Homework 3, due on 11/01

Problem 1. If $\mathbb{E}\,X = -2$ and $Var(X) = 3$, find

(a) $\mathbb{E}(-1+2X)^2$
(b) $Var(5-3X)$

Solution

(a)

$$\begin{aligned}
\mathbb{E}(-1+2X)^2 &= \mathbb{E}(4X^2-4X+1) \\
&= 4\,\mathbb{E}(X^2) - 4\,\mathbb{E}(X) + 1 \\
&= 4(Var(X) + (\mathbb{E}(X))^2) - 4\,\mathbb{E}(X) + 1 \\
&= 4(3+(-2)^2) - 4(-2) + 1 \\
&= 37
\end{aligned}$$

(b)

$$\begin{aligned}
Var(5-3X) &= (-3)^2 Var(X) \\
&= 9 \times 3 \\
&= 27
\end{aligned}$$

Problem 2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cx^2e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

What is the probability that the system functions for at least 10 months?

Solution

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) &= \int_0^{\infty} Cx^2e^{-x}dx \\
&= C\int_0^{\infty} -x^2de^{-x} \\
&= C\{-x^2e^{-x}|_0^{\infty} + \int_0^{\infty} e^{-x}dx^2\} \\
&= C\{0 + \int_0^{\infty} -2xde^{-x}\} \\
&= C\{-2xe^{-x}|_0^{\infty} + 2\int_0^{\infty} e^{-x}dx\} \\
&= C\{0 - 2e^{-x}|_0^{\infty}\} \\
&= 2C
\end{aligned}$$

As $\int_{-\infty}^{\infty} f(x) = 1$, we have $C = \frac{1}{2}$.

$$\begin{aligned}
\mathbb{P}(X \geq 10) &= \int_{10}^{\infty} f(x)dx \\
&= \int_{10}^{\infty} \frac{1}{2}x^2e^{-x}dx \\
&= 61e^{-10}
\end{aligned}$$

Problem 3. If X is uniformly distributed over $(-1, 3)$, find
(a) $\mathbb{P}(|X| > 2)$
(b) The density function of the random variable $|X|$

Solution
Density function for X is

$$f(x) = \begin{cases} \frac{1}{4}, & -1 \leq x < 3 \\ 0, & \text{otherwise} \end{cases}$$

(a)

$$\begin{aligned}
\mathbb{P}(|X| > 2) &= \mathbb{P}(\{X > 2\} \cup \{X < -2\}) \\
&= \mathbb{P}(X > 2) + \mathbb{P}(X < -2) \\
&= \frac{1}{4} + 0 \\
&= \frac{1}{4}
\end{aligned}$$

(b) Let $Y = |X|$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(|X| \leq y) = \begin{cases} \mathbb{P}(-y \leq X \leq y), & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ \frac{y}{2}, & 0 \leq y < 1 \\ \frac{y+1}{4}, & 1 \leq y < 3 \\ 1, & y \geq 3 \end{cases}$$

Therefore, the density function for $Y = |X|$ is

$$\begin{cases} \frac{1}{2}, & 0 \leq y < 1 \\ \frac{1}{4}, & 1 \leq y < 3 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4. If X is an exponential random variable with parameter $\lambda = 2$, compute the probability density function of the random variable Y defined by $Y = \log X + 1$.

Solution As $g(x) = \log(x) + 1$ is strictly monotonic and differentiable, and $g^{-1}(y) = e^{y-1}$, we have

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \\ &= 2e^{-2e^{y-1}}|e^{y-1}| \\ &= 2e^{y-1-2e^{y-1}} \end{aligned}$$

Problem 5. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 50$ and $\sigma = 6$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of less than 30 inches? What assumptions are you making?

Solution Let $X \sim \mathbb{N}(50, 6^2)$,

$$\begin{aligned} \mathbb{P}(X < 30) &= \mathbb{P}\left(\frac{X - 50}{6} < \frac{30 - 50}{6}\right) \\ &= \Phi\left(-\frac{10}{3}\right) \\ &= 0.0004 \end{aligned}$$

The event is equivalent to that for the following 10 years, the annual rainfalls are above 30 inches. Assume the annual rainfalls in different years are independent, we have

$$\mathbb{P}(10 \text{ years, above } 30 \text{ inches}) = (1 - \mathbb{P}(X < 30))^{10} = 0.996$$

A / B	1	2
1	(1,2)	(2,3)
2	(2,3)	(2,4)
3	(3,4)	(3,5)

TABLE 1. Problem 6.(a)

A / B	1	2
1	(1,1)	(1,2)
2	(2,2)	(2,2)
3	(3,3)	(3,3)

TABLE 2. Problem 6.(b)

Problem 6. Urn A contains three balls labeled from 1 to 3 and urn B contains 2 balls labeled from 1 to 2. A ball is randomly selected from urn A and another ball is randomly selected from urn B. Find the joint probability mass function of X and Y when

(a) X is the largest value obtained on the selected balls and Y is the sum of the values;
(b) X is the value on the ball from urn A and Y is the larger of the two values;
(c) X is the smallest and Y is the largest value obtained on the balls.

