Strategic Interaction and Networks

Yann Bramoullé, Rachel Kranton and Martin D'Amours*

April 2009

Abstract: This paper studies a wide class of games, representing many different economic environments. In all games, best replies are linear. We use a graph to capture strategic interactions between different players: a player's payoff is impacted directly by another player if and only if they are linked. Because linked players interact with other linked players, the equilibrium outcomes depend on the entire network structure. We provide a general analysis of Nash and stable equilibria for any network pattern. We construct an algorithm to find all Nash equilibria and show that all equilibrium play is characterized by players' centrality in the network. We derive conditions on the graph structure for unique, corner, and stable equilibria. In strategic substitutes games, equilibria are stable only when the graph connecting active agents is sufficiently absorptive. Except for small payoff impacts, stable equilibria always involve extreme play: some agents take no actions at all. Thus restricting attention to interior equilibria may be misleading. We also tackle comparative statics for strategic substitutes and find aggregate play always decreases as links are added to a network. To derive our results, we use a new combination of optimization, potential games, and spectral graph theory.

*Bramoullé and D'Amours: Université Laval; Kranton: Duke University. We dedicate this paper to the memory of Toni Calvó-Armengol. As will be clear in the subsequent pages, he has made a lasting contribution to our thinking about networks and economics. We deeply miss his insight and his company. We are grateful to Alexander Groves for resesearch assistance, and to Noga Alon, Roland Bénabou, Andrea Galeotti, Sanjeev Goyal, Matthew Jackson, Brian Rogers and participants in seminars and conferences at MEDS (Northwestern), École Polytechnique (Paris), Montréal, Barcelona, and Duke for helpful comments and discussions. Rachel Kranton thanks the Canadian Institute for Advanced Research for support.

I. Introduction

Public goods - criminal activity - quantity-competition - resource extraction - investments and macroeconomic policy - R&D collaborations - tax competition - belief formation - peer effects - team production and communication - a variety of topics and a variety of theoretical treatments.[1] Yet, a large number of models share a common feature. Best reply functions are linear. In many models, other agents' actions enter directly utility/payoff functions. In other models, other agents' actions affect profits or utility through the tax rates or distribution of beliefs. Either way (primitive or reduced form), agents have objective functions which yield best replies that are linear in other agents' play. This paper studies this class of games. We study simultaneous-move games where agents choose from a continuous action space. We use a network, or graph, to capture which player's actions affect whom. We use a payoff impact paramter to capture how much other players' actions matter. The graph can be seen as simply a mathematical device to model direct interactions. The graph can also represent geographic proximity, social links, or market divisions that imply particular players interact with one another directly. While all agents need not interact directly, the equilibrium outcomes depend on the entire pattern of connections: one agent's action affects a second agent whose action affects a third agent, and so on. We study the general properties of equilibria of such games.

This paper characterizes outcomes for any graph and for any level of payoff impacts. We provide an algorithm to find all Nash equilibria and show that all equilibrium play is characterized by players' centrality in the network. We derive conditions on the graph structure for unique, corner, and stable equilibria. In particular, we characterize equilibria for large payoff impacts, which has not yet been previously tackled. We derive comparative statics results and show how aggregate play changes as links are added to the network or as payoff interactions increase. To derive our results we use a new combination of optimization, potential games, and spectral graph theory.

[1] See, for example, Bergstrom, Blume, Varian, (1986), Bramoullé & Kranton (2007), Bloch & Zenginobuz (2006) for public goods; Ballester, Calvó-Armengol & Zenou (2006) for criminal activity; Vives (2001) for quantity competition; İlkiliç (2008) for resource extraction; Angeletos & Pavan (2004, 2007, 2008) for investment and macroeconomic policy; Goyal & Moraga (2001) for R&D collaborations; Kanbur & Keen (1993) for tax competition; Bénabou (2008) for belief formation; Akerlof (1997), Calvó-Armengol, Patacchini, & Zenou (2009) for peer effects; Morris & Shin (2002, 2007), Calvó-Armengol, Martí & Prat (2008), Hagenbach & Koessler (2009) for team production and communication.

Our findings demonstrate the utility of network analysis and identify structural graph features which figure prominently in equilibrium outcomes. A network model allows a general analysis of interactions among multiple players. Most current models assume that all players interact with each other and in exactly the same way. A network structure allows for other possible, empirically relevant, patterns of interaction. Of course, it is often difficult to obtain analytic and intuitive results that apply generally to any network. Here we show that any network can be summarized by two features that give a great deal of information on equilibrium outcomes: the centrality of agents in a graph, and the lowest eigenvalue of a graph. Centrality measures, prominently used in sociology, captures the strength and quantity of paths between agents in the graph. The relationship between Bonacich centrality and interior Nash equilibria - when payoff impacts are small - was a primary finding of Ballester, Calvó-Armengol & Zenou (2006). We study centrality and its role for all payoff impacts and in all equilibria - interior or corner. The lowest eigenvalue characterizes what we call the *absorptive capacity* of a network. A network that has greater absorptive capacity is better able to counter changes, or perturbations, in any one agent's actions. We find necessary and sufficient conditions for unique and stable equilibria that relate payoff impacts to the absorptive capacity of a graph. Unique equilibria exist, for example, if the graph is sufficiently absorptive.

While our findings hold in a general game, we focus on the difficulties that arise in games of strategic substitutes. There are many economic settings where actions are substitutes and social and/or geographic structures can matter, including oligopoly interaction, investment in public goods, research & development, price and information-gathering by consumers, and experimentation with new technology.[2] Yet, the analysis of strategic substitutes game involves complexities not present in the analysis of games with strategic complementarities. Agents' actions tend to go in opposite directions. And while multiplicity is also prominent, the set of equilibria does not generally have a neat lattice structure. To address multiplicity, we derive conditions for stable equilibria as subset of Nash equilibria. Stability depends on the absorptive capacity of the graph connecting active agents. We show that stable equilibria almost always exist and, except for small payoff impacts, always involve extreme play: some agents take no actions at all. We further obtain comparative static results on stable equilibria. When a link is added to a network, some

[2] See, e.g., Foster & Rosensweig (1995), Gladwell (2000), and Grabher (1993).

players increase their actions and other decreases. We show that aggregate actions, however, always decrease - increases in actions are countered by a larger decreases in actions.

Our comparative statics results and stability results yield interesting economic findings for strategic substitutes games. They also show that the common restriction to small payoff impacts and interior equilibria is a significant loss in generality. For example, in private-provision-of-public-goods games, only for very small payoff impacts will all players contribute in stable equilibria. In such games, we also show adding social links will lead to an overall reduction in public good provision. In a model of crime where criminals compete (or competition dominates cooperation), more links connecting criminals leads to lower - not higher - levels of criminal activity.[3] With more links network, we are also likely to see only few active criminals, each doing more crime. Finally, in a model of media markets, when a new firm enters a media market, less news content overall is produced. And as media markets open, so more firms can compete in a variety of markets, a few dominant firms are likely to emerge.

This paper makes at least three contributions to the study of strategic interaction.

First, we advance the recent and active literature on games played on networks. This paper integrates and supercedes findings in two recent lines of inquiry. In this work, the games have quite different payoff functions, but - as we point out here - the best-response functions are identical and linear. Thus, the findings are special cases of our present results. Ballester, Calvó-Armengol & Zenou (2006) study a game where payoffs are quadratic and agents can be hurt by other agents' actions. A central case is where actions are strategic substitutes. They thoroughly analyze the situtation where payoff interactions are weak - that is, actions are weak substitutes. They provide a sufficient condition for a unique interior equilibrium, where agents actions are proportional to their Bonacich centrality measures. In a similar game, and for weak substitutes, Ballester & Calvó-Armengol (2007) find a sufficient condition for a unique equilibrium.[4] Bramoullé & Kranton (2007) study a game where agents' gain from other agents' actions and actions are strategic substitutes. They thoroughly analyze the opposite end of the spectrum, where own and others's actions have strong payoff interactions - actions of linked agents are perfect substitutes. Multiple equilibria are the rule, and there is always an equilibrium where some individuals do no

[3] Calvó-Armengol & Zenou (2004) study a network model of crime with local substitutabilities and global complementarities. They study the case where actions are strategic substitutes and restrict attention to interior equilibria.

[4] Corbo, Calvó-Armengol & Parkes (2007) study network design in the same setting.

actions. Both these research efforts are essentially silent on the general case, when the degree of sustituability can range from weak to strong substitutes.

More generally in the economics network literature, we make progress on the problems of multiplicity and comparative statics. Researchers in this area are often frustrated by the large number of equilibria and, thus, the difficulty of characterizing equilibria and conducting comparative statics. There are several ways of attacking these problems. Galeotti, Goyal, Jackson, Vega-Redondo & Yariv (2006), a prominent example, explore how to overcome these issues by assuming individuals do not know the complete structure of a network. Individuals know only the degree distribution of the graph. With this assumption, they obtain sharp analytical results for symmetric equilibria. Symmetry, however, may not be appropriate for strategic substitutes settings, where players generally take actions in opposite directions. In the present paper, we take a more traditional approach; agents play a simultaneous move game on a graph and we explore Nash and stable equilibria. Importantly, individuals need not have complete information about the network structure to play a best response. An individual simply needs to know who his neighbors are and what they are playing. To attack the problem of characterizing equilibria and comparative statics, we explore different tools and concepts, such as centrality measures (first invoked in such games by Ballester, Calvó-Armengol & Zenou (2006)), potential functions, and structural network features.

Second, we provide new tools to study simultaneous-move games with continuous action spaces. As mentioned above, there is a wide variety of applications that are modeled with payoffs that have linear best responses. The canonical game is one with quadratic payoffs. In many treatments, researchers restrict attention to the cases (a) all agents interact equally with all other agents, and (b) interior equilibria. This paper gives researchers tools to study any possible interation structure, the entire action space, and the full set of equilibrium outcomes. With these tools we can obtain new conclusions, such as the comparative static outcomes mentioned above.

Third, the paper concretely connects techniques and outcomes traditionally used in economics and techniques and outcomes traditionally used by computer scientists in studying network settings. We characterize equilibrium outcomes as a well-defined quadratic programming problem. We construct a simple algorithm which runs in exponential time. While it might be improved upon, we know that any algorithm that identifies all equilibria must run in exponential time.

We also show that important equilibria are related to the largest maximal independent sets of a graph, and finding such sets is a well-known NP hard problem. Computer scientists have recently exploited the theory of potential games, developed by Monderer and Shapley (1996) to study congestion games in networks and other issues in computer networks (e.g., Roughgarden & Tardos (2002), Chien & Sinclair (2007)). In economics, the theory of potential games has generally been applied to finite games, including a few network settings.[5] We provide the first application to network games with continuous action space. We show how to use it, in combination with other formal tools, to illuminate the study of strategic interaction in networks. These cross-overs could have great impact on the economic theory of networks and multiple player games in general.

The paper proceeds as follows. In the next section we present the model and give our basic characterizations of Nash equilibria. Section III reformulates the equilibrium conditions by using a potential function and finds the relationship between the absorptive capacity of the graph and unique and corner equlibria. We then turn to stable equilibria in Section IV and comparative statics in Section V. We show how our results apply to many more games in the conclusion, Section VI.

II. Games with Linear Best Replies and Nash Equilibria

A. The Model

There are n individuals. Each individual i simultaneously chooses an action $x_i \in [0, \infty)$. Let $\mathbf{x}$ denote an action vector for all agents, and let $\mathbf{x}_{-i}$ denote the vector of actions of all agents other than i. To specify payoffs, we use a network, or graph, described by an $n \times n$ adjacency matrix $\mathbf{G} = [g_{ij}]$,[6] where $g_{ij} = 1$ if i impacts j's payoffs directly, $g_{ij} = 0$ otherwise, and it is assumed $g_{ij} = g_{ji}$. Following common usage, we say agent i and j are *linked*, or *connected* when $g_{ij} = g_{ji} = 1$, or i and j are *neighbors*. We normalize $g_{ii} = 0$. We write payoffs as

$$U_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}),$$

[5] Blume (1993) studies finite games when players are situated on a lattice. Young (1998) considers coordination games with two actions and general networks. Bramoullé (2007) studies anti-coordination games with two actions and general networks.

[6] Throughout the paper, we identify a network with its adjacency matrix.

where $\delta \in [0,1]$ is an interaction parameter which captures the extent to which agents' actions directly affect their neighbors' payoffs. Let $\overline{x}_i$ be the action individual i would take in isolation; that is, $\overline{x}_i$ maximizes U_i when $\delta = 0$ or $g_{ij} = 0$ for all j.

We provide a general analysis of pure-strategy Nash equilibria and stable equilibria of games where payoffs $U_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G})$ yield linear best reply functions. As discussed in the Introduction, canonical games in the network literature fall in this class, as well as a wide variety of games from across economics. Since games in this class share the same best reply functions, they share the same set of equilibrium outcomes.

We first present three canonical games. We then fully describe the best-reply functions and our equilibrium concepts. In each of the games that follow $\overline{x}_i = \overline{x}$ for all i. We will focus on this case in the paper; the results hold generally as discussed in Section VI.

