ENRICHED BISIMULATIONS

MATTHEW DI MEGLIO
(Joint work with Bryce Clarke)

APPLIED CATEGORY THEORY 2023

Labelled Transition Systems

Used to describe the possible behaviours of discrete processes.

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S

set of states

A

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transition relation for each $\alpha \in A$

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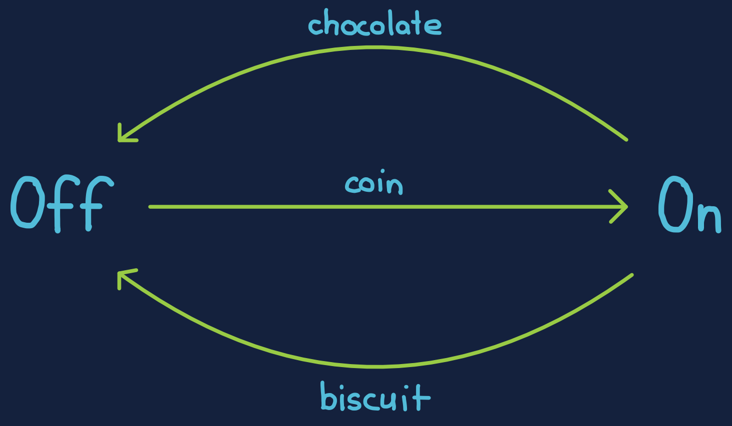
Example: Vending Machine



$S_1 = \{On, Off\}$

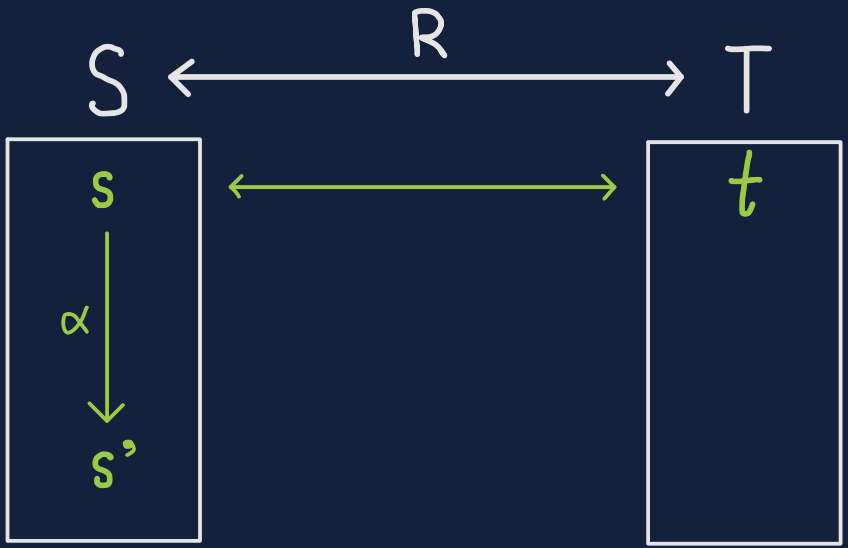
$A = \{coin, biscuit, chocolate\}$

Strong Bisimulation

$$S \xleftrightarrow{\;\;R\;\;} T$$

(S and T are A-labelled transition systems)

Strong Bisimulation



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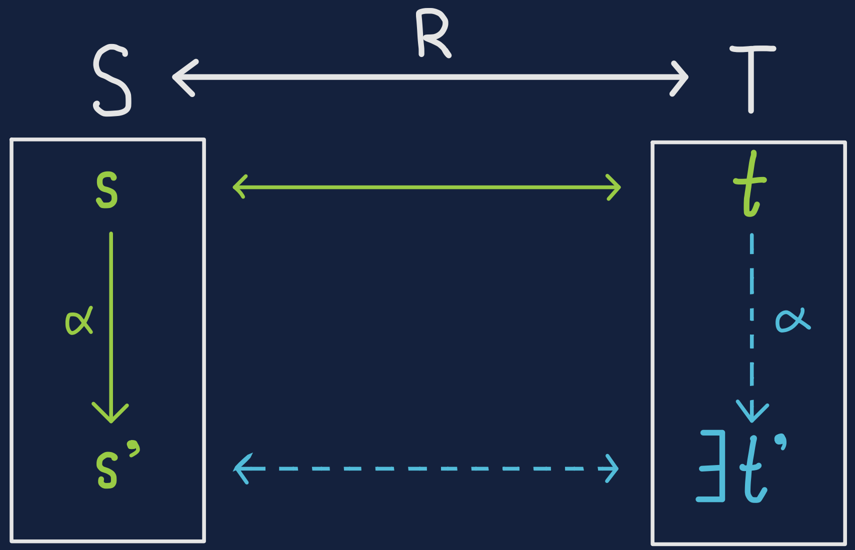

