JOINTLY DISTRIBUTED RANDOM VARIABLES

1. Joint distribution functions

Thus far, we have concerned ourselves only with probability distributions for single random variables. However, we are often interested in probability statements concerning two or more random variables. In order to deal with such probabilities, we define, for any two random variables X and Y, the joint cumulative probability distribution function of X and Y by

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y), \quad x, y \in \mathbb{R}.$$

We can get the cdf of either X or Y from the joint cdf

$$F_X(x) = F(x, +\infty), \quad F_Y(y) = F(+\infty, y),$$

because the event $X \leq +\infty$ or $Y \leq +\infty$ always hold.

Note that the cdf exists for both discrete and continuous random variables. In contrast, the density is defined only for continuous random variables and the probability mass function only for discrete random variables.

1.1. **Discrete case.** In the case when X and Y are both discrete random variables, we define the joint probability mass function of X and Y by

$$p(x, y) = \mathbb{P}(X = x, Y = y).$$

The probability mass function of X (or Y) can be obtained from $p(x, y)$ summing over all possible values y (or x):

$$p(x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y p(x, y).$$

Example 1.1. Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. Let X and Y denote, respectively, the number of red and white balls chosen. Then

$$\begin{aligned} p(0,0) &= \frac{\binom{5}{3}}{\binom{3+4+5}{3}} = \frac{10}{220}, \\ p(1,2) &= \frac{\binom{3}{1} \cdot \binom{4}{2}}{\binom{3+4+5}{3}} = \frac{18}{220}, \\ p(2,1) &= \frac{\binom{3}{2} \cdot \binom{4}{1}}{\binom{3+4+5}{3}} = \frac{12}{220}. \end{aligned}$$

Example 1.2. Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1 child, 35 percent have 2 children, and 30 percent have 3. Suppose further that in each family each child is equally likely (independently) to be a boy or a girl. If a family is chosen at random from this community, let B be the number of boys and G be the number of girls in this family. Then

$$\begin{aligned} p(0,0) &= \mathbb{P}(B=0, G=0) = 0.15, \\ p(1,2) &= \mathbb{P}(B=1, G=2) \\ &= \mathbb{P}(B+G=3) \cdot \mathbb{P}(B=1, G=2 | B+G=3) \\ &= 0.3 \cdot (1/2) \cdot (1/2)^2 \\ &= 0.0375. \end{aligned}$$

1.2. **Continuous case.** We say that X and Y are jointly continuous if there exists a function $f(x,y)$, defined for all real x and y, such that for every set C of pairs of real numbers (that is, C is a set in the two-dimensional plane),

$$\mathbb{P}((X,Y) \in C) = \int\int_{(x,y)\in C} f(x,y)dxdy.$$

The function $f(x,y)$ is called the joint probability density function of X and Y. In particular, if $C = A \times B$ then

$$\mathbb{P}(X \in A, Y \in B) = \int_A \int_B f(x,y)dxdy.$$

If $A = (-\infty, x)$ and $B = (-\infty, y)$ then

$$F(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)dudv.$$

Upon differentiation, we get

$$f(x,y) = \frac{\partial}{\partial x \partial y} F(x,y).$$

To calculate th marginal density of one variable, we integrate the joint density with respect to the other random variable

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy.$$

Example 1.3. Let (X,Y) be a point chosen uniformly at random from a circle of radius one (all regions within the circle of equal area are equally likely to contain the point). Then $f(x,y) = c$ for some constant $c > 0$ if $x^2 + y^2 \leq 1$ and $f(x,y) = 0$ otherwise.

(a) Determine c.
(b) Find the marginal density functions of X and Y.
(c) Let D be the distance from the origin of the point selected. Find the cdf of D.
(d) Find $\mathbb{E}[D]$.

Solution. (a) Since the integral of the joint density over the unit circle is one,

$$1 = c\int\int_{x^2+y^2\leq 1} dxdy = c \cdot Area(\text{circle}) = c \cdot \pi,$$

which implies $c = 1/\pi$.

(b) The density of X is

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} c dy = \frac{2\sqrt{1-x^2}}{\pi}$$

if $x \in (-1,1)$ and $f(x) = 0$ otherwise.

(c) The cdf of $D = \sqrt{X^2+Y^2}$ if $a \in (0,1)$ is

$$\begin{aligned} F(a) &= \mathbb{P}(D \le a) = \mathbb{P}(X^2+Y^2 \le a^2) = \int_{x^2+y^2 \le a^2} f(x,y)dxdy \\ &= c\int_{x^2+y^2\le a^2} dxdy = c \cdot \text{Area(circle of radius } a) = c \cdot \pi \cdot a^2 \\ &= a^2. \end{aligned}$$

Of course $F(a) = 0$ if $a \le 0$ and $F(a) = 1$ if $a \ge 1$. Note that in order to give a complete answer to the question, you must give the formula for F for any $a \in \mathbb{R}$, therefore you have to specify what $F(a)$ is when $a \le 0$ or $a \ge 1$.

