

LIMITS OF SEQUENCES VIA COLIMITS OF CONTRACTIONS

MATTHEW DI MEGLIO
(Joint work with Chris Heunen)

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THM (Heunen and Kornell):

$(-)^{\dagger}: \underline{\mathcal{C}}^{\text{op}} \rightarrow \underline{\mathcal{C}}$
[encodes adjoints]

A monoidal dagger category $\underline{\mathcal{C}}$ with

• finite dagger biproducts } [enrichment in commutative monoids]

$$\begin{array}{ccc} & f, g: X \longrightarrow Y & \\ & X \xrightarrow{f+g} Y & \\ \Delta \downarrow & & \uparrow \Delta^{\dagger} \\ X \oplus X & \xrightarrow{f \oplus g} & Y \oplus Y \end{array}$$

• dagger equalisers

• simple monoidal unit

[the semiring $\mathbb{I} := \underline{\mathcal{C}}(I, I)$
of scalars is a field]

• directed colimits in wide subcategory of dagger monos

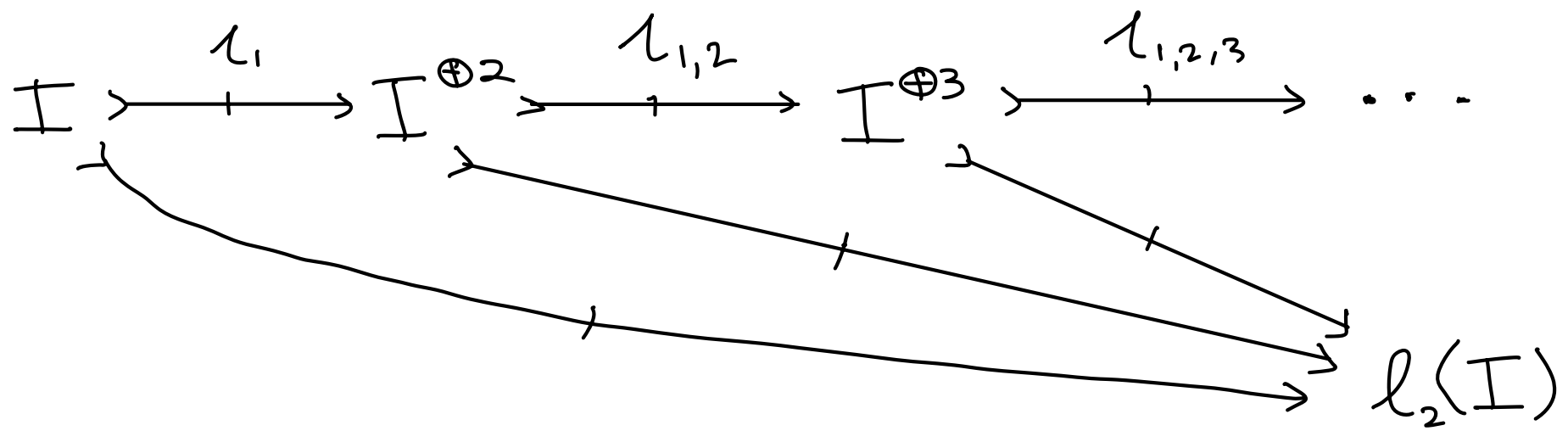
[metric/order completeness of \mathbb{I}]

⋮

is equivalent to Hilb

SOLÉR'S THEOREM: Let X be an orthomodular space over an involutive division ring \mathbb{K} .

If X has an infinite orthonormal subset, then $\mathbb{K} \cong \mathbb{R}, \mathbb{C}$ or \mathbb{H} and X is a Hilbert space



GOAL:

Prove directly that \mathbb{I} is \mathbb{R} or \mathbb{C}

Link directed colimits in category theory
and limits in analysis.

To axiomatise finite-dimensional Hilbert spaces,
can't use Solér's theorem.

