

Lecture 20: Max-Flow Min-Cut

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601.433/633 Introduction to Algorithms

Introduction

Flow Network:

- ▶ Directed graph $G = (V, E)$
- ▶ Capacities $c : E \rightarrow \mathbb{R}_{\geq 0}$ (simplify notation: $c(x, y) = 0$ if $(x, y) \notin E$)
- ▶ Source $s \in V$, sink $t \in V$

Today: flows and cuts

- ▶ Flow: “sending stuff” from s to t
- ▶ Cut: separating t from s

Turn out to be very related!

Today: some algorithms but not efficient. Mostly structure. Better algorithms Tuesday.

Flows

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$$\sum_{u:(u,v) \in E} f(u,v) = \sum_{u:(v,u) \in E} f(v,u)$$

for all $v \in V \setminus \{s, t\}$. This constraint is known as *flow conservation*.

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Value of flow $|f|$: “total amount of stuff sent from s to t ”

$$|f| = \sum_{u:(s,u) \in E} f(s, u) - \sum_{u:(u,s) \in E} f(u, s) = \sum_{u:(u,t) \in E} f(u, t) - \sum_{u:(t,u) \in E} f(t, u)$$

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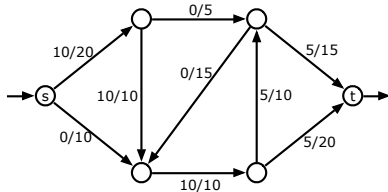
- ▶ An (s, t) -flow satisfying capacity constraints is a *feasible* flow.
- ▶ If $\mathbf{f}(e) = \mathbf{c}(e)$ then \mathbf{f} *saturates* e .
- ▶ If $\mathbf{f}(e) = \mathbf{0}$ then \mathbf{f} *avoids* e .

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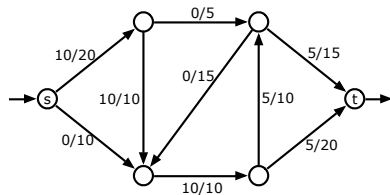
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Problem we'll talk about: find feasible flow of maximum value (max flow)

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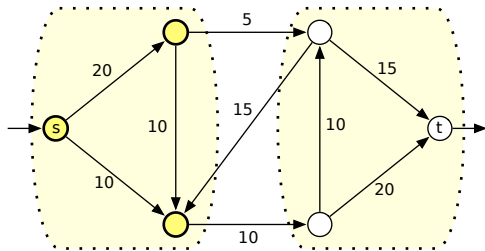
$$\text{cap}(S, \bar{S}) = \sum_{(u,v) \in E: u \in S, v \in \bar{S}} c(u, v) = \sum_{u \in S} \sum_{v \in \bar{S}} c(u, v)$$

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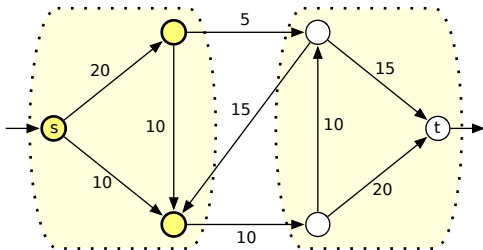


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Problem we'll talk about: find (s, t) -cut of minimum capacity (min cut)

Warmup Theorem

Theorem

Let \mathbf{f} be a feasible (s, t) -flow, and let (S, \bar{S}) be an (s, t) -cut. Then $|\mathbf{f}| \leq \mathbf{cap}(S, \bar{S})$.

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$$\leq \sum_{u \in S} \sum_{v \in \bar{S}} c(u, v) = \mathbf{cap}(S, \bar{S}) \quad (\text{flow is feasible})$$

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Corollary

If \mathbf{f} avoids every $\bar{\mathbf{S}} \rightarrow \mathbf{S}$ edge and saturates every $\mathbf{S} \rightarrow \bar{\mathbf{S}}$ edge, then \mathbf{f} is a maximum flow and $(\mathbf{S}, \bar{\mathbf{S}})$ is a minimum cut.

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Theorem (Max-Flow Min-Cut Theorem)

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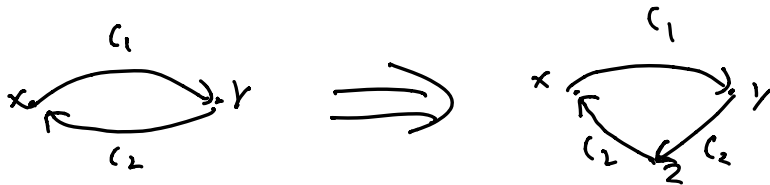
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Spend rest of today proving this.