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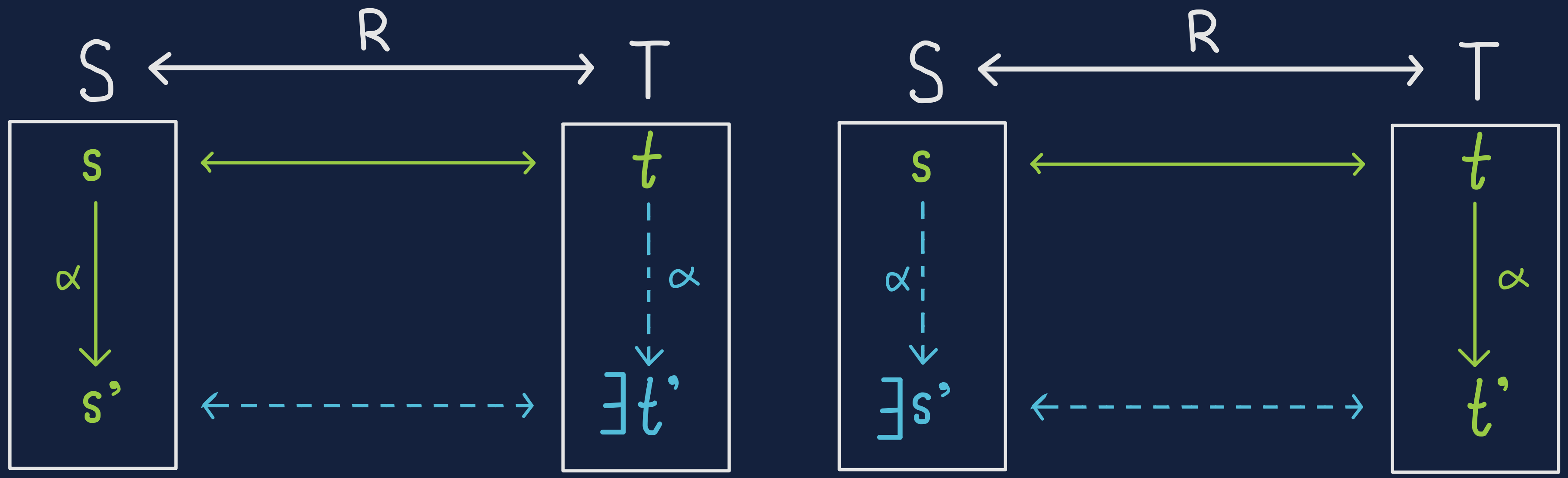
Strong Bisimulation



(S and T are A-labelled transition systems)

Functional Strong Bisimulations*

A function $f : S \to T$ whose graph is a strong bisimulation.

(S and T are A-labelled transition systems)

*Also called abstraction homomorphisms and pure morphisms

Functional Strong Bisimulations*

A function $f: S \to T$ whose graph is a strong bisimulation.

Proposition:

spans of functional strong bisimulations	$\xrightarrow{\text{split surjection}}$	strong bisimulations
$S \xleftarrow{f_1} X \xrightarrow{f_2} T$	$\mapsto$	$\mathrm{Im}\langle f_1, f_2 \rangle \subseteq S \times T$
$S \xleftarrow{\pi_1} R \xrightarrow{\pi_2} T$	$\leftarrow\!\shortmid$	$R \subseteq S \times T$

(S, T and X are A-labelled transition systems)

*Also called abstraction homomorphisms and pure morphisms

$\mathcal{V}$-Enriched Asymmetric Lens

$$F : A \longrightarrow B$$

$$A(a,a') \xrightarrow{F_{a,a'}} B(Fa,Fa') \qquad A_{ob} \xrightarrow{F} B_{ob} \qquad B(Fa,b') \xrightarrow{F_{a,b'}} \sum_{a' \in F^{-1}\{b'\}} A(a,a')$$

hom maps — object map — lifting maps

$\mathcal{V}$	$\mathcal{V}$-ENRICHED CATEGORY	$\mathcal{V}$-ENRICHED ASYMMETRIC LENS
Set	category	delta lens
wSet	weighted category	weighted lens
$([0,\infty],\geqslant)$	metric space	weak submetry

