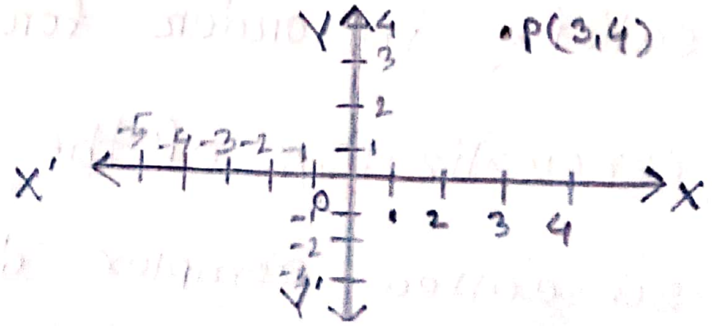
Answer to the question no 1

(a) ⊞ Complex analysis is the study of Complex number together with their derivatives, manipulation and other properties. The key result in complex analysis is the "cauchy integral theorem" which is the reason that single variable complex analysis has so many nice results. A fundamental result of complex analysis is the "Cauchy-Riemann" equations" which give the conditions a function must satisfy in order for a complex generalization of the derivative, the so called complex derivative to exists.

A complex number $x+iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in an xy plane called the complex plane. The complex number represent by P, for example could then be read either (3,4) or $3+4i$.

we refer to the x and y axes as the real and imiginary axes respectively and to the complex plane as the z-plane.



(b) (i) The order pair form of a complex number is an ordered pair (a, b) of two real numbers a and b is represented by the symbol

$a+ib$

(ii) Given,

$$z = (-3+5i)^2$$

or $z = (-3)^2 + 2\cdot(-3)\cdot 5i + (5i)^2$

or, $z = 9 - 30i - 25$

or $z = -16 - 30i$

$\therefore \ |z| = \sqrt{(-16)^2 + (-30)^2}$

$= \sqrt{256 + 900}$

$= 34$

$\therefore$ Arg $z = \pi + \tan^{-1} \dfrac{30}{16}$

Answer to the question no 2

(a) ■ A single valued function is function that for each point in the domain has a unique value in the range.

■ A multiple-valued function is a function that assume two or more distinct values in its range for at least one point in its domain.

■ De Moivre's theorem states that the power of a complex number in polar form is equal to raising the modulus to the same power and multiplying the argument by the same power.

Prove:

If $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

we must show that

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$$z_1/z_2 = r_1/r_2 \{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

Here,

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \{\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\}$$

$$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\}$$

$$= r_1 r_2 \{\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2))\}$$

Aagin

$$z_1/z_2 = \frac{r_1 \cos\theta_1 + i\sin\theta_1}{r_2 \cos\theta_2 + i\sin\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 - \cos\theta_1\, i\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)\}$$

$$= r_1/r_2 \{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

Hence, the theorem is proved.

(b) Given,

$$u = xe^x\cos y - ye^x\sin y$$

$$\frac{\partial u}{\partial x} = \cos y\,(xe^x + e^x) - ye^x\sin y$$

$$\frac{\partial^2 u}{\partial x^2} = \cos y\,(xe^x + 2e^x) - ye^x\sin y$$

$$\frac{\partial u}{\partial y} = -xe^x \sin y - e^x(y\cos y + \sin y)$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x \cos y - e^x(-y\sin y + \cos y + \cos y)$$

$$= -xe^x \cos y + e^x y \sin y - 2e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xe^x \cos y + 2e^x \cos y - ye^x \sin y$$
$$-xe^x \cos y + e^x y \sin y - 2e^x \cos y$$

$$= 0$$

Since, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Hence, u is harmonic.

