

Lecture 9: Priority Queues and Heaps

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September 23, 2025

601.433/633 Introduction to Algorithms

Introduction

Priority Queues / Heaps: Like a queue/stack, but instead of FIFO/LIFO, by priority

- ▶ $\text{Insert}(H, x)$: insert element x into heap H .
- ▶ $\text{Extract-Min}(H)$: remove and return an element with smallest key
- ▶ $\text{Decrease-Key}(H, x, k)$: decrease the key of x to k .
- ▶ $\text{Meld}(H_1, H_2)$: replace heaps H_1 and H_2 with their union

Extra Operations:

- ▶ $\text{Find-Min}(H)$: return the element with smallest key
- ▶ $\text{Delete}(H, x)$: delete element x from heap H

Min-Heap, but can also do Max-Heap.

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Note: x is a *pointer* to an element. No way to lookup, so need a pointer to an element to change it.

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No! Sorting lower bound. But maybe can make one $O(1)$, other $O(\log n)$?

Today and State of the Art

State of the art: *strict Fibonacci Heaps*.

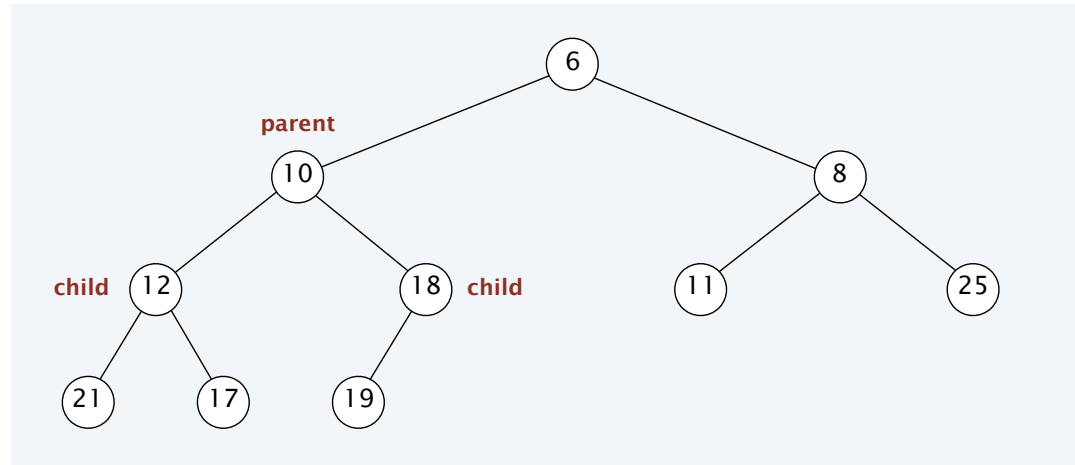
- ▶ Too complicated for this class, not practical. See CLRS 19 for Fibonacci Heaps.

Today: binary heaps (should be review), then binomial heaps

- ▶ Binomial heaps not quite as complicated as Fibonacci heaps, many of same core ideas

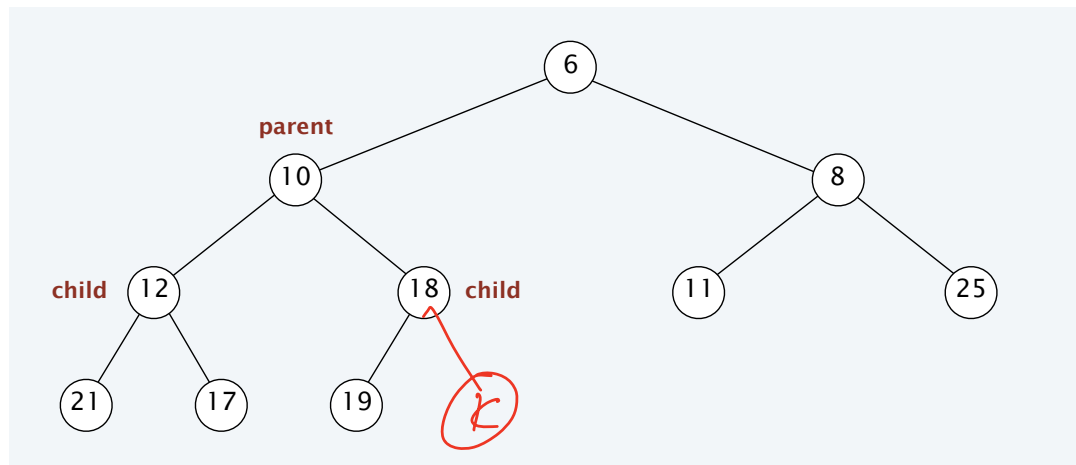
Binary Heaps

- ▶ Complete binary tree, except possibly at bottom level.
- ▶ Heap order: key of any node no larger than key of its children.



