CONTINUOUS RANDOM VARIABLES

There exist random variables whose set of possible values is uncountable. Two examples are the time that a train arrives at a specified stop and the lifetime of a transistor.

Let X be such a random variable. We say that X is a **continuous** random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that, for any set B of real numbers

$$\mathbb{P}(X \in B) = \int_B f(x)dx.$$

The function f is called the *probability density function* of the random variable X. Since X must assume some value, f must satisfy

$$\int_{-\infty}^{\infty} f(x)dx = \mathbb{P}(-\infty < X < \infty) = 1.$$

All probability statements about X can be answered in terms of f. For instance, letting $B = [a, b]$, we obtain

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx.$$

That is, the probability that X is between a and b is the area of the region formed by the x-axis, the graph of function $y = f(x)$ and two vertical lines $x = a$ and $x = b$; see Figure 1. In particular, if $a = b$ then

$$\mathbb{P}(X = a) = \int_a^a f(x)dx = 0.$$

In words, this equation states that the probability that a continuous random variable will assume any fixed value is zero.

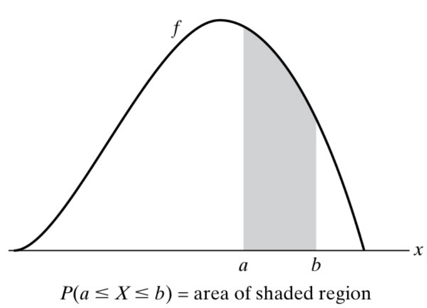

Figure 1. Probability density function f.

Setting $a = -\infty$ and $b = x$, we obtain the **cumulative distribution function** (or cdf) of X:

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

The cdf is a function $F : (-\infty, \infty) \to [0, 1]$ that takes values between 0 and 1 and the domain is the whole real line. By differentiate the cdf, we obtain the density function

$$\frac{d}{dx}F(x) = f(x).$$

Example 0.1. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find C, the cdf and the probability that $X > 1$.

Solution. Since the integral of f over the real line $(-\infty, \infty)$ is one, we have

$$1 = \int_0^2 C(4x - 2x^2)dx = C(2x^2 - 2x^3/3) \Big|_0^2 = 8C/3,$$

or $C = 3/8$. To find the cdf, note first that $F(x) = 0$ if $x \leq 0$ and $F(x) = 1$ if $x \geq 2$ because the density f is zero outside interval $(0, 2)$. For the remaining case when $x \in (0, 2)$,

$$F(x) = \int_0^x f(x)dx = 3/8 \cdot (2x^2 - 2x^3/3) \Big|_0^x = 3x^2/4 - x^3/4.$$

We can check our answer by differentiate $F(x)$:

$$\frac{d}{dx}F(x) = \frac{d}{dx}(3x^2/4 - x^3/4) = 3x/2 - 3x^2/4 = f(x).$$

In summary,

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 3x^2/4 - x^3/4, & \text{if } x \in (0, 2) \\ 1, & \text{if } x \geq 2. \end{cases}$$

It remains to find the probability that $X > 1$:

$$\mathbb{P}(X > 1) = 1 - \mathbb{P}(X \leq 1) = 1 - F(1) = 1 - (3/4 - 1/4) = 1/2.$$

□

1. Expectation and variance

1.1. **Expectation.** The expected value of a continuous random variable with density f is defined by

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx.$$

Similar to the expectation of a discrete random variable, $\mathbb{E}X$ is essentially a weighted sum (or integral, to be exact) of the values x that X can take with the weight being $f(x)$.

Example 1.1. The density function of X is given by

$$f(x) = \begin{cases} 1, & x \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}e^X$.

Solution. We first find the cdf F_Y of $Y = e^X$. Since the range of X is $(0, 1)$, the range of Y is $(1, e)$. If $y \le 1$ then $F_Y(y) = 0$ and if $y \ge e$ then $F_Y(y) = 1$. If $y \in (1, e)$ then

$$F_Y(y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \log y) = \int_0^{\log y} dt = \log y.$$

Therefore the density of e^X when $y \notin (1, e)$ is zero, as the derivative of zero or one is zero. When $y \in (1, e)$, we have

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \log y = \frac{1}{y}.$$

Hence,

$$\mathbb{E}[e^X] = \int_1^e y \cdot \frac{1}{y} dy = e - 1.$$

$\square$

The previous example shows us how to calculate the expected value of a function of a random variable (the function is $g(x) = e^x$ and the new random variable is $Y = g(X)$) by first calculate the density of Y and then find the expectation of Y. Similar to the case of discrete random variables, we can avoid the calculation of the density function by using the following formula

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x) f(x) dx,$$

where f is the density of X.

