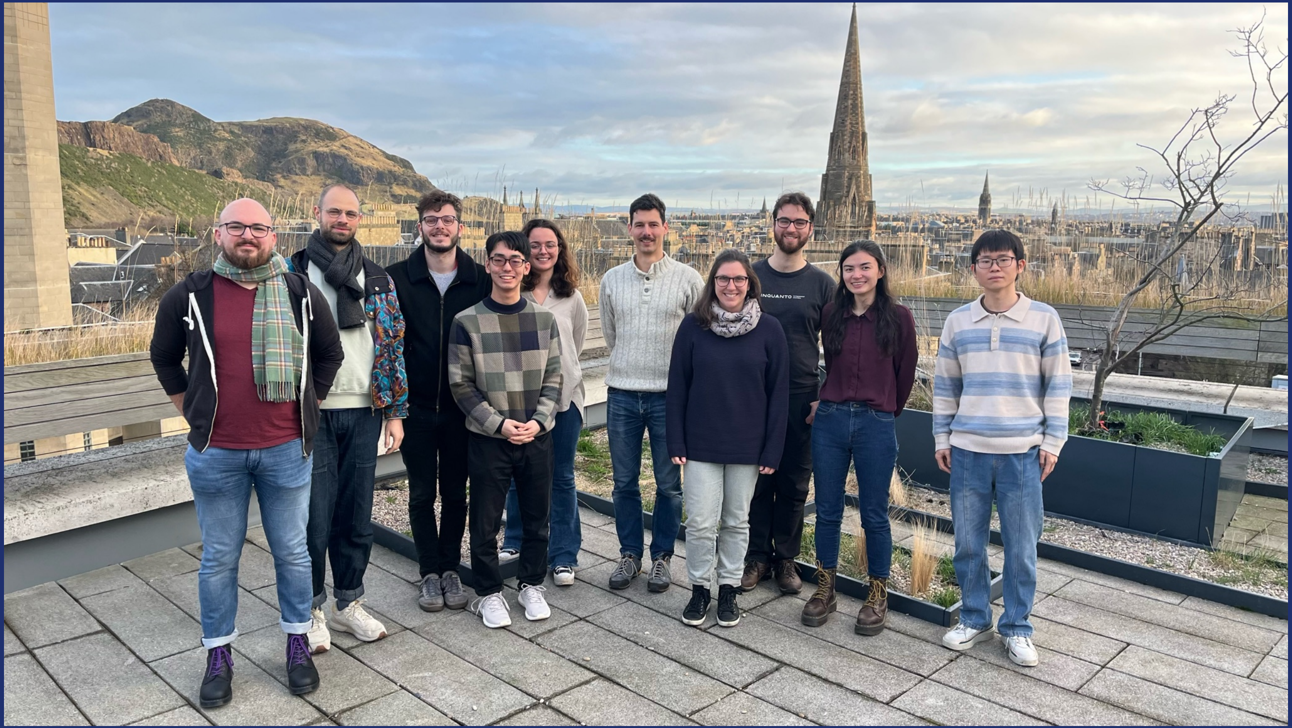
# Categorical Hilbert theory

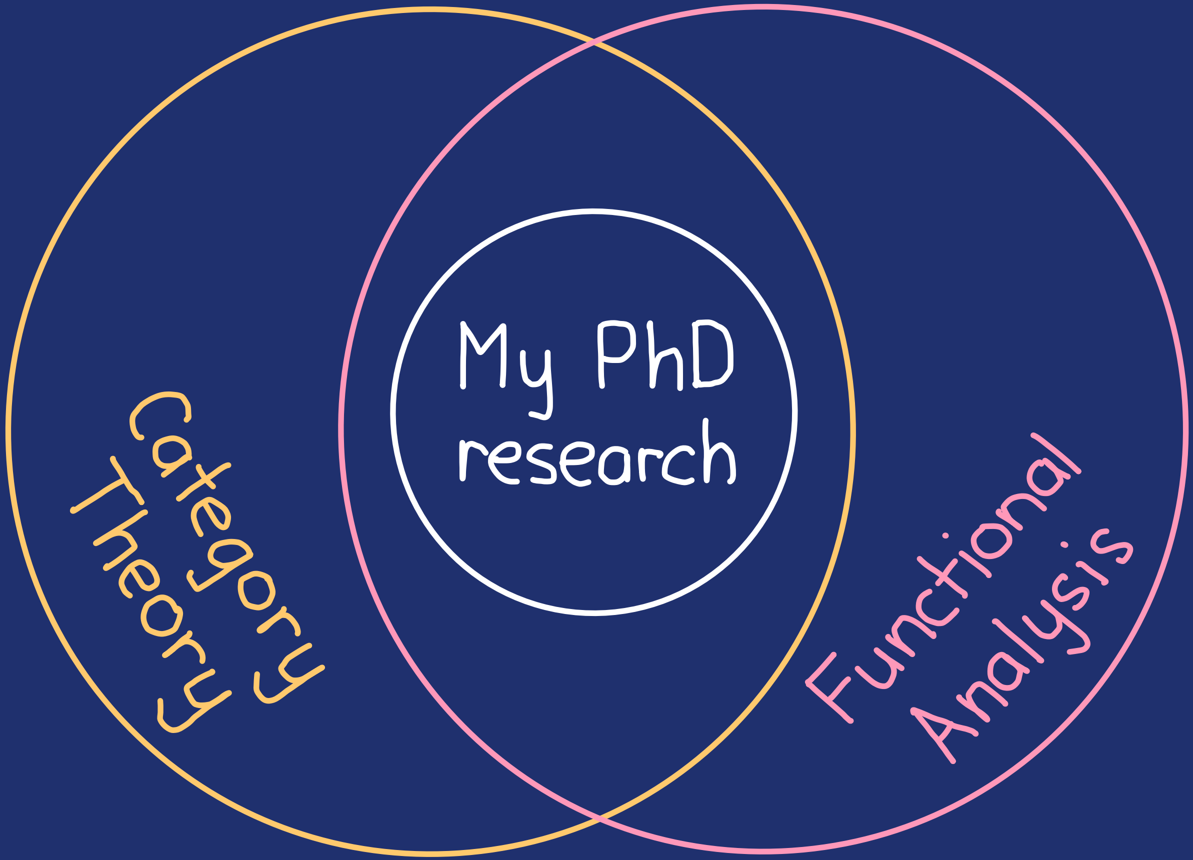
Matthew Di Meglio




PhD student of Chris Heunen at the University of Edinburgh

My PhD research


