HW4 Solutions

November 2019

Problem 1. (a). The integral of joint $p.d.f.$ should be 1.

$$\begin{aligned}
1 &= \int_0^1 \int_0^{2-2x} Cxydydx \\
&= C\int_0^1 x\frac{y^2}{2}|_0^{2-2x}dx \\
&= C\int_0^1 \frac{x(2-2x)^2}{2}dx \\
&= C\int_0^1 2x(1-x)^2dx \\
&= C\int_0^1 (2x^3-4x^2+2x)dx \\
&= C(\frac{x^4}{2}-\frac{4x^3}{3}+x^2)|_0^1 \\
&= C(\frac{1}{2}-\frac{4}{3}+1) \\
&= \frac{1}{6}C.
\end{aligned}$$

Therefore, we have $C = 6$.

(b). We firstly find the marginal density of X.

$$f_x(x) = \int_0^{2-2x} 6xydy = 3x(2-2x)^2, \quad 0 \le x \le 1$$

Therefore,

$$\begin{aligned}
E[1/(1+X)] &= \int_0^1 \frac{1}{1+x} f_X(x)dx \\
&= (\text{let } t = 1+x) \int_1^2 \frac{1}{t} 3(t-1)(4-2t)^2 dt \\
&= 12\int_1^2 \frac{1}{t}(t^3 - 5t^2 + 8t - 4)dt \\
&= 12\int_1^2 (t^2 - 5t + 8 - \frac{4}{t})dt \\
&= 12[\frac{t^3}{3} - \frac{5t^2}{2} + 8t - 4lnt]|_1^2 \\
&= 12[\frac{8}{3} - 10 + 16 - 4ln2 - \frac{1}{3} + \frac{5}{2} - 8] \\
&= 34 - 48ln2
\end{aligned}$$

(c) As $2X + Y < 1$ implies that $X < \frac{1}{2}$, we have

$$\begin{aligned}
P(2X + Y < 1) &= \int_0^{\frac{1}{2}} \int_0^{1-2x} 6xydydx \\
&= 6\int_0^{\frac{1}{2}} \frac{x}{2}(1-2x)^2 dx \\
&= 3\int_0^{\frac{1}{2}} (4x^3 - 4x^2 + x)dx \\
&= 3[x^4 - \frac{4}{3}x^3 + \frac{x^2}{2}]|_0^{\frac{1}{2}} \\
&= \frac{1}{16}
\end{aligned}$$

Problem 2.

$$\begin{aligned}
P(X > Y > Z \text{ or } X < Y < Z) &= 2P(X > Y > Z) \\
&= 2\int_0^1 \int_y^1 \int_0^y 1dzdxdy \\
&= 2\int_0^1 y(1-y)dy \\
&= 2(\frac{y^2}{2} - \frac{y^3}{3})|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

Problem 3. (a) For $X = 1$, conditional mass function of Y are

$$\begin{aligned}
P_{Y|X}(Y=1|X=1) &= \frac{P(X=1,Y=1)}{P(X=1)} = \frac{p(1,1)}{p(1,1)+p(1,2)} = \frac{1}{3} \\
P_{Y|X}(Y=2|X=1) &= \frac{P(X=1,Y=2)}{P(X=1)} = \frac{p(1,2)}{p(1,1)+p(1,2)} = \frac{2}{3}
\end{aligned}$$

For $X = 2$, conditional mass function of Y are

$$\begin{aligned}
P_{Y|X}(Y=1|X=2) &= \frac{P(X=2,Y=1)}{P(X=2)} = \frac{p(2,1)}{p(2,1)+p(2,2)} = \frac{1}{5} \\
P_{Y|X}(Y=2|X=2) &= \frac{P(X=2,Y=2)}{P(X=2)} = \frac{p(2,2)}{p(2,1)+p(2,2)} = \frac{4}{5}
\end{aligned}$$

(b). Since conditional mass function of Y are different given $X = 1$ and $X = 2$, we can conclude that X and Y are not independent.

(c).

$$\begin{aligned}
P(XY \le 5/2) &= P(\{X=1,Y=1\} \cup \{X=1,Y=2\} \cup \{X=2,Y=1\}) \\
&= p(1,1)+p(1,2)+p(2,1) = \frac{1}{2} \\
P(X+Y \ge 7/3) &= P(\{X=1,Y=2\} \cup \{X=2,Y=1\} \cup \{X=2,Y=2\}) \\
&= p(1,2)+p(2,1)+p(2,2) = \frac{7}{8} \\
P(X/Y > 3/2) &= P(\{X=2,Y=1\}) = p(2,1) = \frac{1}{8}
\end{aligned}$$

Problem 4. (a). The integral of joint $p.d.f.$ should be 1.

$$\begin{aligned}
1 &= \int_0^{+\infty}\int_0^{+\infty} Cye^{-y(2+x)}dxdy \\
&= C\int_0^{+\infty} -e^{-y(2+x)}|_{x=0}^{x=+\infty}dy \\
&= C\int_0^{+\infty} e^{-2y}dy = -\frac{1}{2}Ce^{-2y}|_{y=0}^{y=+\infty} \\
&= \frac{1}{2}C.
\end{aligned}$$

Therefore, we have $C = 2$.

