Scatterplots for Unordered Pairs

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Intraclass correlation coefficients (ICC's) measure the strength of the similarity among measurements within a class (or family) relative to the total variability. Classically this index of association has been most often treated as a ratio of variance components, namely the fraction that the family-to-family variability represents of the total variability. For the special case of twins, or any other set of unordered pairs, there has been no authoritative prescription as to which scatterplot one should use to graphically represent the ICC.

We propose that the set of all permutations within pairs of exchangeable measurements forms a legitimate basis for selecting a representative scatterplot. The recommended plot is the one whose interclass coefficient is closest to an ICC. For the product-moment correlation this implies that one selects a plot with r equal to the ANOVA estimator. We prove that Fisher's "symmetrical table" yields a coefficient that is smaller than all of the positive correlations in the permutation set.

KEY WORDS: Graphics; Intraclass correlation; Permutation distribution; Randomization; Twins.

1. INTRODUCTION: HOW SHOULD WE PLOT TWINS?

Imagine that a consultant has a client with data on identical twins, where there is no distinction in the ordering of each twin pair. The client is interested in measuring the similarity between the twins and would like a scatterplot as a visual reinforcement of this similarity. The consultant realizes that, there being no statistical difference between these siblings (i.e., they have the *same marginal* distributions), the individuals within each pair are interchangeable. With n pairs of twins, there are 2^n possible scatterplots of the points. How should the consultant plot these data? For example, Figure 1 contains a variety of such scatterplots from a small data set. If the *X-Y* labels are arbitrary, is one of these plots as good as another?

2. INTRACLASS CORRELATION

The measure of similarity that the client seeks has often been quantified by an intraclass correlation coefficient (ICC). The intraclass correlation, ρ, is defined to be the correlation between two individuals in the same class or family (cf. Stuart and Ord 1991, pp. 989–990). A variety of ICC's have been proposed for different situations. Müller and Büttner (1994) and Karlin, Cameron, and Williams (1981) give good discussions of some of these ICC's. The focus of this paper is on two distinct ICC's first discussed by Fisher [1948, (1st ed. 1925)].

2.1 Fisher's "Symmetrical Table"

In the case of twin data, Fisher (1948) recognized the interchangeability of the twins. In motivating his treatment of intraclass correlation he forced the estimated marginals to be equal by constructing what he called a "symmetrical table". Given n independent pairs of twins, (X_i, Y_i), $i = 1,\ldots,n$, Fisher added n more pairs: (Y_i, X_i), $i = 1,\ldots,n$, and computed the Pearson product-moment correlation of the $2n$ pairs. We denote this statistic by r_s (see the Appendix for more detailed notation). With this device he linked the intraclass correlation to the "interclass" coefficient. When the twins come from a bivariate normal distribution with equal marginals, r_s is the maximum likelihood estimate of ρ (Karlin et al. 1981).

2.2 The ANOVA Setting

Another ICC comes from the one-way random effects ANOVA model. Although Fisher did not actually use such notation in his original discussion, let

$$X_{ij} = \mu + a_i + e_{ij},$$

$i = 1,\ldots,n$, $j = 1$ or 2, where μ is a constant and a_i and e_{ij} are independent random variables with variances σ_a^2 and σ_e^2, respectively. Here a_i is viewed as a family effect with σ_a^2 representing the variability between family means. Under this model the intraclass correlation is

$$\rho = \text{Corr}(X_{i1}, X_{i2}) = \frac{\text{Cov}(X_{i1}, X_{i2})}{\sqrt{\text{Var}(X_{i1})\text{Var}(X_{i2})}}$$

$$= \frac{\text{Cov}(a_i, a_i)}{\text{Var}(X_{ij})} = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2} \geq 0.$$

The most commonly used estimate of ρ — not only for twins, but for variable family sizes as well — is

$$\hat{\rho}_A = \frac{\hat{\sigma}_a^2}{\hat{\sigma}_a^2 + \hat{\sigma}_e^2},$$

where $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ are the usual estimates of σ_a^2 and σ_e^2 from the ANOVA model (cf. Snedecor and Cochran 1989, pp. 242–243). It was known to Fisher that r_s is negatively biased. He went so far as to call it "slightly defective" (Fisher 1948, p. 223). In the next section we shed some more light upon this.

3. IF FISHER HAD HAD A WORKSTATION

If Fisher had access to modern statistical computing facilities, he might have explained intraclass correlation in terms of the permutation set of all 2^n equally likely labelings of the twin pairs. For every labeling of the pairs, r_s and $\hat{\rho}_A$ remain unchanged. But which scatterplot should one use as a visual aid? Which is most "representative" of the relationship between the twins? There is evidence from his other writings that Fisher was of a mind to consider permutation sets, yet he may not have seen any value in discussing all of these interchangeable displays.

It seems reasonable, when considering all permutations of the 2^n labelings, to choose a scatterplot whose Pearson product-moment correlation is close to an accepted measure of the intraclass correlation. It seems safe to say that the most commonly used ICC has been $\hat{\rho}_A$. Hence, it provides a reasonable benchmark for a "representative" scatterplot. It seems ironic that R. A. Fisher, an early advocate of randomization, did not discuss this permutation set.

Table 1. IQ Measurements on Five Pairs of Fraternal Twins

Pair	IQ's
1	(122, 118)
2	(102, 96)
3	(106, 99)
4	(116, 105)
5	(114, 99)

To illustrate this idea consider the IQ measurements on five pairs of twins in Table 1. These were randomly chosen from eight pairs of fraternal twins given by von Eye and Lienert (1980). Figure 1 shows the scatterplots and Pearson correlations, r, for half of the $2^5 = 32$ labelings of the twins. It is sufficient to consider only these, because each has a mirror image over $x = y$ with an identical value of r. Subsequently we will use r' to denote a generic value of r from this permutation distribution.

