The Friend of My Enemy is My Enemy: Alliance Formation and Coercion in Networks

Primary Job Market Paper

Timo Hiller*

European University Institute

November 14, 2010

Abstract

This paper provides a game-theoretic foundation for the notion of structural balance, which dates back to the work of Heider (1946) in social psychology. Structural balance builds on the suggestive notion that friends (or allies) have a tendency to match their attitudes towards third parties.

I develop a model of strategic network formation, where agents may either create friendly or antagonistic links. Under an antagonistic relationship, agents with more allies are able to coerce payoffs from agents with fewer allies. The value extorted depends on a general contest success function, which is strictly increasing in the ratio of the respective agents' number of allies. Nash equilibria are characterized by either a state of utopia, where all agents are allies, or cliques of allies of *different* size arise, with agents in cliques of larger size extorting payoffs from agents in cliques of smaller size.

Key Words: Network formation, economics of conflict, contest success function, structural balance, international relations.

*I am greatful to my supervisors Fernando Vega-Redondo and Massimo Morelli for their invaluable support and guidance. I also thank Matthew O. Jackson, Paolo Pin, Francesco Squintani, participants of the Networks Working Group at the European University Institute, seminar participants at PUC-Rio and participants of the UECE Lisbon Meetings 2010 for helpful comments. All remaining errors are mine. Contact: timo.hiller@eui.eu, Address: Department of Economics, European University Institute, Via della Piazzuola 43, 50133 Florence, Italy

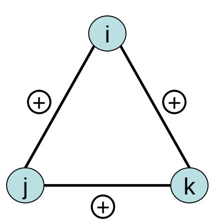
1 Introduction

In much of the literature on networks, links carry a positive connotation and are commonly interpreted as friendship, collaboration or transmission of information. In many contexts, however, links may also be associated with antagonism, coercion or even outright conflict. This paper aims at clarifying the interplay between these two forces by way of a game-theoretic model of network formation. The approach is closely related to the notion of structural balance - an intuitive, yet entirely ad-hoc concept in the social psychology literature, which originated in a seminal contribution by Heider (1946) and continues to be an active field of study until today.

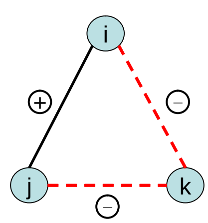
The advantage of the game-theoretic formulation is twofold. First, it shows how the global network properties, as obtained by structural balance, can be derived in a setting where self-interested individuals form their connections strategically. Second, it allows me to address questions concerning relative size and maximum number of groups, which could previously not be answered. Before proceeding, let me briefly explain the main insights of structural balance.

In its most basic form, structural balance assumes a complete network of positive and negative links, with a positive link denoting friendship and a negative one denoting antagonism or enmity. The essential idea is that positively connected nodes have a tendency to match their attitudes relative to third nodes, in order to avoid "cognitive dissonance", or psychological stress.

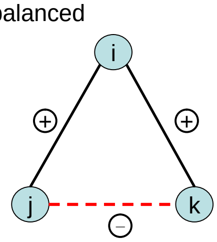
Balanced vs. Unbalanced Triads



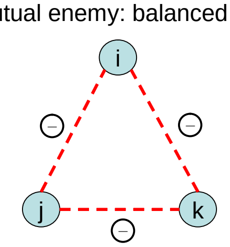
a) i, j, and k are mutual friends: balanced



b) i and j are friends with k as a mutual enemy: balanced



c) i is friends with j and k, but j and k are enemies: unbalanced



d) i, j, and k are mutual enemies: unbalanced, weakly balanced

A signed graph is said to be balanced, if for each triad, either all three links are positive, or one is positive and two are negative. This leaves two triad configurations, which are defined as unbalanced. First, two positive and one negative link. Second, the ambiguous case where all three links are negative. The intuition for the first case - with two positive and one negative link - to be considered unbalanced, is that the node with the two friendship links will either have to choose sides among his friends due to aforementioned "cognitive dissonance", or otherwise the two enemies will have to make peace. The second configuration, where all links are negative, is considered unbalanced, because two of the nodes may have incentives to ally and gang up on their common enemy. Cartwright and Harary (1956) showed that these local properties yield sharp predictions globally. In particular, the only two network configurations, which are balanced are such that either all nodes are friends, or there exist two distinct sets, also called cliques, where nodes in the same set are friends and nodes in different sets sustain antagonistic relationships.[1]

Arguably, the incentives of two nodes to gang up on a common enemy are significantly lower than those of a node having to choose sides among two of his friends. This view has led to the definition of weak structural balance, which assumes triads with only antagonistic relationships among themselves to be balanced (Davis (1967)). A signed graph is weakly balanced, if (and only if) nodes can be divided into distinct sets, such that any two nodes in the same set are friends, while any two nodes in different sets are enemies. In terms of its characterization, the difference between weak structural balance and structural balance in its stronger form, is that weak structural balance allows for more than two antagonistic groups. Put succinctly, while under structural balance "the enemy of my enemy is my friend" must hold, weak structural balance postulates the weaker condition that "the friend of my enemy is my enemy".

I develop a model of strategic network formation, which picks up on the insights obtained from structural balance and accounts for the interplay between friendship or alliances on the one hand and antagonism or enmity on the other. The setup is simple. Each pair of connected agents creates a unit surplus, which may be interpreted as a trade of money, goods, and services.[2] Agents can either extend friendly or antagonistic links, at zero cost. A mutual friendly link creates a friendship or alliance and the surplus is shared in equal parts.

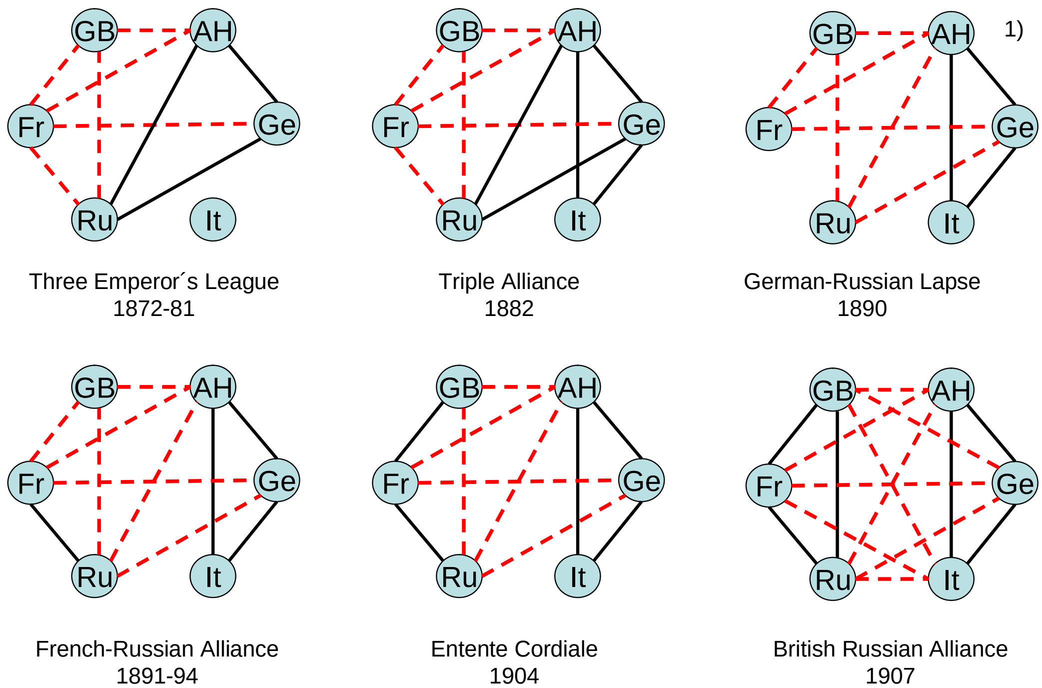
[1] I provide a sketch of the proof. A graph with only positive links is balanced. For a graph with positive and negative links, pick an aribitrary node i and devide the remaining set of nodes into i's friends and i's enemies. All of i's friends must be friends, as otherwise one obtains an unbalanced triad with two positive and one negative link. All of i's enemies must be friends, as otherwise one obtains an unbalanced triads with three negative links. Then, links between i's friends and i's enemies must be negative, as otherwise one obtains an unbalanced triad with two negative and one positive link.

[2] More generally, exchange can be thought of as a general characteristic of social interaction, which is - by and large - based on reciprocation.

If one node extends a friendly link, while the other extends an antagonistic one, then the two nodes enter a coercive relationship and the unit surplus is divided according to a general contest success function. The contest success function is assumed to be strictly increasing in the *ratio* of a node's respective number of allies. Therefore, under a coercive link, nodes with more allies extract payoffs in excess of one half from nodes with fewer allies. A mutual antagonistic link translates into a conflictive relationship and the surplus is lost. The last assumption is, however, not crucial. In fact, any arbitrarily small amount of cost of conflict and a split of the remaining surplus, as implied by the conflict success function, is sufficient for my results to go through. One can then think of the payoffs resulting from a coercive link as settlement in the shadow of conflict.

The characterization of Nash equilibria mirrors the structural properties obtained in the social psychology literature. Equilibrium configurations are such that, either all nodes are friends, or nodes can be divided into distinct sets of *different* size, where nodes within the same set are friends and nodes in different sets sustain coercive relationships. The restriction to asymmetric equilibria is one of the most salient features of the model and at the same time constitutes a departure from previous work on structural balance, which allows for cliques of same size. Outright conflict is never part of an equilibrium, as payoffs under coercion are always positive and either conflict party can profitably deviate by extending a positive link instead.

The range of sustainable equilibrium configurations will depend on the shape of the contest success function, which defines the available coercion or war technology - parametrized by ϕ. Under a coercive relationship, a low value of ϕ is favorable for the node with fewer friends (the potential defender), while a high value of ϕ benefits the node with the higher number of friends (the potential aggressor). I show that, if ϕ is sufficiently high, thereby resulting in a large relative advantage for the aggressor, any configuration with cliques of *different* size (and coercive links between cliques), can be sustained as an equilibrium. This allows for the existence of multiple cliques and the model predictions therefore coincide globally with weak structural balance. Note that I locally (three nodes in isolation) obtain results supporting structural balance in its strong form. Preliminary comparative static results indicate that low values of ϕ - a relative advantage for the defender - force the equilibrium into weakly more asymmetric structures.



Structural balance theory has become an effective tool for the analysis of alliance formation among states. One of the earliest applications is Harary (1961), who examines the rapid shifts of relationships among nations concerning the Middle Eastern situation in 1956 and observes a strong tendency towards balance. Moore (1979) also employs structural balance when explaining the "United States's somewhat surprising support of Pakistan ..." after Bangladesh's separation in 1972. Another, particularly interesting example is provided by Antal, Krapivsky and Redner (2006), who link the formation of alliances in the 19th century - ultimately leading up to WWI - to structural balance. The accompanying graph is depicted above, where alliances are denoted by straight black lines and antagonistic relationships by dashed red lines. Note how the network gradually moves towards a structurally balanced state. A point to be made here is that, although balance appears to be a natural outcome, its implications need not be positive.

Interaction patterns of individuals have, of course, also been examined for structural balance properties. Szell, Lambiotte and Thurner (2010) analyze a vast data-set from a multiplayer online game called Pardus, encompassing more than 300.000 players. The game allows for six types of interactions, of which some have a positive (friendship, communication, trade) and others have a negative association (hostility, aggression, punishment). The authors find strong support for structural balance, favoring its weak specification. Furthermore, positive links display far higher clustering than negative ones and positive links are highly reciprocal, while negative links are not. This is much in line with the results obtained in this paper. Recent research in sociology has examined the evolution of signed network relations. Doreian and Mrvar (1996) and Doreian and Krackhardt (2001) are two such empirical studies. In both cases a perfectly balanced state is not found, but a movement towards balance is evident.

This paper relates to a strand of the economics literature, which allows for appropriation and conflict, recognizing that property rights may not always be perfectly and costlessly enforced. Conflict is modelled in terms of a contest success function (Tullock (1967, 1980)) and Hirshleifer (1989)), where an agent's probability of winning is a function of the resources available for arming. Open conflict, however, does not have to take place, and may instead be used as an instrument for bargaining. Part of this research focuses on coalition and group formation among states in the context of distributional conflict (Wärneryd (1998) and Esteban and Sákovics (2003)). Group structures are, based on the notion of farsighted stability[3] (Chwe (1994)), shown to be symmetric. Jordan (1996) considers coalitional games, where coalitions with more wealth can pillage the wealth of poorer coalitions at no cost. The farsighted core allocations are again symmetric. These findings are in sharp contrast to the results obtained here.