$\mathcal{V}$ is a distributive monoidal category

A-labelled transition systems = $\mathcal{P}(A)$-enriched graphs

S set of states	$S = \mathcal{S}_{ob}$	$\mathcal{S}_{ob}$ set of objects
$\xrightarrow{\alpha} \subseteq S \times S$ transition relation for each $\alpha \in A$	$s \xrightarrow{\alpha} s' \iff \alpha \in \mathcal{S}(s,s')$	$\mathcal{S}(s,s') \in \mathcal{P}(A)$ hom object for all $s, s' \in \mathcal{S}$

A-labelled transition systems = P(A)-enriched graphs

functional strong bisimulations = asymmetric lenses

A function $f : S \to T$ such that

(1) $s \xrightarrow{\alpha} s'$ implies $fs \xrightarrow{\alpha} fs'$,

(2) $fs \xrightarrow{\alpha} t'$ implies exists s' with $s \xrightarrow{\alpha} s'$ and $fs' = t'$.

A function $f : \mathcal{S}_{ob} \to \mathcal{T}_{ob}$ such that

(1) $\mathcal{S}(s, s') \subseteq \mathcal{T}(fs, fs')$,

(2) $\mathcal{T}(fs, t') \subseteq \bigcup_{s' \in f^{-1}\{t'\}} \mathcal{S}(s, s')$.

A-labelled transition systems = P(A)-enriched graphs

functional strong bisimulations = asymmetric lenses

strong bisimulations = ?

$\mathcal{V}$-Enriched Bisimulation

A relation $R \subseteq A_{ob} \times B_{ob}$ together with morphisms

$$A(a,a') \xrightarrow{\vec{R}_{a,a',b}} \sum_{b' : (a',b') \in R} B(b,b') \quad \text{and} \quad \sum_{a' : (a',b') \in R} A(a,a') \xleftarrow{\overleftarrow{R}_{a,b,b'}} B(b,b')$$

that are compatible with identities and composites.

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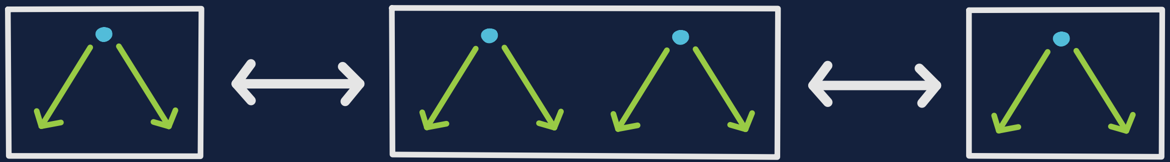
These don't compose!



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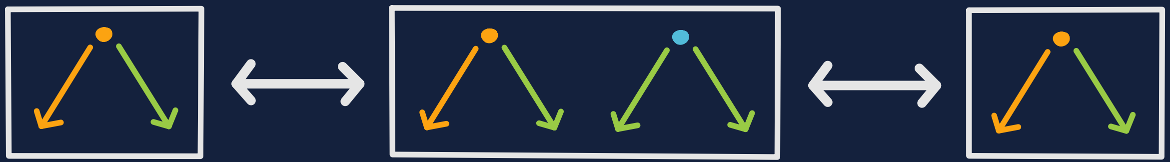

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$\mathcal{V}$-Enriched Bisimulation

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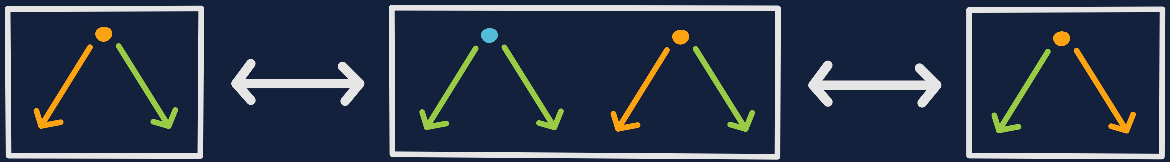

$\mathcal{V}$ is a distributive monoidal category

$\mathcal{V}$-Enriched Bisimulation

A relation $R \subseteq A_{ob} \times B_{ob}$ such that

$$A(a,a') \leqslant \bigvee_{b' : (a',b') \in R} B(b,b') \quad \text{and} \quad \bigvee_{a' : (a',b') \in R} A(a,a') \leqslant B(b,b').$$

These do compose:

$$(a,b) \in R \subseteq A_{ob} \times B_{ob} \qquad (b,c) \in S \subseteq B_{ob} \times C_{ob}$$

$$A(a,a') \leqslant \bigvee_{b' : (a',b') \in R} B(b,b') \leqslant \bigvee_{b' : (a',b') \in R} \; \bigvee_{c' : (b',c') \in S} C(c,c') \leqslant \bigvee_{c' : (a',c') \in R;S} C(c,c')$$

