Dagger Categories of Relations

Matthew Di Meglio

Joint work with Chris Heunen, Paolo Perrone and Dario Stein

Edinburgh Category Seminar
March 2025

Bijection between contractions and relations in category of coisometries

Linear maps $f: X \to Y$ between Hilbert spaces such that $\|fx\| \leqslant \|x\|$ for all $x \in X$

Bijection between contractions and relations in category of coisometries

Linear maps $f: X \to Y$ between Hilbert spaces such that $\|fx\| \leq \|x\|$ for all $x \in X$

Bijection between contractions and relations in category of coisometries

Isomorphism classes of jointly monic spans

Linear maps $f : X \to Y$ between Hilbert spaces such that $\|fx\| \leqslant \|x\|$ for all $x \in X$

Bijection between contractions and relations in category of coisometries

Isomorphism classes of jointly monic spans

Adjointable maps $f : X \to Y$ such that $ff^{\dagger} = 1$ (orthogonal projections onto closed subspaces)

But category of coisometries

is not regular

But category of coisometries

is not regular

It does not have pullbacks

regular categories

$$\mathbf{Reg} \underset{\mathbf{Map}}{\overset{\mathbf{Rel}}{\rightleftarrows}} \mathbf{Tab}$$

tabular allegories

regular categories

$$\begin{tikzcd} \mathbf{Reg} \arrow[r, shift left, "\mathrm{Rel}"] & \mathbf{Tab} \arrow[l, shift left, "\mathrm{Map}"] \end{tikzcd}$$

tabular allegories

$$\begin{tikzcd} ? \arrow[r, shift left, "\mathrm{Rel}"] & \mathbf{Dil} \arrow[l, shift left, "\mathrm{Coisometry}"] \end{tikzcd}$$

$\dagger$-categories with dilators

DEFINITION

A $\dagger$-category is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad f^{\dagger\dagger} = f$$

DEFINITION

A †-category is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad f^{\dagger\dagger} = f$$

EXAMPLES

Con

Hilbert spaces and contractions

$$\langle y | fx \rangle = \langle f^\dagger y | x \rangle$$

DEFINITION

A **†-category** is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad f^{\dagger\dagger} = f$$

EXAMPLES

Con

Hilbert spaces and contractions

$$\langle y | fx \rangle = \langle f^\dagger y | x \rangle$$

PInj

Sets and injective partial functions

$$y = fx \iff x = f^\dagger y$$

DEFINITION

A †-category is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad f^{\dagger\dagger} = f$$

EXAMPLES

Con

Hilbert spaces and contractions

$$\langle y | fx \rangle = \langle f^\dagger y | x \rangle$$

PInj

Sets and injective partial functions

$$y = fx \iff x = f^\dagger y$$

FinPS

A finite probability space X is a finite set X equipped with a function $\mathbb{P}_X : X \to (0,1]$ such that $\sum_{x \in X} \mathbb{P}_X(x) = 1$.

DEFINITION

A †-category is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad f^{\dagger\dagger} = f$$

EXAMPLES

Con

Hilbert spaces and contractions

$$\langle y | fx \rangle = \langle f^\dagger y | x \rangle$$

PInj

Sets and injective partial functions

$$y = fx \iff x = f^\dagger y$$

FinPS

A finite probability space X is a finite set X equipped with a function $\mathbb{P}_X : X \to (0,1]$ such that $\sum_{x \in X} \mathbb{P}_X(x) = 1$.

A stochastic map $f : X \to Y$ is a function $\mathbb{P}_f(-|-) : Y \times X \to [0,1]$ such that

$$\sum_{y \in Y} \mathbb{P}_f(y|x) = 1$$

$$\sum_{x \in X} \mathbb{P}_f(y|x)\, \mathbb{P}_X(x) = \mathbb{P}_Y(y)$$

DEFINITION

A **†-category** is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad f^{\dagger\dagger} = f$$

EXAMPLES

Con

Hilbert spaces and contractions

$$\langle y | fx \rangle = \langle f^\dagger y | x \rangle$$

PInj

Sets and injective partial functions

$$y = fx \iff x = f^\dagger y$$

FinPS

A **finite probability space** X is a finite set X equipped with a function $\mathbb{P}_X : X \to (0,1]$ such that $\sum_{x \in X} \mathbb{P}_X(x) = 1$.

A **stochastic map** $f : X \to Y$ is a function $\mathbb{P}_f(-|-) : Y \times X \to [0,1]$ such that

$$\sum_{y \in Y} \mathbb{P}_f(y|x) = 1$$

$$\sum_{x \in X} \mathbb{P}_f(y|x)\,\mathbb{P}_X(x) = \mathbb{P}_Y(y)$$

The **Bayesian inverse** $f^\dagger$ of f is given by

$$\mathbb{P}_f(y|x)\,\mathbb{P}_X(x) = \mathbb{P}_{f^\dagger}(x|y)\,\mathbb{P}_Y(y)$$

DEFINITION

A morphism f is coisometric if $ff^{\dagger}=1$.

DEFINITION

A morphism f is coisometric if $ff^\dagger = 1$.

Let **Coisometry(C)** be the wide subcategory of coisometries in **C**.

