

# LIMITS OF SEQUENCES VIA COLIMITS OF CONTRACTIONS

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(Joint work with Chris Heunen)

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THM (Heunen and Kornell):

$(-)^{\dagger}: \underline{\mathcal{C}}^{\text{op}} \rightarrow \underline{\mathcal{C}}$   
[encodes adjoints]

A monoidal dagger category  $\underline{\mathcal{C}}$  with

• finite dagger biproducts } [enrichment in commutative monoids]

$$\begin{array}{ccc} & f, g: X \longrightarrow Y & \\ & \xrightarrow{f+g} & \\ X & & Y \\ \Delta \downarrow & & \uparrow \Delta^{\dagger} \\ X \oplus X & \xrightarrow{f \oplus g} & Y \oplus Y \end{array}$$

• dagger equalisers

• simple monoidal unit

[the semiring  $\mathbb{I} := \underline{\mathcal{C}}(I, I)$   
of scalars is a field]

• directed colimits in wide subcategory of dagger monos

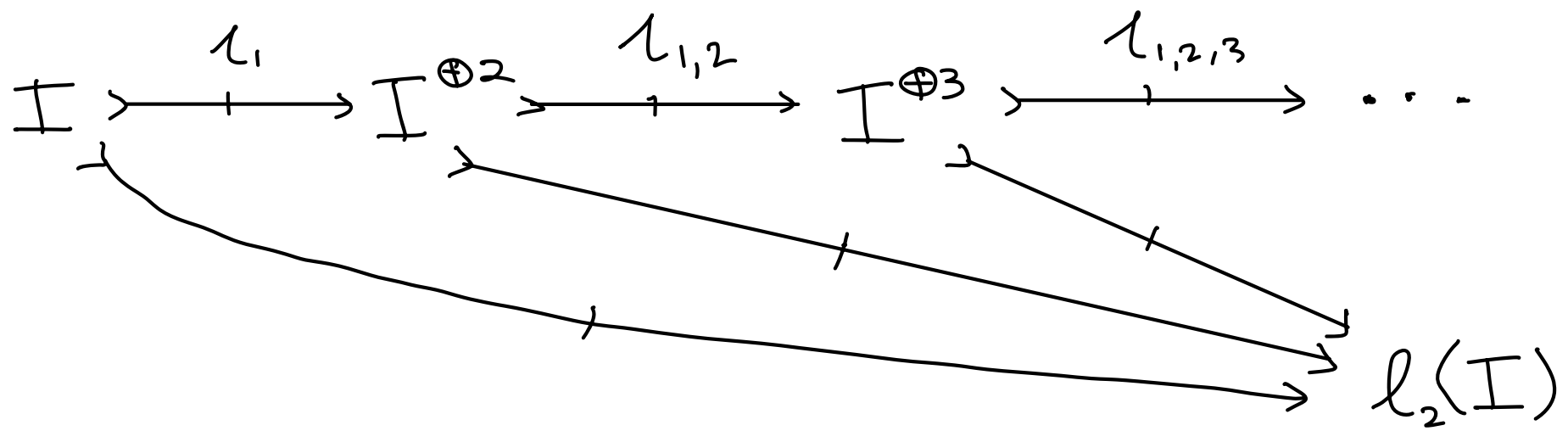
[metric/order completeness of  $\mathbb{I}$ ]

⋮

is equivalent to Hilb

**SOLÉR'S THEOREM:** Let  $X$  be an orthomodular space over an involutive division ring  $\mathbb{K}$ .

If  $X$  has an infinite orthonormal subset, then  $\mathbb{K} \cong \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  and  $X$  is a Hilbert space



## GOAL:

Prove directly that  $\mathbb{I}$  is  $\mathbb{R}$  or  $\mathbb{C}$

Link directed colimits in category theory  
and limits in analysis.

To axiomatise finite-dimensional Hilbert spaces,  
can't use Solér's theorem.

- ③
- Every element is a difference of positives
  - Positives contain 1 and closed under  $+$ ,  $\cdot$ ,  $(-)^{-1}$

**PROP (De Marr 1967):** A partially-ordered field that is Dedekind  $\sigma$ -complete is order isomorphic to  $\mathbb{R}$

Positive decreasing sequences have infima

**LEMMA:** If  $\mathbb{I}_{SA} := \{z \in \mathbb{I} : z = z^+\}$  is  $\mathbb{R}$ , then  $\mathbb{I}$  is  $\mathbb{R}$  or  $\mathbb{C}$ .