My paper also contributes to the theory of network formation, which has been an active area of research in recent years. See for example Aumann and Myerson (1988), Bala and Goyal (2000) and Jackson and Wolinsky (1996). The so called co-author model in Jackson and Wolinsky (1996) and Chwe (2003) obtain similar results in terms of equilibrium structures. Two recent papers, which also feature conflict success functions in a network setting are Goyal and Virgier (2010) and Franke and Öztürk (2009), both with a different focus from mine. Goyal and Virgier (2010) consider a design problem and ask how to optimally structure networks, so that they are robust to attacks in the face of an adversary. Franke and Öztürk (2009), in turn, model a setting where agents are embedded in a network of bilateral conflicts. The authors are concerned with conflict intensity on a fixed network and considerations of link and alliance formation are entirely absent. To the best of my knowledge, the model presented here is the first to incorporate friendship (or alliance) and antagonism (or coercion) in a network formation context.

The remaining part of the paper is organized as follows: Section 2 introduces the model, Section 3 provides an intuitive account of the main results and Section 4 concludes. All formal proofs are confined to the Appendix.

[3] Under farsighted stability agents consider the ultimate outcome of a potential deviation.

2 A Simple Model of Friends and Enemies

Let $N = (1, 2, ..., n)$ be the set of ex-ante identical agents, with $n \geq 3$. A strategy for $i \in N$ is defined as a row vector $\mathbf{g}_i = (g_{i,1}, g_{i,2}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$, where $g_{i,j} \in \{-1, 1\}$ for each $j \in N \backslash \{i\}$. i is said to extend a positive or friendly link to j if $g_{i,j} = 1$ and a negative or coercive link if $g_{i,j} = -1$. The set of strategies of agent i is defined by G_i and the strategy space by $G = G_1 \times ... \times G_n$. The resulting network of relationships is written as $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. Define the undirected network $\bar{\mathbf{g}}$ in the following way. The link between node i and j in the undirected network is positive, if both directed links are positive, so that $\bar{g}_{i,j} = 1$ if $g_{i,j} = g_{j,i} = 1$. The link in the undirected network is negative, if one of the two directed links is positive, while the other is negative, so that $\bar{g}_{i,j} = -1$ if $g_{i,j} * g_{j,i} = -1$. If both nodes extend a negative link to each other, then no link is established in the undirected network, so that $\bar{g}_{i,j} = 0$ if $g_{i,j} = g_{j,i} = -1$. Given a network $\mathbf{g}$, $\mathbf{g} + g^+_{i,j}$ and $\mathbf{g} + g^-_{ij}$ have the following interpretation. When $g_{i,j} = -1$ in $\mathbf{g}$, $\mathbf{g} + g^+_{i,j}$ changes the link $g_{i,j} = -1$ into $g_{i,j} = 1$, while if $g_{i,j} = 1$ in $\mathbf{g}$, then $\mathbf{g} + g^+_{i,j} = \mathbf{g}$. Similarly, if $g_{i,j} = 1$ in $\mathbf{g}$, $\mathbf{g} + g^-_{i,j}$ changes the link $g_{i,j} = 1$ into $g_{i,j} = -1$, while if $g_{i,j} = -1$ in $\mathbf{g}$, then $\mathbf{g} + g^-_{i,j} = \mathbf{g}$.

Define the following sets: $N^+_i(\bar{\mathbf{g}}) = \{k \in N \mid g_{i,k} = 1 \wedge g_{k,i} = 1\}$ is the set of nodes to which node i has a positive link in the undirected network $\bar{\mathbf{g}}$. $N^-_i(\bar{\mathbf{g}}) = \{k \in N \mid g_{i,k} * g_{k,i} = -1\}$ is the set of nodes to which node i has a negative link in the undirected network $\bar{\mathbf{g}}$, while $N^0_i(\bar{\mathbf{g}}) = \{k \in N \mid g_{i,k} = -1 \wedge g_{k,i} = -1\}$ is the set of nodes to which node i has no link in the undirected network $\bar{\mathbf{g}}$. The corresponding cardinalities are denoted by $\eta^+_i(\bar{\mathbf{g}}) =\mid N^+_i(\bar{\mathbf{g}}) \mid$, $\eta^-_i(\bar{\mathbf{g}}) =\mid N^-_i(\bar{\mathbf{g}}) \mid$ and $\eta^0_i(\bar{\mathbf{g}}) =\mid N^0_i(\bar{\mathbf{g}}) \mid$. Call $P_k(\bar{\mathbf{g}}) = \{i \in N \mid \eta^+_i(\bar{\mathbf{g}}) = k\}$ the set of nodes with k positive links in the undirected network $\bar{\mathbf{g}}$.

The payoff function $\Pi_i = G \rightarrow \mathbb{R}^+$ is given by

$$\Pi_i(\mathbf{g}) = \frac{\eta^+_i(\bar{\mathbf{g}})}{2} + \sum_{q \in N^-_i(\bar{\mathbf{g}})} \frac{(\eta^+_i(\bar{\mathbf{g}})+1)^\phi}{(\eta^+_i(\bar{\mathbf{g}})+1)^\phi + (\eta^+_q(\bar{\mathbf{g}})+1)^\phi},$$

with $\phi > 0$. Links in the undirected network are interpreted in the following way. $\bar{g}_{i,j} = 1$ is created by a reciprocated positive link and establishes an alliance or friendship between i and j. The surplus of one is shared in equal parts. $\bar{g}_{i,j} = -1$ stems from one friendly and one coercive link and denotes a coercive relationship. Under a coercive link, a node with more allies can extract payoffs in excess of $\frac{1}{2}$ from a node with fewer allies. The exact extraction value is determined by a general contest success function, explained in more detail below. $\bar{g}_{i,j} = 0$ results from a mutual coercive link and can be thought of a link of conflict, where no exchange occurs and surplus is lost.

I assume that the coercive strength of node i is determined by $\eta_i^+(\bar{\mathbf{g}}) + 1$. In the presence of a coercive link, the relative shares of the surplus of one - denoted here by p_i and p_j - are determined by the *ratio* of i and j's coercive strength. The contest success function is parametrized by ϕ, such that

$$\frac{p_i}{p_j} = \left(\frac{\eta_i^+(\bar{\mathbf{g}}) + 1}{\eta_j^+(\bar{\mathbf{g}}) + 1}\right)^{\phi}.$$

As $p_i + p_j = 1$, I can write

$$p_i = \frac{(\eta_i^+(\bar{\mathbf{g}}) + 1)^{\phi}}{(\eta_i^+(\bar{\mathbf{g}}) + 1)^{\phi} + (\eta_j^+(\bar{\mathbf{g}}) + 1)^{\phi}},$$

which corresponds to the second term in the payoff function Π_i. Note that the contest success function is concave for $0 \leq \phi \leq 1$ and s-shaped for $\phi > 1$, with an inflection point at $\eta_i^+(\bar{\mathbf{g}}) = \eta_j^+(\bar{\mathbf{g}})$.

Definition 1: A *clique* is a set of nodes $C_k(\bar{\mathbf{g}}) \subseteq N$ such that $\bar{\mathbf{g}}_{i,j} = 1\ \forall i, j \in C_k(\bar{\mathbf{g}})$. Call a clique *maximal* and denote with $C_k^m(\bar{\mathbf{g}})$, if for any $l \in N \wedge l \notin C_k^m(\bar{\mathbf{g}})$, $C_k^m(\bar{\mathbf{g}}) \cup \{l\}$ is not a clique.

The equilibrium concept used is Nash Equilibrium. A strategy profile $\mathbf{g}^*$ is a Nash Equilibrium (NE) iff

$$\Pi_i(\mathbf{g}_i^*, \mathbf{g}_{-i}^*) \geq \Pi_i(\mathbf{g}_i, \mathbf{g}_{-i}^*), \forall \mathbf{g}_i \in G_i, \forall i \in N.$$

Throughout the paper, the network after a proposed deviation is denoted with $\bar{\mathbf{g}}^{*\prime}$.

There is one remark I would like to make concerning the assumption that surplus is shared in equal parts under a friendly link. One may object that this split is arbitrary and an alternative model specification could, for example, allow for two different values of ϕ: ϕ_f for the division of surplus under a friendly link and ϕ_c for the division of surplus under a coercive link. Preliminary results indicate that making the sensible assumption that $\phi_f > \phi_c$, thereby allocating a greater portion of the shared surplus to a friend (as compared to an enemy), is sufficient for all my results to hold.

3 The Friend of My Enemy is My Enemy

The aim of the analysis is to obtain a full characterization of Nash equilibria. This section provides fairly detailed sketches of the proofs, while the formal arguments are relegated to the Appendix. In Lemma 1 I first show that outright conflict is never an equilibrium. Lemma 2 then proves that in a coercive relationship, it must always be the node with the higher number of friends extending the negative link. Proposition 1 and Proposition 2 provide existence for my main result in Proposition 3: Nash equilibria are characterized by either a state of utopia, where all nodes are friends, or the undirected network can be partitioned into sets of nodes of *different* size, where nodes in the same set are friends and nodes in different sets sustain coercive relationships. Lemma 3 shows that in any Nash equilibrium, nodes with the same number of friends will also be friends with each other. This observation drives most of the results.

Lemma 1: In any NE $\mathbf{g}^*$, $\nexists \bar{g}_{i,j} = 0$ for some $i, j \in N$.

Proof. See the Appendix.

Lemma 1 states that a conflict link can never be part of a Nash equilibrium. The reason for this is that both nodes involved in the link have an incentive to deviate. By extending a friendly link, both nodes obtain a positive payoff, compared to a payoff of zero under conflict, while payoffs from links to all other nodes remain the same (the deviation does not alter the total number of friends).

Lemma 2: *In any* NE $\bar{g}^*$, *if* $\exists i, j \in N : \bar{g}^*_{i,j} = -1$ *with* $\eta^+_i(\bar{\mathbf{g}}^*) < \eta^+_j(\bar{\mathbf{g}}^*)$ *then* $g^*_{i,j} = 1$.

Proof. See the Appendix.

In Lemma 2 I prove that in equilibrium, for all competitive links in place in the undirected network, it must be the node with more friends extending the directed antagonistic link. This is easy to see, as otherwise the node with fewer friends can profitably deviate by reciprocating the friendly link, thereby increasing his payoff from this specific link to $\frac{1}{2}$. Moreover, he will increase payoffs on all his coercive links, while all friendly links continue to yield him a payoff of $\frac{1}{2}$.

Proposition 1: $\forall \phi > 0 \; \exists NE \; g^* : \bar{g}^*_{i,j} = 1 \; \forall i, j \in N.$

Proof. See the Appendix.

Proposition 1 shows that a state of utopia, where everyone is friends with everyone, is a Nash equilibrium for any value of ϕ. The intuition for this is that no node can unilaterally extract additional payoffs from nodes, who have the same number of friends prior to a potential deviation.

Proposition 2: $\forall \phi > 0 \; \exists NE \; \bar{g}^* : \exists k : \bar{g}^*_{i,k} = -1 \; \forall i \in N \backslash \{k\} \wedge \bar{g}^*_{i,j} = 1 \; \forall i, j \in N \backslash \{k\}.$

Proof. See the Appendix.