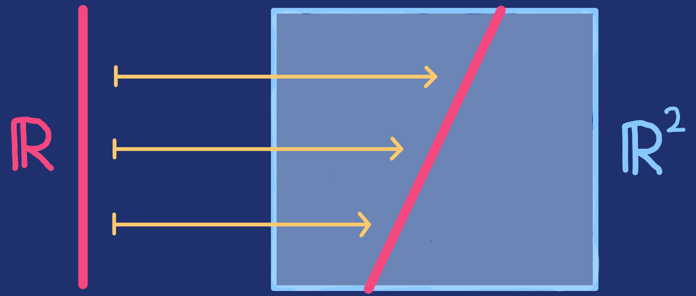
DEFINITION

A morphism f is coisometric if $ff^{\dagger} = 1$.

Let **Coisometry(C)** be the wide subcategory of coisometries in **C**.

EXAMPLES

Isometries in **Con** are inclusions of closed subspaces.



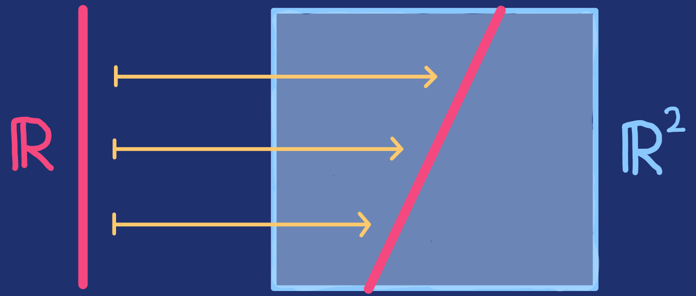
DEFINITION

A morphism f is coisometric if $ff^\dagger = 1$.

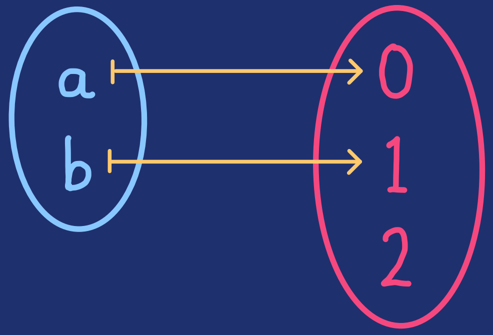
Let **Coisometry(C)** be the wide subcategory of coisometries in **C**.

EXAMPLES

Isometries in **Con** are inclusions of closed subspaces.



Isometries in **PInj** are total.



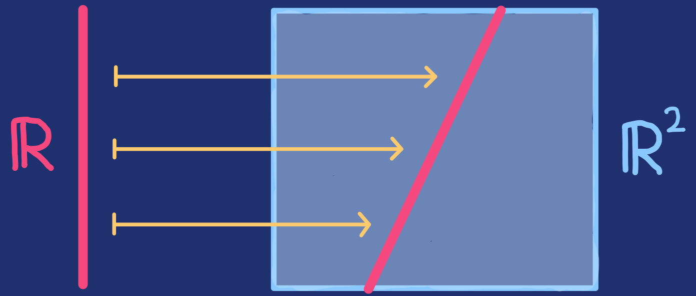
DEFINITION

A morphism f is coisometric if $ff^\dagger = 1$.

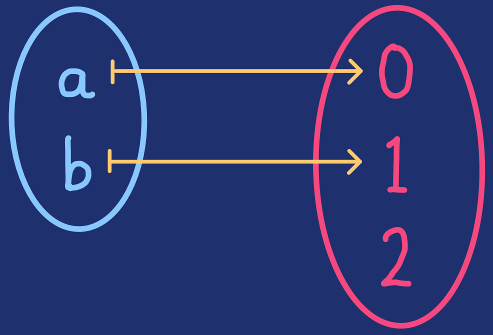
Let **Coisometry(C)** be the wide subcategory of coisometries in **C**.

EXAMPLES

Isometries in **Con** are inclusions of closed subspaces.



Isometries in **PInj** are total.



Coisometries in **FinPS** are deterministic

$$\mathbb{P}_f(y|x) \in \{0, 1\}$$

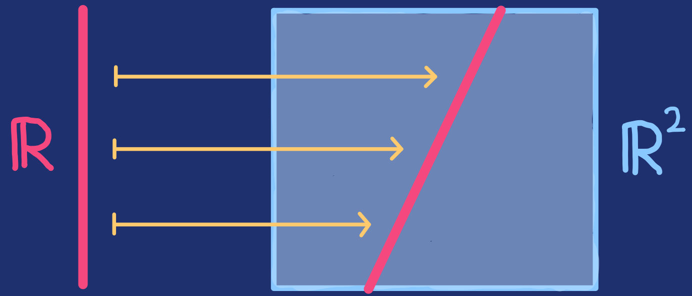
DEFINITION

A morphism f is coisometric if $ff^\dagger = 1$.

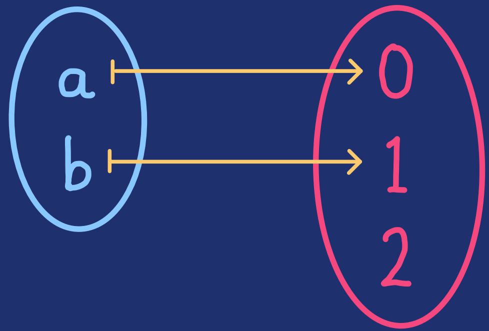
Let **Coisometry(C)** be the wide subcategory of coisometries in **C**.

