ECE 332 Lab 10m
2nd Order Circuit Design

Second-order system design up to this point has been based on making a circuit fit the basic second-order parameters such as the undamped and damped natural frequencies, ω_o and ω_d, and the damping factor ζ. But these aren't what designers want to control when they are designing second-order systems that work with other systems.

Designers often want to specify other parameters. These include overshoot, rise time, and settling time. These parameters are not as easy to design for because they involve messier relationships with the actual circuit elements. But let's get messy and try a design to meet some different specifications.

Goals

This lab has several things to accomplish:

- Understand the use of overshoot, settling time, and rise time as design specifications for second-order systems.
- Design and test in Matlab a second-order system meeting these specfications.

Introduction

Instead of dealing with the common second-order parameters, we are going to work with parameters that are more useful to control-system designers. We'll take a look at the three common ones. If we want exact values for these, we can generally specify only two of the three for any particular design. However, if we specify the maximum acceptable values for all three, we can generally meet all three specifications with our circuit. Figure 1 illustrates the three parameters of interest and we need to develop their relationships to $\alpha, \beta, \zeta, \omega_d$, and ω_o. We'll do this by making use of the s-plane to show how all of this fits together. In the next sections, we will see how the s-plane parameters relate to the system parameters we will be studying.

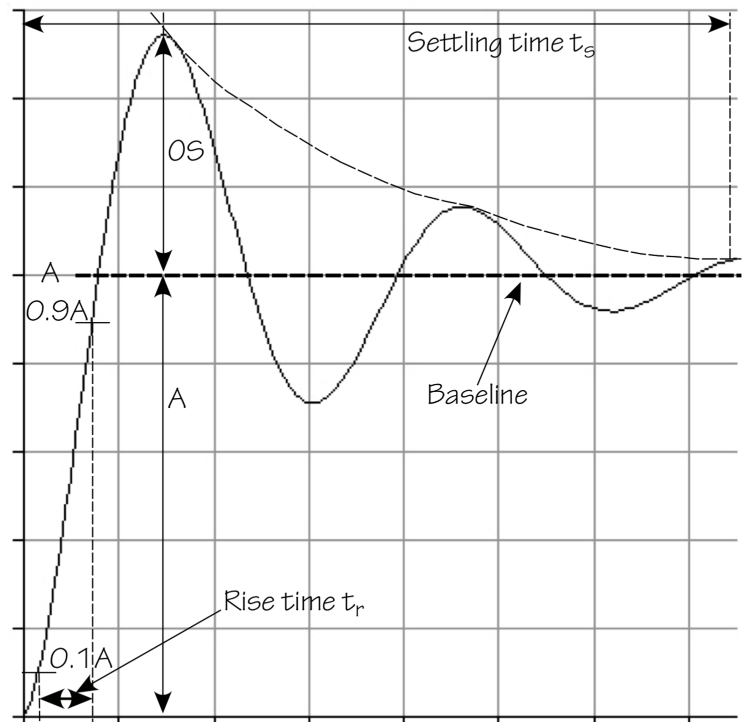

Figure 1: Overshoot, Settling, and Rise Time

Overshoot

Underdamped second-order systems have a response that rises beyond the final value and then oscillates in diminishing wiggles to finally reach a steady-state value. The amount by which the first peak exceeds the final steady-state value is called the overshoot, generally stated as a fraction or a percentage. In Figure 1, the overshoot is OS/A. In other words, it is the ratio of the excess amount of the first peak divided by the final value of the step response.