Proposition 2 provides an existence result for Proposition 3 and states that all nodes being friends with each other, except for one (who in turn is enemies with all remaining nodes), is a Nash equilibrium for any value of ϕ.[4] Call the node without friends node k. There are then at least two other nodes, which I call i and j, who are friends with each other. From Lemma 2 I know that any configuration where k extends competitive links, while any of the remaining nodes extend friendly ones, can not be a Nash equilibrium. Assume therefore that k extends friendly links to all other nodes. A deviation for node k then consists of substituting a subset of these friendly links for competitive ones. This, however, is not profitable, as k then receives a payoff of zero on the resulting conflict links, while payoffs from links to all other nodes remain unchanged. For node i, there are three different types of deviations to be considered and I will show for each of them that no profitable deviation exists. First, i will not find it profitable to extend a friendly link to k, as he will loose out on his payoffs in excess of $\frac{1}{2}$, due to $\eta^+_k(\bar{\mathbf{g}}^*) < \eta^+_i(\bar{\mathbf{g}}^*)$. Second, i is not able to extract additional payoffs by extending competitive links to any subset of $N \backslash \{i, k\}$, as then node i will have weakly less friends than any node in $N \backslash \{i, k\}$. In fact, any such deviation will strictly decrease i's payoffs, as i will have less friends after the deviation and can therefore also extract less from node k. Third, a combination of the above two, that is, reciprocating k's positive link and extending one or more negative link(s) to any of the remaining nodes. For $n = 3$ it is easy to see that there does not exist a profitable deviation, as payoffs remain the same. For $n \geq 4$ payoffs will strictly decrease. I discern two cases. If the deviation involves only one additional negative link, i's payoffs decrease, as node k will have less friends prior to the proposed deviation than j after it. If the deviation involves more than one additional negative link, payoffs will be lower even further, because i can now not only extract less from

[4] In fact, for $n \geq 5$, it can be shown that there always exists an equilibrium configuration where 2 nodes consitute the smallest clique. This result is not yet incorporated in the current version of the paper.

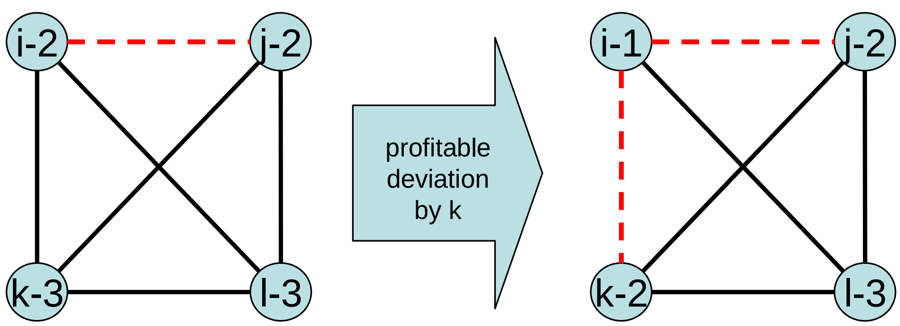
k, but he will also receive payoffs of less than $\frac{1}{2}$ from all other nodes to which he extends negative links.

Lemma 3: *In any NE* $\bar{g}^*$, *if* $\eta_i^+(\bar{g}^*) = \eta_j^+(\bar{g}^*) \Longrightarrow \bar{g}^*_{i,j} = 1$.

Proof. See the Appendix.

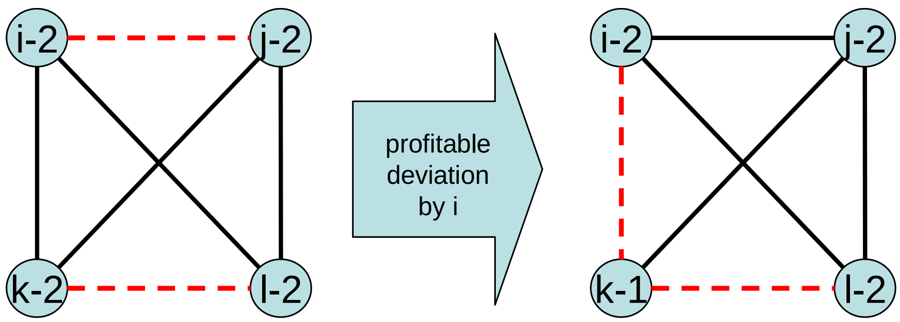
Lemma 3 states that, in any Nash equilibrium, nodes who have the same number of friends are also friends with each other. I will use this result extensively for the characterization of equilibrium. Note that Lemma 3 already rules out any configuration with cliques of equal size as part of a Nash equilibrium. The proof is by contradiction. Assume that there are two nodes, i and j, such that they have the same number of friends, but sustain a coercive relationship. Without loss of generality, assume that node i extends the negative link, while node j extends the positive one. I discern 4 cases, which cover all possible networks, each yielding a contradiction with my initial assumption. In the graphs below, black solid lines signify positive, while dashed red lines stand for negative links. The number next to the node indicates the number of friends of that node.

Lemma 3 - Case 1



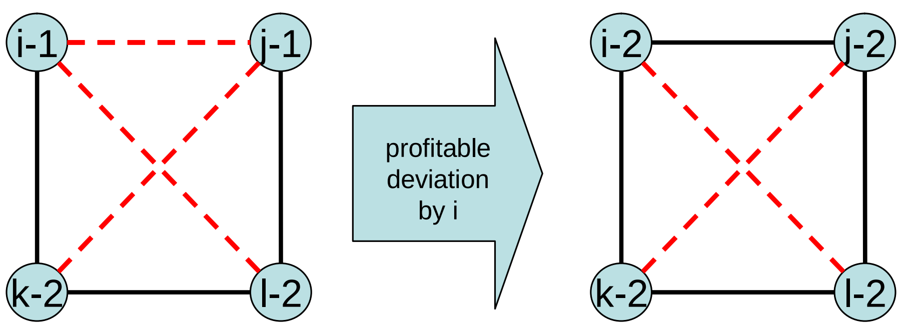
Case 1: The link between i and j is the only negative one in the undirected network, all other links are positive. This can not be an equilibrium, as any node k can profitably deviate by extending a negative link to either i or j. Node k will then extort payoffs in excess of one half from that particular link, while payoffs on all his other links remain unchanged. Note that in a triad, this is analogous to saying that a configuration with two positive links and one negative is not balanced.

Lemma 3 - Case 2



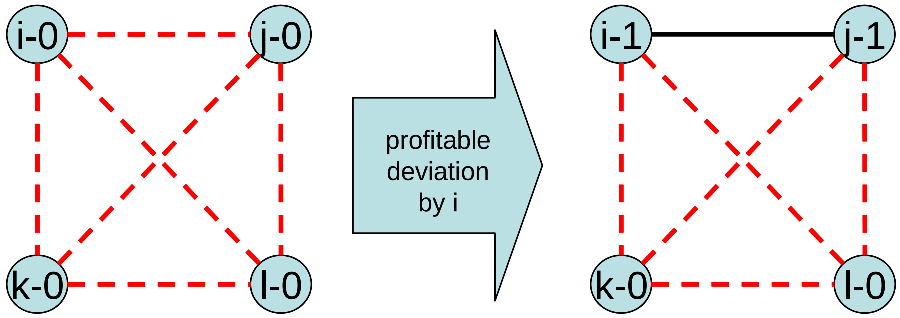
Case 2: There are at least two competitive links in the undirected network and i and j are only involved in the one between themselves (and have $N-2$ friends). There are then two nodes, k and l, which are involved in at least one negative link. Therefore, k has at most as many friends as i prior to the deviation and i can profitably deviate by extending a positive link to j and a negative one to k. i then extorts payoffs in excess of $\frac{1}{2}$ from k, while payoffs from all remaining links stay the same.

Lemma 3 - Case 3



Case 3: Nodes i and j extend both, positive and negative links. i can then deviate profitably by extending a friendly link to j, as i will be able to extract higher payoffs from each of his negative links, while payoffs from all positive links remain constant.

Lemma 3 - Case 4



Case 4: All links in the undirected network are negative. Node i receives a payoff of $(N-1)*\frac{1}{2}$ and can deviate profitably by extending a positive link to j. He then continues to receive a payoff of $\frac{1}{2}$ from his link with j, but receives a payoff in excess of $\frac{1}{2}$ from all other nodes.

Proposition 3: *Any NE $\bar{g}^*$ s.t. $\exists s,t \in N : \eta_s^+(\bar{g}^*) \neq \eta_t^+(\bar{g}^*)$, $\bar{g}^*$ can be partitioned into maximal cliques of different size with $\bar{g}^*_{i,j} = -1$, if $i \in C_k^m(\bar{g}^*)$ and $j \in C_l^m(\bar{g}^*)$.*

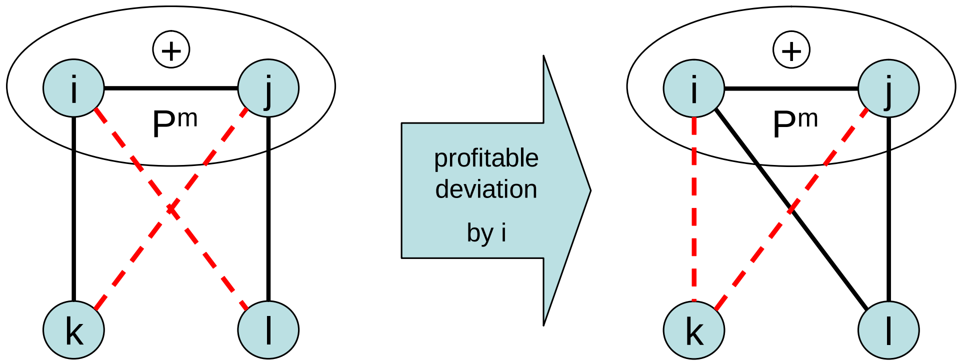
Proof. See the Appendix.

Proposition 3 provides a characterization of equilibrium. It states that, in any Nash equilibrium, such that there are at least two nodes with different numbers of friends, the undirected network can be partitioned into maximal cliques of friends of *different* size. Nodes in larger cliques (with more friends), extend negative links to nodes in smaller cliques (with fewer friends).

The proof for Proposition 3 is by induction. First, I rank the sets of nodes with the same number of friends, $P_k(\bar{\mathbf{g}}^*)$, by their subscripts, where the subscript stands for the number of friends of a node in that set. I call the set of nodes with the highest number of friends $P^m(\bar{\mathbf{g}}^*)$, the set with the second highest number of friends $P^{m-1}(\bar{\mathbf{g}}^*)$ and proceed in this way until the set of nodes with the lowest number of friends. The idea of the proof is to show that the cardinality $\mid P_k(\bar{\mathbf{g}}^*) \mid = k+1$, which - together with Lemma 3 - already implies the result. Remember that by Lemma 2 all nodes, which are not in $P^m(\bar{\mathbf{g}}^*)$, extend positive links to all nodes in $P^m(\bar{\mathbf{g}}^*)$. Therefore, the sign of a link in the undirected network $\bar{\mathbf{g}}^*$ is determined by the sign of the directed link, which the node with more friends extends to the node with fewer friends.

Base Case: In the base case, I prove in four steps that $\mid P^m_x(\bar{\mathbf{g}}^*) \mid= x+1$.

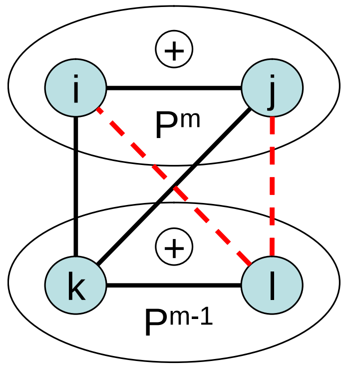
Proposition 3 - Step 1



Step 1: Here I show that in any Nash equilibrium $\mathbf{g}^*$, any two nodes $i, j \in P^m(\bar{\mathbf{g}}^*)$ must be linked to all other nodes in the same way. That is, all nodes to which i has a friendly link in the undirected network $\bar{\mathbf{g}}^*$, j will have a friendly link as well, while all nodes to which i has a antagonistic link in the undirected network $\bar{\mathbf{g}}^*$, j will likewise have a antagonistic link. Assume the contrary. As i and j are both in $P^m(\bar{\mathbf{g}}^*)$, i and j have the same number of friends, $\eta^+_i(\bar{g}^*) = \eta^+_j(\bar{g}^*)$. But then, there must exist a pair of nodes, k and l, such that i has a friendly link with k and an antagonistic one with l, while j has a friendly link with l and an antagonistic one with k. Such a configuration is depicted in the graph above. Note that k and l are not in $P^m(\bar{\mathbf{g}}^*)$, as by Lemma 3 both, i and j, are linked positively to all nodes in $P^m(\bar{\mathbf{g}}^*)$. Furthermore, assume w.l.o.g. that l has at least as many friends as k. This configuration can not be an equilibrium, as node i can profitably deviate by creating a friendly link with l and an antagonistic one with k. i's number of friends remains the same, while k has now strictly less friends than l had before the deviation, i.e. $\eta^+_k(\bar{g}^{*\prime}) < \eta^+_l(\bar{g}^*)$. Therefore, i can extract more from k than he was previously able to extract from l.