EXAMPLES

Isometries in **Con** are inclusions of closed subspaces.



Isometries in **PInj** are total.



Coisometries in **FinPS** are deterministic

$$\mathbb{P}_f(y|x) \in \{0,1\}$$

Write f also for the underlying function

$$\mathbb{P}_f(y|x) = \begin{cases} 1 & \text{if } y = fx \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION

A **dilation** of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

DEFINITION

A dilation of $f : X \to Y$ is a span

$$X \overset{s_1}{\twoheadleftarrow} S \overset{s_2}{\twoheadrightarrow} Y$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

DEFINITION

A dilation of $f : X \to Y$ is a span

$$X \overset{s_1}{\twoheadleftarrow} S \overset{s_2}{\twoheadrightarrow} Y$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

$$X \underset{p_1}{\twoheadleftarrow} X \boxplus_f Y \underset{p_2}{\twoheadrightarrow} Y$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, two heads, "s_1"'] \arrow[dr, two heads, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

DEFINITION

A **dilation** of $f : X \to Y$ is a span

$$X \overset{s_1}{\twoheadleftarrow} S \overset{s_2}{\twoheadrightarrow} Y$$

of coisometries such that $f = s_2 s_1^\dagger$.

A **dilator** of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

DEFINITION

A **dilation** of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^{\dagger}$.

A **dilator** of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, two heads, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, two heads, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd} & X \oplus Y & \\ X \arrow[ur, tail, "{\begin{bmatrix} \sqrt{1-f^{\dagger}f} & f \end{bmatrix}}"] & & Y \arrow[ul, tail, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"'] \end{tikzcd}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd}
 & S \arrow[dl, two heads, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, two heads, "s_2"] & \\
X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd}
 & X \oplus Y & \\
X \arrow[ur, tail, "s_1"] & & Y \arrow[ul, tail, "s_2"']
\end{tikzcd}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] \arrow[d, dashed, two heads, "s"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd} & X \oplus Y & \\ X \arrow[ur, tail, "s_1"] \arrow[r, tail] & X \vee Y \arrow[u, tail, "s_1 \vee s_2"] & Y \arrow[ul, tail, "s_2"'] \arrow[l, tail] \end{tikzcd}$$

DEFINITION

A **dilation** of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A **dilator** of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] \arrow[d, dashed, two heads, "s"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd} & X \oplus Y & \\ X \arrow[ur, tail, "s_1"] \arrow[r, tail] & X \vee Y \arrow[u, tail, "s_1 \vee s_2"] & Y \arrow[ul, tail, "s_2"'] \arrow[l, tail] \end{tikzcd}$$

$$X \vee Y = \overline{\mathrm{Im}\, s_1 + \mathrm{Im}\, s_2}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^{\dagger}$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd} & X \oplus Y & \\ X \arrow[ur, tail, "s_1"] \arrow[r, tail] & X \vee Y \arrow[u, tail, "s_1 \vee s_2"] & Y \arrow[ul, tail, "s_2"'] \arrow[l, tail] \end{tikzcd}$$

$$X \vee Y = \overline{\operatorname{Im} s_1 + \operatorname{Im} s_2}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^{\dagger}$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] \arrow[d, dashed, two heads, "s"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd} & X \oplus Y & \\ X \arrow[ur, tail, "s_1"] \arrow[r, tail] & X \vee Y \arrow[u, tail, "s_1 \vee s_2"] & Y \arrow[ul, tail, "s_2"'] \arrow[l, tail] \end{tikzcd}$$

$$X \vee Y = \overline{\mathrm{Im}\, s_1 + \mathrm{Im}\, s_2}$$

Codilators in **PInj**

$$\begin{tikzcd} & (X \setminus \mathrm{Dom}\, f) \sqcup Y & \\ X \arrow[ur, tail] & & Y \arrow[ul, tail] \end{tikzcd}$$

DEFINITION

A **dilation** of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A **dilator** of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Codilators in **Con**

$$\begin{tikzcd} & X \oplus Y & \\ X \arrow[ur, tail, "s_1"] \arrow[r, tail] & X \vee Y \arrow[u, tail, "s_1 \vee s_2"] & Y \arrow[ul, tail, "s_2"'] \arrow[l, tail] \end{tikzcd}$$

$$X \vee Y = \overline{\operatorname{Im} s_1 + \operatorname{Im} s_2}$$

Codilators in **PInj**

$$\begin{tikzcd} & (X \setminus \operatorname{Dom} f) \sqcup Y \arrow[d, phantom, "\vee"] & \\ X \arrow[ur, tail] & {} \arrow[d, phantom, "\wedge"] & Y \arrow[ul, tail] \\ & \operatorname{Dom} f \arrow[ul, tail] \arrow[ur, tail, "f|_{\operatorname{Dom} f}"'] & \end{tikzcd}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

DEFINITION

A dilation of $f: X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^{\dagger}$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Dilators in **FinPS**

$$\begin{tikzcd} X & X \boxplus_f Y \arrow[l, two heads, "p_1"'] \arrow[r, two heads, "p_2"] & Y \end{tikzcd}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$X \overset{s_1}{\twoheadleftarrow} S \overset{s_2}{\twoheadrightarrow} Y$$

of coisometries such that $f = s_2 s_1^{\dagger}$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\
X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}$$

