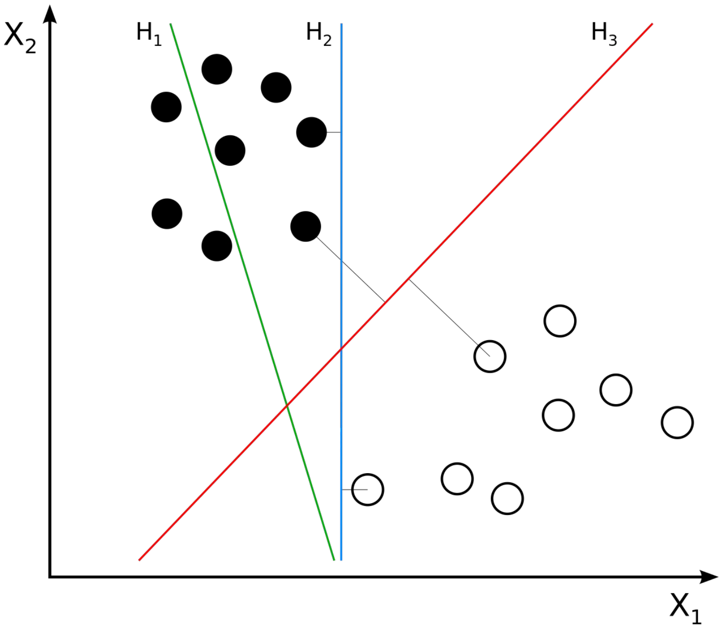
CIS 678 Machine Learning

ML Models: SVM, Kernel Methods

Support Vector Machines

- Maximum margin models

Motivation



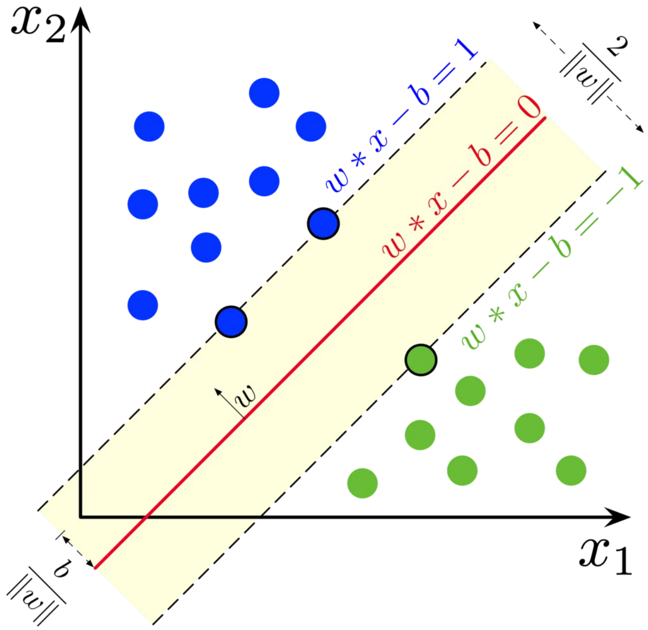
H_1 does not separate the classes. H_2 does, but only with a small margin. H_3 separates them with the maximal margin. (Wiki)

Linear SVM

We are given a training dataset of n points of the form

$$(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n),$$

$$\boxed{y_i \in \{1, -1\}}$$



Maximum-margin hyperplane and margins for an SVM trained with samples from two classes. Samples on the margin are called the support vectors.(Wiki)

Linear SVM

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$$(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n),$$

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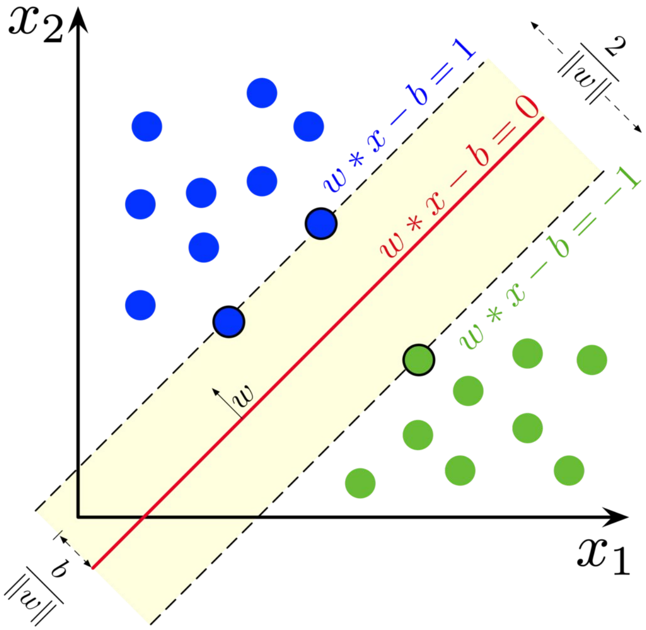
Maximum
margin classifier

$$\boxed{\mathbf{w}^\mathsf{T}\mathbf{x} - b = 0,}$$

Linear SVM: b, $\mathbf{w}$.

