# Exercises in Iterational Asymptotics

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ABSTRACT. The problems and solutions contained here, all associated with nonlinear recurrences and long-term trends, are new (as far as is known).

### 1. Première exercice

Consider initially the famous recurrence

$$x_k = p x_{k-1} (1 - x_{k-1})$$
 for  $k \ge 1$ ;  $0 < x_0 < 1$ 

where  $0 . Quantify the convergence rate of <math>x_k$  as  $k \to \infty$ . Clearly  $0 < x_k < 1$  and

$$x_k$$

thus  $x_k < p^k x_0$  for all k. Observe that

$$x_{k} = p x_{k-1} (1 - x_{k-1})$$

$$= p^{2} x_{k-2} (1 - x_{k-2}) (1 - x_{k-1})$$

$$= p^{3} x_{k-3} (1 - x_{k-3}) (1 - x_{k-2}) (1 - x_{k-1})$$

$$= p^{k} x_{0} \prod_{j=0}^{k-1} (1 - x_{j})$$

hence

$$C = \lim_{k \to \infty} \frac{x_k}{p^k} = x_0 \prod_{j=0}^{\infty} (1 - x_j)$$

exists and is nonzero since

$$\sum_{j=0}^{\infty} x_j < x_0 \sum_{j=0}^{\infty} p^j$$

converges.

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Table 1. Numerical estimates of C, given  $x_0 = 1/2$  and selected values of p > 0: no closed-form expressions are known

p	C	p	C
1/5	0.234690787230465	3/5	0.176983588618567
1/4	0.229832778573153	2/3	0.161059687971223
1/3	0.220577540168322	3/4	0.136649472578135
2/5	0.211947268934865	4/5	0.118823329484862
1/2	0.196453426377889	1	1.767993786136154*

The entry corresponding to p = 1 is starred [1] because C is defined differently than for p < 1:

$$C = -\lim_{k \to \infty} k^2 \left( x_k - \frac{1}{k} + \frac{\ln(k)}{k^2} \right)$$

and details are found in [2, 3, 4].

Consider finally the less-famous recurrence

$$x_k = p x_{k-1} (1 + x_{k-1})$$
 for  $k \ge 1$ ;  $0 < x_0 < \frac{1-p}{p}$ 

where  $0 . Again, quantify the convergence rate of <math>x_k$  as  $k \to \infty$ .

Replacing  $1 - x_{k-1}$  by  $1 + x_{k-1}$  renders the task of bounding  $x_k$  more subtle. Note that (1-p)/p is the only nonzero fixed point of f(x) = p x (1+x):

$$1 = p(1+x)$$
 when  $1 - p = px$ .

If  $x_0 = (1 - p - \varepsilon)/p$  for some  $0 < \varepsilon < 1 - p$ , then

$$p(1 + x_0) = p + (1 - p - \varepsilon) = 1 - \varepsilon < 1$$

hence

$$x_1 = p x_0 (1 + x_0) = p (1 + x_0) x_0 < (1 - \varepsilon) x_0 < x_0.$$

A more geometric approach involves graphing the parabola y = f(x) and the diagonal y = x: since f'(0) = p < 1, the curve dips below the line at the left endpoint and, by continuity, does not cross it again until the right endpoint. More generally,  $x_k < x_{k-1}$  for all k. It follows that

$$x_k  $< (1 - \varepsilon)^2 x_{k-2} < (1 - \varepsilon)^3 x_{k-3} < (1 - \varepsilon)^4 x_{k-4}$$$

thus  $x_k < (1-\varepsilon)^k x_0$  for all k. Observe that

$$x_{k} = p x_{k-1} (1 + x_{k-1})$$

$$= p^{2} x_{k-2} (1 + x_{k-2}) (1 + x_{k-1})$$

$$= p^{3} x_{k-3} (1 + x_{k-3}) (1 + x_{k-2}) (1 + x_{k-1})$$

$$= p^{k} x_{0} \prod_{j=0}^{k-1} (1 + x_{j})$$

hence

$$C = \lim_{k \to \infty} \frac{x_k}{p^k} = x_0 \prod_{j=0}^{\infty} (1 + x_j)$$

exists and is nonzero since

$$\sum_{j=0}^{\infty} x_j < x_0 \sum_{j=0}^{\infty} (1 - \varepsilon)^j$$

converges.