EXAMPLES

Dilators in **FinPS**

$$X \boxplus_f Y = \{(x, y) \in X \times Y : \mathbb{P}_f(y|x) \neq 0\}$$

$$X \overset{p_1}{\twoheadleftarrow} X \boxplus_f Y \overset{p_2}{\twoheadrightarrow} Y$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^{\dagger}$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[d, dashed, two heads, "s"] \arrow[dr, "s_2"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Dilators in **FinPS**

$$X \boxplus_f Y = \{(x, y) \in X \times Y : \mathbb{P}_f(y|x) \neq 0\}$$

$$\mathbb{P}_{X \boxplus_f Y}(x, y) = \mathbb{P}_f(y|x)\,\mathbb{P}_X(x)$$

$$\begin{tikzcd} X & X \boxplus_f Y \arrow[l, two heads, "p_1"'] \arrow[r, two heads, "p_2"] & Y \end{tikzcd}$$

DEFINITION

A dilation of $f : X \to Y$ is a span

$$\begin{tikzcd} X & S \arrow[l, two heads, "s_1"'] \arrow[r, two heads, "s_2"] & Y \end{tikzcd}$$

of coisometries such that $f = s_2 s_1^\dagger$.

A dilator of f is a terminal dilation of f.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] \arrow[d, dashed, two heads, "s"] & \\ X & X \boxplus_f Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y \end{tikzcd}$$

EXAMPLES

Dilators in **FinPS**

$$X \boxplus_f Y = \{(x, y) \in X \times Y : \mathbb{P}_f(y|x) \neq 0\}$$

$$\mathbb{P}_{X \boxplus_f Y}(x, y) = \mathbb{P}_f(y|x)\mathbb{P}_X(x)$$

$$\begin{tikzcd} X & X \boxplus_f Y \arrow[l, two heads, "p_1"'] \arrow[r, two heads, "p_2"] & Y \end{tikzcd}$$

$$p_1(x, y) = x \qquad p_2(x, y) = y$$

Remnants of dilators in Coisometry(C)

Remnants of dilators in Coisometry(C)

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, two heads, "s"] \arrow[dr, "s_2"] & \\
X & X \boxplus_{s_2 s_1^\dagger} Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}$$

factorisation of spans

Remnants of dilators in Coisometry(C)

$$
\begin{tikzcd}
 & S \arrow[dl, two heads, "s_1"'] \arrow[d, two heads, "s"] \arrow[dr, two heads, "s_2"] & \\
X & X \boxplus_{s_2 s_1^\dagger} Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}
$$

factorisation of spans

$$
\begin{tikzcd}
 & X \boxplus_{f_2^\dagger f_1} Y \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_2"] & \\
X \arrow[dr, two heads, "f_1"'] & & Y \arrow[dl, two heads, "f_2"] \\
 & A &
\end{tikzcd}
$$

"pullbacks"

Remnants of dilators in Coisometry(C)

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, two heads, "s"] \arrow[dr, two heads, "s_2"] & \\
X & X \boxplus_{s_2 s_1^\dagger} Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}$$

factorisation of spans

$$\begin{tikzcd}
 & X \boxplus_{f_2^\dagger f_1} Y \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_2"] & \\
X \arrow[dr, two heads, "f_1"'] & & Y \arrow[dl, two heads, "f_2"] \\
 & A &
\end{tikzcd}$$

“pullbacks”

DEFINITION

An independence category is a category equipped with a predicate $\perp$ on commuting squares such that

DEFINITION

An independence category is a category equipped with a predicate $\perp$ on commuting squares such that

(i)
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \text{ and } \begin{tikzcd} \cdot \arrow[r, "b"] \arrow[d, "g"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$$

implies
$$\begin{tikzcd} \cdot \arrow[r, "ba"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "ts"'] & \cdot \end{tikzcd}$$

DEFINITION

An independence category is a category equipped with a predicate $\perp$ on commuting squares such that

(i)
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \cdot \arrow[r, "b"] \arrow[d, "g"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$$

implies
$$\begin{tikzcd} \cdot \arrow[r, "ba"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "ts"'] & \cdot \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

DEFINITION

An *independence category* is a category equipped with a predicate $\perp$ on commuting squares such that

(i)
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \cdot \arrow[r, "b"] \arrow[d, "g"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$$

implies
$$\begin{tikzcd} \cdot \arrow[r, "ba"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "ts"'] & \cdot \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

(iii)
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

DEFINITION

An independence category is a category equipped with a predicate $\perp$ on commuting squares such that

(i)
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \cdot \arrow[r, "b"] \arrow[d, "g"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$$

implies
$$\begin{tikzcd} \cdot \arrow[r, "ba"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "ts"'] & \cdot \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

(iii)
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

(iv)
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

Inspired by Simpson "Equivalence and Independence in Atomic Sheaf Logic"

LEMMA

If **C** is a †-category, then **Coisometry(C)** is an independence category where

$$\begin{tikzcd} {} \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & {} \arrow[d, "g"] \\ {} \arrow[r, "s"'] & {} \end{tikzcd} \quad \text{if} \quad \begin{aligned} ga &= sf \\ af^{\dagger} &= g^{\dagger}s \end{aligned}$$