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Properties:

- ▶ Since (almost) complete binary tree, depth $\Theta(\log n)$
- ▶ Min must be at root

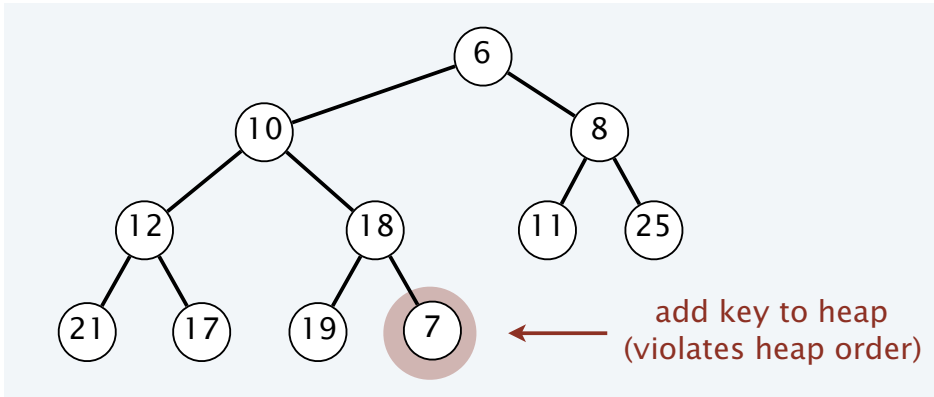
Representation:

- ▶ Pointers to root and rightmost leaf
- ▶ Every node has pointers to parent and children

Insert(H, x)

Preserve heap *structure*: insert x into next open spot (bottom right, or left of new level if bottom level full)

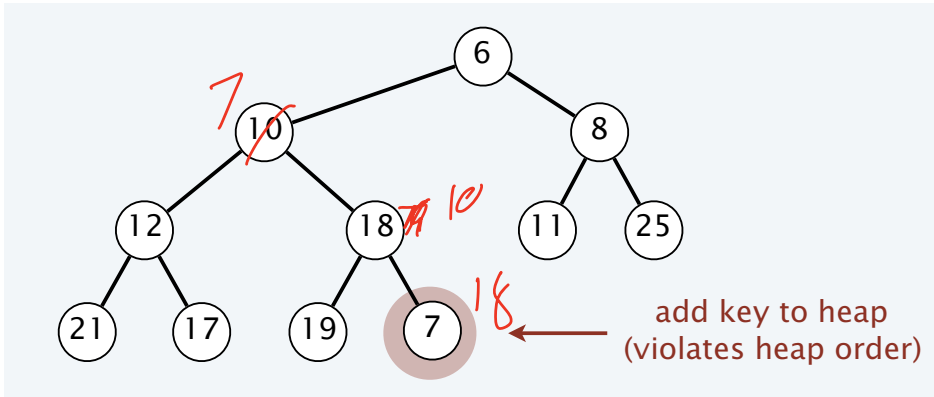
- ▶ Might violate heap *order*!



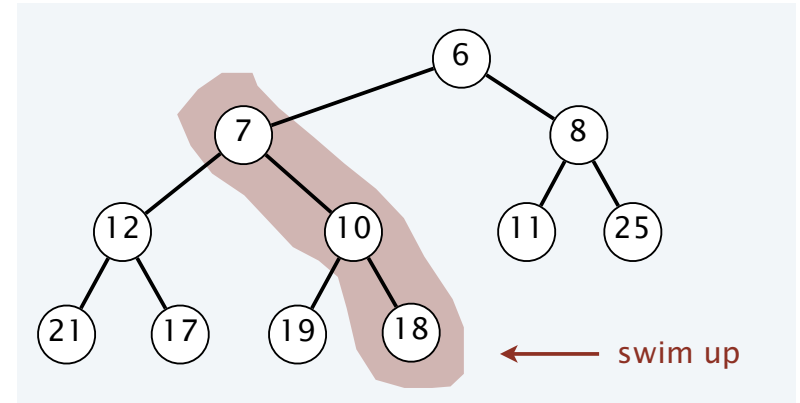
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“Swim up”: as long as x smaller than its parent, swap with parent