Solution

(a) We can write down the values for the pair (X, Y) when different balls are selected. See Table 1.

Based on the table, the joint probability mass function is

$$p(1,2) = \frac{1}{6}$$
$$p(2,3) = \frac{1}{3}$$
$$p(2,4) = \frac{1}{6}$$
$$p(3,4) = \frac{1}{6}$$
$$p(3,5) = \frac{1}{6}$$

A / B	1	2
1	(1,1)	(1,2)
2	(1,2)	(2,2)
3	(1,3)	(2,3)

TABLE 3. Problem 6.(c)

(b) Based on Table 2,

$$p(1,1) = \frac{1}{6}$$
$$p(1,2) = \frac{1}{6}$$
$$p(2,2) = \frac{1}{3}$$
$$p(3,3) = \frac{1}{3}$$

(c)

Based on Table 3,

$$p(1,1) = \frac{1}{6}$$
$$p(1,2) = \frac{1}{3}$$
$$p(1,3) = \frac{1}{6}$$
$$p(2,2) = \frac{1}{6}$$
$$p(2,3) = \frac{1}{6}$$

Problem 7. The joint probability density function of X and Y is given by

$$f(x,y) = C(x^2 + xy), \quad x \in (0,1), \quad y \in (0,2).$$

(a) Find C.
(b) Compute the density of X.
(c) Find $\mathbb{P}(X > Y)$.
(d) Find $\mathbb{E}\,X$ and $\mathbb{E}\,Y$.

Solution

(a)

$$\begin{aligned}
\int_0^2 \int_0^1 C(x^2 + xy)dxdy &= C \int_0^2 [\frac{1}{3}x^3 + \frac{1}{2}x^2 y|_{x=0}^1]dy \\
&= C \int_0^2 \frac{1}{3} + \frac{1}{2}ydy \\
&= C\frac{1}{3}y + \frac{1}{4}y^2|_{y=0}^2 \\
&= \frac{5}{3}C
\end{aligned}$$

As $\int_0^2 \int_0^1 C(x^2 + xy)dxdy = 1$, we have $C = \frac{3}{5}$.

(b)

$$\begin{aligned}
f_X(x) &= \int_0^2 f(x, y)dy \\
&= \int_0^2 \frac{3}{5}(x^2 + xy)dy \\
&= \frac{3}{5}[x^2 y + \frac{1}{2}xy^2]|_{y=0}^2 \\
&= \frac{3}{5}[2x^2 + 2x] \\
&= \frac{6}{5}(x^2 + x), \quad x \in (0, 1)
\end{aligned}$$

(c) See figure 1. The pair (X, Y) takes value from the shaded region. And $X > Y$ holds when (X, Y) takes value from the black triangle region. Therefore,

$$\begin{aligned}
\mathbb{P}(X > Y) &= \int_0^1 \int_0^x f(x, y)dydx \\
&= \int_0^1 \int_0^x \frac{3}{5}(x^2 + xy)dydx \\
&= \frac{3}{5} \int_0^1 [x^2 y + \frac{1}{2}xy^2]|_{y=0}^x dx \\
&= \frac{3}{5} \int_0^1 \frac{3}{2}x^3 dx \\
&= \frac{9}{10} \int_0^1 x^3 dx \\
&= \frac{9}{10} \times \frac{1}{4}x^4|_{x=0}^1 \\
&= \frac{9}{40}
\end{aligned}$$

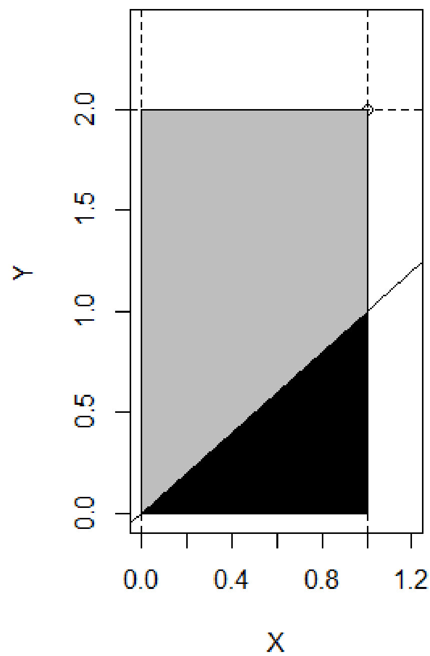

Figure 1. Problem 7

(d)