LEMMA

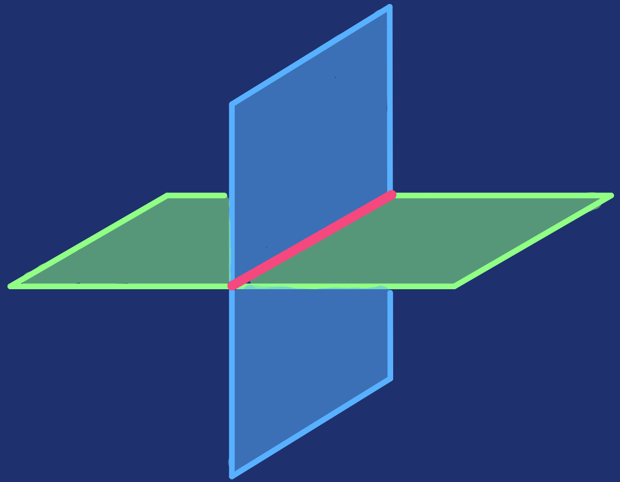
If C is a $\dagger$-category, then **Coisometry(C)** is an independence category where

$$\begin{tikzcd} {} \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & {} \arrow[d, "g"] \\ {} \arrow[r, "s"'] & {} \end{tikzcd}$$

if

$$ga = sf$$
$$af^{\dagger} = g^{\dagger}s$$

$$\begin{tikzcd} & (0,y,z) & (y,z) \arrow[l, mapsto] & \\ (x,0,z) & \mathbb{R}^3 & \mathbb{R}^2 \arrow[l] & (0,z) \\ (x,z) \arrow[u, mapsto] & \mathbb{R}^2 \arrow[u] & \mathbb{R} \arrow[l] \arrow[u] \arrow[ul, phantom, "\perp"] & z \arrow[u, mapsto] \\ & (0,z) & z \arrow[l, mapsto] & \end{tikzcd}$$



EXAMPLES

In **Isometry(Con)**,

$\perp$ captures relative orthogonality.

LEMMA

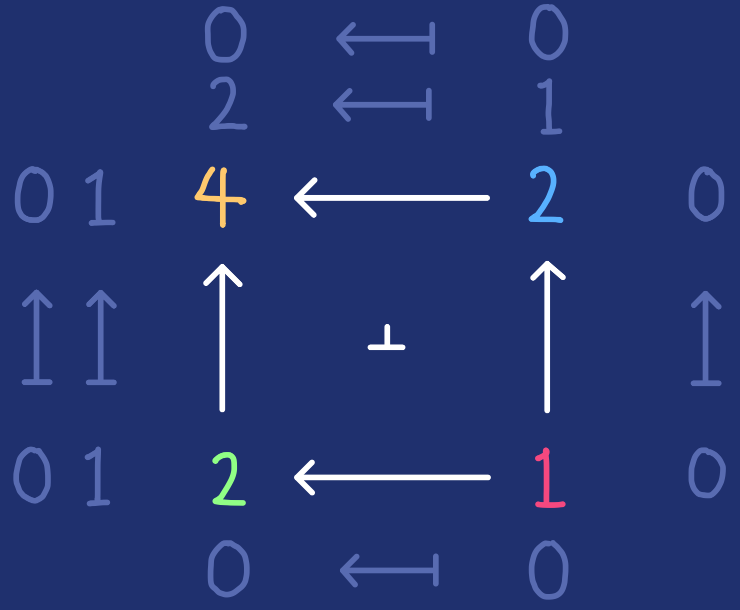
If C is a $\dagger$-category, then **Coisometry(C)** is an independence category where

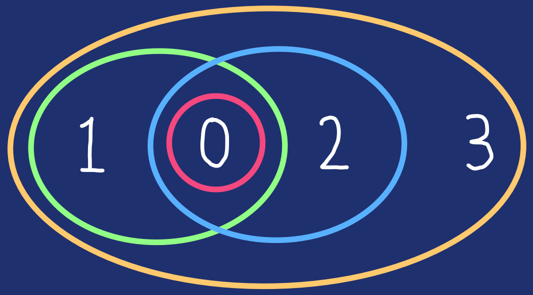
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \quad \text{if} \quad \begin{aligned} ga &= sf \\ af^{\dagger} &= g^{\dagger}s \end{aligned}$$

EXAMPLES

In **Isometry(PInj)**,

$\perp$ captures relative disjointness.





LEMMA

If C is a $\dagger$-category, then **Coisometry(C)** is an independence category where

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \quad \text{if} \quad \begin{array}{l} ga = sf \\ af^\dagger = g^\dagger s \end{array}$$

EXAMPLES

In **Coisometry(FinPS)**,

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & A \end{tikzcd}$$

$$\Updownarrow$$

$sf = tg$ and

$$\mathbb{P}_Z(f^{-1}\{x\} \cap g^{-1}\{y\}) = \frac{\mathbb{P}_X(x)\mathbb{P}_Y(y)}{\mathbb{P}_A(a)}$$

for all $x \in s^{-1}\{a\}$ and $y \in t^{-1}\{a\}$

DEFINITION

In an independence category, an **independent pullback** is a square

$$\begin{tikzcd} {} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & {} \arrow[d, "t"] \\ {} \arrow[r, "s"'] & {} \end{tikzcd}$$

such that for all

$$\begin{tikzcd} {} \arrow[rr, "b"] \arrow[d, "a"'] \arrow[drr, phantom, "\perp"] & & {} \arrow[d, "t"] \\ {} \arrow[r, "r"'] & {} \arrow[r, "s"'] & {} \end{tikzcd}$$

exists unique c such that

$$\begin{tikzcd} {} \arrow[r, "c"] \arrow[d, "a"'] \arrow[dr, phantom, "\perp"] & {} \arrow[d, "f"] \\ {} \arrow[r, "r"'] & {} \end{tikzcd}$$

and $b = gc$.