Let's start by looking at the general solution of a second-order system. We'll use a voltage as our illustration. The general solution is

$$v(t) = A - Ae^{-\alpha t}\left(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t\right)$$

Where is the first peak of $v(t)$? It comes when the cosine term hits -1 for the first time after $t = 0$. This will occur at

$$\beta t = \pi$$

At this point, the sine term is 0, so the value of $v(t)$ at the first peak is

$$v(t)_{peak} = A - Ae^{-\alpha t}\left(\cos\pi + \frac{\alpha}{\beta}\sin\pi\right)$$

$$v(t)_{peak} = \left(A + Ae^{-\alpha t}\right|_{t=\pi/\beta}$$

The final value of $v(t)$ is equal to A, so the amount by which this first peak exceeds A is:

$$Peak = Ae^{-\alpha\pi/\beta}$$

Therefore, the percent overshoot is:

$$\%OS = \frac{Peak}{A} \times 100 = 100e^{-\alpha\pi/\beta}$$

We can simplify this by using what we know about α and β from previous lessons:

$$\alpha = \omega_o\zeta$$

$$\beta = \omega_o\sqrt{1-\zeta^2}$$

The result is

$$\%OS = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

How do we use this information? Generally, designers are looking for a percent overshoot smaller than a certain amount. This means setting a lower limit on ζ because the smaller ζ, the more vigorously the response $v(t)$ oscillates. Look at the s-plane shown in Figure 2.

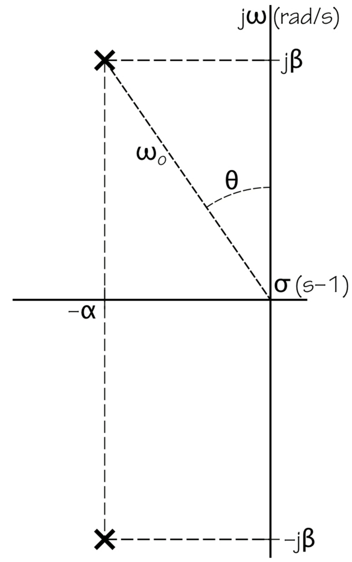

Figure 2: s-Plane Parameters

Note that

$$\frac{\alpha}{\omega_o} = \zeta = \sin\theta$$

Therefore, to limit the percent overshoot, we must staying outside the region of sine θ. The acceptable region is shown in Figure 3.

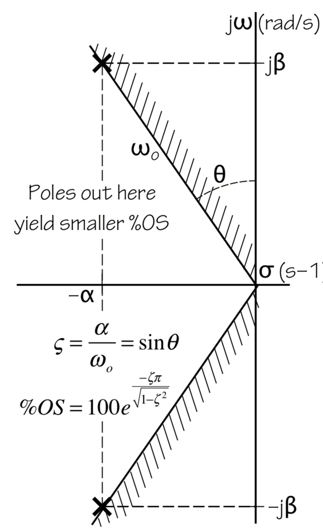

Figure 3: AC Analysis Setup

How does this translate into a design constraint? Suppose the designer specifies a percent overshoot no larger than 25. First, use the %OS equation above to find the value of ζ that will produce a %OS of 25. Second, from the *sine* formula, calculate the angle θ shown in Figure 3. The poles of the system's response must be to the left of the boundary shown. Note they can be anywhere to the left. If they are placed right on the boundary, the %OS will be 25%.

Now we've used one parameter, %OS, to place the poles of the second-order response. But this is only one data point. Since the s-plane is two-dimensional, we need two data points. We can actually have three, but only two are useful in any particular design.

Settling Time

Settling time is a measure of how close the response is to dying out, remembering that technically, it never dies out. Several different measures of settling time are used, but they are all related to the exponential decay of the $e^{-\alpha t}$ term. Common choices are the 5%, 2%, and 1% settling times.

For the 1% settling time, we equate the exponential to 0.01 and see what the exponent must be:

$$e^{-\alpha t_s} = 0.01$$

The natural logarithm solves this for us:

$$\ln\left(e^{-\alpha t_s}\right) = \ln(0.01) \Rightarrow t_s = \frac{-\ln(0.01)}{\alpha}$$

Evaluating the natural logarithm yields

$$t_s = \frac{4.605}{\alpha}$$

Similarly, the 2% settling time is

$$t_s = \frac{3.912}{\alpha}$$

Notice in Figure 4 the smaller settling times are to the left of the shaded area since α is in the denominator of the equation.

