PHYC90010 - Statistical Mechanics Assignment 1

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Contents

Assignment 2 (PHYC90010)
To be handed in by the 26th of April (5pm)

It is never too soon to plan your superannuation strategy! Some respected economists predict that asset prices will decline steadily in real terms (i.e. after inflation) over the next few decades, as the world's population hits environmental capacity constraints and stabilises for the first time since the Industrial Revolution. At the same time, higher volatility is expected to become the norm in investment markets, as the financial industry grows its share of the economy. In this environment, what should you do to ensure that you maintain access to the essentials — fine wine, say, and adventure holidays — as you live to a ripe old age? Below we show that the answer to this question is *be lucky*.

Let $x(t)$ be a random variable denoting the dollar balance of your retirement fund as a function of time, t, measured in years. Every year, you deposit a fixed sum $\lambda > 0$ into the fund. The fund is invested in a mix of stocks and bonds. It earns *interest* at a fluctuating rate $\alpha + \eta(t)$ per annum, where α is the steady, underlying rate of return (which can be positive or negative), and $\eta(\mathrm{t})$ is a stochastic variable of Langevin type which obeys stationary Gaussian statistics with $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$, with $D > 0$.

(a) Justify the following Langevin equation describing the evolution of $x(t)$: [2]

$$\frac{\partial x(t)}{\partial t} = x(t)\,[\alpha + \eta(t)] + \lambda$$

(b) By introducing an integrating factor, or otherwise, explicitly solve Langevin equation (for a fund starting with zero balance) [6]

$$x(t) = \lambda \exp\left[\int_0^t dt'\,(\alpha + \eta(t'))\right] \int_0^t dt'' \exp\left[-\int_0^{t''} dt'\,(\alpha + \eta(t'))\right].$$

(c) Show that the mean balance is given by

$$\langle x(t) \rangle = \frac{\lambda}{\alpha + D}\left\{\exp\left[(\alpha + D)\,t\right] - 1\right\}.$$

Note that the system must self-regulate, such that $\langle x(t) \rangle$ (and hence the total money in the system after inflation) does not diverge in the long term ($t \to \infty$). This requires $\alpha + D < 0$. [6]

(d) Let $q(u,t)$ be the probability distribution function of the logarithmic balances $u = \ln(x)$ of an ensemble of retirement funds. Write down the Langevin equation for u and thereby show that the associated Fokker-Planck equation takes the form [6]

$$\frac{\partial q(u,t)}{\partial t} = -\frac{\partial}{\partial u}\left\{\left[\lambda \exp\left(-u\right) + \alpha\right] q(u,t)\right\} + D\frac{\partial^2 q(u,t)}{\partial u^2}.$$

(e) Explain why the system self-imposes a reflecting boundary at $x = 0$. [2]

(f) Use the boundary condition from part (e) to solve for the steady-state distribution. [4]

(g)

i. Relate $p_s(x)$, the steady-state probability distribution function of x, to $q_s(u)$. [2]

ii. Sketch $p_s(x)$ and comment on its properties. [3]

iii. If the system self-regulates to be marginally stable, i.e. $\alpha + D$ adjusts to be slightly negative, what happens to the dispersion, $\mathrm{var}(x)$, as $t \rightarrow \infty$? [3]

iv. Comment briefly on what all this means in practice for policy makers seeking to design a viable superannuation system. [1]

(h) Write a simple numerical code, based on the Langevin equation used at the start of this assignment, to verify your analytical results in (c), (g(ii)) and (g(iii)). If it is possible [5]. **This is not as easy as it sounds.**

Total Marks 40

1 Question 1

1.1 Question 1 - (a)

The rate of change of the amount in a super fund is by definition proportional to the previous amount in the super fund, i.e.

$$\frac{\partial x(t)}{\partial t} \propto x(t).$$

The proportionality will be dependant on the static growth amount, which in this case is the interest, and some gaussian random noise component, making the strict equivalency

$$\frac{\partial x(t)}{\partial t} = x(t)[\alpha + \eta(t)].$$

However, we also have to factor in the contributions made to the fund, which are some fixed element λ. This makes the final expression

$$\frac{\partial x(t)}{\partial t} = x(t)[\alpha + \eta(t)] + \lambda \tag{1}$$

, as required.

1.2 Question 1 - (b)

To solve this expression using an integrating factor, we need to rearrange the first order differential equation so that it has the form

$$\frac{\partial x(t)}{\partial t} + Px(t) = Q.$$

Doing so, we see that

$$\begin{aligned} P &= -[\alpha + \eta(t)] \\ Q &= \lambda. \end{aligned}$$

Now if introduce an integrating factor, defined as

$$I = \exp\left\{\int Pdx\right\}, \tag{2}$$

we can re-write our differential in the form

$$I\frac{\partial x(t)}{\partial t} + IPx = IQ,$$

and integrate both sides

$$\int \left[I\frac{\partial x(t)}{\partial t} + IPx\right] dt = \int IQdt.$$

Doing so allows us to take advantage of the product rule, whereby $\frac{\mathrm{d}}{\mathrm{d}t}(Ix) = I\frac{\mathrm{d}x}{\mathrm{d}t} + IPx$, which gives the solution to the differential as

$$\begin{aligned} Ix &= \int IQdt \\ \rightarrow x(t) &= \frac{\int IQdt}{I} \end{aligned}$$

Substituting $I = \exp\{\int -[\alpha + \eta(t)]dt\}$ yields the expression

$$\begin{aligned} x(t) &= \frac{\int \lambda \exp\left[-\int_0^{t'} \alpha + \eta(t'')dt''\right]}{I} \\ &= \lambda \exp\left[\int_0^t \alpha + \eta(t')dt'\right] \int_0^t dt'' \exp\left[-\int_0^{t''} \alpha + \eta(t'')dt''\right] \end{aligned}$$

, as required.