DEFINITION

In an independence category, an **independent pullback** is a square

$$\begin{tikzcd} {} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & {} \arrow[d, "t"] \\ {} \arrow[r, "s"'] & {} \end{tikzcd}$$

such that for all

$$\begin{tikzcd} {} \arrow[rr, "b"] \arrow[d, "a"'] \arrow[drr, phantom, "\perp"] & & {} \arrow[d, "t"] \\ {} \arrow[r, "r"'] & {} \arrow[r, "s"'] & {} \end{tikzcd}$$

exists unique c such that

$$\begin{tikzcd} {} \arrow[r, "c"] \arrow[d, "a"'] \arrow[dr, phantom, "\perp"] & {} \arrow[d, "f"] \\ {} \arrow[r, "r"'] & {} \end{tikzcd}$$

and $b = gc$.

Weak independent pullbacks are similar, but with $r = 1$.

DEFINITION

In an independence category, an **independent pullback** is a square

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

such that for all

$$\begin{tikzcd} \cdot \arrow[rr, "b"] \arrow[d, "a"'] \arrow[drr, phantom, "\perp"] & & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "r"'] & \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

exists unique c such that

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "a"'] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "r"'] & \cdot \end{tikzcd}$$

and $b = gc$.

Weak independent pullbacks are similar, but with $r = 1$.

LEMMA

If C is a $\dagger$-category with dilators, then $\mathrm{Coisometry}(C)$ has weak independent pullbacks.

Inspired by Simpson "Equivalence and Independence in Atomic Sheaf Logic"

Remnants of dilators in Coisometry(C)

$$
\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, two heads, "s"] \arrow[dr, "s_2"] & \\
X & X \boxplus_{s_2 s_1^{\dagger}} Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}
$$

factorisation of spans

$$
\begin{tikzcd}
 & X \boxplus_{f_2^{\dagger} f_1} Y \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_2"] & \\
X \arrow[dr, two heads, "f_1"'] & & Y \arrow[dl, two heads, "f_2"] \\
 & A &
\end{tikzcd}
$$

"pullbacks"

Remnants of dilators in Coisometry(C)

$$
\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, two heads, "s"] \arrow[dr, "s_2"] & \\
X & X \boxplus_{s_2 s_1^\dagger} Y \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & Y
\end{tikzcd}
$$

factorisation of spans

$$
\begin{tikzcd}
 & X \boxplus_{f_2^\dagger f_1} Y \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_2"] & \\
X \arrow[dr, "f_1"'] & & Y \arrow[dl, "f_2"] \\
 & A &
\end{tikzcd}
$$

"pullbacks"

DEFINITION

A morphism e is strong epic if it is left orthogonal to the jointly monic spans.

(Non-standard definition)

DEFINITION

A morphism e is strong epic if it is left orthogonal to the jointly monic spans.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, two heads, "e"'] & & X \arrow[dd, tail, "m_1"'] \arrow[ddr, tail, "m_2"] & \\
& & & \\
B \arrow[rr, "g_1"] \arrow[rrr, "g_2"'] & & Y_1 & Y_2
\end{tikzcd}$$

(Non-standard definition)

DEFINITION

A morphism e is strong epic if it is left orthogonal to the jointly monic spans.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "e"'] & X \arrow[d, "m_1"] \arrow[dr, "m_2"] & \\
B \arrow[ur, dashed, "d"] \arrow[r, "g_1"] \arrow[rr, bend right, "g_2"'] & Y_1 & Y_2
\end{tikzcd}
$$

(Non-standard definition)

DEFINITION

A morphism e is **strong epic** if it is left orthogonal to the jointly monic spans.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "e"'] & X \arrow[d, "m_1"] \arrow[ddr, "m_2"] & \\
B \arrow[ur, dashed, "d"] \arrow[r, "g_1"] \arrow[drr, "g_2"'] & Y_1 & \\
& & Y_2
\end{tikzcd}
$$

(Non-standard definition)

LEMMA

Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then

(i) A span in $\mathbf{Coisometry(C)}$ is jointly monic if and only if it is a dilator in $\mathbf{C}$.

DEFINITION

A morphism e is **strong epic** if it is left orthogonal to the jointly monic spans.

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, two heads, "e"'] & & X \arrow[d, tail, "m_1"] \arrow[ddr, "m_2"] & \\
B \arrow[urr, dashed, "d"] \arrow[rr, "g_1"] \arrow[drrr, "g_2"'] & & Y_1 & \\
& & & Y_2
\end{tikzcd}
$$

(Non-standard definition)

LEMMA

Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then

(i) A span in $\mathbf{Coisometry}(\mathbf{C})$ is jointly monic if and only if it is a dilator in $\mathbf{C}$.