**PROOF:** If  $u \in \mathbb{I} \setminus \mathbb{I}_{SA}$ , let  $i = \frac{u - u^+}{\sqrt{-(u - u^+)^2}}$ . Then  $i^2 + 1 = 0$  and  $\{1, i\}$  is basis for  $\mathbb{I}$  over  $\mathbb{I}_{SA}$ .  $\square$

$$a \leq b \iff b - a = x^+ x \text{ for some } x: I \rightarrow X$$

**LEMMA:**  $\mathbb{I}_{SA}$  is a partially-ordered field.

**PROOF:**

$$a^2 = a^+ a$$

$$a = \frac{1}{4}(a+2)^2 - \frac{1}{4}(a^2+4)$$

$$1 = 1^+ 1$$

$$x^+ x + y^+ y = \langle x, y \rangle^+ \langle x, y \rangle$$

$$x^+ x \cdot y^+ y = (x \otimes y)^+ (x \otimes y)$$

$$\frac{1}{x^+ x} = \left( \frac{1}{x^+ x} \right)^2 x^+ x$$

$$a \in \mathbb{I}_{SA}$$

$$x: I \rightarrow X$$

$$y: I \rightarrow Y$$

④

□

GOAL:

~~Prove directly that  $\mathbb{I}$  is  $\mathbb{R}$  or  $\mathbb{C}$~~

Prove that  $\mathbb{I}_{SA}$  is Dedekind  $\sigma$ -complete

PROP (De Marr): Every partially-ordered field that is Dedekind  $\sigma$ -complete is order isomorphic to  $\mathbb{R}$ .

LEMMA: A dagger field with fixed field  $\mathbb{R}$  is  $\mathbb{R}$  or  $\mathbb{C}$ .

LEMMA:  $\mathbb{I}_{SA}$  is a partially-ordered field.

**LEMMA:**  $\mathbb{I}_{\geq 0} = \{y^+y : y: I \rightarrow Y\} = \{x^+x : x: I \rightarrow X \text{ is iso}\}$

**PROOF:**

$$\begin{array}{ccc}
 I & \xrightarrow{y} & Y \\
 \downarrow x & \nearrow k & \\
 & X &
 \end{array}
 \quad
 \begin{array}{c}
 Y \xrightarrow[y]{y(y^+y)^+y^+} Y \\
 \end{array}
 \quad
 \begin{array}{l}
 x^+x = x^+k^+kx \\
 = y^+y
 \end{array}
 \quad \square$$

dagger equaliser

**LEMMA:**  $\mathbb{I}_{SA}$  is Dedekind  $\sigma$ -complete if  $\mathbb{I}_{\geq 0}$  is.

**IDEA:** Addition preserves infima and  $\mathbb{I}_{SA} = \mathbb{I}_{\geq 0} - \mathbb{I}_{\geq 0} \quad \square$

**PROP:**  $\mathbb{I}_{\geq 0}$  is Dedekind  $\sigma$ -complete if the wide subcategory of contractions has directed colimits

$f: X \rightarrow Y$  such that  $f^+f + \bar{f}^+\bar{f} = 1_X$  for some  $\bar{f}: X \rightarrow \bar{Y}$

