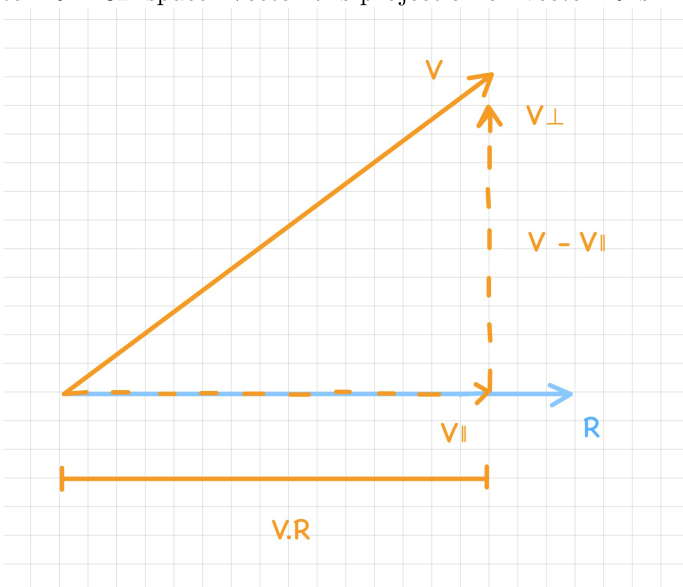
Rotation About Arbitrary Axis

Derivation for the rotation equation

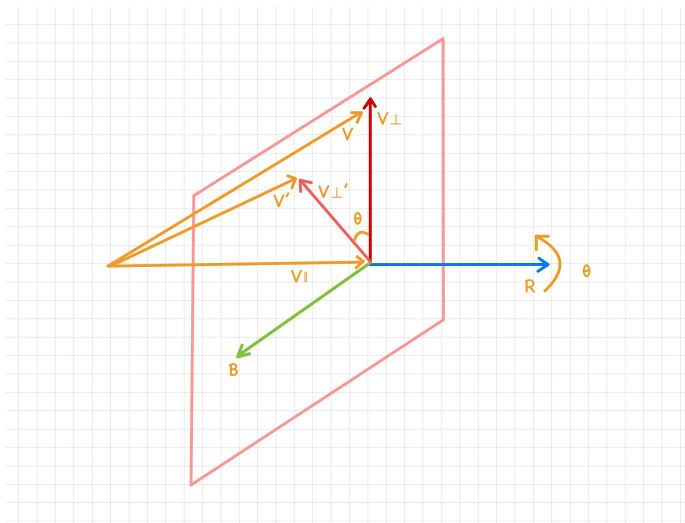
In this section we will derive the equation to rotate a vector V about another vector R in 3D space. Vector V's projection on vector R is



We split V into two vector components $V_{\parallel}$ and $V_{\perp}$ that are parallel and perpendicular to R, respectively.

$$V = V_{\parallel} + V_{\perp} \tag{1}$$
$$V_{\parallel} = (V.R)R \tag{2}$$
$$V_{\perp} = V - (V.R)R \tag{3}$$

We can describe the rotation of V in the plane perpendicular to R



We can see when V is rotated about R by θ, the resulting vector is V'. Considering the vector projection of V' on R

$$V' = V'_{\|} + V'_{\perp} \tag{4}$$

Because $V'_{\|} = V_{\|}$

$$V' = V_{\|} + V'_{\perp} \tag{5}$$

Substituting $V_{\|}$ from 2

$$V' = (V.R)R + V'_{\perp} \tag{6}$$

V and R are known quantities. $V'\perp$ can be calculated by observing that $V_{\perp}$ and B and are two mutually perpendicular plane vectors. Along with R they describe a coordinate frame, $V\perp BR$ that can be used to compute $V'_{\perp}$. Since B is mutally perpendicular to $V'\perp$ and R, it can be expressed as

$$B = R \times V_{\perp} \tag{7}$$

Substituting $V_{\perp}$ from 3

$$B = R \times (V - V_{\|}) \tag{8}$$

Since cross product distributes over vector subtraction and since $V_{\|}$ and R are parallel

$$B = R \times V - R \times V_{\|} \tag{9}$$
$$B = R \times V - 0 \tag{10}$$
$$B = R \times V \tag{11}$$

Using $V_\perp$ and B as basis vectors, $V'_\perp$ can be written as

$$V'_\perp = V_\perp cos\theta + (R \times V)sin\theta \tag{12}$$

Substituting $V'\perp$ into the equation 6, we get the final equation for V' in terms of V and R

$$V' = (V.R)R + V_\perp cos\theta + (R \times V)sin\theta \tag{13}$$
$$V' = (V.R)R + (V - (V.R)R)cos\theta + (R \times V)sin\theta \tag{14}$$