The coefficients range from .414 to .867 as the labeling of the twins changes. The value from Fisher's "symmetrical table", $r_s = .404$ is smaller than any of the correlations in the 16 scatterplots. For these data, $\hat{\rho}_A = .493$. The scatterplot with a correlation closest to $\hat{\rho}_A$ has $r = .497$. We would propose this as our *representative scatterplot*.

Table 2. Numbers of Finger Ridges on Both Hands of 12 Pairs of Female Identical Twins

Pair	No. of ridges	Pair	No. of ridges
1	(71, 71)	7	(114, 113)
2	(79, 82)	8	(57, 44)
3	(105, 99)	9	(114, 113)
4	(115, 114)	10	(94, 91)
5	(76, 70)	11	(75, 83)
6	(83, 82)	12	(76, 72)

A slightly larger data set of 12 pairs of twins comes from Snedecor and Cochran (1989), originally in Newman, Freeman, and Holzinger (1937). These data, given in Table 2, are the numbers of finger ridges on both hands of female identical

twins. There are $2^{11} = 2,048$ relevant scatterplots, too many to examine. Instead, consider a histogram of the values of r' (Figure 2). The dashed line marks the correlation from Fisher's "symmetrical table" ($r_s = .9626$), which again is less than all of the correlations from the 2,048 scatterplots. A proof of a general result to this effect can be found in the Appendix. The solid line marks the ANOVA estimate ($\hat{\rho}_A = .9656$). Figure 3 shows the scatterplot with r' closest to $\hat{\rho}_A$, as well as the scatterplots with the median (.9702), minimum (.9627), and maximum (.9890) values of r' from the permutation distribution. In light of Figure 2 the median scatterplot may also be a reasonable choice for a representative scatterplot. In this case all of the scatterplots in the permutation distribution look very similar because ρ is apparently quite close to 1. Ultimately, the consultant may view a small sample of plots, the plot associated with the median r', or the proposed representative plot with r' closest to $\hat{\rho}_A$.

Perhaps some readers may be concerned that our proposal employs Pearson's r, which uses neither $\hat{\mu}$, the estimate of the common mean, nor the pooled variance. (Detailed notation for this model with equal marginals is in the Appendix.) Even though investigators have been viewing scatterplots for years and seeing r as the index of linear association, it does not embody the constraints that hold for twins. In other words, because r employs $\overline{X}$ and $\overline{Y}$ (and separate variances) some may argue that it is an inappropriate index because the model assumes equal marginals. One reply to such a concern is that $\hat{\rho}_A$ is the appropriate *numerical* value, but most investigators have *visual* experience with the values of r that go with the corresponding patterns in graphs. That is to say, the value of $\hat{\rho}_A$ handles the model in the right way, and yet practitioners are "calibrated" from seeing scatterplots with a corresponding r.

4. SAMPLING THE PERMUTATION DISTRIBUTION

As the number of pairs gets moderately larger, the size of the permutation distribution becomes somewhat unmanageable. For $n = 21$ there are over 1,000,000 scatterplots potentially with distinct correlations. For larger data sets, we must rely on sampling the 2^n scatterplots. Sampling several thousand of these is simplified by uniquely associating each scatterplot with the binary representation of the first 2^n integers. See Venables and Ripley (1994, pp. 144–145) for an S-PLUS function that generates the binary representation of an integer.

In our experience with simulations from bivariate normals for various values of ρ and small n the median of the true permutation distribution of r' agrees well with $\hat{\rho}_A$. Consequently, it appears to be immaterial which of the two corresponding scatterplots is selected to be representative. This will guide us in designing a sampling procedure as an approximation when n is too large.

Enumeration is impractical whenever n is greater than about 20. Indeed most statisticians may prefer random samples to a census for $n > 11$, where there are more than 1,024

candidate values of r'. Dwass (1957) first considered sampling the permutation distribution. This idea was extended by Jöckel (1986). We will randomly sample M scatterplots from the permutation distribution. This sample is to be searched for the value of r' closest to $\hat{\rho}_A$ or sorted to find the median. As a guideline for choosing M the standard error of the sample median is asymptotically $\left[4nf_{r'}^2(\rho')\right]^{-1/2}$, where $f_{r'}$ is the pdf of a smooth version of the relevant permutation distribution and ρ' is its median. Rather than attempt to bound this density or even to model it as a smooth pdf, we rely upon

$$\text{Var}\left[\text{median}(r')\right] \approx \text{Var}\left[\text{mean}(r')\right] < \frac{1}{M}\text{Var}(r) \approx \frac{\left(1-\rho^2\right)^2}{Mn}. \quad (1)$$

The inequality holds easily because the unconditional sampling distribution of r has a greater variance than from conditionally fixing the observed values and interchanging within pairs. The final approximation is from Johnson and Kotz (1970, p. 225). The similarity with Fisher's approximate $\text{Var}(\hat{\rho}_A)$ is addressed by Donner and Koval (1983). Equation (1) suggests that we can achieve $\text{SE}\left[\text{median}(r')\right] \le d$ by choosing $M \ge \left(1-\rho^2\right)^2 / d^2 n$. It is not too ambitious to set $d = .01$ because for $n \ge 20$ the worst case ($\rho = 0$) yields $M \ge 500$. This is our recommended number of subsamples from the permutation distribution, although one may refine it considerably by substituting a desired value of d and $\hat{\rho}_A$ for ρ in the inequality.

