Slow-passage through a Hopf-like bifurcation

September 7, 2020

Abstract

1 Introduction

We use this minimal model to reproduce the slow passage through a Hopf bifurcation in the PWL context

$$\dot{x} = f(x) - y \tag{1}$$
$$\dot{y} = x - a \tag{2}$$

with

$$f(x) = \begin{cases} -mx & x \leq 0, \\ kx & x \geq 0 \end{cases}$$

This system exhibits an unique equilibrium at $(a, f(a))$ and the Jacobian matrix at the equibrium is

$$\begin{pmatrix} f'(a) & -1 \\ 1 & 0 \end{pmatrix}$$

then, the equilibrium point changes its stability at $a = 0$ passing from being stable at $a < 0$ to being unstable at $a > 0$. Assuming $0 < k < m < 2$, then the system exhibits a supercritical Hopf bifurcation at $a = 0$.

We now consider a slow drift on the parameter $a(t) = a_0 + \varepsilon t$ to force a slow passage through the Hopf.

Following we present some simulations showing the effect of the slow passage in the PWL setting, see Figure 1 and Figure 2.

Remark 1.1. *It could be concluded that the slow passage through the Hopf-like in the PWL context does not behave in the same way that it does through the classical Hopf bifurcation. We note that the contraction effect of the slow manifold depends on the real part of the eigenvalues of the Jacobian matrix. Hence, the smaller is the real part of the eigenvalues, the smaller is the contraction effect, and the solutions for different values of* a_0 *remains*

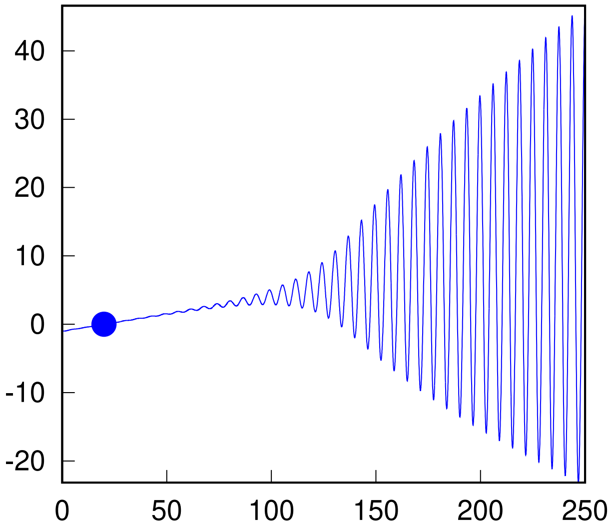

Figure 1: Coordinate $x(t)$ versus t for parameters $m = 0.2, k = 0.1, a_0 = -1, \varepsilon = 0.05$. The point corresponds with the Hopf. It can be observed the delay in the amplitude due to the passage through the Hopf.

different at the Hopf point. Thus the delay is different for different values of a_0*. On the contrary the bigger is the real part of the eigenvalues, the bigger is the contraction, and thus solutions for different values of* a_0 *seems to be equal at the Hopf point. Therefore, the delay is the same for every* a_0*.*

In the smooth context the eigenvalues varies when the parameter does, which implies that close to the fold its real part is small, which implies the existence of different delays for different values of a_0*. On the contrary, in the PWL context, the eigenvalues do not change with the parameter, so when* m *is big enough the contraction makes all solutions to seem the same at the Hopf point, so that the delay is the same for every* a_0*.*

2 Analytical approach

The local expression of the flow is

It is necessary to check that following expressions are correct, both the solutions on the right and the solutions on the left. I suspect that there is any mistake.