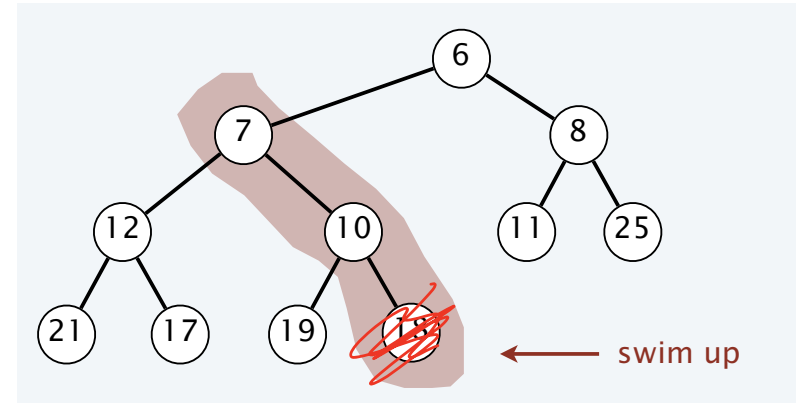
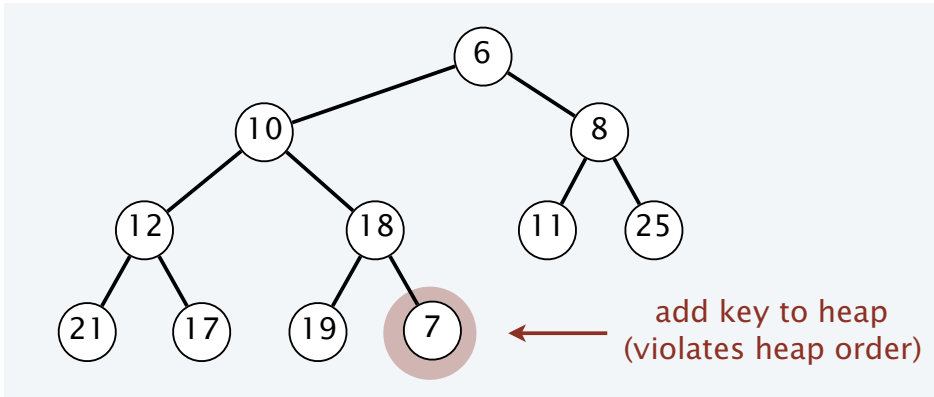


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Running time: $O(\log n)$ worst case (also amortized) via depth

