CIS 635 Knowledge Discovery & Data Mining

Introduction to Linear Algebra

Survey results and project groups

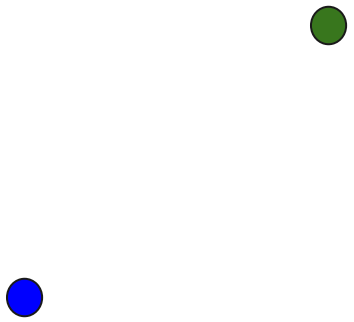
- Summary of the Background survey
- Final project groups formation:
    - Each group will be comprised of two (2) students
    - You are welcome to form your own group, or we will assign groups based on your survey response (background, area of interest, and programming efficiency). Please respond to the survey if you haven't already.
    - If you want to form your group, please let us know by: **09/04/2024**. The point of contact on the following TA: **Sridevi Bommidi (**bommidis@mail.gvsu.edu**)**

Outline

- Proximity vs Distance Metric
- k-NN, our first ML model
- Concept of Vectors and Vector operations
- Digital data, their encodings, and their representations through Vectors
- NumPy basics

Proximity vs Distance metric

- Let's we are give two data points: one, the **blue**, and the other is the **green** circle.



Proximity vs Distance metric

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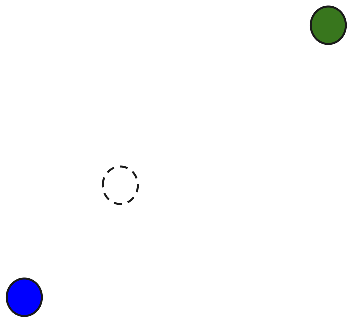
- Now lets, we are given a new circle, and we are given the task to label it with either blue or green.



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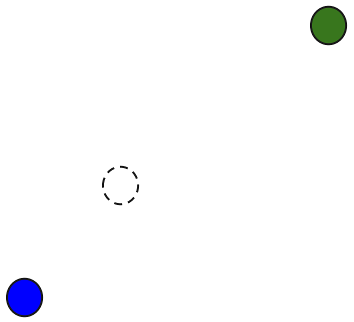

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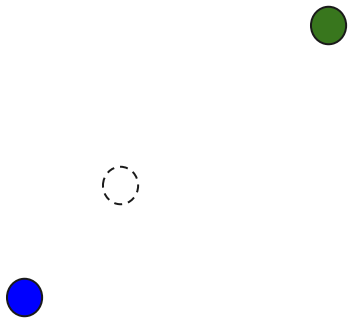
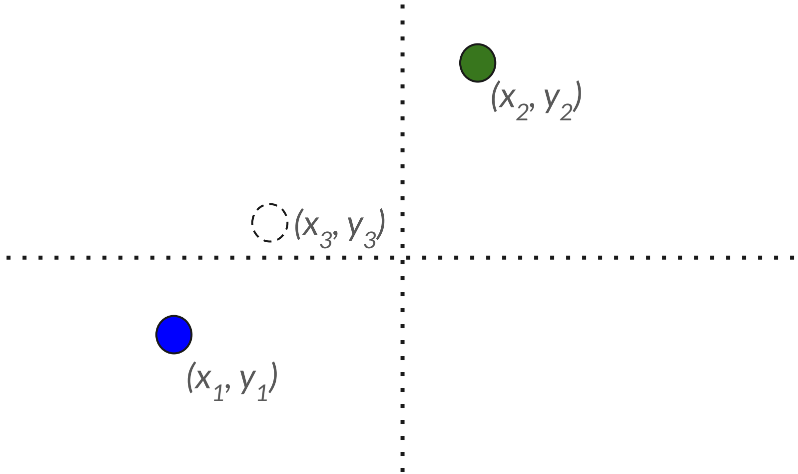
- Visually, it looks to be closer to the blue, dot; but how do we quantify it?
- We can use a proximity or distance metric.



Proximity vs Distance metric

- We can use a proximity or distance metric.
- These three points are depicted on a 2D plane; right?



