

CATEGORICAL HILBERT THEORY

MATTHEW DI MEGLIO

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Hilbert spaces

Unitary
representations

Hilbert spaces

Hilbert modules

C^* -algebras

Hilbert spaces

Unitary
representations

Hilbert theory

Hilbert modules

C^* -algebras

① Abstract frameworks

② Limits

③ Relations

①

Abstract frameworks

Theory

homological algebra

probability theory

differential geometry

Hilbert theory

Categorical setting

abelian categories

Markov categories

tangent categories

?

A $*$ -category is equivalent to $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ or $\mathbf{Hilb}_{\mathbb{H}}$ iff

- has zero object
- has binary *orthonormal* products
- has *isometric* kernels
- $\Delta: X \rightarrow X \oplus X$ are kernels
- subcategory of *isometries* has directed colimits
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R^* -categories and M^* -categories

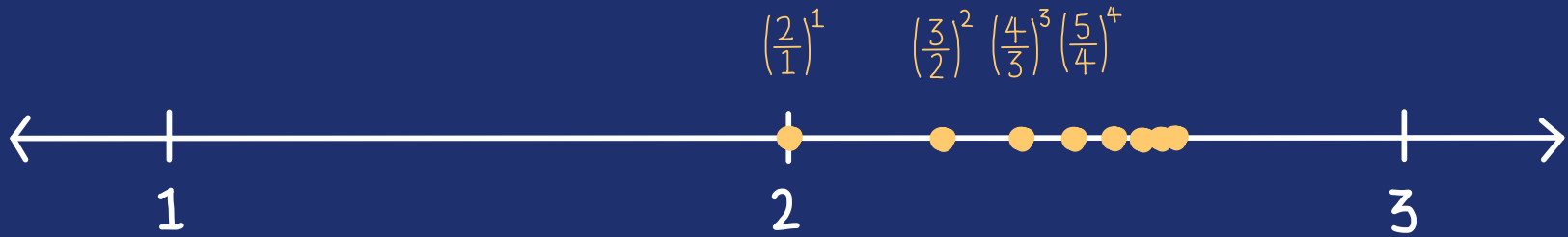
- capture key features of Hilbert spaces:
 - order and inner products
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 - suprema and sums
- also include \mathbf{Hilb}_A and \mathbf{URep}_C

②

Limits

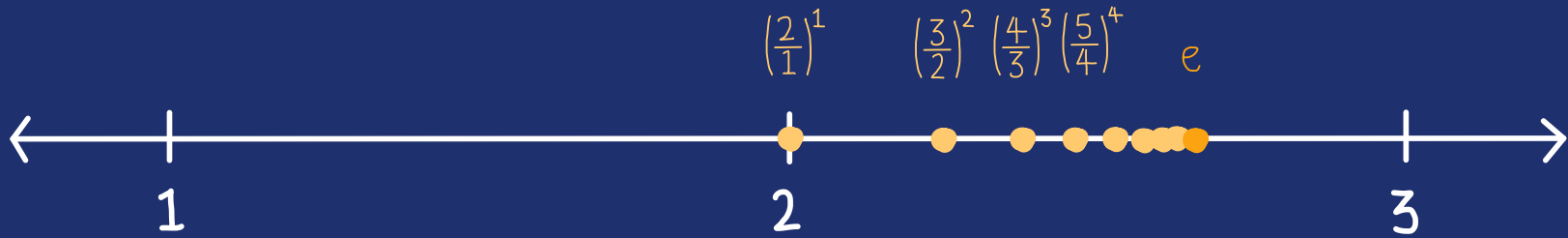
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$$\begin{array}{ccccccc} x_1 & & \longleftarrow & & (x_1, x_2) & & \longleftarrow & & (x_1, x_2, x_3) \\ \mathbb{C} & \longleftarrow & & \mathbb{C}^2 & \longleftarrow & & \mathbb{C}^3 & \longleftarrow & \dots \end{array}$$