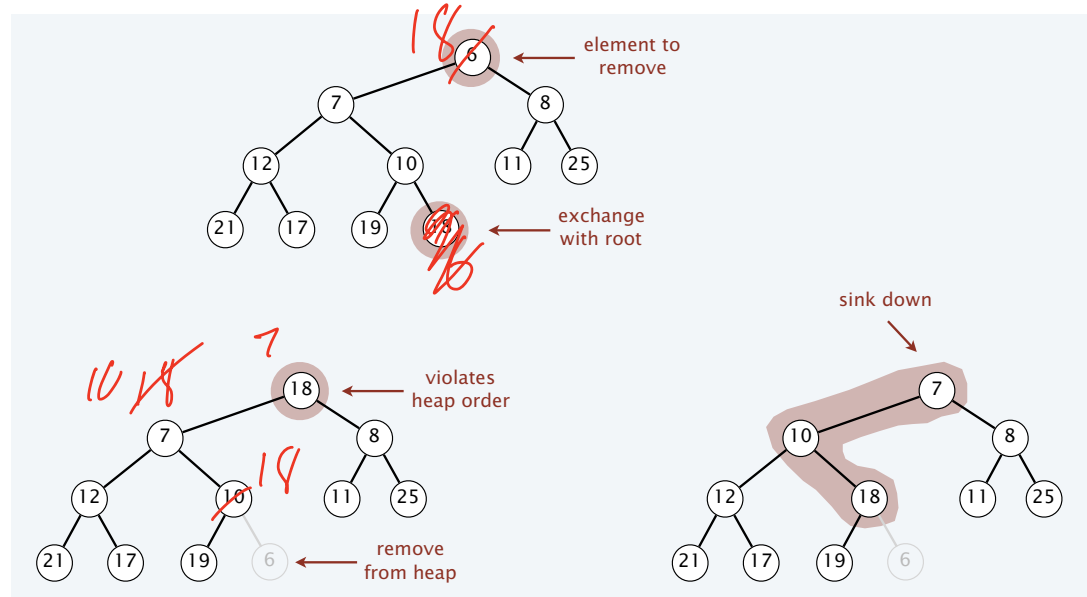
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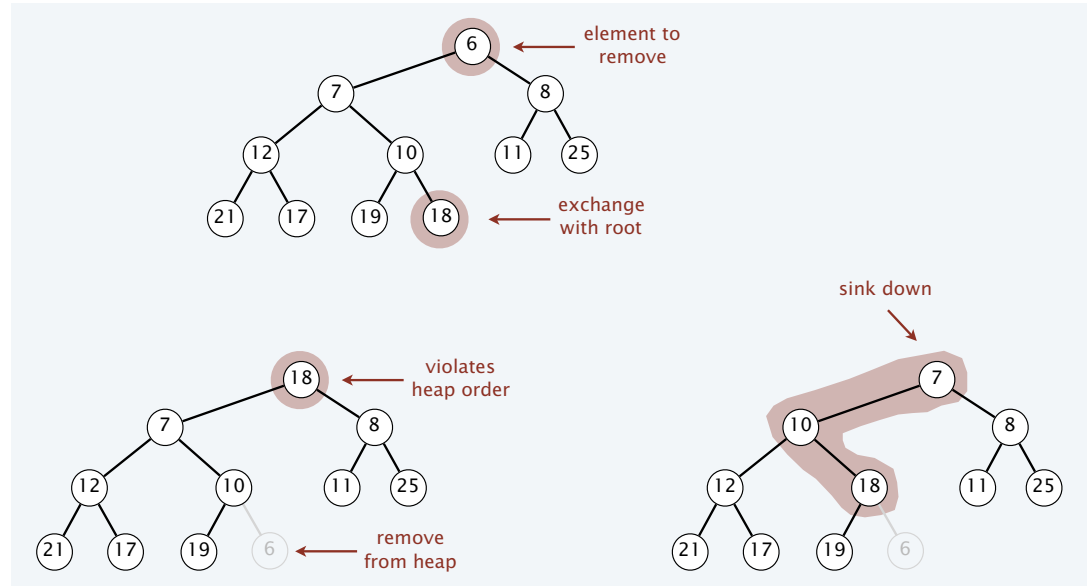
- ▶ Swap root with final heap element, remove former root.
- ▶ Sink down: swap root with smaller of its children until heap order restored



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Running time: $O(\log n)$ worst case (via depth). Amortized: $O(1)$ (not obvious)

Decrease-Key(H, x, k)

Decrease key of x to k , “swim up” until heap order restored.

Running time: $O(\log n)$ (depth)

Meld(H_1, H_2)

Assume both heaps have size n .

- ▶ Obvious approach: insert each element of H_2 into H_1 . Time: $O(n \log n)$

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 - ▶ At most $n/2^h$ nodes at height h

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$$\sum_{h=0}^{\log n} h \left(\frac{n}{2^h} \right) = n \sum_{h=0}^{\log n} \frac{h}{2^h} \leq O(n)$$

Amortized Extract-Min

Weights: $w(x) = \text{depth of } x$

- ▶ Root has weight **0**, its children have weight **1**, etc.

Potential: $\Phi(H) = \sum_x w(x)$

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- ▶ True cost: height $h = \Theta(\log n)$ of tree, plus $O(1)$ (for initial swap).
- ▶ $\Delta\Phi$: one less node at depth $h \implies \Delta\Phi = -h$
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Uses Inserts to “pay for” Extract-Mins.

Improvements

Downsides of binary heaps:

- ▶ Do at least as many Inserts as Extract-Mins! Want $O(1)$ Insert, $O(\log n)$ Extract-Min
- ▶ Meld in $O(n)$ is better than trivial, but still not great.