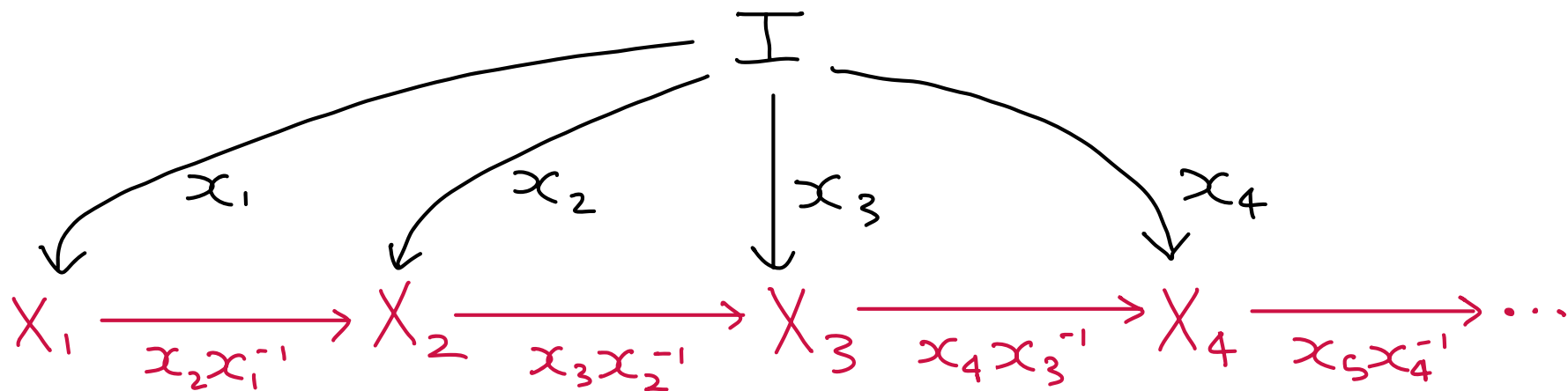
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**PROOF:**  $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

$x_j: I \rightarrow X_j$   
isomorphism

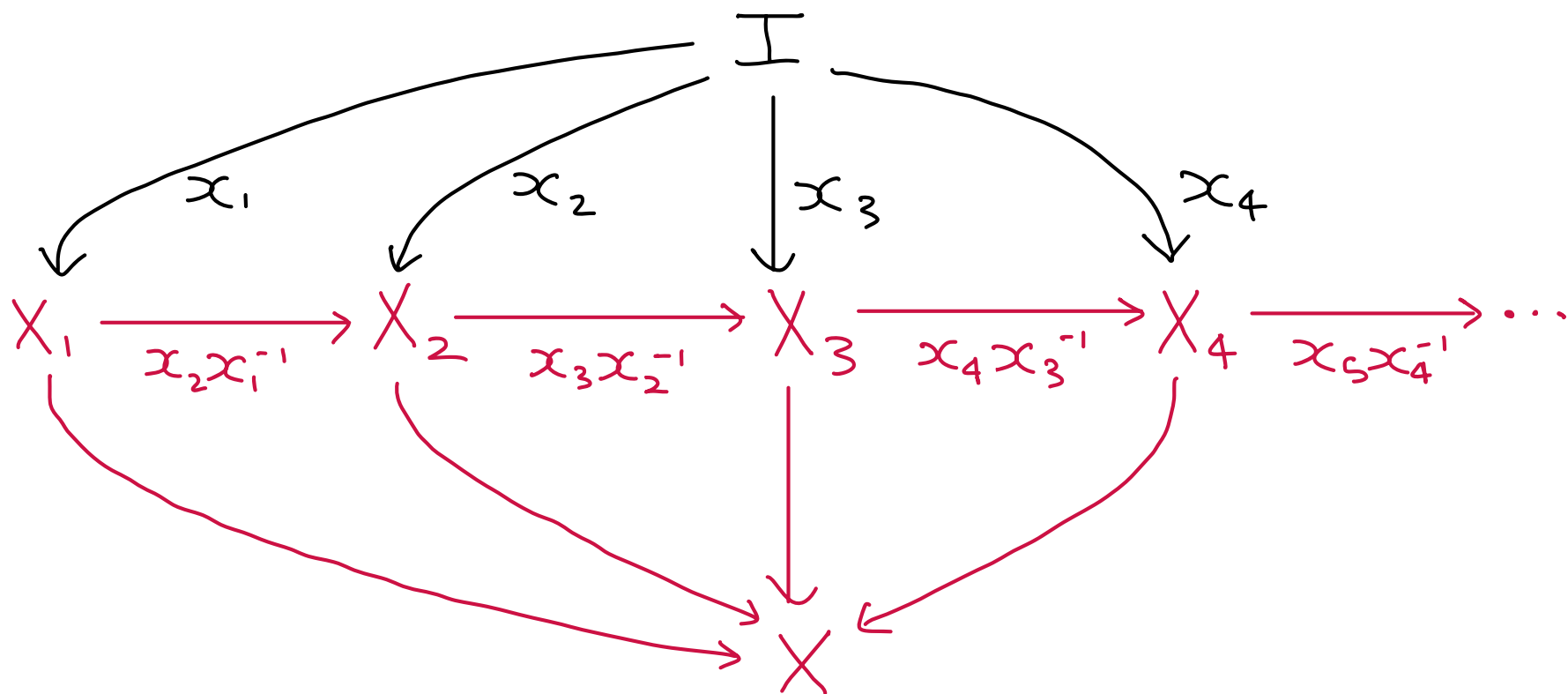
$$1 = x_j^{-+} x_j^+ x_j x_j^{-1} \geq x_j^{-+} x_{j+1}^+ \underbrace{x_{j+1} x_j^{-1}}_{\text{contraction}}$$