My PhD research
Category Theory


My PhD research
Category Theory
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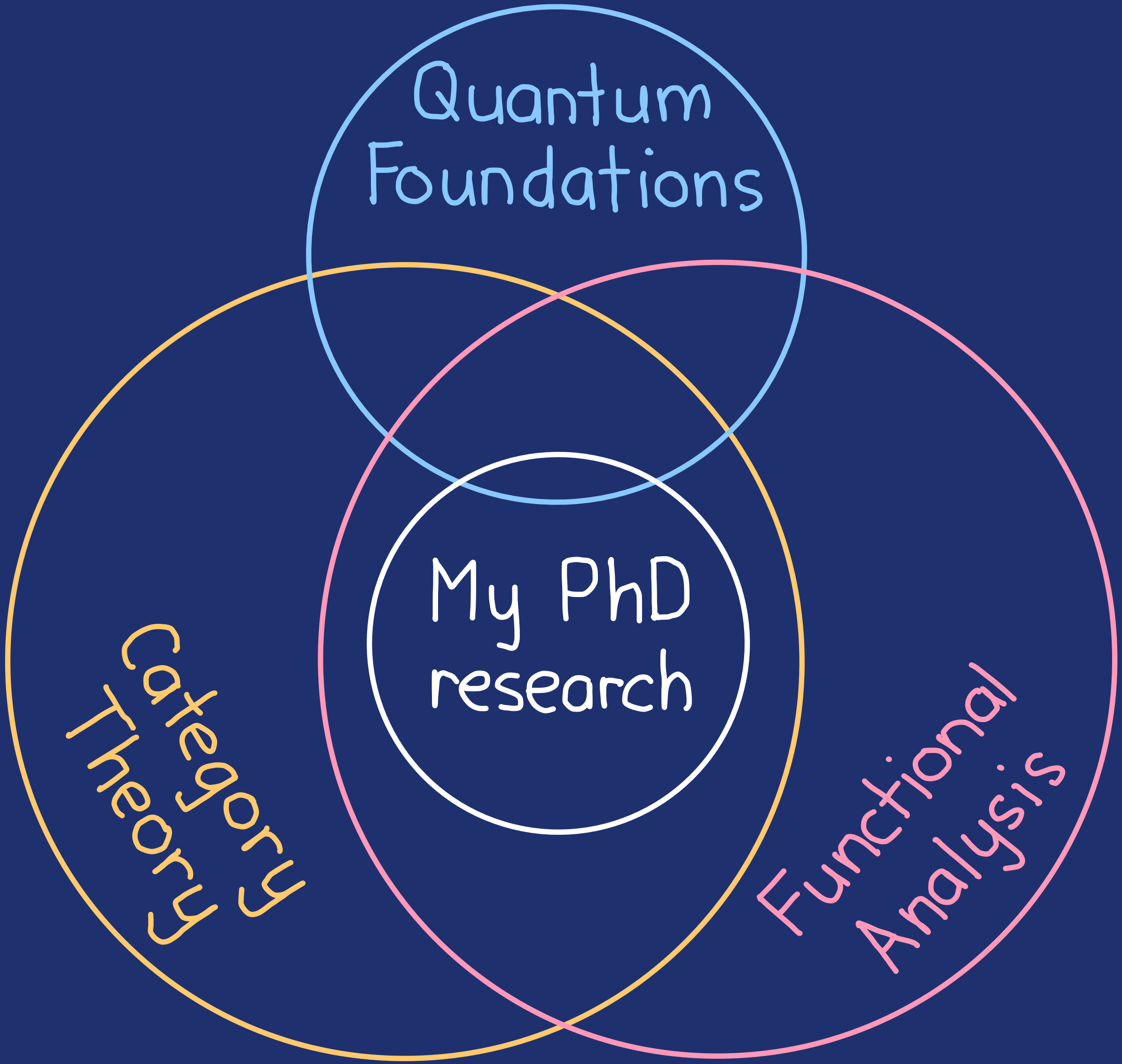
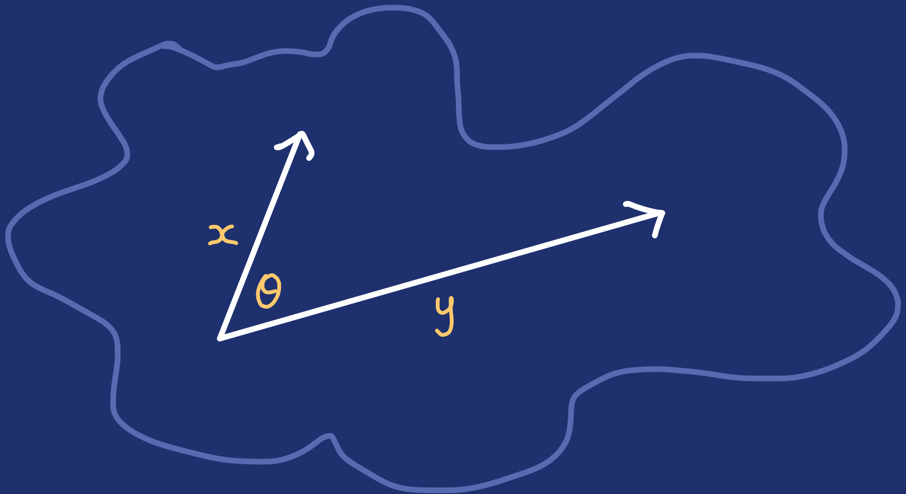