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- ▶ Get Insert down to $O(1)$ (amortized)
- ▶ Meld in $O(\log n)$ (worst-case and amortized)
- ▶ Downside: $O(\log n)$ Extract-Min, $O(\log n)$ Find-Min

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Fibonacci Heaps:

- ▶ Everything $O(1)$ (amortized) except $O(\log n)$ Extract-Min (amortized)

Binomial Heaps

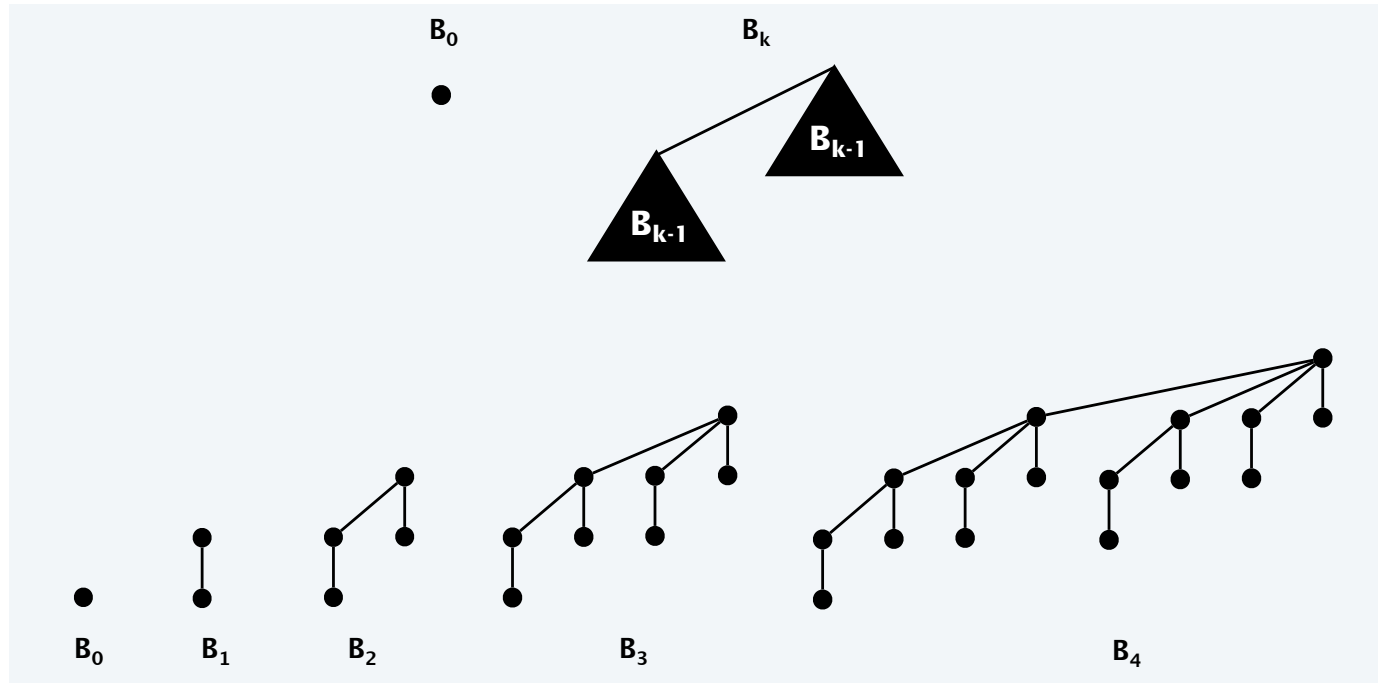
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- ▶ B_0 = single node.
- ▶ B_k = one B_{k-1} linked to another B_{k-1} .

