A Multivariate Generalized Laplace Distribution

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Summary

A multivariate generalized Laplace distribution is described that is contained in the class of elliptically contoured distributions. It includes multivariate versions of the normal, Laplace, and uniform distributions. Variate generation is discussed and some bivariate density plots are provided.

Keywords: elliptically contoured distribution; error distribution; exponential power family; generalized gamma distribution.

1 Introduction

Elliptically contoured distributions provide a useful alternative to the multivariate normal distribution in statistical simulation and modeling. Johnson (1987) encourages the use of elliptically contoured distributions in Monte Carlo simulations to assess the robustness of certain statistical procedures under multivariate non-normality. In particular, he mentions the Pearson Type II and Type VII distributions because of their ease of variate generation and wide range within the class of elliptically contoured distributions. See Chmielewski (1981) for an annotated bibliography about elliptically contoured distributions or Fang and Anderson (1990) for a more advanced treatment.

One reasonable alternative to the multivariate normal distribution is a multivariate Laplace distribution. Referencing McGraw and Wagner (1968), Johnson (1987, pg. 121) mentions that the bivariate Laplace and bivariate generalized Laplace distributions have complicated parametric forms involving Bessel functions. Furthermore, he notes that variate generation has not been worked out in the literature making these distributions rather impractical for researchers. Using the properties of elliptically contoured distributions, this note describes a multivariate generalized Laplace distribution whose marginal distributions are univariate generalized Laplace distributions. Variate generation is shown to be straightforward and some bivariate density plots are provided showing the wide range of possible distributions.

2 Results for Elliptically Contoured Distributions

Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ be a random vector, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)' \in \mathbb{R}^p$, and $\boldsymbol{\Sigma} = ((\sigma_{ij}))$ be a $p \times p$ symmetric positive definite matrix. If the random variable $\mathbf{X}$ has a density function, then $\mathbf{X}$ has an elliptically contoured distribution if and only if its density function is of the form

$$f(\mathbf{x}) = k_p |\boldsymbol{\Sigma}|^{-\frac{1}{2}} g\big[(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\big], \tag{2.1}$$

where g is a non-negative real-valued function independent of p and k_p is a positive proportionality constant. We denote the distribution of $\mathbf{X}$ as $EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. The following three results for $\mathbf{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ will be used throughout this note and can be found in Johnson (1987, pp. 108-110).

(a) The density function of $R = \sqrt{(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})}$ is

$$h(r) = \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} k_p r^{p-1} g\big(r^2\big), \qquad r > 0. \tag{2.2}$$

(b) If $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')'$, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')'$, and $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$, where $\mathbf{X}_1$ and $\boldsymbol{\mu}_1$ are $m \times 1$ and $\boldsymbol{\Sigma}_{11}$ is $m \times m$, then

$$\mathbf{X}_1 \sim EC_m(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}; g). \tag{2.3}$$

(c) If R is a random variable with density (2.2), $\mathbf{U}^{(p)}$ is a p-vector independent of R and uniformly distributed on the unit hypersphere, and $\boldsymbol{B}$ is a $p \times p$ matrix such that $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}'$, then

$$\mathbf{X} \stackrel{d}{=} R\boldsymbol{B}\mathbf{U}^{(p)} + \boldsymbol{\mu}. \tag{2.4}$$

We may now introduce a multivariate generalized Laplace distribution.

3 A Multivariate Generalized Laplace Distribution

Suppose $\mathbf{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$, and specifically let $g(t) = e^{-t^{\lambda/2}}$ with $\lambda > 0$. It follows from (2.1) that the density of **X** is

$$f(\mathbf{x}) = k_{p,\lambda}|\boldsymbol{\Sigma}|^{-\frac{1}{2}} e^{-\left[(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{\lambda/2}}.$$

From (2.2), the density of $R = \sqrt{(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})}$ is

$$h(r) = \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} k_{p,\lambda} r^{p-1} e^{-r^{\lambda}}, \qquad r > 0. \tag{3.1}$$