(ii) Every morphism in $\mathbf{Coisometry}(\mathbf{C})$ is strong epic.

DEFINITION

A morphism e is **strong epic** if it is left orthogonal to the jointly monic spans.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"', two heads] & X \arrow[d, "m_1"'] \arrow[dr, "m_2"] & \\
B \arrow[ur, dashed, "d"] \arrow[r, "g_1"] \arrow[rr, bend right=15, "g_2"'] & Y_1 & Y_2
\end{tikzcd}$$

(Non-standard definition)

LEMMA

Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then

(i) A span in $\mathbf{Coisometry}(\mathbf{C})$ is jointly monic if and only if it is a dilator in $\mathbf{C}$.

(ii) Every morphism in $\mathbf{Coisometry}(\mathbf{C})$ is strong epic.

(iii) Every span in $\mathbf{Coisometry}(\mathbf{C})$ has a (strong epic, jointly monic) factorisation.

DEFINITION

An independence category is regular-ish if

DEFINITION

An independence category is regular-ish if

(i) it has weak independent pullbacks,

DEFINITION

An independence category is regular-ish if

(i) it has weak independent pullbacks,

(ii) every span factors as a strong epi followed by a jointly monic span,

DEFINITION

An independence category is **regular-ish** if

(i) it has weak independent pullbacks,

(ii) every span factors as a strong epi followed by a jointly monic span,

(iii) every morphism is strong epic

DEFINITION

An independence category is regular-ish if

(i) it has weak independent pullbacks,

(ii) every span factors as a strong epi followed by a jointly monic span,

(iii) every morphism is strong epic

THEOREM

Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then $\mathbf{Coisometry}(\mathbf{C})$ is regular-ish.

DEFINITION

An independence category is regular-ish if

(i) it has weak independent pullbacks,

(ii) every span factors as a strong epi followed by a jointly monic span,

(iii) every morphism is strong epic

THEOREM

Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then $\mathbf{Coisometry}(\mathbf{C})$ is regular-ish.

LEMMA

In a regular-ish independence category every weak independent pullback is an independent pullback.

DEFINITION

Let D be a regular-ish independence category. Define Rel(D) as follows:

- objects are objects of D
- morphisms are relations in (isomorphism classes of jointly monic spans)
- composition is by independent pullback and span factorisation

DEFINITION

Let D be a regular-ish independence category. Define $\mathrm{Rel}(D)$ as follows:

- objects are objects of D
- morphisms are relations in D (isomorphism classes of jointly monic spans)
- composition is by independent pullback and span factorisation

THEOREM

- $\mathrm{Rel}(D)$ is a dagger category with dilators.
- $\mathrm{Coisometry}(\mathrm{Rel}(D)) \cong D$
- $\mathrm{Rel}(\mathrm{Coisometry}(C)) \cong C$

What is the connection between dilators and tabulators?

Rel is a poset-enriched $\dagger$-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

Rel is a poset-enriched †-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Rel is a poset-enriched $\dagger$-category.

$$f \leq g \quad \text{implies} \quad f^\dagger \leq g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leq 1$.

Let $f : X \to Y$ be a relation.

Rel is a poset-enriched $\dagger$-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Let $f : X \to Y$ be a relation.

$$X \xleftarrow[{[1\ \ p_1\ \ \perp]}]{} X \sqcup f \sqcup Y \xrightarrow[{[\perp\ \ p_2\ \ 1]}]{} Y$$

Rel is a poset-enriched †-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Let $f : X \to Y$ be a relation.

$$s_2 s_1^\dagger \leqslant f$$

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & X \sqcup f \sqcup Y \arrow[l, "{[1\ \ p_1\ \ \bot]}"] \arrow[r, "{[\bot\ \ p_2\ \ 1]}"'] & Y
\end{tikzcd}$$

Rel is a poset-enriched $\dagger$-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Let $f : X \to Y$ be a relation.

$$s_2 s_1^\dagger \leqslant f$$

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, dashed, "s"] \arrow[dr, "s_2"] & \\
X & X \sqcup f \sqcup Y \arrow[l, "{[1 \;\; p_1 \;\; \perp]}"] \arrow[r, "{[\perp \;\; p_2 \;\; 1]}"'] & Y
\end{tikzcd}$$

Rel is a poset-enriched $\dagger$-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Let $f : X \to Y$ be a relation.

$s_2 s_1^\dagger \leqslant f$

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, dashed, "s"] \arrow[dr, "s_2"] & \\
X & X \sqcup f \sqcup Y \arrow[l, "{[1 \;\; p_1 \;\; \perp]}"] \arrow[r, "{[\perp \;\; p_2 \;\; 1]}"'] & Y
\end{tikzcd}$$

A morphism $f : (X, x) \to (Y, y)$ in **Rel**$_*$ is a relation $f : X \to Y$ such that $(x, y) \in f$.

Rel is a poset-enriched $\dagger$-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Let $f : X \to Y$ be a relation.

$$s_2 s_1^\dagger \leqslant f$$

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, dashed, "s"] \arrow[dr, "s_2"] & \\
X & X \sqcup f \sqcup Y \arrow[l, "{[1 \;\; p_1 \;\; \perp]}"] \arrow[r, "{[\perp \;\; p_2 \;\; 1]}"'] & Y
\end{tikzcd}$$

A morphism $f : (X, x) \to (Y, y)$ in $\mathbf{Rel}_*$ is a relation $f : X \to Y$ such that $(x, y) \in f$.

