EQUATION SHEET

First-Order RL or RC Circuit Response

$v_C(t) \text{ or } i_L(t) = (IV - FV)e^{-t/\tau} + FV, t \geq 0$

Second-Order Differential Equation

Nonhomogeneous	Homogeneous
$\frac{d^2y_N(x)}{dx^2} + 2\zeta\omega_o\frac{dy_N(x)}{dx} + \omega_o^2y_N(x) = g(x)$	$\frac{d^2y_N(x)}{dx^2} + 2\zeta\omega_o\frac{dy_N(x)}{dx} + \omega_o^2y_N(x) = 0$

General Forms of Particular Responses

g(x)	**Form of $y_P(x)$**
1 (any constant)	A
5x+7	Ax+B
$3x^2$-2	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+D
sin(4x)	Acos(4x)+Bsin(4x)
cos(4x)	Acos(4x)+Bsin(4x)
e^{5x}	Ae^{5x}

Series RLC ODE

$\frac{d^2v_{CN}(t)}{dt^2} + \frac{R_T}{L}\frac{dv_{CN}(t)}{dt} + \frac{1}{LC}v_{CN}(t) = \frac{v_T(t)}{LC}$	$\zeta = \frac{R_T}{2}\sqrt{\frac{C}{L}} \quad \omega_o = {}^1/_{\sqrt{LC}}$

Parallel RLC ODE

$\frac{d^2i_{LN}(t)}{dt^2} + \frac{1}{R_TC}\frac{di_{LN}(t)}{dt} + \frac{1}{LC}i_{LN}(t) = \frac{i_N(t)}{LC}$	$\zeta = \frac{1}{2R_T}\sqrt{\frac{L}{C}} \quad \omega_o = {}^1/_{\sqrt{LC}}$

General Roots of Characteristic Equation and Natural Response

$s_{1,2} = \omega_o(-\zeta \pm \sqrt{\zeta^2 - 1})$	$y_N(t) = K_1e^{s_1t} + K_2e^{s_2t}, t \geq 0$

Case A

$s_{1,2} = -\alpha_1, -\alpha_2$	$y_N(t) = K_1e^{-\alpha_1t} + K_2e^{-\alpha_2t}, t \geq 0$

Case B

$s_{1,2} = -\alpha$	$y_N(t) = K_1e^{-\alpha t} + K_2te^{-\alpha t}, t \geq 0$

Case C

$s_{1,2} = -\alpha \pm j\beta$ $\alpha = \zeta\omega_o \text{ and } \beta = \omega_d = \omega_o\sqrt{1 - \zeta^2}$	$y_N(t) = K_1e^{-\alpha t}\cos(\beta t) + K_2e^{-\alpha t}\sin(\beta t), t \geq 0$

Some ways to represent the characteristic equation of second-order circuits

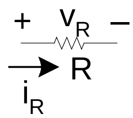
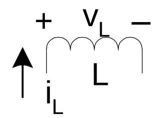
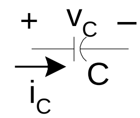
$s^2 + 2\zeta\omega_os + \omega_o^2=0$	$(s + \alpha)^2 + \beta^2=0$	$(s - p_1)(s - p_2) = 0$
$s^2 + Bs + \omega_0^2 = 0$	$s^2 + 2\alpha s + \omega_o^2 = 0$	$s^2 + \frac{\omega_0}{Q}s + \omega_0^2 = 0$

How to determine parameters from a response plot

$\omega_d = \beta = 2\pi\frac{1}{T}$	$\delta = \ln\left(\frac{y_1 - y_\infty}{y_2 - y_\infty}\right)$	$\zeta = \frac{1}{\sqrt{1 + \frac{4\pi^2}{\delta^2}}}$

Initial and Final Value Theorems

$\lim_{t\to0} f(t) = \lim_{s\to\infty} sF(s)$	$\lim_{t\to\infty} f(t) = \lim_{s\to0} sF(s)$