The first game is a model of public goods in networks (Bramoullé & Kranton (2007)). Individual i benefits from his neighbors' actions, and individual i's payoff are

$$\widehat{U}_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = b(x_i + \delta \sum_j g_{ij} x_j) - \kappa x_i,$$

where $b(.)$ gives the benefits from own and neighbors' actions and is a strictly increasing and strictly concave function on $\mathbb{R}_+$, and $\kappa > 0$ is the constant marginal cost of own action, such that $b'(0) > \kappa$ and $b'(+\infty) < \kappa$. Bramoullé & Kranton (2007) primarily study the case of $\delta = 1$.

The second game is a case in Ballester, Calvó-Armengol, & Zenou (2006) where agents are hurt, not helped by neighbors' effort. Individual i's payoff is

$$\widetilde{U}_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = \bar{x} x_i - \frac{1}{2} x_i^2 - \delta \sum_j g_{ij} x_i x_j,$$

For future references, we will call these payoffs "quadratic payoffs." Ballester, Calvó-Armengol, & Zenou (2006) and Ballester & Calvó-Armengol (2007) study this game for δ sufficiently low.

The third game is a textbook Cournot oligopoly with linear demand and constant marginal cost. With marginal cost d, firm i facing (inverse) demand curve $P_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = a - b \cdot (x_i + \delta \sum_j g_{ij} x_j)$ has payoffs:

$$\Pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = x_i \left(a - b \cdot (x_i + \delta \sum_j g_{ij} x_j) \right) - d x_i$$

In a standard treatment $g_{ij} = 1$ for all i and j, and δ is the extent to which products from different firms' are substitutes. In a network treatment $g_{ij} \in \{0, 1\}$, and $g_{ij} = 1$ indicates i and j's output affect each others' prices. An example would be media markets, where x_i represents media content, and the graph represents which firms are competiting with each other in the different markets.

These three games, where agents care about the *sum* of linked players actions, yield exactly the best response functions that follow. In other models, widely used in micro and macroeconomics, agents are concerned about the *average* play of a group of other players. We can modify our results and techniques to study these games as well, and we discuss these games in Section VI.

In our base case, the best reply for agent i is

$$\begin{aligned} x_i &= \bar{x} - \delta \sum_j g_{ij} x_j \qquad && \text{if } \delta \sum_j g_{ij} x_j < \bar{x} \qquad \text{and} \\ x_i &= 0 && \text{if } \delta \sum_j g_{ij} x_j \geq \bar{x}. \end{aligned}$$

An agent will achieve a total of $\bar{x}$ either through his own or his neighbors' actions. If the weighted sum of his neighbors' actions is less than $\bar{x}$, the agent's action makes up the difference. If the weighted sum is higher, the agent simply chooses 0. Without loss of generality, we normalize $\bar{x}$ to be equal to one. This normalization implies a best reply function $f_i(\mathbf{x}) \in [0, 1]$:

$$f_i(\mathbf{x}, \delta, \mathbf{G}) = \max(0, 1 - \delta \sum_{j=1}^{n} g_{ij} x_j).$$

We will solve for Nash equilibria and stable equilibria for games with this best-reply function. A Nash equilibrium is, of course, a vector $\mathbf{x} \in [0, 1]^n$ that satisfies - simultaneously - the best reply functions for all agents $i \in \{1, \ldots, n\}$. As is well-known, not all Nash equilibria are "stable," in the sense that a small change in one agent's action could lead - through an adjustment process - to a different vector, possibly far from the initial vector of actions. In our study of stable equilibria, we adopt a continuous adjustment process. We thus follow early work on Cournot games with multiple sellers which shows existence of stable equilibria with continuous but not discrete adjustment (a finding replicated here, see Fisher (1961))

Our objective is to solve for and characterize all Nash equilibria and all stable equilibria as they depend on the interaction parameter δ and on the graph $\mathbf{G}$. Throughout the paper, for a given graph $\mathbf{G}$, we say that some property holds for *almost every* δ if it holds for every δ except possibly for a finite number of values.[7]

The games we study are strategic substitutes games. Our results incorporate the much easier case of strategic complement games, which would assume $g_{ij} \in \{-1, 0\}$. Our model directly extends to more complex strategic substitutes games including games with weighted graphs, $g_{ij} \in [0, 1]$, and models where agents have heterogeneous values $\bar{x}_i$, as long as $g_{ij} = g_{ji}$. Section Section VI contains these extensions as well as how our results and tools can be used to analyze particular games where $g_{ij} \neq g_{ji}$.

We use the following terminology and notation. We say a vector $\mathbf{x}$ is *interior* if and only if $\forall i, x_i > 0$. A vector $\mathbf{x}$ is a *corner* if and only if for some agent $i, x_i = 0$. A corner vector $\mathbf{x}$ is *specialized* if and only if $\forall i, x_i \in \{0, 1\}$. Let $\mathbf{I}$ denote the n-square identity matrix, let $\mathbf{1}$ denote a vector of ones, $\mathbf{o}$ a vector of zeros, and $\mathbf{O}$ a square matrix of zeros. Let k_i denote the number of i's neighbors, also called i's *degree*: $k_i = \sum_j g_{ij}$. Let $k_{\min}(\mathbf{G})$ and $k_{\max}(\mathbf{G})$ respectively, denote the smallest and largest degree in graph $\mathbf{G}$. The empty network is then $\mathbf{G} = \mathbf{O}$. In complete network, every agent is connected to every other agent, denoted by $\mathbf{C} = [c_{ij}]$, where $\forall i \neq j$, $c_{ij} = 1$.

B. Two Key Graph Features

Two features of a network $\mathbf{G}$ will be key in our analysis: the vector of Bonacich centralities of agents in the graph, and the lowest eigenvalue of the graph.

Centrality measures have long been used in sociology to capture differences in agents' network positions. Given a graph $\mathbf{G}$ and a scalar a such that $[\mathbf{I} - a\mathbf{G}]$ is invertible, the vector of *Bonacich centralities* $\mathbf{c}(a, \mathbf{G})$ is defined by $\mathbf{c}(a, \mathbf{G}) = [\mathbf{I} - a\mathbf{G}]^{-1}\, \mathbf{G}\mathbf{1}$, see Bonacich (1987). The centrality measure for agent i, $c_i(a, \mathbf{G})$, can be seen as a weighted sum of the paths in $\mathbf{G}$ that start with i.[8] We use the original measure proposed by Bonacich rather than the transformation of this

[7] Therefore, for a graph $\mathbf{G}$, the measure of the set of values of δ for which the property does not hold is equal to zero.

[8] To see this, note that when $|a| < 1/\rho(\mathbf{G})$, $\mathbf{c}(a, \mathbf{G}) = \mathbf{G}\mathbf{1} + a\mathbf{G}^2\mathbf{1} + a^2\mathbf{G}^3\mathbf{1} + \cdots$; the centrality of individual i counts the number of paths in $\mathbf{G}$ that start at i; paths of length k are weighted by a^{k-1}.

measure used by Ballester, Calvó-Armengol & Zenou (2006), who were the first to relate Bonacich centrality to Nash equilibria. The original measure allows us to capture important economic and theoretical features that emerge when $a < 0$, as discussed below.

The lowest eigenvalue of a graph is the key attribute of overall network structure in our analysis, and we equate it with what we call *absorptive capacity* of a graph. Given a square matrix $\mathbf{M}$, denote by $\lambda_{\min}(\mathbf{M})$ and $\lambda_{\max}(\mathbf{M})$ the smallest and largest eigenvalues of $\mathbf{M}$, respectively, and denote by $\rho(\mathbf{M}) \equiv \max(|\lambda_{\min}(\mathbf{M})|, |\lambda_{\max}(\mathbf{M})|)$ its spectral radius. For any non empty graph $\mathbf{G}$, $\lambda_{\min}(\mathbf{G}) < 0$, $\lambda_{\max}(\mathbf{G}) > 0$ and $\lambda_{\max}(\mathbf{G}) \geq -\lambda_{\min}(\mathbf{G})$.[9] The value $\lambda_{\min}(\mathbf{G})$ features prominently in our results. As $\lambda_{\min}(\mathbf{G})$ increases, we say the graph has more *absorptive capacity*. We will see that an absorptive graph dampens - rather than magnifies - the impact of players' actions on others.

C. Nash Equilibria

Existence of Nash equilibria in these games is guaranteed thanks to a fixed-point argument.[10] The equilibrium set has been well-studied when δ is low (Ballester, Calvó-Armengol, & Zenou (2006)) and when $\delta = 1$ (Bramoullé & Kranton (2007)). In contrast, little has been known in the general case.

Our first proposition gives the general characteristics of Nash equilibria for any δ and $\mathbf{G}$. For a vector $\mathbf{x}$, let S denote the set of agents whose actions are strictly positive: $S = \{i : x_i > 0\}$. We call this set the *support* of the vector $\mathbf{x}$, and we call agents in the support *active* agents. We call agents outside the support *inactive* agents. We denote by $\mathbf{x}_S$ the vector of actions of active agents, by $\mathbf{G}_S$ the subgraph of $\mathbf{G}$ connecting the active agents, and by $\mathbf{G}_{N-S,S}$ the subgraph of $\mathbf{G}$ connecting active agents to inactive agents. Parsing the network in this way and using matrix notation to express the best-reply conditions leads directly to the following result.

Proposition 1. *A profile* $\mathbf{x}$ *with support* S *is a Nash equilibrium if and only if*

$$(1) \qquad (\mathbf{I} + \delta\mathbf{G}_S)\mathbf{x}_S = \mathbf{1}$$

$$(2) \qquad \delta\mathbf{G}_{N-S,S}\mathbf{x}_S \geq \mathbf{1}$$

[9] Since the trace of $\mathbf{G}$ is equal to zero, $\lambda_{\min}(\mathbf{G}) < 0$ and $\lambda_{\max}(\mathbf{G}) > 0$. Since $\mathbf{G}$ is a nonnegative matrix, $\lambda_{\max}(\mathbf{G}) \geq -\lambda_{\min}(\mathbf{G})$ by Perron's theorem (see Theorem 0.13 in Cvetković et al. (1979)).

[10] The best-response function $\mathbf{f}$ is continuous from $[0, 1]^n$ to itself. By Brouwer's theorem, $\mathbf{f}$ has a fixed point.

From Proposition 1 we derive two basic results for Nash equilibria: First, we derive a straightforward algortihm to find all the equilibria for any graph $\mathbf{G}$ and (almost every) δ. Second, we show that all equilibria for any graph $\mathbf{G}$ and (almost every) δ are a mixture of Bonacich centrality and free-riding.

Algorithm. *Given a graph $\mathbf{G}$, consider a subset Q of individuals. If $\det(\mathbf{I}+\delta\mathbf{G}_Q) \neq 0$, compute the profile $\mathbf{x}_Q = (\mathbf{I}+\delta\mathbf{G}_Q)^{-1}\mathbf{1}$. Check whether $\mathbf{x}_Q \geq 0$ and $\delta\mathbf{G}_{N-Q,Q}\mathbf{x}_Q \geq 1$. If these two sets of linear inequalities are satisfied, $\mathbf{x}$ is an equilibrium. If either condition fails, it is not. Repeating this procedure for every subset of N yields all the equilibria.*

The algorithm proves that for any graph $\mathbf{G}$, the equilibria are finite for almost every delta.[11] This follows simply from the observation that for a given graph $\mathbf{G}$, $\forall Q$, $\det(\mathbf{I} + \delta\mathbf{G}_Q) \neq 0$ for almost every δ.

In terms of performance, the algorithm runs in exponential time. While it might be improved, note that the number of equilibria can be exponential.[12] So any algorithm describing the whole set of equilibria must run in exponential time.

We next show all equilibria for any $\mathbf{G}$ and (almost every) δ have a similar configuration. They combine Bonacich centrality among active agents and free-riding for inactive agents.

Equilibrium and centrality. *Consider an equilibrium $\mathbf{x}$ with its set of active agents S such that* $\det(\mathbf{I} + \delta\mathbf{G}_S) \neq 0$. Then

$$\mathbf{x}_S = \mathbf{1} - \delta\mathbf{c}(-\delta, \mathbf{G}_S)$$

The active agents who are more central do lower actions.

The original Bonacich measure captures important economic features of equilibrium outcomes. The centrality measure for agent i can be seen as a weighted sum of paths leading from agent i. The weight depends on the length of the path and a scalar a, which corresponds here to minus the payoff impact factor, that is $-\delta$. Thus, for $\delta = 0$, $c_i(0, \mathbf{G}) = k_i$; an individual's centrality is just equal to her number of neighbors. For very low δ, centrality is aligned with degree and individuals with more neighbors are more central.[13] We show in the Appendix that this extends to arbitrary δ in the following sense. For any $\delta < 1$, individuals with more neighbors in terms

[11] Even when not finite, the equilibrium set has a well-defined mathematical structure. For any δ and $\mathbf{G}$ the set of equilibria is a finite union of compact convex sets. The Appendix provides the proof.

[12] Bramoullé & Kranton (2007) show that the number of specialized equilibria can be exponential when $\delta = 1$.

[13] See Bloch & Quérou (2008) for a thorough analysis of this case.

of set inclusion are always more central.[14] This property means, for instance, that the center of a star network always does lower action in equilibrium if $\delta < 1$, as seen in Example 1 below. In general, with the scalar $a = -\delta < 0$, the measure puts positive weight on paths of odd length, and negative weight on paths of even length.[15] These alternating signs match the logic of strategic substitutes. An agent i will do less when his neighbors do more, and his neighbors will do more when their neighbors do less, and so on. In general, an agent's centrality and equilibrium play depends on the entire network architecture. We will see, for example, that an agent situated in the middle of the graph is not necessarily the most central according to this measure. The agent who is most central may rather be one whose neighbors have less alternatives.