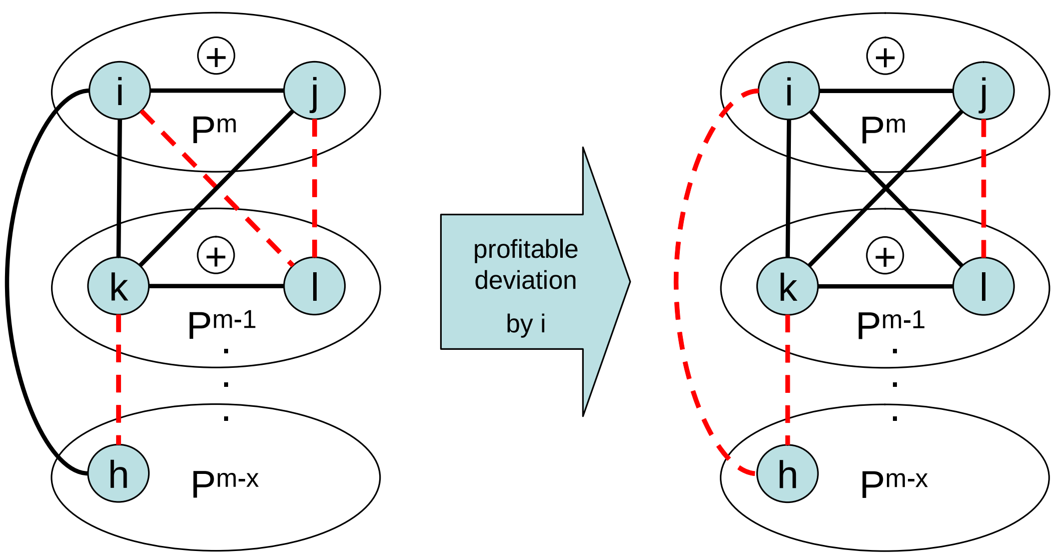
Step 2: I prove that any configuration, which is not of the following two types, can not be a Nash equilibrium $\mathbf{g}^*$. First, all nodes $i \in P^m(\bar{\mathbf{g}}^*)$ have positive links with all nodes $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ in the undirected network $\bar{\mathbf{g}}^*$. Second, all nodes $i \in P^m(\bar{\mathbf{g}}^*)$ have negative links with all nodes $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ in the undirected network $\bar{\mathbf{g}}^*$. I assume to the contrary that there are positive and negative links between nodes in $P^m(\bar{\mathbf{g}}^*)$ and $P^{m-1}(\bar{\mathbf{g}}^*)$ and discern two cases.

Proposition 3 - Step 2, Case 1



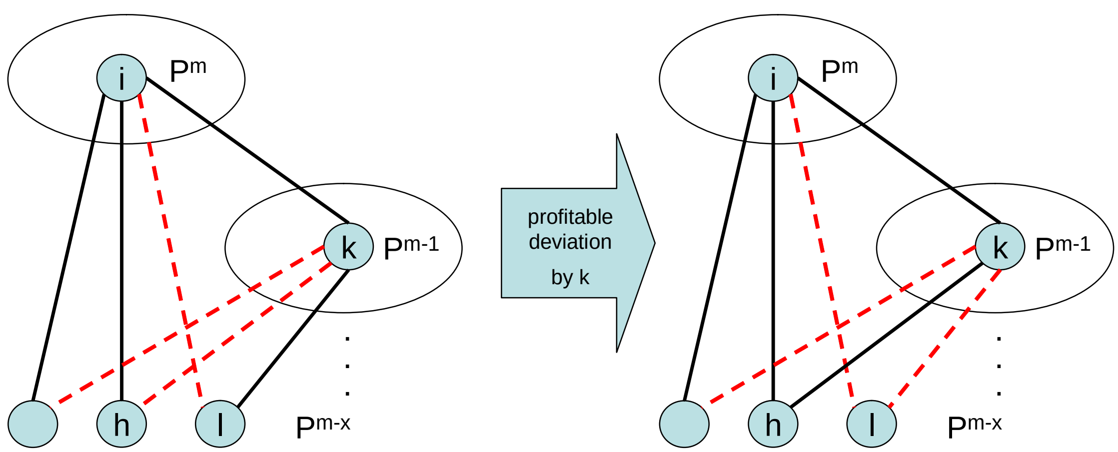
First, there does not exist a set $P^{m-2}(\bar{\mathbf{g}}^*)$. I know from Step 1 that nodes in $P^m(\bar{\mathbf{g}}^*)$ are linked to all remaining nodes in the same way. Assume now, w.l.o.g., that $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ is linked positively with all $i \in P^m(\bar{\mathbf{g}}^*)$, while $l \in P^{m-1}(\bar{\mathbf{g}}^*)$ is linked negatively with all $i \in P^m(\bar{\mathbf{g}}^*)$ in the undirected network $\bar{\mathbf{g}}^*$. This can not be an equilibrium, as from Lemma 3 I know that all nodes in $P^{m-1}(\bar{\mathbf{g}}^*)$ are connected to each other in a positive way and therefore node k has more friends than node l. But then k and l can not both be in $P^{m-1}(\bar{\mathbf{g}}^*)$.

Proposition 3 - Step 2, Case 2



Second, there exists a set $P^{m-2}(\bar{\mathbf{g}}^*)$. Recall that any two nodes in $P^m(\bar{\mathbf{g}}^*)$ are linked to all remaining nodes in the same way. Assume again that $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ is linked positively with all $i \in P^m(\bar{\mathbf{g}}^*)$, while $l \in P^{m-1}(\bar{\mathbf{g}}^*)$ is linked negatively with all $i \in P^m(\bar{\mathbf{g}}^*)$ in the undirected network $\bar{\mathbf{g}}^*$. But then k is linked positively to all nodes in $P^{m-1}(\bar{\mathbf{g}}^*)$ and to all nodes in $P^m(\bar{\mathbf{g}}^*)$, while for all $i \in P^m(\bar{\mathbf{g}}^*)$ there exists a node $l \in P^{m-1}(\bar{\mathbf{g}}^*)$, for which the link in the undirected network $\bar{\mathbf{g}}^*$ is negative. In order for $i \in P^m(\bar{\mathbf{g}}^*)$ to have more friends than $k \in P^{m-1}(\bar{\mathbf{g}}^*)$, there must exist a node $h \in P^{m-x}(\bar{\mathbf{g}}^*)$, with $x \geq 2$, such that all $i \in P^m(\bar{\mathbf{g}}^*)$ are linked positively to $h \in P^{m-x}(\bar{\mathbf{g}}^*)$, while node k is linked negatively to $h \in P^{m-x}(\bar{\mathbf{g}}^*)$ in the undirected network $\bar{\mathbf{g}}^*$. But this can not be a Nash equilibrium, as then i can profitably deviate by extending a negative link to $h \in P^{m-x}(\bar{\mathbf{g}}^*)$, with $x \geq 2$, and a positive one to $l \in P^{m-1}(\bar{\mathbf{g}}^*)$. This deviation is profitable, as h has less friends after the deviation than l had prior to it.

Proposition 3 - Step 3 with cardinality of l's friends smaller/equal h's



Step 3: I show that, if all nodes $i \in P^m(\bar{\mathbf{g}}^*)$ are linked positively to all $k \in P^{m-1}(\bar{\mathbf{g}}^*)$, then it must be that the set of friends of node $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ are a strict subset of the friends of node $i \in P^m(\bar{\mathbf{g}}^*)$. The underlying argument is very similar to what I already used in the first step.

First, note that there must exist a node h, which is neither in $P^m(\bar{\mathbf{g}}^*)$, nor in $P^{m-1}(\bar{\mathbf{g}}^*)$, such that h is connected positively with all $i \in P^m(\bar{\mathbf{g}}^*)$ and there exists some $k \in P^{m-1}(\bar{\mathbf{g}}^*)$, such that h and k are connected negatively in the undirected network $\bar{\mathbf{g}}^*$. Otherwise k would have at least as many friends as i, contradicting the assumption that $i \in P^m(\bar{\mathbf{g}}^*)$ and $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. Assume now that k's friends are not a strict subset of i's friends. Then there must exist a node l, which is neither in $P^m(\bar{\mathbf{g}}^*)$, nor in $P^{m-1}(\bar{\mathbf{g}}^*)$, such that l is linked negatively with all $i \in P^m(\bar{\mathbf{g}}^*)$ and positively with some node $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. This is depicted in the graph above. I can discern two cases. First, h has at most as many friends as l, that is, $\eta_h^+(\bar{\mathbf{g}}^*) \leq \eta_l^+(\bar{\mathbf{g}}^*)$. Then, i can deviate profitably by extending a positive link to l and a negative one to h - the deviation shown in the above graph. If l has at most as many friends as h, $\eta_l^+(\bar{\mathbf{g}}^*) \leq \eta_h^+(\bar{\mathbf{g}}^*)$, then node k can deviate profitably by extending a friendly link to h and a competitive one to l. Both cases yield a contradiction.

Step 4: I can now at last establish that, in any Nash equilibrium $\mathbf{g}^*$, links between all $i \in P^m(\bar{\mathbf{g}}^*)$ and all $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ must be negative in the undirected network $\bar{\mathbf{g}}^*$. Assume the contrary. If there does not exist a set $P^{m-2}(\bar{\mathbf{g}}^*)$, it is easy to see that the statement must hold, as otherwise i and k would have the same number of friends, i.e. $\eta_i^+(\bar{\mathbf{g}}) = \eta_k^+(\bar{\mathbf{g}})$, yielding an immediate contradiction. For the case where there exists a set $P^{m-2}(\bar{\mathbf{g}}^*)$, I can distinguish two cases.

First, payoffs of $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ are at least as high as payoffs of $i \in P^m(\bar{\mathbf{g}}^*) : \Pi_k(g^*) \geq \Pi_i(g^*)$. Recall from Step 3 that k's friends are a subset of i's friends and consider a deviation where i imitates k's strategy. Under the resulting undirected network after the deviation $\bar{\mathbf{g}}^{*\prime}$, i and j are linked in the same way and i obtains higher payoffs in $\bar{\mathbf{g}}^{*\prime}$ than k did in $\bar{\mathbf{g}}^*$. To see this, note that no enemy of k has more friends after the deviation and at least one has fewer friends. Therefore, $\Pi_i(g^{*\prime}) > \Pi_k(g^*) \geq \Pi_i(g^*)$ and i can profitably deviate.

Second, payoffs of $i \in P^m(\bar{\mathbf{g}}^*)$ are higher than payoffs of $k \in P^{m-1}(\bar{\mathbf{g}}^*) : \Pi_i(g^*) > \Pi_k(g^*)$. Now k can profitably deviate by imitating i's strategy and payoffs of k after the deviation are equal to i's, i.e. $\Pi_i(g^*) = \Pi_k(g^{*\prime}) > \Pi_k(g^*)$. Note that from Step 3 - that k's friends are a subset of i's friends - I know that k imitating i's strategy entails linking positively to all nodes to which i is linked positively, while k is linked negatively in $\bar{\mathbf{g}}^*$. Then, k and i are linked in the same way in the undirected network after the deviation $\bar{\mathbf{g}}^{*\prime}$, while all nodes to which i extends antagonistic links have the same number of friends in $\bar{\mathbf{g}}^*$ and $\bar{\mathbf{g}}^{*\prime}$. Therefore, k's payoffs after the deviation are equal to i's payoffs prior to it, resulting in a profitable deviation.

I have now shown that all links between all $i \in P^m(\bar{\mathbf{g}}^*)$ and all $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ must be negative. But then, all links between all $i \in P^m(\bar{\mathbf{g}}^*)$ and all $h \in P^{m-x}(\bar{\mathbf{g}}^*)$, for $x \geq 2$, must be negative as well. Assume the contrary. Then i could profitably deviate by extending a positive link to k and a negative one to h. As all nodes in $i \in P_x^m(\bar{\mathbf{g}}^*)$ (where the subscript indicates the number of friends of a node in set $P_x^m(\bar{\mathbf{g}}^*)$) extend negative links to all other nodes $k \notin P_x^m(\bar{\mathbf{g}}^*)$, it must be that $\mid P_x^m(\bar{\mathbf{g}}^*) \mid = x + 1$. This concludes the description of the base case of the proof.