Quantum Foundations
Category Theory
Functional Analysis
My PhD research

1. Limits are limits
2. Categorical Hilbert theory
3. Dilators

# Background

Hilbert spaces are vector spaces with geometry (encoded by a complete innerproduct)



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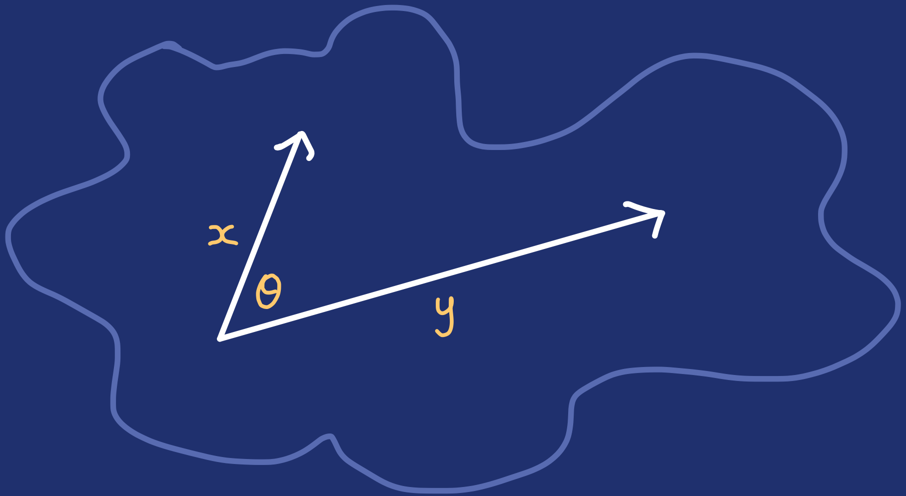
$\|x\| = \sqrt{\langle x | x \rangle}$

Lengths



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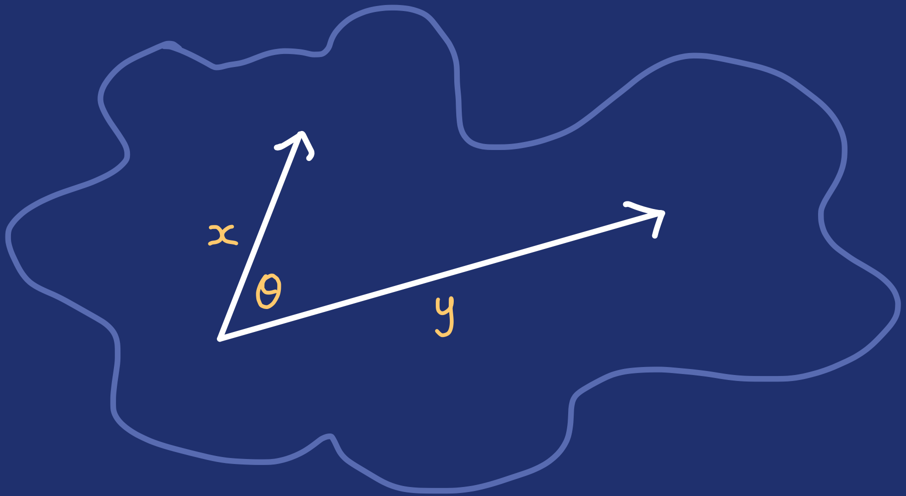


