

# CATEGORICAL HILBERT THEORY

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Hilbert spaces

Hilbert spaces

Unitary  
representations

Hilbert modules

$C^*$ -algebras

Hilbert spaces

Unitary  
representations

# Hilbert theory

Hilbert modules

$C^*$ -algebras

1

Abstract frameworks

2

Limits

3

Relations

1

Abstract frameworks

# Theory

homological algebra

probability theory

differential geometry

Hilbert theory

# Categorical setting

abelian categories

Markov categories

tangent categories

?

A  $*$ -category is equivalent to  $\mathbf{Hilb}_{\mathbb{R}}$ ,  $\mathbf{Hilb}_{\mathbb{C}}$  or  $\mathbf{Hilb}_{\mathbb{H}}$  iff

- has zero object
- has binary orthonormal products
- has isometric kernels
- $\Delta: \mathbb{X} \rightarrow \mathbb{X} \oplus \mathbb{X}$  are kernels
- subcategory of isometries has directed colimits
- has simple separator

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A category is equivalent to  $\mathbf{Mod}_R$  for some ring  $R$  iff

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- monos are kernels and epis are cokernels
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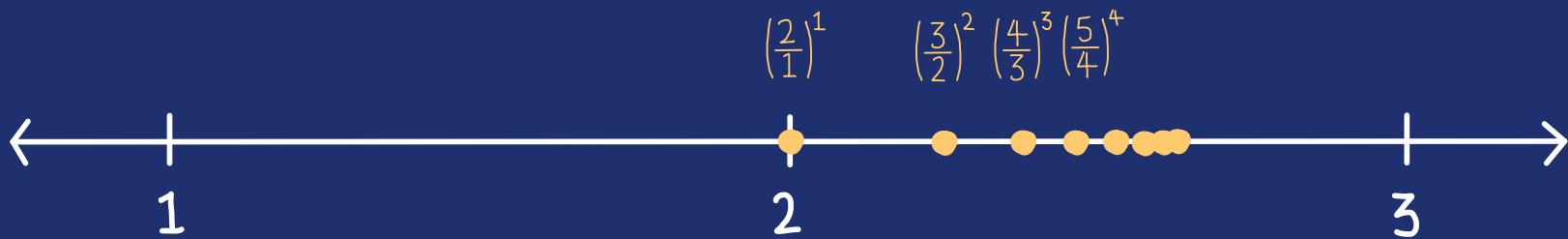
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- also include  $\mathbf{Hilb}_A$  and  $\mathbf{URep}_G$

2

Limits

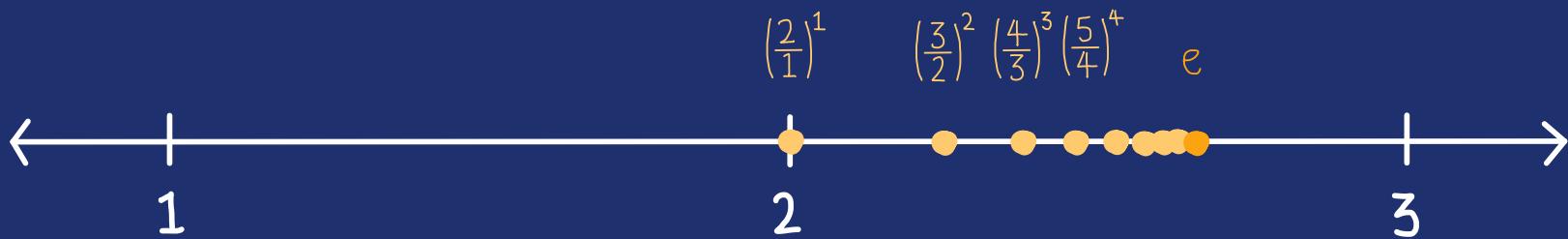
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$$\lim_{n \in \mathbb{N}} \mathbb{C}^n =$$

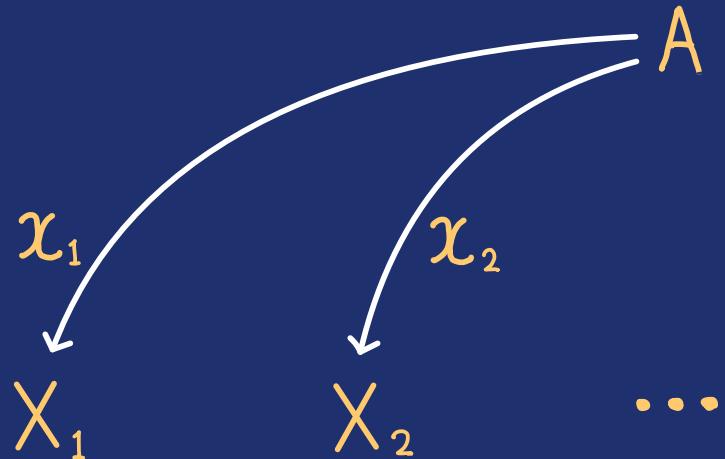
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$$\lim_{n \in \mathbb{N}} \mathbb{C}^n = \ell^2(\mathbb{N}) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \mid |x_1|^2 + |x_2|^2 + \dots < \infty \right\}$$