Inductive Step: For the inductive steps I define the set $\tilde{P}^r(\bar{\mathbf{g}}^*) = \{P^m(\bar{\mathbf{g}}^*), P^{m-1}(\bar{\mathbf{g}}^*), ..., P^{m-r}(\bar{\mathbf{g}}^*)\}$, which contains all nodes that have the $r+1$ highest number of friends. The base case showed that the statement of Propositions 2 holds for $\tilde{P}^0(\bar{\mathbf{g}}^*) = P^m(\bar{\mathbf{g}}^*)$. In the inductive step I assume that the statement holds for the set $\tilde{P}^r(\bar{\mathbf{g}}^*)$ and then show that it must also hold for $\tilde{P}^{r+1}(\bar{\mathbf{g}}^*)$. Note first that, given the statement holds for $\tilde{P}^r(\bar{\mathbf{g}}^*)$, all nodes $q \in \tilde{P}^r(\bar{\mathbf{g}}^*)$ extend negative links to all nodes $z \notin \tilde{P}^r(\bar{\mathbf{g}}^*)$. By Lemma 1 I know that in any Nash equilibrium $\mathbf{g}^*$, all $z \notin \tilde{P}^r(\bar{\mathbf{g}}^*)$ extend friendly links to $q \in \tilde{P}^r(\bar{\mathbf{g}}^*)$. Assuming that the statement holds for $\tilde{P}^r(\bar{\mathbf{g}}^*)$, I can then relabel $P^m(\bar{\mathbf{g}}^*)$ with $P^{m-(r+1)}(\bar{\mathbf{g}}^*)$ and $P^{m-1}(\bar{\mathbf{g}}^*)$ with $P^{m-(r+2)}(\bar{\mathbf{g}}^*)$ and repeat the Steps 1 through 4 from the base case to establish that all links between all $h \in P^{m-(r+1)}(\bar{\mathbf{g}}^*)$ and all $z \in P^{m-(r+2)}(\bar{\mathbf{g}}^*)$ are negative and therefore $h \in P^{m-(r+1)}(\bar{\mathbf{g}}^*)$ is linked negatively to all nodes $w \notin P^{m-(r+1)}(\bar{\mathbf{g}}^*)$. $\mid P_y^{m-(r+1)}(\bar{\mathbf{g}}^*) \mid = y+1$ and above statement holds for $\tilde{P}^{r+1}(\bar{\mathbf{g}}^*)$. This concludes the equilibrium characterization.

Proposition 4: If $\bar{\mathbf{g}}$ can be partitioned into maximal cliques of different size, with $g_{i,j} = -1$ and $g_{j,i} = 1$ for $i \in P^{m-x}(\bar{\mathbf{g}}^*)$ and $j \in P^{m-y}(\bar{\mathbf{g}}^*)$ with $y > x$, then $\exists \tilde{\phi} : \forall \phi > \tilde{\phi}$, $\bar{\mathbf{g}}$ is a Nash equilibrium $\bar{\mathbf{g}}^*$.

Proof. See the Appendix.

Proposition 4 states that there exists a value $\tilde{\phi}$, such that for all $\phi > \tilde{\phi}$ any configuration that is in accordance with Proposition 3 can be sustained as a Nash equilibrium. The intuition is simple. As ϕ becomes sufficiently large, payoffs from a coercive link (to a node with fewer allies) approach 1. Therefore, in the limit, payoffs of a deviation consisting of extending positive links to z enemies with fewer friends yields a deviation payoff of $-\frac{z}{2}$. As payoffs are continuous in ϕ, the described threshold value $\tilde{\phi}$ exists.

4 Conclusion and Future Work

In this paper I present a simple model of network formation, where agents enter into positive (friendship or alliances) and negative relationships (antagonism or coercion). The coercive power of an agent, relative to another agent, is determined by the ratio of their respective allies. That is to say, an agent with more allies may exploit another agent with fewer allies under a coercive relationship. There are three main insights to be drawn.

First, the model shows how in this context self-interested behavior of agents yield the following sharp structural predictions under Nash equilibrium. Either all nodes are friends, or cliques of allies emerge, with antagonistic relationships among distinct cliques. This mirrors results on signed networks obtained in the structural balance literature of social psychology.

Second, cliques are of *different* size. This result is interesting, because it constitutes a departure from structural balance, where balanced outcomes allow for cliques of same size. It is also in contrast to models of alliance and group formation in the literature of economics of conflict and coalitional games of pillage, where group structures are shown to be symmetric.

Third, the game theoretic approach allows me to address questions concerning the relative size and number of cliques, which could previously not be answered. A conflict technology, which is favorable for the node with a higher number of allies, permits cliques to be of similar (yet different) size, while a conflict technology relatively favouring the node with fewer allies, forces equilibrium configurations into more asymmetrical structures.

This paper is the first to incorporate friendly and antagonistic links in a game-theoretic model of network formation. There are many interesting questions yet to be addressed in the future and in the following I list a few. The so called trade-off between guns vs. butter is not part of the current specification and one could, for example, allow for investment into production on the one hand and arming (the coercive technology) on the other. Moreover, it might be worthwhile to introduce incomplete or asymmetric information, as outright conflict may then arise as part of an equilibrium. It seems natural to add a cost of linking and/or a cost to coercion. A slightly different model setup might also be promising, where surplus is not generated by the link, but a resource specific to the agent.

5 Appendix

Lemma 1: In any NE $\mathbf{g}^*$, $\nexists \bar{g}_{i,j} = 0$ for some $i, j \in N$.

Proof. Assume there exist two nodes i and j such that $\bar{g}_{i,j} = 0$. To see that this can not be a Nash equilibrium, note that i and j obtain a payoff of zero from $\bar{g}_{i,j} = 0$. But $\frac{(\eta_i^+(\bar{\mathbf{g}})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}})+1)^\phi+(\eta_j^+(\bar{\mathbf{g}})+1)^\phi} > 0$ for $\phi > 0$ and i can deviate profitably with the following strategy: $g_i + g_{i,j}^+$.
Q.E.D

Lemma 2: *In any NE $\bar{g}^*$, if $\exists i, j \in N : \bar{g}_{i,j}^* = -1$ with $\eta_i^+(\bar{\mathbf{g}}^*) < \eta_j^+(\bar{\mathbf{g}}^*)$ then $g_{i,j}^* = 1$.*

Proof. Assume the contrary, i.e. $\bar{g}_{i,j}^* = -1$ with $g_{i,j}^* = -1$ and $g_{j,i}^* = 1$. But then i can profitably deviate with $g_i^* + g_{i,j}^+$, yielding $\bar{g}_{i,j}^* = 1$ in the undirected network. This strictly increases payoffs for i from his link with j, as $\frac{1}{2} > \frac{(\eta_i^+(\bar{\mathbf{g}}^*)+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^*)+1)^\phi+(\eta_j^+(\bar{\mathbf{g}}^*)+1)^\phi}$ for $\eta_i^+(\bar{\mathbf{g}}^*) < \eta_j^+(\bar{\mathbf{g}}^*)$ and payoffs from all $k \in N\backslash\{i, j\}$ are non-decreasing in $\eta_i^+(\bar{\mathbf{g}}^*)$.
Q.E.D.

Proposition 1: $\forall \phi > 0$ $\exists NE$ $g^* : \bar{g}_{i,j}^* = 1$ $\forall i, j \in N$.

Proof. A deviation for node i consists of extending negative links to some subset of $N\backslash\{i\}$. Denote the undirected network after a deviation with $\bar{\mathbf{g}}^{*\prime}$. If the deviation strategy of i consists of extending a negative link to only one other node j, then payoffs remain the same, as $\eta_i^+(\bar{\mathbf{g}}^{*\prime}) = \eta_j^+(\bar{\mathbf{g}}^{*\prime}) = N - 2$ and i and j will continue to share the surplus of one in equal parts. If the deviation strategy of i consists of extending two or more negative links, payoffs will be strictly lower for i under the deviation, as then $\eta_i^+(\bar{\mathbf{g}}^{*\prime}) < \eta_k^+(\bar{\mathbf{g}}^{*\prime})$ $\forall k \in N\backslash\{i\}$ while $\bar{g}_{i,j}^* = -1$ for some $j \in N\backslash\{i\}$.
Q.E.D.

Proposition 2: $\forall \phi > 0$ $\exists NE$ $\bar{g}^* : \exists k : \bar{g}_{i,k}^* = -1$ $\forall i \in N\backslash\{k\} \wedge \bar{g}_{i,j}^* = 1$ $\forall i, j \in N\backslash\{k\}$.

Proof. Recall first from Lemma 1 that in any NE $\bar{g}^*$, $\nexists s, t \in N : \bar{g}_{s,t}^* = 0$. As above I will check in the following for profitable deviations. Denote the undirected network after a proposed deviation with $\bar{\mathbf{g}}^{*\prime}$. First, node k. From Lemma 2 I know that, as $\eta_k^+(\bar{\mathbf{g}}^*) < \eta_i^+(\bar{\mathbf{g}}^*)$ $\forall i \in N\backslash\{k\}$, $g_{k,i}^* = 1$ $\forall i \in N\backslash\{k\}$ and a deviation of k therefore consists of extending negative links to some subset of $N\backslash\{k\}$. To see that any such deviation decreases k's payoffs strictly, note that for $\bar{g}_{k,i}^* = -1$ with $g_{k,i}^* = 1$, it must be that $g_{i,k}^* = -1$. If k were to extend a negative link to i, the undirected link between k and i in $\bar{\mathbf{g}}^{*\prime}$ becomes $\bar{g}_{k,i}^{*\prime} = 0$ and payoffs for k are zero. Under $\bar{g}_{k,i}^* = -1$, however, payoffs for k from its link to i are positive,

while payoffs from nodes $j \in N \backslash \{k, i\}$ remain the same in both networks, $\bar{\mathbf{g}}^*$ and $\bar{\mathbf{g}}^{*\prime}$ (as $\eta_k^+(\bar{\mathbf{g}}^*) = \eta_k^+(\bar{\mathbf{g}}^{*\prime}) = 0$ and $\eta_j^+(\bar{\mathbf{g}}^*) = \eta_j^+(\bar{\mathbf{g}}^{*\prime}) = N - 2 \ \forall j \in N \backslash \{k, i\}$). Second, node i. There are three types of possible deviations. First, i extends a positive link to k. But this decreases i's payoffs strictly as $\eta_k^+(\bar{\mathbf{g}}^*) < \eta_i^+(\bar{\mathbf{g}}^*)$, while payoffs from all other links remain the same. Second, i extends a negative link to some subset of $N \backslash \{k, i\}$. This deviation strictly decreases payoffs: i's payoffs will decrease from the link with k, as $\eta_i^+(\bar{\mathbf{g}}^{*\prime}) < \eta_i^+(\bar{\mathbf{g}}^*)$ and will at most yield constant payoffs from links with $j \in N \backslash \{i, k\}$, by an argument analogous to Lemma 2. Third, a combination of the above two deviations. Assume first i extends a positive link to k and one negative link to some $j \in N \backslash \{k, i\}$. i will not increase payoffs if $n = 3$, as then $\eta_k^+(\bar{\mathbf{g}}^*) = \eta_j^+(\bar{\mathbf{g}}') = 0$ and will strictly decrease payoffs for $n \geq 4$. To see this, note that then $\eta_k^+(\bar{\mathbf{g}}^*) = 0$ while $\eta_j^+(\bar{\mathbf{g}}') \geq 1 \ \forall j \in N \backslash \{k, i\}$. Furthermore, for $n \geq 4$, extending more than one negative link to $N \backslash \{k, i\}$ yields even lower payoffs than when linking negatively to only one node in $N \backslash \{k, i\}$, again by the argument used in Lemma 2.
Q.E.D.

Lemma 3: *In any NE $\bar{g}^*$, if $\eta_i^+(\bar{g}^*) = \eta_j^+(\bar{g}^*) \Longrightarrow \bar{g}_{i,j}^* = 1$*

Proof. Assume there exists a Nash equilibrium strategy profile $\mathbf{g}^* : \eta_i^+(\bar{\mathbf{g}}^*) = \eta_j^+(\bar{\mathbf{g}}^*) \wedge \bar{g}_{i,j}^* = -1$. I distinguish four cases.