$$\lim_{n \in \mathbb{N}} \mathbb{C}^n =$$

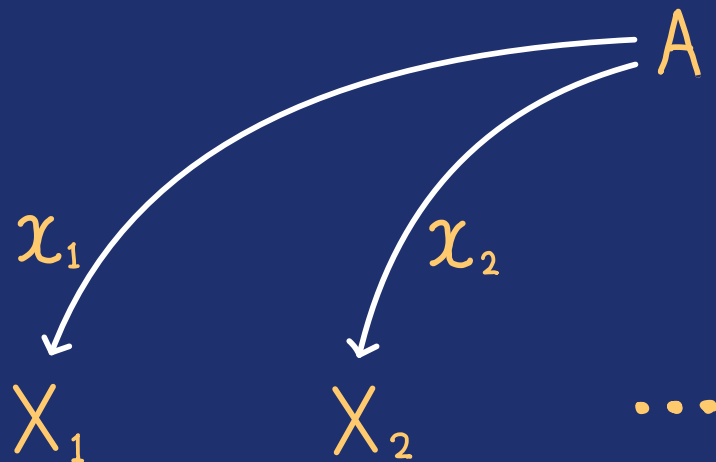
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$$\lim_{n \in \mathbb{N}} \mathbb{C}^n = \ell^2(\mathbb{N}) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \mid |x_1|^2 + |x_2|^2 + \dots < \infty \right\}$$

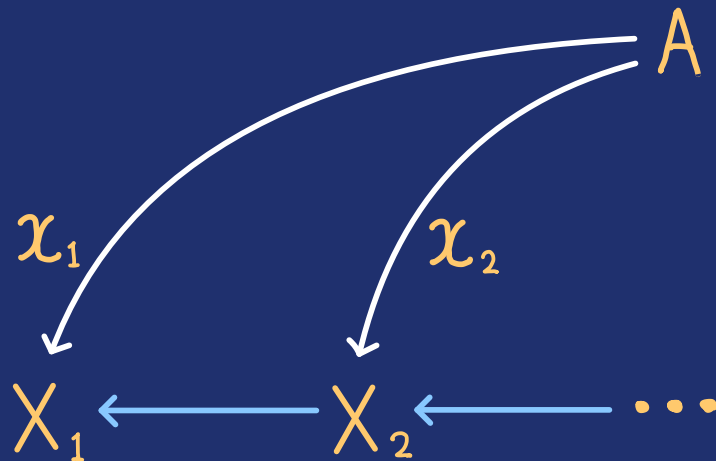
Given operators $0 < a_1 \leq a_2 \leq \dots \leq 1$ on a Hilbert space \mathcal{H}

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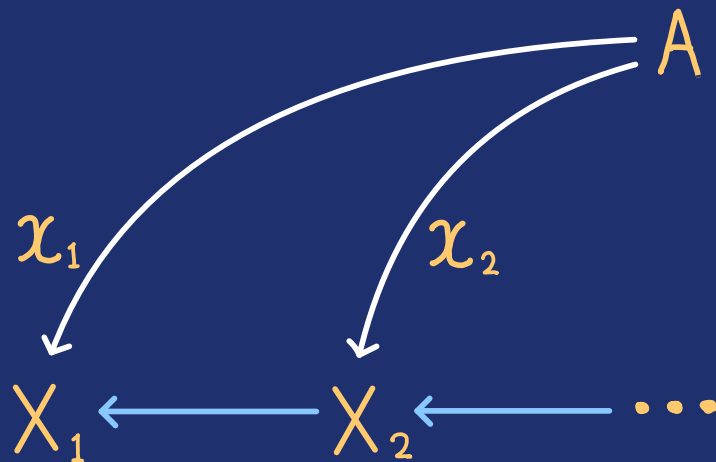
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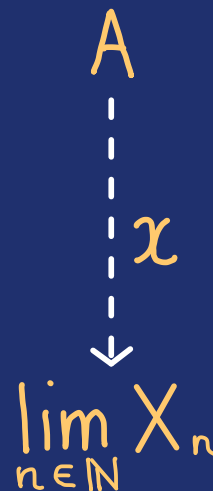
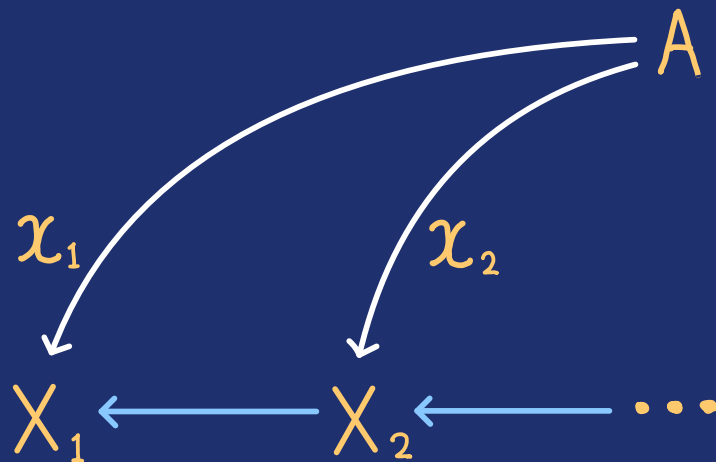
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$$\lim_{n \in \mathbb{N}} X_n$$