Device/Model	Resistance – R	Inductance – L	Capacitance – C
Units	ohms, Ω	Henrys, H	Farads, F
Circuit Symbol			
Voltage Equation	$v_R = i_R R$	$v_L = L\, di_L/dt$	$v_C = v_C(0^+) + (1/C)\int i_C\, dt$
Current Equation	$i_R = v_R G = v_R/R$	$i_L = i_L(0^+) + (1/L)\int v_L\, dt$	$i_C = C\, dv_C/dt$
Power Equation	$p_R = i_R \times v_R$	$p_L = i_L \times v_L$	$p_C = i_C \times v_C$
Energy Equation	$w_R = \int p_R\, dt$	$w_L = ½\, L\, i_L^2$	$w_C = ½\, C\, v_C^2$
Energy Storage	None	Magnetic Field	Electric Field
Continuity Equation	N/A	$i_L(\tau^-) = i_L(\tau^+)$	$v_C(\tau^-) = v_C(\tau^+)$
Typical Range	1 kΩ – 10 MΩ	1 μH – 10 H	10 pF – 100 μF
Series	$R_{EQ}=R_1+R_2+\ldots$	$L_{EQ}=L_1+L_2+\ldots$	$C_{EQ} = \dfrac{1}{\frac{1}{C_1}+\frac{1}{C_2}+\cdots}$
Parallel	$R_{EQ} = \dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\cdots}$	$L_{EQ} = \dfrac{1}{\frac{1}{L_1}+\frac{1}{L_2}+\cdots}$	$C_{EQ}=C_1+C_2+\ldots$
Impedance	Z=R	Z=1/(jωC)	Z=jωL
Impedance @ ω=0 (dc)	R	behaves like a short	behaves like an open
Impedance @ ω=∞ (very high freq)	R	behaves like an open	behaves like a short

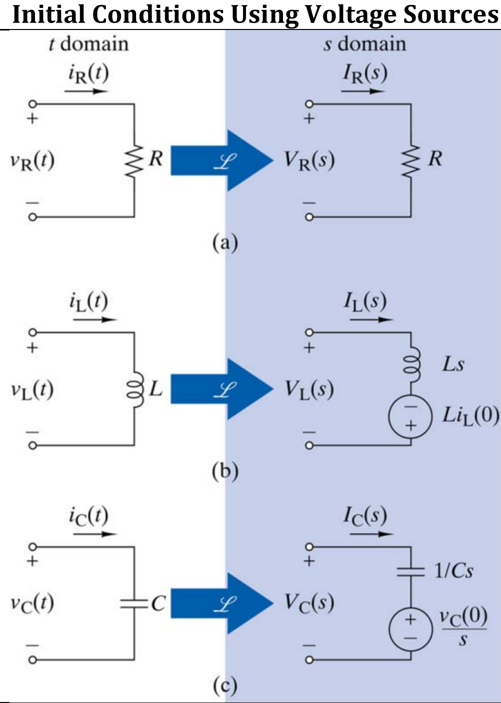
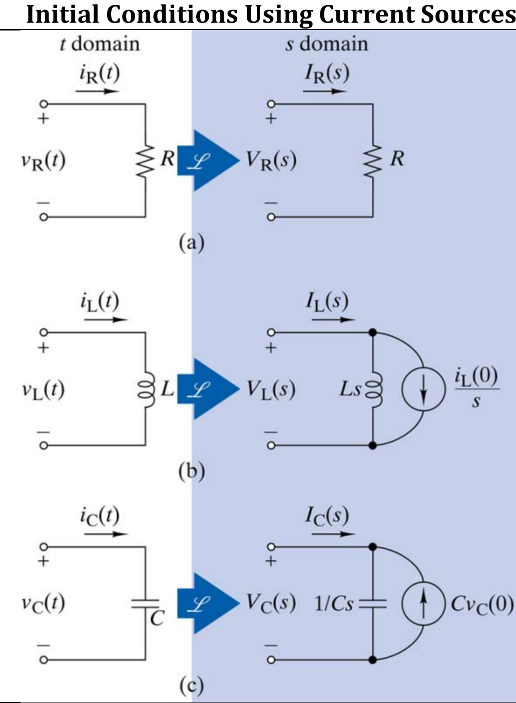
Initial Conditions Using Voltage Sources	Initial Conditions Using Current Sources
	

TABLE 9–2 BASIC LAPLACE TRANSFORM PAIRS

Signal	Waveform $f(t)$	Transform $F(s)$
Impulse	$\delta(t)$	1
Step function	$u(t)$	$\frac{1}{s}$
Ramp	$tu(t)$	$\frac{1}{s^2}$
Exponential	$[e^{-\alpha t}]u(t)$	$\frac{1}{s+\alpha}$
Damped ramp	$[te^{-\alpha t}]u(t)$	$\frac{1}{(s+\alpha)^2}$
Sine	$[\sin \beta t]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$[\cos \beta t]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped sine	$[e^{-\alpha t}\sin \beta t]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$[e^{-\alpha t}\cos \beta t]u(t)$	$\frac{(s+\alpha)}{(s+\alpha)^2+\beta^2}$