Table 3. LDL Cholesterol Measurements on 31 Pairs of Brothers

Family	Brother Older	Brother Younger	Family	Brother Older	Brother Younger
1	104	93	17	62	68
2	85	112	18	112	86
3	86	101	19	88	72
4	90	82	20	87	55
5	65	66	21	73	94
6	103	124	22	80	100
7	63	74	23	104	102
8	59	101	24	95	108
9	120	92	25	108	105
10	73	56	26	92	114
11	100	87	27	99	102
12	53	63	28	134	76
13	77	104	29	58	82
14	97	102	30	84	73
15	81	77	31	98	136
16	97	89			

As an example we consider a data set (Table 3) concerning low density lipoprotein cholesterol (LDL-C), known to be positively associated with heart disease. The data are a subset of a larger database collected to investigate the genetic determinants of LDL-C. We gratefully acknowledge Dr. Jonathan Cohen, University of Texas Southwestern Medical Center, for allowing us to use these data which consist of LDL-C measurements (mg/dl) on 31 pairs of full brothers whose ages range from 5 to 15 years. Following fairly standard practice,

ordering the siblings according to their age (x = oldest, y = youngest) yields $r = .353$, while $\hat{\rho}_A = .362$. In the above formula for the number of representative samples required to accurately obtain an estimate of the median r' we substitute $\hat{\rho}_A$ for ρ and obtain 244 as lower bound for M. Figure 4(c) shows a histogram of $M = 301$ sample values of r'. The scatterplot associated with the median r' (.362 from this sample) is shown in Figure 4(a). The closeness of these values suggests that the scatterplot corresponding to the age ordering should be similar to that shown in Figure 4(a) and fortunately in this case it is. In general, however, arbitrarily selecting an ordering, for example, according to age, and basing an association on the subsequent scatterplot and r may be misleading. For instance, suppose we choose to order the pairs according to the magnitude of LDL-C measurements: $x_1 = \min(x, y)$ and $x_2 = \max(x, y)$, as was done in Table 1. The resulting scatterplot [Figure 4(b)] visually implies a much larger correlation ($r = .719$) than $\hat{\rho}_A$ indicates ($\hat{\rho}_A = .362$).

5. PERMUTATION TEST OF INTERCHANGEABILITY

One of the assumptions we made from the beginning was that the order of each twin pair is arbitrary. This implies that the marginal distributions are identical. Hence we can test the hypothesis that paired data come from a distribution with identical marginals (i.e., that the pairs are interchangeable) by comparing the observed correlation to the permutation distribution of 2^n correlations. There is more to consider, but this is an important feature implied by the null. See Hsu (1940) and Bradley and Blackwood (1989) for the likelihood ratio test of equal means and variances under bivariate normality. These papers were spurred by papers of Morgan (1939) and Pitman (1939), respectively, on testing the equality of bivariate normal variances. Other approaches to simultaneously testing means and variances can be found in Kepner and Randles (1984) and the references therein.

Table 4. Measurements by two different observers of cardiac output of 23 patients

Patient	Observer A	Observer B	Patient	Observer A	Observer B
1	4.8	5.8	13	7.7	8.5
2	5.6	5.1	14	7.7	9.5
3	6.0	7.7	15	8.2	9.1
4	6.4	7.8	16	8.2	10.0
5	6.5	7.6	17	8.3	9.1
6	6.6	8.1	18	8.5	10.8
7	6.8	8.0	19	9.3	11.5
8	7.0	8.1	20	10.2	11.5
9	7.0	6.6	21	10.4	11.2
10	7.2	8.1	22	10.6	11.5
11	7.4	9.5	23	11.4	12.0
12	7.6	9.6			

As an example, consider the data in Table 4 given by Müller and Büttner (1994). These data are measurements by two different observers of cardiac output of 23 patients using Doppler echocardiography. It is clear here that the order of the

pairs is *not* arbitrary. One may be interested in testing whether the measurements by observer A and observer B are consistent with each other. Because the permutation distribution has more than 4,000,000 correlations, we must rely upon sampling from this distribution for an approximate randomization test of interchangeability. Figure 5 shows the original scatterplot of the data, with a correlation of .924, along with 19 randomly selected scatterplots from the permutation distribution, with their correlations. Our observed correlation exceeds the correlation in all 19 of the other scatterplots. This is evidence against the hypothesis of interchangeability with $p = \frac{1}{20} = .05$, indicating a violation of the interchangeability assumption. In practice, we would produce many more than 19 correlation coefficients from the permutation distribution and we would not look at all the scatterplots, but it is instructive to see what a fair selection of scatterplots look like under the assumption of interchangeability. A histogram of the exact null distribution is given in Figure 6.

6. CONCLUDING REMARKS

When dealing with bivariate observations that have identical marginals, the exchangeability within the pairs suggests a permutation set of 2^n scatterplots. Our proposal to choose a "representative" scatterplot as the one whose Pearson correlation is closest to an accepted measure of the intraclass correlation seems reasonable because practitioners are comfortable seeing scatterplots with associated values of r. The ICC from the ANOVA model, $\hat{\rho}_A$, has long been a widely accepted measure of ρ, but any reasonable measure could be used instead. The median value of r' from the permutation distribution of scatterplots may have some merit as a new estimator of ρ. The choice of Pearson's r to measure the interclass correlation in each of the 2^n scatterplots is somewhat arbitrary. Other measures, such as Kendall's τ and Spearman's ρ, can easily be used if desired. The principle is the same, but the corresponding ICC is required, for example, see Whitfield (1949) and Rothery (1979), respectively.

When collecting family data as described in the LDL-C example, there may be, and typically are, more than two siblings per family. In this case it is not obvious what (scatter) plot best graphically illustrates the meaning and magnitude of an ICC. Because $\hat{\rho}_A$ is a function of the variance components in the one-way random effects model, it is natural to conjecture that a plot involving the best linear unbiased predictors (BLUP's) (Robinson 1991) could be developed to graphically interpret the correlation coefficient. An approach in this direction appears promising.

Finally, the permutation distribution of the 2^n scatterplots suggests a test for the interchangeability of two paired variables. If the observed value of the interclass correlation is extreme, when compared to its value computed on each scatterplot in the permutation set, then we can reject the hypothesis of interchangeability.