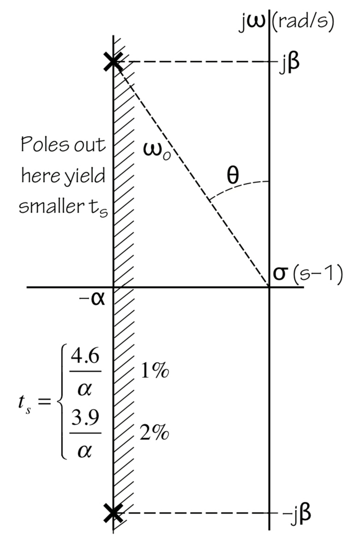

Figure 4: Settling Time Range

How does this translate into a design constraint? If the designer specifies a certain settling time, this automatically establishes the minimum value of α.

Rise Time

Overshoot and settling time are pretty straight forward. Overshoot is a rather messy equation but readily leads to a value of ζ that will provide a certain overshoot. Settling time is a simple equation that leads to α. Just when you think things are simple and neat, we introduce rise time.

The concept of rise time is very easy to understand. It is the time it takes a signal to go from 10% of its final value to 90% of its final value. Figure 1 shows this on a waveform. Note some authors and designers use 5% and 95% as their ranges when defining rise time, but 10% and 90% are the most common.

Finding the rise time should be easy, right? If $v(t)$ is going from 0 to A as in Figure 1, just write an equation that says $v(t) = 0.1A$ and solve for the time this happens. Do the same thing for 0.9A, and then subtract one time from the other. Easy, right? The problem is this equation becomes a transcendental equation with both an exponential and a sum of a sine and a cosine. Solving this is not simple.

Because of this, there are a number of methods of making a good estimate of rise time without solving the equations. While we were writing this lab, we looked at what ten different authors used for their estimate. Our search found **10 different ways** of estimating rise time.

Figure 5 shows what we think is a middle ground for rise time estimation.

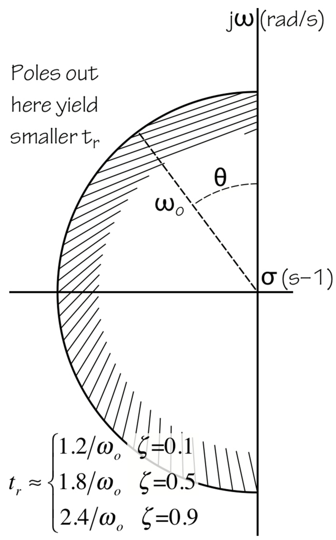

Figure 5: Rise Time Range

Three different equations are given for three different values of ζ. All have ω_o in their denominators.

Figure 5 shows the region of the s-plane where the rise time will be smaller than a specified amount, a region outside a circle of radius ω_o, since ω_o is in the denominator of our estimators.

How does this translate into a design constraint? Specifying the rise time establishes the minimum value of ω_o. If your ζ is not one of the three values shown, interpolate between the equations' numerators.

Design Example

To show how these design parameters work, here is an example. Only two of the three parameters can be specified because there are only two independent second-order system values, α and β. Note ω_o and ω_d can be computed from these two variables.

Here is the example. Design a second-order system with a rise time of 120 μs and a 2% settling time of 1.5 ms. Here are the steps I would use:

- Use the 2% settling time equation to derive α.
- Use the rise-time equation to derive ω_o. Choose the formula for $\zeta = 0.1$ as a starting guess.
- Determine the value of ζ and readjust the rise-time equation if necessary.
- Write the time-domain response of a second-order system to a unit-step in the form:

$$v(t) = 1 - e^{-\alpha t}\left(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t\right)$$

- Use Matlab to plot the response as a function of time and measure the settling time and the rise time to check the results.
- If the results do not meet specifications, use the s-plane diagrams of Figures 3, 4, and 5 to decide how to adjust α and ω_o.