Table 2. Numerical estimates of C, given  $x_0 = (p-1)/(2p)$ : no closed-form expressions are known

p	C	p	C
1/5	24.539007835941751	3/5	1.015970842139591
1/4	13.119009853937092	2/3	0.690744393761287
1/3	5.896477923507413	3/4	0.415551960439528
2/5	3.529895194705441	4/5	0.295525160728184
1/2	1.832010583354543	1	1.597910218031873*

The entry corresponding to p = 1 is starred [5] because again C is defined differently than for p < 1:

$$C = \lim_{k \to \infty} x_k^{(2^{-k})} \quad \text{where} \quad x_0 = 1 \quad (\text{not } 0).$$

This constant appears elsewhere, although thinly veiled [6, 7]:

$$\sqrt{C} = \lim_{k \to \infty} y_k^{(2^{-k-1})} = 1.264084735305301...$$

where  $y_k = 1 + x_k$ ; we have

$$y_k = y_{k-1}^2 - y_{k-1} + 1$$
 for  $k \ge 1$ ;  $y_0 = 2$ 

and the latter is known as Sylvester's sequence [8]. Also, letting  $z_k = 1 + 2x_k$ , we have

$$z_k = \frac{1}{2} (z_{k-1}^2 + 1)$$
 for  $k \ge 1$ ;  $z_0 = 3$ 

and the latter possesses an interesting connection with Pythagorean triples [9]. Not only is C irrational [10], it is also transcendental [11].

#### 2. Deuxième exercice

For two distinct starting values  $0 < x_0 < 1$ , determine numerically

$$C = C(x_0) = -\frac{1}{8} \lim_{k \to \infty} k^3 \left( x_k - \frac{4}{k^2} + 12 \frac{\ln(k)}{k^3} \right)$$

where

$$x_k = x_{k-1} \left( 1 - \sqrt{x_{k-1}} \right)$$
 for  $k \ge 1$ .

Using C, find the asymptotic expansion of  $x_k$  to order  $1/k^6$ .

We shall accomplish the steps in reverse order. This work builds on [12, 13, 14]. To conserve space, formulaic knowledge of sections 1, 2, 3 of [3] is assumed. For the function  $f(x) = x (1 - \sqrt{x})$ , we have  $\tau = 1/2$ ,

$$\{a_m\}_{m=1}^7 = \{-1, 0, 0, 0, 0, 0, 0\}$$

and  $\lambda = 2$ ; consequently

$$\{b_j\}_{j=1}^6 = \left\{\frac{3}{2}, \frac{5}{2}, \frac{35}{8}, \frac{63}{8}, \frac{231}{16}, \frac{429}{16}\right\},$$

$$\{a_{0j}\}_{j=1}^6 = \left\{1, 1, \frac{4}{3}, 2, \frac{16}{5}, \frac{16}{3}\right\},$$

$$\{c_i\}_{i=1}^5 = \left\{1, \frac{15}{16}, \frac{35}{24}, \frac{14}{5}, \frac{448}{75}\right\},$$

$$T_2 = \frac{3}{2}X - 1,$$

$$T_3 = -\frac{3}{4}X^2 + \frac{13}{4}X - \frac{39}{16},$$

$$T_4 = \frac{1}{2}X^3 - \frac{35}{8}X^2 + \frac{39}{4}X - \frac{587}{96},$$

$$T_5 = -\frac{3}{8}X^4 + \frac{41}{8}X^3 - \frac{339}{16}X^2 + \frac{1055}{32}X - \frac{5451}{320},$$

$$T_6 = \frac{3}{10}X^5 - \frac{91}{16}X^4 + \frac{575}{16}X^3 - \frac{3127}{32}X^2 + \frac{37629}{320}X - \frac{245957}{4800}$$

$$P_2 = \frac{3}{4}X^2 + \frac{3}{2}X + 2,$$

$$P_3 = \frac{1}{2}X^3 + \frac{21}{8}X^2 + \frac{25}{4}X + \frac{39}{8},$$

and

$$P_4 = \frac{5}{16}X^4 + \frac{47}{16}X^3 + \frac{47}{4}X^2 + \frac{345}{16}X + \frac{731}{48},$$

$$P_5 = \frac{3}{16}X^5 + \frac{171}{64}X^4 + \frac{517}{32}X^3 + \frac{1599}{32}X^2 + \frac{2497}{32}X + \frac{7791}{160},$$

$$P_6 = \frac{7}{64}X^6 + \frac{1377}{640}X^5 + \frac{4645}{256}X^4 + \frac{2641}{32}X^3 + \frac{27073}{128}X^2 + \frac{11499}{40}X + \frac{3091081}{19200}.$$