APPENDIX: NOTATION AND PROOFS

Let (X_i, Y_i), $i = 1, \ldots, n$, be independent and identically distributed bivariate random variables, and let $(X'_i, Y'_i) = (X_i, Y_i)$ or (Y_i, X_i), that is, either interchange of the X-Y labels within each pair, $i = 1, \ldots, n$. Define

$$r_s = \frac{\sum_{i=1}^n (X_i - \hat{\mu})(Y_i - \hat{\mu})}{\frac{1}{2}\left[\sum_{i=1}^n (X_i - \hat{\mu})^2 + \sum_{j=1}^n (Y_j - \hat{\mu})^2\right]},$$

where

$$\hat{\mu} = \frac{\overline{X} + \overline{Y}}{2},$$

and

$$r' = \frac{\sum_{i=1}^n (X'_i - \overline{X}')(Y'_i - \overline{Y}')}{\left[\sum_{i=1}^n (X'_i - \overline{X}')^2 \sum_{i=1}^n (Y'_j - \overline{Y}')^2\right]^{1/2}},$$

where

$$\overline{X}' = \frac{1}{n}\sum_{i=1}^n X'_i \quad \text{and} \quad \overline{Y}' = \frac{1}{n}\sum_{i=1}^n Y'_i.$$

Finally, define $r'_{\min} = \min r'$, where the minimum is taken over all 2^n sets of $\{(X'_i, Y'_i) \mid i = 1, \ldots, n\}$.

Lemma. Let A and B be real numbers. Then $[(A+B)/2]^2 \geq AB$.

Proof.
$$\begin{aligned}
[(A+B)/2]^2 &= \tfrac{1}{4}[A^2 + 2AB + B^2] \\
&= \tfrac{1}{4}[A^2 - 2AB + B^2] + AB \\
&= [(A-B)/2]^2 + AB \geq AB.
\end{aligned}$$

Note that if A and B are nonnegative, this is the familiar inequality between the arithmetic and geometric means.

Theorem. If $r'_{\min} \geq 0$, then $r_s \leq r'_{\min}$.

Proof. Note that the numerators of r_s and $r'_{\min}$ can be written as $\sum_{i=1}^n X_i Y_i - n\hat{\mu}^2$ and $\sum_{i=1}^n X_i Y_i - n\overline{X}'\overline{Y}'$, respectively. From the lemma,

$$\hat{\mu}^2 = \left(\frac{\overline{X} + \overline{Y}}{2}\right)^2 = \left(\frac{1}{2n}\sum_{i=1}^n (X_i + Y_i)\right)^2$$
$$= \left(\frac{1}{2n}\sum_{i=1}^n (X'_i + Y'_i)\right)^2 = \left(\frac{\overline{X}' + \overline{Y}'}{2}\right)^2 \geq \overline{X}'\overline{Y}'.$$

Hence

$$\sum_{i=1}^n X_i Y_i - n\hat{\mu}^2 \leq \sum_{i=1}^n X_i Y_i - n\overline{X}'\overline{Y}'. \tag{A.1}$$

Properties of the sample mean and the lemma give the following string of inequalities:

$$\frac{1}{2}\left[\sum_{i=1}^{n}\left(X_i - \hat{\mu}\right)^2 + \sum_{j=1}^{n}\left(Y_j - \hat{\mu}\right)^2\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n}\left(X_i' - \hat{\mu}\right)^2 + \sum_{j=1}^{n}\left(Y_j' - \hat{\mu}\right)^2\right]$$

$$\geq \frac{1}{2}\left[\sum_{i=1}^{n}\left(X_i' - \overline{X}'\right)^2 + \sum_{j=1}^{n}\left(Y_j' - \overline{Y}'\right)^2\right]$$

$$\geq \left[\sum_{i=1}^{n}\left(X_i' - \overline{X}'\right)^2 \sum_{j=1}^{n}\left(Y_j' - \overline{Y}'\right)^2\right]^{1/2}.$$

Combining this with (A.1) and the fact that $r'_{\min} \geq 0$, we see that $r_s \leq r'_{\min}$.

In the twin context positive correlation is the case of practical interest. If $r'_{\min} < 0$, then we cannot be sure about its relationship to r_s. However, it does follow that $r_s < 0$ as well.

REFERENCES

Bradley, E. L. and Blackwood, L. G. (1989), "Comparing Paired Data: A Simultaneous Test for Means and Variances," *The American Statistician*, 43, 234–235.

Donner, A. and Koval, J. J. (1983), "A note on the accuracy of Fisher's approximation to the large sample variance of an intraclass correlation," *Communications in Statistics – Simulation and Computation*, 12, 443–449.

Dwass, M. (1957), "Modified Randomization Tests for Nonparametric Hypotheses," *The Annals of Mathematical Statistics*, 28, 181–187.

Fisher, R. A. (1948), *Statistical Methods for Research Workers* (10th ed.), New York: Hafner Publishing Company Inc.

Hsu, C. T. (1940), "On Samples from a Normal Bivariate Population," *The Annals of Mathematical Statistics*, 11, 410–426.

Jöckel, K.-H. (1986), "Finite Sample Properties and Asymptotic Efficiency of Monte Carlo Tests," *The Annals of Statistics*, 14, 336–347.

Johnson, N. L. and Kotz, S. (1970), *Distributions in Statistics: Continuous Univariate Distributions – 2*, Boston: Houghton Mifflin Company.

Karlin, S., Cameron, E. C., and Williams, P. T. (1981), "Sibling and parent-offspring correlation estimation with variable family size," *Proceedings of the National Academy of Sciences of the United States of America*, 78, 2664–2668.

Kepner, J. L. and Randles, R. H. (1984), "Comparison of Tests for Bivariate Symmetry versus Location and/or Scale Alternatives," *Communications in Statistics – Theory and Methods*, 13, 915–930.

