Slow passage through a Hopf bifurcation: minimal example

We consider the following canonical model for a supercritical Hopf bifurcation

$$\begin{aligned} x' &= -y + x(\mu - x^2 - y^2) \\ y' &= \quad x + y(\mu - x^2 - y^2). \end{aligned} \tag{1}$$

It is easy to show that system (1) has a supercritical Hopf bifurcation at $\mu = 0$. We can rewrite this system in polar coordinates via the transformation $x = r\cos(\theta), y = r\sin(\theta)$. We obtain the following canonical form in the (r, θ) coordinate system

$$\begin{aligned} r' &= r(\mu - r^2) \\ \theta' &= 1. \end{aligned} \tag{2}$$

We recover the Hopf bifurcation at $\mu = 0$ and also the fact that limit cycle are in fact circles of equation $r = \sqrt{\mu}$.
We now add a slow drift on parameter μ by appending to system (1) or equivalently to system (2) the constant slow drift of equation $\dot{\mu} = \varepsilon$ where $0 < \varepsilon \ll 1$ is a small parameter. It is well known that the resulting 3D slow-fast system (2 fast, 1 slow) exhibit a ***slow passage through Hopf bifurcation*** or a ***delayed-Hopf bifurcation*** or as well a ***dynamic Hopf bifurcation***. Practically, it means that a trajectory of this 3D system, starting at $\mu_0 < 0$, follows slowly the line of equilibria of the (r, θ) (fast sub-) system at $r = 0$, passes the value $\mu = 0$ at which the unperturbed system has a Hopf bifurcation and then gets repelled only at a certain $\mu_1 > 0$ to get attracted to the family of the limit cycles of the (r, θ). We want to quantify this delay to the switch from quasi-equilibrium behaviour to quasi-periodic behaviour. To do so, we first consider individually the (decoupled) r-equation in (2) and rewrite it by separating variables in the following way

$$\frac{dr}{r} \quad = \quad (\mu - r^2)dt, \tag{3}$$

with: $\mu(t) = \mu_0 + \varepsilon t$. Therefore we can integrate the left-hand and right-hand sides of equation (3) and we obtain

$$\log r \quad = \quad \mu_0 t + \varepsilon \frac{t^2}{2} - \int r^2(t)dt, \tag{4}$$

to which we apply the exponential function on both sides in order to get

$$r(t) \quad = \quad \exp\left(\mu_0 t + \varepsilon \frac{t^2}{2}\right) \cdot \exp\left(-\int r^2(t)dt\right). \tag{5}$$

Hence, the behaviour of $r(t)$ depends upon the behaviour of the two exponential terms that appeared in (5). The left exponential terms stays small for all t, since $r > 0$, therefore the change of behaviour intervenes when the other exponential terms goes from small to large. This happens as the argument of this exponential goes through 0. Therefore we solve for t the equation of this argument being equal to 0 and easily find that this happens when

$$\mu_0 t + \varepsilon \frac{T^2}{2} \quad = \quad 0 \quad \Longleftrightarrow \quad T = -\frac{2\mu_0}{\varepsilon}. \tag{6}$$

Therefore the solution leaves the vicinity of $r = 0$ passed the Hopf bifurcation point at

$$\mu(T) = \mu_0 + \varepsilon\left(-\frac{2\mu_0}{\varepsilon}\right) = -\mu_0. \tag{7}$$