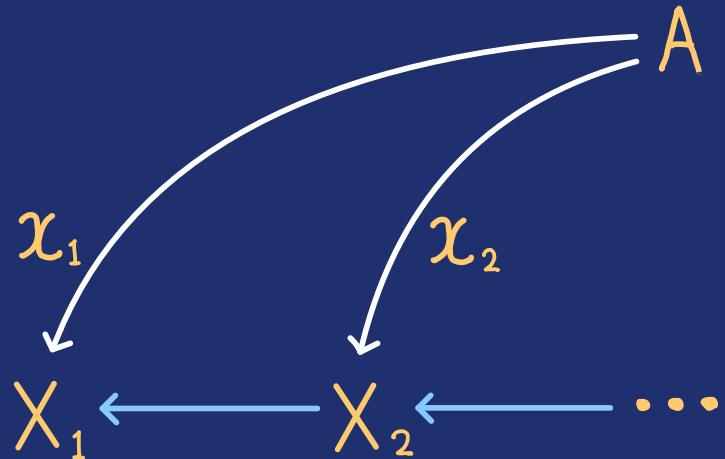
Given operators  $0 < a_1 \leq a_2 \leq \dots \leq 1$  on a Hilbert space  $A$

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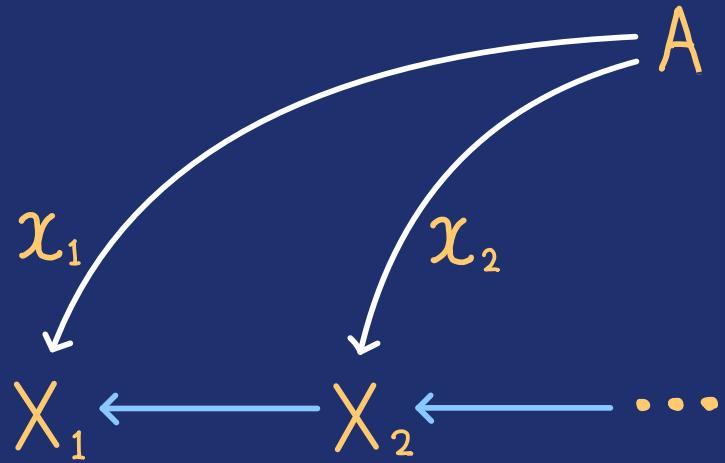
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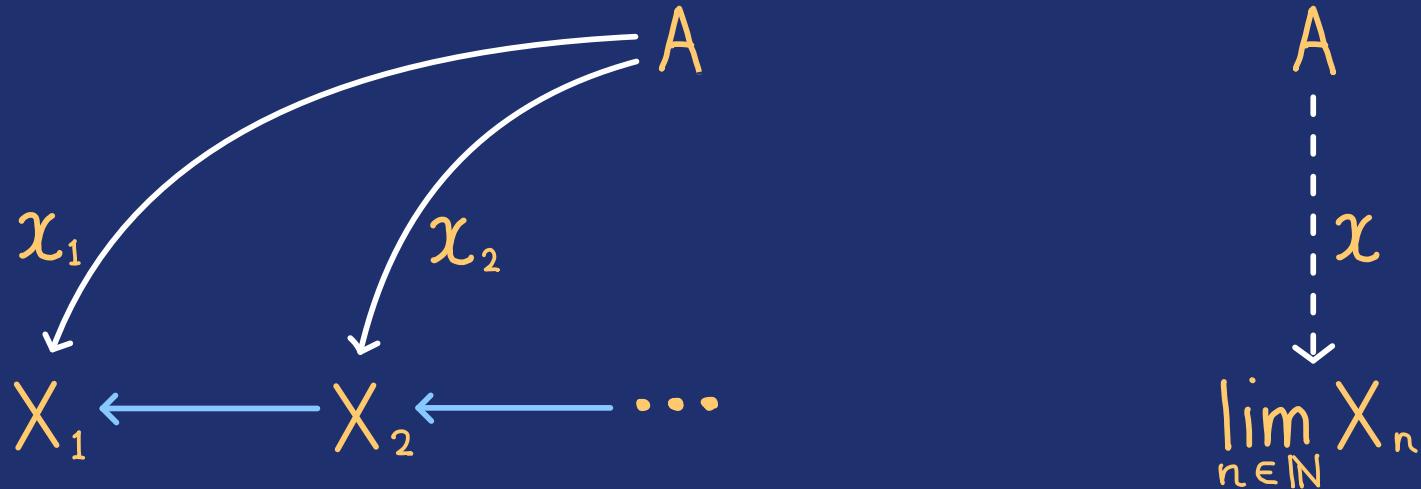
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$$\lim_{n \in \mathbb{N}} X_n$$