$$\cos\theta = \frac{\mathrm{Re}\,\langle x|y\rangle}{\|x\|\,\|y\|}$$

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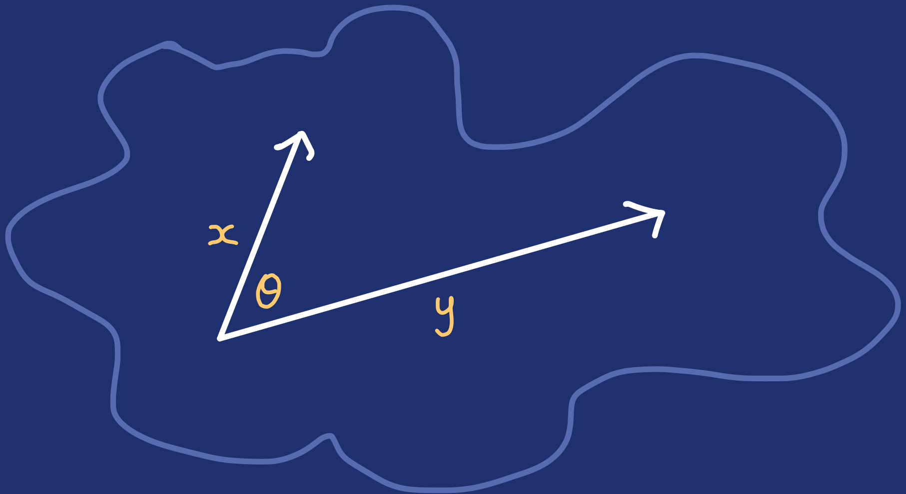


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Angles

e.g. $\mathbb{C}, \mathbb{C}^2, \dots, \ell^2(\mathbb{N})$

Adjointables are maps $f : X \to Y$ between Hilbert spaces with an adjoint $f^* : Y \to X$

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They form a category **Adjointable**

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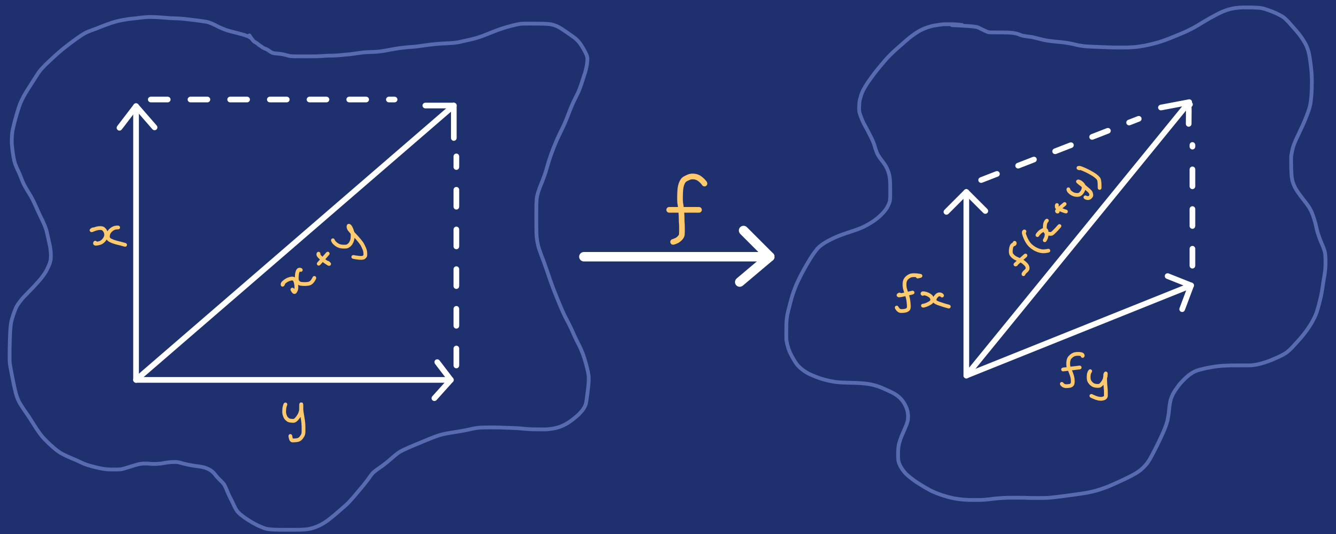


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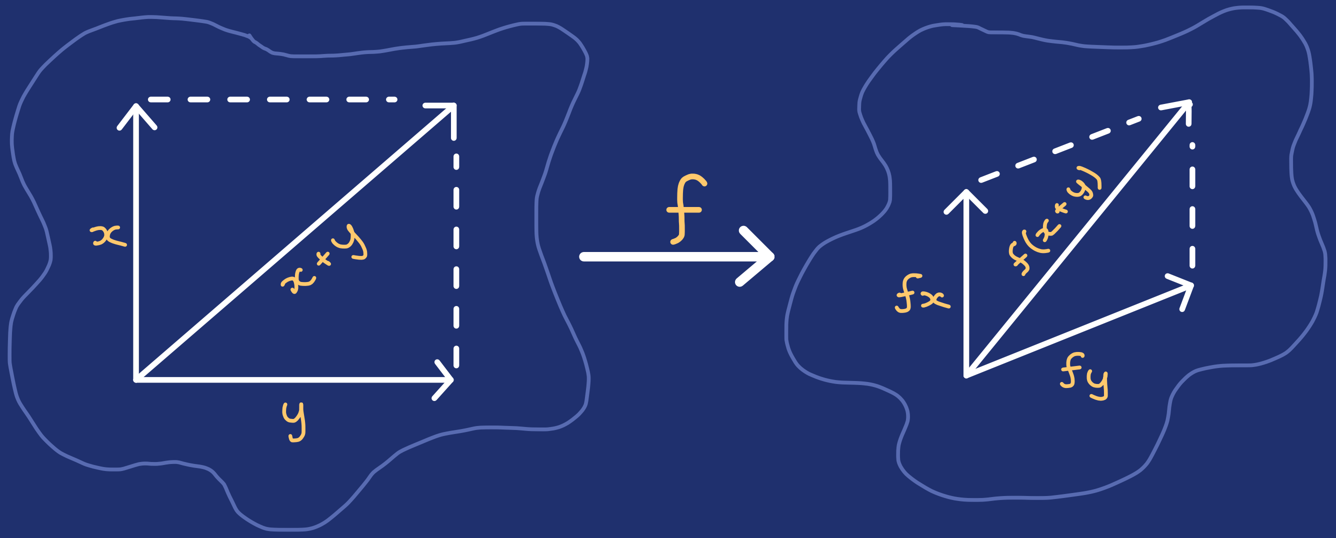