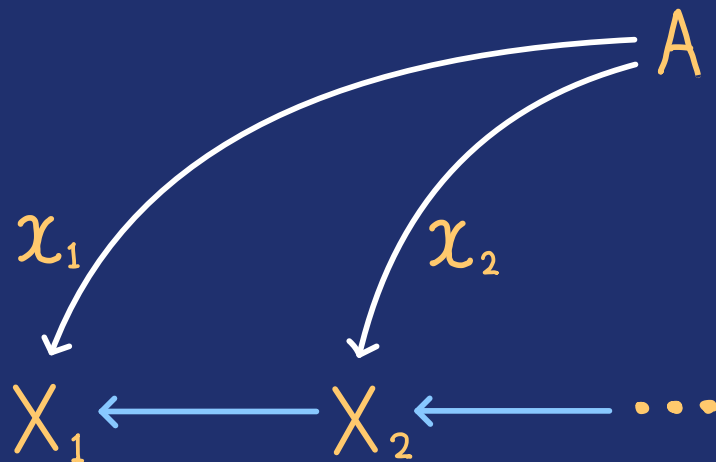
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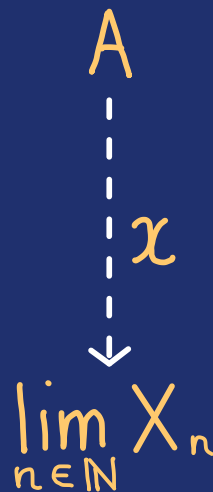


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$$x^* x = \sup_{n \in \mathbb{N}} a_n$$

This idea yields

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- the first characterisation of a category of finite-dimensional Hilbert spaces

③

Relations

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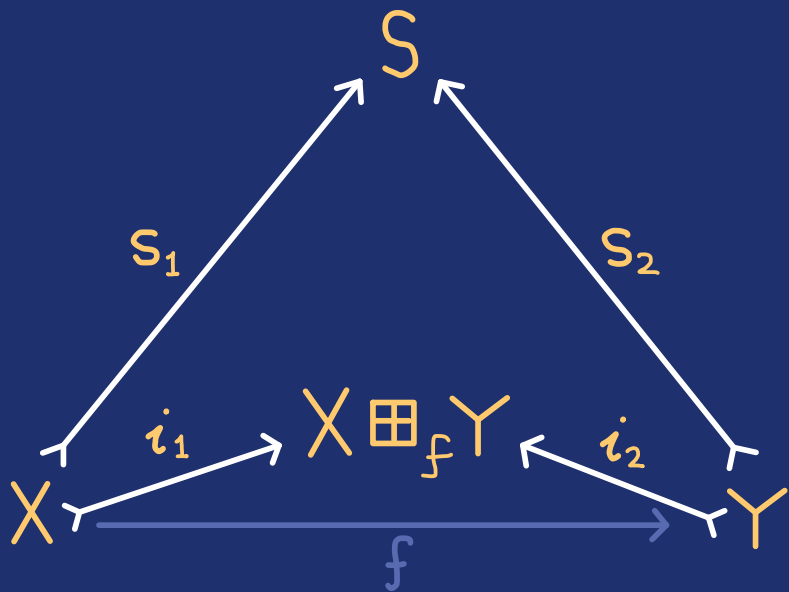