Let v its conjugate harmonic function, then,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \{-(-xe^x \sin y - e^x y\cos y - e^x \sin y)\} dx$$

$$+ (\cos y . xe^x + \cos y e^x - ye^x \sin y) dy$$

$$= (xe^x \sin y + e^x y\cos y + e^x \sin y) dx + (xe^x \cos y$$

$$+ \cos y\, e^x - ye^x \sin y) dy$$

Integrating, we obtain,

$$v = \sin y (xe^x - e^x) + ye^x \cos y + e^x \sin y + c$$

Answer to the question no 3

ⓐ Given, $\int_{(0,1)}^{(2,5)} (3x+y)\, dx + (2y-x)\, dy$

The given line integral equal

$$\int_{x=0}^{2} \{(3x + x^2 + 1)\, dx + (2x^2 + 2 - x)\, 2x\} dx$$

$$= \int_{x=0}^{2} \{(3x + x^2 + 1) dx + (4x^3 + 4x - 2x^2)\} dx$$

$$\int_{x=0}^{2} (3x+x^2+1)\,dx + \int_{x=0}^{2} (4x^3+4x-2x^2)\,dx$$

$$= \left[\frac{3x^2}{2} + \frac{x^3}{3} + x\right]_0^2 + \left[\frac{4x^4}{4} + \frac{4x^2}{2} - 2\frac{x^3}{3}\right]_0^2$$

$$= \frac{3\cdot\cancel{4}^{2}}{\cancel{2}} + \frac{8}{3} + 2 + \frac{4\times\cancel{16}^{4}}{\cancel{4}} + \frac{4\times\cancel{4}^{2}}{\cancel{2}} - \frac{2\times 8}{3}$$

$$= \frac{18+8+6+48+24-16}{3}$$

$$= \frac{88}{3}$$

☐ Along the straight line from $(0,1)$ to $(2,1)$, $y=1$, $dy=0$, then given integral equals.

$$\int_{x=0}^{2} (3x+1)\,dx + (2\cdot 1-x)\cdot 0$$

$$= \left[3\cdot\frac{x^2}{2} + x\right]_0^2$$

$$= 3\cdot\frac{\cancel{4}^{2}}{\cancel{2}} + 2$$

$$= 8$$

Along the straigh line from $(2,1)$ to $(2,5)$, $x=2$, $dx=0$ and the line integral equals

$$\int_{y=1}^{5} (6+y)0 + (2y-2)\,dy$$

$$= \int_{y=1}^{5} 2y-2\,dy$$

$$= \left[\cancel{2}\frac{y^2}{\cancel{2}} - 2y\right]_1^5$$

$$= (5^2-10) - (1-2)$$

$$= 15+1$$

$$= 16$$

Then the required value $= 8+16$
$= 24$

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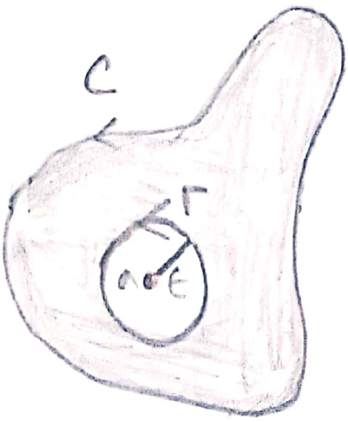
⑥ Cauchy's integral formulae is

If $f(z)$ be analytic inside and on a simple closed curve C and a is any point inside C then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

prove: The function $f(z)/z-a$ is analytic inside and on C except at the point $z = a$, we have

$$\oint_C \frac{f(z)}{z-a} dz = \oint_\Gamma \frac{f(z)}{z-a} dz$$



where we can choose Γ as a circle of radius ϵ with centre at a. Then an equation for Γ is $|z-a| = \epsilon$. substituting $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, so

$$\oint_\Gamma \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})\, i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta$$

$$= i\int_0^{2\pi} f(a+\epsilon e^{i\theta})$$

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Taking limit both side

$$\oint_C \frac{f(z)}{z-a}\,dz = \lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} f(a)\,d\theta = 2\pi i f(a)$$