Proximity vs Distance metric

- We can use a proximity or distance metric.
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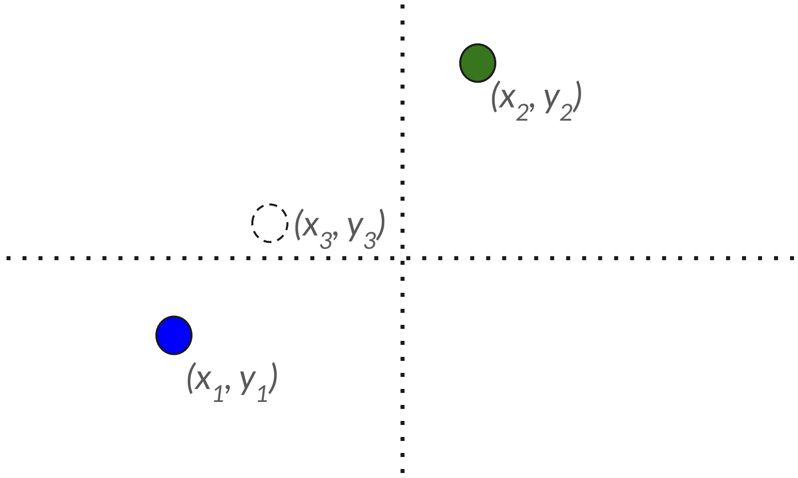
- We can use the **Cartesian coordinate system** to quantify the location, and measure their distance; more specifically the Euclidean distance that we learned in our high-school math.



Proximity vs Distance metric

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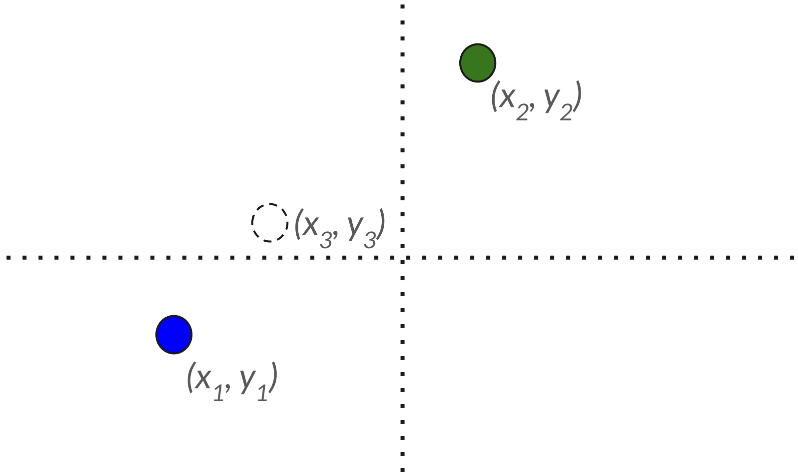
- These three points are depicted on a 2D plane; right?
- We can use the Cartesian coordinate system to quantify the location, and measure their distance; more specifically the Euclidean distance that we learned in our high-school math.
- The Euclidean distance is also known as L2 distance in the DS community



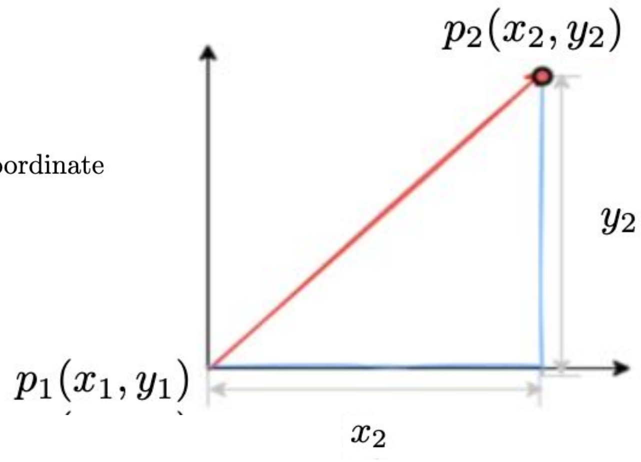
Distance metrics

L2 (or Euclidean) distance: The L2 distance between point $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ is :