**PROOF:**  $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

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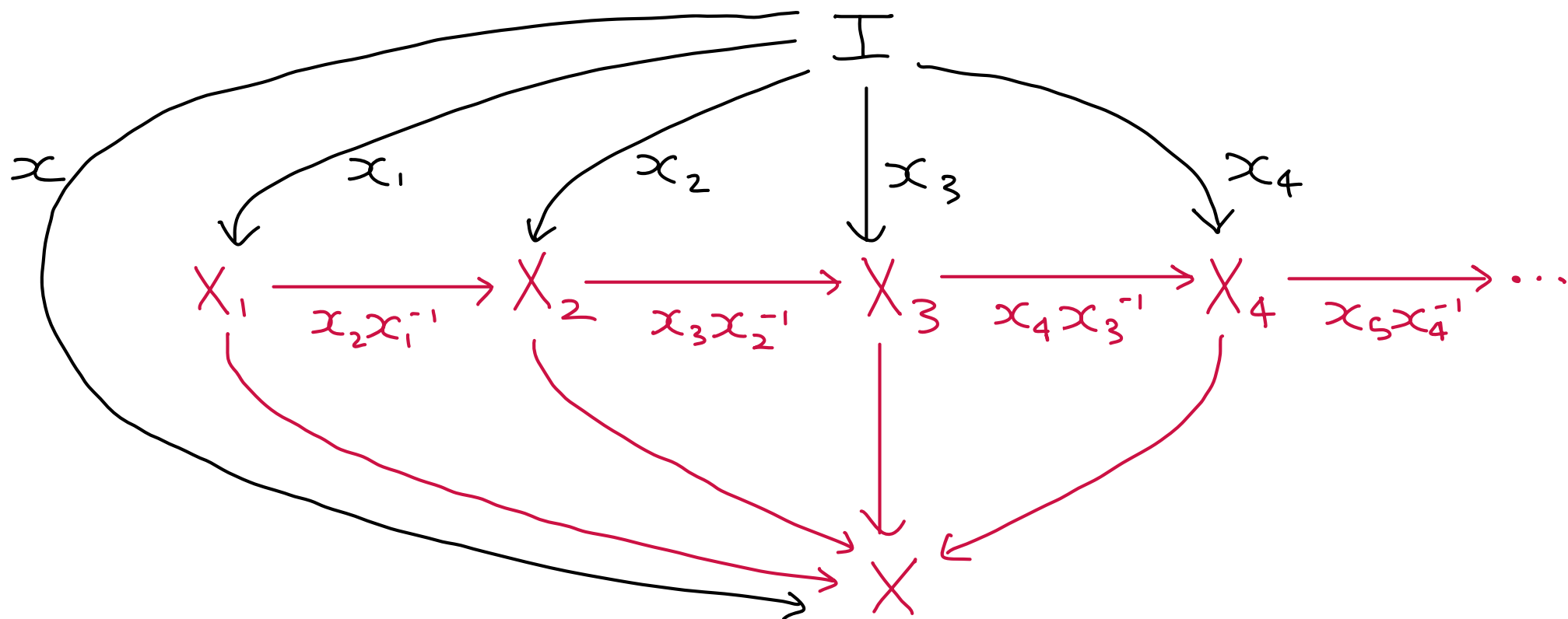
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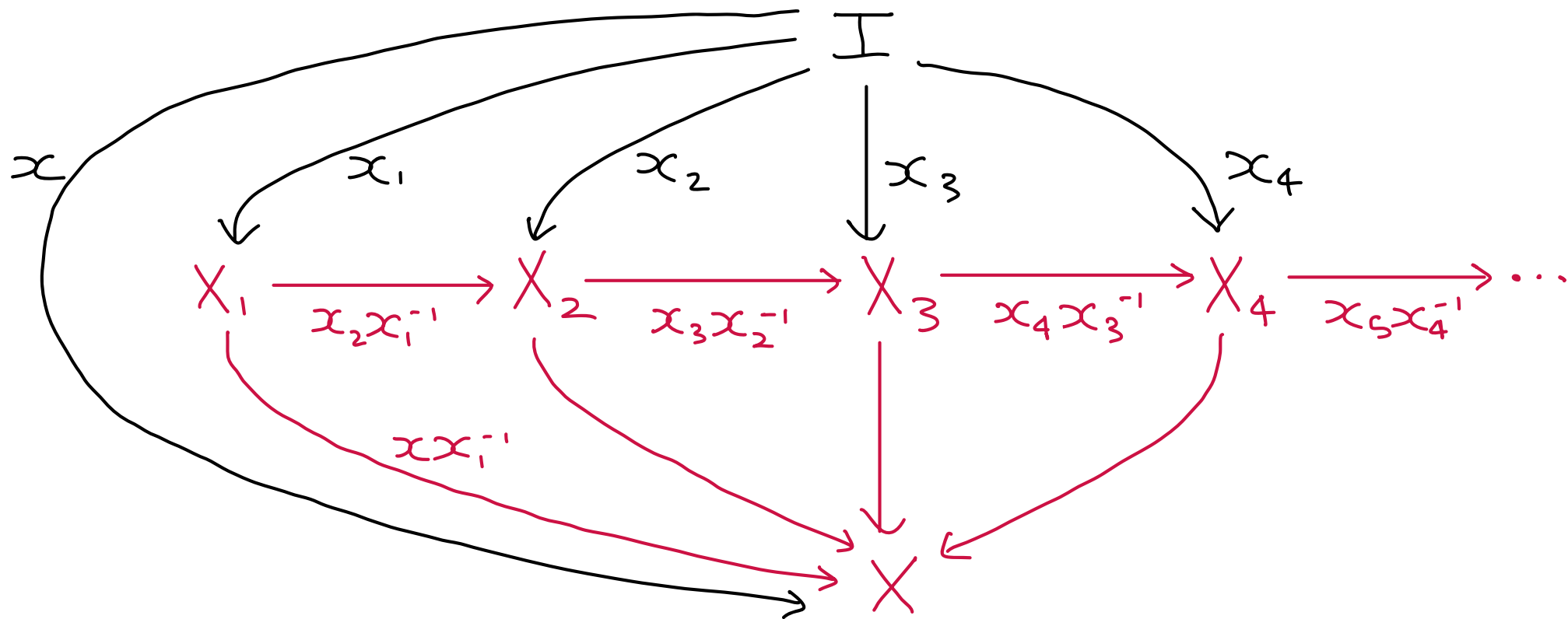