So that we have, as required

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Answer to the question no 4

ⓐ The Residue theorem

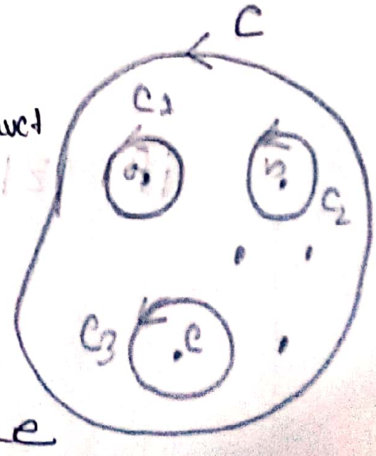
Let $f(z)$ be single valued and analytic inside and on a simple closed curve C except at the singularities a, b, c inside C which have residues a_{-1}, b_{-1}, c_{-1}. Then the theorem states that

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \dots)$$

i.e the integral of $f(z)$ around C is $2\pi i$ times the sum of residues of $f(z)$ at the singularities enclosed by C

Prove:

With centers at a, b, c construct circle $C_1, C_2, C_3 \dots$ that lie entirely inside C. This can be done since $a, b, c \dots$ are



interior points. We have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \dots$$

$$\oint_{C_1} f(z)\,dz = 2\pi i\, a_{-1}$$

$$\oint_{C_2} f(z)\,dz = 2\pi i\, b_{-1}$$

$$\oint_{C_3} f(z)\,dz = 2\pi i\, c_{-1}$$

So, $\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \dots)$

Hence the theorem is proved.

(b) Here, $$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\,\frac{1}{z+3}$$

if $|z| > 1$, $$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right)$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \dots$$

if $|z| < 3$, $$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right)$$

$$= \frac{1}{6} - \frac{z}{18} + \frac{z^2}{56} - \frac{z^3}{162} + \dots$$

then the required Laurent expansion valid for both $|z|>1$ and $|z|<3$ i.e $1<|z|<3$ is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{56} + \frac{z^3}{162} - \cdots$$

C

(i) If a is outside C, then $f(z) = \frac{1}{z-a}$ is analytic every where inside and on C. Hence by cauchy's theorem,

$$\oint_C \frac{dz}{z-a} = 0$$

(ii) Suppose a is inside C and let Γ be a circle of radius ϵ with center at $z=a$ so that Γ is inside C.

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a}$$

Now on, Γ, $|z-a| = \epsilon$ on $z-a = \epsilon e^{i\theta}$ i.e $z = a + \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$

Thus, since $dz = i\epsilon e^{i\theta} d\theta$ the right side becomes.

$$\int_{\theta=0}^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

which is required value.

section-B

Answer to the question no 5

ⓐ. If the kernel $K(S,t)$ is defined

as

$$K(S,t) = \begin{cases} 0 & \text{for } t<0 \\ e^{-St} & \text{for } t \geq 0 \end{cases}$$

then $f(S) = \int_0^\infty e^{-St} F(t)\, dt$

The function $f(s)$ defined above integral is called the laplace transform of the function $F(t)$ and is denoted by $\mathcal{L}(F(t))$.

ⓑ Given,

$$F(t) = \begin{cases} t, & 0<t<2 \\ 3, & t>2 \end{cases}$$

By definition of laplace transform of a function $F(t)$, we have

$$\mathcal{L}(F(t)) = \int_0^\infty e^{-St} F(t)\, dt$$

$$= \int_0^2 e^{-st} \cdot F(t)\, dt + \int_2^\infty e^{-st} F(t)\, dt$$

$$= \int_0^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= \left[t\, e^{-st}/_{-s} \right]_0^2 + \int_0^2 \frac{e^{-st}}{s}\, dt + 3 \cdot \left[\frac{e^{-st}}{-s} \right]_2^\infty$$

$$= -\frac{2}{s} e^{-2s} + \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^2 + 3 \left[0 - \frac{e^{-2s}}{-s} \right]$$

$$= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} + 3 \cdot \frac{e^{-2s}}{s}$$

$$= \frac{1}{s^2} + \frac{e^{-2s}}{s} + - \frac{e^{-2s}}{s^2}$$

(b) First shifting property is

If $\mathcal{L}\{F(t)\} = f(s)$ then

$\mathcal{L}\{e^{at} F(t)\} = f(s-a)$.