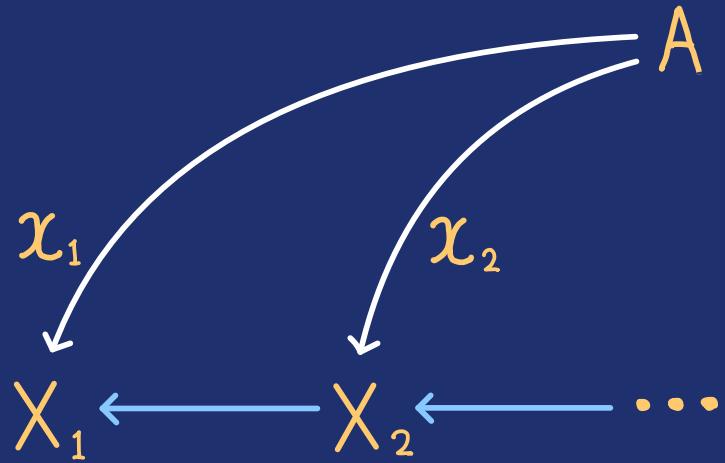
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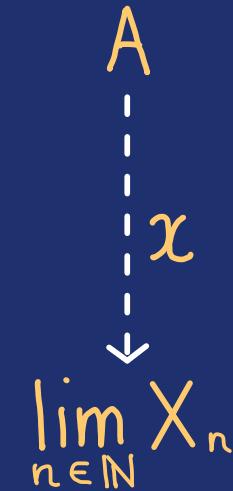


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$$\chi^* \chi = \sup_{n \in \mathbb{N}} a_n$$

This idea yields

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- the first characterisation of a category of finite-dimensional Hilbert spaces

3

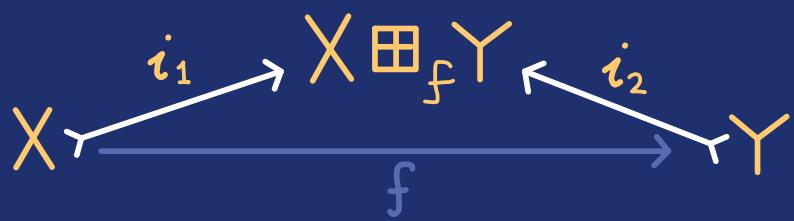
# Relations

Every morphism in  $\mathbf{Con}$  has a codilator

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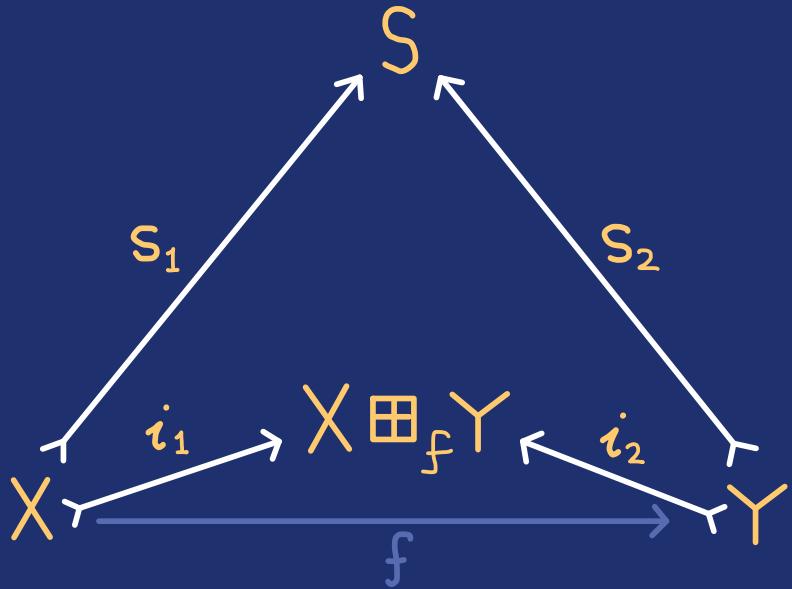
$$X \xrightarrow{f} Y$$