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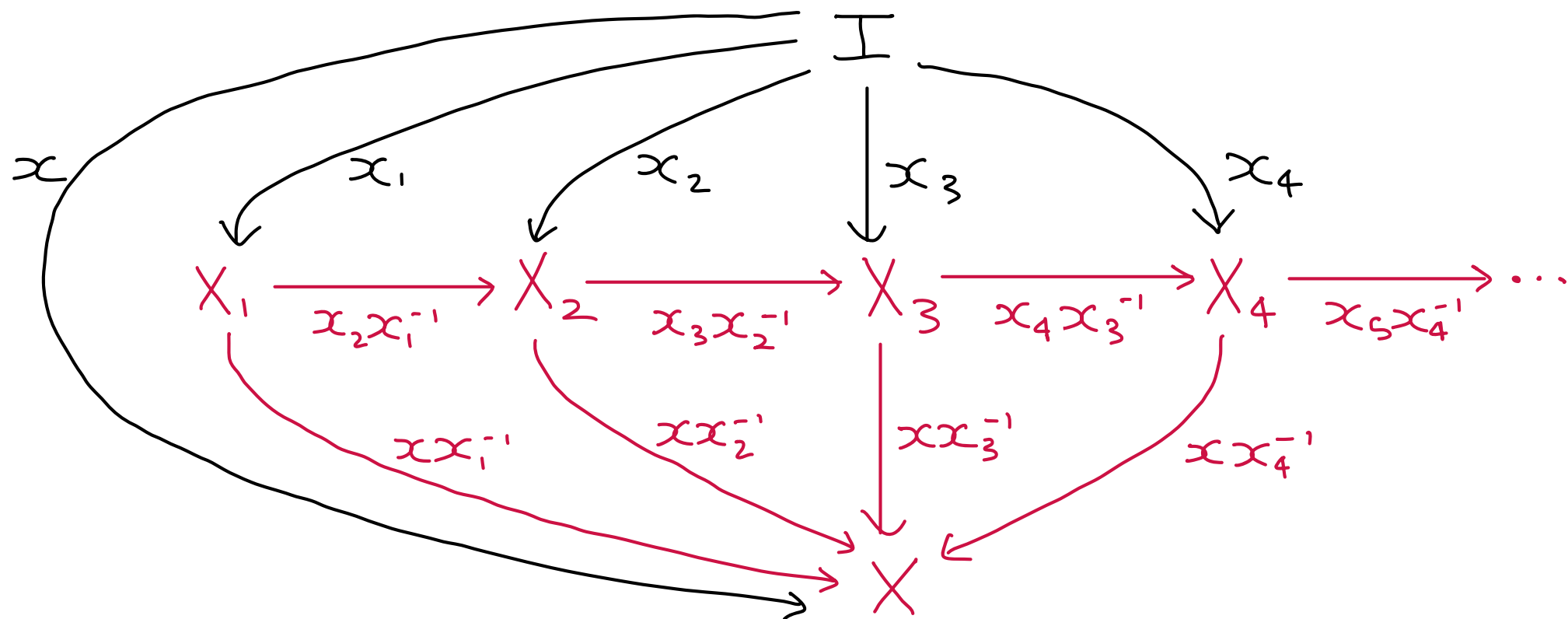
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**PROOF:**  $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