(d) We first find the density of D:

$$f(a) = \frac{d}{da}F(a) = 2a$$

if $a \in (0,1)$ and $f(a) = 0$ otherwise. Hence

$$\mathbb{E}D = \int_0^1 a \cdot 2a da = 2/3.$$

□

We can also define joint probability distributions for n random variables in exactly the same manner as we did for $n = 2$.

1.3. **Independent random variables.** If X and Y are independent then

$$F(x,y) = \mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x) \cdot \mathbb{P}(Y \le y) = F_X(x) \cdot F_Y(y).$$

By differentiating the joint cdf, we get

$$f(x,y) = \frac{\partial}{\partial x \partial y} F_X(x) \cdot F_Y(y) = \frac{\partial}{\partial x} F_X(x) \cdot \frac{\partial}{\partial y} F_Y(y) = f_X(x) \cdot f_Y(y).$$

Example 1.4. Let X, Y be independent exponential random variables with parameter $\lambda = 1$. Find the density of $Z = X/Y$.

Solution. Since X, Y are independent, the joint density of X and Y is

$$f(x,y) = f_X(x) f_Y(y) = e^{-x-y}, \quad 0 < x, y < \infty.$$

We first find the cdf of Z

$$F_Z(z) = \mathbb{P}(X/Y \le z) = \int\int_{x/y \le z} e^{-x-y} dxdy.$$

We integrate e^{-x-y} over the set of (x,y) such that $x/y \le z$ and $x, y > 0$, which equal

$$\{(x,y) : 0 < y < \infty, 0 < x < zy\}.$$

Therefore

$$\begin{aligned} F_Z(z) &= \int_0^\infty dy \int_0^{zy} e^{-x-y} dx = \int_0^\infty e^{-y} dy \int_0^{zy} e^{-x} dx \\ &= \int_0^\infty e^{-y}(1-e^{-zy}) dy = \left(-e^{-y} + \frac{e^{-(z+1)y}}{z+1}\right)\Big|_0^\infty \\ &= \frac{z}{z+1}. \end{aligned}$$

Differentiating the cdf, we find that the density of Z is $f_Z(z) = 1/(z+1)^2$ for $0 < z < \infty$. □

It is often important to be able to calculate the distribution of $X+Y$ from the distributions of X and Y when X and Y are independent. Suppose that X and Y are independent, continuous random variables having probability density functions f_X and f_Y. Then the joint density of X and Y is $f(x,y) = f_X(x) \cdot f_Y(y)$. The cumulative distribution function of $Z = X + Y$ is obtained as follows:

$$F_Z(z) = \mathbb{P}(X + Y \le z) = \int\int_{x+y\le z} f_X(x) f_Y(y) dx dy$$

We integrate over the set of all (x, y) such that $x + y \le z$. This set can be described as the set of all (x, y) such that $y \in \mathbb{R}$ and $x \in (-\infty, z - y)$. Therefore we can rewrite the integral as

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-y} f_X(x) dx \right) f_Y(y) dy = \int_{-\infty}^{\infty} F_X(z-y) f_Y(y) dy.$$

Now, differentiate both sides with respect to z, we obtain

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy.$$

This is a general formula for calculating the density of a sum of independent continuous random variables. The integral on the right is known as the convolution of f_X and f_Y.

Example 1.5. If X and Y are independent random variables, both uniformly distributed on $(0, 1)$, calculate the probability density of $Z = X + Y$.

Solution. Note first that $f_X(x) = 1$ if $x \in (0, 1)$ and $f_X(x) = 0$ otherwise; also, $f_X = f_Y$. Therefore

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy = \int_0^1 f_X(z-y) dy.$$

Now, $f_X(z - y) = 1$ if $z - y \in (0, 1)$ and $f_X(z - y) = 0$ otherwise. The condition $z - y \in (0, 1)$ is equivalent to $z \in (y, y + 1)$.

If $z \le 0$ then $z \notin (y, y + 1)$, so $f_Z(z) = 0$; similarly, if $z > 2$ then $z \notin (y, y + 1)$ so $f_Z(z) = 0$. If $z \in (0, 1)$ then

$$f_Z(z) = \int_0^z dy = z.$$

If $z \in [1,2)$ then

$$f_Z(z) = \int_{z-1}^{1} dy = 2 - z.$$

Therefore

$$f_Z(z) = \begin{cases} z, & z \in (0,1) \\ 2-z, & z \in [1,2) \\ 0, & \text{otherwise.} \end{cases}$$

□

Example 1.6. Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ be independent normal random variables. Then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$. For the proof of this fact, please see the textbook.