Every morphism in  $\mathbf{Con}$  has a codilator



$$i_1^* i_1 = 1 \quad i_2^* i_1 = f \quad i_2^* i_2 = 1$$

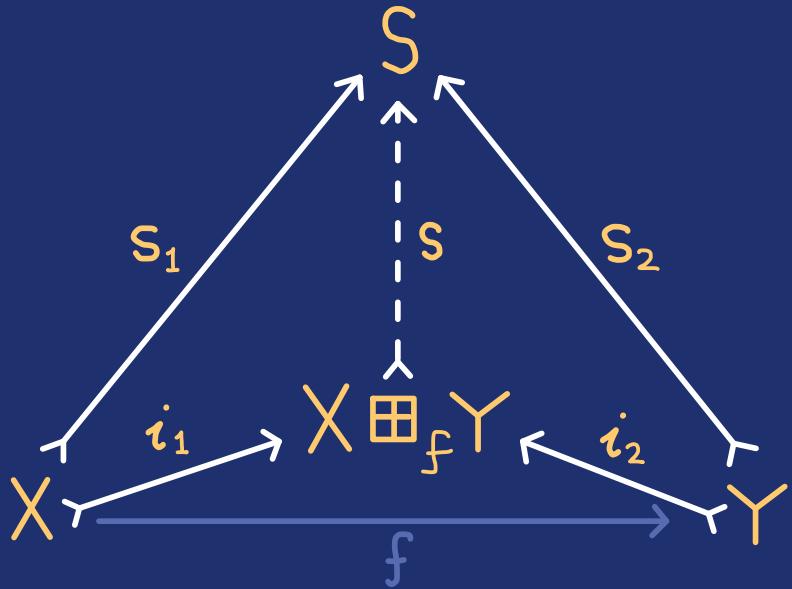
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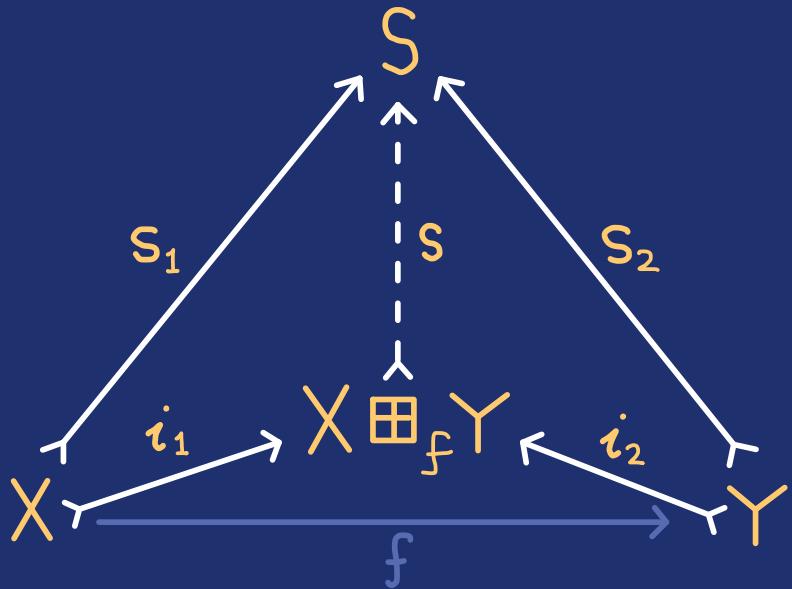
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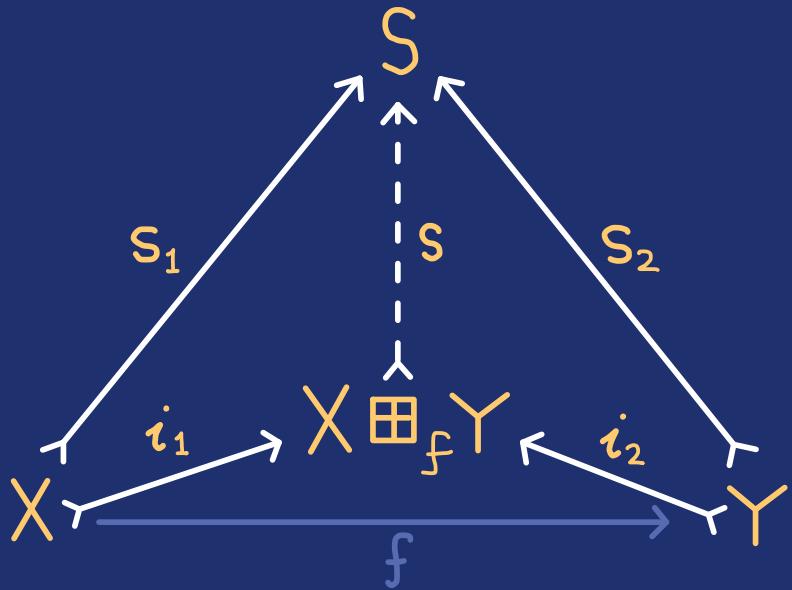