Morgan, W. A. (1939), "A Test for the Significance of the Difference between the Two Variances in a Sample from a Normal Bivariate Population," *Biometrika*, 31, 13–19.

Müller, R., and Büttner, P. (1994), "A Critical Discussion of Intraclass Correlation Coefficients," *Statistics in Medicine*, 13, 2465–2476.

Newman, H. H., Freeman, F. N., and Holzinger, K. J. (1937), *Twins: A Study of Heredity and Environment*, Chicago: The University of Chicago Press.

Pitman, E. J. G. (1939), "A Note on Normal Correlation," *Biometrika*, 31, 9–12.

Robinson, G. K. (1991), "That BLUP is a good thing – the estimation of random effects," *Statistical Science*, 6, 15–51.

Rothery, P. (1979), "A nonparametric measure of intraclass correlation," *Biometrika*, 66, 629–639.

Snedecor, G. W., and Cochran, W. G. (1989), *Statistical Methods* (8th ed.), Ames, IA: Iowa State University Press.

Stuart, A., and Ord, J. K. (eds.) (1991), *Kendall's Advanced Theory of Statistics* (Vol. 2, 5th ed.), New York: Oxford University Press.

Venables, W. N., and Ripley, B. D. (1994), *Modern Applied Statistics with S-Plus*, New York: Springer-Verlag.

von Eye, A. and Lienert, G. A. (1980), "Nonparametric evaluation of heritability by intra-twin differences of mono- and dicygotic twins," *Biometrical Journal*, 22, 505–511.