Our result characterizes equilibria for (almost every) $\delta \in [0, 1]$ and, thus, supercedes previous findings for low δ and high $\delta = 1$. Taking a different approach altogether, Ballester, Calvó-Armengol, & Zenou (2006, Theorem 1) shows that if $\delta < 1/(1 + \lambda_{\max}(\mathbf{C} - \mathbf{G}))$, then there is a unique interior equilibrium $\mathbf{x}$ where actions are determined by the Bonacich centralities of individuals in the graph $\mathbf{C} - \mathbf{G}$ for the scalar $\delta/(1 - \delta)$.[16] For future reference, we label this critical value as $\underline{\delta}_{BCAZ} \equiv 1/(1 + \lambda_{\max}(\mathbf{C} - \mathbf{G}))$. In the present paper, applying Proposition 1, $S = N$ in an interior equilibrium and individual i's effort is $x_i = 1 - \delta c_i(-\delta, \mathbf{G}) > 0$.[17] Second, Theorem 1 in Bramoullé & Kranton (2007) show for $\delta = 1$, multiple specialized equilibria exist for any non empty graph. The active players constitute a maximal independent set of the graph $\mathbf{G}$.[18] In the present paper, applying Proposition 1, $\mathbf{G}_S = \mathbf{O}$ and $\mathbf{c}(-\delta, \mathbf{G}_S) = \mathbf{o}$.

The following examples illustrate equilibria for the full range of δ for particular graph structures. These examples show how the interaction parameter and the graph interplay to determine the equilibria. Especially, uniqueness and interiority break down as δ increases and in a way that crucially depends on the network. Our results in the next section clarify this intuition.

[14] More generally, if $\delta < 1$ and i is connected to j and to j's neighbors then $x_i \leq x_j$ in any equilibrium $\mathbf{x}$.

[15] To see this, recall that if $\delta < 1/\rho(\mathbf{G})$, then $\mathbf{c}(-\delta, \mathbf{G}) = \mathbf{G}\mathbf{1} - \delta\mathbf{G}^2\mathbf{1} + \delta^2\mathbf{G}^3\mathbf{1} - \cdots$.

[16] Precisely, introducing the vector $\mathbf{y} = \mathbf{1} + \frac{\delta}{1-\delta}\mathbf{c}\left(\frac{\delta}{1-\delta}, \mathbf{C} - \mathbf{G}\right)$, their result states that if $\delta < 1/(1+\lambda_{\max}(\mathbf{C}-\mathbf{G}))$, the unique, interior equilibrium action vector is $\mathbf{x} = \frac{1}{1-\delta+\delta\sum_j y_j}\mathbf{y}$.

[17] Putting the two results together shows that centrality in the original graph $\mathbf{G}$ is an affine negative transformation of centrality in the complementary graph $\mathbf{C} - \mathbf{G}$. This graph captures the local complementarities that remain once global substituabilities have been accounted for.

[18] A set of agents I is an *independent set* of the graph $\mathbf{G}$ if no two agents in I are linked; that is, for all $i, j \in I, g_{ij} = 0$. An independent set I is *maximal* if I is not a subset of another independent set. For any maximal independent set, agents are divided into two groups: (i) those in the set and (ii) those linked to an agent in the set.

Example 1. ***Stars.*** *Consider a star with n players. This example shows the equilibrium can be unique for a large range of δ above $\underline{\delta}_{BCAZ}$ and demonstrates the unique equilibrium need not be interior. If $\delta < 1/(n-1) = \underline{\delta}_{BCAZ}$ there is a unique, interior equilibrium where the center plays $[1-(n-1)\delta]/[1-(n-1)\delta^2]$ and each peripheral agent plays $(1-\delta)/[1-(n-1)\delta^2]$. If $1/(n-1) \leq \delta < 1$, there is a unique, specialized equilibrium where the center plays 0 and peripheral agents play 1. For $\delta = 1$, there are two specialized equilibria: (i) the center plays 0 and peripheral agents play 1, and (ii) the center plays 1 and the peripheral agents play 0. These equilibria are pictured in Figure 1 for $n = 4$.*

Example 2. ***Line with Four Players***. *This example shows that an interior equilibrium can exist for any value of δ even though it may not be unique, see Figure 2. For this graph, $\underline{\delta}_{BCAZ} \simeq 0.382$. For $\delta < \frac{\sqrt{5}-1}{2} \simeq 0.618$, this interior equilibrium is unique; players 1 and 4 on the ends play $\frac{1}{(1+\delta-\delta^2)}$, and players 2 and 3 in the middle of the line play $\frac{1-\delta}{(1+\delta-\delta^2)}$ For $\frac{\sqrt{5}-1}{2} < \delta < 1$, two corner equilibria emerge in addition to the interior one. In the corner equilibria, one player is inactive. To preview later results, in this range only the corner equilibria are stable. Finally for $\delta = 1$, we depict the multiple specialized equilibria, where active agents are a maximal independent set of the graph.*

Example 3. ***Line with Five Players.*** *The line with five players illustrates the value of the original Bonacich centrality measure. Players 2 and 4, in all equilibria, have the lowest action levels, see Figure 3. They are not in the middle of the line, but they have the highest centrality since they are connected to agents 1 and 5 who have no other neighbors. For $\delta < 0.5$, there is a unique interior equilibrium, where players 1 and 5 on the ends play $\frac{(1-\delta-\delta^2)}{(1-3\delta)}$, players 2 and 4, next to the ends, play $\frac{(1-2\delta)}{(1-3\delta)}$, and the middle player, 3, plays $\frac{(1-2\delta+\delta^2)}{(1-3\delta)}$. For $0.5 < \delta < 1$, there is a unique specialized equilibrium where agents 1,3, and 5 play 1 and agents 2 and 4 play 0.*[19]

Example 4. ***Complex network.*** *Figure 4 depicts equilibria on a complex network linking 20 agents for 3 values of δ. This network is a particular realization of a Poisson random graph with connection probability equal to 0.75. Higher degree nodes are depicted closer to the center of the figure. The color of a node is proportional to action on a fixed Black-White scale (Black=1,*

[19] *For this graph $\underline{\delta}_{BCAZ} \simeq 0.287$.*

White=0). For $\delta = 0.15$, there is a unique interior equilibrium represented in panel (a). Individuals with higher degrees tend to do less even if the correspondence between centrality and degree is not perfect. In contrast, there are 3 equilibria for $\delta = 0.35$ and 31 equilibria for $\delta = 0.55$. In both cases, we only represent the equilibrium with highest aggregate action (panels (b) and (c)). It has many inactive agents at $\delta = 0.35$ and becomes specialized at $\delta = 0.55$.

III. The Shape of Nash Equilibria

In this section we address two major challenges in the analysis of strategic substitute games and games on networks: uniqueness and the shape of equilibria for a given graph. Section IV studies stable equilibria, and Section V engages comparative statics. In all, we derive sufficient conditions and necessary conditions for unique, interior, corner, and stable outcomes as well as changes in aggregate outcomes - all as functions of the network structure and interaction parameter δ. To do so, we combine tools from the theory of potential games, the theory of maximization, and spectral graph theory.

We begin our analysis by reformulating the equilibrium conditions and representing the set of Nash equilibria as a maximization problem. Following Monderer & Shapley (1996), we show one game in our class is a potential game and thus has a potential function. Since all the games have the same equilibria, we can use this potential function to solve for equlibria for all games in the class.

A. A potential function

Following Monderer & Shapley (1996), consider a game where player i chooses an action $x_i \in A_i$, $A = \prod_i A_i$, and payoffs are $V_i(x_i, \mathbf{x}_{-i})$. A function $\varphi(x_i, \mathbf{x}_{-i})$ is a *potential function* for this game if and only if for all i

$$\varphi(x_i, \mathbf{x}_{-i}) - \varphi(x_i', \mathbf{x}_{-i}) = V_i(x_i, \mathbf{x}_{-i}) - V_i(x_i', \mathbf{x}_{-i}) \text{ for all } x_i, x_i' \in A_i \text{ and all } \mathbf{x}_{-i} \in A_{-i}$$

For continuous, twice-differentiable payoffs, there exists a potential function if and only if $\frac{\partial^2 V_i(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 V_i(\mathbf{x})}{\partial x_j \partial x_i}$ for all $i \neq j$.

Examining the quadratic payoffs, $\widetilde{U}_i$, we see that $\frac{\partial^2 \widetilde{U}_i}{\partial x_i \partial x_j} = -\delta g_{ij} = \frac{\partial^2 \widetilde{U}_i}{\partial x_j \partial x_i}$. Hence it is a potential game. The following quadratic function

$$\varphi(\mathbf{x};\delta,\mathbf{G}) = \sum_{i=1}^{n}\left(x_i - \frac{1}{2}x_i^2\right) - \frac{1}{2}\delta\sum_{i,j=1}^{n} g_{ij}x_ix_j.$$

is a potential function for this game. In matrix notations, $\varphi(\mathbf{x};\delta,\mathbf{G}) = \mathbf{x}^T\mathbf{1} - \frac{1}{2}\mathbf{x}^T(\mathbf{I}+\delta\mathbf{G})\mathbf{x}$. Notice that the potential function is strictly concave in each argument, and $\frac{\partial^2\widetilde{U}_i}{\partial x_i^2} = \frac{\partial^2\varphi}{\partial x_i^2} = -1$.

We use this function to characterize the equilibria for all games in our class. Consider the following constrained optimization problem:

$$\max_{\mathbf{x}} \varphi(\mathbf{x};\delta,\mathbf{G}) \qquad \text{s.t.} \qquad \forall i, x_i \geq 0 \tag{P}$$

Proposition 2. *A profile* $\mathbf{x}$ *is a Nash equilibrium of any game with the best reply function* $f_i(\mathbf{x}) = \max(0, 1-\delta\sum_{j=1}^{n} g_{ij}x_j)$ *if and only if* $\mathbf{x}$ *satisfies the Kuhn-Tucker conditions of problem (P).*

To see this, take a vector $\mathbf{x}$ that satisfies the Kuhn-Tucker conditions of problem (P). No agent i has an incentive to deviate by any amount, since the first order conditions of agent i's problem are either $\frac{\partial\varphi}{\partial x_i} = \frac{\partial\widetilde{U}_i}{\partial x_i} = 0$ or $\frac{\partial\varphi}{\partial x_i} = \frac{\partial\widetilde{U}_i}{\partial x_i} < 0$ and $x_i = 0$, and second-order conditions are satisfied, $\frac{\partial^2\varphi}{\partial x_i^2} = \frac{\partial^2\widetilde{U}_i}{\partial x_i^2} < 0$.

Finding the set of equilibria for these games is thus a particular instance of quadratic programming. Quadratic optimization has been well-studied in mathematics and computer science, see e.g. Lee, Tam & Yen (2005). Especially, problem (P) is known to be NP-hard in general, see Pardalos & Vavasis (1991).[20] While solving the problem is hard, knowledge about the problem gives us enormous tractability on characterizing equilibria, as we discuss next.

B. Unique and Corner Equilibria

We use the potential function to obtain a strong, intuitive sufficient condition for the uniqueness of equilibria. Previous work (Ballester & Calvo-Armengol (2007)) find that unique equilibria exist for sufficiently low δ. Clearly if δ is low enough, individuals react little to others' actions. Especially, we show in Appendix that the best-reply function is contracting if and only if $\delta < 1/\rho(\mathbf{G})$. As

[20] An important special case where problem (P) is solvable in polynomial time is when the potential is concave.

ever, contraction of the best-replies guarantees uniqueness. However, our next result shows that this classic sufficient condition is not necessary. Indeed, we identify unique equilibria even when the best replies are not contracting. The key insight is that individual reactions to others' actions are geometrically structured by the potential function. The whole action profile moves along the surface of the potential. It is the curvature of the potential function, and the network graph which gives the potential its shape, which determines the number and character of equilibrium outcomes.

We provide a uniqueness condition that depends on the absorptive capacity of the graph, $\lambda_{\min}(\mathbf{G})$, which relates directly to the shape of the potential. Consider $\delta < -1/\lambda_{\min}(\mathbf{G})$. In this case, the graph absorbs and concentrates the payoffs interactions. The potential function $\varphi(\mathbf{x};\delta,\mathbf{G})$ is strictly concave if and only if $\delta < -1/\lambda_{\min}(\mathbf{G})$. This follows from the Hessian of the potential which is equal to $\nabla^2\varphi = -(\mathbf{I}+\delta\mathbf{G})$, and $(\mathbf{I}+\delta\mathbf{G})$ is positive definite if and only if $\delta < -1/\lambda_{\min}(\mathbf{G})$. We can then invoke the theory of convex optimization to analyze problem (P). There is a unique global maximum, and the Kuhn-Tucker conditions are necessary and sufficient.[21]

Proposition 3. *If for a graph $\mathbf{G}$, $\delta < -1/\lambda_{\min}(\mathbf{G})$, there is a unique Nash equilibrium.*

This condition is both necessary and sufficient for unique equilibria in many graphs. The condition is necessary and sufficient for an important class of graphs, regular graphs - as shown below in Corollary 1. We can also see this property in particular graphs, such as the line with four players in Example 2, where $\lambda_{\min}(\mathbf{G}) = -\frac{\sqrt{5}-1}{2}$.