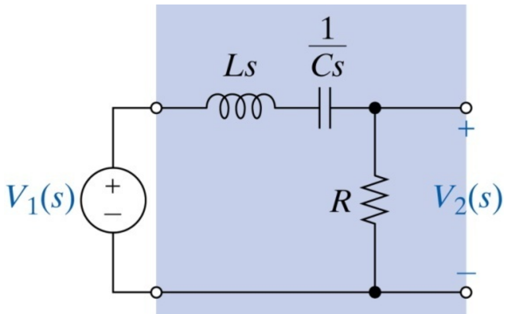
TABLE 9–1 BASIC LAPLACE TRANSFORMATION PROPERTIES

Properties	Time Domain	Frequency Domain
Independent variable	t	s
Signal representation	$f(t)$	$F(s)$
Uniqueness	$\mathscr{L}^{-1}\{F(s)\}(=)[f(t)]u(t)$	$\mathscr{L}\{f(t)\} = F(s)$
Linearity	$Af_1(t) + Bf_2(t)$	$AF_1(s) + BF_2(s)$
Integration	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
Differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f'(0-)$
	$\frac{d^3f(t)}{dt^3}$	$s^3F(s) - s^2f(0-) - sf'(0-) - f''(0-)$
s-Domain translation	$e^{-\alpha t}f(t)$	$F(s+\alpha)$
t-Domain translation	$f(t-a)u(t-a)$	$e^{-as}F(s)$

Form of F(s)	**Technique**	**Residues**
real distinct roots	PFE	$k_i = (s - p_i)F(s)\|_{s=p_i}$
complex roots	determine residue k using PFE	$f(t) = 2\|k\|e^{-\alpha t}\cos(\omega t + \angle k)$
real repeated roots	factor repeated root then PFE	$k_i = (s - p_i)F(s)\|_{s=p_i}$
improper function	long division then PFE	$k_i = (s - p_i)F(s)\|_{s=p_i}$

Passive RLC Filter Topologies

Series RLC Circuit (Output across R)



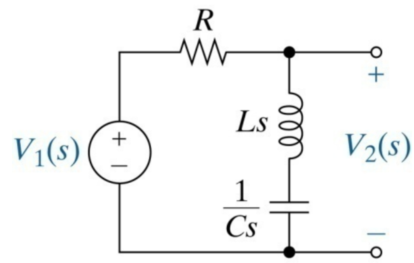
$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\omega_{c1}, \omega_{c2} = \mp\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$B = \frac{R}{L}$$

$$Q = \frac{\omega_o}{B} = \frac{1}{R}\sqrt{\frac{L}{C}}, \ \zeta = \frac{1}{2Q} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Series RLC Circuit (Output across L and C)



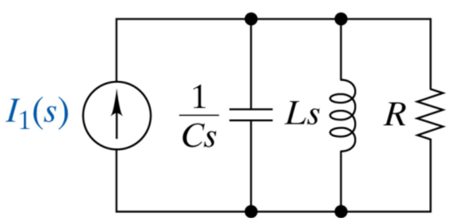
$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\omega_{c1}, \omega_{c2} = \mp\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$B = \frac{R}{L}$$

$$Q = \frac{\omega_o}{B} = \frac{\sqrt{L/C}}{R}, \ \zeta = \frac{1}{2Q} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Parallel RLC Circuit (Output thru R)



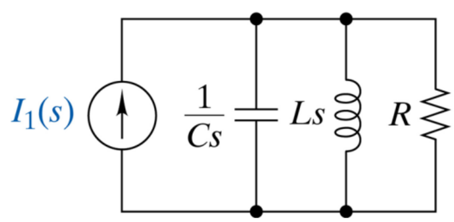
$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\omega_{c1}, \omega_{c2} = \mp\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \frac{1}{RC}$$

$$Q = \frac{\omega_o}{B} = R\sqrt{\frac{C}{L}}, \ \zeta = \frac{1}{2Q} = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Parallel RLC Circuit (Output thru L or C)