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They form a subcategory **Contraction** of **Adjointable**

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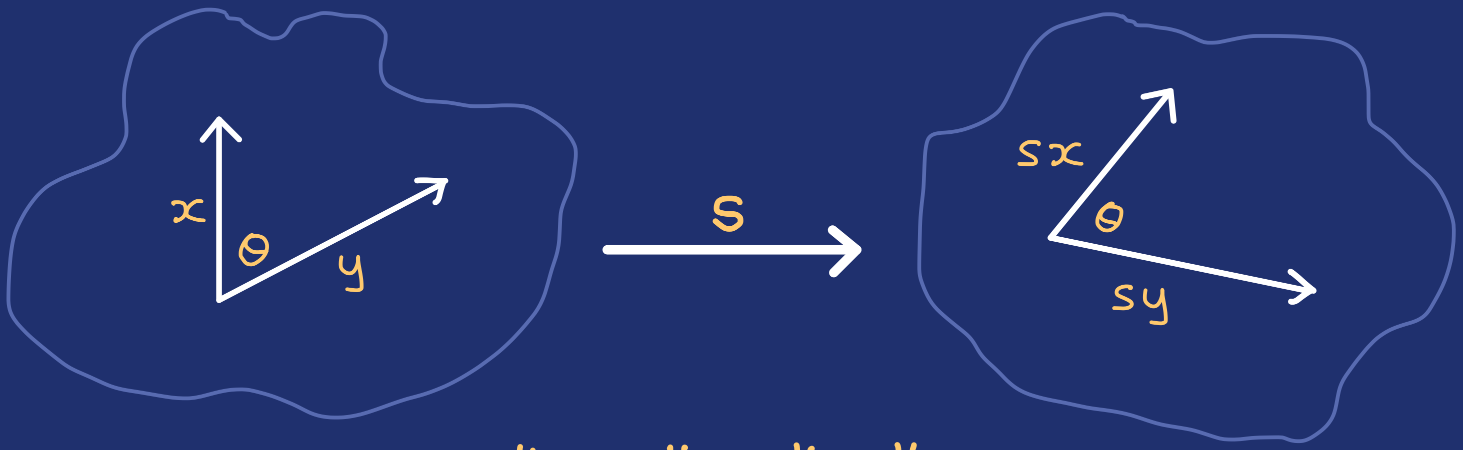


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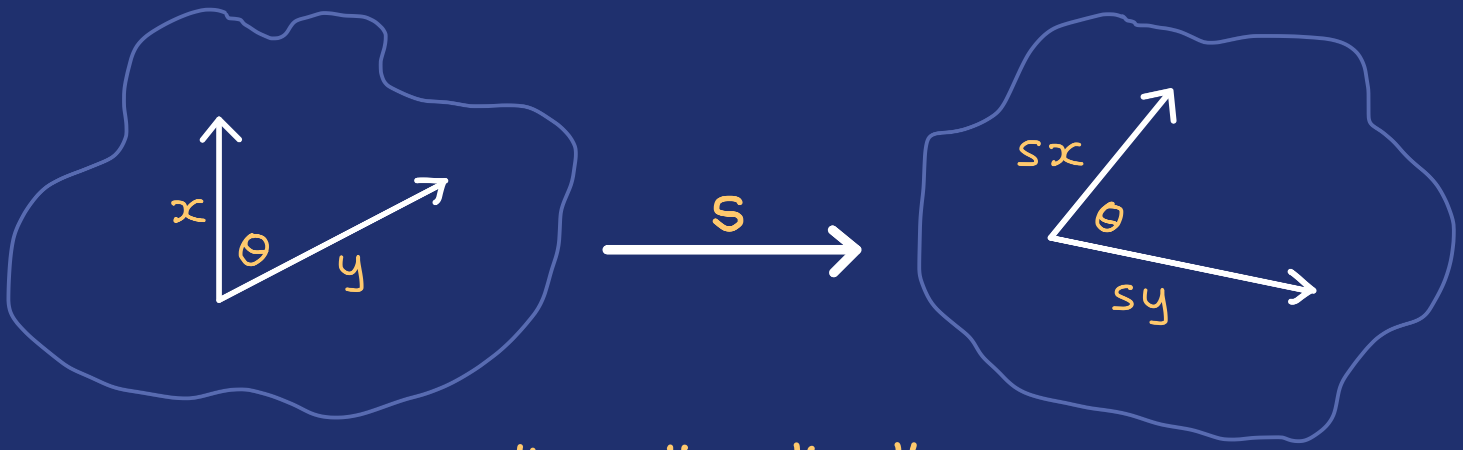