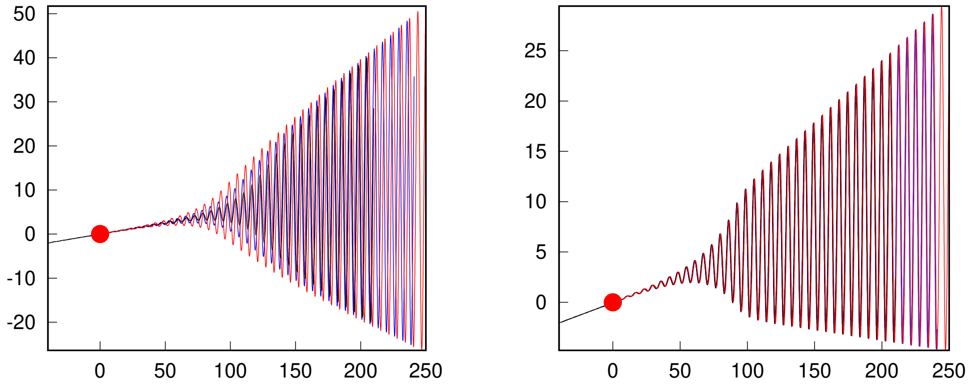

Figure 2: Coordinate $x(t)$ versus t for parameters $k = 0.1, \varepsilon = 0.05$ and different values of m and a_0. (a) For $m = 0.2$, blue curve corresponds to $a_0 = -1$, black curve to $a_0 = -0.2$ and red curve to $a_0 = 0$. The delay in the amplitude seems not to be dependent on the the nevative parameter a_0 is. (b) For $m = 1$, black curve corresponds to $a_0 = -2$, blue curve to $a_0 = -0.45$ and red curve to $a_0 = 0$. In this case, the delay in the amplitude due to the passage through the Hopf is the same for the three solutions It is probably due to the increase in the contraction as we have increased parameter m, which force that three solutions seems the same as they arrive at the Hopf point. Moreover, the delay is smaller than the delay in the previous examples, note that solutions achieve the expected amplitude at time 100 approx.

$$
\begin{aligned}
x_L(t) &= a(t) - m\varepsilon + e^{-\frac{m}{2}t}\left(C_1\cos\left(\frac{\sqrt{4-m^2}}{2}t\right) + C_2\sin\left(\frac{\sqrt{4-m^2}}{2}t\right)\right),\\
y_L(t) &= -m(a(t) - m\varepsilon) - \varepsilon\\
&\quad + e^{-\frac{m}{2}t}\left(-\frac{C_2\sqrt{4-m^2}+C_1 m}{2}\cos\left(\frac{\sqrt{4-m^2}}{2}t\right) + \frac{C_1\sqrt{4-m^2}-C_2 m}{2}\sin\left(\frac{\sqrt{4-m^2}}{2}t\right)\right),
\end{aligned}
$$

on the left region, and

$$
\begin{aligned}
x_R(t) &= a(t) - k\varepsilon + e^{\frac{k}{2}t}\left(D_1\cos\left(\frac{\sqrt{4-k^2}}{2}t\right) + D_2\sin\left(\frac{\sqrt{4-k^2}}{2}t\right)\right),\\
y_R(t) &= k(a(t) - k\varepsilon) - \varepsilon\\
&\quad + e^{\frac{k}{2}t}\left(\frac{kD_1 - D_2\sqrt{4-k^2}}{2}\cos\left(\frac{\sqrt{4-k^2}}{2}t\right) + \frac{kD_2 + D_1\sqrt{4-k^2}}{2}\sin\left(\frac{\sqrt{4-k^2}}{2}t\right)\right)
\end{aligned}
$$

on the right region.

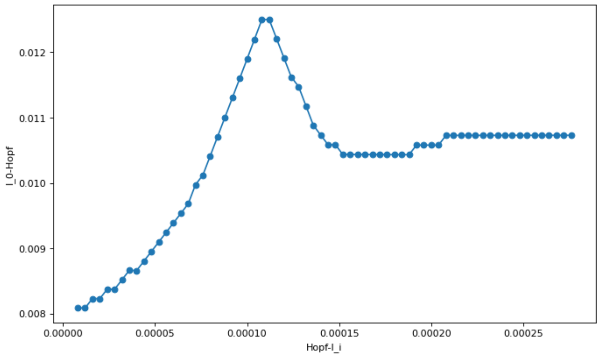

Figure 3: Representation of the delay versus the a_0 parameter in the PWL version of the Morris-Lecar model. It can be seen that after a maximun in the delay, the delay saturates and remains constant. This fact could be explained from Figure 3.