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^2 \int_0^1 x f(x,y) dx dy \\
&= \int_0^2 \int_0^1 \frac{3}{5}(x^3 + x^2 y) dx dy \\
&= \frac{7}{10}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^2 \int_0^1 y f(x,y) dx dy \\
&= \int_0^2 \int_0^1 \frac{3}{5}(x^2 y + x y^2) dx dy \\
&= \frac{6}{5}
\end{aligned}$$

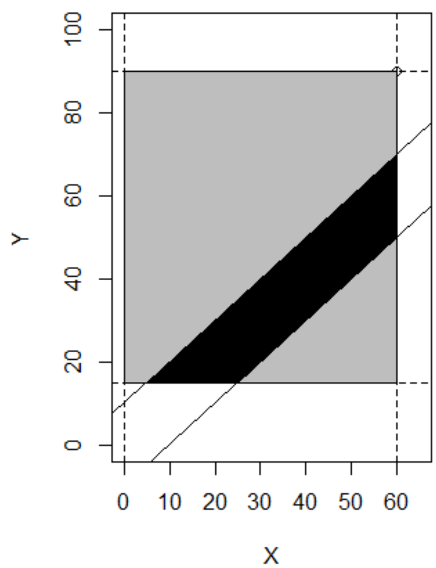

Figure 2. Problem 8.(a)

Problem 8. A man and a woman agree to meet at a certain location about 10:00 a.m. If the man arrives at a time uniformly distributed between 9:30 and 10:30 a.m., and if the woman independently arrives at a time uniformly distributed between 9:45 and 11:00 a.m., find the probability that the first to arrive waits no longer than 10 minutes. What is the probability that the woman arrives first?

Solution

(a) Let X be the number of minutes between the time the man arrives and 9:30 a.m. For example, $X = 45$ if the man arrives at 10:15 a.m. Then $X \sim \text{Unif}(0, 60)$. Similarly, let Y be the number of minutes between the time the woman arrives and 9:30 a.m. So $Y \sim \text{Unif}(15, 90)$. As X and Y are independent, the density for (X, Y) is

$$f(x, y) = \frac{1}{4500}, \quad x \in (0, 60), \quad y \in (15, 90).$$

The first to arrive waits no longer than 10 minutes is equivalent to $|X - Y| \leq 10$. So the goal is to find $\mathbb{P}(|X - Y| \leq 10)$.

See figure 2, $|X - Y| \leq 10$ holds when (X, Y) takes value from the black region. The area of the region is

$$\frac{(60-5)^2}{2} - \frac{(60-25)^2}{2} = 900$$

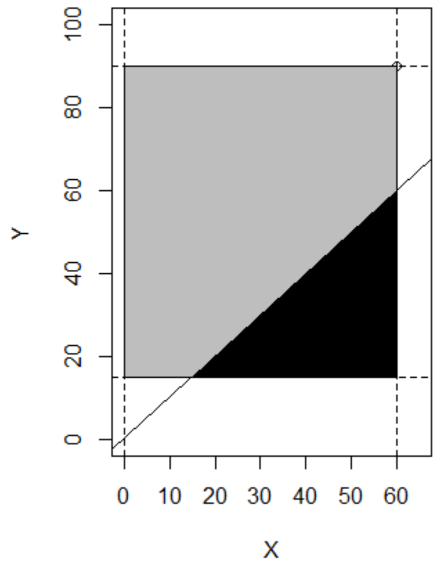

Figure 3. Problem 8.(b)

The overall area is $60 \times 75 = 4500$. As (X, Y) is uniformly distributed in the shaded region,

$$\mathbb{P}(|X - Y| \leq 10) = \frac{900}{4500} = \frac{1}{5}$$

(b) The woman arrives first means $X > Y$. See figure 3, $X > Y$ holds when (X, Y) takes value from the black triangle region. The area of the region is

$$\frac{(60 - 15)^2}{2} = 1012.5$$

Then

$$\mathbb{P}(X > Y) = \frac{1012.5}{4500} = \frac{9}{40}$$