Every morphism in **Con** has a **codilator**



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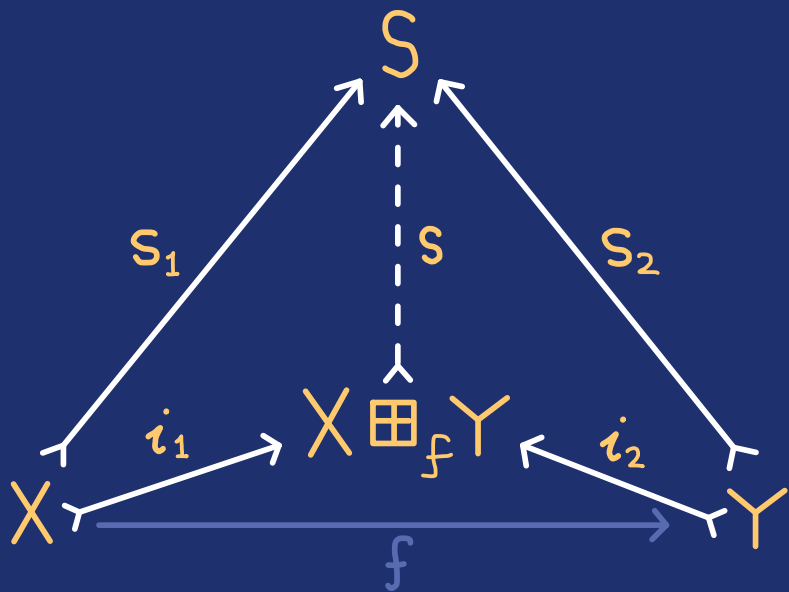
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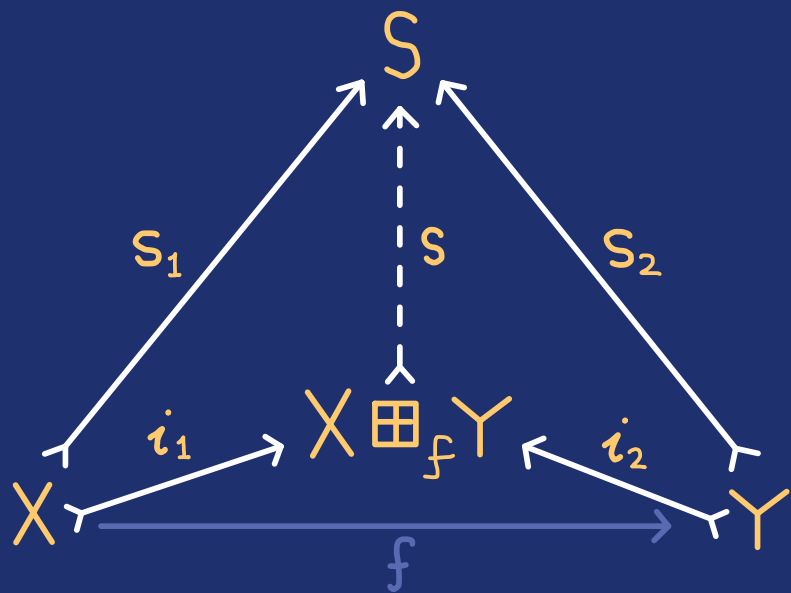
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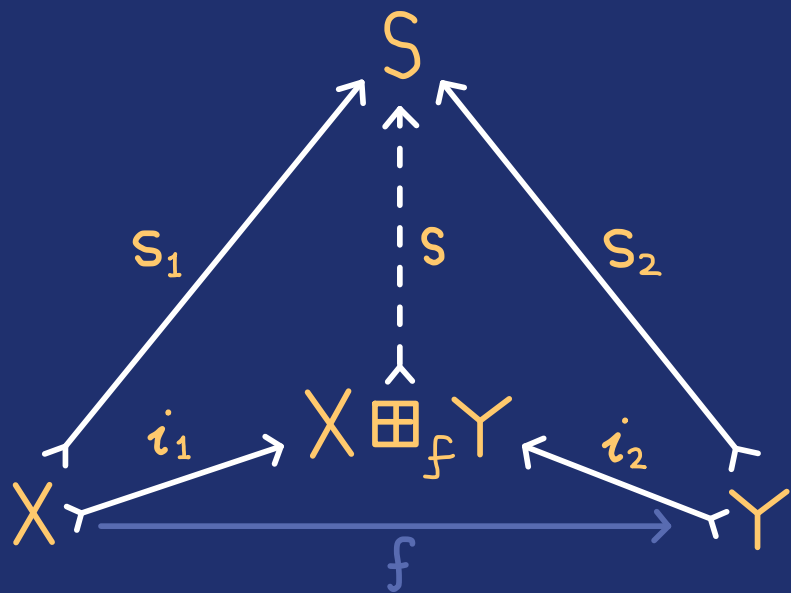