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They form a subcategory **Isometry** of **Contraction**

1

# Limits are limits

Limits in analysis are about approximating elements

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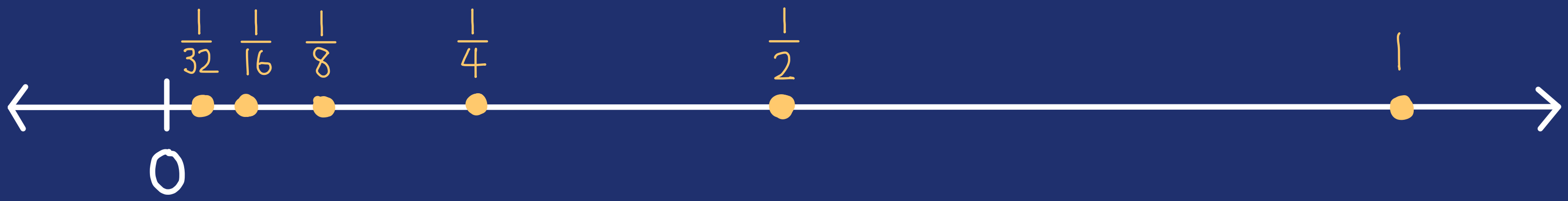

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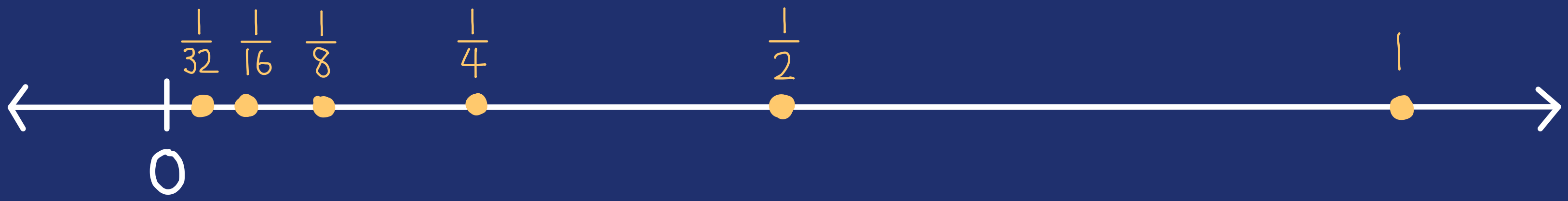


$$\lim_{n\to\infty} 2^{-n} = 0$$

Codirected limits in category theory are about approximating objects

in the category Contraction

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$$\begin{array}{ccccc} x_1 & \longleftarrow\!\shortmid & (x_1, x_2) & \longleftarrow\!\shortmid & (x_1, x_2, x_3) \\ \mathbb{C} & \longleftarrow & \mathbb{C}^2 & \longleftarrow & \mathbb{C}^3 \longleftarrow \cdots \end{array}$$

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$$\lim_{n \in \mathbb{N}} \mathbb{C}^n = \ell^2(\mathbb{N}) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \;\middle|\; |x_1|^2 + |x_2|^2 + \cdots < \infty \right\}$$

in the category Contraction

# AXIOMS FOR THE CATEGORY OF HILBERT SPACES

CHRIS HEUNEN AND ANDRE KORNELL

ABSTRACT. We provide axioms that guarantee a category is equivalent to that of continuous linear functions between Hilbert spaces. The axioms are purely categorical and do not presuppose any analytical structure. This addresses a question about the mathematical foundations of quantum theory raised in reconstruction programmes such as those of von Neumann, Mackey, Jauch, Piron, Abramsky, and Coecke.

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What is the deeper connection between these two kinds of limits?

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Given real numbers $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant 1$

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$$\begin{tikzcd}
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X_1 & X_2 \arrow[l] & \cdots \arrow[l] &
\end{tikzcd}$$

$$\|x_1\|^2 = a_1 \qquad \|x_2\|^2 = a_2 \qquad \cdots$$

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Given operators $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant 1$ on a Hilbert space $A$

$$\begin{tikzcd}
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$$x_1^* x_1 = a_1 \qquad x_2^* x_2 = a_2 \qquad \cdots \qquad\qquad x^* x = \sup_{n\in\mathbb{N}} a_n$$

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ABSTRACT. We characterise the category of finite-dimensional Hilbert spaces and linear contractions using simple category-theoretic axioms that do not refer to norms, continuity, dimension, or real numbers. Our proof directly relates limits in category theory to limits in analysis, using a new variant of the classical characterisation of the real numbers instead of Solèr's theorem.

This idea yields

- insightful new proofs of the characterisations of the categories of Hilbert spaces
- the first characterisation of a category of finite-dimensional Hilbert spaces

Relevant for quantum computing

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# 2 Categorical Hilbert theory

# CATEGORICAL REFORMULATIONS

| Theory | Categorical setting |
| --- | --- |
| homological algebra | abelian categories |
| probability theory | Markov categories |
| differential geometry | tangent categories |

Reformulating a theory category theoretically can

- unify and generalise known results,
- reveal new results,
- simplify it, making it more accessible.