- ▶ Many different valid proofs.
- ▶ We'll see a classical proof which will naturally lead to algorithms for these problems.

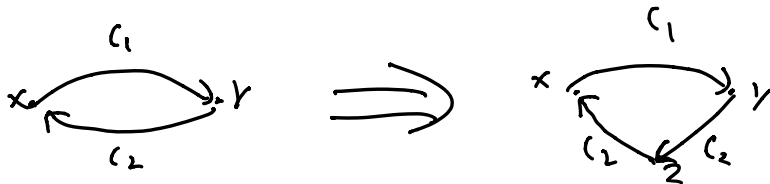
One Direction

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- ▶ Doesn't change max-flow or min-cut
- ▶ Increases #edges by constant factor, # nodes to original # edges.

Residual

Let \mathbf{f} be feasible (\mathbf{s}, \mathbf{t}) -flow. Define *residual capacities*:

$$c_f(\mathbf{u}, \mathbf{v}) = \begin{cases} c(\mathbf{u}, \mathbf{v}) - \mathbf{f}(\mathbf{u}, \mathbf{v}) & \text{if } (\mathbf{u}, \mathbf{v}) \in E \\ 0 & \text{otherwise} \end{cases}$$

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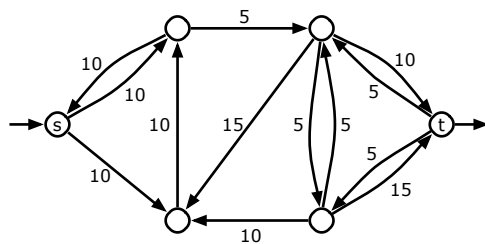
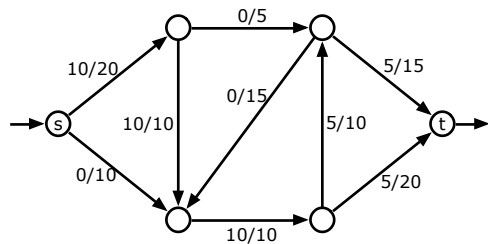
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Residual Graph: $G_f = (V, E_f)$ where $(u, v) \in E_f$ if $c_f(u, v) > 0$.



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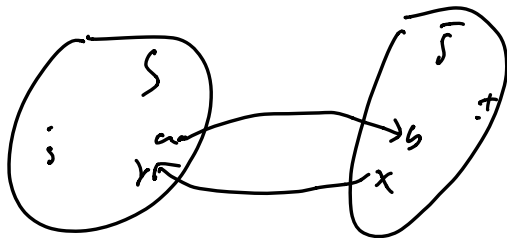
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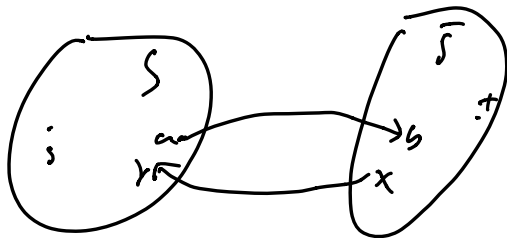
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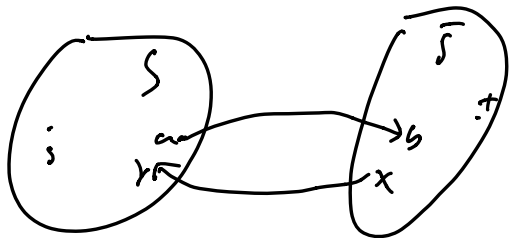
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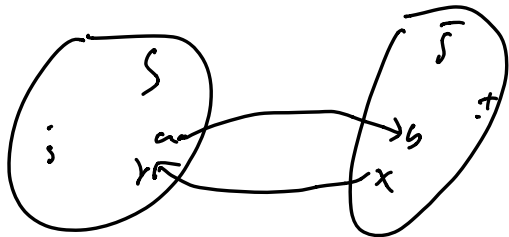
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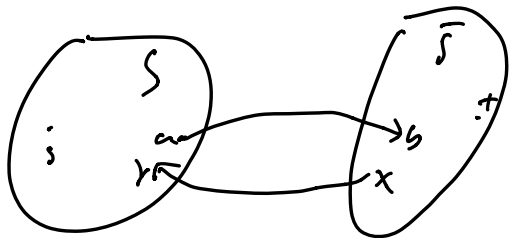
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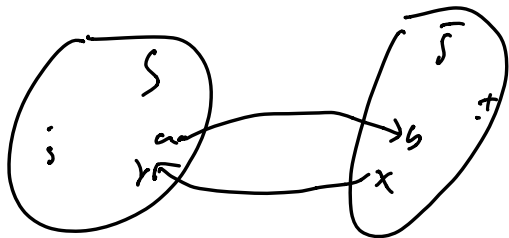
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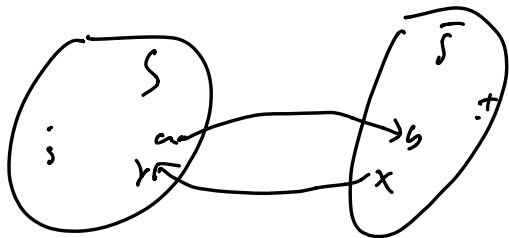
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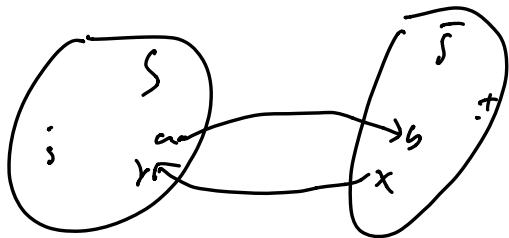
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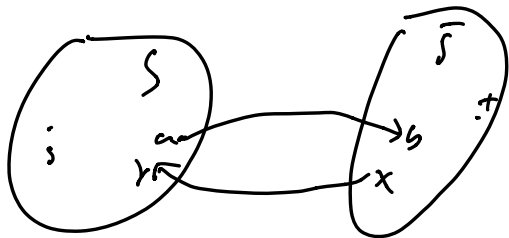
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f saturates $S \rightarrow \bar{S}$ edges, avoids $\bar{S} \rightarrow S$ edges $\implies \text{cap}(S, \bar{S}) = |f|$ by corollary