Proposition 3 provides the best known sufficient condition - applicable to any graph - for unique equilibria in these games. It improves on the two previous uniqueness conditions in the literature. Ballester & Calvó-Armengol (2007, Proposition 1) show that there exists a unique equilibrium if $\delta < 1/\rho(\mathbf{G})$. For any $\mathbf{G}$, $-1/\lambda_{\min}(\mathbf{G}) \geq 1/\rho(\mathbf{G})$, and this inequality is strict when no component of $\mathbf{G}$ is bipartite.[22] Ballester, Calvó-Armengol, & Zenou (2006, Theorem 1) show there exists a unique interior equilibrium for $\delta < 1/(1+\lambda_{\max}(\mathbf{C}-\mathbf{G}))$. For any $\mathbf{G}$, $-1/\lambda_{\min}(\mathbf{G}) \geq 1/(1+\lambda_{\max}(\mathbf{C}-\mathbf{G}))$, and this inequality is strict when $k_{\max}(\mathbf{G}) < n/2$.

[21] In addition, we know that the unique equilibrium varies continuously with δ when $\delta < -1/\lambda_{\min}(\mathbf{G})$.

[22] See Theorem 0.13 in Cvetković et al. (1979).

We next obtain a sharp result for the shape of equilibria when $\delta > -1/\lambda_{\min}(\mathbf{G})$. When the graph absorptive capacity is lower, $\delta > -1/\lambda_{\min}(\mathbf{G})$, and the potential $\varphi(\mathbf{x}; \delta, \mathbf{G})$ is not concave.[23] Problem (P) then falls in the domain of non-convex optimization, and multiple equilibria are possible.

Yet, multiplicity does not mean that "anything goes." Proposition 2 partitions the equilibria in two categories: maxima and saddle points of the potential.[24] Maxima always exist.[25] We call the equilibria which correspond to the maxima of the potential *max equilibria*. For instance, when unique, the Nash equilibrium is a max equilibrium. The equilibria which yield the highest level of the potential - the global maxima - we call the *peak equilibria*. Let $\mathbf{x}^*(\delta, \mathbf{G})$ denote such an equilibrium.

When $\delta > -1/\lambda_{\min}(\mathbf{G})$, we show all max equilibria - including peak equilibria - are corners, precisely because the graph does not absorb payoff impacts and does not concentrate the players' reactions:

Proposition 4. *If for a graph* $\mathbf{G}$, $\delta > -1/\lambda_{\min}(\mathbf{G})$, *all max equilibria are corners.*

To prove this result, we study the second-order conditions of problem (P). We show in the Appendix that a necessary condition for a profile $\mathbf{x}$ with support S to be a maximum of the potential is that $\mathbf{I}+\delta\mathbf{G}_S$ is positive semi-definite. It is interior only if $S = N$ and $\delta \leq -1/\lambda_{\min}(\mathbf{G})$. Since a maximum exists, it must be that for $\delta > -1/\lambda_{\min}(\mathbf{G})$, $S \neq N$.

Intuitively, when δ is high relative to the absorptive capacity of the graph, individual responses are amplified, not dampened by the graph. Under strategic substitutes, as an individual's actions increase, his neighbor's decrease.[26] If graph cannot absorb the payoff impact and reactions rebound, some boundaries of the action space must be reached. This feature depends on how links connect agents in the graph. Following results in spectral graph theory, roughly speaking, $\lambda_{\min}(\mathbf{G})$ is smaller - the graph is less absorptive - when (a) the links divide the agents into two

[23] When $\delta = -1/\lambda_{\min}(\mathbf{G})$, the potential is concave but not strictly concave. While multiple equilibria may emerge, the equilibrium set is strongly structured. Equilibria form a convex set and all equilibria yield the same aggregate action. This happens, for instance, on the complete graph when $\delta = 1$. In addition, one equilibrium is the limit of the unique equilibrium for $\delta < -1/\lambda_{\min}(\mathbf{G})$ as δ tends to $-1/\lambda_{\min}(\mathbf{G})$ from below.

[24] The study of second-order conditions, in Appendix, shows that minima are excluded.

[25] Problem (P) always has a solution since we can easily find some $M > 0$ such that $\varphi(\mathbf{x}) < 0$ if $\mathbf{x} \notin [0, M]^n$.

[26] Under strategic complements, such amplification could lead to divergence and nonexistence of Nash equilibria, see section VI.

distinct sets and (b) there are more connections between sets than within the sets.[27] Connections between sets of agents amplify shocks. Connections within sets dampen shocks. Thus, with more connections between than within, absorption decreases. To illustrate, for a given number of agent n, consider *the least absorptive graph.* The graph $\mathbf{G}$ with lowest $\lambda_{\min}(\mathbf{G})$ is a complete bipartite graph. It divides the agents in two sets with sizes as equal as possible ($\frac{n}{2}$ in each set for n even; $\frac{n+1}{2}$ and $\frac{n-1}{2}$ for n odd). Agents are not connected to any agent in their set but are connected to every agent in the other set.[28] This structure maximizes the amplification of shocks over all possible graphs.

We illustrate how the absorptive capacity of the graph affects the equilibrium set in our next example. We contrast the equilibria for two regular graphs, each with six agents, $k = 3$. In the least absorptive graph, extreme play is an equilibrium outcome for a wider range of δ.

Example 5. ***Unique and Corner Equilibria and the Absorptive Capacity of a Graph.*** *Consider two regular graphs for $n = 6$, where $k = 3$. The first graph is a complete bipartite graph, and thus has the lowest $\lambda_{\min}(\mathbf{G})$ for all graphs with $n = 6$. This graph is pictured in Figure 5. The second graph looks like a (flat) pyramid and is shown in Figure 6. Since these are both regular graphs with $k = 3$, in both these graphs, for $\delta \in [0, 1]$, there is a interior, symmetric equilibrium where $x_i = \frac{1}{1+3\delta}$. In the complete bipartite graph, $\lambda_{\min}(\mathbf{G}) = -3$, hence this is the unique (and peak) equilibrium for $\delta \leq 1/3$. For the pyramid graph, $\lambda_{\min}(\mathbf{G}) = -2$. Hence, this is the unique (and peak) equilibrium for $\delta \leq 1/2$. In both these graphs - as we will show is true for all regular graphs below - the condition $\delta \leq -1/\lambda_{\min}(\mathbf{G})$ is necessary and sufficient for a unique equilibrium. Hence, for $1/3 \leq \delta \leq 1/2$, there are corner equilibria in the complete bipartite graph, but not for the pyramid graph. To intuitively see why, consider how agents best respond to each other's actions. Consider a change in agent 1's play: $x_1 + \varepsilon$ In the bipartite graph, all three agents on the other side of the graph adjust by the full amount ε. In the pyramid graph, the agents connected to agent 1 are also linked to each other and thus "share" the adjustment. For the complete bipartite graph, all the equilibrium outcomes are: for $\delta < 1/3$, a unique, interior equilibrium where each agent plays $\frac{1}{1+3\delta}$; for $1/3 \leq \delta \leq 1$ three equilibria: the interior equilibrium where each players chooses where each agent plays $\frac{1}{1+\delta 3}$, and two specialized equilibrium where*

[27] See Desai & Rao (1994), Alon & Sudokov (2000), and Trevisan (2008) on the relationship between the smallest eigenvalue and bipartite subgraphs of an adjacency matrix.

[28] Proof provided by Noga Alon, Tel Aviv University, in personal correspondence, January 2009.

agents on one side play 1 *and agents on the other side play* 0*. The interior equilibrium is not a max equilibrium in the range* $1/3 \leq \delta \leq 1$*. For the pyramid graph, the equilibrium outcomes are: for* $\delta < 1/2$*, a unique, interior equilibrium where each agent plays* $\frac{1}{1+3\delta}$*; for* $1/2 \leq \delta \leq \frac{\sqrt{5}-1}{2}$ *the interior equilibrium and two corner equilibria where agents in the middle play* $\frac{1-\delta}{1+\delta-\delta^2}$ *and the agents in the corner play* $\frac{1}{1+\delta-\delta^2}$ *(these equilibria mimic the equilibria in the line with four agents seen above); for* $\frac{\sqrt{5}-1}{2} \leq \delta \leq 1$*, the interior equilibrium and two corner equilibria where one agent in the corner plays* 0*, the neighboring agent in the middle plays* 0*, and the other two agents play* $\frac{1}{1+\delta}$ *(again, mimicing the line with four agents); for* $\delta = 1$*, the interior equilibrium and specialized equilibria where agents in the corner and the middle alternate 1 and 0 (again mimicing the line with four agents). The interior equilibrium is not a max equilibrium for all* $\delta \geq 1/2$*.*

More generally, we can see that Propositions 3 and 4 are both necessary and sufficient on regular graphs. Observe that when every agent has k neighbors, a symmetric interior equilibrium always exists where every agent plays $x_i = 1/(1 + \delta k)$. If $\delta < -1/\lambda_{\min}(\mathbf{G})$, this is the unique equilibrium. And if $\delta > -1/\lambda_{\min}(\mathbf{G})$, it cannot maximize the potential, hence other, corner equilibria emerge . To summarize,[29]

Corollary 1. *On regular graphs, there is a unique interior equilibrium if* $\delta < -1/\lambda_{\min}(\mathbf{G})$ *while both interior and corner equilibria emerge if* $\delta > -1/\lambda_{\min}(\mathbf{G})$*.*

Our last result in this section concerns the highest level of payoff impact $\delta = 1$. At this extreme, regardless of the absorptive capacity of the graph, there is always a peak equilibrium which is a corner. Active agents in this equilibrium form a largest maximal independent set of the graph.

Proposition 5. *When* $\delta = 1$*, any specialized equilibrium with a largest maximal independent set of agents is a peak equilibrium.*

Thus, specialized equilibria with most specialists yield largest total effort among *all* equilibria when $\delta = 1$. This finding generalizes a result obtained in Bramoullé & Kranton (2007, Example 5) for the circle. It confirms the important role played by specialization under perfect substitutes. Example 4 above shows that specialized peak equilibria may actually appear at lower values of δ.

[29] At $\delta = -1/\lambda_{\min}(\mathbf{G})$, there are multiple equilibria (interior and corner) and they all maximize the potential.

Combining Propositions 3-5, and previous results, for a graph $\mathbf{G}$ we can divide the range of δ into four parts. In the lowest part, $0 \leq \delta < \underline{\delta}_{BCAZ}$ there is a unique and interior equilibrium. The equilibrium vector is $\mathbf{x} = (\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{1} = \mathbf{1} - \delta\mathbf{c}(-\delta, \mathbf{G})$ and corresponds to the unique vector $\mathbf{x}^*(\delta, \mathbf{G})$ which maximizes the potential function $\varphi(\mathbf{x}; \delta, \mathbf{G})$. In the lower-middle range, $\underline{\delta}_{BCAZ} \leq \delta < -1/\lambda_{\min}(\mathbf{G})$, there is a unique equilibrium, which is either interior or corner.[30] The equilibrium is again the vector $\mathbf{x}^*(\delta, \mathbf{G})$. For the set of agents taking positive action, $\mathbf{x}_S = (\mathbf{I} + \delta\mathbf{G}_S)^{-1}\mathbf{1} = \mathbf{1} - \delta\mathbf{c}(-\delta, \mathbf{G}_S)$. In the upper-middle range, $-1/\lambda_{\min}(\mathbf{G}) \leq \delta < 1$, multiple equilibria are possible. All maxima equilibria are corners if $-1/\lambda_{\min}(\mathbf{G}) < \delta$. In any equilibrium, the actions taken by active agents are $\mathbf{x}_S = (\mathbf{I} + \delta\mathbf{G}_S)^{-1}\mathbf{1} = \mathbf{1} - \delta\mathbf{c}(-\delta, \mathbf{G}_S)$. At the extreme value, $\delta = 1$, for any graph, there is an equilibrium vector of active agents where $\mathbf{x}_S = \mathbf{1}$, and such an equilibrium with the largest maximal independent set of agents is a peak equilibrium. We see these ranges in the examples above

IV. Stable Equilibria

In this section we delve more deeply into the set of equilibria and identify those equilibria which are stable. We show the key to stability - as it is key to maxima of the potential - is a graph's absorptive capacity. Thus there is a direct relationship between maxima of the potential and stable equilibria.

We adopt a continuous adjustment process. Consider the following system of differential equations:

$$\begin{aligned}
\dot{x}_1 &= h_1(\mathbf{x}) = f_1(\mathbf{x};\delta, \mathbf{G}) - x_1 \\
&\vdots \\
\dot{x}_n &= h_n(\mathbf{x}) = f_n(\mathbf{x};\delta, \mathbf{G}) - x_n
\end{aligned}$$

where $f_i(\mathbf{x};\delta, \mathbf{G})$, from above, is agent i's best response function. In vector notation $\dot{\mathbf{x}} = H(\mathbf{x})$, where $H = (h_1, \ldots h_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$. By construction, $\mathbf{x}$ is a stationary state (Weibull (1995), Definition 6.4, pg. 242) of this system if and only if $\mathbf{x}$ is a Nash equilibrium of our games. We consider when a Nash equilibrium $\mathbf{x}$ is an *asymptotically stable* stationary state of the system

[30] This range turns out to be empty when $\underline{\delta}_{BCAZ} = -1/\lambda_{\min}(\mathbf{G})$.

