Lecture 6: Sorting Lower Bound and “Linear-Time” Sorting

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601.433/633 Introduction to Algorithms

Introduction

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No: every algorithm in the comparison model must have worst-case running time $\Omega(n \log n)$.

Yes: If we assume extra structure for the elements, can do sorting in $O(n)$ time*

Sorting Lower Bound

Statement

Theorem

Any sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Lower bound on the number of comparisons – running time could be even worse!

Allows algorithm to reorder elements, copy them, move them, etc. for free.

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Why is this hard?

- Lower bound needs to hold for *all* algorithms
- How can we simultaneously reason about algorithms as different as mergesort, quicksort, heapsort, ...?

Sorting as Permutations

Think of an array A as a *permutation*: $A[i]$ is the $\pi(i)$'th smallest element

$$A = [23, 14, 2, 5, 76]$$

Corresponds to $\pi = (3, 2, 0, 1, 4)$:

$$\pi(0) = 3 \qquad \pi(1) = 2 \qquad \pi(2) = 0 \qquad \pi(3) = 1 \qquad \pi(4) = 4$$

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Lemma

Given A with $|A| = n$, if can sort in $T(n)$ comparisons then can find π in $T(n)$ comparisons

Sorting As Permutations (cont'd)

Lemma

Given A *with* $|A| = n$*, if can sort in* $T(n)$ *comparisons then can find* π *in* $T(n)$ *comparisons*

Proof Sketch.

- "Tag" each element of A with index:
  $[23, 14, 2, 5, 76] \rightarrow [(23, 0), (14, 1), (2, 2), (5, 3), (76, 4)]$
- Sort tagged A into tagged B with $T(n)$ comparisons:
  $[(2, 2), (5, 3), (14, 1), (23, 0), (76, 4)]$
- Iterate through to get π: $\pi(2) = 0, \pi(3) = 1, \pi(1) = 2, \pi(0) = 3, \pi(4) = 4$ $\square$

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Corollary

If need at least $T(n)$ comparisons to find π, need at least $T(n)$ comparisons to sort!

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Want to show that it takes $\Omega(n \log n)$ comparisons to find π in comparison model.

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- Starts with some comparison (e.g., compares $A[0]$ to $A[1]$)
- Rules out some possible permutations!
  - If $A[0] < A[1]$ then $\pi(0) < \pi(1)$
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Remind you of anything?

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Model any algorithm as a *binary decision tree*

- Internal nodes: comparisons
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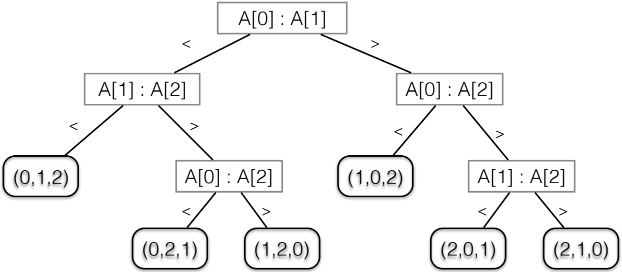
Example: $n = 3$. Six possible permutations.



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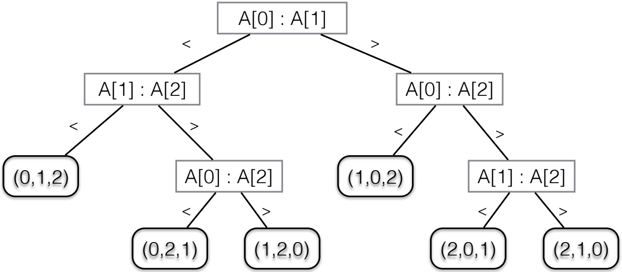

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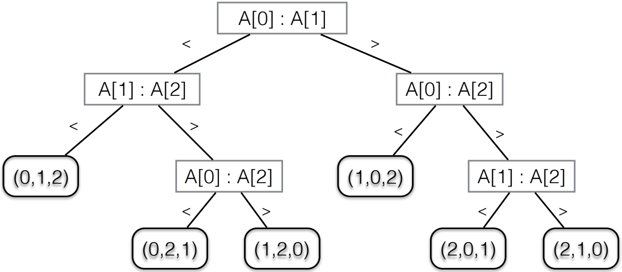
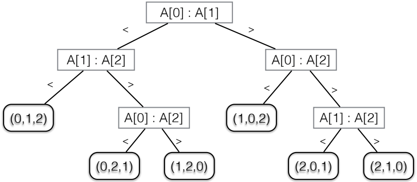


Max # comparisons: **3**

Finishing Up



Scale to general n. Consider arbitrary decision tree.