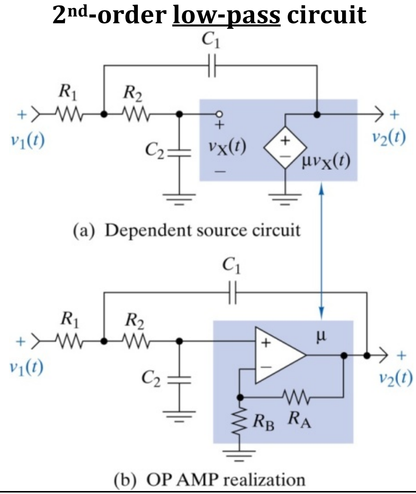
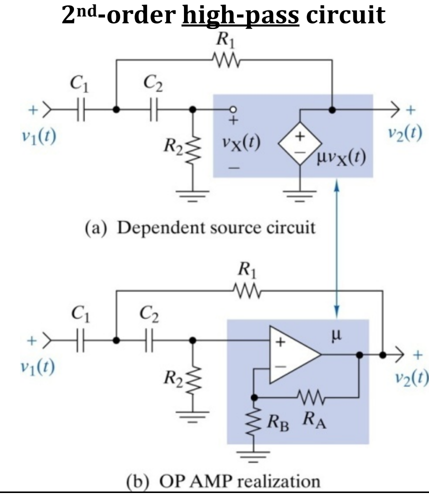
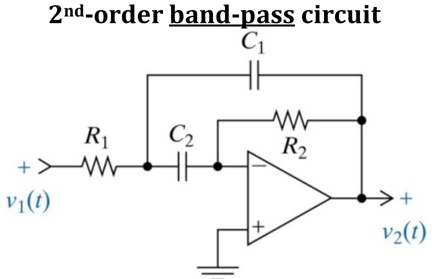
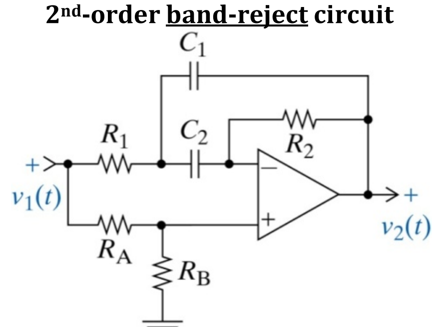
$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\omega_{c1}, \omega_{c2} = \mp\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \frac{1}{RC}$$

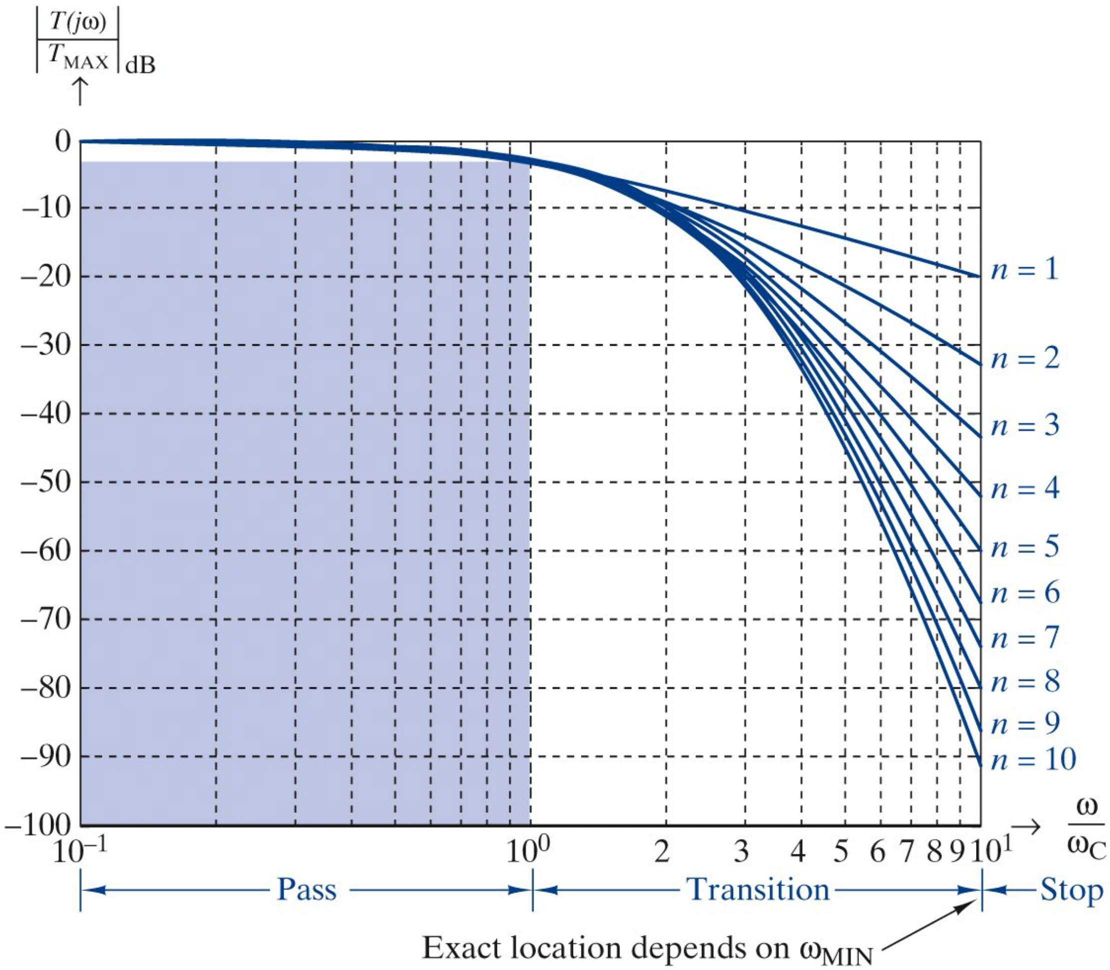
$$Q = \frac{\omega_o}{B} = R\sqrt{C/L}, \ \zeta = \frac{1}{2Q} = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Active RC Filter Topologies