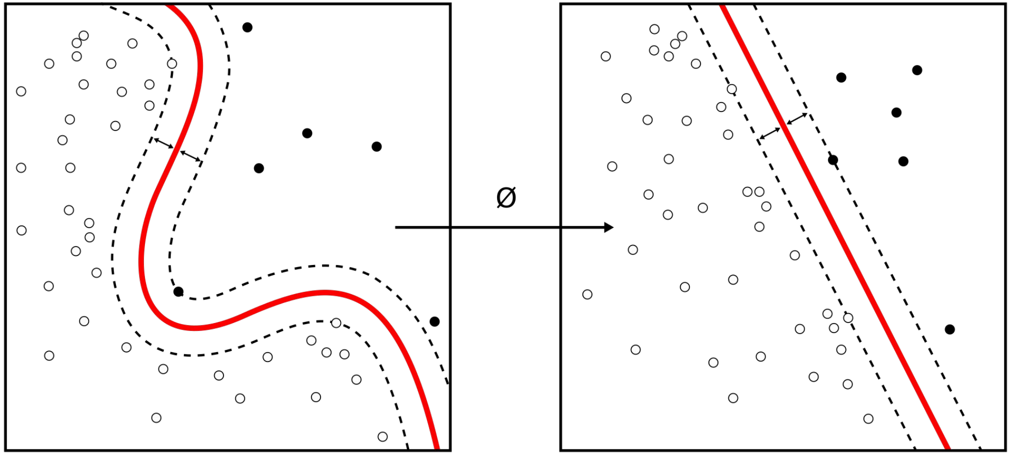
Margin : $\dfrac{2}{\|\mathbf{w}\|}$,

Maximize



Maximum-margin hyperplane and margins for an SVM trained with samples from two classes. Samples on the margin are called the support vectors.([Wiki])

Nonlinearity through Kernels



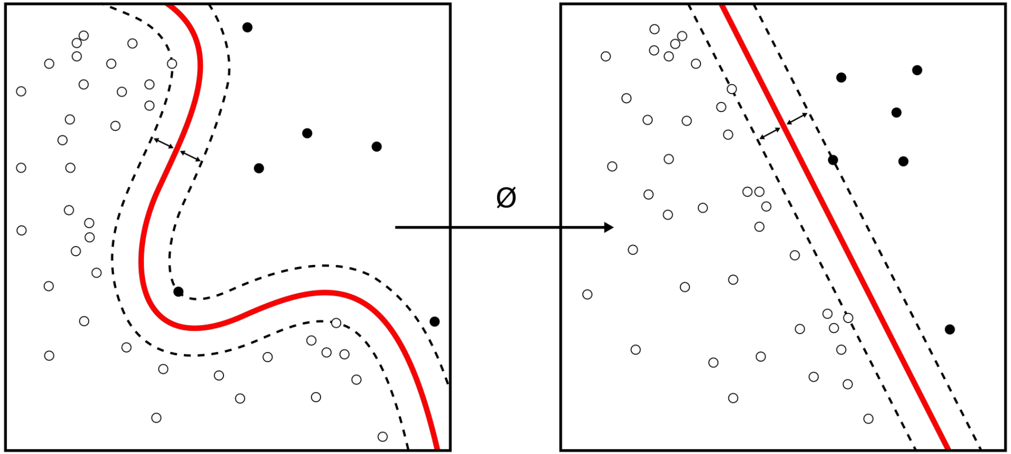
Kernel Machine(Wiki)

$$y(\mathbf{x}) = \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}) + b$$

Nonlinearity through Kernels

$$
\begin{aligned}
k(\mathbf{x}, \mathbf{z}) &= \left(\mathbf{x}^{\mathrm{T}}\mathbf{z}\right)^2 = (x_1 z_1 + x_2 z_2)^2 \\
&= x_1^2 z_1^2 + 2x_1 z_1 x_2 z_2 + x_2^2 z_2^2 \\
&= (x_1^2, \sqrt{2}x_1 x_2, x_2^2)(z_1^2, \sqrt{2}z_1 z_2, z_2^2)^{\mathrm{T}} \\
&= \phi(\mathbf{x})^{\mathrm{T}}\phi(\mathbf{z}).
\end{aligned}
$$