For example, in the previous example, we can calculate the expected value of e^X as

$$\mathbb{E}e^X = \int_0^1 e^x \cdot 1 dx = e^x \Big|_0^1 = e - 1.$$

Similar to the case of discrete random variables, if X, Y are continuous random variables and a, b are some real numbers then

$$\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y, \quad \mathbb{E}(aX + b) = a\mathbb{E}X + b.$$

1.2. **Variance.** The variance of a continuous random variable is defined exactly as before

$$Var(X) = \mathbb{E}(X - \mu)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2, \quad \mu = \mathbb{E}X.$$

For any real number a, we have

$$Var(aX) = \mathbb{E}(aX)^2 - (\mathbb{E}aX)^2 = a^2\mathbb{E}X^2 - a^2(\mathbb{E}X)^2 = a^2 Var(X).$$

Example 1.2. Let X be the random variable in Example 1.1 and $Y = e^X$. Then $\mathbb{E}Y = e - 1$ and using integral by parts, we have

$$\mathbb{E}Y^2 \quad = \quad \mathbb{E}e^{2X} = \int_0^1 e^{2x} dx = e^{2x}/2\Big|_0^1 = (e^2 - 1)/2.$$

Therefore

$$Var(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = (e^2 - 1)/2 - (e - 1)^2 = (-e^2 + 4e - 3)/2.$$

Similar to the case of discrete random variables, the variance of a sum of independent continuous random variables is equal to the sum of their variances. That is, if $X_1, ..., X_n$ are independent then

$$Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n).$$

Here, $X_1, X_2, ..., X_n$ are said to be independent if for any sets $A_1, A_2, ..., A_n \subset \mathbb{R}$, we have

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, ..., X_n \in A_n) = \mathbb{P}(X_1 \in A_1) \cdot \mathbb{P}(X_2 \in A_2) \cdots \mathbb{P}(X_n \in A_n).$$

1.3. **The distribution of a function of a random variable.** Often, we know the density of X and are interested in determining the density of $Y = g(X)$, not just the expected value of Y.

Example 1.3. If X is a continuous random variable with probability density f, then the distribution of $Y = X^2$ is obtained as follows: For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \mathbb{P}(X \leq \sqrt{y}) - \mathbb{P}(X \leq -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

Differentiating the cdf gives

$$f_Y(y) = \frac{1}{2\sqrt{y}}\Big[f_X(\sqrt{y}) - f_X(-\sqrt{y})\Big].$$

When the dependence of Y on X is strictly monotone, we have the following general formula to calculate the density of Y. Let us first recall the definition of a monotone function.

Function $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing if $g(x) < g(y)$ for any $x < y$; g is strictly decreasing if $g(x) > g(y)$ for any $x < y$. We say that g is strictly monotone if it is either strictly increasing or strictly decreasing.

Proposition 1.4. *Let X be a continuous random variable with density f_X. Suppose that $Y = g(X)$ where g is a strictly monotonic and differentiable function. Then the density f_Y of Y is given by*

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy}g^{-1}(y)\right|.$$

The proof of this formula is quite simple. Consider the case when g is strictly increasing (the proof for the other case is similar). We first find the cdf of Y:

$$F_Y(y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X\left(g^{-1}(y)\right).$$

When g is strictly increasing g^{-1} is also increasing and the derivative of g^{-1} is positive. Therefore

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \frac{d}{dy}g^{-1}(y) = f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy}g^{-1}(y)\right|.$$

2. Common continuous random variables

2.1. **Uniform random variable.** A random variable is said to be uniformly distributed on interval (a, b) if its density is constant on (a, b) and zero outside (a, b). Since the integral of the density over (a, b), it follows that

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

A simple calculation shows that

$$\mathbb{E}X = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}.$$

2.2. **Normal random variable.** We say that X is a normal random variable, or simply that X is normally distributed, with parameters μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

The expectation and variance of X are

$$\mathbb{E}X = \mu, \quad Var(X) = \sigma^2.$$

When $\mu = 0$ and $\sigma^2 = 1$, X is said to be standard normal random variable. The cdf of X is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx.$$

Although $\Phi(x)$ can not be calculated exactly for most values of x, we can approximate it with any accuracy. The values of Φ are often given by a lookup table.

2.3. **Exponential random variable.** A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

is said to be an exponential random variable with parameter λ. The expected value, variance and cdf are given by

$$\mathbb{E}X = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}, \quad F(x) = 1 - e^{-\lambda x}.$$