Whitfield, J. W. (1949), "Intra-class rank correlation," *Biometrika*, 36, 463–467.
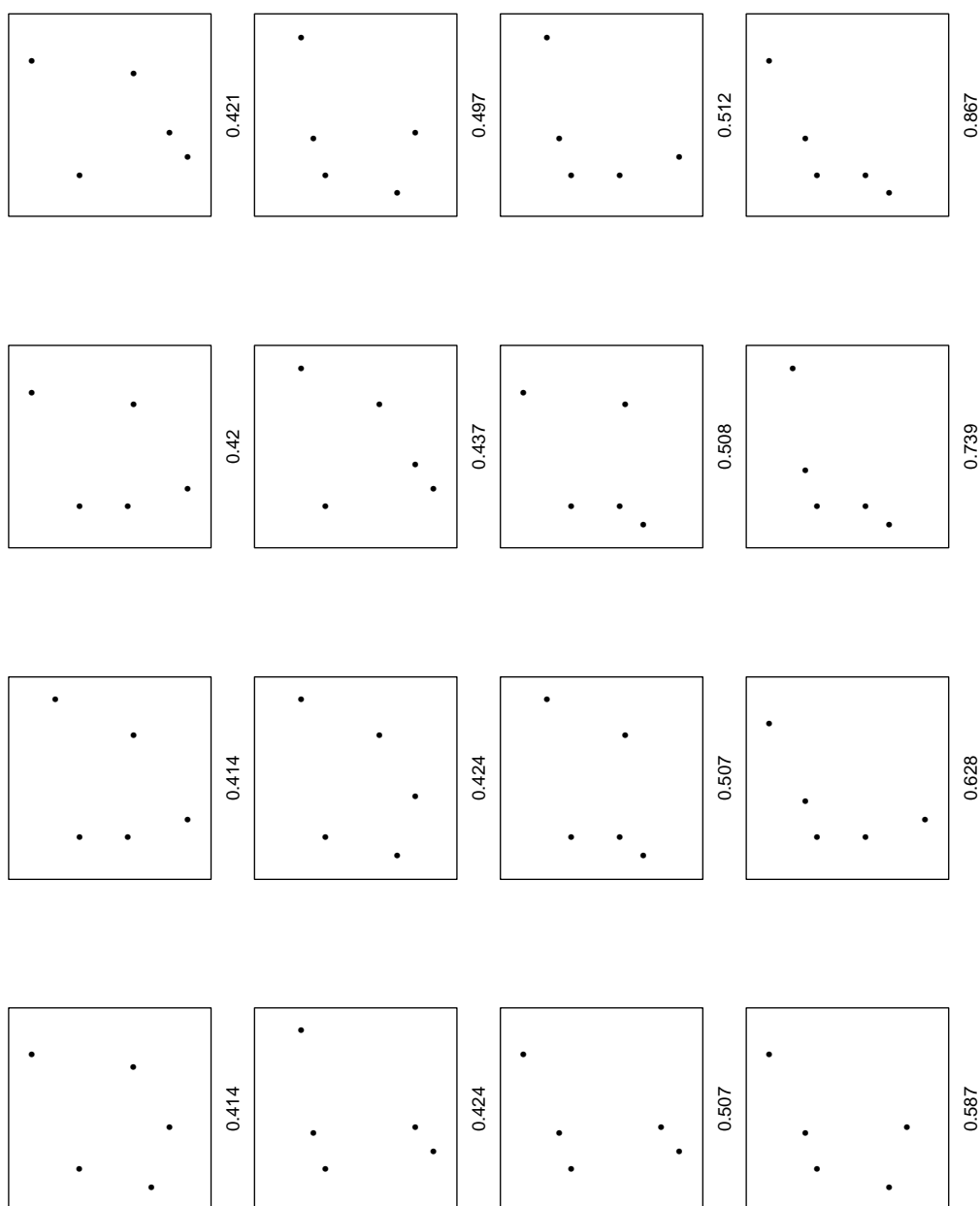
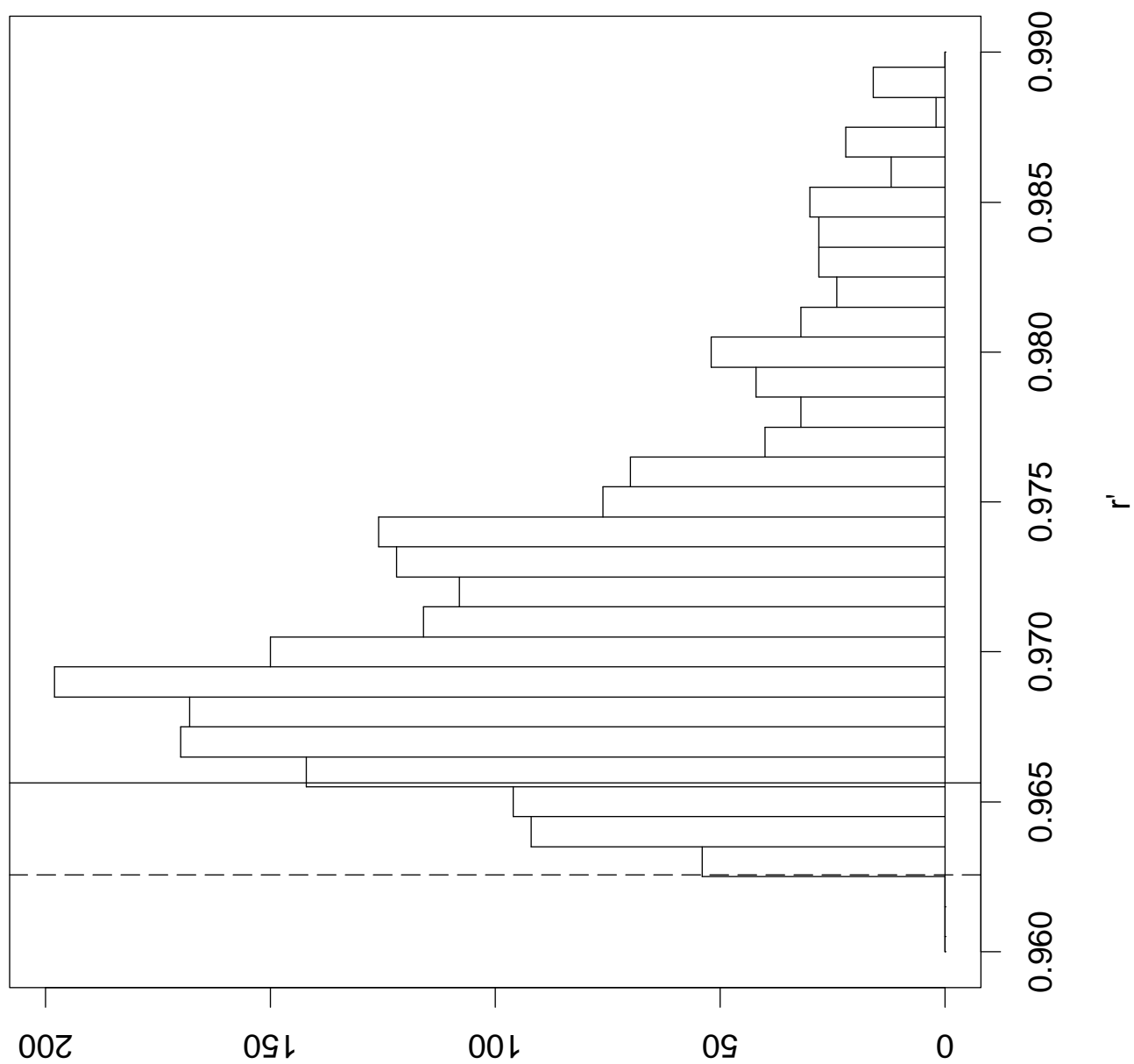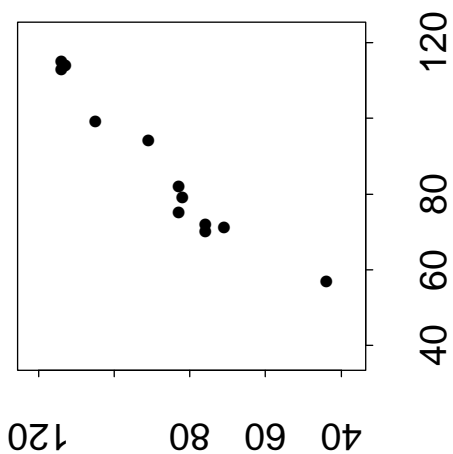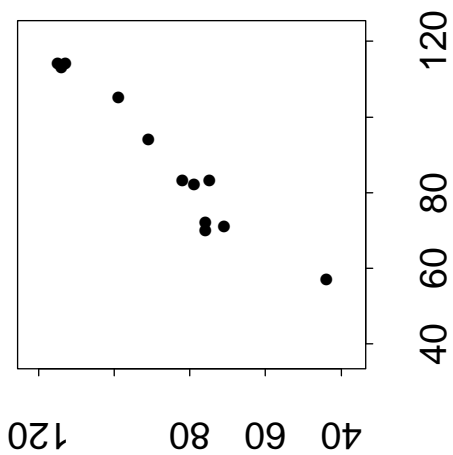Figure 1. 16 Scatterplots and Their Correlations from the Permutation Distribution of the Data in Table 1.