(b). We firstly find the marginal density of Y.

$$\begin{aligned}
f_Y(y) &= \int_0^{+\infty} Cye^{-y(2+x)}dx \\
&= -Ce^{-y(2+x)}|_{x=0}^{x=+\infty} \\
&= Ce^{-2y} = 2e^{-2y},\ y > 0.
\end{aligned}$$

Therefore, the conditional density of X given Y is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{Cye^{-y(2+x)}}{Ce^{-2y}} = ye^{-xy},\ x,y > 0.$$

(c). We first find the cdf of Z.

$$\begin{aligned}
F_Z(z) &= P(Z \le z) = P(XY \le z) = \int_0^{+\infty}\int_0^{\frac{z}{y}} 2ye^{-y(2+x)}dxdy \\
&= 2\int_0^{+\infty} -e^{-y(2+x)}|_{x=0}^{x=\frac{z}{y}} dy = 2\int_0^{+\infty}\left(e^{-2y} - e^{-(z+2y)}\right)dy \\
&= 2(1-e^{-z})\int_0^{+\infty} e^{-2y}dy = -(1-e^{-z})e^{-2y}|_{y=0}^{y=+\infty} \\
&= 1-e^{-z},\ z > 0.
\end{aligned}$$

Therefore, $f_Z(z) = \frac{dF_Z(z)}{dz} = e^{-z}$ when $z > 0$.

Problem 5. (a). The integral of joint *p.d.f.* should be 1.

$$\begin{aligned}
1 &= C\int_1^{+\infty}\int_1^{+\infty}\frac{1}{x^3y^2}dydx \\
&= C\int_1^{+\infty} -\frac{1}{x^3y}|_{y=1}^{y=+\infty}dx \\
&= C\int_1^{+\infty}\frac{1}{x^3}dx = -\frac{1}{2}Cx^{-2}|_{x=1}^{x=+\infty} \\
&= \frac{1}{2}C.
\end{aligned}$$

Therefore, we have $C = 2$.

(b). We first write out the Jacobian,

$$J(x,y) = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ y & x \end{vmatrix} = \frac{2x}{y}.$$

Notice that $X = \sqrt{UV} \ge 1$ and $Y = \sqrt{\frac{V}{U}} \ge 1$, so $U > 0$, $V \ge 1$ and $\frac{1}{V} \le U \le V$. The joint density of U, V is

$$f_{U,V}(u,v) = f_{X,Y}(x,y)|J(x,y)|^{-1} = \frac{2}{x^3y^2}\frac{y}{2x} = \frac{1}{x^4y} = \frac{1}{u^{\frac{3}{2}}v^{\frac{5}{2}}},$$

with $u > 0$, $v \ge 1$ and $\frac{1}{v} \le u \le v$.

(c). We find marginal of X, Y, U, V below:

$$\begin{aligned}
f_X(x) &= 2\int_1^{+\infty}\frac{1}{x^3y^2}dy \\
&= \frac{-2}{x^3y}|_{y=1}^{y=+\infty} \\
&= \frac{2}{x^3},\ x \ge 1.
\end{aligned}$$

$$\begin{aligned} f_Y(y) &= 2\int_1^{+\infty} \frac{1}{x^3y^2}dx \\ &= \frac{-1}{x^2y^2}|_{x=1}^{x=+\infty} \\ &= \frac{1}{y^2},\ y \geq 1. \end{aligned}$$

Notice that we have $\frac{1}{v} \leq u \leq v$, this means $v \geq \frac{1}{u}$ and $v \geq u$. Let's discuss two cases. The first one is that $0 < u \leq 1$, this means $\frac{1}{u} \geq 1$, so in this case

$$\begin{aligned} f_U(u) &= \int_{\frac{1}{u}}^{+\infty} \frac{1}{u^{\frac{3}{2}}v^{\frac{5}{2}}}dv \\ &= -\frac{1}{u^{\frac{3}{2}}}\frac{2}{3}v^{-\frac{3}{2}}|_{v=\frac{1}{u}}^{v=+\infty} \\ &= \frac{2}{3},\ 0 < u \leq 1. \end{aligned}$$

The second case is that $u \geq 1$, this means $\frac{1}{u} \leq 1$, so in this case

$$\begin{aligned} f_U(u) &= \int_{u}^{+\infty} \frac{1}{u^{\frac{3}{2}}v^{\frac{5}{2}}}dv \\ &= -\frac{1}{u^{\frac{3}{2}}}\frac{2}{3}v^{-\frac{3}{2}}|_{v=u}^{v=+\infty} \\ &= \frac{2}{3}u^{-3},\ u \geq 1. \end{aligned}$$

For the density of V, much simpler.

$$\begin{aligned} f_V(v) &= \int_{\frac{1}{v}}^{v} \frac{1}{u^{\frac{3}{2}}v^{\frac{5}{2}}}du \\ &= -2v^{-\frac{5}{2}}u^{-\frac{1}{2}}|_{\frac{1}{v}}^{v} \\ &= 2(v^{-2} - v^{-3}),\ v \geq 1. \end{aligned}$$

To summarize, we have

$$\begin{aligned} f_{X,Y}(x,y) &= 2x^{-3}y^{-2}\mathcal{I}(x \geq 1)\mathcal{I}(y \geq 1), \\ f_X(x) &= \frac{2}{x^3}\mathcal{I}(x \geq 1), \\ f_Y(y) &= \frac{1}{y^2}\mathcal{I}(y \geq 1), \\ f_{U,V}(u,v) &= u^{-\frac{3}{2}}v^{-\frac{5}{2}}\mathcal{I}(u \geq 0)\mathcal{I}(v \geq 1)\mathcal{I}(\frac{1}{v} \leq u \leq v), \\ f_V(v) &= 2(v^{-2} - v^{-3})\mathcal{I}(v \geq 1), \\ f_U(u) &= \begin{cases} \frac{2}{3}u^{-3}, & u \geq 1 \\ 2/3, & 0 < u \leq 1. \end{cases} \end{aligned}$$

Here the indicator function $\mathcal{I}(A) = 1$ if A happens, otherwise, it equals 0. Therefore, from the above, we can see that X and Y are independent. However, U and V are not independent. (The reason is that the indicator function in $f_{U,V}(u, v)$ cannot be written into the product of two functions of u and v only.)