- ③
- Every element is a difference of positives
 - Positives contain 1 and closed under $+$, \cdot , $(-)^{-1}$

PROP (De Marr 1967): A partially-ordered field that is Dedekind σ -complete is order isomorphic to \mathbb{R}

Positive decreasing sequences have infima

LEMMA: If $\mathbb{I}_{SA} := \{z \in \mathbb{I} : z = z^+\}$ is \mathbb{R} , then \mathbb{I} is \mathbb{R} or \mathbb{C} .

PROOF: If $u \in \mathbb{I} \setminus \mathbb{I}_{SA}$, let $i = \frac{u - u^+}{\sqrt{-(u - u^+)^2}}$. Then $i^2 + 1 = 0$ and $\{1, i\}$ is basis for \mathbb{I} over \mathbb{I}_{SA} . \square

$$a \leq b \iff b - a = x^+ x \text{ for some } x: I \rightarrow X$$

LEMMA: \mathbb{I}_{SA} is a partially-ordered field.

PROOF:

$$a^2 = a^+ a$$

$$a = \frac{1}{4}(a+2)^2 - \frac{1}{4}(a^2+4)$$

$$1 = 1^+ 1$$

$$x^+ x + y^+ y = \langle x, y \rangle^+ \langle x, y \rangle$$

$$x^+ x \cdot y^+ y = (x \otimes y)^+ (x \otimes y)$$

$$\frac{1}{x^+ x} = \left(\frac{1}{x^+ x} \right)^2 x^+ x$$

$$a \in \mathbb{I}_{SA}$$

$$x: I \rightarrow X$$

$$y: I \rightarrow Y$$

④

□

GOAL:

~~Prove directly that \mathbb{I} is \mathbb{R} or \mathbb{C}~~

Prove that \mathbb{I}_{SA} is Dedekind σ -complete

PROP (De Marr): Every partially-ordered field that is Dedekind σ -complete is order isomorphic to \mathbb{R} .

LEMMA: A dagger field with fixed field \mathbb{R} is \mathbb{R} or \mathbb{C} .

LEMMA: \mathbb{I}_{SA} is a partially-ordered field.

LEMMA: $\mathbb{I}_{\geq 0} = \{y^+y : y: I \rightarrow Y\} = \{x^+x : x: I \rightarrow X \text{ is iso}\}$

PROOF:

$$\begin{array}{ccc}
 I & \xrightarrow{y} & Y \\
 \downarrow x & \nearrow k & \\
 & X &
 \end{array}
 \quad
 \begin{array}{c}
 Y \xrightarrow[y]{y(y^+y)^+y^+} Y \\
 \end{array}
 \quad
 \begin{array}{l}
 x^+x = x^+k^+kx \\
 = y^+y
 \end{array}
 \quad \square$$

dagger equaliser

LEMMA: \mathbb{I}_{SA} is Dedekind σ -complete if $\mathbb{I}_{\geq 0}$ is.

IDEA: Addition preserves infima and $\mathbb{I}_{SA} = \mathbb{I}_{\geq 0} - \mathbb{I}_{\geq 0} \quad \square$

PROP: $\mathbb{I}_{\geq 0}$ is Dedekind σ -complete if the wide subcategory of contractions has directed colimits