For graphs: $\mathcal{V}$ is a suplattice

For categories: $\mathcal{V}$ is a quantale

Examples

strong bisimulation of A-labelled transition systems	=	bisimulation of $P(A)$-enriched graphs
weak bisimulation of $A \sqcup \{\tau\}$-labelled transition systems	=	bisimulation of $P(A^*)$-enriched categories
bisimulation of Kripke frames	=	bisimulation of 2-enriched graphs

Proposition:

spans of $\mathcal{V}$-enriched asymmetric lenses $\xrightarrow{\text{split surjection}}$ $\mathcal{V}$-enriched bisimulations

$$A_{ob} \xleftarrow{f_1} \mathcal{X}_{ob} \xrightarrow{f_2} B_{ob} \quad \mapsto \quad \mathrm{Im}\langle f_1, f_2 \rangle$$

$$A_{ob} \xleftarrow{\pi_1} R_{ob} \xrightarrow{\pi_2} B_{ob} \quad \leftarrow\!\shortmid \quad R \subseteq A_{ob} \times B_{ob}$$

$$R_{ob} = R \qquad R\left(\binom{a}{b}, \binom{a'}{b'}\right) = A(a, a') \wedge B(b, b')$$

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$$R_{ob} = R \qquad R\left(\binom{a}{b}, \binom{a'}{b'}\right) = A(a,a') \wedge B(b,b')$$

$$(a,b) \in \mathrm{Im}\langle f_1, f_2 \rangle \iff f_1 x = a \text{ and } f_2 x = b \text{ for some } x \in \mathcal{X}_{ob}$$

$$A(a,a') = A(f_1 x, a') \leqslant \bigvee_{x' \in f_1^{-1} a'} \mathcal{X}(x,x') \leqslant \bigvee_{x' \in f_1^{-1} a'} B(f_2 x, f_2 x') = \bigvee_{\substack{b' : (a',b') \\ \in \mathrm{Im}\langle f_1, f_2 \rangle}} B(b,b')$$

Proposition:

spans of $\mathcal{V}$-enriched asymmetric lenses $\xrightarrow{\text{split surjection}}$ $\mathcal{V}$-enriched bisimulations

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$$R(r, r') = A(\pi_1 r, \pi_1 r') \wedge B(\pi_2 r, \pi_2 r') \leqslant A(\pi_1 r, \pi_1 r')$$

$$A(\pi_1 r, a') = A(\pi_1 r, a') \wedge A(\pi_1 r, a') \leqslant A(\pi_1 r, a') \wedge \bigvee_{b' : (a', b') \in R} B(\pi_2 r, b')$$

$$= \bigvee_{b' : (a', b') \in R} A(\pi_1 r, a') \wedge B(\pi_2 r, b') = \bigvee_{r' \in \pi_1^{-1} a'} R(r, r')$$

Proposition:

$$\begin{array}{ccc}
\text{spans of } \mathcal{V}\text{-enriched asymmetric lenses} & \xrightarrow{\text{split surjection}} & \mathcal{V}\text{-enriched bisimulations} \\
A_{ob} \xleftarrow{f_1} \mathcal{X}_{ob} \xrightarrow{f_2} B_{ob} & \mapsto & \mathrm{Im}\langle f_1, f_2 \rangle \\
A_{ob} \xleftarrow{\pi_1} R_{ob} \xrightarrow{\pi_2} B_{ob} & & \\
R_{ob} = R \quad R\left(\binom{a}{b}, \binom{a'}{b'}\right) = A(a,a') \wedge B(b,b') & \mapsfrom & R \subseteq A_{ob} \times B_{ob}
\end{array}$$

local reflection*

$$\begin{tikzcd}
\underbrace{\mathcal{V}\text{-}\underline{\mathrm{LensSpan}}}_{\text{bicategory when } \mathcal{V} \text{ is locally completely distributive}}(A,B) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\underline{\mathrm{Bisim}}(A,B) \arrow[l, hook', shift left]
\end{tikzcd}$$

*See Walker's PhD thesis

Summary

Enriched bisimulations

- generalise several common kinds of bisimulation
- are equivalence classes of spans of enriched asymmetric lenses
- are only defined for thin bases of enrichment

Questions

- What are the bisimulations for other common quantales?
- What other parts of bisimulation theory generalise?

https://mdimeglio.github.io
https://bryceclarke.github.io