a portfolio composed of α units of S and β units of cash in the bank (money market account) that replicates the return of the European call Option. We assume that both α and β can take negative values

We build a portfolio of α units of the stock and β units in the money market account as to *replicate the European Option payoff at time 1.* In both states of the world our portfoli gives us the same payoff of the European Call Option.

Up state

$$\alpha * S_{1,up} + \beta * (1 + r) = 3$$

$$\alpha * 8 + \beta * (1.25) = 3$$

Down state:

$$\alpha * S_{1,down} + \beta * (1 + r) = 0$$

$$\alpha * 2 + \beta * (1.25) = 0$$

Solving for α and β we determine the portfolio that replicates the option payoff at period 1 .

Notice that

$$\alpha * 2 + \beta * (1.25) = 0$$

implies

$$\alpha = -\beta \frac{1.25}{2}$$

Then

$$-\beta \frac{1.25}{2} * 8 + \beta * (1.25) = 3$$

which implies

$$\beta = \frac{1}{1.25} = -0.8$$

And hence

$$\alpha = -\frac{-0.8 * 1.25}{2} = \frac{1}{2}$$

European price option at period 0: Notice that a portfolio with $= \frac{1}{2}$ units of the stock and $\beta = -0.8$ in the money market account is worth 1.2 at period 0.

$$\alpha S_0 + m = \frac{1}{2} * 4 - 0.8 = 1.2$$

This portfolio matches the option payoff in both states of the world. Consequently our European option should be worth, in period 0, the same of this portfolio. Therefore the price of the European Option at period 0 is 1.2. Defining with c the price of the European Call

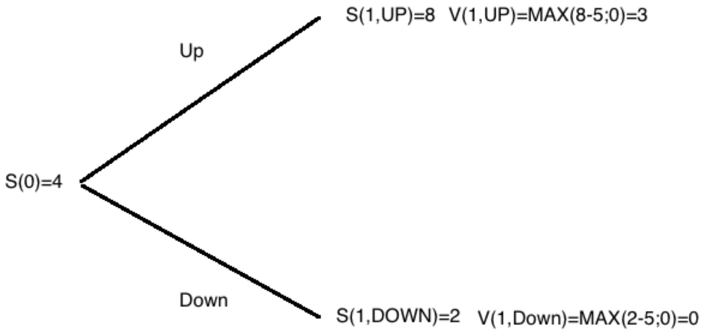
$$c = 1.2$$

we have Let's show that if $c > 1.2$ or $c < 1.2$ arbitrage opportunities arise

	Portfolio	Period 0, Cash-flows	Up State	Down State
α	$\frac{1}{2}$	-2	$8 * \frac{1}{2}$	$4 * \frac{1}{2}$=2
β	-0.8	+0.8	$-(0.8 * (1.25)) = -1$	$-(0.8 * (1.25)) = -1$
		-1.2	3	0

Notice that the cash flows of this portfolio exactly matches the payoff that the ownership of the European call option with strike $K = 5$ guarantees. **Assumptions**: we can lend and borrow at the same rate, r is the risk free interest rate, r is constant, no transaction costs, short selling of securities is allowed!

Binomial Tree: 2 period, Graph



? The price of the call at time zero, V(0)

Let assume that c is traded at a price equal to 1.25 in the market. Can we make money? Remind we shall sell the over-priced portfolio and buy the one that is under-priced. Hence

- at period 1 we sell the option. we receive 1.25 dollars. Cash inflow of 1.25
- we buy $\frac{1}{2}$ of the stock. we spend 2 dollars. Hence this is a cash outflow of -2
- we borrow 0.8 dollars from the bank. This means a cash inflow
- Summing $+1.25 - 2 + 0.8 = 0.05$. We leave this money in the bank account.. which will give us $0.05 * (1 + 0.25)$

At period 1 the price can go either up or down; but we end up with a positive amount of money in both states of the world.

Asset goes up, $S_{1,up} = 8$

- we sold the option. Hence we must give 3 dollars to the owner. Cash outflow of -3
- the stock is now worth 8. Our half of the stock is worth 4. We sell it out and make 4 dollars
- we need to pay back our 0.8 at the bank.. cash outflow of $-0.8 * (1.25) = -1$
- Hence we have $-3 + 4 - 1 = 0$.. However we had 0.05 dollars in our bank account.. hence we started with nothing and we end with $0.05 * (1.25)$ dollars at period 1, a positive amount!