The tabulator of $f : (X, x) \to (Y, y)$ is

$$(X, x) \xleftarrow{p_1} (f, (x, y)) \xrightarrow{p_2} (Y, y)$$

Rel is a poset-enriched †-category.

$$f \leq g \quad \text{implies} \quad f^\dagger \leq g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leq 1$.

Let $f : X \to Y$ be a relation.

$s_2 s_1^\dagger \leq f$

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, dashed, "s"] \arrow[dr, "s_2"] & \\
X & X \sqcup f \sqcup Y \arrow[l, "{[1 \;\; p_1 \;\; \perp]}"] \arrow[r, "{[\perp \;\; p_2 \;\; 1]}"'] & Y
\end{tikzcd}$$

A morphism $f : (X, x) \to (Y, y)$ in $\mathbf{Rel}_*$ is a relation $f : X \to Y$ such that $(x, y) \in f$.

The tabulator of $f : (X, x) \to (Y, y)$ is

$$(X, x) \xleftarrow{p_1} (f, (x, y)) \xrightarrow{p_2} (Y, y)$$

Consider $\mathbf{Rel} \to \mathbf{Rel}_*$:

Rel is a poset-enriched †-category.

$$f \leq g \quad \text{implies} \quad f^\dagger \leq g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leq 1$.

Let $f : X \to Y$ be a relation.

$$s_2 s_1^\dagger \leq f$$

$$\begin{tikzcd}
& & S \arrow[dll, "s_1"'] \arrow[d, dashed, "s"] \arrow[drr, "s_2"] & & \\
X & & X \sqcup f \sqcup Y \arrow[ll, "{[1 \;\; p_1 \;\; \perp]}"] \arrow[rr, "{[\perp \;\; p_2 \;\; 1]}"'] & & Y
\end{tikzcd}$$

A morphism $f : (X, x) \to (Y, y)$ in $\mathbf{Rel}_*$ is a relation $f : X \to Y$ such that $(x, y) \in f$.

The tabulator of $f : (X, x) \to (Y, y)$ is

$$(X, x) \xleftarrow{p_1} (f, (x, y)) \xrightarrow{p_2} (Y, y)$$

Consider $\mathbf{Rel} \to \mathbf{Rel}_*$:

$$\begin{gathered} X \\ \downmapsto \\ (X \sqcup \{*\}, *) \end{gathered}$$

Rel is a poset-enriched †-category.

$$f \leqslant g \quad \text{implies} \quad f^\dagger \leqslant g^\dagger$$

A partial map is a morphism f such that $ff^\dagger \leqslant 1$.

Let $f : X \to Y$ be a relation.

$$s_2 s_1^\dagger \leqslant f$$

$$\begin{tikzcd}
 & S \arrow[dl, "s_1"'] \arrow[d, dashed, "s"] \arrow[dr, "s_2"] & \\
X & X \sqcup f \sqcup Y \arrow[l, "{[1 \;\; p_1 \;\; \perp]}"] \arrow[r, "{[\perp \;\; p_2 \;\; 1]}"'] & Y
\end{tikzcd}$$

A morphism $f : (X, x) \to (Y, y)$ in $\mathbf{Rel}_*$ is a relation $f : X \to Y$ such that $(x, y) \in f$.

The tabulator of $f : (X, x) \to (Y, y)$ is

$$(X, x) \xleftarrow{p_1} (f, (x, y)) \xrightarrow{p_2} (Y, y)$$

Consider $\mathbf{Rel} \to \mathbf{Rel}_*$:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, mapsto] & & Y \arrow[d, mapsto] \\
(X \sqcup \{*\}, *) \arrow[rr, "{\begin{bmatrix} f & \{*\} \times Y \\ X \times \{*\} & \{(*,*)\} \end{bmatrix}}"] & & (Y \sqcup \{*\}, *)
\end{tikzcd}$$

PROJECTS

With Heunen, Perrone and Stein

m.dimeglio@ed.ac.uk https://mdimeglio.github.io

PROJECTS

With Heunen, Perrone and Stein

Characterise the category of Hilbert spaces and coisometries

m.dimeglio@ed.ac.uk https://mdimeglio.github.io

All coisometries have a quantum interpretation

PROJECTS

With Heunen, Perrone and Stein

Characterise the category of Hilbert spaces and coisometries

Not a $\dagger$-category

m.dimeglio@ed.ac.uk https://mdimeglio.github.io

PROJECTS

With Heunen, Perrone and Stein

All coisometries have a quantum interpretation

Characterise the category of Hilbert spaces and coisometries

Characterise a category of probability spaces

Not a $\dagger$-category

m.dimeglio@ed.ac.uk https://mdimeglio.github.io

PROJECTS

With Heunen, Perrone and Stein

All coisometries have a quantum interpretation

The first characterisation in categorical probability

Characterise the category of Hilbert spaces and coisometries

Characterise a category of probability spaces

Not a †-category

Not Markov categories

m.dimeglio@ed.ac.uk https://mdimeglio.github.io