The remarkable formula connecting  $P_m$  and asymptotics of  $x_k = f(x_{k-1})$  is

$$x_k \sim \left(\frac{\lambda}{k}\right)^{1/\tau} \left\{ 1 + \sum_{m=1}^6 P_m \left( -\frac{1}{\tau} \left[ b_1 \ln(k) + C \right] \right) \frac{1}{k^m} \right\}$$

which implies

$$x_k \sim \frac{4}{k^2} - 12 \frac{\ln(k)}{k^3} - \frac{8C}{k^3} + 27 \frac{\ln(k)^2}{k^4} + (36C - 18) \frac{\ln(k)}{k^4} + (12C^2 - 12C + 8) \frac{1}{k^4}$$

$$- 54 \frac{\ln(k)^3}{k^5} - \left(108C - \frac{189}{2}\right) \frac{\ln(k)^2}{k^5} - \left(72C^2 - 126C + 75\right) \frac{\ln(k)}{k^5}$$

$$- \left(16C^3 - 42C^2 + 50C - \frac{39}{2}\right) \frac{1}{k^5} + \frac{405 \ln(k)^4}{4 \ln(k)^6} + \left(270C - \frac{1269}{4}\right) \frac{\ln(k)^3}{k^6}$$

$$+ \left(270C^2 - \frac{1269}{2}C + 423\right) \frac{\ln(k)^2}{k^6} + \left(120C^3 - 423C^2 + 564C - \frac{1035}{4}\right) \frac{\ln(k)}{k^6}$$

$$+ \left(20C^4 - 94C^3 + 188C^2 - \frac{345}{2}C + \frac{731}{12}\right) \frac{1}{k^6}.$$

Assuming  $x_0 = 1/2$  (the midpoint), the constant C is estimated to be

$$C = 1.98803983644549695008812308629512...$$

by a simple numerical method [2] using the preceding expansion. Assuming  $x_0 = 4/9$  (the argument at which C is minimal), we have instead

$$C = 1.96846882098495471088450855794395...$$

A similar procedure applies to the recurrence

$$x_k = x_{k-1} \left( 1 - x_{k-1}^2 \right) \quad \text{for } k \ge 1.$$

We did this earlier [15] but using a different approach; see also [16]. Values of C were found for  $x_0 = 1/2$  and for  $x_0 = 1/\sqrt{3}$  (again, the argument that minimizes C). We leave analyses of

$$u_k = u_{k-1} \left( 1 - \frac{1}{2} u_{k-1}^2 \right), \quad v_k = v_{k-1} \cos(v_{k-1}), \quad w_k = w_{k-1} \exp\left( -\frac{w_{k-1}^2}{2} \right)$$

for an interested reader.

#### 3. Troisième exercice

For two distinct parameter values q > 1, determine numerically

$$C = C(q) = q \lim_{k \to \infty} k^{(q-1)/q} \left( q^{(q-1)/q} x_k - q k^{1/q} - \frac{q-1}{2q} \frac{\ln(k)}{k^{(q-1)/q}} \right)$$

where

$$x_k = x_{k-1} + \frac{1}{x_{k-1}^{q-1}}$$
 for  $k \ge 1$ ;  $x_0 = 1$ .

Using C, find the asymptotic expansion of  $x_k$  to order  $1/k^{(3q-1)/q}$ . Let  $y_k = x_k^q$ . From

$$\left(\frac{y_k}{y_{k-1}}\right)^{1/q} = \frac{x_k}{x_{k-1}} = 1 + \frac{1}{x_{k-1}^q} = 1 + \frac{1}{y_{k-1}}$$

we have

$$y_k = y_{k-1} \left( 1 + \frac{1}{y_{k-1}} \right)^q = y_{k-1} \varphi \left( \frac{1}{y_{k-1}} \right)$$

where  $\varphi(z) = (1+z)^q$ . Note that  $\varphi(z) > 1$  for all z > 0,  $\varphi(0) = 1$  and the derivative  $\varphi'(z)$  satisfies  $\varphi'(0) \neq 0$ ; also

$$\alpha = \varphi'(0) = q,$$

$$\begin{split} \beta &= \frac{\varphi''(0)}{2\varphi'(0)} = -\frac{1}{2} + \frac{q}{2}, \\ \gamma &= \frac{\varphi''(0)^2}{4\varphi'(0)^3} = \frac{1}{4q} - \frac{1}{2} + \frac{q}{4}, \\ \delta &= -\frac{\varphi''(0)}{4\varphi'(0)} - \frac{\varphi'''(0) - 3C\,\varphi''(0)}{6\varphi'(0)^2} + \frac{\varphi''(0)^2}{4\varphi'(0)^3} = -\frac{1}{12q} - \frac{C}{2q} + \frac{1}{4} + \frac{C}{2} - \frac{q}{6}. \end{split}$$