Hence in this state we end up with a positive amount of money

Asset goes down, $S_{1,down} = 2$

- the option is not going to be exercised.
- the asset is now worth 2. Our half of the stock is worth 1. We sell it out and receive 1 dollars
- we need to pay back our 0.8 dollars at the bank.. cash outflow of $-0.8 * (1.25) = -1$
- Hence we have $0 + 1 - 1 = 0$. However we still have 0.05 dollars in our bank account.. hence we started with nothing and we end with $0.05 * (1.25)$ dollars

Hence in this state we end up with a positive amount of money

Summing up: thanks to the arbitrage opportunity we make money out of nothing, in both states of the world we end up with a positive amount of money!

Exercise: what would you do if the the European option is traded at 0 at a price equal to 1.15. Describe what you buy and sell to take advantage of the arbitrage opportunities.. Can you make money out of nothing? How much?

- This simple model is full insights. It gives a way to determine the price of an European call option
- Notice that to determine the price the probability p does not enter in the calculation... what is important is how much the price can go either up or down but not the probabilities!.. *volatility*
- **Assumptions**: risk free interest rate is constant, you can borrow and lend at the same rate any quantity, short selling is allowed and no transaction cost

- In this part we consider the model of stock price behavior of Black, Scholes and Merton
- For now we skip the theoretical foundations of the model interested students may refer to either the Hull's book on derivatives or the book wrote by Wilmott on the same argument or the Shreve book on Stochastic calculus
- Main idea... stock prices are stochastic variables!
- This model assumes that percentage changes in the stock price in a short period of time are normally distributed.

Stock Prices (2)

- Define with μ the expected return on stock per year
- Define with σ Volatility of the stock price per year.
  The mean of the return in time Δ_t is $\mu\Delta t$ and the standard deviation is $\sigma\sqrt{\Delta t}$ so that

$$\frac{\Delta S}{S} \sim \phi\left(\mu\Delta t, \sigma^2\Delta t\right)$$

where ΔS is the change in the stock rice S in time Δt and $\phi\left(m, v\right)$ denotes a normal distribution with mean m and variance v

Stock Prices (3)

- This model implies

$$ln\left(S_T\right) - ln\left(S_0\right) \sim \phi\left[\left(\frac{\mu - \sigma^2}{2}\right)T, \sigma^2 T\right]$$

- which implies

$$ln\left(\frac{S_T}{S_0}\right) \sim \phi\left[\left(\frac{\mu - \sigma^2}{2}\right)T, \sigma^2 T\right]$$

- and

$$ln\left(S_T\right) \sim \phi\left[ln\left(S_0\right) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

- The equation shows that $ln\left(S_T\right)$ is normally distributed, so that $\left(S_T\right)$ has a log-normal distribution. Namely this implies that the value of the stock cannot be lower than zero. A variable that has a log normal distribution can take any value between zero and infinity. Unlike the normal distribution the log-normal is skewed so that mean, median and mode are all different

Example

- Consider a Stock with an initial price of \$40, and expected return of 16%, and a volatility of 20%per annum. Then the probability distribution of the stock price S_T in 6 months time is given by

$$ln\left(S_T\right) \sim \phi\left[ln\left(40\right) + \left(0.16 - \frac{0.2^2}{2}\right), 0.2^2 * 0.5\right] = \phi\left[3.759, 0.02\right]$$

- There is 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. In this case the standard deviation is $\sqrt{0.02} = 0.141$. Hence with a 95% confidence

$$3.759 - (1.96 * 0.141) < ln\left(S_T\right) < 3.759 + (1.96 * 0.141),$$

- this can be written

$$e^{(3.759-(1.96*0.141))} < S_T < e^{3.759+(1.96*0.141)} \longleftrightarrow 32.55 < S_T < 56.56$$

- Therefore, according to our model, there is 95% probability that the stock price in 6 months will lie between 32.55 and 56.56

The expected value of the stock

$$E\left[S_t\right] = S_0 e^{\mu T}$$

The variance

$$var\left(S_T\right) = S_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)$$

Consider a stock where the current price is \$20, the expected return is 20% per annum, and the volatility is 40% per annum. $E\left[S_t\right]$ and $VAR\left[S_T\right]$, in 1 year are given by

- $E\left[S_t\right] = 20e^{0.2*1}$
- $var\left(S_T\right) = 400 * e^{2*0.2*1}\left(e^{0.2^2*1} - 1\right) = 103.54$, the standard deviation of the stock price in one year $\sqrt{103.54} = 10.18$