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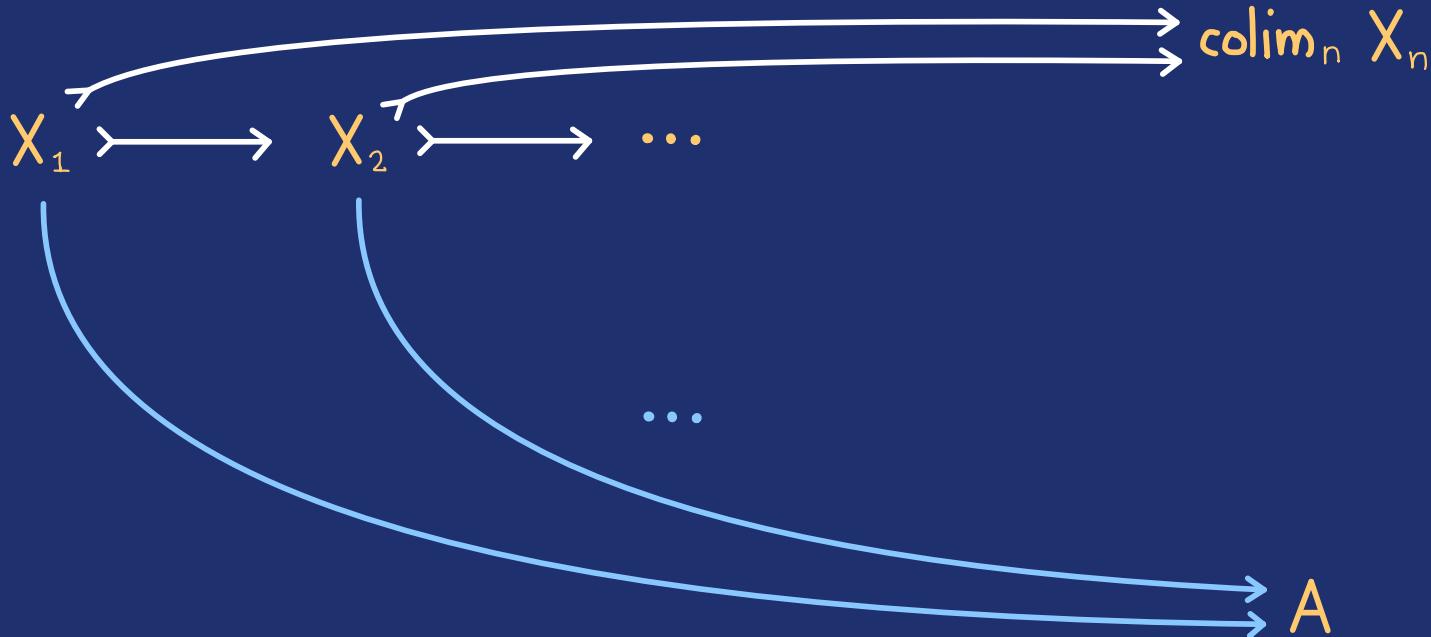
Inspired by Sz.-Nagy's minimal unitary dilations  
Reminiscent of cotabulators

Isom $\rightarrow$ Con preserves directed colimits

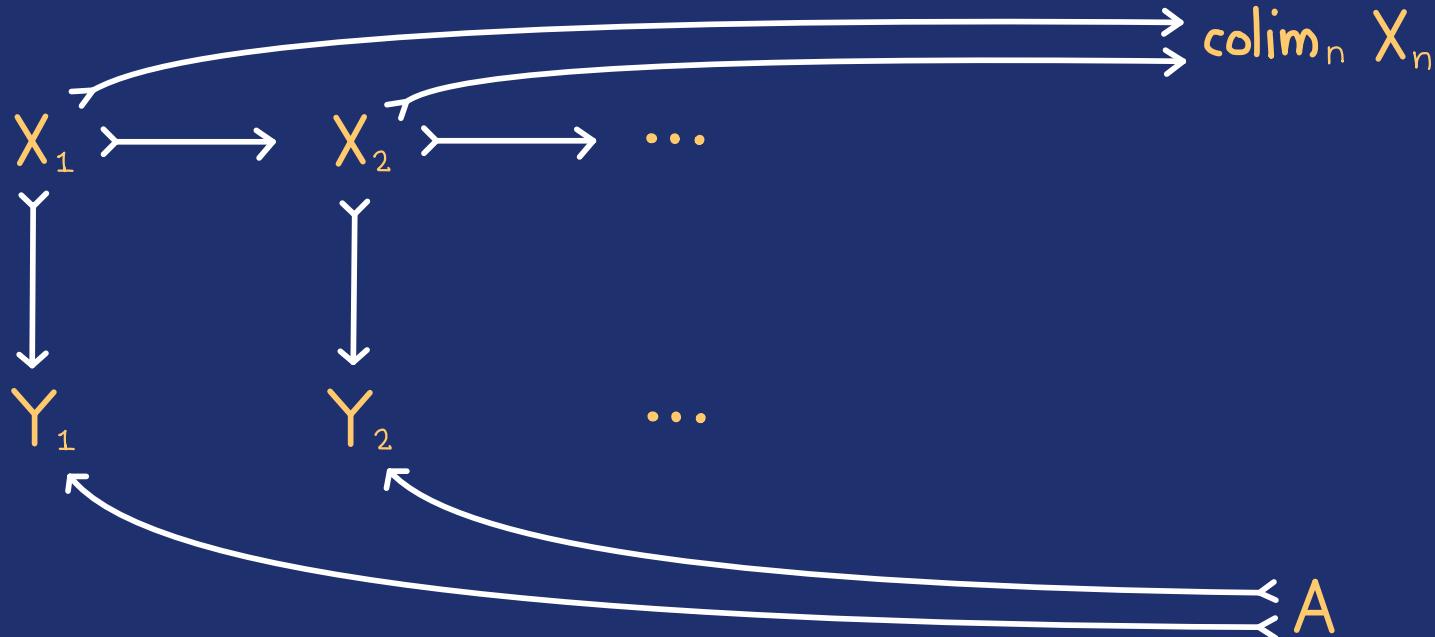
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$$\begin{array}{c} X_1 \rightarrow X_2 \rightarrow \dots \end{array} \xrightarrow{\text{colim}_n X_n}$$

