Final Solution

Class: **CSC431** Winter 2013
Professor: Massimo Di Pierro
Books Allowed: **Yes**
Handwritten Notes Allowed: **No**
Calculator Allowed: **Yes**
Computer Allowed: **No**
Total Problems **10**

Problems are organized by topic. They are not in order of difficulty. Some of them have multiple True or multiple False False answers. For every option circle True or False. Each problem counts for 4 points. An unaswered True/False question will be graded as a wrong answer.

Student Name: ....
Date: ....
Signature: ....

1 Problem 1

In floating point arithmetic $a + b == a$ when $|b| < \epsilon|a|$.

- **True**: In single precision floating point arithmetic $\epsilon \simeq 10^{-7}$
- **True**: In single precision $a + b == a$ can be true even when $b = 1$.
- **True**: If $a + b == a$ then $n * a + n * b == n * a$ for every n
- **False**: If $a + b == a$ then $a + 2 * b - b == a$

2 Problem 2

Consider the following functions:

$$f_1(x) = \frac{2}{x^2 - 1} - \frac{1}{x - 1}$$

$$f_2(x) = \frac{-1}{1 + x}$$

Assuming floating point arithmetics, which of the following statments is False:

- **True**: They are equivalent up to numerical issues
- **False**: They both diverge when $x = 1$
- **False**: f_2 is better than f_1 around $x = -1$
- **False**: f_1 is better than f_2 around $x = 1$

3 Problem 3

Consider these numbers: $x_0 = 1 + a$, $x_1 = 1 - a$ for $x_i = 1$ for $i = 2...N - 1$ and $a = 10^{-3}$. The average is $\mu = 1$ and the variance is $\sigma^2 = 2a^2/N$. Assume single precision floating point arithmetics and assume you compute the variance using the formula:

$$\sigma^2 = (\frac{1}{N}\sum_i x^2) - \mu^2$$

For which value of N will you get an incorrect value for the variance? Explain your answer.

$$\begin{aligned} \sigma^2 &< \epsilon\mu^2 && (1) \\ 2a^2/N &< \epsilon\mu^2 && (2) \\ N &> 2a^2/(\epsilon\mu^2) && (3) \\ N &> 20 && (4) \end{aligned}$$

4 Problem 4

Consider the following matrix:

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Compute the inverse in 6 steps. Show your steps.

$$\begin{pmatrix} 3/4 & 1/4 & -3/4 \\ -1/4 & 1/4 & 1/4 \\ 1/12 & -1/12 & 1/4 \end{pmatrix}$$

5 Problem 5

Consider the following 2×2 matrix:

$$\begin{pmatrix} 1 & \mathrm{a} \\ \mathrm{a} & 2 \end{pmatrix}$$

For which values of a is it positive definite? Show your steps.

It is symmetric. It is positive definite if the Chlwesky exists. If we call the matrix M:

$$\begin{aligned} l_{00} &= \sqrt{m_{00}} = 1 && (5) \\ l_{01} &= m_{01}/l_{01} = a && (6) \\ l_{11} &= \sqrt{m_{11} - l_{01}^2} = \sqrt{2 - a^2} && (7) \end{aligned}$$

Requires $|a| < \sqrt{2}$.

6 Problem 6

Consider the following algorithms:

```
def D(f,h=1e-4):
    return lambda x: (f(x+h)-f(x-h))/(2.0*h)

def P(f,x,ns=10):
    for k in range(ns): x = x - f(x)/D(f)(x)
    return x

def Q(f,x,ns=10):
    for k in range(ns): x = x - f(x)/D(D(f))(x)
    return x
```

- **True**: $P(f)$ finds x which solves $f(x) = 0$ using Newton method.
- **False**: $Q(f)$ finds x which solves $f'(x) = 0$ using Newton method.
- **False**: $P(f)$ always converges and converges to the correct result.
- **False**: Each call to P calls the function f, $2 \times \text{ns}$ times.

7 Problem 7

What is the output of

```
def f(x): return x**3-x-1
print P(f,x=0,ns=2)
```

Show your steps.

```
x1 = x0 - f(x0)/f'(x0) = -1
x2 = x1 - f(x1)/f'(x1) = -0.5
```

8 Problem 8

Consider the following algorithms:

```
def R(f,x,ap=1e-5,rp=1e-5,h=1e-4):
    while True:
        (x_old, x) = (x, x - f(x)*h/(f(x)-f(x-h)))
        if abs(x-x_old)<max(ap,rp*abs(x)): return x

def T(f,x,ap=1e-5,rp=1e-5,h=1e-4):
    fx = f(x)
    (x_old, fx_old, x) = (x, fx, x - fx*h/(fx-f(x-h)))
    while True:
        fx = f(x)
        (x_old, fx_old, x) = (x, fx, x - fx*(x-x_old)/(fx-fx_old))
        if abs(x-x_old)<max(ap,rp*abs(x)): return x
    return x
```

- **True**: The two algorithms solve the same problems in different ways
- **False**: The two algorithms always return a solution.
- **True**: The two algorithms can go into an infinite loop
- **False**: `R` is more efficient than `T` (less calls to function f for same precision).

9 Problem 9

Consider the following three points:

t_i	y_i
0	2
1	5
2	6

You perform a linear fit using the linear least square algorithm with $y(t) = c_0 + c_1 t$. The output is given by $c = (A^t A)^{-1} A^t y$. Where $A_{i0} = 1$, $A_{i1} = t_i$. Using the fact that

$$\begin{pmatrix} \text{a} & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{a-1} \begin{pmatrix} 1 & \text{-1} \\ \text{-1} & \text{a} \end{pmatrix}$$

determine the value of the coefficients c. Show your steps.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{8}$$

$$y = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \tag{9}$$

$$A^t A = 3 \begin{pmatrix} 1 & 1 \\ 1 & 5/3 \end{pmatrix} \tag{10}$$

$$(A^t A)^{-1} = \frac{1}{3}\frac{3}{/}2 \begin{pmatrix} 5/3 & \text{-1} \\ \text{-1} & 1 \end{pmatrix} \tag{11}$$

$$A^t y = \begin{pmatrix} 13 \\ 17 \end{pmatrix} \tag{12}$$

$$c = (A^t A)^{-1} A^t y = \begin{pmatrix} 7/3 \\ 2 \end{pmatrix} \tag{13}$$

10 Problem 10

- **True**: The Newton solver is likely to fail when $f'(x) \simeq 0$ in proximity of of the solution.
- **False**: The Newton optimizer is likely to fail when $f''(x) >> 0$ in proximity of the solution.
- **True**: $f'(x) = 0$ and $f''(x) > 0$ is a relative minimum of the function f.
- **False**: $f'(x) = 0$ is either relative minimum or a maximum of the function f.