Example 1.7. If X and Y are independent Poisson random variables with respective parameters λ_X and λ_y then $X + Y$ is a Poisson random variable with parameter $\lambda_X + \lambda_Y$.

2. Conditional distributions

2.1. **Discrete case.** Hence, if X and Y are discrete random variables, it is natural to define the conditional probability mass function of X given that $Y = y$, by

$$p_{X|Y}(x|y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \frac{p(x,y)}{p_Y(y)}$$

for all values of y such that $p_Y(y) > 0$. Similarly, the conditional cdf of X given that $Y = y$ is defined

$$F_{X|Y}(x|y) = \mathbb{P}(X \le x | Y = y) = \sum_{a \le x} \mathbb{P}(X = a | Y = y).$$

If X and Y are independent then $p_{X|Y}(x|y) = p_X(x)$ and $F_{X|Y}(x|y) = F_X(x)$.

Example 2.1. If X and Y are independent Poisson random variables with respective parameters α_X and λ_Y, calculate the conditional distribution of X given that $Z = X+Y = n$.

Solution. Recall that Z has a Poisson distribution with parameter $\lambda_X + \lambda_Y$. Therefore

$$\begin{aligned} \mathbb{P}(X = k | Z = n) &= \frac{\mathbb{P}(X = k, Z = n)}{\mathbb{P}(Z = n)} = \frac{\mathbb{P}(X = k, Y = n - k)}{\mathbb{P}(Z = n)} = \frac{\mathbb{P}(X = k) \cdot \mathbb{P}(Y = n - k)}{\mathbb{P}(Z = n)} \\ &= \frac{e^{-\lambda_X} \lambda_X^k}{k!} \cdot \frac{e^{-\lambda_Y} \lambda_Y^{n-k}}{(n-k)!} \cdot \left(\frac{e^{-\lambda_X - \lambda_Y} (\lambda_X + \lambda_Y)^n}{n!} \right)^{-1} \\ &= \binom{n}{k} \cdot \left(\frac{\lambda_X}{\lambda_X + \lambda_Y} \right)^k \cdot \left(\frac{\lambda_Y}{\lambda_X + \lambda_Y} \right)^{n-k}. \end{aligned}$$

In other words, the conditional distribution of X given that $X + Y = n$ is the binomial distribution with parameters n and $\lambda_X / (\lambda_X + \lambda_Y)$. □

2.2. **Continuous case.** If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X given that $Y = y$ is defined, for all values of y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

If X and Y are independent then $f(x, y) = f_X(x)f_Y(y)$, so $f_{X|Y}(x|y) = f_X(x)$.

Example 2.2. Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y-y}}{y}, & 0 < x, y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathbb{P}[X > 1|Y = y]$.

Solution. We first find the marginal density of Y by integrating the joint density with respect to x:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{\infty} \frac{e^{-x/y-y}}{y}dx = e^{-y}, \quad y \in (0, \infty).$$

Therefore the conditional density of X given $Y = y \in (0, \infty)$ is

$$f_X|Y(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{e^{-x/y-y}}{y}}{e^{-y}} = \frac{e^{-x/y}}{y}, \quad x \in (0, \infty).$$

Hence

$$\mathbb{P}[X > 1|Y = y] = \int_1^{\infty} \frac{e^{-x/y}}{y}dx = e^{-x/y}\Big|_1^{\infty} = e^{-1/y}.$$

$\square$

3. Joint distribution of functions of random variables

Let X_1 and X_2 be jointly continuous random variables with joint probability density function f_{X_1,X_2}. It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2, which arise as functions of X_1 and X_2. Specifically, suppose that

$$Y_1 = g_1(X_1, X_2), \quad Y_2 = g_2(X_1, X_2).$$

Assume that the functions g_1 and g_2 satisfy the following conditions:

(1) Equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2 with solutions given by $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$.

(2) The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant

$$J(x_1, x_2) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} \neq 0$$

at all points (x_1, x_2).

Under these two conditions, it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}.$$

Example 3.1. Let X_1 and X_2 be independent uniform $(0, 1)$ random variables, $Y_1 = X_1+X_2$ and $Y_2 = X_1 - X_2$. Find the joint distribution of Y_1 and Y_2.

Solution. In in example, $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$. Solving the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ for x_1 and x_2, we get

$$x_1 = h_1(y_1, y_2) = \frac{y_1 + y_2}{2}, \quad x_2 = h_2(y_1, y_2) = \frac{y_1 - y_2}{2}.$$

Therefore

$$J(x_1, x_2) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2.$$

Since X_1, X_2 are independent uniform random variables on $(0, 1)$, their joint density is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1, & 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

It then follows that

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 1/2, & 0 < y_1 + y_2 < 2, \quad 0 < y_1 - y_2 < 2 \\ 0, & \text{otherwise.} \end{cases}$$

$\square$