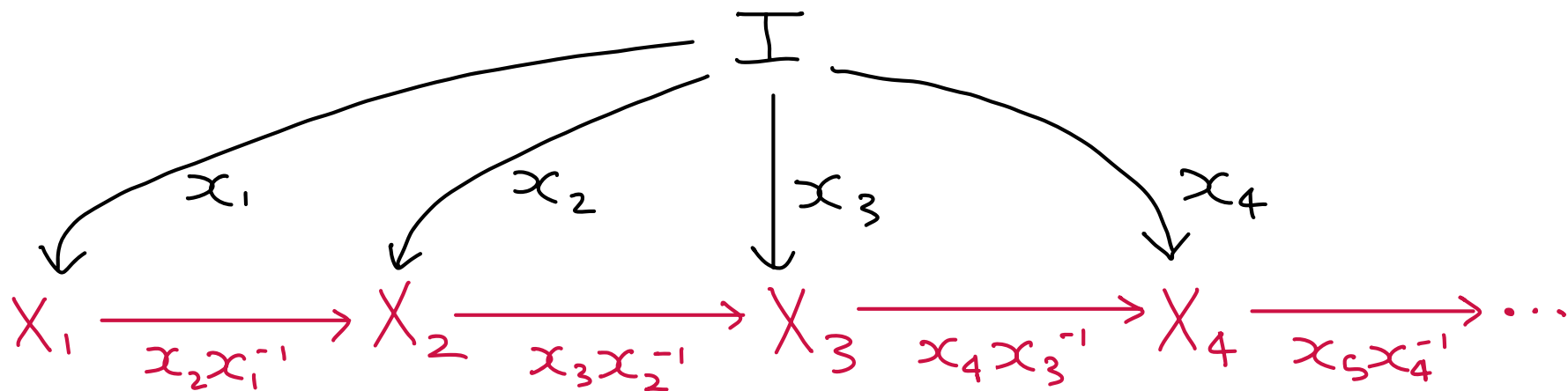
$f: X \rightarrow Y$ such that $f^+f + \bar{f}^+\bar{f} = 1_X$ for some $\bar{f}: X \rightarrow \bar{Y}$

5

PROOF: $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