Polynomial kernel



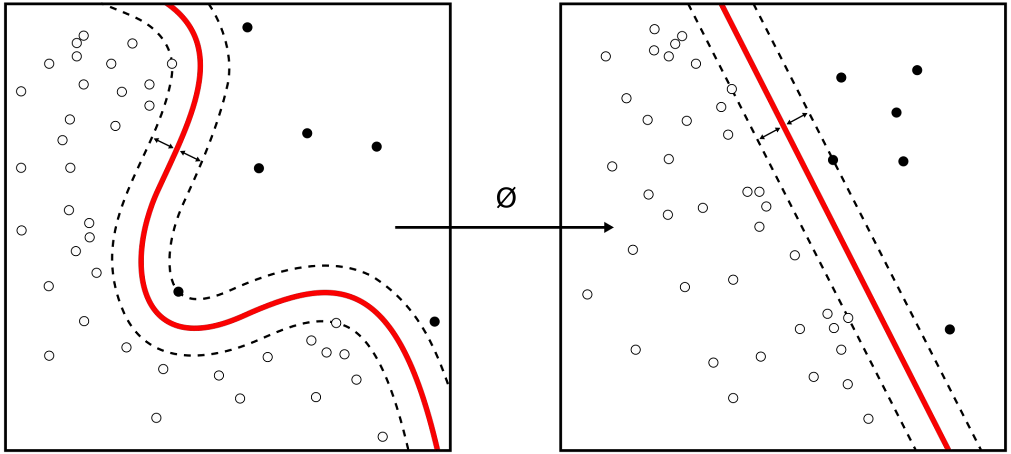
Kernel Machine(Wiki)

$$y(\mathbf{x}) = \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}) + b$$

Nonlinearity through Kernels

$$\begin{aligned} k(\mathbf{x}, \mathbf{z}) &= \left(\mathbf{x}^{\mathrm{T}}\mathbf{z}\right)^2 = (x_1 z_1 + x_2 z_2)^2 \\ &= x_1^2 z_1^2 + 2x_1 z_1 x_2 z_2 + x_2^2 z_2^2 \\ &= (x_1^2, \sqrt{2}x_1 x_2, x_2^2)(z_1^2, \sqrt{2}z_1 z_2, z_2^2)^{\mathrm{T}} \\ &= \phi(\mathbf{x})^{\mathrm{T}}\phi(\mathbf{z}). \end{aligned}$$

Polynomial kernel



Kernel Machine(Wiki)

$$y(\mathbf{x}) = \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}) + b$$

Nonlinearity through Kernels

Some common kernels include:

- Polynomial (homogeneous): $k(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i \cdot \mathbf{x}_j)^d$. Particularly, when $d = 1$, this becomes the linear kernel.
- Polynomial (inhomogeneous): $k(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i \cdot \mathbf{x}_j + r)^d$.
- Gaussian radial basis function: $k(\mathbf{x}_i, \mathbf{x}_j) = \exp\left(-\gamma \|\mathbf{x}_i - \mathbf{x}_j\|^2\right)$ for $\gamma > 0$. Sometimes parametrized using $\gamma = 1/(2\sigma^2)$.
- Sigmoid function (Hyperbolic tangent): $k(\mathbf{x_i}, \mathbf{x_j}) = \tanh(\kappa \mathbf{x}_i \cdot \mathbf{x}_j + c)$ for some (not every) $\kappa > 0$ and $c < 0$.

The kernel is related to the transform $\varphi(\mathbf{x}_i)$ by the equation $k(\mathbf{x}_i, \mathbf{x}_j) = \varphi(\mathbf{x}_i) \cdot \varphi(\mathbf{x_j})$. The value $\mathbf{w}$ is also in the transformed space, with $\mathbf{w} = \sum_i \alpha_i y_i \varphi(\mathbf{x}_i)$. Dot products with $\mathbf{w}$ for classification can again be computed by the kernel trick, i.e. $\mathbf{w} \cdot \varphi(\mathbf{x}) = \sum_i \alpha_i y_i k(\mathbf{x}_i, \mathbf{x})$.

Kernel Machine(Wiki)