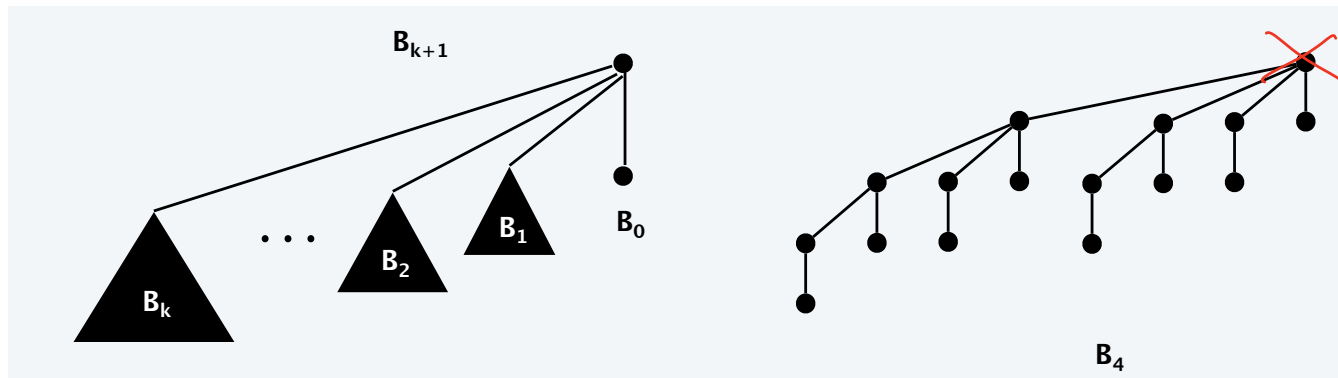


Structure Lemma

Lemma

The order k binomial tree B_k has the following properties:

1. Its height is k .
2. It has 2^k nodes
3. The degree of the root is k
4. If we delete the root, we get k binomial trees B_{k-1}, \dots, B_0 .

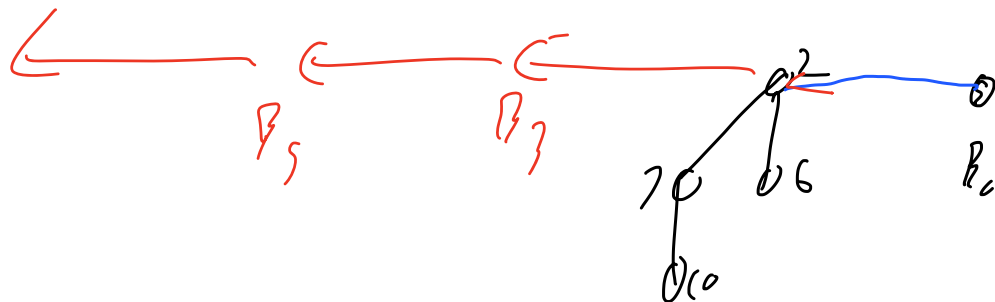


Binomial Heap

Definition

A *binomial heap* is a collection of binomial trees so that each tree is heap ordered, and there is exactly **0** or **1** tree of order **k** for each integer **k** .

Keep roots of trees in linked list, from smallest order (not key!) to largest

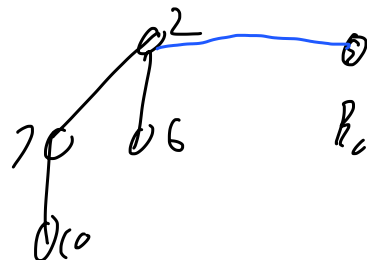


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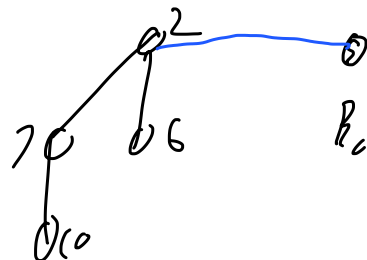
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\implies at most **$\log n$** trees, and by lemma each has height $\leq \log n$

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Analyze all operations both worst-case and amortized.

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- ▶ Correct: each tree heap-ordered, so global min one of the roots
- ▶ Worst-case: $O(\log n)$
- ▶ Amortized: doesn't change potential, also $O(\log n)$.

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Warmup: H_1, H_2 both single trees of same order k .