<table>
<tr><td>2nd-order <u>low-pass</u> circuit

</td><td>2nd-order <u>high-pass</u> circuit

</td></tr>
<tr><td>Transfer Function
$T(s) = \frac{K\omega_o^2}{s^2+2\zeta\omega_o s+\omega_o^2}$ $|T(j\omega_o)| = {}^{|K|}/_{2\zeta}$</td><td>Transfer Function
$T(s) = \frac{Ks^2}{s^2+2\zeta\omega_o s+\omega_o^2}$ $|T(j\omega_o)| = {}^{|K|}/_{2\zeta}$</td></tr>
<tr><td>Equal Element
$RC = \frac{1}{\omega_o}$ and $\mu = 3 - 2\zeta = \frac{R_A+R_B}{R_B}$</td><td>Equal Element
$RC = \frac{1}{\omega_o}$, $\mu = 3 - 2\zeta = \frac{R_A+R_B}{R_B}$</td></tr>
<tr><td>Unity Gain
$R\sqrt{C_1C_2} = \frac{1}{\omega_o}$ and $\frac{C_2}{C_1} = \zeta^2$</td><td>Unity Gain
$C\sqrt{R_1R_2} = \frac{1}{\omega_o}$ and $\frac{R_1}{R_2} = \zeta^2$</td></tr>
<tr><td>2nd-order <u>band-pass</u> circuit

</td><td>2nd-order <u>band-reject</u> circuit

</td></tr>
<tr><td>Transfer Function
$T(s) = \frac{K\omega_o s}{s^2+2\zeta\omega_o s+\omega_o^2}$ $|T(j\omega_o)| = {}^{|K|}/_{2\zeta}$</td><td>Transfer Function
$T(s) = \frac{K(s^2+\omega_o^2)}{s^2+2\zeta\omega_o s+\omega_o^2}$ $|T(j\omega_o)| = 0$</td></tr>
<tr><td>Equal Capacitor
$C\sqrt{R_1R_2} = \frac{1}{\omega_o}$ and $\frac{R_1}{R_2} = \zeta^2$ and $|T(j\omega_o)| = {}^{1}/_{2\zeta^2}$</td><td>Equal Capacitor
$C\sqrt{R_1R_2} = \frac{1}{\omega_o}, \frac{R_1}{R_2} = \zeta^2$ and $\frac{R_A}{R_B} = \frac{2R_1}{R_2}$</td></tr>
</table>