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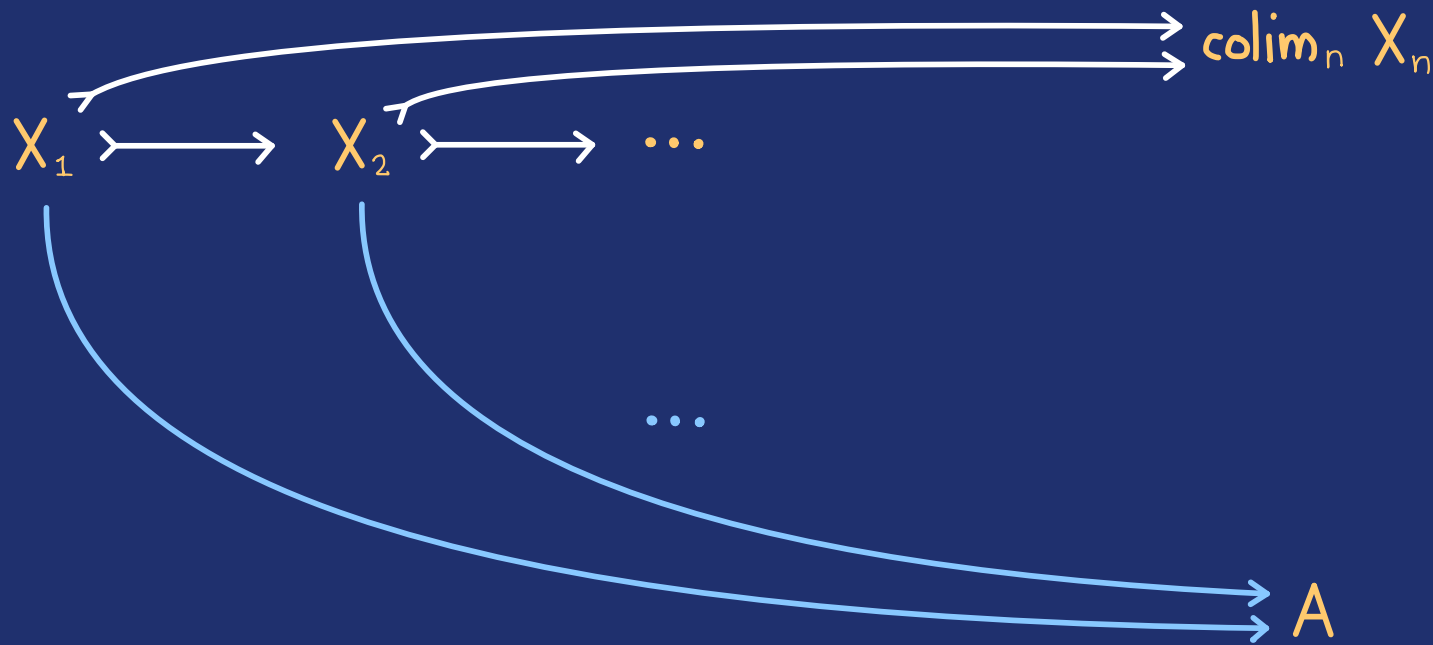
Inspired by Sz.-Nagy's minimal unitary dilations
Reminiscent of cotabulators

