Abelian groups are to abelian categories as Hilbert spaces are to what?

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The category of Hilbert spaces and bounded linear maps

Hilbert spaces

An ***inner product*** on a vector space encodes geometry.

$$\|x\| = \sqrt{\langle x|x\rangle} \qquad\qquad \cos\theta = \frac{\langle x|y\rangle}{\|x\|\,\|y\|}$$

A ***Hilbert space*** is a vector space with a ***complete*** inner product.

Every n-dimensional (complex) Hilbert space is isomorphic to $\mathbb{C}^n$ with

$$\big\langle (x_1, x_2, \ldots, x_n) \big| (y_1, y_2, \ldots, y_n) \big\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

$\ell_2(\mathbb{N}) = \big\{ (x_1, x_2, \ldots) \in \mathbb{C}^{\mathbb{N}} \;\big|\; |x_1|^2 + |x_2|^2 + \cdots < \infty \big\}$ with

$$\big\langle (x_1, x_2, \ldots) \big| (y_1, y_2, \ldots) \big\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots$$

Bounded linear maps

A map $T\colon X \to Y$ is ***bounded*** if there is a $C > 0$ such that

$$\|Tx\| \leqslant C\|x\|.$$

A linear map is ***continuous*** if and only if it is bounded.

$\mathbf{Hilb}_{\mathbb{K}}$ is the category of Hilbert spaces and bounded linear maps over $\mathbb{K}$ where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Adjoints

The ***adjoint*** of a bounded linear map $T\colon X \to Y$ is the unique bounded linear map $T^\dagger\colon Y \to X$ such that

$$\langle y|Tx\rangle = \langle T^\dagger y|x\rangle.$$

The matrix of $T^\dagger\colon \mathbb{C}^m \to \mathbb{C}^n$ is the conjugate-transpose of the matrix of $T\colon \mathbb{C}^n \to \mathbb{C}^m$.

Dagger categories

A ***dagger category*** is a category equipped with a choice of $f^\dagger\colon Y \to X$ for each $f\colon X \to Y$, such that

$$1^\dagger = 1, \qquad (gf)^\dagger = f^\dagger g^\dagger, \qquad (f^\dagger)^\dagger = f.$$

Examples include $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ and $\mathbf{Hilb}_{\mathbb{H}}$.

Characterisation of $\mathbf{Hilb}_{\mathbb{K}}$

Theorem (Heunen–Kornell[1], Tobin[2])

A dagger category is equivalent to $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ *or* $\mathbf{Hilb}_{\mathbb{H}}$ *if and only if*

- *it has a zero object,*
- *it has binary dagger products,*
- *it has dagger equalisers,*
- *every dagger mono is normal,*
- *the wide subcategory of dagger monos has directed colimits, and*
- *it has a simple separator.*

[1] Heunen and Kornell, "Axioms for the category of Hilbert spaces".
[2] Tobin, "Characterisations for the category of Hilbert spaces".

Dagger monos

A linear map $f \colon X \to Y$ is an ***isometry*** if $\|fx\| \leqslant \|x\|$.

Isometries represent ***closed*** subspaces.

A morphism $f \colon X \to Y$ is ***dagger monic*** if $f^\dagger f = 1$.

A bounded linear map is an isometry if and only if it is dagger monic.

Dagger kernels and equalisers

The ***kernel*** of a bounded linear map $f \colon X \to Y$ is the subspace

$$\mathrm{Ker}\, f = \{x \in X \mid fx = 0\}.$$

The restricted inner product makes $\mathrm{Ker}\, f$ a Hilbert space and the canonical inclusion $\mathrm{Ker}\, f \hookrightarrow X$ an isometry.

A ***dagger kernel/equaliser*** is a dagger monic kernel/equaliser.

Hilb has dagger kernels/equalisers.