Proof: By definition of Laplace transform we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, ds = f(s)$$

$$= \int_0^\infty \{e^{-(s-a)t} F(t)\} \, dt$$

$$= f(s-a)$$

Hence, the theorem is proved.

Answer to the question no 7

(a) same as

(b) Given $tY'' + Y' + 4tY = 0$

$$\mathcal{L}(tY'') + \mathcal{L}(Y') + 4\mathcal{L}(tY) = 0$$

or, $-\dfrac{d}{ds}\{s^2y - sY(0) - Y'(0)\} + sy - Y(0) - \dfrac{4dy}{ds} = 0$

or $-\dfrac{d}{ds}\{s^2y - 3s\} + sy - 3 - 4\dfrac{dy}{ds} = 0$

or $(s^2+4)\dfrac{dy}{ds} + sy = 0$

or $\dfrac{dy}{y} + \dfrac{s}{s^2+4} = 0$

Integrating, $\log y + \frac{1}{2}\log(s^2+4) = \log c$

or $\log y\sqrt{s^2+4} = \log c$

$\therefore\ y = \dfrac{c}{\sqrt{s^2+4}}$

$\mathcal{L}^{-1}(y) = c\,\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{s^2+2^2}}\right\}$

$Y = c\,J_0\,2t$

Here $Y(0) = c\,J_0(0)$

$\therefore\ c = 1$

So, $Y = 3J_0 2t$

which is required solution.

Answer to the question no: 8

ⓐ Here,

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

or, $5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$ — ①

putting $s=1$ in (1)

$8 = 8A \therefore A = 1$

Equating co-efficient of s^2 from both side of 1

$A + B = 0$

$\therefore B = -A = -1$

Putting $s=0$ in 1

$3 = 5A - C$

$C = 5A - 3 = 2$

So, $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} + \frac{-s+2}{s^2+2.s.1+1^2+4}$$

$$= \frac{1}{s-1} + \frac{-(s+1)+3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t.$$

Ans

(b) Let $F(t)$ and $G(t)$ be ~~be~~ two functions, then the convolution of the two function $F(t)$ and $G(t)$ denoted by $F*G$ is defined by the relation

$$F * G = \int_0^t F(u) \cdot G(t-u)\, du$$

Here

$$\frac{3}{s^2(s+2)} = \frac{1}{s+2} \cdot \frac{3}{s^2}$$

$$f(s) = \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= e^{-2t} = F(t)$$

$$g(s) = \frac{3}{s^2}$$

$$\mathcal{L}^{-1}\{g(s)\} = 3\mathcal{L}^{-1}\frac{1}{s^2}$$

$$= 3t = G(t)$$

Now, By using convolution theorem.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\} = \int_0^t e^{-2u}\, 3(t-u)\, du$$

$$= \int_0^t e^{-2u} \cdot 3t\, du - \int_0^t e^{-2u}\, 3u\, du$$

$$= 3t\int_0^t e^{-2u}\, du - 3\int_0^t u e^{-2u}\, du$$

$$= 3t \cdot \left[\frac{e^{-2u}}{-2}\right]_0^t - 3\left[\left[\frac{ue^{-2u}}{-2}\right]_0^t - \left[\frac{e^{-2u}}{4}\right]_0^t\right]$$

$$= -\frac{3t}{2}e^{-2t} + \frac{3t}{2} + \frac{3t}{2}e^{-2t} + \frac{3}{4}e^{-2t} - \frac{3}{4}$$

$$= \frac{3t}{2} + \frac{3}{4}e^{-2t} - \frac{3}{4}$$

Ans