Figure 2. Histogram of the Values of r′ from the Permutation Distribution of the Data in Table 2. The solid line marks $\hat{\rho}_A$ and the dashed line marks r_s.
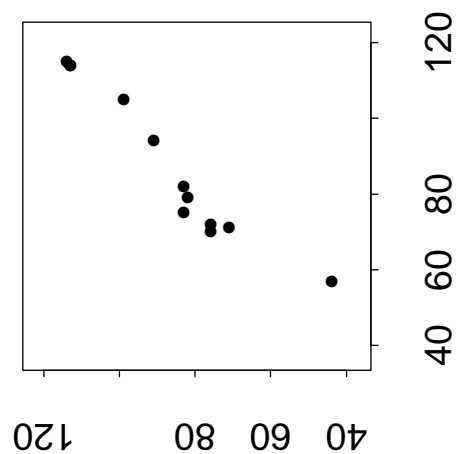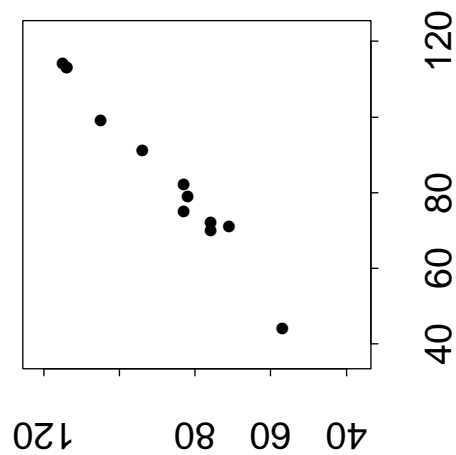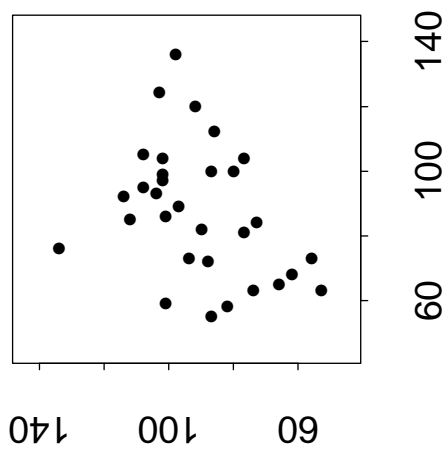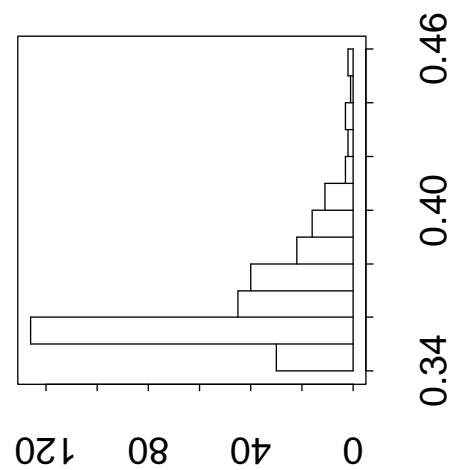
Figure 3. Four Scatterplots from the Permutation Distribution of the Data in Table 2. (a) Scatterplot with r′ closest to $\hat{\rho}_A$. (b) Scatterplot with median r′. (c) Scatterplot with minimum r′. (d) Scatterplot with maximum r′.
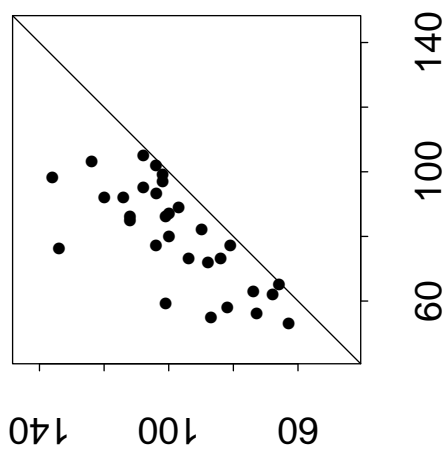
Plot with Y > X

(b) r = .719

Plot with Median r'

(a) r = .362

301 sampled values of r'

(c)