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# R*-categories are a new categorical abstraction of algebraic aspects of Hilbert spaces

## R*-CATEGORIES

### THE HILBERT-SPACE ANALOGUE OF ABELIAN CATEGORIES

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Abstract. This article introduces R*-categories—an abstraction of categories exhibiting the "algebraic" aspects of the theory of Hilbert spaces. Notably, finite biproducts in R*-categories can be orthogonalised using the Gram–Schmidt process, and generalised notions of positivity and contraction support a variant of Sz.-Nagy's unitary dilation theorem. Underpinning these generalisations is the structure of an involutive identity-on-objects contravariant endofunctor, which encodes adjoints of morphisms. The R*-category axioms are otherwise inspired by those for abelian categories, comprising a few simple properties of products and kernels. Additivity is not assumed, but nevertheless follows. In fact, the similarity with abelian categories runs deeper—R*-categories are quasi-abelian and thus homological. Examples include the category of unitary representations of a group, the category of finite-dimensional inner product modules over a partially ordered division ring, and the category of self-dual Hilbert modules over a W*-algebra.

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M*-categories also include analytic aspects

Articles in preparation

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(2) binary orthonormal biproducts, — $X \oplus Y = \{(x,y) : x \in X, y \in Y\}$
$\langle (x,y) \mid (u,v) \rangle = \langle x \mid u \rangle + \langle y \mid v \rangle$

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$\mathrm{Eq}(f,g) = \{x \in X : fx = gx\}$
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$Ker(f) = Eq(f, 0)$

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e.g. $\ell^2(\mathbb{N})$

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R*-category

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R*-category

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## THEOREM:

If $\mathbf{C}$ has a simple separator $A$ then $\mathbf{C}(A,A)$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and

$$\begin{tikzcd}
 & \mathbf{Adjointable}_{\mathbf{C}(A,A)} \arrow[d] \\
\mathbf{C} \arrow[ur, dashed, "\simeq"] \arrow[r, "\mathbf{C}(A,-)"'] & \mathbf{Set}
\end{tikzcd}$$

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  $f^*f \geqslant 0$ $\langle f | g \rangle = f^*g$
- Symmetry (If $a \geqslant 1$ then $a$ invertible)
- Contractions (Morphisms $f$ with $f^*f \leqslant 1$)
- Monotone completeness
  (Bounded increasing nets have suprema)

## 3 Dilators

# DEFINITION:

A codilation of $f : X \to Y$ is a cospan of isometries

$$\begin{tikzcd} & S & \\ X \arrow[ur, tail, "s_1"] & & Y \arrow[ul, tail, "s_2"'] \end{tikzcd}$$

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such that $s_2^* s_1 = f$.

A codilator of $f$ is an initial codilation of $f$.

$$\begin{tikzcd} & S & \\ & X \boxplus_f Y \arrow[u, dashed, tail, "s"] & \\ X \arrow[uur, tail, "s_1"] \arrow[ur, tail, "i_1"'] & & Y \arrow[uul, tail, "s_2"'] \arrow[ul, tail, "i_2"] \end{tikzcd}$$

# PROPOSITION:

Every morphism in **Contraction** has a codilator

$$X \xrightarrow{\quad f \quad} Y$$

Inspired by minimal unitary dilations

## PROPOSITION:

Every morphism in **Contraction** has a codilator

$$
\begin{tikzcd}
 & X \oplus Y & \\
X \arrow[ur, tail, "{\begin{bmatrix} \sqrt{1-f^*f} & f \end{bmatrix}}"] \arrow[rr, "f"'] & & Y \arrow[ul, tail, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"']
\end{tikzcd}
$$

Inspired by minimal unitary dilations

# PROPOSITION:

Every morphism in **Contraction** has a codilator

$$
\begin{tikzcd}
 & X \oplus Y & \\
 & X \vee Y \arrow[u, tail] & \\
X \arrow[uur, tail, "\begin{bmatrix}\sqrt{1-f^*f} & f\end{bmatrix}"] \arrow[ur, tail, "s_1"'] \arrow[rr, "f"'] & & Y \arrow[uul, tail, "\begin{bmatrix}0 & 1\end{bmatrix}"'] \arrow[ul, tail, "s_2"]
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\end{tikzcd}
$$

Inspired by minimal unitary dilations

Other $*$-categories with codilators of all morphisms:

- Sets and partial bijections
- Sets and bitotal relations
- Finite probabity spaces and stochastic maps
  ($f^*$ is the Bayesian inverse of $f$)

Monotone completeness of M*-categories

Universal property for $\oplus$ in Contraction

# Dilators are useful

Louis Lemonnier

Semantics for symmetric pattern matching

Monotone completeness of M*-categories

Universal property for $\oplus$ in Contraction

# Dilators are useful

Louis Lemonnier
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# PROPOSITION:

Isometry $\longrightarrow$ Contraction preserves directed colimits

# PROPOSITION:

**Isometry $\longrightarrow$ Contraction** preserves directed colimits

# PROOF:

$$X_1 \rightarrowtail X_2 \rightarrowtail \cdots$$

# PROPOSITION:

**Isometry → Contraction** preserves directed colimits

# PROOF:

$$\begin{tikzcd}
X_1 \arrow[r, tail] \arrow[rrr, bend left, tail] & X_2 \arrow[r, tail] \arrow[rr, bend left, tail] & \cdots & \operatorname{colim}_n X_n
\end{tikzcd}$$