(Weibull (1995), Definition 6.5, pg. 243.).[31]

To study asymptotically stable states, we relate this system of differential equations to the potential equation $\varphi(\mathbf{x})$. Our characterization builds on the following lemma. We show the function $-\varphi$ provides a natural Lyapunov function for the system of differential equations, see Weibull (1995), Chapter 6. To see this, note the potential always increases along the trajectories of the system: $\frac{d}{dt}\varphi(\mathbf{x}(t)) = \sum_i \frac{\partial \varphi}{\partial x_i}\dot{x}_i = \nabla\varphi(\mathbf{x}).H(\mathbf{x})$.

Lemma 1. *$\forall \mathbf{x} \in \mathbb{R}^n_+$, $\nabla\varphi(\mathbf{x}).H(\mathbf{x}) \geq 0$ and $\nabla\varphi(\mathbf{x}).H(\mathbf{x}) = 0$ if and only if $\mathbf{x}$ is a Nash equilibrium.*

Next, say that a maximum $\mathbf{x}$ of the potential is *locally unique* if there is an open neighborhood of $\mathbf{x}$ in $\mathbb{R}^n_+$ on which φ does not have any other maximum. For instance, any maximum is locally unique when the number of equilibria is finite. We then prove the following one-to-one relationship between locally unique maxima of the potential and stable equilibria:

Proposition 6. *An equilibrium $\mathbf{x}$ is asymptotically stable if and only if $\mathbf{x}$ is a locally unique maximum of the potential.*

The result follows from the theory of the Lyapunov functions, as well as the following three cases. If $\mathbf{x}$ is not a maximum of the potential, some arbitrarily small perturbation leads the system to a profile with higher potential. Because the potential increases along the trajectories, the system cannot converge back to $\mathbf{x}$. If $\mathbf{x}$ is a maximum but is not locally unique, some small perturbation leads to another maximum with the same potential. This, again, prevents convergence back to $\mathbf{x}$. If $\mathbf{x}$ is a locally unique maximum, we show in Appendix that it must also be a locally unique equilibrium.[32]

Thus, the asymptotically stable equilibria are the subset of Nash equilibria which correspond to locally unique maxima of the potential. Stability eliminates the Nash equilibria that are saddle points and the maxima which are not locally unique. The absorptive capacity of a graph plays a

[31] We also looked at discrete Nash tatonnement, as in Bramoullé & Kranton (2007). We can show that equilibria stable to Nash tatonnement are (almost) always asymptotically stable. However, the reverse is not true and Nash tatonnement stable equilibria may fail to exist in large regions of the parameter space. (Proofs are available upon request). In contrast, asymptotically stable equilibria exist generically, as shown below.

[32] When $\mathbf{x}$ is a potential maximum that is not locally unique, we can still find a set of equilibria which includes $\mathbf{x}$ and which is asymptotically stable as a set.

key role in stability, just as it does in identifying maxima of the potential. Building on Proposition 6, we obtain:

Proposition 7. *For a graph* $\mathbf{G}$*, for almost every* δ*, an equilibrium* $\mathbf{x}$ *with support* S *is asymptotically stable if and only if* $\delta < -1/\lambda_{\min}(\mathbf{G}_S)$.

The following example, again using the two regular graphs in Figure 6, illustrates.

Example 6. *Stable Equilibria and the Absorptive Capacity of a Graph.* *Consider the interior, symmetric equilibrium where* $x_i = \frac{1}{1+3\delta}$ *which exists in both graphs for* $\delta \in [0,1]$ *In the complete bipartite graph,* $\lambda_{\min}(\mathbf{G}) = -3$*, hence this equilibrum is stable for* $\delta \leq 1/3$*. For the pyramid graph,* $\lambda_{\min}(\mathbf{G}) = -2$*. Hence, this equilibrium is stable for* $\delta \leq 1/2$*. We have already seen some of the intuition. Consider agent 1 playing* $\frac{1}{1+3\delta} + \varepsilon$ *In the bipartite graph, all three agents on the other side of the graph will adjust their play by the full amount* ε*. In the pyramid graph, the agents "share" the adjustment, since two impacted agents are also linked to each other. For the complete bipartite graph, the stable equilibria are* $\delta < 1/3$*, a unique vector where each agent plays* $\frac{1}{1+3\delta}$*; for* $1/3 \leq \delta \leq 1$ *the two specialized vectors where agents on one side play 1 and agents on the other side play 0. Stability eliminates the interior equilibrium in this range. For the pyramid graph, the stable outcomes are: for* $\delta < 1/2$*, a unique vector where each agent plays* $\frac{1}{1+3\delta}$*; for* $1/2 \leq \delta \leq \frac{\sqrt{5}-1}{2}$ *the two corner vectors where agents in the middle of the pyramid play* $\frac{1-\delta}{1+\delta-\delta^2}$ *and the agents in the adjacent corners play* $\frac{1}{1+\delta-\delta^2}$*, since* $\lambda_{\min}(\mathbf{G}_S) = -\frac{\sqrt{5}-1}{2}$ *as in the line for four agents; for* $\frac{\sqrt{5}-1}{2} \leq \delta \leq 1$*, the two corner vectors where where one agent in the corner plays 0, the neighboring agent in the middle plays 0, and the other two agents play* $\frac{1}{1+\delta}$*, since* $\lambda_{\min}(\mathbf{G}_S) = -1$*; for* $\delta = 1$*, the specialized equilibria where agents in the corner and the middle alternate 1 and 0.*

Observe that inactive agents do not appear in the condition of Proposition 7. We can also see that the inactive agents do not play a role in the stability conditions in the corner equilibria for the graphs in the example. The inactive agents - at the equilibrium vector - are (almost always) strictly better off by taking no action than taking a positive action. Hence, the only case when an inactive agent i might change his action in response to a perturbation is when he obtains exactly 1 as a weighted sum from his neighbors' actions: $\delta \sum_j g_{ij} x_j = 1$. We show in Appendix that this knife-edge situation is nongeneric.[33]

Thus, when $\delta < -1/\lambda_{\min}(\mathbf{G})$, the unique equilibrium is asymptotically stable, while when $\delta > -1/\lambda_{\min}(\mathbf{G})$ any asymptotically stable equilibrium is a corner. On regular graphs, the interior equilibrium is onyl stable for low δ. The stability conditions focus our attention on particular equilibria, and in the following section we conduct comparative statics on these equilibria.

V. Comparative Statics

Comparative statics are notoriously difficult in games of strategic substitutes. The reason is that direct and indirect effects generally pull in opposite directions. Consider a parameter's change inducing a positive shock on i's action. In reaction, i's neighbors decrease their action. In turn, i's neighbors' neighbors increase theirs. If some neighbors are also neighbors' neighbors, a trade-off appears. More generally, indirect effects through paths of odd length lower actions; indirect effects through paths of even length increase them. The resulting impact on any individual in the network is usually complicated and nonmonotonic. In this section, we use the potential function to overcome these difficulties.

We obtain clean comparative statics for peak equilibria, which are also the equilibria which yield the highest aggregate effort. As we have seen above, these equilibria are almost always stable. We show that an increase in δ or an additional link to the graph always leads to a decrease in the highest aggregate actions. Thus, despite the increases of some agents, overall action levels fall. Previous comparative statics results in the literature are special cases.

Proposition 8. *Consider a δ and $\mathbf{G}$ and the corresponding peak equilibrium $\mathbf{x}^*(\delta, \mathbf{G})$. And consider a δ' and $\mathbf{G}'$ and corresponding peak equilibrium $\mathbf{x}^*(\delta', \mathbf{G}')$ where $\delta' \geq \delta$ and $\mathbf{G}$ is a subgraph of $\mathbf{G}'$. Then, $\sum_{i=1}^n x_i^*(\delta', \mathbf{G}') \leq \sum_{i=1}^n x_i^*(\delta, \mathbf{G})$.*

To understand this result, notice that for a given vector $\mathbf{x}$, $\varphi(\mathbf{x}; \delta', \mathbf{G}') \leq \varphi(\mathbf{x}; \delta, \mathbf{G})$; that is, the value of the potential at any given action profile is lower when social interactions expand. Therefore, $\varphi(\mathbf{x}^*(\delta', \mathbf{G}'); \delta', \mathbf{G}') \leq \varphi(\mathbf{x}^*(\delta', \mathbf{G}'); \delta, \mathbf{G}) \leq \varphi(\mathbf{x}^*(\delta, \mathbf{G}); \delta, \mathbf{G})$ where the second inequality holds because $\mathbf{x}^*(\delta, \mathbf{G})$ is a peak equilibrium. Combining this observation with the

[33] We also derive the corresponding stability condition which turns out to be significantly more complicated.

fact that the potential of an equilibrium is the same proportion to total effort yields the result.[34] We show in the Appendix that the decrease in aggregate actions is usually strict.[35]

Previous comparative statics results are special cases. For low δ, Theorem 2 in Ballester, Calvó-Armengol, & Zenou (2006) says if $\mathbf{G}$ is a subgraph of $\mathbf{G}'$, $\delta \leq \delta'$, and if there is a unique interior equilibrium $\mathbf{x}$ for $(\delta, \mathbf{G})$ and $\mathbf{x}'$ for $(\delta', \mathbf{G}')$, then $\sum_i x_i' \leq \sum_i x_i$. Total effort is lower when social interactions are stronger. When $\delta = 1$, combining Propositions 5 and 8 tells us that the number of nodes in a largest maximal independent set decreases when the graph expands. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) noted this fact.

Our result above gives information on changes in the equilibria with largest aggregate effort. Our next result shows that in some circumstances, comparative statics can also be obtained locally by fixing the set of agents who take positive actions in specific equilibria. We again look at max equilibria.

Proposition 9. *Suppose that $\delta \leq \delta'$ and $\mathbf{G}$ is a subgraph of $\mathbf{G}'$. For δ and $\mathbf{G}$, consider a max equilibrium $\mathbf{x}$ with support S. Now consider for δ' and $\mathbf{G}'$ any equilibrium $\mathbf{x}$ with support S' such that $S' \subset S$. Then, $\sum_{i=1}^n x_i' \leq \sum_{i=1}^n x_i$.*

Thus, even when considering some arbitrary equilibrium, aggregate effort may drop when δ or $\mathbf{G}$ expands. This property holds when two conditions are met. First, the equilibrium must be a maximum of the potential. This especially applies to asymptotically stable equilibria. The second condition is simply that an equilibrium for the new parameter values must be "close enough" to the original equilibrium - in the sense that no new agents enter the support. The existence of such a close-by equilibrium when $\mathbf{G} = \mathbf{G}'$, for example, is guaranteed for almost any δ if the increase in δ is small.[36]

On arbitrary networks, individual actions may be strongly non-monotonic in the parameters. In special circumstances, however, we can recover monotonicity at the individual level. This happens for specific structures. We can show that direct and indirect effects are perfectly aligned if and only if $\mathbf{G}$ is bipartite. This comes from the classic property that a graph is bipartite if

[34] In contrast, we can build examples where the *lowest* total effort in equilibrium is non-monotonic in δ or $\mathbf{G}$.

[35] Precisely, suppose that $\delta < \delta'$ and $\mathbf{G} = \mathbf{G}'$. Then, aggregate actions decrease strictly when no peak equilibrium on δ and $\mathbf{G}$ is specialized. In contrast, if $\mathbf{G} \subsetneq \mathbf{G}'$ and δ is unchanged, aggregate effort decreases strictly as soon as for any peak equilibrium $\mathbf{x}^*(\delta, \mathbf{G})$, the new links connect individuals in the equilibrium support.

[36] For almost any δ, if S is the support of an equilibrium for δ and $\mathbf{G}$ there exists $\varepsilon > 0$ such that if $|\delta' - \delta| \leq \varepsilon$, S is also the support of an equilibrium for δ' and $\mathbf{G}$.

and only if it has no odd cycles. This is related to the fact that the absorptive capacity may be lowest in these graphs. On connected bipartite graphs, a positive shock on i's action eventually leads to an increase in the action of all agents in his group and to a decrease in the action of all agents in the other group.[37]

VI. Conclusion: Many More Games

We conclude the paper with a discussion of future research and how our analysis and results apply to many more games and more general formulations. Extending the results to strategic substitutes games with weighted graphs and to games with heterogeneous autarkic play is straightforward as long as $g_{ij} = g_{ji}$. We further show that our analysis covers a widely used set of models in economics where, in a network formulation, $g_{ij} \neq g_{ji}$. In these models, agents care about the average of other players' actions, and we call them *linear-in-means* games. Our analysis also subsumes the simple case of strategic complements games.