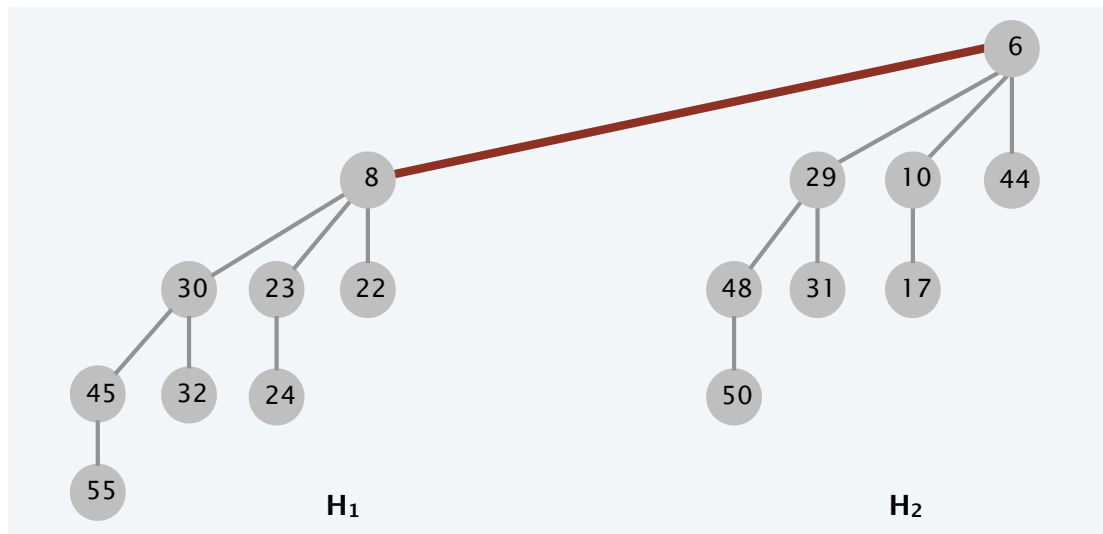
- ▶ Union has size $2^k + 2^k = 2^{k+1}$: just a single B_{k+1}
- ▶ Easy to make a B_{k+1} out of two B_k 's!

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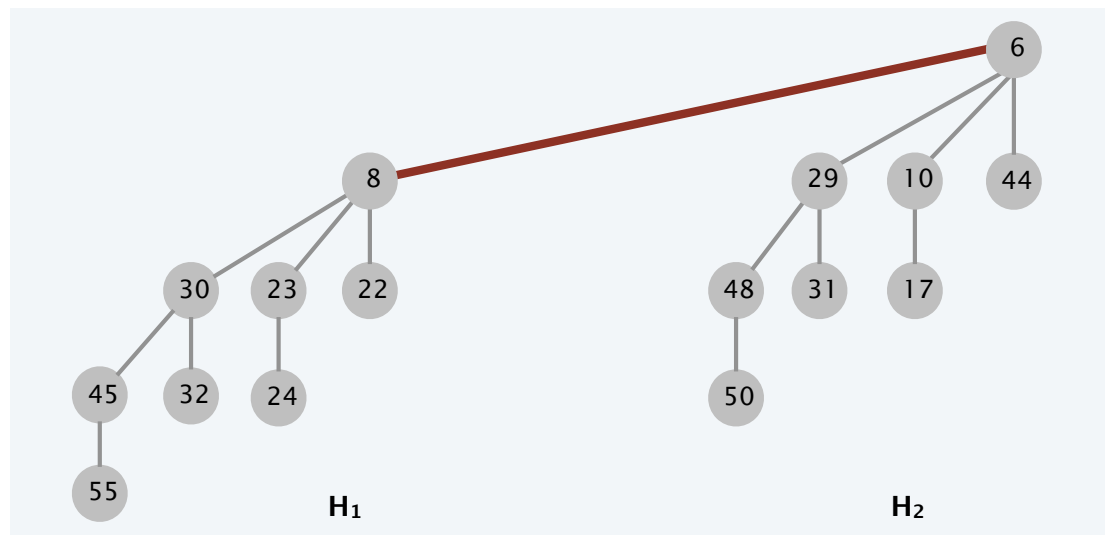