3 Geometrical approach

Consider the 3D version of the problem given by

$$\dot{\mathbf{u}} = \begin{cases} A_-\mathbf{u} + \mathbf{b} & \text{if } u_1 \leq 0 \\ A_+\mathbf{u} + \mathbf{b} & \text{if } u_1 \geq 0 \end{cases},$$

where

$$A_- = \begin{pmatrix} -m & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_+ = \begin{pmatrix} k & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix},$$

For $\varepsilon = 0$, it is known that the system exhibits two invariant sets. One of these invariant sets is given by a poligonal line

$$\begin{cases} (x, -mx, x) & \text{if } x \leq 0 \\ (x, kx, x) & \text{if } x \geq 0 \end{cases}$$

which is the critical manifold $\mathcal{S}_0$ of the problem and has two branches one attracting (for $x \leq 0$) and one repelling (for $x \geq 0$). The other invariant set is a stable cone foliated by periodic orbits, see Figure **??**(a).

After perturbation $\varepsilon > 0$ see Figure **??**(b), the critical manifold perturbs in a slow manifold $\mathcal{S}_\varepsilon$ formed by two segments which are the attracting

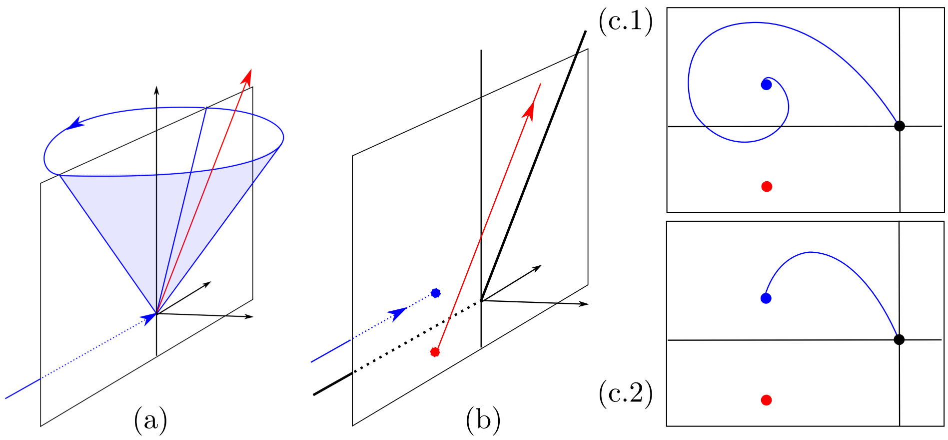

Figure 4: **Invariant objects before and after perturbation:** (a) Slow flow defined over the critical manifold and stable invariant manifold under the fast flow. (b) Attracting and repelling branches of the slow manifold appearing after perturbation of the critical manifold and their intersection points, $\mathbf{p}_- = (0, -\varepsilon, m\varepsilon)$ (blue point) and $\mathbf{p}_+ = (0, -\varepsilon, -k\varepsilon)$ (red point), with the switching plane $\{x = 0\}$. (c) Image on the switching plane of the attracting branch of the critical manifold by the flow in the half-space $x < 0$, (c.1) under focus conditions, (c.2) under node conditions

branch $\mathcal{S}_\varepsilon^a$ and the repelling branch $\mathcal{S}_\varepsilon^r$ given by the solutions

$$u_-(t) = \begin{pmatrix} 0 \\ -\varepsilon \\ m\varepsilon \end{pmatrix} + \varepsilon t \begin{pmatrix} 1 \\ -m \\ 1 \end{pmatrix}, \quad t \leq 0$$

$$u_+(t) = \begin{pmatrix} 0 \\ -\varepsilon \\ -k\varepsilon \end{pmatrix} + \varepsilon t \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}, \quad t \geq 0$$

intersecting the plane $\{x = 0\}$ are at a distance $O(\varepsilon)$ of the intersection point of critical manifold at the origin.

Solution $u_+(t)$ should be written in the form given by x_R and y_R, and viceversa, solutions (x_R, y_R) should be written as an straight line plus an oscillation. Considering initial conditions at $x = 0$ this give us restictions on the initial conditions to remain along a time τ in a neighbourhood δ from the $u_+(t)$, which provides a relation between the distance to the slow manifold and the delay.

