Gamma Domain Conversion and Smith Chart Equations

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1 Introduction

Wanted to write down all the equations that convert Impedance $\leftrightarrow$ Gamma, and Admittance $\leftrightarrow$ Gamma. The equations for constant R and X in the Impedance smith chart, and constant G and B circles in the Admittance smith chart are also shown.

2 Impedance Smith Chart

$$Z = R + jX \qquad \text{Normalized } \overline{z} = \frac{Z}{Z_o} = r + jx$$
$$\Gamma = P + jQ$$

2.1 Impedance to Gamma

$$\begin{aligned}
\Gamma &= P + jQ \\
&= \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{\overline{z_L} - 1}{\overline{z_L} + 1} = \frac{r + jx - 1}{r + jx + 1} \\
&= \frac{\left(r^2 - 1\right) - (r-1)\,jx + (r+1)\,jx + x^2}{(r+1)^2 + x^2} \\
\implies P &= \frac{\left(r^2 - 1\right) + x^2}{(r+1)^2 + x^2} \qquad Q = \frac{2x}{(r+1)^2 + x^2}
\end{aligned} \tag{1}$$

2.2 Gamma to Impedance

$$\begin{aligned}
\overline{z_L} &= r + jx \\
&= \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + P + jQ}{1 - P - jQ} \\
&= \frac{\left(1 - P^2\right) + (1+P)\,jQ + (1-P)\,jQ - Q^2}{(1-P)^2 + Q^2} \\
\implies r &= \frac{1 - P^2 - Q^2}{(1-P)^2 + Q^2} \qquad x = \frac{2Q}{(1-P)^2 + Q^2}
\end{aligned} \tag{2}$$

2.3 Impedance Circles

Manipulate r and x equations from above by moving over the denominators, and completing the square.

Constant R circle

$$r = \frac{1 - P^2 - Q^2}{(1-P)^2 + Q^2}$$

$$r - 2rP + rP^2 + rQ^2 = 1 - P^2 - Q^2$$

$$P^2(1+r) + Q^2(1+r) - 2rP = 1 - r$$

$$P^2 + Q^2 - 2P\frac{r}{1+r} = \frac{1-r}{1+r}$$

Complete the square

$$P^2 + Q^2 - 2P\frac{r}{1+r} + \frac{r^2}{(1+r)^2} = \frac{1-r}{1+r} + \frac{r^2}{(1+r)^2}$$

$$\left(P - \frac{r}{1+r}\right)^2 + Q^2 = \left(\frac{1}{1+r}\right)^2 \tag{3}$$

Constant X Circle

$$x = \frac{2Q}{(1-P)^2 + Q^2}$$

$$x - 2xP + xP^2 + xQ^2 = 2Q$$

$$xP^2 - 2xP + xQ^2 - 2Q = -x$$

$$P^2 - 2P + Q^2 - \frac{2Q}{x} = -1$$

Complete the square

$$P^2 - 2P + 1 + Q^2 - \frac{2Q}{x} + \left(\frac{1}{x}\right)^2 = -1 + 1 + \left(\frac{1}{x}\right)^2$$

$$(P-1)^2 + \left(Q - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{4}$$

Circle Type	Center Point	Radius
R	$\left(\frac{r}{r+1},\ 0\right)$	$\frac{1}{r+1}$
X	$\left(1,\ \frac{1}{x}\right)$	$\left\lvert\frac{1}{x}\right\rvert$

Note that the Constant X circles can have a positive or negative x as the center point, while the constant r circle is only positive.

3 Admittance Smith Chart

3.1 Admittance to Gamma

$$\begin{aligned}
\Gamma &= P + jQ \\
&= \frac{1 - \overline{y_L}}{1 + \overline{y_L}} = \frac{(1-g) - jb}{(1+g) + jb} \\
&= \frac{\left(1 - g^2\right) + (1-g)\,jb - (1+g)\,jb - b^2}{(1-g^2) + b^2} \\
\implies P &= \frac{1 - g^2 - b^2}{(1-g^2) + b^2} \qquad Q = \frac{-2b}{(1-g^2) + b^2}
\end{aligned} \tag{5}$$

3.2 Gamma to Admittance

$$\begin{aligned}
\overline{y_L} &= g + jb \\
&= \frac{1 - \Gamma}{1 + \Gamma} = \frac{1 - P - jQ}{1 + P + jQ} \\
&= \frac{\left(1 - P^2\right) - (1-P)\,jQ - (1+P)\,jQ - Q^2}{(1+P)^2 + Q^2} \\
\implies g &= \frac{1 - P^2 - Q^2}{(1+P)^2 + Q^2} \qquad b = \frac{-2Q}{(1+P)^2 + Q^2}
\end{aligned} \tag{6}$$

3.3 Admittance Circle Equations

Constant G circle

$$
\begin{aligned}
g &= \frac{1-P^2-Q^2}{(1+P)^2+Q^2} \\
g+2gP+gP^2+gQ^2 &= 1-P^2-Q^2 \\
P^2(1+g)+Q^2(1+g)+2gP &= 1-g \\
P^2+Q^2+2P\frac{g}{1+g} &= \frac{1-g}{1+g}
\end{aligned}
$$

Complete the square

$$
\begin{aligned}
P^2+2P\frac{g}{1+g}+\frac{g^2}{(1+g)^2}+Q^2 &= \frac{1-g}{1+g}+\frac{g^2}{(1+g)^2} \\
\left(P+\frac{g}{1+g}\right)^2+Q^2 &= \left(\frac{1}{1+g}\right)^2 \qquad (7)
\end{aligned}
$$

Constant B Circle

$$
\begin{aligned}
b &= \frac{-2Q}{(1+P)^2+Q^2} \\
b+2bP+bP^2+bQ^2 &= 2Q \\
bP^2+2bP+bQ^2-2Q &= -b \\
P^2+2P+Q^2+\frac{2Q}{b} &= -1
\end{aligned}
$$

Complete the square

$$
\begin{aligned}
P^2+2P+1+Q^2+\frac{2Q}{b}+\left(\frac{1}{b}\right)^2 &= -1+1+\left(\frac{1}{b}\right)^2 \\
(P+1)^2+\left(Q+\frac{1}{b}\right)^2 &= \left(\frac{1}{b}\right)^2 \qquad (8)
\end{aligned}
$$

Circle Type	Center Point	Radius
G	$\left(\frac{-g}{g+1},\ 0\right)$	$\frac{1}{g+1}$
B	$\left(-1,\ -\frac{1}{b}\right)$	$\left\lvert\frac{1}{g}\right\rvert$

Note the polarity difference of the center point of the B circles. This means that, for positive $Q = Im\{\Gamma\}$, the suspectance is negative.