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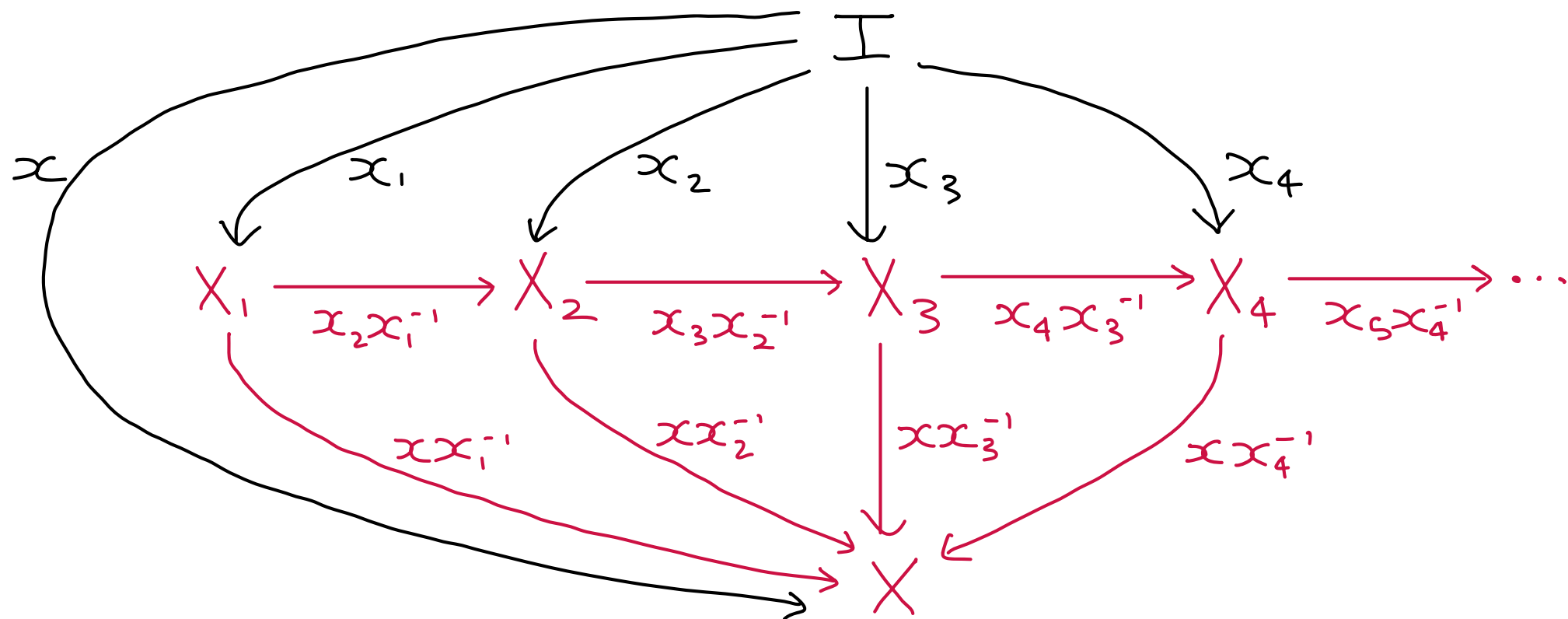
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**PROOF:**  $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