# PROPOSITION:

**Isometry $\to$ Contraction** preserves directed colimits

# PROOF:

$$\begin{tikzcd}
 & & & \operatorname{colim}_n X_n \\
X_1 \arrow[r, tail] \arrow[urrr, tail, bend left] \arrow[drrr, bend right] & X_2 \arrow[r, tail] \arrow[urr, tail, bend left] \arrow[dr, bend right] & \cdots & \\
 & & & A
\end{tikzcd}$$

# PROPOSITION:

**Isometry $\rightarrow$ Contraction** preserves directed colimits

# PROOF:

$$
\begin{tikzcd}
 & & & \mathrm{colim}_n\, X_n \\
X_1 \arrow[r, tail] \arrow[d, tail] \arrow[urrr, tail, bend left] & X_2 \arrow[r, tail] \arrow[d, tail] \arrow[urr, tail, bend left] & \cdots & \\
Y_1 & Y_2 & \cdots & \\
 & & & A \arrow[ulll, tail, bend left] \arrow[ull, tail, bend left]
\end{tikzcd}
$$

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Isometry $\rightarrow$ Contraction preserves directed colimits

# PROOF:

$$
\begin{tikzcd}
 & & & \mathrm{colim}_n\, X_n \\
X_1 \arrow[r, tail] \arrow[d, tail] \arrow[urrr, tail, bend left] & X_2 \arrow[r, tail] \arrow[d, tail] \arrow[urr, tail, bend left] & \cdots & \\
Y_1 \arrow[r, tail] & Y_2 \arrow[r, tail] & \cdots & \\
 & & & A \arrow[ulll, tail, bend left] \arrow[ull, tail, bend left]
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\begin{tikzcd}
X_1 \arrow[r, tail] \arrow[d, tail] \arrow[rrrr, bend left, tail] & X_2 \arrow[r, tail] \arrow[d, tail] \arrow[rrr, bend left, tail] & \cdots & & \operatorname{colim}_n X_n \\
Y_1 \arrow[r, tail] \arrow[rrrr, bend left, tail] & Y_2 \arrow[r, tail] \arrow[rrr, bend left, tail] & \cdots & & \operatorname{colim}_n Y_n \\
& & & & A \arrow[ullll, bend left, tail] \arrow[ulll, bend left, tail]
\end{tikzcd}
$$

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$$
\begin{tikzcd}
X_1 \arrow[r, tail] \arrow[d, tail] \arrow[rrr, tail, bend left] & X_2 \arrow[r, tail] \arrow[d, tail] \arrow[rr, tail, bend left] & \cdots & \operatorname{colim}_n X_n \arrow[d, tail] \\
Y_1 \arrow[r, tail] \arrow[rrr, tail, bend left] & Y_2 \arrow[r, tail] \arrow[rr, tail, bend left] & \cdots & \operatorname{colim}_n Y_n \\
 & & & A \arrow[ulll, tail, bend left] \arrow[ull, tail, bend left]
\end{tikzcd}
$$

# PROPOSITION:

**Isometry $\to$ Contraction** preserves directed colimits

# PROOF:

$$
\begin{tikzcd}
X_1 \arrow[r, tail] \arrow[d, tail] \arrow[rrr, bend left, tail] & X_2 \arrow[r, tail] \arrow[d, tail] \arrow[rr, bend left, tail] & \cdots & \mathrm{colim}_n\, X_n \arrow[d, tail] \\
Y_1 \arrow[r, tail] \arrow[rrr, bend left, tail] & Y_2 \arrow[r, tail] \arrow[rr, bend left, tail] & \cdots & \mathrm{colim}_n\, Y_n \\
 & & & A \arrow[u, tail] \arrow[ulll, bend left, tail] \arrow[ull, bend left, tail]
\end{tikzcd}
$$

# SUMMARY

- Limits are limits
- R*- and M*-categories for Hilbert theory
- Dilators relate isometries and contractions

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## SUMMARY

- Limits are limits
- $R^*$- and $M^*$-categories for Hilbert theory
- Dilators relate isometries and contractions

## NEXT STEPS

- Contractions as relations in Isometry

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## SUMMARY

- Limits are limits
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## NEXT STEPS

- Contractions as relations in **Isometry**
- Axioms for **Isometry**

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## SUMMARY

- Limits are limits
- $R^*$- and $M^*$-categories for Hilbert theory
- Dilators relate isometries and contractions

## NEXT STEPS

- Contractions as relations in **Isometry**
- Axioms for **Isometry**
- Axioms for a category of probability spaces

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Reformulating a theory category theoretically can

- unify and generalise known results,

  e.g. Discrete and continuous cases of the Blackwood-Sherman-Stein theorem on statistical experiments unified for the first time via Markov categories

- reveal new results,

  e.g. convergence in mean for backward martingales indexed by an arbitrary net was proved for the first time (according to the authors) using dagger categories

- simplify it, making it more accessible.

  e.g. to theoretical computer scientists who already know category theory

# PROPOSITION: Every morphism in FinPS has a dilator

bloom-shriek factorisation
"The information loss of a stochastic map"

coisometric
$\Leftrightarrow$
deterministic