$x_j: I \rightarrow X_j$
isomorphism

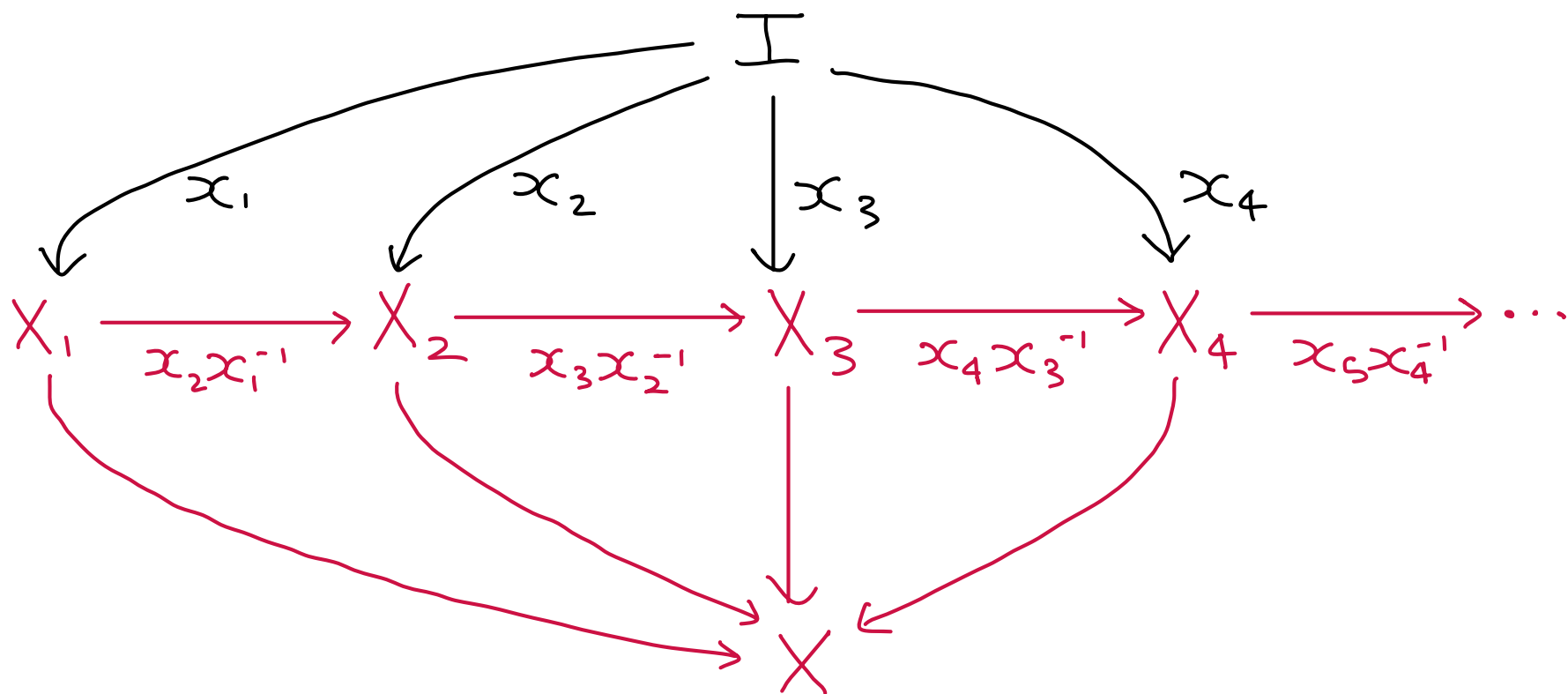
$$1 = x_j^{-+} x_j^+ x_j x_j^{-1} \geq x_j^{-+} x_{j+1}^+ \underbrace{x_{j+1} x_j^{-1}}_{\text{contraction}}$$