Active Filter Design

First-order Cascade



$$T_{LP}(s) = \frac{|K|^n \alpha^n}{(s+\alpha)^n}$$

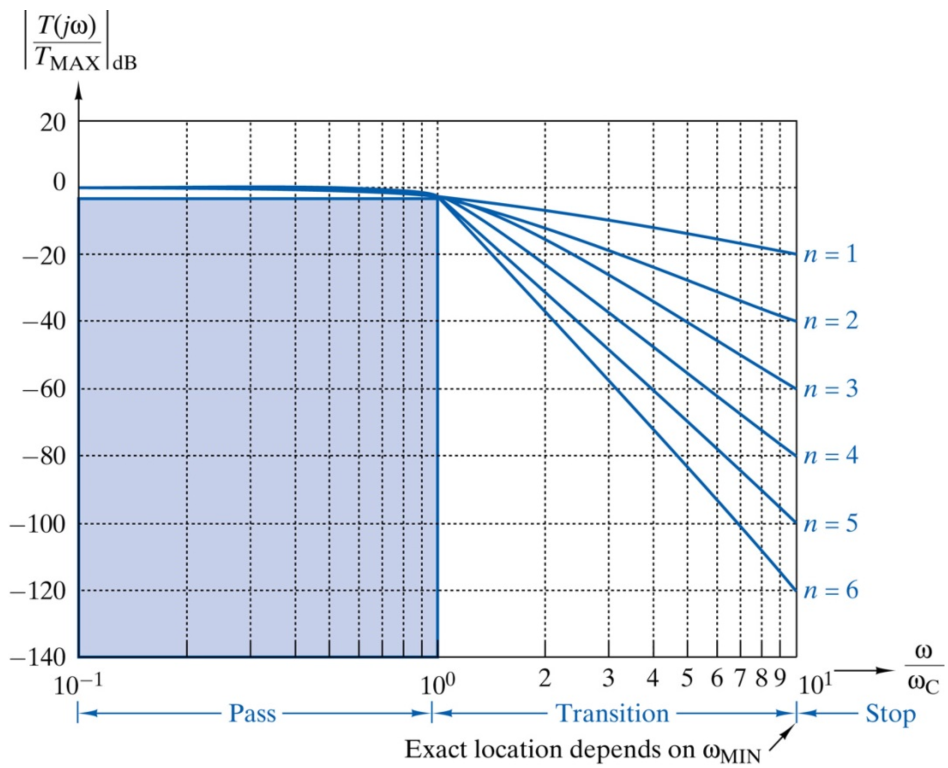
$$T_{HP}(s) = \frac{|K|^n s^n}{(s+\alpha)^n}$$

$$\alpha_{LP} = \frac{\omega_c}{\sqrt{2^{1/n} - 1}}$$

$$\alpha_{HP} = \omega_c\sqrt{2^{1/n} - 1}$$

Note: K is the gain of each individual stage where
$K = T_{max}^{1/n}$.

Butterworth



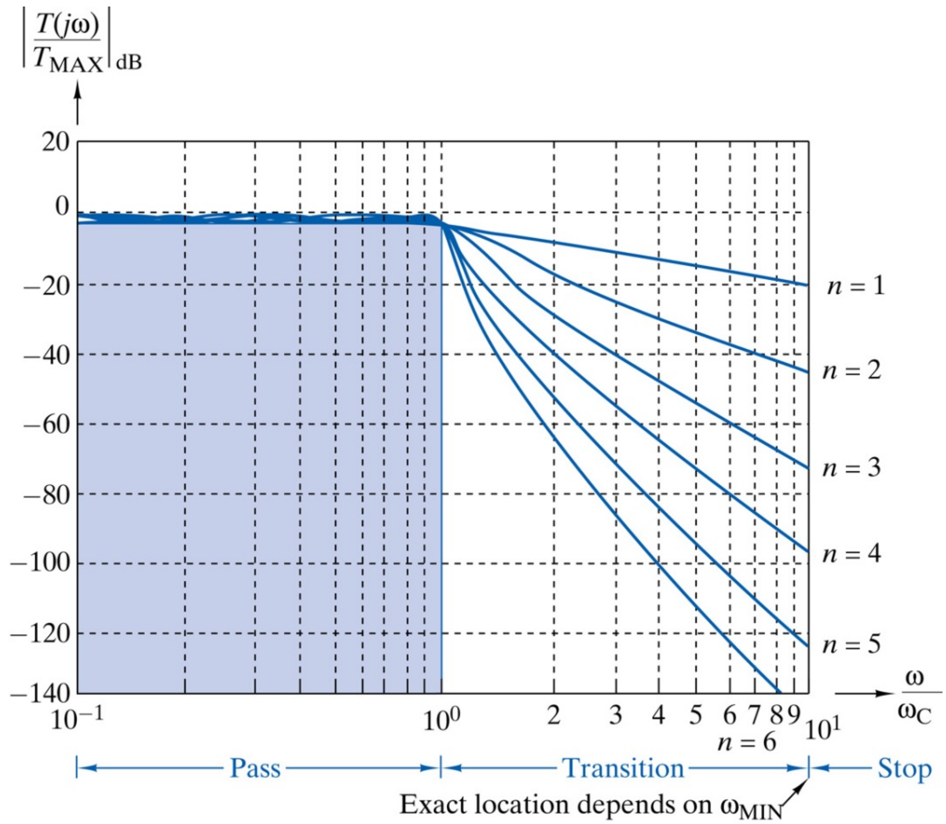
$n \geq \frac{1}{2}\frac{\ln[(T_{max}/T_{min})^2 - 1]}{\ln[\omega_{min}/\omega_c]}$ (Note: T_{max} & T_{min} in V/V, **not** dB)