Case 1: $\bar{g}_{i,j}^* = -1 \wedge \bar{g}_{i,k}^* = \bar{g}_{j,k}^* = \bar{g}_{k,l}^* = 1 \ \forall k, l \in N \backslash \{i, j\}$.

This can not be a Nash Equilibrium, as $\exists k \in N : \eta_k^+(\bar{\mathbf{g}}^*) > \eta_i^+(\bar{\mathbf{g}}^*)$ and k can profitably deviate with the following strategy $\mathbf{g}_k^* + g_{k,i}^-$. To see this, note first that $\eta_i^+(\bar{\mathbf{g}}^* + g_{k,i}^-) = \eta_i^+(\bar{\mathbf{g}}^*) - 1$.Then, cancelling out terms, $\Pi_k(\mathbf{g}_k^* + g_{k,i}^-, \mathbf{g}_{-k}^*) > \Pi_k(\mathbf{g}_k^*, \mathbf{g}_{-k}^*)$
can be written as

$$\frac{(\eta_k^+(\bar{\mathbf{g}}^*)+1)^\phi}{(\eta_k^+(\bar{\mathbf{g}}^*)+1)^\phi + (\eta_i^+(\bar{\mathbf{g}}^*)+1)^\phi} > \frac{1}{2}$$

and the inequality follows from $\eta_k^+(\bar{\mathbf{g}}^*) > \eta_i^+(\bar{\mathbf{g}}^*)$ and $\phi > 0$.

Case 2: $\bar{g}_{i,j}^* = -1 \wedge \bar{g}_{i,k}^* = \bar{g}_{j,k}^* = 1 \ \forall k \in N \backslash \{i, j\} \wedge \exists \bar{g}_{k,l}^* = -1$ for some $k, l \in N \backslash \{i, j\}$.

Without loss of generality assume that $g_{i,j}^* = -1$ and $g_{j,i}^* = 1$. To see that this can not be a Nash Equilibrium, note that if $\bar{g}_{k,l}^* = -1$ for some $k, l \in N \backslash \{i, j\}$, then $\eta_k^+(\bar{\mathbf{g}}^*) \leq \eta_l^+(\bar{\mathbf{g}}^*) \leq \eta_i^+(\bar{\mathbf{g}}^*)$. But then i can deviate profitably with the following strategy $\mathbf{g}_i^* + g_{i,j}^+ + g_{i,k}^-$. Denote the undirected network, resulting from the proposed deviation with $\bar{\mathbf{g}}^{*\prime}$. Note that $\eta_i^+(\bar{\mathbf{g}}^*) = \eta_i^+(\bar{\mathbf{g}}^{*\prime})$, while $\eta_k^+(\bar{\mathbf{g}}^{*\prime}) = \eta_k^+(\bar{\mathbf{g}}^*) - 1$. $\Pi_i(\mathbf{g}_i^* + g_{i,j}^+ + g_{i,k}^-, \mathbf{g}_{-i}^*) > \Pi_i(\mathbf{g}^*)$
follows from

$$\frac{\eta_i^+(\bar{\mathbf{g}}^{*\prime})}{2} + \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*\prime})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi} > \frac{\eta_i^+(\bar{\mathbf{g}}^{*})}{2} + \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*})+1)^\phi}.$$

Cancelling out terms yields

$$\sum_{q \in N_i^-(\bar{\mathbf{g}}^{*\prime})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi} > \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*})+1)^\phi}.$$

Note that the only links that differ in the undirected network $\bar{\mathbf{g}}^{*\prime}$, relative to $\bar{\mathbf{g}}^{*}$, are $\bar{g}_{i,j}$ and $\bar{g}_{i,k}$. I can therefore write the above condition as

$$\frac{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi + (\eta_k^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi} > \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi + (\eta_k^+(\bar{\mathbf{g}}^{*})+1)^\phi}.$$

The inequality then holds from $\eta_i^+(\bar{\mathbf{g}}^{*}) = \eta_i^+(\bar{\mathbf{g}}^{*\prime})$, while $\eta_k^+(\bar{\mathbf{g}}^{*\prime}) = \eta_k^+(\bar{\mathbf{g}}^{*}) - 1$.

Case 3: $\bar{g}_{i,j}^* = -1 \wedge N_i^-(\bar{\mathbf{g}}^{*}) \neq \emptyset \wedge N_j^-(\bar{\mathbf{g}}^{*}) \neq \emptyset$.

Without loss of generality assume below that $g_{i,j}^* = -1$ and $g_{j,i}^* = 1$. This can not be a Nash equilibrium, as i can profitably deviate with $\mathbf{g}_i^* + g_{i,j}^+$. Denote again the undirected network after the deviation with $\bar{\mathbf{g}}^{*\prime}$. Note that $\eta_i^+(\bar{\mathbf{g}}^{*})+1 = \eta_i^+(\bar{\mathbf{g}}^{*\prime})$ and that $N_i^-(\bar{\mathbf{g}}^{*\prime}) = N_i^-(\bar{\mathbf{g}}^{*})\backslash\{j\}$. I can then write $\Pi_i(\mathbf{g}_i^* + g_{i,j}^+, \mathbf{g}_{-i}^*) > \Pi_i(\mathbf{g}_i^*)$ as

$$\frac{\eta_i^+(\bar{\mathbf{g}}^{*\prime})}{2} + \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*\prime})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi} > \frac{\eta_i^+(\bar{\mathbf{g}}^{*})}{2} + \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*})+1)^\phi}$$

which can be rewritten as

$$\frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)}{2} + \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*\prime})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^m}{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^m + (\eta_q^+(\bar{\mathbf{g}}^{*\prime})+1)^m} > \frac{\eta_i^+(\bar{\mathbf{g}}^{*})}{2} + \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*})\backslash\{j\}} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^m}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^m + (\eta_q^+(\bar{\mathbf{g}}^{*})+1)^m} + \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^m}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^m + (\eta_j^+(\bar{\mathbf{g}}^{*})+1)^m}$$

from $\eta_i^+(\bar{\mathbf{g}}^{*}) = \eta_j^+(\bar{\mathbf{g}}^{*})$ and $N_i^-(\bar{\mathbf{g}}^{*\prime}) = N_i^-(\bar{\mathbf{g}}^{*})\backslash\{j\}$ I can write the above as

$$\sum_{q \in N_i^-(\bar{\mathbf{g}}^{*\prime})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*\prime})+1)^\phi} > \sum_{q \in N_i^-(\bar{\mathbf{g}}^{*\prime})} \frac{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi}{(\eta_i^+(\bar{\mathbf{g}}^{*})+1)^\phi + (\eta_q^+(\bar{\mathbf{g}}^{*})+1)^\phi}$$

and the inequality simply follows from $\eta_i^+(\bar{\mathbf{g}}^{*}) + 1 = \eta_i^+(\bar{\mathbf{g}}^{*\prime})$ and $\eta_k^+(\bar{\mathbf{g}}^{*}) = \eta_k^+(\bar{\mathbf{g}}^{*\prime})$ $\forall k \in N_i^-(\bar{\mathbf{g}}^{*\prime})$.

Case 4: $\bar{g}^*_{k,l} = -1\ \forall k,l \in N$.

Without loss of generality assume below that $g^*_{i,j} = -1$ and $g^*_{j,i} = 1$. This can not be a NE, as then node i could profitably deviate with the following strategy $g^*_i + g^+_{i,j}$, thereby obtaining payoffs larger than $\frac{1}{2}$ from all nodes $k \in N \backslash \{i,j\}$.
Q.E.D.

Proposition 3: *Any NE $\bar{g}^*$ s.t. $\exists s,t \in N : \eta^+_s(\bar{g}^*) \neq \eta^+_t(\bar{g}^*)$, $\bar{g}^*$ can be partitioned into maximal cliques of different size with $\bar{g}^*_{i,j} = -1$, if $i \in C^m_k(\bar{g}^*)$ and $j \in C^m_l(\bar{g}^*)$.*

The proof is by induction and requires several steps. First, rank the sets $P_k(\bar{\mathbf{g}}^*)$ by its subscript and call the set with the highest subscript $P^m_k(\bar{\mathbf{g}}^*)$ for some $k \in \mathbb{N}^+$, the one with the second highest subscript with $P^{m-1}_l(\bar{\mathbf{g}}^*)$ for some $l < k$ with $l,k \in \mathbb{N}^+$. Proceed in this way until the set with the lowest subscript. For ease of notation, I will drop the subscripts in the following.

Base Case: In any NE $\bar{\mathbf{g}}^* : \exists s,t \in N : \eta^+_s(\bar{\mathbf{g}}) \neq \eta^+_t(\bar{\mathbf{g}})$, then $\bar{g}_{i,j} = -1\ \forall i \in P^m(\bar{\mathbf{g}}^*) \wedge \forall j \notin P^m(\bar{\mathbf{g}}^*)$.

Step 1: I will first show that in any NE $\bar{\mathbf{g}}^* : \exists s,t \in N : \eta^+_s(\bar{\mathbf{g}}^*) \neq \eta^+_t(\bar{\mathbf{g}}^*)$, $N^+_i(\bar{\mathbf{g}}^*) \backslash \{j\} = N^+_j(\bar{\mathbf{g}}^*) \backslash \{i\} \wedge N^-_i(\bar{\mathbf{g}}^*) = N^-_j(\bar{\mathbf{g}}^*)\ \forall i,j \in P^m(\bar{\mathbf{g}}^*)$.

For $\mid P^m(\bar{\mathbf{g}}^*) \mid = 1$, the statement holds trivially. Assume $\mid P^m(\bar{\mathbf{g}}^*) \mid \geq 2$, and - contrary to the above - that $\exists i,j \in P^m(\bar{\mathbf{g}}^*) : N^+_i(\bar{\mathbf{g}}^*) \backslash \{j\} \neq N^+_j(\bar{\mathbf{g}}^*) \backslash \{i\} \wedge N^-_i(\bar{\mathbf{g}}^*) \neq N^-_j(\bar{\mathbf{g}}^*)$. This configuration, however, can not be a Nash equilibrium. To see this, take two nodes k and l, such that $k \in N^+_i(\bar{\mathbf{g}}^*) \backslash \{j\} \wedge k \notin N^+_j(\bar{\mathbf{g}}^*) \backslash \{i\}$ and $l \notin N^+_i(\bar{\mathbf{g}}^*) \backslash \{j\} \wedge l \in N^+_j(\bar{\mathbf{g}}^*) \backslash \{i\}$. Note that under the above assumption such a pair of nodes k and l must always exist, as otherwise $\eta^+_i(\bar{\mathbf{g}}^*) \neq \eta^+_j(\bar{\mathbf{g}}^*)$. Without loss of generality, assume $\eta^+_k(\bar{\mathbf{g}}^*) \leq \eta^+_l(\bar{\mathbf{g}}^*)$. From Lemma 2 I know that, in order for $\bar{\mathbf{g}}^*$ to be a Nash equilibrium, $g^*_{i,l} = -1$ while $g^*_{l,i} = 1$. But then i can profitably deviate with the following strategy: $\bar{\mathbf{g}}^*_i + g^+_{i,l} + g^-_{i,k}$. Call the network after the proposed deviation $\bar{\mathbf{g}}^{*\prime}$. This deviation is profitable for i, as $\eta^+_i(\bar{\mathbf{g}}^*) = \eta^+_i(\bar{\mathbf{g}}^{*\prime})$ but $\eta^+_k(\bar{\mathbf{g}}^{*\prime}) < \eta^+_l(\bar{\mathbf{g}}^*)$, yielding strictly higher payoffs for i. I know now that $N^+_i(\bar{\mathbf{g}}^*) \backslash \{j\} = N^+_j(\bar{\mathbf{g}}^*) \backslash \{i\} \wedge N^-_i(\bar{\mathbf{g}}^*) = N^-_j(\bar{\mathbf{g}}^*)$ holds $\forall i,j \in P^m(\bar{\mathbf{g}}^*)$.

Step 2: I will next show that any configuration $\bar{\mathbf{g}} : \exists s,t \in N : \eta^+_s(\bar{\mathbf{g}}) \neq \eta^+_t(\bar{\mathbf{g}})$, which is not one of the following two types, can not be a Nash equilibrium $\bar{\mathbf{g}}^*$. First, $\bar{g}_{i,k} = 1\ \forall i \in P^m(\bar{\mathbf{g}})$, $\forall k \in P^{m-1}(\bar{\mathbf{g}})$. Second, $\bar{g}_{i,k} = -1\ \forall i \in P^m(\bar{\mathbf{g}})$, $\forall k \in P^{m-1}(\bar{\mathbf{g}})$.