Note

There are three terms in 14

$$T1 = (V.R)R \tag{15}$$
$$T2 = (V - (V.R)R)cos\theta \tag{16}$$
$$T3 = (R \times V)sin\theta \tag{17}$$

Notice that $T2$ is actually $(V - T1)cos\theta$. We will be using these three terms in the derivation of the rotation matrix in the next section

Derivation for the rotation matrix

A column-major rotation matrix is of the form

$$\begin{bmatrix} B1 & B2 & B3 \end{bmatrix}$$

The columns $B1$,$B2$, and $B3$ are the coordinates of the basis vectors after rotation. To derive the rotation matrix for rotating V and producing V', we start with the basis vectors of the right-handed coordinate system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We first rotate these basis vectors using the rotation equation (12), and to form the rotation matrix, we add the rotated coordinates of the basis vectors to the rotation matrix

Rotation of X-axis

Substituting x-axis for V in 14, we can get the $B1$ vector after rotation. Instead of solving for V' in one-shot, for simplicity's sake we will solve the three terms $T1, T2, andT3$ separately and assemble the results

$$T1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{18}$$

$$T2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} cos\theta \tag{19}$$

$$T3 = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} sin\theta \tag{20}$$

Solving for $T1$

$$T1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{21}$$

$$T1 = R_x \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{22}$$

$$T1 = \begin{bmatrix} R_x^2 \\ R_x R_y \\ R_x R_z \end{bmatrix} \tag{23}$$

Solving for $T2$

$$T2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} sin\theta \tag{24}$$

$$T2 = \begin{bmatrix} 1 - R_x^2 \\ -R_x R_y \\ -R_x R_z \end{bmatrix} cos\theta \tag{25}$$

$$T2 = \begin{bmatrix} (1 - R_x^2) cos\theta \\ -R_x R_y cos\theta \\ -R_x R_z cos\theta \end{bmatrix} \tag{26}$$

Solving for $T3$

$$T3 = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} sin\theta \tag{27}$$

$$T3 = \begin{bmatrix} 0 \\ R_z \\ -R_y \end{bmatrix} sin\theta \tag{28}$$

$$T3 = \begin{bmatrix} 0 \\ R_z sin\theta \\ -R_y sin\theta \end{bmatrix} \tag{29}$$

Summing the terms

$$V' = \begin{bmatrix} R_x^2 \\ R_xR_y \\ R_xR_z \end{bmatrix} + \begin{bmatrix} (1-R_x^2)cos\theta \\ -R_xR_ycos\theta \\ -R_xR_zcos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ R_zsin\theta \\ -R_ysin\theta \end{bmatrix} \tag{30}$$

$$V' = \begin{bmatrix} R_x^2 + (1-R_x^2)cos\theta \\ R_xR_y - R_xR_ycos\theta + R_zsin\theta \\ R_xR_z - R_xR_zcos\theta - R_ysin\theta \end{bmatrix} \tag{31}$$

$$V' = \begin{bmatrix} R_x^2(1-cos\theta) + cos\theta \\ R_xR_y(1-cos\theta) + R_zsin\theta \\ R_xR_z(1-cos\theta) - R_ysin\theta \end{bmatrix} \tag{32}$$

Rotation of Y-axis

Terms $T1, T2, and T3$ for $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is

$$T1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{33}$$

$$T2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} cos\theta \tag{34}$$

$$T3 = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} sin\theta \tag{35}$$

Solving for $T1$

$$T1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{36}$$

$$T1 = R_y \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{37}$$

$$T1 = \begin{bmatrix} R_xR_y \\ R_y^2 \\ R_yR_z \end{bmatrix} \tag{38}$$

Solving for $T2$

$$T2 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} . \begin{bmatrix}R_x\\R_y\\R_z\end{bmatrix} \begin{bmatrix}R_x\\R_y\\R_z\end{bmatrix} cos\theta \tag{39}$$

$$T2 = \begin{bmatrix}-R_xR_y\\(1-R_y^2)\\-R_yR_z\end{bmatrix} cos\theta \tag{40}$$

$$T2 = \begin{bmatrix}-R_xR_ycos\theta\\(1-R_y^2)cos\theta\\-R_yR_zcos\theta\end{bmatrix} \tag{41}$$