Heterogeneous thresholds. Consider the model in the text, except that agents have heterogeneous autarkic play. The best-reply is

$$f_i(\mathbf{x}) = \left(0, \bar{x}_i - \delta \sum_{j=1}^{n} g_{ij} x_j\right)$$

where $\bar{x}_i > 0 \; \forall i$ and $g_{ij} = g_{ji} \in \{0, 1\}$. Let $\bar{\mathbf{x}} = (\bar{x}_1, \ldots \bar{x}_n)$ be the vector of individuals' *thresholds*. A key preliminary observation is that the existence of the potential and Propositions 2-3 hold in this model as long as $g_{ij} = g_{ji}$. The potential function is now equal to $\varphi(\mathbf{x}) = \mathbf{x}^T \bar{\mathbf{x}} - \frac{1}{2}\mathbf{x}^T(\mathbf{I} + \delta\mathbf{G})\mathbf{x}$ and $\varphi(\mathbf{x}) = \frac{1}{2}\sum_i \bar{x}_i x_i$ if $\mathbf{x}$ is an equilibrium. Hetereogeneous thresholds do not affect Propositions 4-6, which apply direcly. Comparative statics results, on the other hand, have to be modified to account for the individual heterogeneity. We now have: $\sum_{i=1}^{n} \bar{x}_i x_i^*(\delta', \mathbf{G}') \leq \sum_{i=1}^{n} \bar{x}_i x_i^*(\delta, \mathbf{G})$ when $\delta' \geq \delta$ and $\mathbf{G}$ is a subgraph of $\mathbf{G}'$. Individual actions are weighted by the individual thresholds when computing the aggregate action index. Furthermore, we can obtain clean comparative statics with respect to $\bar{\mathbf{x}}$. The logic behind our results applies, and the weighted

[37] In the model with heterogenous thresholds (see section VI), this would follow from an increase in $\bar{x}_i$. We present a formal result in the Appendix. Changes in δ or $\mathbf{G}$ may have ambiguous effects on individual actions even on bipartite graphs, since they a priori affect several individuals.

sum of actions in a peak equilibrium increases weakly when individual thresholds increase.

This model captures at least two well-known games, written in a network form. It gives us immediately a linear-demand Cournot model where firms have different marginal costs. It captures the strategic private provision of public goods as in Bergstrom, Blume and Varian (1986). Suppose that individuals with Cobb-Douglas utility functions allocate their resources between private good consumption and public good provision and that benefits from the public good flow through the network structure. Let w_i the income of individual i, let q_i his level of private good consumed, and let p the relative price of the public good. Individual i's payoff is equal to $U_i = q_i^{\alpha}(x_i + \beta \sum_i g_{ij} x_j)^{1-\alpha}$ with $q_i + px_i = w_i$. Simple computations show that the best-reply function of this game can be written as follows:

$$f_i(\mathbf{x}) = \max \left(0, \frac{1-\alpha}{p} w_i - \alpha\beta \sum_{j=1}^{n} g_{ij} x_j \right)$$

which fits the previous model with $\bar{x}_i = \frac{1-\alpha}{p} w_i$ and $\delta = \alpha\beta$.

Linear-in-means. In many economic applications, agents are concerned about the *average* play of a group of other players. Here we study such games, in a network formulation. The coordination/beauty contest/peer effects/investments etc.. models often have quadratic payoffs and arise frequently in macroeconomics, as well as microeconomics. While many treatments involve coordination - where agents, for example, want to invest in the same the technology, other treatments consider strategic substitutes. The standard treatments do not include networks, but a network treatment could transform the analysis. For example, Angeletos & Pavan (2007), pose the following payoff function to represent firms in a deep recesssion, where firms want to invest in the opposite direction from the average of other firms. We have written the function with a network form:

$$\Pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = (\theta_1 + \theta_2)\, x_i - \frac{x_i^2}{2} - \delta\theta_2 \left(\frac{1}{\sum_j g_{ij}} \right) \sum_j g_{ij} x_j$$

where the private return to investment x_i is $(\theta_1 + \theta_2)$. In Angeletos and Pavan (2007), $g_{ij} = 1$, so the final term is simply the mean of all other agents' investments. Here, with $g_{ij} = \{0, 1\}$, the model could represent possibly disjoint and overlapping sectors in the economy. We call such a model a *linear-in-means* game, and they are standard stock in macroeconomics.

Consider a linear-in-means version of our baseline model with the best response function:

$$f_i(\mathbf{x}) = \max\left(0, 1 - \delta\frac{1}{k_i}\left(\sum_{j=1}^{n} g_{ij}x_j\right)\right)$$

with $g_{ij} \in \{0,1\}$ and $\forall i, k_i \neq 0$. That is, an agent's payoffs directly depend on the average play of linked players. When the graph is not regular, $\frac{1}{k_i}g_{ij} \neq \frac{1}{k_j}g_{ji}$ and the game with (modified) quadratic payoffs $\tilde{U}_i$ does not have a potential. In this case, however, the potential can be recovered by an appropriate rescaling of the payoffs. Define $\widetilde{\Pi}_i = k_i\tilde{U}_i$. Obviously, the game with payoffs $\widetilde{\Pi}_i$ has the same best-replies and equilibria as the game with payoffs $\tilde{U}_i$. And we can easily see that $\frac{\partial^2\widetilde{\Pi}_{ii}(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2\widetilde{\Pi}_i(\mathbf{x})}{\partial x_j \partial x_i}$ for all $i \neq j$. So this modified game is a potential game with potential function $\varphi(\mathbf{x}) = \sum_i(k_i x_i - \frac{1}{2}k_i x_i^2) - \frac{1}{2}\delta\sum_j g_{ij}x_i x_j$.[38] Introduce the network $\tilde{\mathbf{G}}$ such that $\tilde{g}_{ij} = \frac{g_{ij}}{\sqrt{k_i}\sqrt{k_j}}$. Then, the potential is strictly concave if and only if $\delta < -1/\lambda_{\min}(\tilde{\mathbf{G}})$. The uniqueness condition now includes the absorptive capacity of the original graph "normalized" by the numbers of agents' neighbors, which of course affect the magntitude of the impact of average payoffs. Similarly, the potential is not concave if and only if $\delta > -1/\lambda_{\min}(\tilde{\mathbf{G}})$, and the results on multiple and corner equilibria apply. The stability results in Section IV all hold, again using $\tilde{\mathbf{G}}$. As for comparative statics, observe that if $\mathbf{x}$ is an equilibrium, $\varphi(\mathbf{x}) = \frac{1}{2}\sum_i k_i x_i$. Thus, if δ increases, the sum of individual actions *weighted by degrees* in a peak equilibrium decreases weakly.[39] In contrast, the effect of adding links to the original graph $\mathbf{G}$ is not immediate. Connecting two agents increases their degrees which changes the weights used to compute the aggregate index. We show in Appendix that this new positive effect dominates the negative ones. If $\mathbf{G}$ is a subgraph of $\mathbf{G}'$ then $\sum_{i=1}^n k_i' x_i^*(\delta, \mathbf{G}') \geq \sum_{i=1}^n k_i x_i^*(\delta, \mathbf{G})$. In a linear-in-means formulation, the sum of actions weighted by degrees increases weakly when the graph gets more connections.

Strategic complements. Strategic complements games are straightforward and have none of the complications of the strategic substitutes games analyzed in this paper. Consider a best

[38] A similar method works for any directed network $\mathbf{G}$ with $g_{ij} \in \mathbb{R}$ such that $\forall i, j, \alpha_i g_{ij} = \alpha_j g_{ij}$ for some profile $\boldsymbol{\alpha} \in \mathbb{R}^n$.

[39] The similarity with the model with heterogeneous thresholds is not a coincidence. Apply the change of variables $y_i = x_i/\sqrt{k_i}$. Then $V_i = (k_i)^{3/2}y_i - \frac{1}{2}y_i^2 - \delta\sum_j \tilde{g}_{ij}y_i y_j$. This corresponds to a model with thresholds $\bar{y}_i = (k_i)^{3/2}$. Then, $\sum_i \bar{y}_i y_i = \sum_i k_i x_i$. These thresholds, however, are endogenously determined by the network.

response

$$f_i(\mathbf{x}) = \left(0, 1 - \delta \sum_{j=1}^{n} g_{ij} x_j\right)$$

where $g_{ij} \in \{0, -1\}$. As noted in Corbo et al. (2007), the formal analysis of these games is relatively simple: if $\delta < 1/\lambda_{\max}(-\mathbf{G})$ there is a unique interior equilibrium, while equilibria fail to exist if $\delta > 1/\lambda_{\max}(-\mathbf{G})$, since complementarities then have an explosive effect on actions. Since $\lambda_{\max}(-\mathbf{G}) = -\lambda_{\min}(\mathbf{G})$, this result matches our result in Proposition 3. Our results also show this unique, interior equilibrium is stable. None of the questions concerning shape of equilibria, multiplicity, or stability arise. Comparative statics are also straightforward. Following an increase in one agent i's action, i's neighbors increase their actions, then their neighbors increase theirs, and so on. No trade-off appears. When the equilibrium exists, individual efforts x_i are an increasing function of δ or $\mathbf{G}$. Obviously, then aggregate actions are increasing in δ or $\mathbf{G}$. This outcome is of course in line with standard results of monotone comparative statics under strategic complements, see e.g. Milgrom & Roberts (1990). In this extreme case, our analysis confirms what we already know, via a different route.[40] However, these clear-cut results break down as soon as *any* substituability is present.

Our future research concerns games with mixes of strategic substitutes and strategic complements. The missing piece is when equilibria exist. When equilibria exist, our techniques hold. Consider the best-reply $f_i(\mathbf{x}) = \max\left(0, 1 - \delta \sum_{j=1}^{n} g_{ij} x_j\right)$ with $g_{ij} = g_{ji} \in \mathbb{R}$. A potential exists for quadratic payoffs. The potential is strictly concave if and only if $\delta < -1/\lambda_{\min}(\mathbf{G})$. In this case, any equilibrium would be unique. The results concerning Nash equilibria for $\delta > -1/\lambda_{\min}(\mathbf{G})$ would apply. The stability analysis would also be valid. In particular, if $\delta > -1/\lambda_{\min}(\mathbf{G})$, any stable equilibrium would be a corner. Comparative statics results would also apply to changes in $\mathbf{G}$.[41] For any network $\mathbf{G}'$ with $\mathbf{G}' \geq \mathbf{G}$, $\sum_{i=1}^{n} x_i^*(\delta, \mathbf{G}') \leq \sum_{i=1}^{n} x_i^*(\delta, \mathbf{G})$ and largest aggregate action in equilibrium decreases weakly. Here, $\mathbf{G}' \geq \mathbf{G}$ means that the game under $\mathbf{G}'$ has more substitutabilities and/or less complementarities than under $\mathbf{G}$.

[40] The potential is strictly concave if and only if $\delta < 1/\lambda_{\max}(-\mathbf{G})$. Proposition 3 shows that uniqueness holds in that range. Because of the change of sign, Proposition 8 now says that aggregate action increases when social interactions expand.

[41] Since g_{ij} may be positive or negative, $\varphi(\mathbf{x}; \delta, \mathbf{G})$ may be nonmonotonic in the global interaction parameter δ and comparative statics with respect to δ may be ambiguous.

APPENDIX

Proof of statements in section II.

The equilibrium set if a finite union of compact convex sets. Given a subset of individuals Q, define E_Q as the set of equilibria such that $(\mathbf{I} + \delta\mathbf{G}_Q)\mathbf{x}_Q = \mathbf{1}$ and $\forall i \notin Q, x_i = 0$. (The equilibrium support is included in Q but may be smaller). Let E be the set of equilibria. Proposition 1 implies that $E = \cup_Q E_Q$. Next, show that E_Q is compact and convex. Note that $\mathbf{x} \in E_Q$ iff $(\mathbf{I} + \delta\mathbf{G}_Q)\mathbf{x}_Q = \mathbf{1}$, $\forall i \notin Q, x_i = 0$ and $\delta\mathbf{G}_{N-Q,S}\mathbf{x}_Q \geq \mathbf{1}$. Given a converging sequence of elements of E_Q, these three sets of conditions still hold when taking the limit. Thus, E_Q is closed hence compact. Second, let $\mathbf{x}, \mathbf{x}' \in E_Q$, $\lambda \in [0,1]$ and $\mathbf{x}^\lambda = \lambda\mathbf{x} + (1-\lambda)\mathbf{x}'$. By linearity, $(\mathbf{I}+\delta\mathbf{G}_Q)\mathbf{x}^\lambda_Q = \mathbf{1}$, $\forall i \notin Q, x^\lambda_i = 0$ and $\delta\mathbf{G}_{N-Q,S}\mathbf{x}^\lambda_Q \geq \mathbf{1}$. Thus $\mathbf{x}^\lambda \in E_Q$ and E_Q is convex. QED.

Equilibrium and centrality. If $\mathbf{I}+\delta\mathbf{G}_S$ is invertible, Proposition 1 shows that $\mathbf{x}_S = (\mathbf{I}+\delta\mathbf{G}_S)^{-1}\mathbf{1}$. Next, write $(\mathbf{I} + \delta\mathbf{G}_S)^{-1}(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{1} = \mathbf{1}$. Developing yields $\mathbf{x}_S + \delta\mathbf{c}(-\delta, \mathbf{G}_S) = \mathbf{1}$. QED.

Individuals with more neighbors are more central when $\delta < 1$. Suppose that $\delta < 1$ and that i is connected to j and j's neighbors. Suppose that $\mathbf{I} + \delta\mathbf{G}$ is invertible and let $\mathbf{x} = (\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{1} = \mathbf{1} - \delta\mathbf{c}$. Then, $x_i + \delta(x_j + \sum_{k\neq i,j} g_{ik}x_k) = 1$ and $x_j + \delta(x_i + \sum_{k\neq i,j} g_{jk}x_k = 1$. Subtracting both equalities yields $(1-\delta)(x_i - x_j) = -\sum_{k\neq i,j}(g_{ik} - g_{jk})x_k$ hence $x_i \leq x_j$ and $c_i \geq c_j$. More generally, conisder any equilibrium $\mathbf{x}$. If $x_i, x_j > 0$, the previous argument applies. If $x_j = 0$, then $\delta(x_i + \sum_{k\neq i,j} g_{jk}x_k) \geq 1$ hence $\delta(x_i + \sum_{k\neq i,j} g_{ik}x_k) \geq 1$. If $x_i > 0$, then $(1-\delta)x_i + \delta(x_i + \sum_{k\neq i,j} g_{ik}x_k) = 1$ which is impossible. Thus, $x_i = 0$. QED.