Order	Normalized Denominator Polynomial
1	$(s+1)$
2	$(s^2 + 1.414s + 1)$
3	$(s+1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.848s + 1)$
5	$(s+1)(s^2 + 0.6180s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.414s + 1)(s^2 + 1.932s + 1)$

$$T_{LP}(s) = \frac{K}{q_n\left(\frac{s}{\omega_c}\right)}$$

$$T_{HP}(s) = \frac{K}{q_n\left(\frac{\omega_c}{s}\right)}$$

Chebyshev



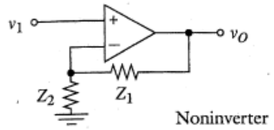
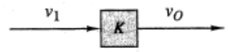
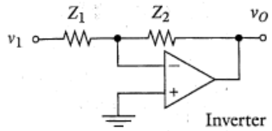
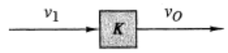
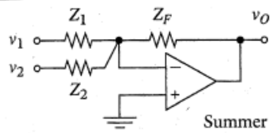
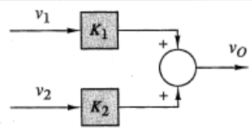
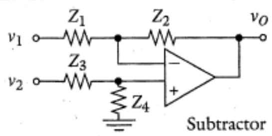
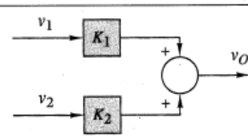
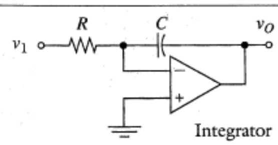
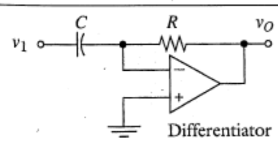
$$n \geq \frac{\cosh^{-1}\sqrt{(T_{max}/T_{min})^2-1}}{\cosh^{-1}(\omega_{min}/\omega_c)} \quad \text{(Note: } T_{max} \text{ \& } T_{min} \text{ in V/V, } \textbf{not} \text{ dB)}$$

Order	Normalized Denominator Polynomial
1	$(s+1)$
2	$\left(\frac{s}{0.8409}\right)^2 + 0.7654\frac{s}{0.8409} + 1$
3	$\left(\frac{s}{0.2980}+1\right)\left(\left(\frac{s}{0.9159}\right)^2 + 0.3254\left(\frac{s}{0.9159}\right)+1\right)$
4	$\left(\left(\frac{s}{0.9502}\right)^2 + 0.1789\left(\frac{s}{0.9502}\right)+1\right)\left(\left(\frac{s}{0.4425}\right)^2 + 0.9276\left(\frac{s}{0.4425}\right)+1\right)$
5	$\left(\frac{s}{0.1772}+1\right)\left(\left(\frac{s}{0.9674}\right)^2 + 0.1132\left(\frac{s}{0.9674}\right)+1\right)\left(\left(\frac{s}{0.6139}\right)^2 + 0.4670\left(\frac{s}{0.6139}\right)+1\right)$
6	$\left(\left(\frac{s}{0.9771}\right)^2 + 0.0781\left(\frac{s}{0.9771}\right)+1\right)\left(\left(\frac{s}{0.7223}\right)^2 + 0.2886\left(\frac{s}{0.7223}\right)+1\right)\left(\left(\frac{s}{0.2978}\right)^2 + 0.9562\left(\frac{s}{0.2978}\right)+1\right)$