Case 2

Suppose \exists an $s \rightarrow t$ path P in G_f .

- ▶ Called an *augmenting path*

Idea: show that we can “push” more flow along P , so f not a max flow. Contradiction, can't be in this case.

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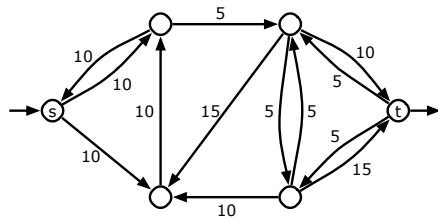
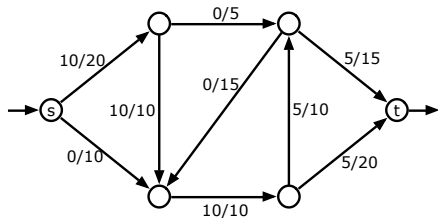
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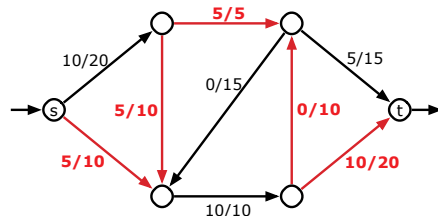
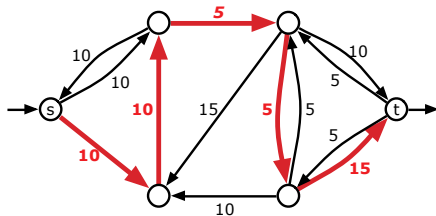
Idea: show that we can “push” more flow along \mathbf{P} , so \mathbf{f} not a max flow. Contradiction, can't be in this case.

- ▶ Foreshadowing: augmenting path allows us to send more flow. Algorithm to increase flow!

Intuition



A flow f in a weighted graph G and the corresponding residual graph G_f .



An augmenting path in G_f with value $F = 5$ and the augmented flow f' .

Formalities

Let \mathbf{P} be (simple) augmenting path in \mathbf{G}_f . Let $\mathbf{F} = \min_{e \in \mathbf{P}} c_f(e)$.

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Define new flow f' : for all $(u, v) \in E$, let

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Claim: f' is a feasible (s, t) -flow with $|f'| > |f|$.