1.3 Question 1 - (c)

We know that $\exp\left[\int_0^t \alpha + \eta(t')dt'\right]$ doesn't depend on dt'', so we can move it inside the integral. Defining $\zeta(t) = \alpha + \eta(t)$, we see that the expression becomes

$$\begin{aligned} x(t) = \lambda \int_0^t dt'' \exp\left[\int_0^t \zeta(t')dt'\right] \exp\left[-\int_0^{t''} \zeta(t'')dt''\right] \\ =\lambda \int_0^t dt'' \exp\left[\int_{t''}^t \zeta(t')dt'\right]. \end{aligned}$$

Now if we take the expecatation value of this expression, it becomes

$$\langle x(t) \rangle = \left\langle \lambda \int_0^t dt'' \exp\left[\int_{t''}^t \zeta(t')dt'\right] \right\rangle.$$

We can pass the expectation value through the integral function, and all constants, since the expecation value of a number is just a number

$$= \lambda \int_0^t dt'' \left\langle \exp\left[\int_{t''}^t \zeta(t')dt'\right] \right\rangle \tag{3}$$

In order to calculate this, we need to first characterise the argument of the exponent. Now, we know that $\zeta(t)$ is some shifted random gaussian variable. This essentially means that by taking the integral over some finite domain, we are in essence sampling the distribution, which is another gaussian !!!!! This is basically the expecation value of the moment generating function. For some random gaussian variable, we know that this takes the form

$$\left\langle e^{pX} \right\rangle = \int_{-\infty}^{\infty} e^{px'} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-(x' - \mu)^2}{2\sigma^2}\right\} dx'.$$

If we make a change of variable, letting $x = x' - \mu$, and complete the square inside the exponential, this becomes

$$\begin{aligned} &= \frac{\exp(p\mu)}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp(px) \exp\left(\frac{-x^2}{2\sigma^2}\right) dx \\ &= \frac{\exp(p\mu)}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\sigma^2} x^2 - 2\sigma^2 px + \sigma^4 p^2 - \sigma^4 p^2\right) dx \\ &= \exp\left(p\mu + \frac{\sigma^2}{2} p\right) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\infty}^{\infty} \exp\left\{\frac{-(x - \sigma^2 p^2)^2}{2\sigma^2}\right\} dx \\ &= \exp\left(p\mu + \frac{\sigma^2}{2} p\right). \end{aligned}$$

Given that we know the form of the solution, we can see that in this case,

$$\begin{aligned} &\mu = \alpha(t - t''), \\ &\sigma^2 = 2D(t - t''), \text{ and} \\ &p = 1. \end{aligned}$$

This makes (3) into

$$\begin{aligned} &= \lambda \int_0^t dt'' \exp\big(\alpha(t - t'') + D\big) \\ &= \frac{\lambda}{\alpha + D} \left[\exp\big((\alpha + D)(t - t'')\big)\right]_0^{t''=t} \\ &= \frac{\lambda}{\alpha + D} [\exp((\alpha + D)t) - 1] \end{aligned}$$

as required.

1.4 Question 1 - (d)

1.5 Question 1 - (e)

If we consider that you cannot have a negative balance in a superannuation fund, the amount of money in the fund cannot go below zero, and so the system self-imposes a reflecting boundary at $x = 0$

1.6 Question 1 - (f)

1.7 Question 1 - (g)

1.8 Question 1 - (h)

```python
import numpy as np
# import scipy as sp
import matplotlib.pyplot as plt


class super_simulation:
    def __init__(self, lam, alpha, runtime_years, step_size, num_sims):
        self.lam = lam
        self.alpha = alpha
        self.runtime_years = runtime_years
        self.step_size = step_size
        self.num_sims = num_sims
        self.data = np.zeros(shape=(runtime_years, num_sims))


    def __repr__(self):
        return f'Superannuation Simulation with Yearly Deposit ({self.lam!r}), Interest rate,{self.alpha!r}), run over ({self.runtime_years!r}) years, and ({self.num_sims!r}) simulations containing ({self.data!r})'

    def run_sim(self):
        for i in range(self.runtime_years):
                for j in range(self.num_sims):
                        self.data[i][j] = self.data[i-1][j] + self.step_size * (self.data[i-1][j]*(self.alpha + np.random.normal()) + self.lam)

    def plot_super(self):
            for j in range(self.num_sims):
                    plt.plot(self.data[:][j])
            plt.show()

    def plot_mean(self):
            means = np.mean(self.data, axis=1)
            plt.plot(means)
            plt.show()

    def plot_variance(self):
            variances = np.var(self.data, axis=1)
            plt.plot(variances)
            plt.show()

# def plot_super(array, num_sims):
#         for j in range(num_sims):
#                 plt.plot(array[:][j])
#         plt.show()


# def plot_stats(array, num_sims):
# for j in range(num_sims):
#         plt.plot(np.mean(array[:][j]))
# plt.show()

if __name__ == '__main__':
        lam = float(input("Enter a yearly contribution amount (lambda): "))
        alpha = float(input("Enter an interest rate (alpha): "))
```

```python
        assert -1 < alpha < 1, "Interest rate must be between -100 percent and 100 percent (-1 and 1)"
        h = float(input("Enter the step size (h) : "))
        years = int(input("Enter the number of years to run the simulation by : "))
        num_sims = int(input("Enter the number of simulations to run: "))
        runtime = int(years/h)

        simulation = super_simulation(lam,alpha,years,h,num_sims)
        simulation.run_sim()
        simulation.plot_super()
        simulation.plot_mean()
```

Listing 1: Python Code for Automaton