Assume the contrary (and in accordance with *Step 1*) that $\exists k \in P^{m-1}(\bar{\mathbf{g}}^*) : \bar{g}^*_{k,i} = 1$ $\forall i \in P^m(\bar{\mathbf{g}}^*)$ and $\exists l \in P^{m-1}(\bar{\mathbf{g}}^*) : \bar{g}^*_{l,i} = -1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$. I distinguish two cases. First,

$\nexists P^{m-2}(\bar{\mathbf{g}}^*)$. Then the contradiction is immediate, as $\bar{g}^*_{k,l} = 1\ \forall k, l \in P^{m-1}(\bar{\mathbf{g}}^*)$ while $\bar{g}^*_{k,i} = 1$ $\forall i \in P^m(\bar{\mathbf{g}}^*)$ and $\bar{g}^*_{l,i} = -1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$ yields $\eta^+_k(\bar{\mathbf{g}}^*) \neq \eta^+_l(\bar{\mathbf{g}}^*)$ and either $k \notin P^{m-1}(\bar{\mathbf{g}}^*)$ or $l \notin P^{m-1}(\bar{\mathbf{g}}^*)$, resulting in a contradiction. Second, assume $\exists P^{m-2}(\bar{\mathbf{g}}^*)$ and $\exists k \in P^{m-1}(\bar{\mathbf{g}}^*)$: $\bar{g}^*_{k,i} = 1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$ and $\exists l \in P^{m-1}(\bar{\mathbf{g}}^*)$: $\bar{g}^*_{l,i} = -1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$. Note that for node $k \in P^{m-1}(\bar{\mathbf{g}}^*)$, $\bar{g}^*_{k,l} = 1\ \forall l \in P^{m-1}(\bar{\mathbf{g}}^*)\backslash\{k\} \wedge \bar{g}^*_{k,i} = 1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$, while for node $i \in P^m(\bar{\mathbf{g}}^*)$, $\exists l \in P^{m-1}(\bar{\mathbf{g}}^*)$: $\bar{g}^*_{i,l} = -1 \wedge\ \bar{g}^*_{i,j} = 1\ \forall j \in P^m(\bar{\mathbf{g}}^*)\backslash\{i\}$. Therefore, in order for $\eta^+_i(\bar{\mathbf{g}}^*) > \eta^+_k(\bar{\mathbf{g}}^*)$, there must exist a node $h \in P^{m-x}(\bar{\mathbf{g}}^*)$ with $x \geq 2$, such that $\bar{g}^*_{i,h} = 1$. This, however, can not be a Nash equilibrium, as i can then profitably deviate with the following strategy: $g^*_i + g^+_{i,l} + g^-_{i,h}$. Denote again the network after the proposed deviation by $\bar{\mathbf{g}}^{*\prime}$. The deviation is profitable for node i, as $\eta^+_i(\bar{\mathbf{g}}^*) = \eta^+_i(\bar{\mathbf{g}}^{*\prime})$, but $\eta^+_h(\bar{\mathbf{g}}^{*\prime}) < \eta^+_l(\bar{\mathbf{g}}^*)$ (and it is also profitable for node l, as $\eta^+_l(\bar{\mathbf{g}}^*) < \eta^+_i(\bar{\mathbf{g}}^*)$, which is relevant for the Bilateral Equilibrium case). I have so far shown that $\forall i \in P^m(\bar{\mathbf{g}}^*)$, either $\bar{g}^*_{i,k} = 1\ \forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$, or $\bar{g}^*_{i,k} = -1$ $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$.

Step 3: In any NE $\bar{\mathbf{g}}^*$: $\exists s, t \in N$: $\eta^+_s(\bar{\mathbf{g}}) \neq \eta^+_t(\bar{\mathbf{g}})$ and $\bar{g}^*_{i,k} = 1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$, then $N^+_k(\bar{\mathbf{g}}^*)\backslash\{i\} \subset N^+_i(\bar{\mathbf{g}}^*)\backslash\{k\}\ \forall i \in P^m(\bar{\mathbf{g}}^*)$ and $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$ (and, conversely, $N^-_k(\bar{\mathbf{g}}^*) \subset N^-_i(\bar{\mathbf{g}}^*)$).

First, note that there must $\exists h \notin \{P^m(\bar{\mathbf{g}}^*), P^{m-1}(\bar{\mathbf{g}}^*)\}$: $\bar{g}^*_{i,h} = 1\ \forall i \in P^m(\bar{\mathbf{g}}^*) \wedge \bar{g}^*_{h,k} = -1$ for some $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. For, otherwise, $\eta^+_k(\bar{\mathbf{g}}) \geq \eta^+_i(\bar{\mathbf{g}})$, contradicting the initial assumption that $i \in P^m(\bar{\mathbf{g}}^*)$ and $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. Assume now that $N^+_k(\bar{\mathbf{g}})\backslash\{i\} \subset N^+_i(\bar{\mathbf{g}})\backslash\{k\}$ does not hold. Then there must $\exists l \in N$: $\bar{g}^*_{i,l} = -1\ \forall i \in P^m(\bar{\mathbf{g}}^*) \wedge \bar{g}^*_{k,l} = 1$ for some $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. This, however, can not be a NE. Assume $\eta^+_h(\bar{\mathbf{g}}) \leq \eta^+_l(\bar{\mathbf{g}})$. Then, a profitable deviation exists of the following form: $g^*_i + g^+_{i,l} + g^-_{i,h}$. If $\eta^+_l(\bar{\mathbf{g}}) \leq \eta^+_h(\bar{\mathbf{g}})$ the profitable deviation is of the form $g^*_k + g^+_{k,h} + g^-_{k,l}$. Both cases yield a contradiction.

Before proceeding to Step 4, I will define some further sets, which will prove to be useful.

Definition 2: Define the following set(s) $\tilde{N}^m(\bar{\mathbf{g}}^*) = \{k \in N \mid N^+_i(\bar{\mathbf{g}}^*) \cap N^-_j(\bar{\mathbf{g}}^*)$ with $i \in P^m(\bar{\mathbf{g}}^*)$ and $j \in P^{m-1}(\bar{\mathbf{g}}^*)\}$ and $\tilde{N}^{m-1}(\bar{\mathbf{g}}^*) = \{k \in N \mid N^+_i(\bar{\mathbf{g}}^*) \cap N^-_j(\bar{\mathbf{g}}^*)$ with $i \in P^{m-1}(\bar{\mathbf{g}}^*)$ and $j \in P^{m-2}(\bar{\mathbf{g}}^*)\}$. Proceed in this way until the two sets $P(\bar{\mathbf{g}}^*)$ with the lowest number of allies.

Step 4: I am now in the position to show that in any NE $\bar{\mathbf{g}}^*$: $\exists s, t \in N$: $\eta^+_s(\bar{\mathbf{g}}) \neq \eta^+_t(\bar{\mathbf{g}})$, then $\bar{g}^*_{i,k} = -1\ \forall i \in P^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$.

If $\nexists P^{m-2}(\bar{\mathbf{g}}^*)$, the statement must hold, as otherwise $\eta^+_i(\bar{\mathbf{g}}) = \eta^+_k(\bar{\mathbf{g}})$. Assume now $\exists P^{m-2}(\bar{\mathbf{g}}^*)$ and contrary to the above statement, but in accordance with the statement in Step 2, that

$\bar{g}^*_{i,k} = 1$ for $\forall i \in P^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$. I discern two cases. First, $\Pi_k(g^*) \geq \Pi_i(g^*)$ for $i \in P^m(\bar{\mathbf{g}}^*)$, $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. Second, $\Pi_i(g^*) > \Pi_k(g^*)$, for $i \in P^m(\bar{\mathbf{g}}^*)$, $k \in P^{m-1}(\bar{\mathbf{g}}^*)$.

Case 1: $\Pi_k(g^*_k, g^*_{-k}) \geq \Pi_i(g^*_i, g^*_{-i})$ for $\forall i \in P^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$.

First note that by *Step 3*, if $\bar{g}^*_{i,k} = 1$ $\forall i \in P^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$ then the following has to hold $N^+_k(\bar{\mathbf{g}})\backslash\{i\} \subset N^+_i(\bar{\mathbf{g}})\backslash\{k\}$ $\forall i \in P^m(\bar{\mathbf{g}}^*)$ and $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$ (and, conversely, $N^-_i(\bar{\mathbf{g}}^*) \subset N^-_k(\bar{\mathbf{g}}^*)$). Note that $i \in P^m(\bar{\mathbf{g}}^*)$ can then unilaterally imitate the strategy of $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ in the following way: $g^*_i + \sum_{k \in \tilde{N}(\bar{\mathbf{g}}^*)} g^-_{i,k}$. Then $N^+_k(\bar{\mathbf{g}}^*)\backslash\{i) = N^+_i(g^*_i + \sum_{l \in \tilde{N}(\bar{\mathbf{g}}^*)} g^-_{i,l}, g^*_{-i})\backslash\{k\}$ holds for any pair $i \in P^m(\bar{\mathbf{g}}^*)$, $k \in P^{m-1}(\bar{\mathbf{g}}^*)$. Note, however, that $\Pi_i(g^*_i + \sum_{l \in \tilde{N}(\bar{\mathbf{g}}^*)} g^-_{i,l}, g^*_{-i}) > \Pi_k(g^*) \geq \Pi_i(g^*)$, as now $N^-_i(g^*_i + \sum_{l \in \tilde{N}(\bar{\mathbf{g}}^*)} g^-_{i,k}, g^*_{-i}) = N^-_k(\bar{\mathbf{g}}^*)$, but $\eta^+_h(g^*_i + \sum_{l \in \tilde{N}(\bar{\mathbf{g}}^*)} g^-_{i,l}, g^*_{-i}) < \eta^+_h(\bar{\mathbf{g}}^*)$ $\forall h \in \tilde{N}^m(\bar{\mathbf{g}}^*)$.

Case 2: $\Pi_i(g^*) > \Pi_k(g^*)$, for $i \in P^m(\bar{\mathbf{g}}^*)$, $k \in P^{m-1}(\bar{\mathbf{g}}^*)$.

From Lemma 2 - together with $\eta^+_h(\bar{\mathbf{g}}^*) < \eta^+_k(\bar{\mathbf{g}}^*)$ $\forall h \in \tilde{N}^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$ - I know that $g^*_{h,k} = 1$ while $g^*_{k,h} = -1$. Similarly to Case 1, node $k \in P^{m-1}(\bar{\mathbf{g}}^*)$ can therefore unilaterally imitate the strategy of $i \in P^m(\bar{\mathbf{g}}^*)$, such that $\bar{g}^*_{h,k} = 1$ $\forall h \in \tilde{N}^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$. Denote again the undirected network after the deviation with $\bar{\mathbf{g}}^{*\prime}$. To see that $\Pi_i(\mathbf{g}^*) = \Pi_k(\mathbf{g}^{*\prime})$, note that $N^+_k(\bar{\mathbf{g}}^*)\backslash\{i\} = N^+_i(\bar{\mathbf{g}}^{*\prime})\backslash\{k\}$ $\forall i \in P^m(\bar{\mathbf{g}}^*)$ and $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$ (and $N^-_i(\bar{\mathbf{g}}^*) = N^-_k(\bar{\mathbf{g}}^{*\prime})$). Therefore, k can profitably deviate by extending friendly links to all nodes in $\tilde{N}^m(\bar{\mathbf{g}}^*)$ and I have reached a contradiction.

From the above it follows that in any NE $\bar{\mathbf{g}}^* : \exists s, t \in N : \eta^+_s(\bar{\mathbf{g}}^*) \neq \eta^+_t(\bar{\mathbf{g}}^*)$, then $g^*_{i,j} = -1$ $\forall i \in P^m(\bar{\mathbf{g}}^*)$, $\forall k \in P^{m-1}(\bar{\mathbf{g}}^*)$. But then $g^*_{i,v} = -1$ $\forall v \notin P^m(\bar{\mathbf{g}}^*)$ and therefore $\mid P^m_x(\bar{\mathbf{g}}^*) \mid = x + 1$.