PROOF: $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

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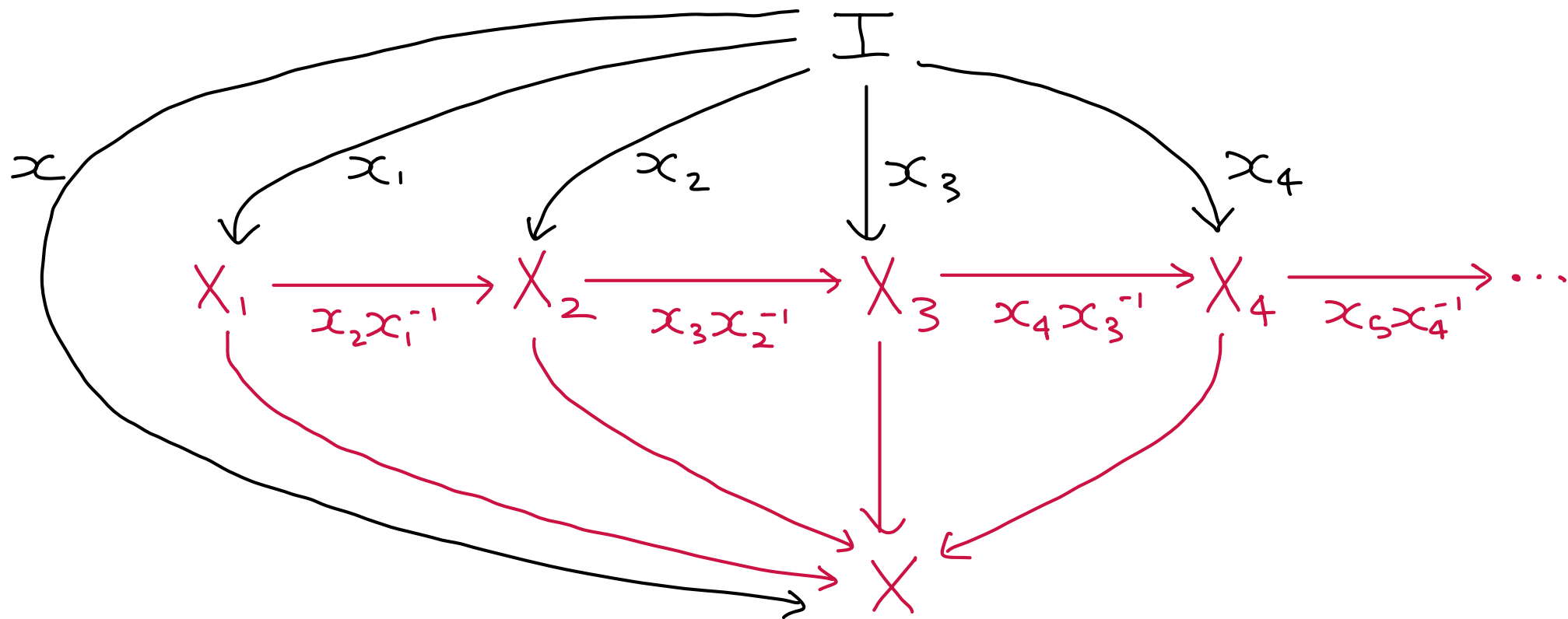