Dagger coproducts

The ***direct sum*** of X and Y is

$$X \oplus Y = \{(x,y) \,|\, x \in X, y \in Y\},$$
$$\big\langle (x,y) \big| (x',y') \big\rangle = \langle x|x'\rangle + \langle y|y'\rangle.$$

The injections

$$i_1 \colon X \to X \oplus Y \qquad X \oplus Y \leftarrow Y : i_2$$
$$x \mapsto (x,0) \qquad (0,y) \leftarrow\!\shortmid\, y$$

are ***orthogonal*** isometries.

A ***dagger coproduct*** is a coproduct whose injections are dagger monic and pairwise orthogonal.

$$i_j^\dagger i_k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Dagger coproducts are biproducts.

Hilb has finite dagger coproducts.

Characterisations of $\mathbf{Hilb}_{\mathbb{K}}$ and $\mathbf{Mod}_R$

Theorem (Heunen–Kornell, Tobin)

A dagger category is equivalent to $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ or $\mathbf{Hilb}_{\mathbb{H}}$ if and only if

- *it has a zero object,*
- *it has binary dagger products,*
- *it has dagger kernels,*
- *binary diagonals are normal,*
- *the subcategory of dagger monos has directed colimits,*
- *it has a simple separator.*

Theorem

A category is equivalent to $\mathbf{Mod}_R$ for some ring R if and only if

- *it has a zero object,*
- *it has binary products/coproducts,*
- *it has kernels/cokernels,*
- *all monos/epis are normal,*
- *it has small coproducts,*
- *it has a compact projective separator.*[3]

[3] See Freyd, "Abelian Categories", p. 106.

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Rational dagger categories

Definition

A dagger category is ***rational*** if

- it has a zero object,
- it has binary dagger products,
- it has dagger kernels, and
- all $\Delta\colon X \to X \oplus X$ are normal.

A category is ***abelian*** if

- it has a zero object,
- it has binary products/coproducts,
- it has kernels/cokernels, and
- all monos/epis are normal.[4]

[4]See Borceux, *Handbook of Categorical Algebra*.

Examples of rational dagger categories

- **Hilb**$_{\mathbb{K}}$ where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.
- For each W*-algebra A, the category of self-dual Hilbert A-modules and bounded A-linear maps.
- For each partially semiordered involutive division ring R, the category of finite-dimensional inner-product R-modules and R-linear maps.
- For each involutive division ring R that is formally complex and quadratically closed, the category of R-valued matrices.

Similar properties

Rational dagger category properties:

- monic if and only if zero kernel,
- (epi, dagger mono) factorisations,
- additive, so have finite dagger biproducts and dagger equalisers,
- normal monos are pushout stable,
- pushouts along normal monos are pullbacks.

Abelian category properties:

- monic if and only if zero kernel,
- (epi, mono) factorisations,
- additive, so have finite biproducts and equalisers,
- monos are pushout stable,
- pushouts along monos are pullbacks.

Additivity

Theorem

In a semiadditive category, an object X is abelian if and only if $\nabla\colon X \oplus X \to X$ is the cokernel of a split mono.

Let $e\colon X \oplus X \to X \oplus X$ be the induced idempotent. Then

$$(p_1 + p_2)e = 0 \qquad \text{and} \qquad fe = 0 \implies fi_1 = fi_2.$$

Observe that

$$\begin{aligned}(1 + p_1ei_2 + p_2ei_1)p_1e &= p_1e + p_1ei_2p_1e + p_2ei_1p_1e + p_2ei_2(p_1 + p_2)e \\ &= p_1e + (p_1 + p_2)ei_2p_1e + p_2e(i_1p_1 + i_2p_2)e = p_1e + p_2e^2 = 0.\end{aligned}$$

Hence $1 + p_1ei_2 + p_2ei_1 = (1 + p_1ei_2 + p_2ei_1)p_1i_1 = (1 + p_1ei_2 + p_2ei_1)p_1i_2 = 0.$

Partial semiordering

Let $\mathbf{C}$ be a rational dagger category.