$x_j: I \rightarrow X_j$   
isomorphism

$$1 = x_j^{-+} x_j^+ x_j x_j^{-1} \geq x_j^{-+} x_{j+1}^+ \underbrace{x_{j+1} x_j^{-1}}_{\text{contraction}}$$



$$x_j^+ x_j \geq x_j^+ (x x_j^{-1})^+ (x x_j^{-1}) x_j = \underbrace{x^+ x}_{\text{lower bound}}$$

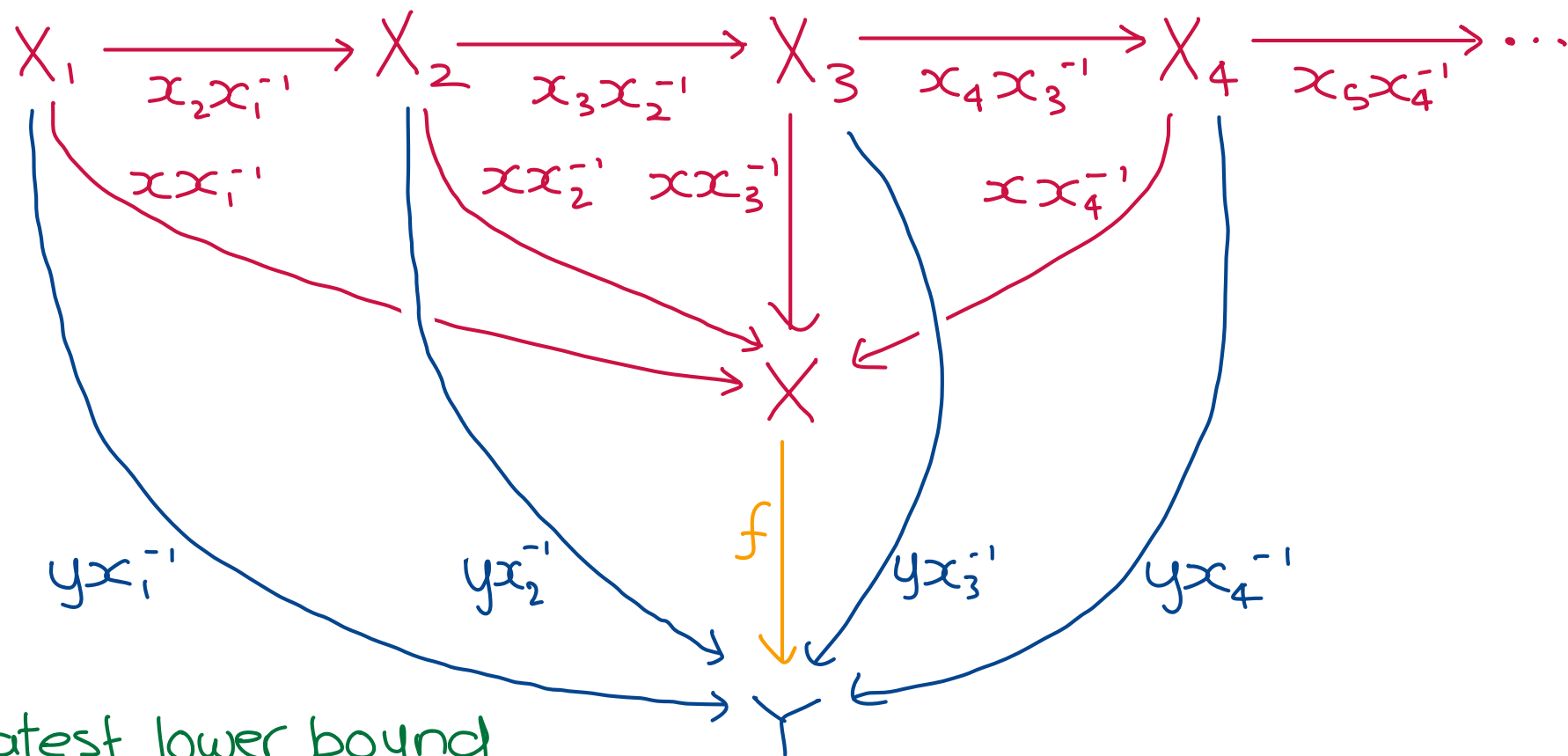
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$$x_j^\dagger x_j \geq y^\dagger y$$