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{x_2^2 + y_2^2} \quad \text{given that } p_1(x_1, y_1) = (0, 0)\text{, the origin of the coordinate}$$

I.e. **L2 distance** is the **diagonal** side of a triangle at the right, also known as **Euclidean distance**



Distance metrics

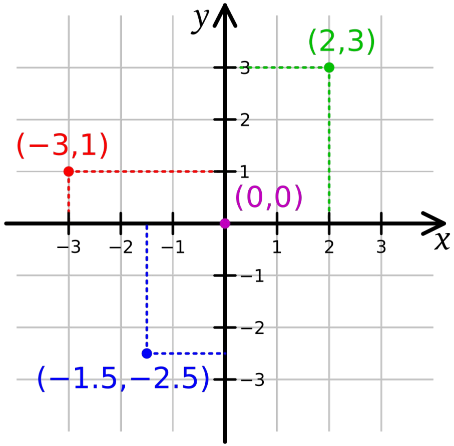
- **L2 (or Euclidean) distance:**

  - L2 distance between vectors [2, 3] and [0, 0] is:

    $$\sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13} = 3.61$$

  - L2 distance between vectors [2, 3] and [-3, 1] is:

    $$\sqrt{(2-(-3)^2 + (3-1)^2} = \sqrt{29} = 5.39$$



Distance metrics

- We have other distance metrics, such as
- **L1 distance**

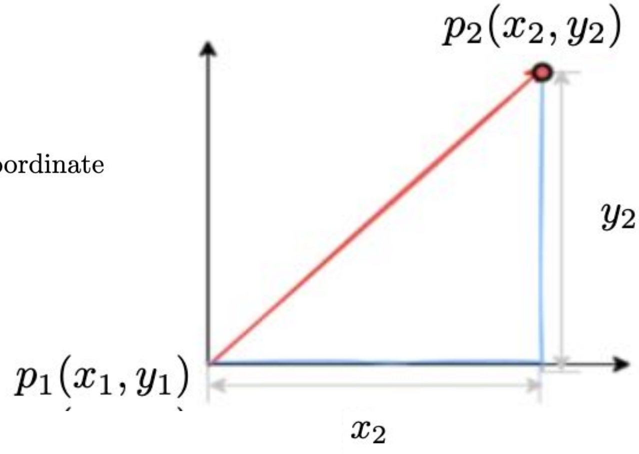
Distance metrics

L1 distance: The L1 distance between point $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ is :

$$|x_2 - x_1| + |y_2 - y_1|$$

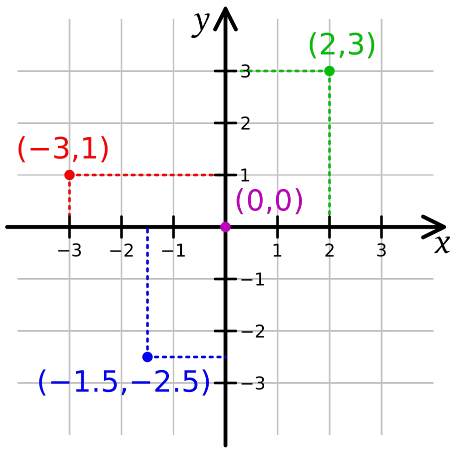
$$= x_2 + y_2 \quad \text{given that } p_1(x_1, y_1) = (0, 0)\text{, the origin of the coordinate}$$

I.e. L1 distance is the summation of the **horizontal** and the **vertical** sides of a triangle at the right.