For each object A of $\mathbf{C}$, define $\leqslant$ on the self-adjoint endomorphisms of A by

$$a \leqslant b \qquad \iff \qquad b - a = x^\dagger x \text{ for some } x\colon A \to X.$$

Then $\leqslant$ is a partial order with the following properties:

$$0 \leqslant 1, \qquad a \leqslant b \implies a + c \leqslant b + c, \qquad a \leqslant b \implies f^\dagger a f \leqslant f^\dagger b f.$$

Each endohomset of $\mathbf{C}$ is thus a *partially semiordered involutive ring*.

Isomorphisms

Theorem

In a rational dagger category, if $a \geqslant 1$ then a is invertible.[5]

Theorem

Rational dagger categories are uniquely enriched in the category of rational vector spaces.

[5] Similar to Handelman, "Rings with involution as partially ordered abelian groups", Proposition 1.13.

Orthogonal complements

The ***orthogonal complement*** of a mono $m\colon A \to X$ is the dagger mono $m^\perp\colon X \ominus A \to X$ defined by

$$m^\perp = \ker m^\dagger = (\operatorname{coker} m)^\dagger.$$

Well-known properties of kernels and cokernels imply that

$$m^{\perp\perp\perp} = m^\perp, \qquad 0^\perp = 1, \qquad 1^\perp = 0, \qquad m \leqslant n^\perp \iff m^\perp \leqslant n.$$

Theorem

In a rational dagger category, if $m\colon A \to X$ is dagger monic, then $(X, m, m^\perp)$ is a dagger coproduct of A and $X \ominus A$.

Gram–Schmidt procedure

In a rational dagger category, for each biproduct

$$\left(A_k \underset{r_k}{\overset{s_k}{\rightleftarrows}} X\right)_{k=1}^{n},$$

the equations

$$t_1 = s_1 \qquad \text{and} \qquad t_{m+1} = s_{m+1} - \sum_{k=1}^{m} t_k ({t_k}^{\dagger} t_k)^{-1} {t_k}^{\dagger} s_{m+1}.$$

define an ***orthogonal*** biproduct

$$\left(A_k \underset{({t_k}^{\dagger} t_k)^{-1} {t_k}^{\dagger}}{\overset{t_k}{\rightleftarrows}} X\right)_{k=1}^{n}$$

where $\bigcup_{k=1}^{m} t_k = \bigcup_{k=1}^{m} s_k$ for each m.

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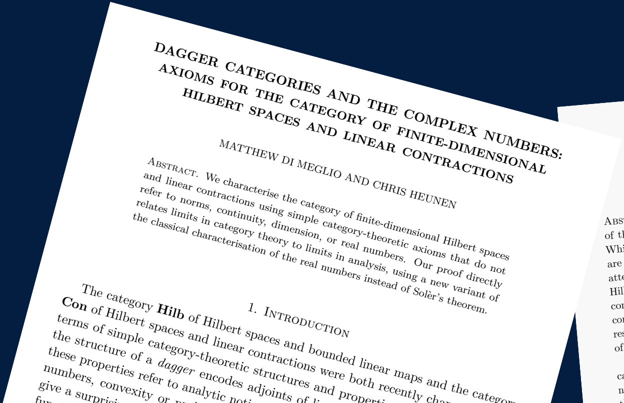
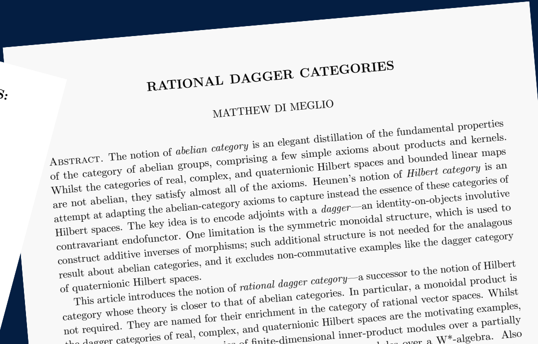
Conclusion

Contact me at m.dimeglio@ed.ac.uk





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