This is the density of a generalized gamma random variable introduced by Stacy (1962) and described by Johnson and Kotz (1970a, pg. 197). Equating the constant terms, we find that $k_{p,\lambda} = \frac{\lambda\Gamma(p/2)}{2\pi^{p/2}\Gamma(p/\lambda)}$. Hence, the density function of **X** is completely specified as

$$f(\mathbf{x}) = \frac{\lambda\Gamma\left(\frac{p}{2}\right)}{2\pi^{\frac{p}{2}}\Gamma\left(\frac{p}{\lambda}\right)}|\boldsymbol{\Sigma}|^{-\frac{1}{2}} e^{-\left[(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{\lambda/2}}. \tag{3.2}$$

A random variable **X** with this density function will be called a *multivariate generalized Laplace* random variable. Some justification for this name is given in the next section. We shall denote this as $\mathbf{X} \sim MGL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda)$. Notice that when $\lambda = 2$, **X** has a multivariate normal distribution. The model (3.2) was used by Kuwana and Kariya (1991) to develop a test of multivariate normality ($\lambda = 2$ versus $\lambda < 2$ or $\lambda > 2$) which allows for a variety of elliptically contoured alternatives.

4 Marginal Distributions

Suppose $\mathbf{X} \sim MGL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda)$. Applying (2.3) to an individual element X_i of **X** yields $X_i \sim MGL_1(\mu_i, \sigma_{ii}, \lambda)$. That is, the marginal distributions of **X** have densities

$$f(x_i) = \frac{\lambda\Gamma\left(\frac{1}{2}\right)}{2\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{\lambda}\right)}|\sigma_{ii}|^{-\frac{1}{2}} e^{-\left[\frac{(x_i-\mu_i)^2}{\sigma_{ii}}\right]^{\lambda/2}} = \frac{\lambda}{2\sigma_i\Gamma\left(\frac{1}{\lambda}\right)} e^{-\left|\frac{x_i-\mu_i}{\sigma_i}\right|^{\lambda}},$$

where $\sigma_i = \sqrt{\sigma_{ii}}$. This is the density of the generalized Laplace distribution [also called the exponential power family or the error distribution, see Johnson and Kotz (1970b, pg. 33) or Evans, Hastings, and Peacock (1993, pg. 57)], a family of univariate distributions that includes the Laplace ($\lambda = 1$), the normal ($\lambda = 2$), and the uniform$[\mu_i - \sigma_i, \mu_i + \sigma_i]$ $(\lambda \to \infty)$ distributions.

5 Variate Generation

Johnson (1987, pg. 121) notes that "variate generation has not been explicitly worked out ... in the literature" for the bivariate Laplace and bivariate generalized Laplace distributions. Since the density of the multivariate generalized Laplace distribution in (3.2) is elliptically contoured, the task of variate generation is made simpler by means of (2.4). To generate a random variate $\mathbf{X} \sim MGL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda)$, we need only generate a random p-vector $\mathbf{U}^{(p)}$ on the unit hypersphere and, independently, a random variate R with density (3.1).

Methods for generating $\mathbf{U}^{(p)}$ can be found in Johnson (1987). The simplest of these is to form the components of $\mathbf{U}^{(p)}$ as

$$U_i = \frac{Z_i}{\sqrt{Z_1^2 + Z_2^2 + \cdots + Z_p^2}}, \qquad i = 1, 2, \ldots, p,$$

where $Z_1, Z_2, \ldots, Z_p$ are iid standard normal random variables. Furthermore, it is easy to check that if $Y \sim \text{gamma}\left(\frac{p}{\lambda}, 1\right)$, then $Y^{1/\lambda}$ is a generalized gamma random variable with density (3.1). Methods for generating gamma random variates can be found in Ahrens and Dieter (1974). Hence from (2.4),