$\text{Isom} \rightarrow \text{Con}$ preserves directed colimits

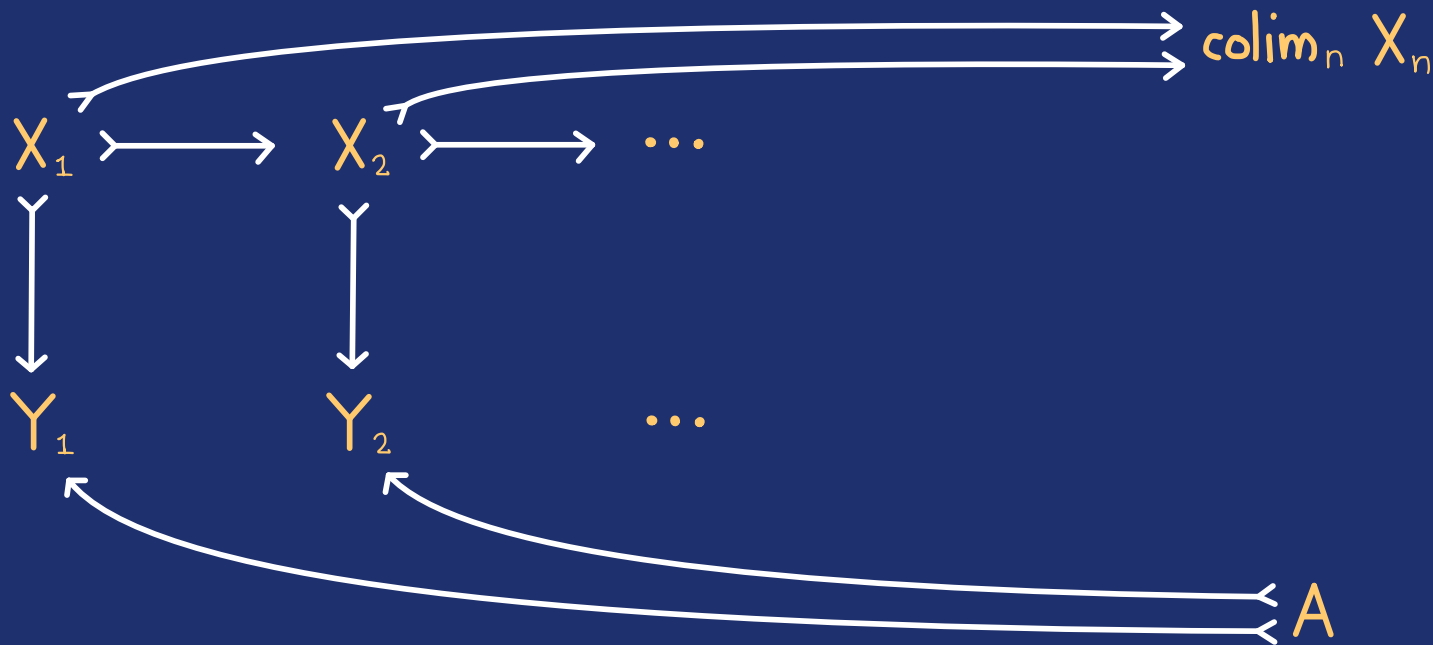
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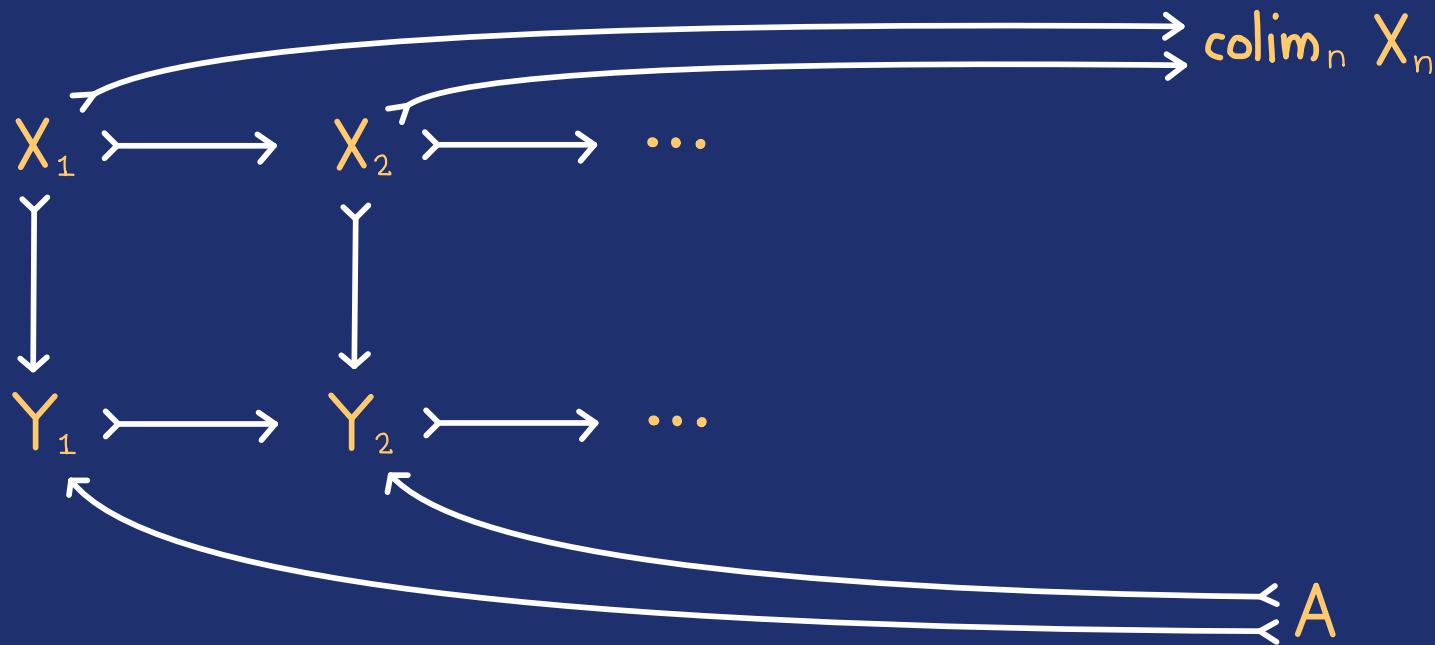
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