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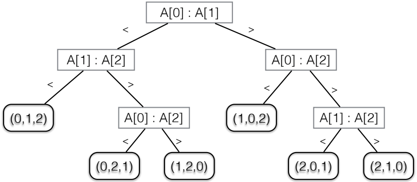

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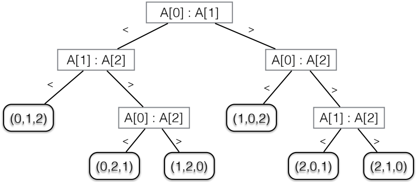

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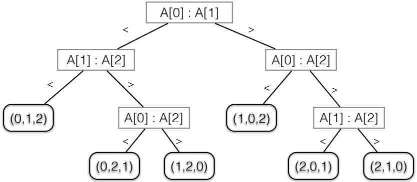

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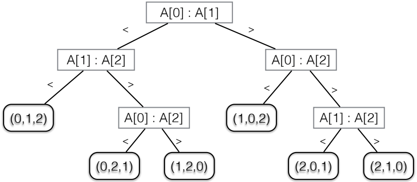

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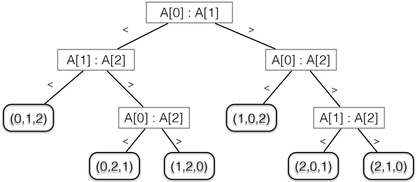

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$$\begin{aligned} \text{Max } \# \text{ comparisons} &= \text{depth of tree} \\ &\geq \log_2(\# \text{ leaves}) \\ &= \log_2(n!) \\ &= \Theta(n \log n) \end{aligned}$$

Sorting Lower Bound Summary

Theorem

Every sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Proof Sketch.

1. Lower bound on finding permutation $\pi \implies$ lower bound on sorting
2. Any algorithm for finding π is a binary decision tree with $n!$ leaves.
3. Any binary decision tree with $n!$ leaves has depth $\geq \log(n!) = \Theta(n \log n)$

$\implies$ Every algorithm has worst case number of comparisons at least $\Theta(n \log n)$. $\square$

“Linear-Time” Sorting

Bypassing the Lower Bound

What if we're *not* in the comparison model?

- Can do more than just compare elements.

Main example: *integers*.

- What is the $\mathbf{3}$rd bit of $\boldsymbol{A}[\mathbf{0}]$?
- Is $\boldsymbol{A}[\mathbf{0}] \ll \boldsymbol{k}$ larger than $\boldsymbol{A}[\mathbf{1}] \gg \boldsymbol{c}$?
- Is $\boldsymbol{A}[\mathbf{0}]$ even?

Same ideas apply to letters, strings, etc.

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Stable: if two objects have same key, order between them after sorting is same as before.

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If you were sorting cards, with a number on each card, what might you do?

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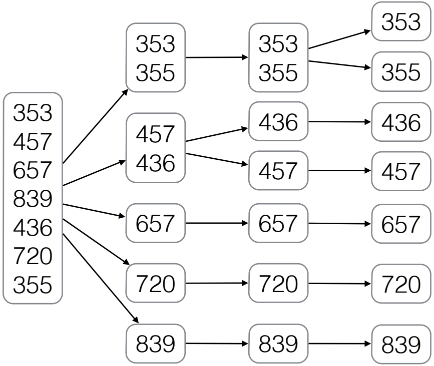

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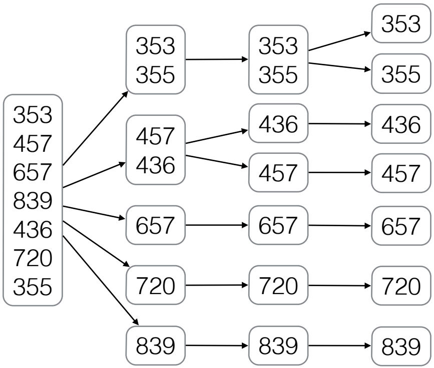

Works, but clunky

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For iteration $\boldsymbol{i}$, use bucket sort where key is $\boldsymbol{i}$'th digit and object is number.

Theorem

Radix sort from least significant to most significant is correct if the sort used on each digit is stable.

Least-Significant Radix Sort: Correctness

Proof.

Claim: After $\boldsymbol{i}$'th iteration, correctly sorted by last $\boldsymbol{i}$ digits (interpreted as # in $[\mathbf{0}, \mathbf{10}^{\boldsymbol{i}} - \mathbf{1}]$).

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- Suppose correct for i
- After $i + 1$ sort:
  - If two numbers have different $i + 1$ digits, now correct.
  - If two number have same $i + 1$ digit, were correct and still correct by stability.

□

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Improve to $O(n)$?

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Example: sorting integers between 0 and n^{10}. Then d should be about $\log_{10} n^{10} = 10 \log_{10} n$, as required.