Limits of Sequences via Colimits of Contractions

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(Joint work with Chris Heunen)

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THM (Heunen and Kornell):

$(-)^\dagger : \underline{C}^{op} \longrightarrow \underline{C}$
encodes adjoints

A monoidal dagger category $\underline{C}$ with

- finite dagger biproducts — enrichment in commutative monoids

$f, g : X \longrightarrow Y$

$$\begin{tikzcd} X \arrow[r, "f+g"] \arrow[d, "\Delta"'] & Y \\ X \oplus X \arrow[r, "f \oplus g"'] & Y \oplus Y \arrow[u, "\Delta^\dagger"'] \end{tikzcd}$$

- dagger equalisers
- simple monoidal unit

— the semiring $\mathbb{I} := \underline{C}(I, I)$ of scalars is a field

- directed colimits in wide subcategory of dagger monos — metric/order completeness of $\mathbb{I}$

⋮

is equivalent to $\underline{\mathrm{Hilb}}$

SOLÉR'S THEOREM: Let X be an orthomodular space over an involutive division ring $\mathbb{K}$.
If X has an infinite orthonormal subset, then $\mathbb{K} \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and X is a Hilbert space

$$
\begin{tikzcd}
I \arrow[r, tail, "\iota_1"] \arrow[drrr, tail, bend right] & I^{\oplus 2} \arrow[r, tail, "\iota_{1,2}"] \arrow[drr, tail] & I^{\oplus 3} \arrow[r, tail, "\iota_{1,2,3}"] \arrow[dr, tail] & \cdots \\
 & & & \ell_2(I)
\end{tikzcd}
$$

GOAL:

Prove directly that $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{C}$

Link directed colimits in category theory and limits in analysis.

To axiomatise finite-dimensional Hilbert spaces, can't use Solér's theorem.

3

- Every element is a difference of positives
- Positives contain 1 and closed under $+, \cdot, (-)^{-1}$

PROP (De Marr 1967): A partially-ordered field that is Dedekind σ-complete is order isomorphic to $\mathbb{R}$

Positive decreasing sequences have infima

LEMMA: If $\mathbb{I}_{SA} := \{z \in \mathbb{I} : z = z^\dagger\}$ is $\mathbb{R}$, then $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{C}$.

PROOF: If $u \in \mathbb{I} \setminus \mathbb{I}_{SA}$, let $i = \frac{u - u^\dagger}{\sqrt{-(u-u^\dagger)^2}}$. Then $i^2 + 1 = 0$ and $\{1, i\}$ is basis for $\mathbb{I}$ over $\mathbb{I}_{SA}$. $\square$

$$a \leqslant b \iff b - a = x^\dagger x \text{ for some } x : I \to X$$

LEMMA: $\mathbb{I}_{SA}$ is a partially-ordered field.

PROOF:

$$a^2 = a^\dagger a$$

$$a = \tfrac{1}{4}(a+2)^2 - \tfrac{1}{4}(a^2+4)$$

$a \in \mathbb{I}_{SA}$

$$1 = 1^\dagger 1$$

$$x^\dagger x + y^\dagger y = \langle x, y\rangle^\dagger \langle x, y\rangle$$

$$x^\dagger x \cdot y^\dagger y = (x \otimes y)^\dagger (x \otimes y)$$

$$\frac{1}{x^\dagger x} = \left(\frac{1}{x^\dagger x}\right)^2 x^\dagger x$$

$x : I \to X$

$y : I \to Y$

$\square$

GOAL:

~~Prove directly that $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{C}$~~

Prove that $\mathbb{I}_{SA}$ is Dedekind σ-complete

PROP (De Marr): Every partially-ordered field that is Dedekind σ-complete is order isomorphic to $\mathbb{R}$.

LEMMA: A dagger field with fixed field $\mathbb{R}$ is $\mathbb{R}$ or $\mathbb{C}$.

LEMMA: $\mathbb{I}_{SA}$ is a partially-ordered field.