Figure 4. Two Scatterplots and a Histogram from the Permutation Distribution of the LDL Cholesterol Data in Table 3. (a) Scatterplot with median r'. (b) Scatterplot with the Y value greater than the X value in each pair. (c) Histogram of the permutation distribution of r'.
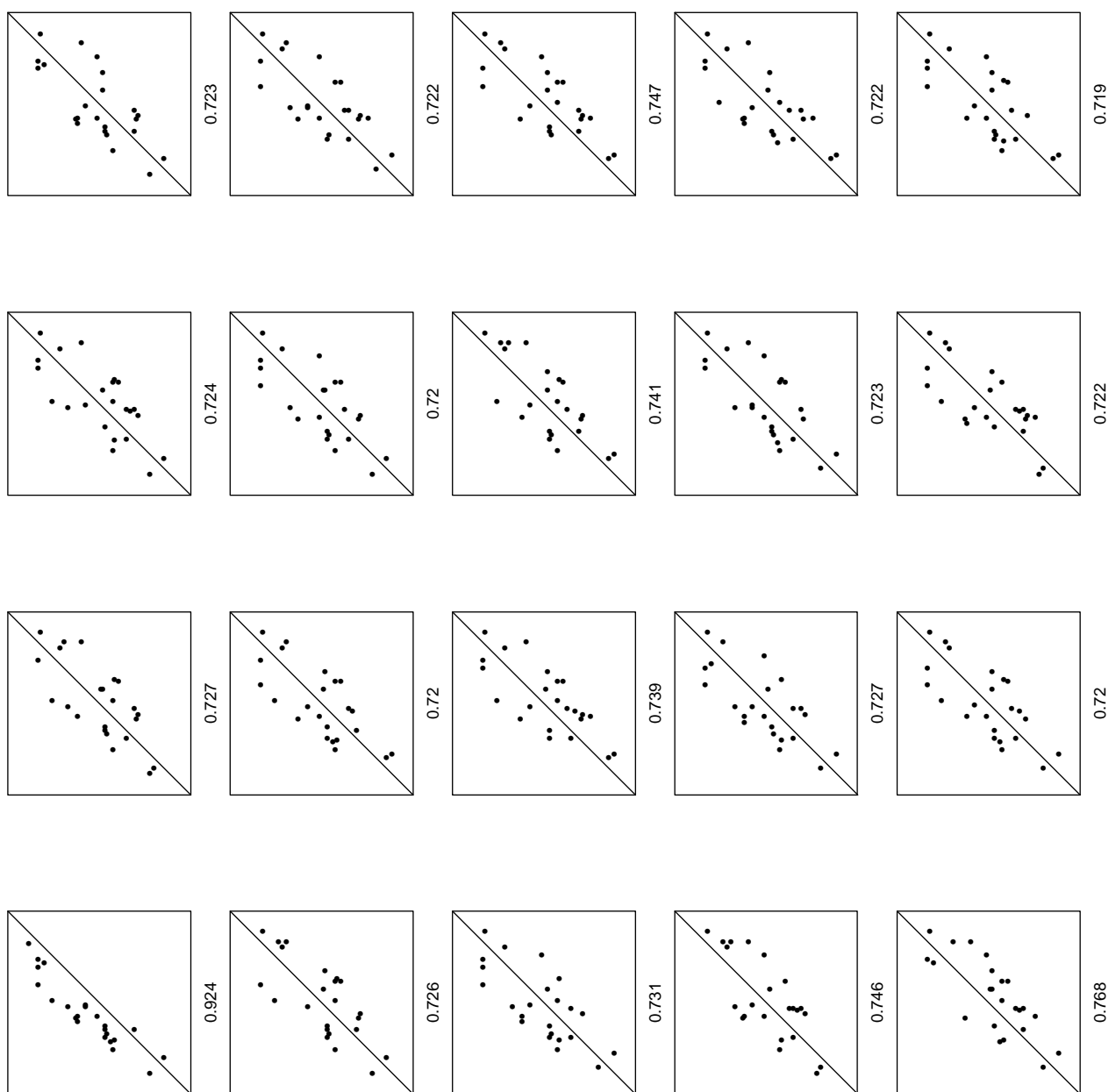
Figure 5. Scatterplot of the Data in Table 4 (Upper Left) and 19 Randomly Selected Scatterplots from the Permutation Distribution with Correlations.
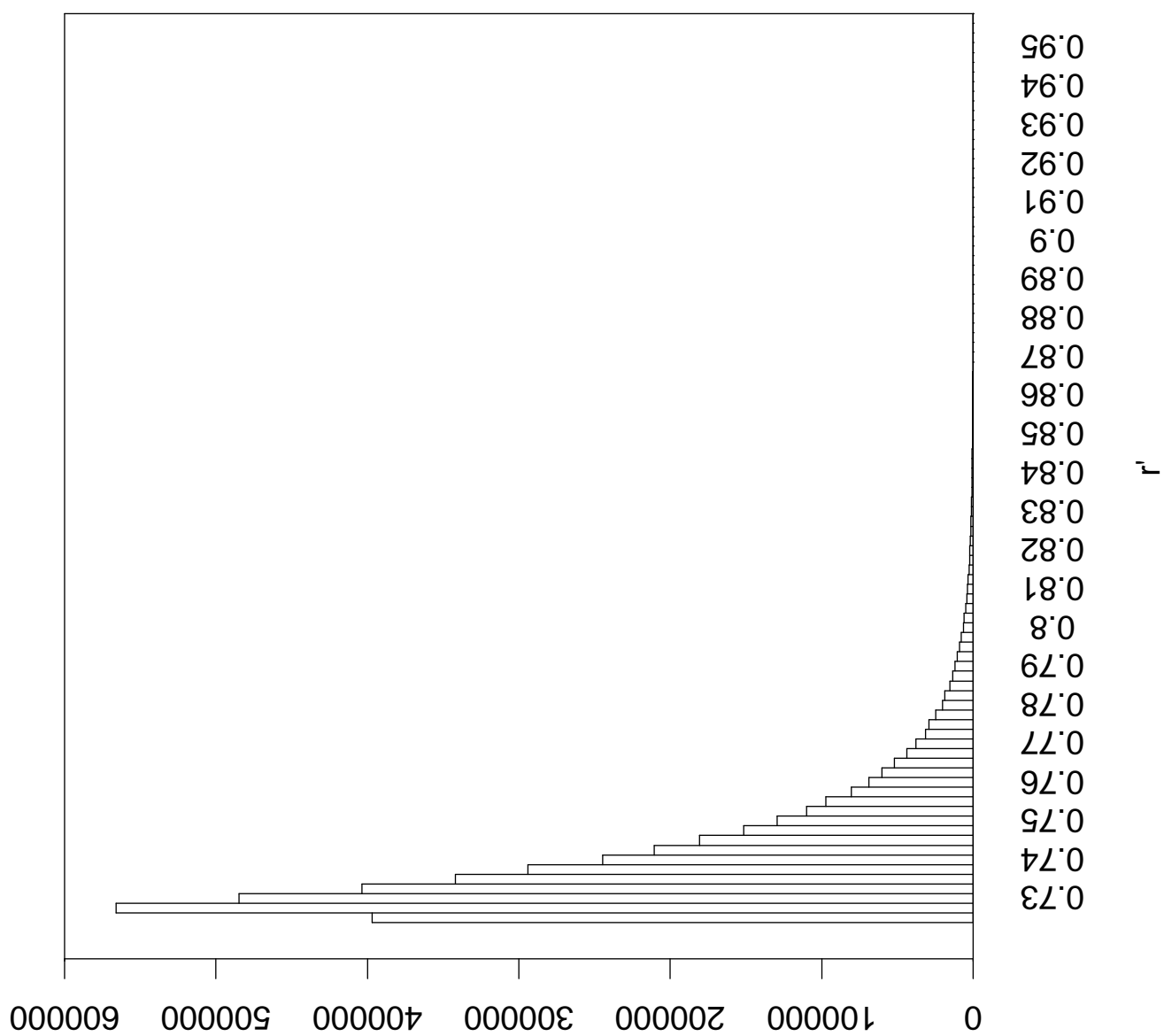
Figure 6. Histogram of the Values of r′ from the Permutation Distribution of the Data in Table 4.