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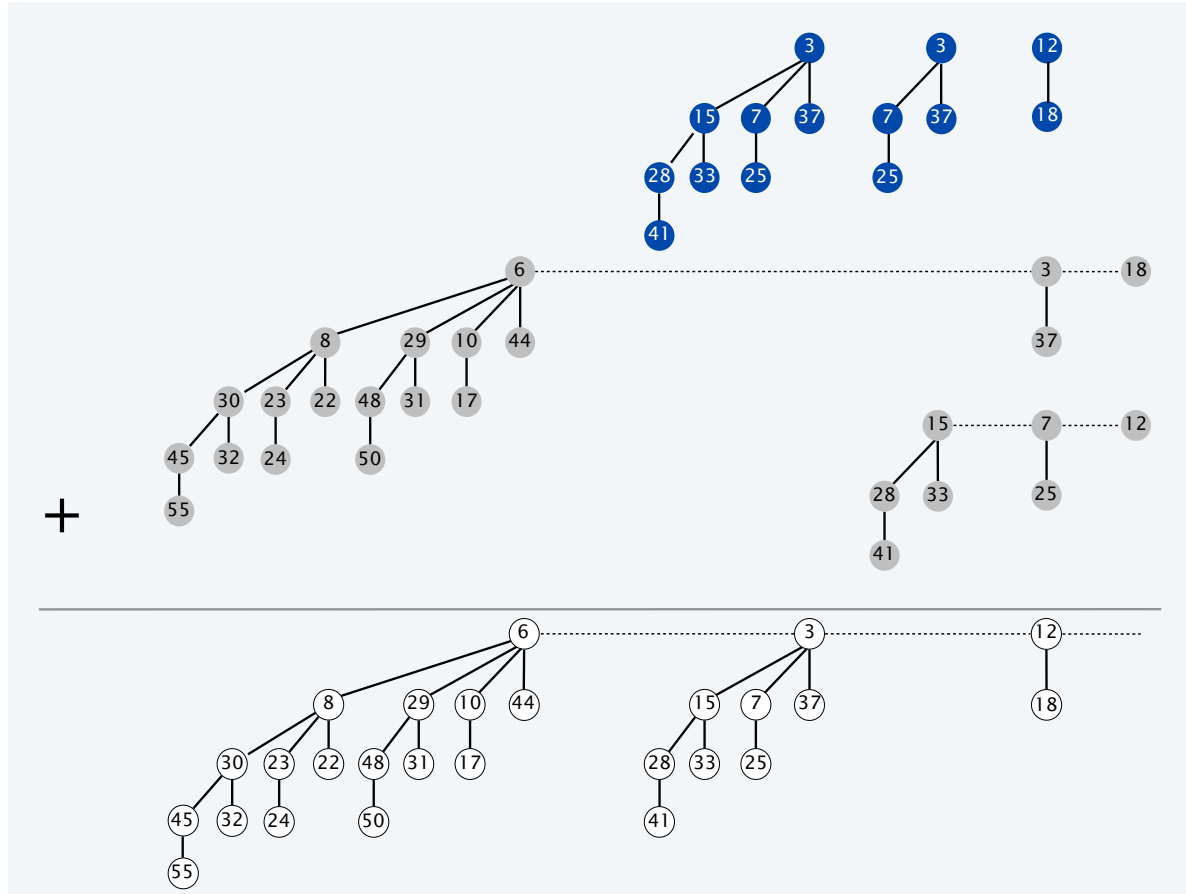


Link of two trees.

- ▶ Worst-case time: $O(1)$ (create a single link). Normalize: call 1
- ▶ $\Delta\Phi$: two trees to one: -1
- ▶ Amortized cost:
 $1 - 1 = 0 = O(1)$.

Meld(H_1, H_2): General Case

(Almost) just like binary addition!



H_1

H_2

Meld(H_1, H_2): Analysis

Easy to prove correct (exercise for home).

Running time:

- ▶ Worst case: $O(1)$ per “order” $k \implies \leq O(\log n)$
- ▶ Amortized: Potential does not go up, but could stay the same $\implies O(\log n)$ amortized

Insert(H, x)

Use Meld:

- ▶ Create new heap H' with one B_0 consisting of just x
- ▶ Meld(H, H')

Correctness: Obvious

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- ▶ Worst case: $O(\log n)$ (via Meld)

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 - ▶ Like incrementing a binary counter!

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Correctness: Obvious

Running Time:

- ▶ Worst case: $O(\log n)$ (via Meld)
- ▶ Amortized:
 - ▶ Like incrementing a binary counter!
 - ▶ If we link k trees, potential goes down by $k - 1$
 - ▶ Cost = # links plus 1 (for making new heap)
 - ▶ Amortized cost = $k + 1 + \Delta\Phi = k + 1 - (k - 1) = 2 = O(1)$

Extract-Min(H)

Use Meld again!

- ▶ $O(\log n)$ to Find-Min: one of the roots.
- ▶ Delete and return this root: tree turns into a new heap!
- ▶ Meld with original heap (minus the tree)

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Correctness: Obvious

Running Time:

- ▶ Worst-Case: $O(\log n)$ from creating new heap, Meld
- ▶ Amortized:
 - ▶ Potential can go up! But by at most $\log n$
 - ▶ Amortized time at most $O(\log n) + \log n = O(\log n)$