Proof of statements in section III.

The potential of an equilibrium. Let $\mathbf{x}$ be an equilibrium with support S. Since $\mathbf{x}_{N-S} = \mathbf{o}$, $\mathbf{x}^T(\mathbf{I} + \delta\mathbf{G})\mathbf{x} = \mathbf{x}_S^T(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{x}_S$. By Proposition 1, $(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{x}_S = \mathbf{1}$. Since $\mathbf{x}_S^T\mathbf{1} = \mathbf{x}^T\mathbf{1}$, $\varphi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{1}$. QED.

The best-reply are contracting iff $\delta < 1/\rho(\mathbf{G})$. Introduce $\mathbf{e}(\mathbf{x}) = \mathbf{1} - \delta\mathbf{G}\mathbf{x}$ such that $f_i(\mathbf{x}) = \max(0, e_i(\mathbf{x}))$. We can see that $|f_i(\mathbf{x}) - f_i(\mathbf{x}')| \leq |e_i(\mathbf{x}) - e_i(\mathbf{x}')|$. For instance, if $f_i(\mathbf{x}) > 0$ and $f_i(\mathbf{x}') = 0$, then $|f_i(\mathbf{x}) - f_i(\mathbf{x}')| = f_i(\mathbf{x})$ while $|e_i(\mathbf{x}) - e_i(\mathbf{x}')| = f_i(\mathbf{x}) - e_i(\mathbf{x}')$ with $e_i(\mathbf{x}') < 0$. Therefore, $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')|| \leq ||\mathbf{e}(\mathbf{x}) - \mathbf{e}(\mathbf{x}')||$ under the Euclidean norm. Next, $\mathbf{e}(\mathbf{x}) - \mathbf{e}(\mathbf{x}') = -\delta\mathbf{G}(\mathbf{x} - \mathbf{x}')$, and $||\mathbf{e}(\mathbf{x}) - \mathbf{e}(\mathbf{x}')||^2 = \delta^2(\mathbf{x} - \mathbf{x}')^T\mathbf{G}^2(\mathbf{x} - \mathbf{x}')$. We then apply the Rayleigh-Ritz theorem, see Horn & Johnson (1985, p. 176). Since $\lambda_{\max}(\delta^2\mathbf{G}^2) = \delta^2\rho(\mathbf{G})^2$, this yields:

$||\mathbf{e}(\mathbf{x}) - \mathbf{e}(\mathbf{x}')||^2 \leq \delta^2 \rho(\mathbf{G})^2 ||\mathbf{x} - \mathbf{x}'||^2$. Thus, $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')|| \leq \delta\rho(\mathbf{G})||\mathbf{x} - \mathbf{x}'||$ and $\mathbf{f}$ is contracting if $\delta\rho(\mathbf{G}) < 1$. Reciprocally, since $\mathbf{G}$ is nonnegative, it has a nonnegative eigenvector $\mathbf{x}_0$ associated to the eigenvalue $\rho(\mathbf{G})$, see Horn & Johnson (1985, p. 503). Let $\mathbf{x} = \lambda\mathbf{x}_0$ and $\mathbf{x}' = \lambda'\mathbf{x}_0$ with $\lambda, \lambda' > 0$. If λ, λ' are small enough, $\mathbf{f}(\mathbf{x}) = \mathbf{e}(\mathbf{x}) = \mathbf{1} - \delta\rho(\mathbf{G})\mathbf{x}$ and the same holds for $\mathbf{f}(\mathbf{x}')$. Thus, $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}') = \delta\rho(\mathbf{G})(\mathbf{x} - \mathbf{x}')$ and $\mathbf{f}$ is not contracting if $\delta\rho(\mathbf{G}) \geq 1$. QED.

Proof of Propositions 4 and 7. We analyze the second-order conditions of problem (P). Given an equilibrium $\mathbf{x}$ with support S, introduce the *quasi-support* of $\mathbf{x}$ as $\bar{S} = \{i : x_i + \delta \sum_j g_{ij} x_j = 1\}$. Clearly, $S \subset \bar{S}$ and $\bar{S} - S = \{i : x_i = 0 \text{ and } \delta \sum_j g_{ij} x_j = 1\}$. The application of Theorems 3.4 and 3.6 in Lee, Tam & Yen (2005) to our setting yields the following result.

Lemma 2. *Second-order conditions of problem (P).*
$\mathbf{x}$ *is a local maximum of* φ *on* $\mathbb{R}^n_+$ *iff* $\forall \mathbf{v} \in \mathbb{R}^{\bar{S}}$ *such that* $\mathbf{v}_{\bar{S}-S} \geq \mathbf{o}$, $\mathbf{v}^T(\mathbf{I} + \delta\mathbf{G}_{\bar{S}})\mathbf{v} \geq 0$
$\mathbf{x}$ *is a locally unique maximum of* φ *on* $\mathbb{R}^n_+$ *iff* $\forall \mathbf{v} \in \mathbb{R}^{\bar{S}} - \{\mathbf{o}\}$ *such that* $\mathbf{v}_{\bar{S}-S} \geq \mathbf{o}$, $\mathbf{v}^T(\mathbf{I} + \delta\mathbf{G}_{\bar{S}})\mathbf{v} > 0$

We derive necessary conditions. Observe that when $\mathbf{v}_{\bar{S}-S} = \mathbf{o}$, $\mathbf{v}^T(\mathbf{I} + \delta\mathbf{G}_{\bar{S}})\mathbf{v} = \mathbf{v}_S^T(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{v}_S$. Therefore, if $\mathbf{x}$ is a local maximum then $\mathbf{I} + \delta\mathbf{G}_S$ is positive semi-definite. Especially, if $\mathbf{x}$ is an interior local maximum, $S = N$ and $\mathbf{I} + \delta\mathbf{G}$ is positive semi-definite, which proves Proposition 4. Similarly, if $\mathbf{x}$ is a local minimum, then $\mathbf{I} + \delta\mathbf{G}_S$ is negative semi-definite which is impossible.

When $S = \bar{S}$, the second-order conditions take a simple form: $\mathbf{x}$ is a local maximum iff $\mathbf{I} + \delta\mathbf{G}_S$ is positive semi-definite and it is locally unique iff $\mathbf{I} + \delta\mathbf{G}_S$ is positive definite. We next show that this situation is generic.

Lemma 3. *Given a graph* $\mathbf{G}$*, for all equilibria the support is equal to the quasi-support for almost every* δ*.*

Proof: Take a graph $\mathbf{G}$ such that $\mathbf{I} + \delta\mathbf{G}$ is invertible. Define $P_i(\delta) = \det(\mathbf{I} + \delta\mathbf{G})[(\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{1}]_i$. Consider the classic relation between a matrix's inverse and its cofactors (e.g. Horn & Johnson (1985, p.20)). Let m_{ij} be the determinant of the submatrix of $\mathbf{I} + \delta\mathbf{G}$ obtained by removing the ith row and jth column. Then, $P_i(\delta) = \sum_{j=1}^n (-1)^{i+j} m_{ji}$ hence P_i is a polynomial of degree less than or equal to $n-1$ in δ. In addition, $P_i(0) = 1$ so P_i cannot have more than $n-1$ zeros. Thus, for almost any δ, $\forall i, x_i \neq 0$. Next, for any equilibrium $\mathbf{x}$, $\mathbf{x}_{\bar{S}} = (\mathbf{I}+\delta\mathbf{G}_{\bar{S}})^{-1}\mathbf{1}$ if $\det(\mathbf{I}+\delta\mathbf{G}_{\bar{S}}) \neq \mathbf{0}$, hence for almost any δ, $\forall i \in \bar{S}$, $x_i > 0$ and $S \neq \bar{S}$. QED.

Proof of Proposition 5. Suppose that $\delta = 1$ and let $\mathbf{x}$ be an equilibrium with support S. Let N_i be the set of neighbors of i in S. Then, $\forall i \in S$, $x_i + \sum_{j\in N_i} x_j = 1$. Take a maximal independent set I of $\mathbf{G}_S$. Any agent in S is either in I or connected to an agent in I. Thus, $S = \bigcup_{i\in I}\{i\} \cup N_i$. Therefore, $\sum_{i\in N} x_i = \sum_{i\in S} x_i \leq \sum_{i\in I}(x_i + \sum_{j\in N_i} x_j) = |I|$. Next, observe that I is an independent set of the graph $\mathbf{G}$, hence is included in a maximal independent set I'. This yields: $\sum_{i\in N} x_i \leq |I'|$ and aggregate action in equilibrium is always lower than or equal to the size of the largest maximal independent set. QED.

Proof of Lemma 1. We have: $\frac{\partial\varphi}{\partial x_i} = 1 - \delta\sum_j g_{ij}x_j - x_i$ and $h_i(x) = 1 - \delta\sum_j g_{ij}x_j - x_i$ if $\delta\sum_j g_{ij}x_j \leq 1$ and $-x_i$ if $\delta\sum_j g_{ij}x_j > 1$. Then, $\frac{\partial\varphi}{\partial x_i}h_i = (1 - \delta\sum_j g_{ij}x_j - x_i)^2$ in the first case and $x_i(\delta\sum_j g_{ij}x_j - 1 + x_i)$ in the second case. Therefore, $\nabla\varphi(\mathbf{x}).H(\mathbf{x}) \geq 0$. Equality occurs iff $x_i = 1 - \delta\sum_j g_{ij}x_j$ if $\delta\sum_j g_{ij}x_j \leq 1$ and $x_i = 0$ if $\delta\sum_j g_{ij}x_j > 1$. QED.

Proof of Proposition 6. First, suppose that $\mathbf{x}$ is not a local maximum of the potential. There is a sequence $\mathbf{x}^k \in \mathbb{R}^n_+$ converging to $\mathbf{x}$ such that $\varphi(\mathbf{x}^k) > \varphi(\mathbf{x})$. Since φ cannot decrease along trajectories, starting at $\mathbf{x}^k$ the system cannot converge back to $\mathbf{x}$. Hence $\mathbf{x}$ is not asymptotically stable. Second, suppose that $\mathbf{x}$ is a local maximum but is not locally unique. There is a sequence $\mathbf{x}^k$ of local potential maxima converging to $\mathbf{x}$ such that $\mathbf{x}^k \neq \mathbf{x}$ and $\varphi(\mathbf{x}^k) = \varphi(\mathbf{x})$. Since $\mathbf{x}^k$ is an equilibrium, it is a steady state of the system of differential equations. Thus, starting at $\mathbf{x}^k$ the system does not converge back to $\mathbf{x}$ and $\mathbf{x}$ is not asymptotically stable.

Third, suppose that $\mathbf{x}$ is a locally unique maximum of the potential. If there exists $\varepsilon > 0$ such that $\mathbf{x}$ is the unique equilibrium on $B_O(\mathbf{x}, \varepsilon) \cap \mathbb{R}^n_+$, then we can apply Theorem 6.4 in Weibull (1995). Precisely, the Lyapunov function is equal to $\varphi(\mathbf{x}) - \varphi(\mathbf{y})$ for any $\mathbf{y} \in B_O(\mathbf{x}, \varepsilon) \cap \mathbb{R}^n_+$. Hence $\mathbf{x}$ is asymptotically stable. Thus, we only need to show that $\mathbf{x}$ is a locally unique equilibrium. Suppose the contrary. There exists a sequence of equilibria $\mathbf{x}^k$ converging to $\mathbf{x}$ such that $\mathbf{x}^k$ is not a potential maximum. Without loss of generality, we can assume that $\forall k, \mathbf{x}^k$ has the same support S since there is a finite number of subsets of N. Hence $(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{x}^k_S = \mathbf{1}$ and $\delta\mathbf{G}^k_{N-S,S}\mathbf{x}_S \geq 1$. Taking the limit shows that $(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{x}_S = \mathbf{1}$ and $\delta\mathbf{G}_{N-S,S}\mathbf{x}_S \geq 1$, hence that the support of $\mathbf{x}$ is included in S. Therefore, we can write $\mathbf{x}^k = \mathbf{x} + \mathbf{y}^k$ with $\mathbf{y}^k$ such that $\mathbf{y}^k_{N-S} = \mathbf{o}$ and $(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{y}^k_S = \mathbf{o}$. Next, compute $\varphi(\mathbf{x}^k)$ in two different ways. First, $\varphi(\mathbf{x}^k) = \frac{1}{2}(\mathbf{x}^k)^T\mathbf{1} = \varphi(\mathbf{x}) + \frac{1}{2}(\mathbf{y}^k)^T\mathbf{1}$ since $\mathbf{x}^k$ and $\mathbf{x}$ are equilibria. Second, by definition, $\varphi(\mathbf{x}^k) = \varphi(\mathbf{x} + \mathbf{y}^k) = \mathbf{x}^T\mathbf{1} + (\mathbf{y}^k)^T\mathbf{1} - \frac{1}{2}(\mathbf{x} + \mathbf{y}^k)^T(\mathbf{I} + \delta\mathbf{G})(\mathbf{x} + \mathbf{y}^k)$. The last expression is equal to $(\mathbf{x}_S + \mathbf{y}^k_S)^T(\mathbf{I} + \delta\mathbf{G}_S)(\mathbf{x}_S + \mathbf{y}^k_S) = \mathbf{x}_S^T(\mathbf{I} + \delta\mathbf{G}_S)\mathbf{x}_S$. Thus, $\varphi(\mathbf{x}^k) = \varphi(\mathbf{x}) + (\mathbf{y}^k)^T\mathbf{1}$. This shows that $(\mathbf{y}^k)^T\mathbf{1} = \mathbf{o}$. Hence $\varphi(\mathbf{x}^k) = \varphi(\mathbf{x})$ which contradicts the fact that $\mathbf{x}^k$ is not a potential maximum. QED.