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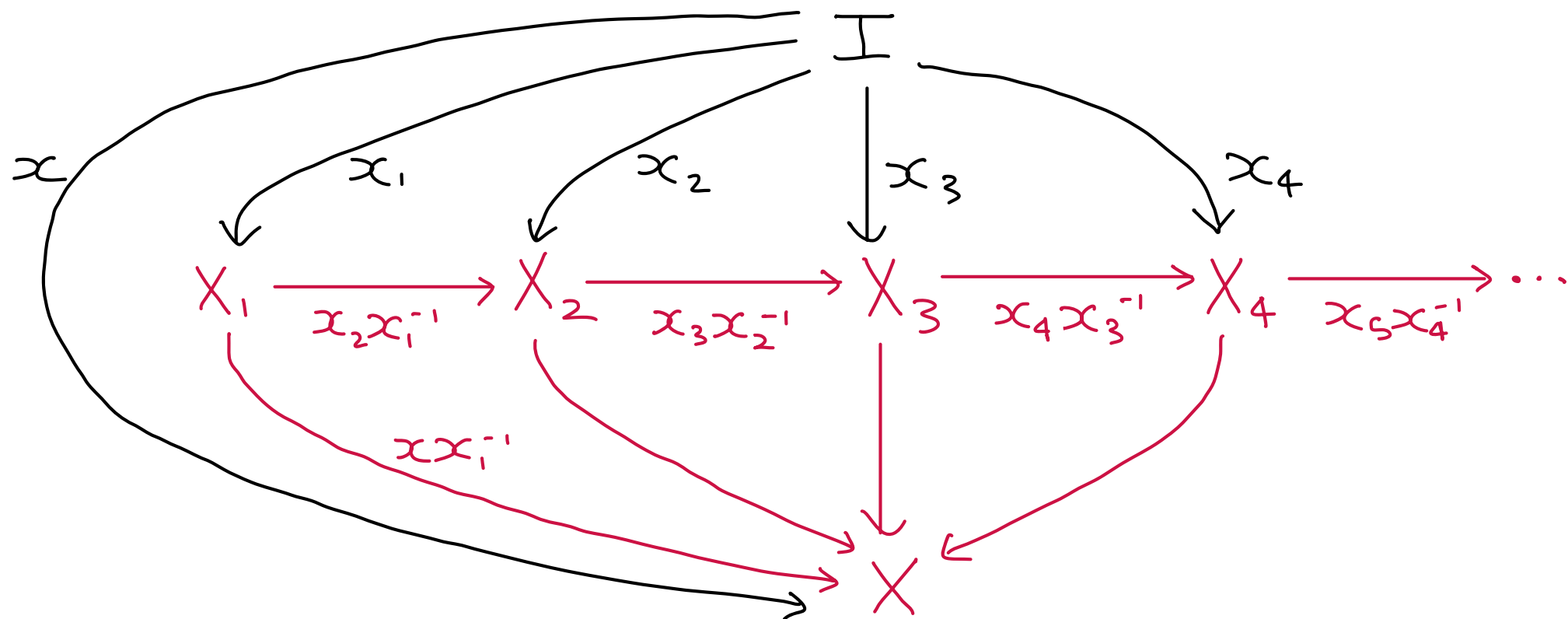
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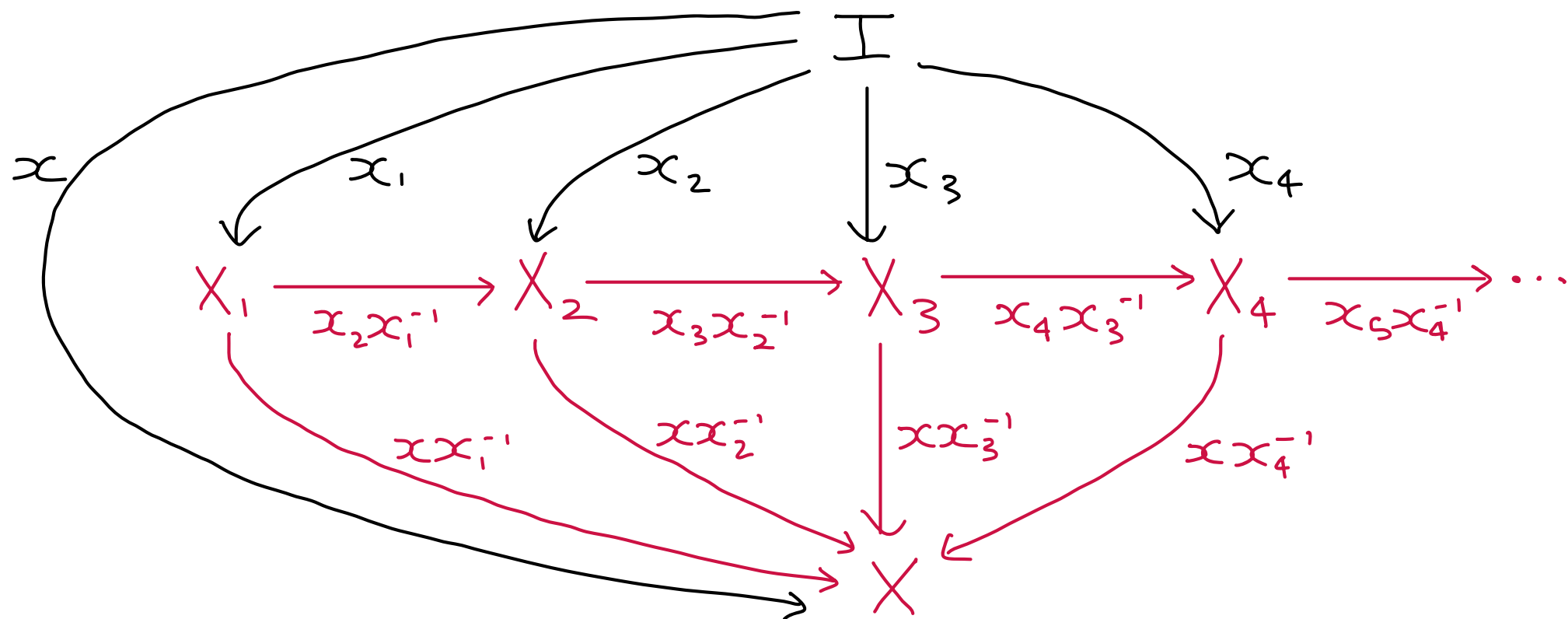