Remark 3.1. *The point $\mathbf{p}_- = (0, -\varepsilon, m\varepsilon)$ gives the asymptotic value of the graph of the delay.*

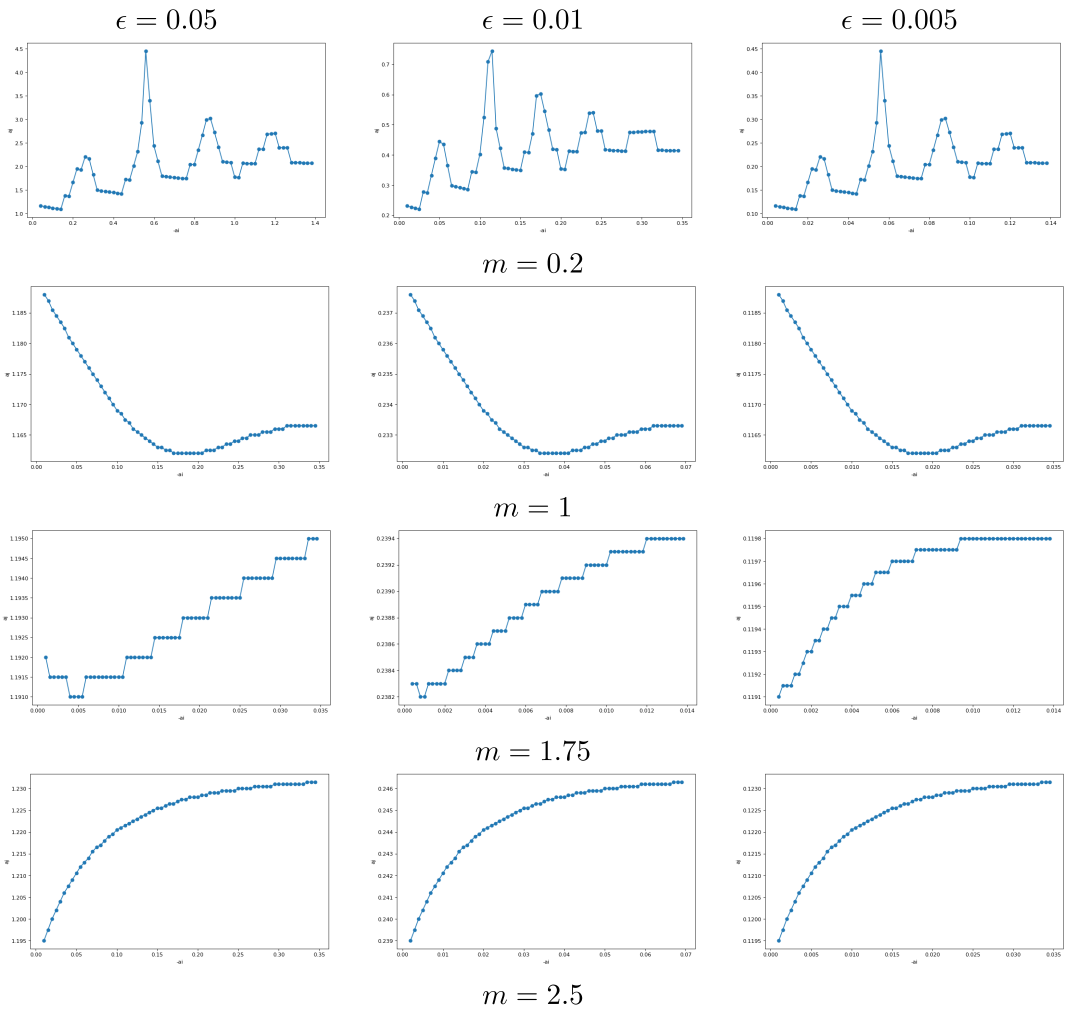

Table 1: **Slow passage through a PWL Hopf for** $k = 0.1$ **and different values of** m **and** ϵ. From these simulations it can be concluded that parameter m, that is, the divergence of the system, is related with the shape of the graph of the function input-output. Moreover, the parameter ε, that is the velocity of the passage, is related with the size of the delay but also with a translation of the graph. Both efect seem to be $O(\varepsilon)$.