Proof of statements in section V.

Strict comparative statics. Suppose that $\delta' > \delta$, $\mathbf{G}' = \mathbf{G}$ and $\varphi(\mathbf{x}; \delta', \mathbf{G})$ and $\varphi(\mathbf{x}; \delta, \mathbf{G})$ have the same maximum value over $\mathbb{R}^n_+$. Since $\varphi(\mathbf{x}; \delta', \mathbf{G}) \leq \varphi(\mathbf{x}; \delta, \mathbf{G})$, both functions have a common maximum $\mathbf{x}^*$. This profile $\mathbf{x}^*$ is an equilibrium at δ and at δ' and satisfies $\varphi(\mathbf{x}^*; \delta', \mathbf{G}) = \varphi(\mathbf{x}^*; \delta, \mathbf{G})$. Since $\varphi(\mathbf{x}; \delta, \mathbf{G}) = \mathbf{x}^T\mathbf{1} - \frac{1}{2}\mathbf{x}^T(\mathbf{I} + \delta\mathbf{G})\mathbf{x}$, we have $(\mathbf{x}^*)^T\mathbf{G}\mathbf{x}^* = \mathbf{o}$. Hence the support of $\mathbf{x}^*$ is an independent set of $\mathbf{G}$ and $\mathbf{x}^*$ is specialized. Next, suppose that $\delta' = \delta$, $\mathbf{G} \neq \mathbf{G}'$ and $\varphi(\mathbf{x}; \delta, \mathbf{G})$ and $\varphi(\mathbf{x}; \delta, \mathbf{G}')$ have the same maximum value over $\mathbb{R}^n_+$. Again, both functions have a common maximum $\mathbf{x}^*$. It now satisfies $(\mathbf{x}^*)^T(\mathbf{G}' - \mathbf{G})\mathbf{x}^* = \mathbf{o}$ and no link in $\mathbf{G}' - \mathbf{G}$ connects two individuals in the support of $\mathbf{x}^*$. QED.

Comparative statics on bipartite graphs. Consider the model with heterogeneous thresholds and suppose that $\mathbf{G}$ is connected and bipartite. Consider an interior equilibrium $\mathbf{x}$ with $\delta < -1/\lambda_{\min}(\mathbf{G})$. Then, for any j in i's group, $\partial x_j/\partial \bar{x}_i > 0$ while for any j in the other group $\partial x_j/\partial \bar{x}_i < 0$. To see why, note that $\mathbf{x} = (\mathbf{I} + \boldsymbol{\delta}\mathbf{G})^{-1}\bar{\mathbf{x}}$, hence $\partial \mathbf{x}/\partial \bar{x}_i = (\mathbf{I} + \boldsymbol{\delta}\mathbf{G})^{-1}\mathbf{u}_i$ where $u_{ii} = 1$ and $u_{ij} = 0$ if $j \neq i$. Since $\mathbf{G}$ is bipartite, $\lambda_{\min}(\mathbf{G}) = -\rho(\mathbf{G})$ and we can write $(\mathbf{I} + \delta\mathbf{G})^{-1} = \sum_{k=0}^{+\infty}(-\delta)^k\mathbf{G}^k$. Thus, $\partial x_j/\partial \bar{x}_i = \sum_{k=0}^{+\infty}(-\delta)^k(\mathbf{G}^k)_{ji}$ where $(\mathbf{G}^k)_{ji} = \sum_{l_1,\ldots,l_{k-1}} g_{jl_1}g_{l_1l_2}\cdots g_{l_{k-1}i}$ counts the number of paths of length k connecting j to i. On bipartite graphs, all cycles have even length. Any path is the union of a shortest path and of cycles. Therefore, if the shortest distance between j and i in $\mathbf{G}$ is even, all paths connecting them have even length. This happens when j and i are in the same group. Terms for k odd in the sum are equal to zero and $\partial x_j/\partial \bar{x}_i > 0$. Conversely, if the distance between them is odd they are in different groups and all paths connecting them have odd length. Terms for k even are equal to zero and $\partial x_j/\partial \bar{x}_i < 0$. Also, since the subgraph of a bipartite graph is bipartite, the result extends to any equilibrium $\mathbf{x}$ with support S when $\delta < -1/\lambda_{\min}(\mathbf{G}_S)$. And if $\mathbf{G}$ is not bipartite, it has an odd cycle and direct and indirect effects are not perfectly aligned. QED.

Proof of Proposition 9. Let $s = |S|$ and for any $\mathbf{y} \in \mathbb{R}^s$, define $\varphi_S(\mathbf{y};\delta,\mathbf{G}) = \varphi(\hat{\mathbf{y}};\delta,\mathbf{G})$ where $\hat{\mathbf{y}}$ is such that $\hat{\mathbf{y}}_S = \mathbf{y}$ and $\hat{\mathbf{y}}_{N-S} = \mathbf{o}$. Since $\mathbf{x}$ is a local maximum of φ, $\mathbf{I} + \delta\mathbf{G}_S$ is positive semi-definite. Therefore, φ_S is concave and $\mathbf{x}_S$ is a global maximum of φ_S on $\mathbb{R}^s_+$. Since $S' \subset S$, $\sum_{i\in N} x'_i = \varphi_S(\mathbf{x}'_S;\delta',\mathbf{G}') \leq \varphi_S(\mathbf{x}'_S;\delta,\mathbf{G}) \leq \varphi_S(\mathbf{x}_S;\delta,\mathbf{G}) = \sum_{i\in N} x_i$. The first inequality comes from the monotonicity of the potential. The second inequality holds because $\mathbf{x}_S$ is a global maximum of φ_S. QED.

Proof of statements in section VI.

Linear-in-means. Let $\mathbf{D}$ be the diagonal matrix such that $d_{ii} = k_i$. Then, the hessian of the potential is $-(\mathbf{D} + \delta\mathbf{G})$. For any $\mathbf{x} \in \mathbb{R}^n$, define $\mathbf{y}$ such that $y_i = \sqrt{k_i}x_i$. We can see that $\mathbf{x}^T(\mathbf{D} + \delta\mathbf{G})\mathbf{x} = \mathbf{y}^T(\mathbf{I} + \delta\tilde{\mathbf{G}})\mathbf{y}$. Therefore, φ is strictly concave iff $\mathbf{I} + \delta\tilde{\mathbf{G}}$ is positive definite. QED.

REFERENCES

Akerlof, George (1996). "Social Distance and Social Decisions," *Econometrica* 65 (5), pp. 1005-1027.

Alon, Noga and Benny Sudakov (2000). "Bipartite Subgraphs and the Smallest Eigenvalue," *Combinatorics, Probability, and Computing*, vol. 9, pp. 1-12.

Angeletos, George-Marios and Alessandro Pavan (2004). "Transparency of Information and Coordination in Economies with Investment Complementarities," *American Economic Review Papers and Proceedings* 94 (2), pp. 91-98.

Angeletos, George-Marios and Alessandro Pavan (2007). "Efficient Use of Information and Social Value of Information," *Econometrica* 75 (4), pp. 1103-1142.

Angeletos, George-Marios and Alessandro Pavan (2008). "Policy with Dispersed Information," working paper.

Ballester, Coralio, and Antoni Calvó-Armengol (2007). "Moderate Interactions in Games with Induced Complementarities," mimeo, Universitat Autonoma de Barcelona.

Ballester, Coralio, Calvó-Armengol, Antoni, and Yves Zenou (2006). "Who's who in networks. Wanted: the key player ," *Econometrica*, 74(5), 1403-1417.

Bénabou, Roland (2008). "Groupthink: Collective Delusions in Organizations and Markets," working paper.

Bergstrom, Theodore, Blume, Lawrence, and Hal Varian (1986). "On the Private Provision of Public Goods," *Journal of Public Economics,* 29, pp. 25-49.

Bloch, Francis and Nicolas Quérou (2008). "Pricing in Networks," working paper.

Bloch, Francis and Unal Zenginobuz (2007). "The Effect of Spillovers on the Provision of Local Public Goods," *Review of Economic Design*, 11, pp. 199-216.

Bramoullé, Yann (2007). "Anti-coordination and social interactions," *Games and Economic Behavior*, 58(1), 30-49.

Bramoullé, Yann, and Rachel Kranton (2007). "Public Goods in Networks," *Journal of Economic Theory*, 135 (1), pp. 478-494.

Bryant, John (1983). "A Simple Rational Expectations Keynes-Type Model," *Quaterly Journal of Economics* 98 (3), pp. 525-28.

Blume, Larry (1993). "The statistical mechanics of strategic interaction," *Games and Economic Behavior*, 5(3), 387-424.

Calvó-Armengol, Antoni, de Marti, Joan, and Andrea Prat (2009). "Endogenous Communication in Complex Organizations," working paper.

Calvó-Armengol, Antoni, Patacchini, Eleanora and Yves Zenou (2009). "Peer Effects and Social Networks in Education," forthcoming *Review of Economic Studies*.

Calvó-Armengol, Antoni, and Yves Zenou (2004). "Social Networks and Crime Decisions: The Role of Social Structure in Facilitating Delinquent Behavior," *International Economic Review.* vol 45, no 3, pp. 939-957.

Cooper, Russell (1988). "Coordinating Coordination Failures in Keynesian Models," *Quarterly Journal of Economics* 103, pp. 441-463.

Corbo, Jacomo, Calvó-Armengol, Antoni, and David Parkes (2007). "The importance of network topology in local contribution games." In *Proc. of the international workshop on Internet and Network Economics*.

Chien, S and A. Sinclair (2007). "Convergence to approximate Nash equilibria in congestion games." In *Proc. 18th ACM-SIAM Symp. on Discrete Algorithms (SODA)*.

Cvetković, Dragos, Doob, Michael, and Horst Sachs. (1979). *Spectra of Graphs, Theory and Applications,* New York: Academic Press.

Desai, Madhav and Vasant Rao (1994)."A Characterization of the Smallest Eigenvalue of a Graph," *Journal of Graph Theory,* vol. 18, no. 2, pp. 181-194.

Fisher, Franklin M. (1961) "The Stability of the Cournot Oligopoly Solution: The Effects of Speeds of Adjustment and Increasing Marginal Costs," *The Review of Economic Studies,* Vol. 28, No. 2 Februrary, pp. 125-135

Foster, Andrew and Mark Rosenzweig (2006) "Learning by Doing and Learning from Others: Human Capital and Technical Change," *Journal of Political Economy*, 103 (6), pp. 1176-1209.

Galeotti, Andrea, Goyal, Sanjeev, Jackson, Matthew, Vega-Redondo, Fernando, and Leeat Yariv. (2006) "Network Games," mimeo, Stanford University.

Gladwell, Malcolm. (2000). *The Tipping Point.* Boston: Little Brown.

Goyal, Sanjeev and Jose Luis Moraga. (2001). "R&D Networks," *Rand Journal of Economics,* 32(4), pp. 686-707.

Grabher, G. (1993). *The embedded firm: On the socioeconomics of industrial networks.* London: Routledge.

Hagenbach, Jeanne and Frédéric Koessler (2009). "Strategic Communication in Networks," working paper.

Horn, Roger, and Charles, Johnson (1985). *Matrix Analysis.* Cambridge: Cambridge University Press.

İlkiliç, Rahmi (2008). "Network of Commons," working paper.

Kanbur, Ravi and Michael Keen (1993). "Jeux Sans Frontieres: Tax Competition and Tax

Coordination when Countries Differ in Size" *American Economic Review* 83 (4), pp. 877-892.

Lee, Gue Myung, Tam, Nguyen Nang, and Nguyen Dong Yen. (2005) *Quadratic Programming and Affine Variational Inequalities.* New York: Springer Science.

Milgrom, Paul, and John Roberts (1990). "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrics*, 58(6), pp. 1255-1277.

Monderer, Dov, and Lloyd Shapley (1996). "Potential Games," *Games and Economic Behavior*, 14: 124-143.

Morris, Stephen and Hyun Song Shin (2002). "Social Value of Public Information," *American Economic Review* 92 (5), pp. 1521-1534.

Morris, Stephen and Hyun Song Shin (2007). "Optimal Communication," *Journal of the European Economic Association* 5 (2-3), pp. 594-602.

Pardalos, Panos and Stephen Vavasis (1991). "Quadratic Programming with One Negative Eigenvalue is NP-Hard," *Journal of Global Optimisation*, 1, pp. 15-22.

Roughgarden, T and É. Tardos (2002). "How bad is selfish routing?," *Journal of the ACM*, 49 (2), pp. 236-259.

Trevisan, Luca (2008). "Max Cut and the Smallest Eigenvalue," working paper, UC Berkeley Computer Science.

Vives, Xavier (2001). *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge: MIT Press, 2001.

Young, Peyton (1998). *Individual Strategy and Social Structure*, Princeton: Princeton University Press.