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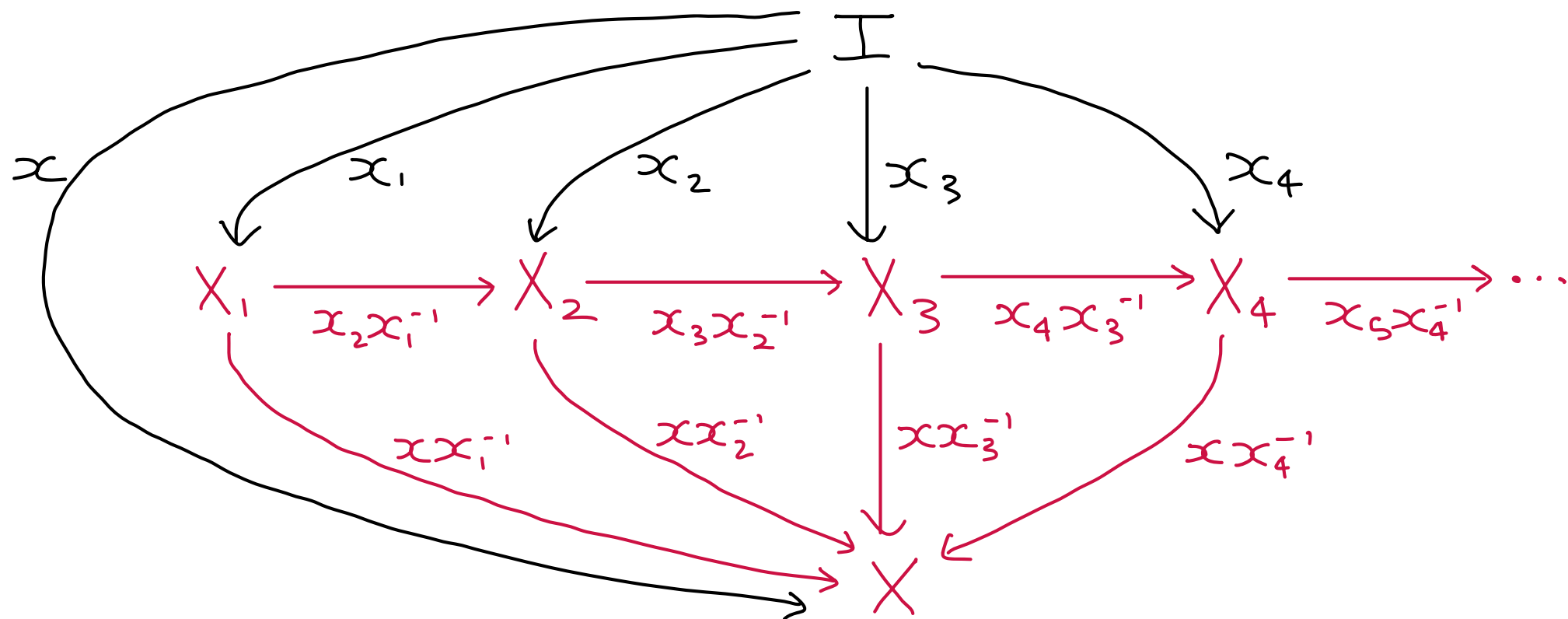
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PROOF: $x_1^+ x_1 \geq x_2^+ x_2 \geq \dots$