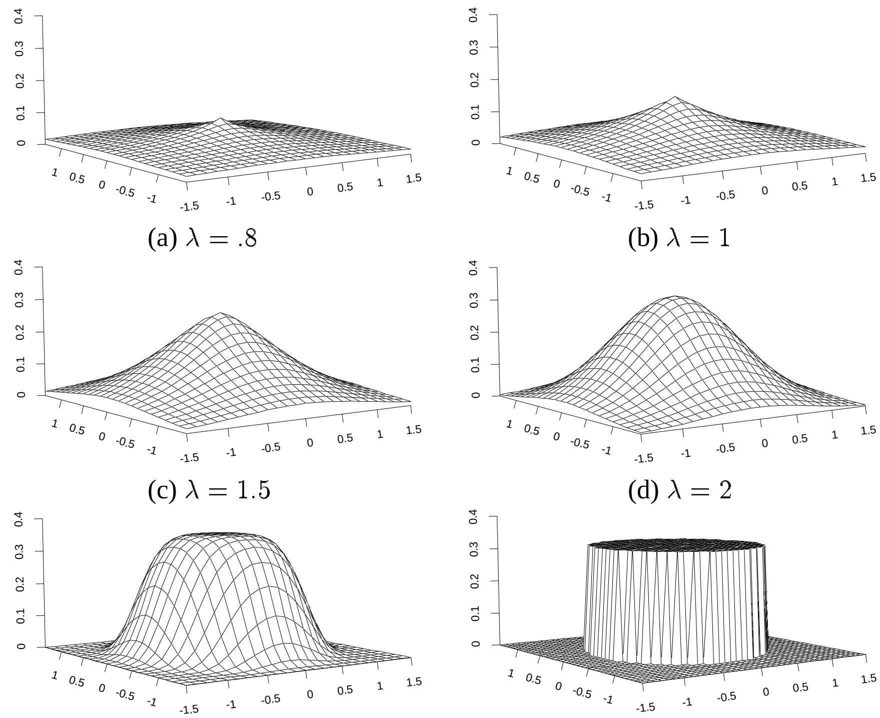
$$Y^{1/\lambda} \boldsymbol{B} \mathbf{U}^{(p)} + \boldsymbol{\mu} \sim MGL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda), \tag{5.1}$$

where $\boldsymbol{B}$ is a $p \times p$ matrix such that $\boldsymbol{B}\boldsymbol{B}' = \boldsymbol{\Sigma}$.

Giri (1996, pp. 70-71) shows that for $\mathbf{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$, $\mathrm{E}(\mathbf{X}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{X}) = \frac{1}{p}\mathrm{E}(R^2)\boldsymbol{\Sigma}$. From Johnson and Kotz (1970a, pg. 197) it is easy to see that $\mathrm{E}(R^2) = \Gamma\left(\frac{p+2}{\lambda}\right) \Big/ \Gamma\left(\frac{p}{\lambda}\right)$. Hence, to produce a MGL variate with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, simply replace $\boldsymbol{B}$ in (5.1) with $\sqrt{p/\mathrm{E}(R^2)}\boldsymbol{B} = \sqrt{p\Gamma\left(\frac{p}{\lambda}\right) \Big/ \Gamma\left(\frac{p+2}{\lambda}\right)}\boldsymbol{B}$.

6 Bivariate Density Plots

Figure 1 shows some bivariate density plots ($p = 2$) for the pdf in (3.2) with $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$. The values of λ used are .8, 1, 1.5, 2, 5, and ∞. Values of λ less than 2 correspond to distributions with heavier tails than the normal, while values of λ greater than 2 correspond to distributions with lighter tails than the normal. In general, $\lambda = \infty$ corresponds to a uniform distribution in a p-dimensional ellipsoid. The specific case in Figure 1(f) is uniform in a circle.



(a) $\lambda = .8$ (b) $\lambda = 1$

(c) $\lambda = 1.5$ (d) $\lambda = 2$

(e) $\lambda = 5$ (f) $\lambda = \infty$

Figure 1. Density Plots for the Bivariate Generalized Laplace Distribution with (a) $\lambda = .8$, (b) $\lambda = 1$, (c) $\lambda = 1.5$, (d) $\lambda = 2$, (e) $\lambda = 5$, (f) $\lambda = \infty$.

7 Summary

The density function (3.2) has a simple parametric form and defines a family of multivariate distributions with generalized Laplace marginals. This family contains a variety of multivariate distributions including the normal, the Laplace, and the uniform distributions. Being in the elliptically contoured family of distributions, variate generation is straightforward from (5.1). This broad family, with its easy variate generation, provides a useful alternative to the multivariate normal distribution for many multivariate simulation studies including distributions with both lighter and heavier tails.

Acknowledgments

The author would like to thank Professors Mark E. Johnson and William R. Schucany for their helpful comments and encouragement in preparing this note.

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