LEMMA: $I_{\geq 0} = \{ y^\dagger y : y : I \to Y \} = \{ x^\dagger x : x : I \to X \text{ is iso} \}$

PROOF:

$$\begin{tikzcd} I \arrow[r, "y"] \arrow[dr, dashed, "x"', "\cong"] & Y \arrow[r, shift left, "y(y^\dagger y)^{-1} y^\dagger"] \arrow[r, shift right, "1"'] & Y \\ & X \arrow[u, "k"'] & \end{tikzcd}$$

dagger equaliser

$$x^\dagger x = x^\dagger k^\dagger k x = y^\dagger y \quad \square$$

LEMMA: $\mathbb{I}_{SA}$ is Dedekind σ-complete if $\mathbb{I}_{\geq 0}$ is.

IDEA: Addition preserves infima and $\mathbb{I}_{SA} = \mathbb{I}_{\geq 0} - \mathbb{I}_{\geq 0}$ $\square$

PROP: $\mathbb{I}_{\geq 0}$ is Dedekind σ-complete if the wide subcategory of $\underbrace{\text{contractions}}$ has directed colimits

$f : X \to Y$ such that $f^\dagger f + \bar{f}^\dagger \bar{f} = 1_X$ for some $\bar{f} : X \to \bar{Y}$

PROOF: $x_1^\dagger x_1 \geqslant x_2^\dagger x_2 \geqslant \cdots$

$x_j : I \longrightarrow X_j$ isomorphism

$$1 = x_j^{-\dagger} x_j^\dagger x_j x_j^{-1} \geqslant x_j^{-\dagger} x_{j+1}^\dagger \underbrace{x_{j+1} x_j^{-1}}_{\text{contraction}}$$

$$\begin{tikzcd}
 & & I \arrow[dll, bend right, "x_1"'] \arrow[dl, bend right, "x_2"'] \arrow[d, "x_3"] \arrow[dr, bend left, "x_4"] & & \\
X_1 \arrow[r, "x_2 x_1^{-1}"'] & X_2 \arrow[r, "x_3 x_2^{-1}"'] & X_3 \arrow[r, "x_4 x_3^{-1}"'] & X_4 \arrow[r, "x_5 x_4^{-1}"'] & \cdots
\end{tikzcd}$$

PROOF: $x_1^{\dagger} x_1 \geqslant x_2^{\dagger} x_2 \geqslant \cdots$

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X_1 \arrow[r, "x_2 x_1^{-1}"'] \arrow[drr, bend right] & X_2 \arrow[r, "x_3 x_2^{-1}"'] \arrow[dr, bend right] & X_3 \arrow[r, "x_4 x_3^{-1}"'] \arrow[d] & X_4 \arrow[r, "x_5 x_4^{-1}"'] \arrow[dl, bend left] & \cdots \\
 & & X & &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & & I \arrow[dll, bend right, "x_1"'] \arrow[dl, bend right, "x_2"'] \arrow[d, "x_3"] \arrow[dr, bend left, "x_4"] \arrow[dd, bend right=80, "x"'] & & \\
X_1 \arrow[r, "x_2x_1^{-1}"'] \arrow[drr, bend right] & X_2 \arrow[r, "x_3x_2^{-1}"'] \arrow[dr, bend right] & X_3 \arrow[r, "x_4x_3^{-1}"'] \arrow[d] & X_4 \arrow[r, "x_5x_4^{-1}"'] \arrow[dl, bend left] & \cdots \\
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 & & I \arrow[dll, "x_1"'] \arrow[dl, "x_2"'] \arrow[d, "x_3"] \arrow[dr, "x_4"] \arrow[dd, bend right=80, "x"'] & & \\
X_1 \arrow[r, "x_2 x_1^{-1}"'] \arrow[drr, bend right, "x x_1^{-1}"'] & X_2 \arrow[r, "x_3 x_2^{-1}"'] \arrow[dr, bend right] & X_3 \arrow[r, "x_4 x_3^{-1}"'] \arrow[d] & X_4 \arrow[r, "x_5 x_4^{-1}"'] \arrow[dl, bend left] & \cdots \\
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$$\begin{tikzcd}
& & I \arrow[dll, bend right, "x_1"'] \arrow[dl, bend right, "x_2"'] \arrow[d, "x_3"] \arrow[dr, bend left, "x_4"] \arrow[dd, bend right=120, "x"'] & & \\
X_1 \arrow[r, "x_2 x_1^{-1}"'] \arrow[drr, bend right, "x x_1^{-1}"'] & X_2 \arrow[r, "x_3 x_2^{-1}"'] \arrow[dr, bend right, "x x_2^{-1}"'] & X_3 \arrow[r, "x_4 x_3^{-1}"'] \arrow[d, "x x_3^{-1}"] & X_4 \arrow[r, "x_5 x_4^{-1}"'] \arrow[dl, bend left, "x x_4^{-1}"] & \cdots \\
& & X & &
\end{tikzcd}$$