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isomorphism

$$1 = x_j^{-+} x_j^+ x_j x_j^{-1} \geq x_j^{-+} x_{j+1}^+ \underbrace{x_{j+1} x_j^{-1}}_{\text{contraction}}$$



$$x_j^+ x_j \geq x_j^+ (x x_j^{-1})^+ (x x_j^{-1}) x_j = \underbrace{x^+ x}_{\text{lower bound}}$$

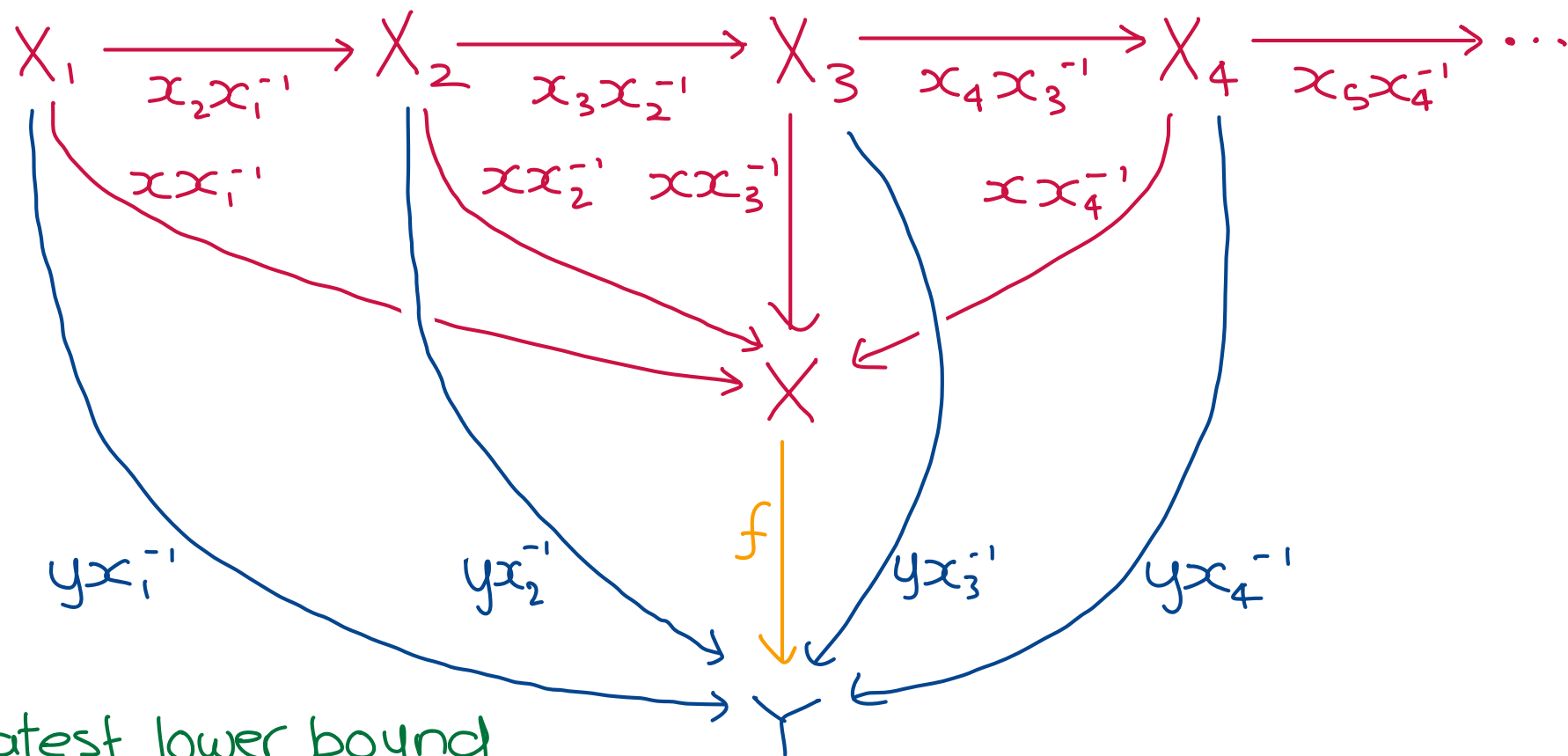
6

$$x_j^\dagger x_j \geq y^\dagger y$$

$y: I \rightarrow Y$ isomorphism

7

$$1 = x_j^{-\dagger} x_j^\dagger x_j x_j^{-1} \geq x_j^{-\dagger} y^\dagger \underbrace{y x_j^{-1}}_{\text{contraction}}$$



$$\underbrace{x^\dagger x}_{\text{greatest lower bound}} \geq x^\dagger f^\dagger f x = x_1^\dagger (x x_1^{-1})^\dagger f^\dagger f (x x_1^{-1}) x_1 = x_1^\dagger (y x_1^{-1})^\dagger (y x_1^{-1}) x_1 = y^\dagger y$$

THM: If wide subcategory of contractions
has directed colimits, then \mathbb{I} is \mathbb{R} or \mathbb{C} .

8

OPEN QUESTIONS:

- ① Can we construct directed colimits of contractions from those of dagger monos?
- ② If we drop symmetric monoidal structure and let \mathbb{I} be a simple projective separator, can we deduce that \mathbb{I} is \mathbb{R} , \mathbb{C} or \mathbb{H} ?

WORK IN PROGRESS:

9

- Dagger-category analogue of abelian categories
(Includes Hilb, Monoidal structure not needed)
- Axioms for FdHilb_{con} (with Chris & André)
- Axioms for FdHilb (with Chris)
- Axioms for Hilb_{isometry} (with Chris, Robert)

(Ultimate goal is quantum-relevant categories like

FdHilb_{unitary}, but this is hard)