Here are my calculations. First, use the settling time to get α:

$$t_s = \frac{3.912}{\alpha} = 1.5 \times 10^{-3}$$

$$\alpha = 2608 \;\; s^{-1}$$

Next, use the rise time equations, trying the $\zeta = 0.1$ equation:

$$t_r = \frac{1.2}{\omega_o} = 120 \times 10^{-6}$$

$$\omega_o = 10,000 \;\; rad/s$$

Check ζ to see if my choice of rise time is ok:

$$\zeta = \frac{\alpha}{\omega_o} = \frac{2608}{10000} = 0.26$$

Hmmm. For the rise-time equation, I chose $\zeta = 0.1$. This is pretty far off. I should use a slightly different rise-time equation. It is helpful to create an Excel spread-sheet to perform these calculations. If I want to determine ω_o for $\zeta = 0.2$ and $\zeta = 0.25$, I get the following:

$$\omega_o^{\zeta=0.2} = \frac{1.35}{120 \times 10^{-6}} = 11,250 \ \ rad/s$$

$$\omega_o^{\zeta=0.25} = \frac{1.425}{120 \times 10^{-6}} = 11,875 \ \ rad/s$$

I calculate ζ using my value for ω_o and α. I get $\zeta = 0.2318$ and $\zeta = 0.2196$ using the values obtained above. These values are very close, so I can conclude I am in the ballpark. I can split the difference and use $\omega_o = 11,562$ rad/s and $\zeta = 0.2257$.

I can solve for β and write the time response of this system:

$$\beta = \omega_o \sqrt{1 - \zeta^2} = 11,263 \ \ rad/s$$

$$v(t) = 1 - e^{-2608t} \left(\cos 11263t + 0.231 \sin 11263t\right)$$

for $t > 0$. I used Matlab to plot this and used a full-page printout of the graph to measure rise-time and settling time. I used the following Matlab code and the data cursor option in the Figure menu to come up with my times:

```
t=0:0.000001:0.002;
vt = 1 - exp(-2608*t).*(cos(11263*t) +
0.231*sin(11263*t));
plot(t,vt)
grid
```

Using the data cursor, I obtained time at 0.1A to be 40 μs. For the time at 0.9A, I obtained a time of 145.5 μs. Therefore, the rise time was 105.5 μs, which is within specifications. For the settling time, use the data cursor, and find the largest time where the amplitude is either less than 0.98 or greater than 1.02. I get $t_s = 1.463$ ms.

- Rise time = 110 μs, which is within specifications.
- Settling time = 1.463 ms, which is within specifications.

Suppose we get a change to our specifications and want to have a smaller settling time. Looking at the s-plane range for settling time in Figure 4, I decided the poles will need to be placed further to the left, so α needs to be larger to make the settling time smaller. I chose to increase α by a ratio of 1.7/1.5. Therefore, $\alpha = 2956$. I then have to recalculate ω_o, since because changing α will change ζ. Since we increased α, we know our damping coefficient will also increase. I calculated the new value for ω_o assuming ζ is approximately 0.25:

$$\omega_o = \frac{1.425}{120 \times 10^{-6}} = 11,875 \ \ rad/s$$

Calculating ζ, I get:

$$\zeta = \frac{2956}{11875} = 0.249$$

This is almost exactly correct, so no change needs to be made to the numbers. I can calculate a new value for β:

$$\beta = 11875\sqrt{1 - 0.25^2} = 11497$$

Rewriting the time response yields:

$$v(t) = 1 - e^{-2956t} \left(\cos 11497t + 0.257 \sin 11497t\right)$$

When I plotted this in Matlab, I obtained a rise time approximately 105 μs and a settling time of 119 μs.

Tasks

Design a second-order system with:

- Overshoot of 55% within 5 percentage points.
- Settling time of 2.5 ms within 5%.

Test your design in Matlab by printing a large plot and measuring the parameters. Then, adjust your design parameters if necessary to meet the specifications.

The End

Report your results on the hand-in page. Fill in the blanks with your final design values, sketch the s-plane that shows the location of your poles (see Figure 2), record the measured values of %OS and settling time, and attach your Matlab plot.

Report of Lab 10m Results

Name ____________________________________

Final Design Values

α_______________ ω_o_______________ ζ_______________ β_______________

s-Plane Sketch

Measured values from Matlab plot

%OS_______________ Settling time _______________

Attach Matlab plot