BYPASSING SOLÈR'S THEOREM

The key to axiomatising dagger categories of finite-dimensional Hilbert spaces

MATTHEW DI MEGLIO

(Joint work with Chris Heunen)

APPLIED CATEGORY THEORY 2023

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These should eventually inform the design of programming languages for quantum computers.

THEOREM (Heunen and Kornell):

A monoidal dagger category in which

- finite dagger biproducts exist
- dagger equalisers exist
- monoidal unit is simple
- wide subcategory of dagger monos has directed colimits

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enrichment in commutative monoids

$$\begin{tikzcd} X \arrow[r, "f+g"] \arrow[d, "\Delta"'] & Y \\ X \oplus X \arrow[r, "f \oplus g"'] & Y \oplus Y \arrow[u, "\nabla"'] \end{tikzcd}$$

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$\mathbb{I} := \mathrm{Hom}(I, I)$ is a field and $\mathrm{char}(\mathbb{I}) = 0$

$\mathbb{I}$ is Dedekind complete

Solèr's Theorem:

Let X be an orthomodular space over an involutive division ring $\mathbb{K}$. If X has an infinite orthonormal subset, then $\mathbb{K} \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and X is a Hilbert space.

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I \arrow[r, "i_1"] \arrow[drrrrr, bend right] & I^{\oplus 2} \arrow[r, "i_{1,2}"] \arrow[drrrr, bend right] & I^{\oplus 3} \arrow[r, "i_{1,2,3}"] \arrow[drrr, bend right] & I^{\oplus 4} \arrow[r] \arrow[drr] & \cdots & \\
 & & & & & \ell_2(I)
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This idea will not work for <u>FDHilb</u>.

GOAL:

Prove that $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{C}$ without Solèr's theorem.

THEOREM (De Marr 1967):
All Dedekind σ-complete partially ordered fields are isomorphic to $\mathbb{R}$.

Non-negatives contain 1 and closed under $+, \cdot, (-)^{-1}$

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All Dedekind σ-complete partially ordered fields are isomorphic to $\mathbb{R}$.

Non-negatives have infima of non-increasing sequences

Proposition:

$\mathbb{I}_{SA} := \{z \in \mathbb{I} : z = z^{\dagger}\}$ is a partially ordered field with $a \leqslant b \Leftrightarrow b - a = x^{\dagger}x$ for some $x : I \to X$.

PROPOSITION:

$\mathbb{I}_{SA} := \{z \in \mathbb{I} : z = z^\dagger\}$ is a partially ordered field with $a \leqslant b \Leftrightarrow b - a = x^\dagger x$ for some $x : I \to X$.

PROOF:

$a \in \mathbb{I}_{SA}$

$a^2 = a^\dagger a$

$x^\dagger x \cdot y^\dagger y = (x \otimes y)^\dagger (x \otimes y)$

$x : I \to X$

$a = \frac{1}{4}(a+2)^2 - \frac{1}{4}(a^2+4)$

$x^\dagger x + y^\dagger y = \langle x, y\rangle^\dagger \langle x, y\rangle$

$y : I \to Y$

$1 = 1^\dagger 1$

$(x^\dagger x)^{-1} = (x^\dagger x)^{-2} x^\dagger x$

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Lemma: If $a > 0$, then $a = x^\dagger x$ for some isomorphism $x : I \to X$.

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$f : X \to Y$ such that $f^\dagger f + \bar{f}^\dagger \bar{f} = 1$ for some $\bar{f} : X \to \bar{Y}$

Lemma:

If $a > 0$, then $a = x^\dagger x$ for some isomorphism $x : I \to X$.

Proof:

$a = y^\dagger y$ for some $y : I \to Y$

$$\begin{tikzcd} I \arrow[r, "y"] \arrow[dr, dashed, "x"', "\cong"] & Y \arrow[r, shift left, "y(y^\dagger y)^{-1} y^\dagger"] \arrow[r, shift right, "1"'] & Y \\ & X \arrow[u, "e"'] & \end{tikzcd}$$

dagger equaliser

$$\begin{aligned} x^\dagger x &= x^\dagger e^\dagger e x \\ &= y^\dagger y \\ &= a \end{aligned}$$

Proof:

$$x_1^\dagger x_1 \geqslant x_2^\dagger x_2 \geqslant \cdots$$

$$x_j : I \to X_j \text{ iso}$$

$$\begin{tikzcd}
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greatest lower bound

$$\underbrace{x^\dagger x}_{\text{greatest lower bound}} \geqslant x^\dagger f^\dagger f x$$

$$= x_1^\dagger (x x_1^{-1})^\dagger f^\dagger f (x x_1^{-1}) x_1$$

$$= x_1^\dagger (y x_1^{-1})^\dagger (y x_1^{-1}) x_1$$

$$= y^\dagger y$$

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lower bound

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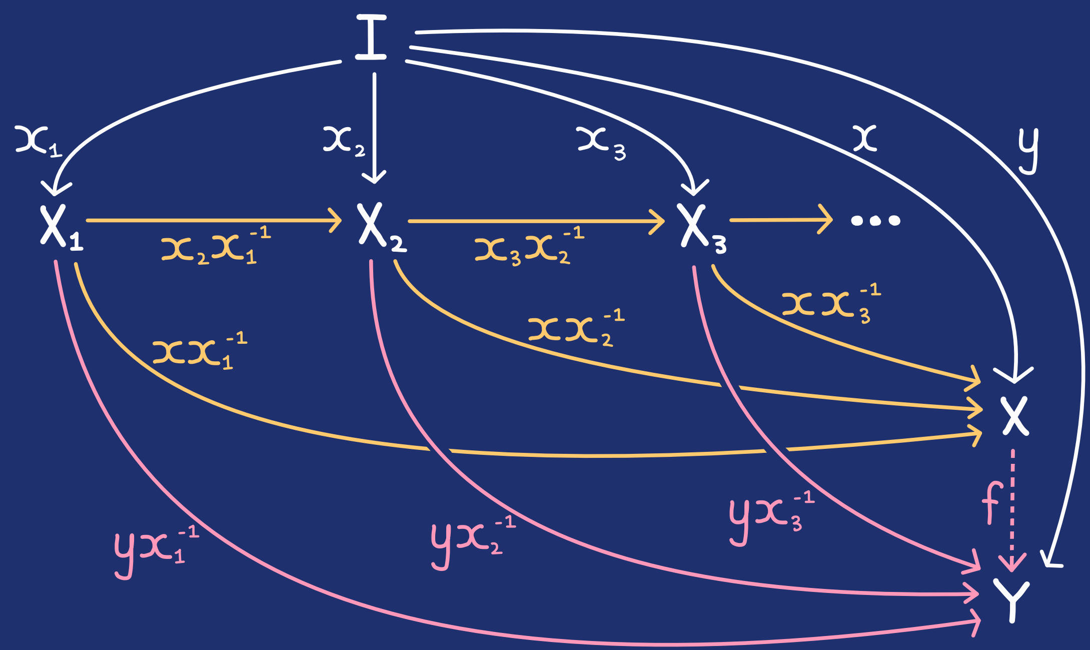

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Proof:

$\mathbb{I}_{SA} \cong \mathbb{R}$ by De Marr's theorem. Suppose that $u \in \mathbb{I} \setminus \mathbb{I}_{SA}$. Let

$$i = \frac{u - u^\dagger}{\sqrt{-(u - u^\dagger)^2}}.$$

Then $\{1, i\}$ is a basis for $\mathbb{I}$ over $\mathbb{I}_{SA}$ and $i^2 = -1$.

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Question:

Can directed colimits of contractions be constructed from those of dagger monos?