Assignment 3 (PHYC90010)
To be handed in by the 11th of May (5pm)

It is never too soon to plan your superannuation strategy! Some respected economists predict that asset prices will decline steadily in real terms (i.e. after inflation) over the next few decades, as the world's population hits environmental capacity constraints and stabilises for the first time since the Industrial Revolution. At the same time, higher volatility is expected to become the norm in investment markets, as the financial industry grows its share of the economy. In this environment, what should you do to ensure that you maintain access to the essentials — fine wine, say, and adventure holidays — as you live to a ripe old age? Below we show that the answer to this question is *be lucky*.

Let $x(t)$ be a random variable denoting the dollar balance of your retirement fund as a function of time, t, measured in years. Every year, you deposit a fixed sum $\lambda > 0$ into the fund. The fund is invested in a mix of stocks and bonds. It earns *interest* at a fluctuating rate $\alpha + \eta(t)$ per annum, where α is the steady, underlying rate of return (which can be positive or negative), and $\eta(\mathrm{t})$ is a stochastic variable of Langevin type which obeys stationary Gaussian statistics with $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$, with $D > 0$.

(a) Justify the following Langevin equation describing the evolution of $x(t)$: [2]

$$\frac{\partial x(t)}{\partial t} = x(t)\left[\alpha + \eta(t)\right] + \lambda$$

(b) By introducing an integrating factor, or otherwise, explicitly solve Langevin equation (for a fund starting with zero balance) [6]

$$x(t) = \lambda \exp\left[\int_0^t dt'\,(\alpha + \eta(t'))\right] \int_0^t dt'' \exp\left[-\int_0^{t''} dt'\,(\alpha + \eta(t'))\right].$$

(c) Show that the mean balance is given by

$$\langle x(t) \rangle = \frac{\lambda}{\alpha + D}\left\{\exp\left[(\alpha + D)\,t\right] - 1\right\}.$$

Note that the system must self-regulate, such that $\langle x(t) \rangle$ (and hence the total money in the system after inflation) does not diverge in the long term ($t \to \infty$). This requires $\alpha + D < 0$. [6]

(d) Let $q(u,t)$ be the probability distribution function of the logarithmic balances $u = \ln(x)$ of an ensemble of retirement funds. Write down the Langevin equation for u and thereby show that the associated Fokker-Planck equation takes the form [6]

$$\frac{\partial q(u,t)}{\partial t} = -\frac{\partial}{\partial u}\left\{\left[\lambda \exp\left(-u\right) + \alpha\right] q(u,t)\right\} + D\frac{\partial^2 q(u,t)}{\partial u^2}.$$

(e) Explain why the system self-imposes a reflecting boundary at $x = 0$. [2]

(f) Use the boundary condition from part (e) to solve for the steady-state distribution. [4]

(g)

i. Relate $p_s(x)$, the steady-state probability distribution function of x, to $q_s(u)$. [2]

ii. Sketch $p_s(x)$ and comment on its properties. [3]

iii. If the system self-regulates to be marginally stable, i.e. $\alpha + D$ adjusts to be slightly negative, what happens to the dispersion, $\mathrm{var}(x)$, as $t \to \infty$? [3]

iv. Comment briefly on what all this means in practice for policy makers seeking to design a viable superannuation system. [1]

(h) Write a simple numerical code, based on the Langevin equation used at the start of this assignment, to verify your analytical results in (c), (g(ii)) and (g(iii)). [5]

Total Marks 40