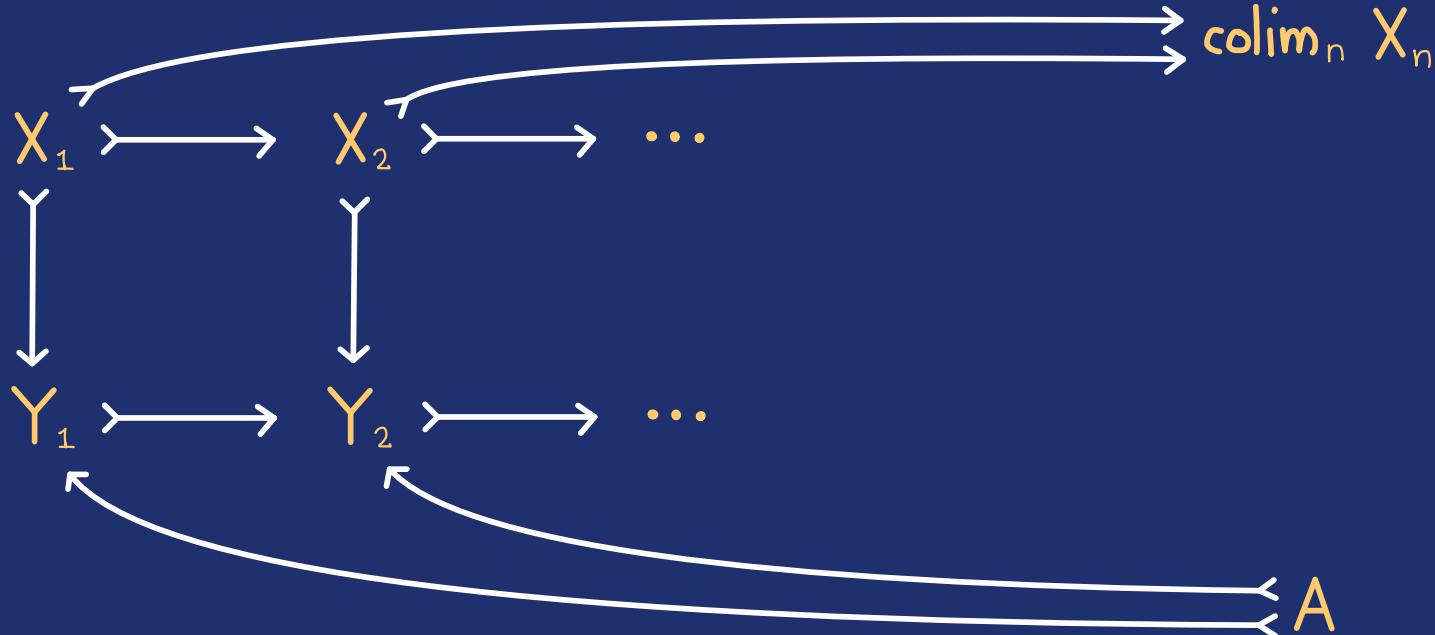
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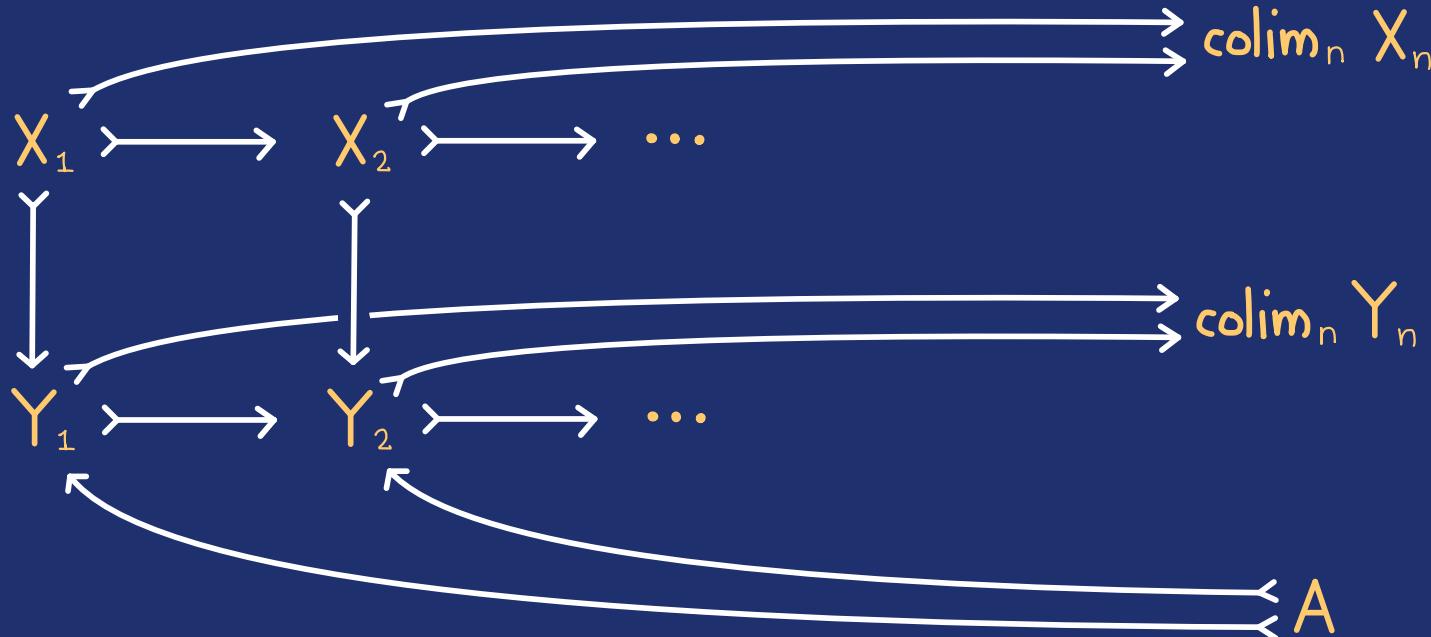
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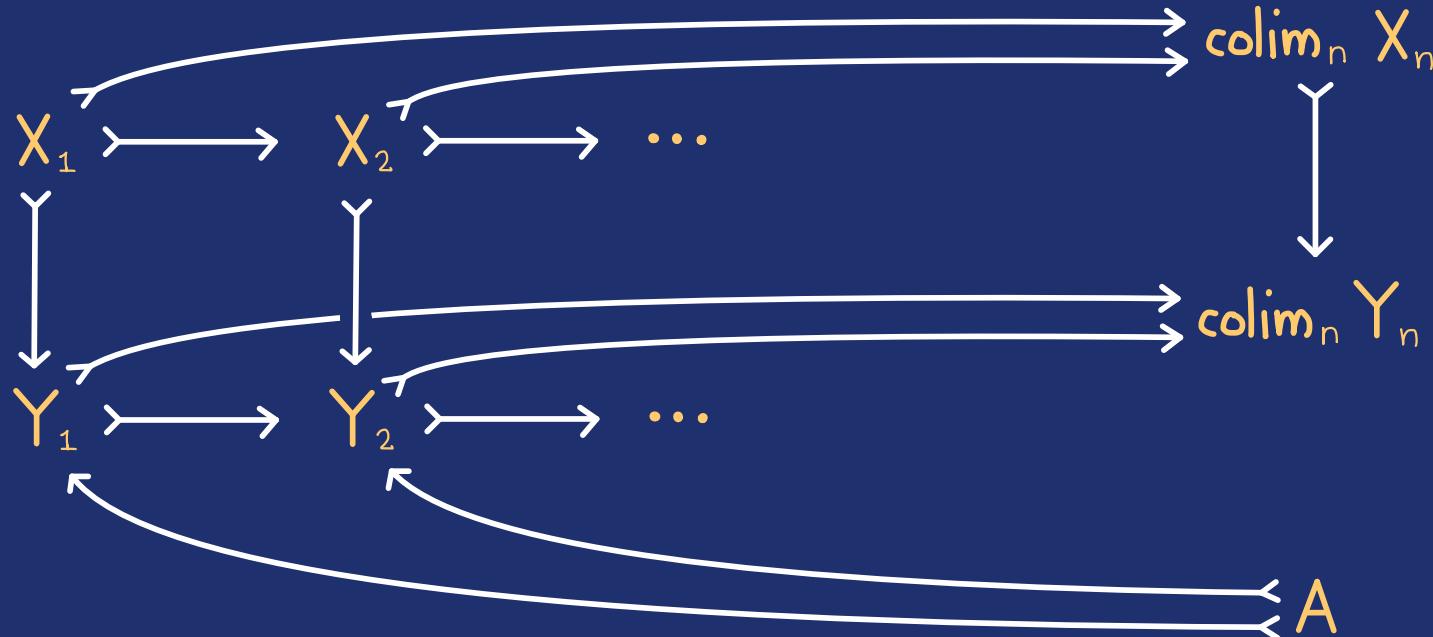
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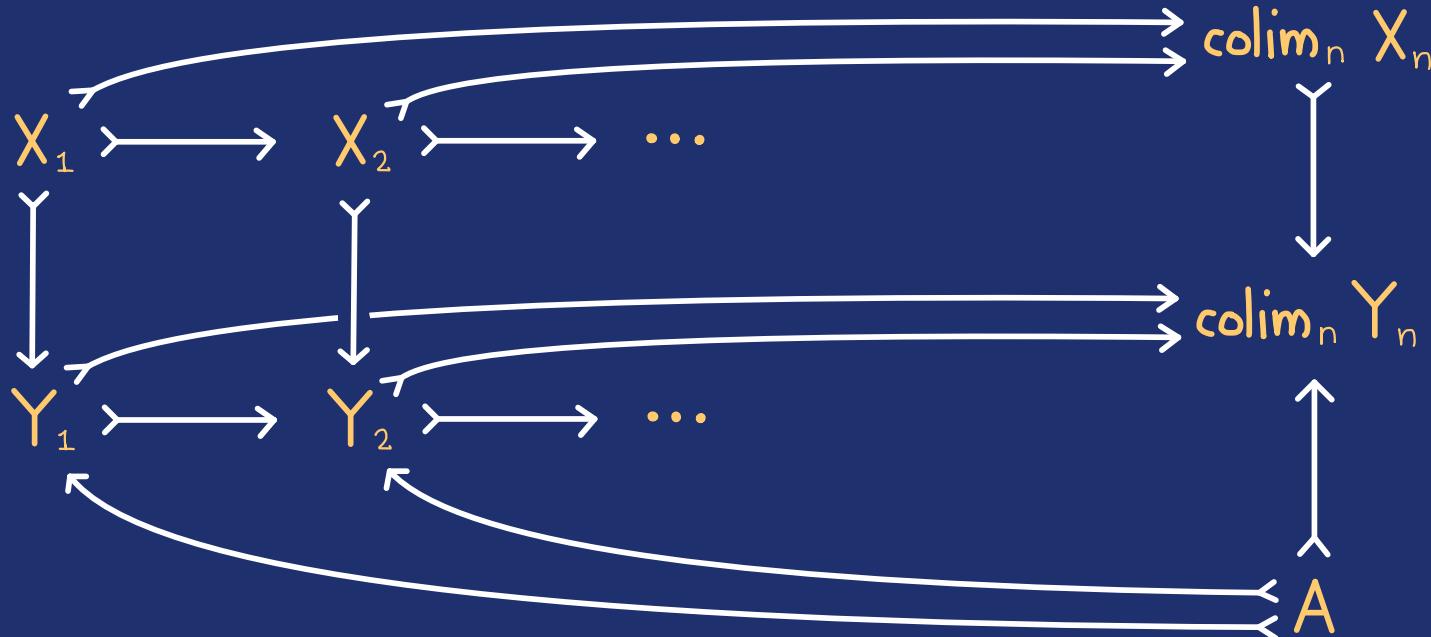
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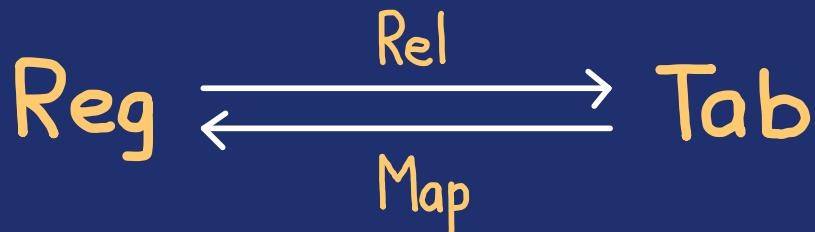
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regular  
categories



tabular  
allegories

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Con      Plnj      MSurj      FinProb      StdBorProb

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regular  
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Reg  $\begin{array}{c} \xrightarrow{\text{Rel}} \\ \xleftarrow{\text{Map}} \end{array}$  Tab

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?  $\begin{array}{c} \xrightarrow{\text{Rel}} \\ \xleftarrow{\text{Coisom}} \end{array}$  Dil

$*$ -categories  
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epi-regular  
independence  
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