By Theorem 6 of [17] and Theorem 5 of [18].

$$y_k \sim \alpha k + \beta \ln(k) + C + \gamma \frac{\ln(k)}{k} + \delta \frac{1}{k}$$

as  $k \to \infty$  (beware: the lead coefficient 1/2 of  $\delta$  in [18] should be 1/4). Let r = 1/q. By Proposition 7 of [18],

$$x_k \sim \alpha^r k^r + \frac{r \beta}{\alpha^{1-r}} \frac{\ln(k)}{k^{1-r}} + \frac{r C}{\alpha^{1-r}} \frac{1}{k^{1-r}} + \frac{r(r-1)\beta^2}{2\alpha^{2-r}} \frac{\ln(k)^2}{k^{2-r}} + \frac{r(r-1)\beta C + r \alpha \gamma}{\alpha^{2-r}} \frac{\ln(k)}{k^{2-r}} + \frac{r(r-1)C^2 + 2r \alpha \delta}{2\alpha^{2-r}} \frac{1}{k^{2-r}}$$

which, after multiplying both sides by  $\alpha^{1-r}$ , becomes

$$q^{1-r} x_k \sim q \, k^r + \left( -\frac{1}{2q} + \frac{1}{2} \right) \frac{\ln(k)}{k^{1-r}} + \frac{C}{q} \frac{1}{k^{1-r}} - \left( -\frac{1}{8q^3} + \frac{3}{8q^2} - \frac{3}{8q} + \frac{1}{8} \right) \frac{\ln(k)^2}{k^{2-r}}$$

$$+ \left[ \left( \frac{1}{4q^2} - \frac{1}{2q} + \frac{1}{4} \right) - \left( \frac{1}{2q^3} - \frac{1}{q^2} + \frac{1}{2q} \right) C \right] \frac{\ln(k)}{k^{2-r}}$$

$$+ \left[ -\left( \frac{1}{12q^2} - \frac{1}{4q} + \frac{1}{6} \right) + \left( -\frac{1}{2q^2} + \frac{1}{2q} \right) C - \left( -\frac{1}{2q^3} + \frac{1}{2q^2} \right) C^2 \right] \frac{1}{k^{2-r}}.$$

In particular,

$$2^{1/2} x_k \sim 2k^{1/2} + \frac{1}{4} \frac{\ln(k)}{k^{1/2}} + \frac{C}{2} \frac{1}{k^{1/2}} - \frac{1}{64} \frac{\ln(k)^2}{k^{3/2}} + \left(\frac{1}{16} - \frac{C}{16}\right) \frac{\ln(k)}{k^{3/2}} + \left(-\frac{1}{16} + \frac{C}{8} - \frac{C^2}{16}\right) \frac{1}{k^{3/2}}$$

when q=2;

$$3^{2/3} x_k \sim 3k^{1/3} + \frac{1}{3} \frac{\ln(k)}{k^{2/3}} + \frac{C}{3} \frac{1}{k^{2/3}} - \frac{1}{27} \frac{\ln(k)^2}{k^{5/3}} + \left(\frac{1}{9} - \frac{2C}{27}\right) \frac{\ln(k)}{k^{5/3}} + \left(-\frac{5}{54} + \frac{C}{9} - \frac{C^2}{27}\right) \frac{1}{k^{5/3}}$$

when q = 3;

$$\left(\frac{3}{2}\right)^{1/3} x_k \sim \frac{3}{2} k^{2/3} + \frac{1}{6} \frac{\ln(k)}{k^{1/3}} + \frac{2C}{3} \frac{1}{k^{1/3}} - \frac{1}{216} \frac{\ln(k)^2}{k^{4/3}} + \left(\frac{1}{36} - \frac{C}{27}\right) \frac{\ln(k)}{k^{4/3}} + \left(-\frac{1}{27} + \frac{C}{9} - \frac{2C^2}{27}\right) \frac{1}{k^{4/3}}.$$

when q = 3/2.