Solving for $T3$

$$T3 = \begin{bmatrix}R_x\\R_y\\R_z\end{bmatrix} \times \begin{bmatrix}0\\1\\0\end{bmatrix} sin\theta \tag{42}$$

$$T3 = \begin{bmatrix}-R_z\\0\\R_x\end{bmatrix} sin\theta \tag{43}$$

$$T3 = \begin{bmatrix}-R_zsin\theta\\0\\R_xsin\theta\end{bmatrix} \tag{44}$$

Summing the terms

$$V' = \begin{bmatrix}R_xR_y\\R_y^2\\R_yR_z\end{bmatrix} + \begin{bmatrix}-R_xR_ycos\theta\\(1-R_y^2)cos\theta\\-R_yR_zcos\theta\end{bmatrix} \begin{bmatrix}-R_zsin\theta\\0\\R_xsin\theta\end{bmatrix} \begin{bmatrix}0\\R_zsin\theta\\-R_ysin\theta\end{bmatrix} \tag{45}$$

$$V' = \begin{bmatrix}R_xR_y - R_xR_ycos\theta - R_zsin\theta\\R_y^2 + cos\theta - R_y^2cos\theta\\R_yR_z - R_zR_ycos\theta + R_xsin\theta\end{bmatrix} \tag{46}$$

$$V' = \begin{bmatrix}R_xR_y(1-cos\theta) - R_zsin\theta\\R_y^2(1-cos\theta) + cos\theta\\R_yR_z(1-cos\theta) + R_xsin\theta\end{bmatrix} \tag{47}$$

Rotation of Z-axis

Terms $T1, T2, and T3$ for $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is

$$T1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{48}$$

$$T2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} cos\theta \tag{49}$$

$$T3 = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} sin\theta \tag{50}$$

Solving for $T1$

$$T1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{51}$$

$$T1 = R_z \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \tag{52}$$

$$T1 = \begin{bmatrix} R_xR_z \\ R_yR_z \\ R_zR_z \end{bmatrix} \tag{53}$$

Solving for $T2$

$$T2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} cos\theta \tag{54}$$

$$T2 = \begin{bmatrix} -R_xR_z \\ -R_yR_z \\ (1 - R_z^2) \end{bmatrix} cos\theta \tag{55}$$

$$T2 = \begin{bmatrix} -R_xR_zcos\theta \\ -R_yR_zcos\theta \\ (1 - R_z^2)cos\theta \end{bmatrix} \tag{56}$$

Solving for $T3$

$$T3 = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} sin\theta \tag{57}$$

$$T3 = \begin{bmatrix} R_y \\ -R_x \\ 0 \end{bmatrix} sin\theta \tag{58}$$

$$T3 = \begin{bmatrix} R_ysin\theta \\ -R_xsin\theta \\ 0 \end{bmatrix} \tag{59}$$

Summing the terms

$$V' = \begin{bmatrix} R_xR_z \\ R_yR_z \\ R_z^2 \end{bmatrix} + \begin{bmatrix} -R_xR_zcos\theta \\ -R_yR_zcos\theta \\ (1-R_z^2)cos\theta \end{bmatrix} \begin{bmatrix} R_ysin\theta \\ -R_xsin\theta \\ 0 \end{bmatrix} \tag{60}$$

$$V' = \begin{bmatrix} R_xR_z - R_xR_zcos\theta + R_ysin\theta \\ R_yR_z - R_yR_zcos\theta - R_xsin\theta \\ R_z^2 + (1-R_z^2)cos\theta \end{bmatrix} \tag{61}$$

$$V' = \begin{bmatrix} R_xR_z(1-cos\theta) + R_ysin\theta \\ R_yR_z(1-cos\theta) - R_xsin\theta \\ R_z^2(1-cos\theta) + cos\theta \end{bmatrix} \tag{62}$$

Finally, assembling the three transformed basis vectors into the rotation matrix we get the matrix that transforms any arbitrary vector about a different arbitrary vector R

$$\begin{bmatrix} R_x^2(1-cos\theta)+cos\theta & R_xR_y(1-cos\theta)-R_zsin\theta & R_xR_z(1-cos\theta)+R_ysin\theta \\ R_xR_y(1-cos\theta)+R_zsin\theta & R_y^2(1-cos\theta)+cos\theta & R_yR_z(1-cos\theta)-R_xsin\theta \\ R_xR_z(1-cos\theta)-R_ysin\theta & R_yR_z(1-cos\theta)+R_xsin\theta & R_z^2(1-cos\theta)+cos\theta \end{bmatrix} \tag{63}$$