$y: I \rightarrow Y$  isomorphism

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$$1 = x_j^{-\dagger} x_j^\dagger x_j x_j^{-1} \geq x_j^{-\dagger} y^\dagger \underbrace{y x_j^{-1}}_{\text{contraction}}$$



$$\underbrace{x^\dagger x}_{\text{greatest lower bound}} \geq x^\dagger f^\dagger f x = x_1^\dagger (x x_1^{-1})^\dagger f^\dagger f (x x_1^{-1}) x_1 = x_1^\dagger (y x_1^{-1})^\dagger (y x_1^{-1}) x_1 = y^\dagger y$$

**THM:** If wide subcategory of contractions  
has directed colimits, then  $\mathbb{I}$  is  $\mathbb{R}$  or  $\mathbb{C}$ .

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## OPEN QUESTIONS:

- ① Can we construct directed colimits of contractions from those of dagger monos?
- ② If we drop symmetric monoidal structure and let  $\mathbb{I}$  be a simple projective separator, can we deduce that  $\mathbb{I}$  is  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$ ?



# WORK IN PROGRESS:

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- Dagger-category analogue of abelian categories  
(Includes Hilb, Monoidal structure not needed)
- Axioms for FdHilb<sub>con</sub> (with Chris & André)
- Axioms for FdHilb (with Chris)
- Axioms for Hilb<sub>isometry</sub> (with Chris, Robert)

(Ultimate goal is quantum-relevant categories like

FdHilb<sub>unitary</sub>, but this is hard)



Let  $F$  be an involutive field with fixed field  $\mathbb{R}$ .

Suppose that  $F \neq \mathbb{R}$ . Then there is an  $e \in F$

with  $e \neq \bar{e}$ . We will show that  $\{1, e\}$  is

a basis for  $F$  as a vector space over  $\mathbb{R}$ .

If  $a + be = 0$  for  $a, b \in \mathbb{R}$ , then  $be = -a$ .

If  $b \neq 0$ , then  $e = -\frac{a}{b} \in \mathbb{R}$ , contradicting  $e \neq \bar{e}$ .

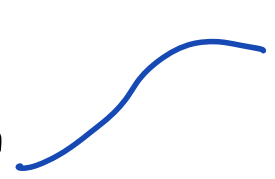
So  $b = 0$ , and thus  $a = -be = 0$  as well.

Each  $x \in \mathbb{I}$  is of the form

$$x = \frac{x\bar{e} - \bar{x}e}{\bar{e} - e} + \frac{\bar{x} - x}{\bar{e} - e} e$$

where  $\frac{x\bar{e} - \bar{x}e}{\bar{e} - e}$  and  $\frac{\bar{x} - x}{\bar{e} - e}$  are self adjoint.

Hence  $[\mathbb{F} : \mathbb{R}] = 2$ , and  $\mathbb{F} = \mathbb{R}(e)$ .

Let  $i = \frac{e - \bar{e}}{\sqrt{-(e - \bar{e})^2}}$   uses self adjoint is  $\mathbb{R}$

Then

$$i^2 + 1 = \frac{(e - \bar{e})^2}{-(e - \bar{e})^2} + 1 = -1 + 1 = 0$$

$$\overline{i} = -i$$