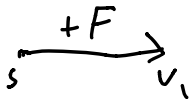
Plan: prove (sketch) each subclaim individually

- ▶ $|f'| > |f|$
- ▶ f' an (s, t) -flow (flow conservation)
- ▶ f' feasible (obeys capacities)

$$|f'| > |f|$$

Consider first edge of P (out of s), say (s, v_1)

- ▶ If $(s, v_1) \in E$, then $f'(s, v_1) = f(s, v_1) + F$
- ▶ If $(v_1, s) \in E$ then $f'(v_1, s) = f(v_1, s) - F$



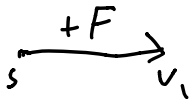
or



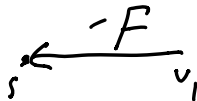
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$$|f'| = \sum_u f'(s, u) - \sum_u f'(u, s) = |f| + F > |f|$$

f' obeys flow conservation

Consider some $u \in V \setminus \{s, t\}$.

f' obeys flow conservation

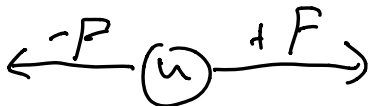
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f' obeys flow conservation

Consider some $u \in V \setminus \{s, t\}$.

- ▶ If $u \notin P$, no change in flow at $u \implies$ still balanced.
- ▶ If $u \in P$, four possibilities:



f' obeys capacity constraints

Let $(u, v) \in E$

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► If $(v, u) \in P$:

$$\begin{aligned} f'(u, v) &= f(u, v) - F \\ &\geq f(u, v) - c_f(v, u) \\ &= f(u, v) - f(u, v) \\ &= 0 \end{aligned}$$

Ford-Fulkerson Algorithm and Integrality

FF Algorithm

Obvious algorithm from previous proof: keep pushing flow!

```
 $f = \vec{0}$   
while( $\exists s \rightarrow t$  path  $P$  in  $G_f$ ) {  
     $F = \min_{e \in P} c_f(e)$   
    Push  $F$  flow along  $P$  to get new flow  $f'$   
     $f = f'$   
}  
return  $f$ 
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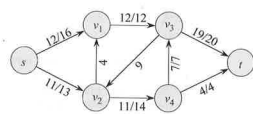
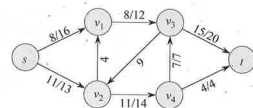
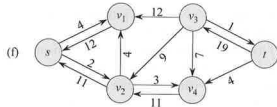
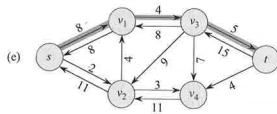
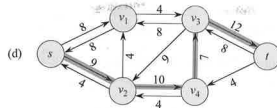
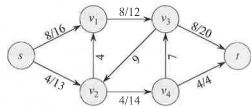
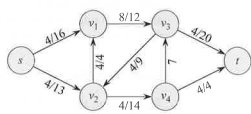
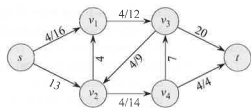
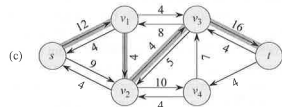
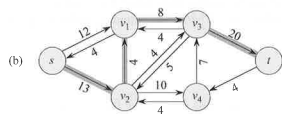
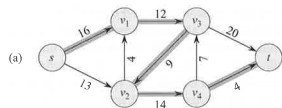
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Correctness: directly from previous proof

Example



Integrality

Corollary

If all capacities are integers, then there is a max flow such that the flow through every edge is an integer

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If all capacities are integers, then there is a max flow such that the flow through every edge is an integer

Proof.

Induction on iterations of the Ford-Fulkerson algorithm: initially true, stays true \implies true at end. □

Running Time

Theorem

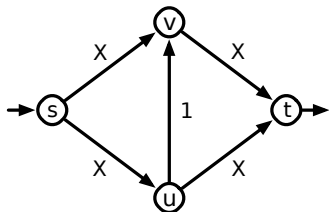
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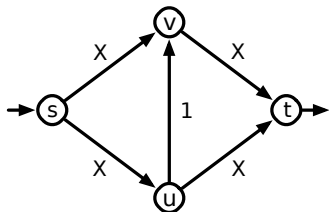
A bad example for the Ford-Fulkerson algorithm.

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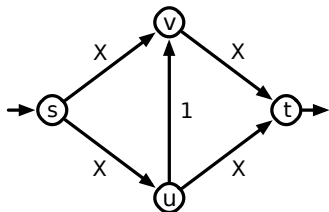
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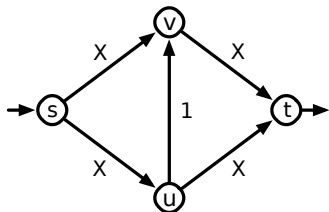
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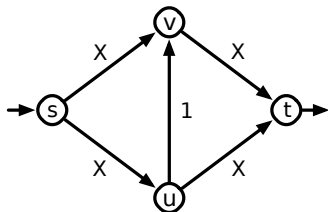
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\implies Exponential time!