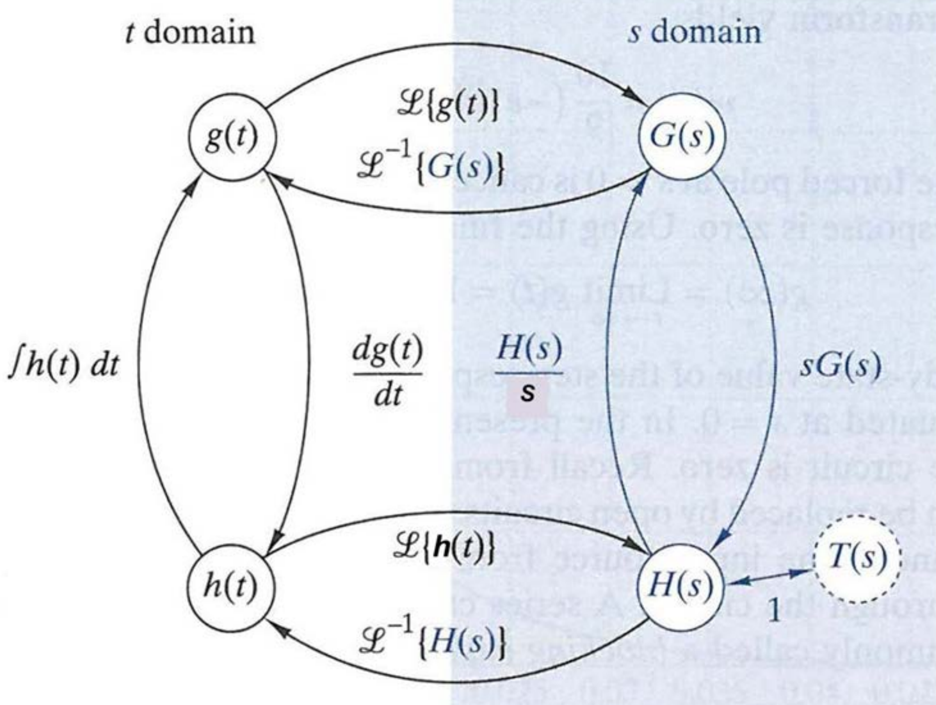
$$T_{LP}(s) = \frac{K}{q_n\left(\frac{s}{\omega_c}\right)}, \text{ n odd; } T_{LP}(s) = \frac{K/\sqrt{2}}{q_n\left(\frac{s}{\omega_c}\right)}, \text{ n even}$$

$$T_{HP}(s) = \frac{K}{q_n\left(\frac{\omega_c}{s}\right)}, \text{ n odd; } T_{HP}(s) = \frac{K/\sqrt{2}}{q_n\left(\frac{\omega_c}{s}\right)}, \text{ n even}$$

BASIC OP AMP MODULES

Circuit	Block Diagram	Gains
 Noninverter		$K = \frac{Z_1 + Z_2}{Z_2}$
 Inverter		$K = -\frac{Z_2}{Z_1}$
 Summer		$K_1 = -\frac{Z_F}{Z_1}$ $K_2 = -\frac{Z_F}{Z_2}$
 Subtractor		$K_1 = -\frac{Z_2}{Z_1}$ $K_2 = \left(\frac{Z_1 + Z_2}{Z_1}\right)\left(\frac{Z_4}{Z_3 + Z_4}\right)$
 Integrator		$K = -\frac{1}{RC}$
 Differentiator		$K = -RC$

Time-Domain and s-Domain Relationships



State Matrix Equations

Here are some steps to help develop the state matrix.

1. Choose capacitor voltages and inductor currents as the state variables.
2. For each capacitor, write a KCL equation, expressing the capacitor current in terms of the state variables and other currents, as necessary.
3. For each inductor, write a KVL equation, expressing the inductor voltage in terms of state variables and other voltages, as necessary.
4. Write other KVL and KCL equations and use element relations as necessary to eliminate the "other" currents and voltages shown in steps 2 and 3. The resulting equations should be your state equations. Express your state equations in matrix form $\dot{X} = AX$ with appropriate state variables.

$$\begin{bmatrix} v'_{C,L}(t) \\ i'_{C,L}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_{C,L}(t) \\ i_{C,L}(t) \end{bmatrix}$$

5. To find the eigen values λ of A, use

$$\det(A - \lambda I) = 0$$

6. To find the eigen vectors K for each λ, use

$$(A - \lambda I)K = 0$$

7. The general solution is $X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$ where the constants c_1 and c_2 are determined from initial conditions.