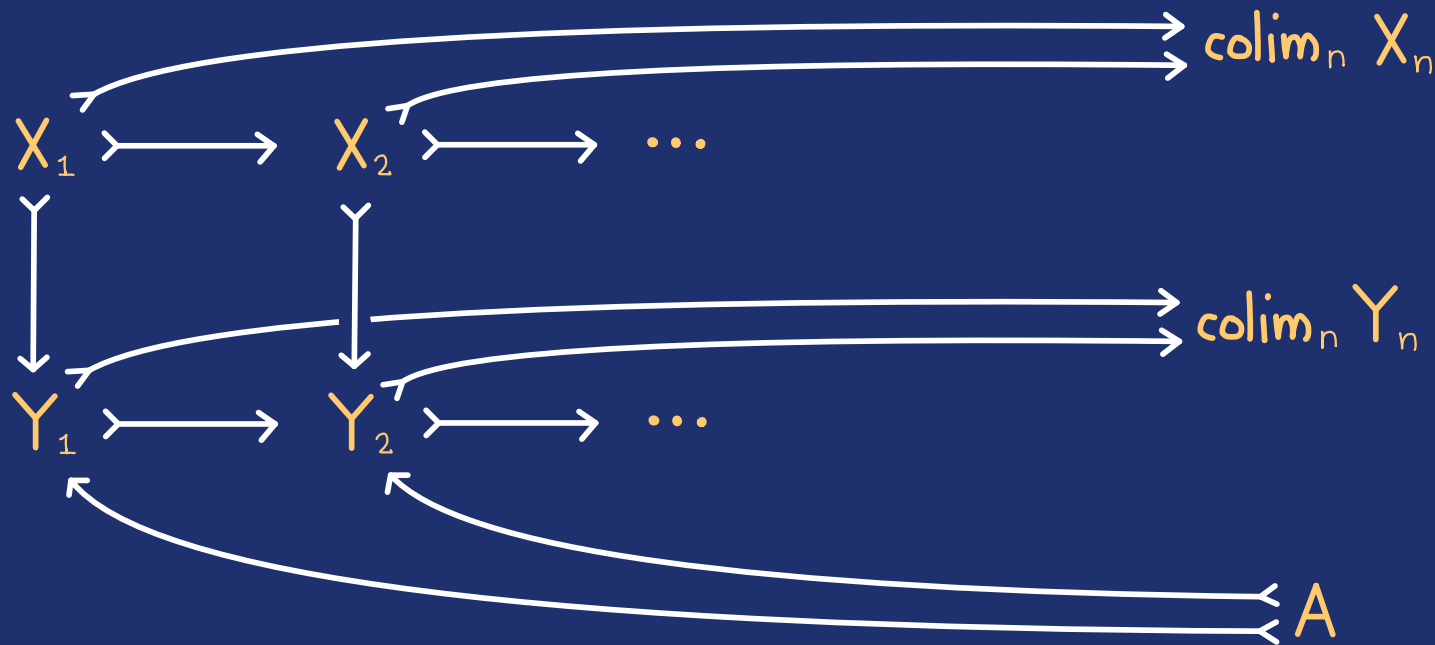
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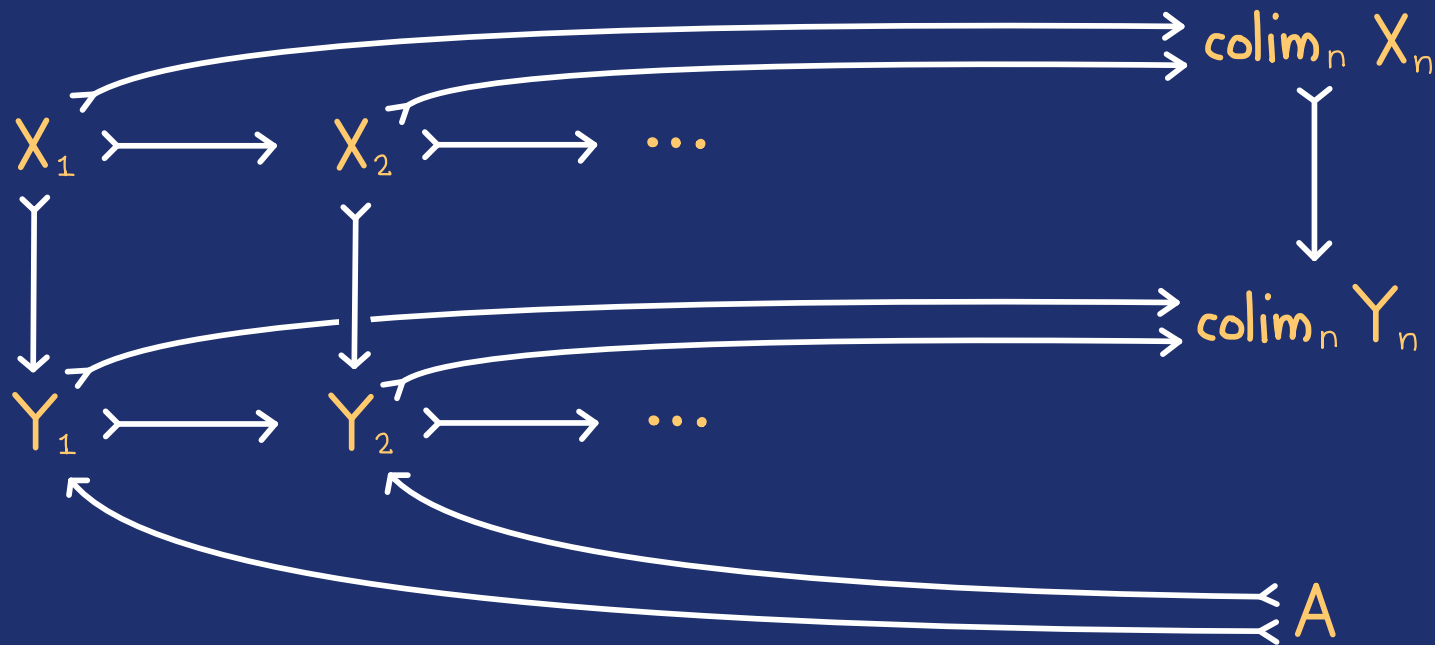
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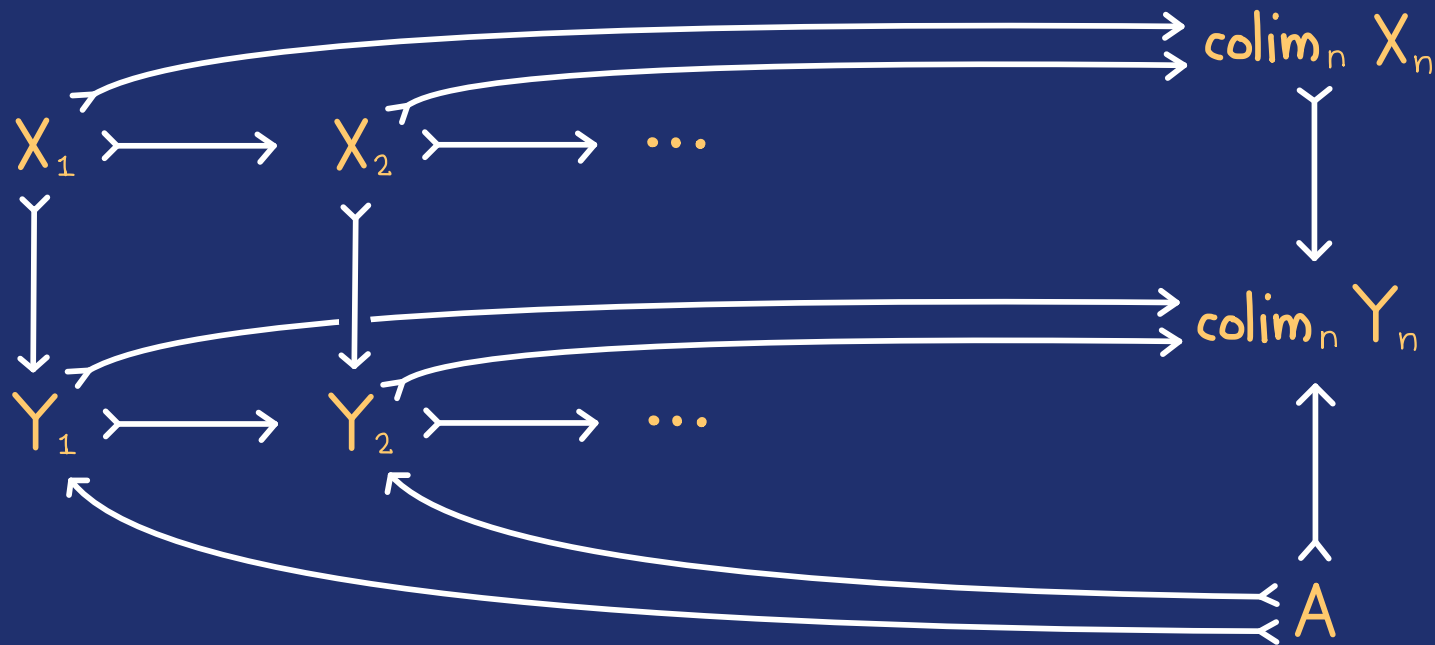
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tabular
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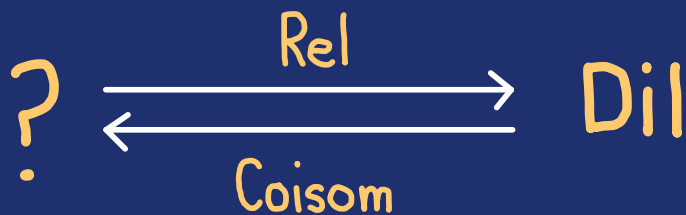
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$*$ -categories
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