To obtain C to high numerical precision requires more terms in the above expansions. The elegant technique from [13, 14], useful in Section 2, does not apply here (since  $x_k \to 0$ ). We turn to a brute-force matching-coefficient method, demonstrated in [2, 4, 15, 19]. For q = 2, we expand

$$(k+1)^{1/2}$$
,  $\frac{\ln(k+1)^{\ell}}{(k+1)^{1/2}}$ ,  $\frac{\ln(k+1)^m}{(k+1)^{3/2}}$ ,  $\frac{\ln(k+1)^n}{(k+1)^{5/2}}$ 

for  $\ell=1,0; m=2,1,0; n=3,2,1,0$  and compare a series for  $x_{k+1}$  with a series for  $x_k+x_k^{-1}$ , yielding additional terms

$$\frac{1}{512} \frac{\ln(k)^3}{k^{5/2}} + \left( -\frac{1}{64} + \frac{3C}{256} \right) \frac{\ln(k)^2}{k^{5/2}} + \left( \frac{5}{128} - \frac{C}{16} + \frac{3C^2}{128} \right) \frac{\ln(k)}{k^{5/2}} + \left( -\frac{11}{384} + \frac{5C}{64} - \frac{C^2}{16} + \frac{C^3}{64} \right) \frac{1}{k^{5/2}}.$$

For q = 3, we expand

$$(k+1)^{1/3}$$
,  $\frac{\ln(k+1)^{\ell}}{(k+1)^{2/3}}$ ,  $\frac{\ln(k+1)^m}{(k+1)^{5/3}}$ ,  $\frac{\ln(k+1)^n}{(k+1)^{8/3}}$ 

for  $\ell$ , m, n identical with before and compare a series for  $x_{k+1}$  with a series for  $x_k + x_k^{-2}$ , yielding additional terms

$$\frac{5}{729} \frac{\ln(k)^3}{k^{8/3}} + \left( -\frac{7}{162} + \frac{5C}{243} \right) \frac{\ln(k)^2}{k^{8/3}} + \left( \frac{43}{486} - \frac{7C}{81} + \frac{5C^2}{243} \right) \frac{\ln(k)}{k^{8/3}} + \left( -\frac{1}{18} + \frac{43C}{486} - \frac{7C^2}{162} + \frac{5C^3}{729} \right) \frac{1}{k^{8/3}}.$$

From such enhancements, letting c = C/q, we obtain

$$c(2) = 0.8615711875687117305317813... = (0.6092228292047829402293060...)\sqrt{2},$$

$$c(3) = 1.3784186157718345713984647...$$

Finding c(3/2) = 0.8010888849... accurately is more difficult, owing to the square root in the iteration. Introducing a power series approximation for it in the brute-force method might be an option; augmenting the theory in [17, 18] is another. Most preferable, however, would be a generalization of the algorithm presented in [13, 14] to encompass certain cases where  $x_k \to \infty$ . The latter would be a powerful broadening of available tools for asymptotic expansion.

#### 4. Acknowledgements

Robert Israel and Anthony Quas gave the simple proof that C exists for the recurrence  $x_k = (1-p) + p x_{k-1}^2$  when  $0 ; their technique was extended in [2] to <math>1/2 and applied to <math>x_k = \sqrt{1 + x_{k-1}}$  in [19]. The creators of Mathematica earn my gratitude every day: this paper could not have otherwise been written.

### 5. Addendum

A certain reciprocity has been found, permitting a calculation (left unfinished in Section 3) to be completed:

$$c(3/2) = \Lambda = 0.8010888849039666437110775...$$

where

$$-\lim_{k \to \infty} k^{5/3} \left( \left( \frac{3}{2} \right)^{2/3} \xi_k - \frac{1}{k^{2/3}} + \frac{1}{9} \frac{\ln(k)}{k^{5/3}} \right) = \frac{2\Lambda}{3},$$
$$\xi_k = \frac{\xi_{k-1}}{1 + \xi_{k-1}^{3/2}} \quad \text{for } k \ge 1; \quad \xi_0 = 1$$

and this iteration (fortunately!) may be treated via [13, 14]. In the same way,

$$-\lim_{k \to \infty} k^{4/3} \left( 3^{1/3} \eta_k - \frac{1}{k^{1/3}} + \frac{1}{9} \frac{\ln(k)}{k^{4/3}} \right) = \frac{c(3)}{3},$$
$$-\lim_{k \to \infty} k^{3/2} \left( 2^{1/2} \zeta_k - \frac{1}{k^{1/2}} + \frac{1}{8} \frac{\ln(k)}{k^{3/2}} \right) = \frac{c(2)}{2}$$

where

$$\eta_k = \frac{\eta_{k-1}}{1 + \eta_{k-1}^3} \text{ and } \zeta_k = \frac{\zeta_{k-1}}{1 + \zeta_{k-1}^2} \text{ for } k \ge 1; \quad \eta_0 = \zeta_0 = 1.$$

We plan to report on this phenomenon in greater depth later [20].

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