Distance metrics

- **L1 distance**



- L1 distance between vectors [2, 3] and [0, 0] is:

  |2-0| + |3-0| = 5

- L1 distance between vectors [2, 3] and [-3, 1] is :

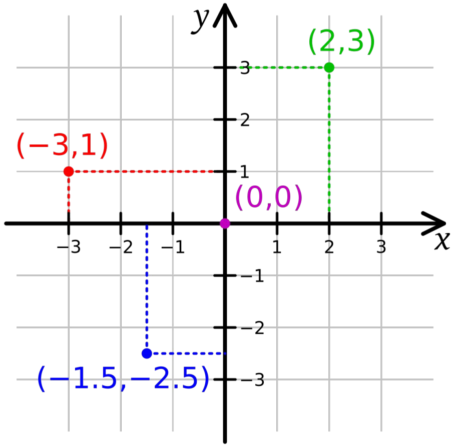
  |2 - (-3)| + |3 - 1| = 5 + 2 = 7

Our first ML Model

- k-NN

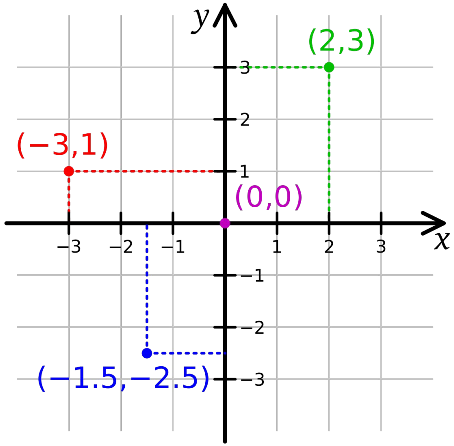
k-NN model

- k-nearest neighbors (k-NN)
  - Supervised learning



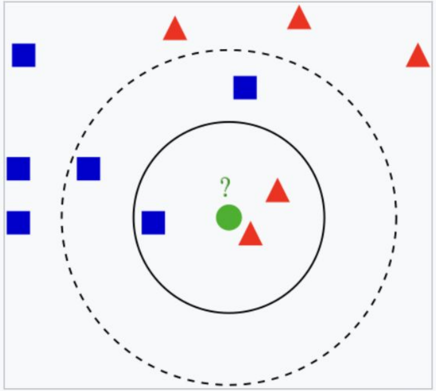
k-NN model

- k-nearest neighbors (k-NN)
  - Supervised learning
  - Non parametric



k-NN model

- k-nearest neighbors (k-NN)
  - Supervised learning
  - Non parametric (distance based method)
  - Both for Classification and Regression solutions



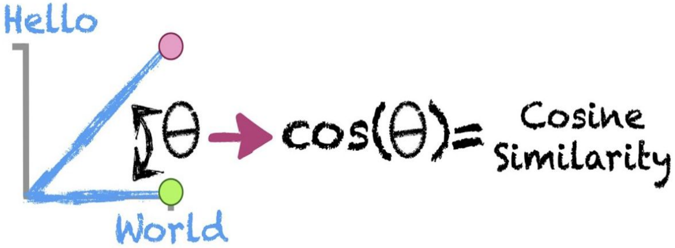
Example of *k*-NN classification. The test sample (green dot) should be classified either to blue squares or to red triangles. If $k = 3$ (solid line circle) it is assigned to the red triangles because there are 2 triangles and only 1 square inside the inner circle. If $k = 5$ (dashed line circle) it is assigned to the blue squares (3 squares vs. 2 triangles inside the outer circle).

src: https://en.wikipedia.org/wiki/K-nearest_neighbors_algorithm

Another unique Distance metric

- We have other distance metrics, such as
- L1 distance, and
- **Cosine distance**

Cosine distance

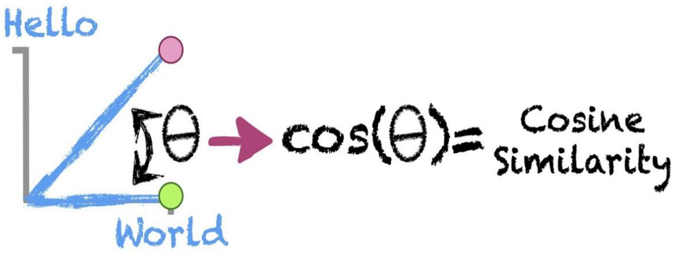


$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}$$

$$\|\vec{b}\| = \sqrt{b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2}$$

Cosine distance (angular)