Define the set $\tilde{P}^r(\bar{\mathbf{g}}^*) = \{P^m(\bar{\mathbf{g}}^*), P^{m-1}(\bar{\mathbf{g}}^*), ..., P^{m-r}(\bar{\mathbf{g}}^*)\}$. Note $\tilde{P}^0(\bar{\mathbf{g}}^*) = P^m(\bar{\mathbf{g}}^*)$.

Inductive Step: In any NE $\bar{\mathbf{g}}^*$, if $\bar{g}^*_{i,j} = 1$ $\forall i, j \in P^{m-t}(\bar{\mathbf{g}}^*) \wedge \bar{g}^*_{i,v} = -1$ $\forall v \notin P^{m-t}(\bar{\mathbf{g}}^*)$ holds $\forall t \in \mathbb{N} : 0 \leq t \leq r$, then $\bar{g}^*_{h,l} = 1$ $\forall h, l \in P^{m-(r+1)}(\bar{\mathbf{g}}^*) \wedge \bar{g}^*_{h,w} = -1$ $\forall w \notin P^{m-(r+1)}(\bar{\mathbf{g}}^*)$.

In the base case I proved that $\bar{g}^*_{i,j} = 1$ $\forall i, j \in P^m(\bar{\mathbf{g}}^*) \wedge \bar{g}^*_{i,v} = -1$ $\forall v \notin P^m(\bar{\mathbf{g}}^*)$, i.e., that the first part of the statement holds for $r = 0$. To prove that this must hold $\forall P_k(\bar{\mathbf{g}}^*)$, $k \in \mathbb{N}_+$, I assume the induction hypothesis holds for some $r \in \mathbb{N}_+$ and then show that it also holds for $r + 1$. Note first that, assuming the induction hypothesis holds, $g^*_{q,z} = -1$ $\forall q \in \tilde{P}^r(\bar{\mathbf{g}}^*), \forall z \notin \tilde{P}^r(\bar{\mathbf{g}}^*)$. But then I also know that that in any Nash equilibrium $g^*_{z,q} = 1$ $\forall q \in \tilde{P}^r(\bar{\mathbf{g}}^*), \forall z \notin \tilde{P}^r(\bar{\mathbf{g}}^*)$, as $\nexists \bar{g}^*_{i,j} = 0$. I can therefore repeat steps 1 through 4 from the

base step, relabeling $P^m(\bar{\mathbf{g}}^*)$ with $P^{m-(r+1)}(\bar{\mathbf{g}}^*)$ and $P^{m-1}(\bar{\mathbf{g}}^*)$ with $P^{m-(r+2)}(\bar{\mathbf{g}}^*)$ to establish that $g^*_{h,l} = 1$ $\forall h, l \in P^{m-(r+1)}(\bar{\mathbf{g}}^*)$, $g^*_{h,z} = -1$ $\forall z \in P^{m-(r+2)}(\bar{\mathbf{g}}^*)$ and therefore $\bar{g}^*_{h,w} = -1$ $\forall w \notin P^{m-r-1}(\bar{\mathbf{g}}^*)$. I have now shown that the statement holds for $\tilde{P}^{r+1}(\bar{\mathbf{g}}^*)$, which concludes the proof.
Q.E.D.

Proposition 4: If $\bar{\mathbf{g}}$ can be partitioned into maximal cliques of different size, with $g_{i,j} = -1$ and $g_{j,i} = 1$ for $i \in P^{m-x}(\bar{\mathbf{g}}^*)$ and $j \in P^{m-y}(\bar{\mathbf{g}}^*)$ with $y > x$, then $\exists \tilde{\phi} : \forall \phi > \tilde{\phi}$, $\bar{\mathbf{g}}$ is a Nash equilibrium $\bar{\mathbf{g}}^*$.

First note that the relevant deviation of node i consists of extending positive links to a total number of z nodes with fewer friends than i. I discern two types of deviations. First, deviations where, if $i \in P^{m-x}(\bar{\mathbf{g}}^*)$ extends a positive link to some $j \in P^{m-y}(\bar{\mathbf{g}}^*)$ with $y > x$, then i also extends positive links to all other $k \in P^{m-y}(\bar{\mathbf{g}}^*)$ with $y > x$. Two, where there exists a set $P^{m-y}(\bar{\mathbf{g}}^*)$ with $y > x$, such that i links positively to only some nodes in $P^{m-y}(\bar{\mathbf{g}}^*)$. From the above I know that any configuration where i extends a positive link to a node l and a negative one to a node k with $\eta_i(\bar{\mathbf{g}}) > \eta_k(\bar{\mathbf{g}})$ and $\eta_k(\bar{\mathbf{g}}) > \eta_l(\bar{\mathbf{g}})$, can not be a Nash equilibrium $\bar{\mathbf{g}}^*$, as i could strictly increase his payoffs by extending a positive link to k and a negative one to l. Rank and label the set of nodes as follows:
$P^m(\bar{\mathbf{g}}^*), P^{m-1}(\bar{\mathbf{g}}^*), ..., P^{m-x}(\bar{\mathbf{g}}^*), P^{m-(x+1)}(\bar{\mathbf{g}}^*), ..., P^{m-(x+q)}(\bar{\mathbf{g}}^*), P^{m-(x+q+1)}(\bar{\mathbf{g}}^*), ..., P^{m-(x+p)}(\bar{\mathbf{g}}^*)$.
For the first type of deviation considered, $P^{m-(x+q)}(\bar{\mathbf{g}}^*)$ is the set of nodes with the smallest number of number of friends to which node $i \in P^{m-x}(\bar{\mathbf{g}}^*)$ extends positive links to in the network after the deviation, $\bar{\mathbf{g}}^{*\prime}$. For the second type of deviation, i will also extend positive links to r nodes in the set $P^{m-(x+q+1)}(\bar{\mathbf{g}}^*)$, with $r \in [1, \mid P^{m-(x+q+1)}(\bar{\mathbf{g}}^*) \mid -1]$. For the first type of deviation I can then write $z = \sum\limits_{j=x+1}^{x+q} \mid P^{m-j}(\bar{\mathbf{g}}^*) \mid$, while for the second type of deviation $z = \sum\limits_{j=x+1}^{x+q} \mid P^{m-j}(\bar{\mathbf{g}}^*) \mid +r$. In the expression below the first type of deviation corresponds to $r = 0$. Taking the limit of the payoffs of a deviation then yields

$$\lim_{\phi \longrightarrow \infty} \{ \sum_{j=x+q+1}^{p} \mid P^{m-j}(\bar{\mathbf{g}}^*) \mid * \left\{ \tfrac{(|P^{m-x}(\bar{\mathbf{g}}^*)|+z)^\phi}{|P^{m-j}(\bar{\mathbf{g}}^*)|^\phi + (|P^{m-x}(\bar{\mathbf{g}}^*)|+z)^\phi} - \tfrac{|P^{m-x}(\bar{\mathbf{g}}^*)|^\phi}{|P^{m-j}(\bar{\mathbf{g}}^*)|^\phi + |P^{m-x}(\bar{\mathbf{g}}^*)|^\phi} \right\}$$

$$+ \sum_{j=x-1}^{m} \mid P^{m-j}(\bar{\mathbf{g}}^*) \mid * \left\{ \tfrac{(|P^{m-x}(\bar{\mathbf{g}}^*)|+z)^\phi}{|P^{m-j}(\bar{\mathbf{g}}^*)|^\phi + (|P^{m-x}(\bar{\mathbf{g}}^*)|+z)^\phi} - \tfrac{|P^{m-x}(\bar{\mathbf{g}}^*)|^\phi}{|P^{m-j}(\bar{\mathbf{g}}^*)|^\phi + |P^{m-x}(\bar{\mathbf{g}}^*)|^\phi} \right\}$$

$$+ \sum_{j=x+1}^{x+q} \mid P^{m-j}(\bar{\mathbf{g}}^*) \mid * \left\{ \tfrac{1}{2} - \tfrac{|P^{m-x}(\bar{\mathbf{g}}^*)|^\phi}{|P^{m-j}(\bar{\mathbf{g}}^*)|^\phi + |P^{m-x}(\bar{\mathbf{g}}^*)|^\phi} \right\}$$

$$+r * \left\{ \frac{1}{2} - \frac{|P^{m-(x+q+1)}(\bar{\mathbf{g}}^*)|^\phi}{|P^{m-j}(\bar{\mathbf{g}}^*)|^\phi + |P^{m-(x+q+1)}(\bar{\mathbf{g}}^*)|^\phi} \right\} \Bigg\} = -\frac{\sum_{j=x+1}^{x+q} |P^{m-j}(\bar{\mathbf{g}}^*)| + r}{2} = -\frac{z}{2}$$

Note that the first term corresponds to the additional payoffs i receives from all enemies with fewer friends in the network after the deviation, $\bar{\mathbf{g}}^{*\prime}$. The second term stands for the payoffs foregone from linking positively in $\bar{\mathbf{g}}^{*\prime}$ to nodes, who were previously enemies with fewer allies than node i in $\bar{\mathbf{g}}^*$. Finally, the third term reflects the additional payoffs i receives from all enemies with more friends in the network after the deviation, $\bar{\mathbf{g}}^{*\prime}$. The expression is continuous in ϕ and therefore there exists a value $\tilde{\phi}$, such that the expression is negative for all $\phi > \tilde{\phi}$.

Q.E.D.

6 References

Antal, T., P. Krapivsky and S. Redner, (2006), "Social balance on networks: The dynamics of friendship and enmity", Physica D, 224(130).

Aumann, R. and R. Myerson, (1988), "Endogenous Formation of Links Between Players and Coalitions: An Application of the Shapley Value" in The Shapley Value, A. Roth (ed.), Cambridge University Press, pp. 175-191.

Bala, V. and S. Goyal, (2000), "A non-cooperative model of network formation", Econometrica, Vol. 68, pp. 1181–1230.

Davis, J. A., (1967), "Clustering and structural balance in graphs", Human Relations, 20(2), pp. 181–187.

Cartwright D. and F. Harary, (1956), "Structural balance: A generalization of Heider's theory", Psychological Review, 63(5), pp. 277–293.

Chwe, M., (1994), "Farsighted Coalitional Stability", Journal of Economic Theory, Vol. 63, pp. 299-325.

Chwe, M., (2000), "Communication and coordination in social networks", Review of Economic Studies, Vol. 67 (1), pp. 1–17.

Doreian, P. and D. Krackhardt, (2001), "Pre-transitive mechanisms for signed networks", Journal of Mathematical Sociology, Vol. 25, pp. 43–67.

Doreian, P. and A. Mrvar, (1996), "A partitioning approach to structural balance", Social Networks, Vol. 18, pp. 149–168.

Esteban, J.M. and J. Sákovics, (2003), "Olson vs. Coase: Coalition worth in conflict", Theory and Decision ,Vol. 55, pp. 339-357.

Frank, J. and T. Öztürk, (2009), "Conflict Networks", Ruhr Economic Papers.

Goyal, S. and A. Vigier, (2010), "Robust networks", working paper.

Harary, F., (1961), "A structural analysis of the situation in the Middle East in 1956", Journal of Conflict Resolution, Vol. 5, pp. 167-178.

Heider, F., (1946), "Attitudes and cognitive organization", Journal of Psychology, Vol. 21, pp. 107–112.

Jordan, J.S., (2006), "Pillage and property", Journal of Economic Theory, Vol. 131, Issue 1.

Jackson, M.O. and A. Wolinsky, (1996)," A Strategic Model of Social and Economic Networks", Journal of Economic Theory, Vol. 71, No. 1, pp. 44–74.

Moore, M., (1979), "Structural balance and international relations", European Journal of Social Psychology, Vol. 9 (3), pp. 323–326.

Szell, M., R. Lambiotte and S. Thurner, (2010), "Multirelational organization of large-scale social networks in an online world", PNAS, Vol. 107, No. 31, pp. 13636–13641.

Tullock, G., (1967), "The Welfare Cost of Tariffs, Monopoly, and Theft", Western Economic Journal, pp. 224-232.

Tullock, G., (1980), "Efficient Rent Seeking", in: J. Buchanan, R. Tollison and G. Tullock, eds.,Toward a Theory of Rent Seeking Society, Texas A&M University Press, pp. 97-112.

Wärneryd, K., (1998), "Distributional conflict and jurisdictional organization", Journal of Public Economics, Vol. 69, pp. 435-450.