PROOF: $x_1^\dagger x_1 \geq x_2^\dagger x_2 \geq \cdots$

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$$\begin{tikzcd}
 & & I \arrow[dll, "x_1"'] \arrow[dl, "x_2"'] \arrow[d, "x_3"] \arrow[dr, "x_4"] \arrow[dd, bend right=80, "x"'] & & \\
X_1 \arrow[r, "x_2 x_1^{-1}"'] \arrow[drr, "x x_1^{-1}"'] & X_2 \arrow[r, "x_3 x_2^{-1}"'] \arrow[dr, "x x_2^{-1}"] & X_3 \arrow[r, "x_4 x_3^{-1}"'] \arrow[d, "x x_3^{-1}"] & X_4 \arrow[r, "x_5 x_4^{-1}"'] \arrow[dl, "x x_4^{-1}"] & \cdots \\
 & & X & &
\end{tikzcd}$$

$$x_j^\dagger x_j \geq x_j^\dagger (x x_j^{-1})^\dagger (x x_j^{-1}) x_j = \overbrace{x^\dagger x}^{\text{lower bound}}$$

$x_j^\dagger x_j \geq y^\dagger y$ $\qquad$ $y : I \longrightarrow Y$ isomorphism

$$1 = x_j^{-\dagger} x_j^\dagger x_j x_j^{-1} \geq x_j^{-\dagger} y^\dagger \overbrace{y x_j^{-1}}^{\text{contraction}}$$

$$\begin{tikzcd}
X_1 \arrow[r, "x_2 x_1^{-1}"'] \arrow[drr, bend right, "x x_1^{-1}"] \arrow[ddrr, bend right, "y x_1^{-1}"'] & X_2 \arrow[r, "x_3 x_2^{-1}"'] \arrow[dr, bend right, "x x_2^{-1}"] \arrow[ddr, bend right, "y x_2^{-1}"'] & X_3 \arrow[r, "x_4 x_3^{-1}"'] \arrow[d, "x x_3^{-1}"'] \arrow[dd, bend left, "y x_3^{-1}"] & X_4 \arrow[r, "x_5 x_4^{-1}"'] \arrow[dl, bend left, "x x_4^{-1}"'] \arrow[ddl, bend left, "y x_4^{-1}"] & \cdots \\
 & & X \arrow[d, "f"'] & & \\
 & & Y & &
\end{tikzcd}$$

$$\overbrace{x^\dagger x}^{\text{greatest lower bound}} \geq x^\dagger f^\dagger f x = x_1^\dagger (x x_1^{-1})^\dagger f^\dagger f (x x_1^{-1}) x_1 = x_1^\dagger (y x_1^{-1})^\dagger (y x_1^{-1}) x_1 = y^\dagger y$$

THM: If wide subcategory of contractions has directed colimits, then $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{C}$.

OPEN QUESTIONS:

1. Can we construct directed colimits of contractions from those of dagger monos?
2. If we drop symmetric monoidal structure and let I be a simple projective separator, can we deduce that $\mathbb{I}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$?

WORK IN PROGRESS:

- Dagger-category analogue of abelian categories (Includes <u>Hilb</u>, Monoidal structure not needed)
- Axioms for $\underline{\mathrm{FdHilb}}_{\mathrm{con}}$ (with Chris & André)
- Axioms for <u>FdHilb</u> (with Chris)
- Axioms for $\underline{\mathrm{Hilb}}_{\mathrm{isometry}}$ (with Chris, Robert)

(Ultimate goal is quantum-relevant categories like $\underline{\mathrm{FdHilb}}_{\mathrm{unitary}}$, but this is hard)