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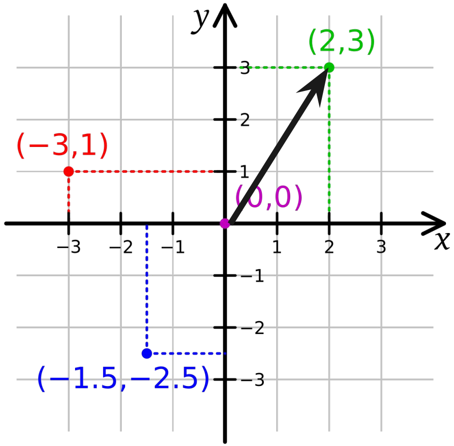
Cosine distance (angular)

- Distances: L1, L2, Cosine

- Cosine distance between vectors [2, 3] and [0, 0] is:

  0.00

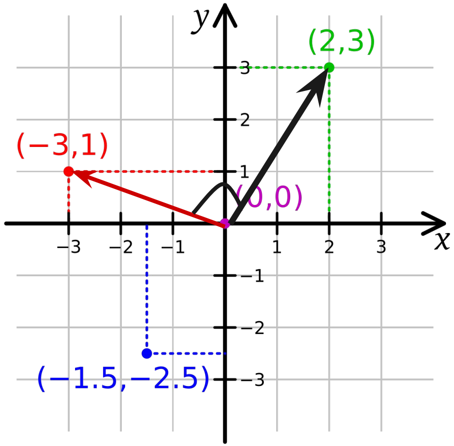


Cosine distance (angular)

- Distances: L1, L2, Cosine

  - Cosine distance between vectors [2, 3] and [-3, 1] is :

$$\frac{-3}{\sqrt{13}\sqrt{10}} = -0.26$$



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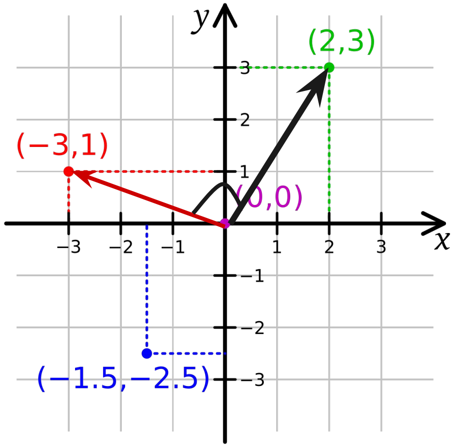
- Negative distance



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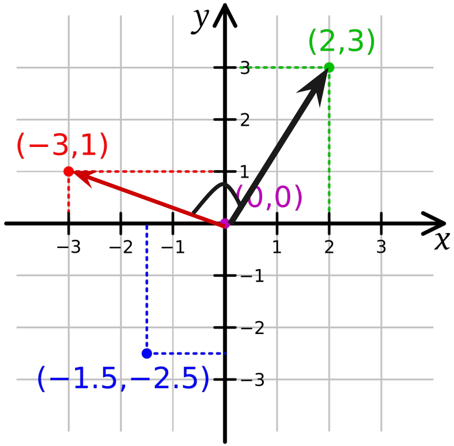
  - Negative distance, **which is unique**



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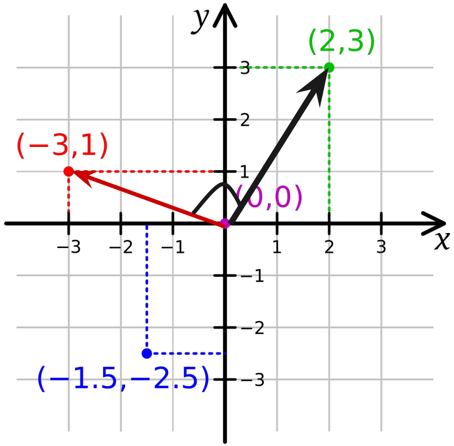

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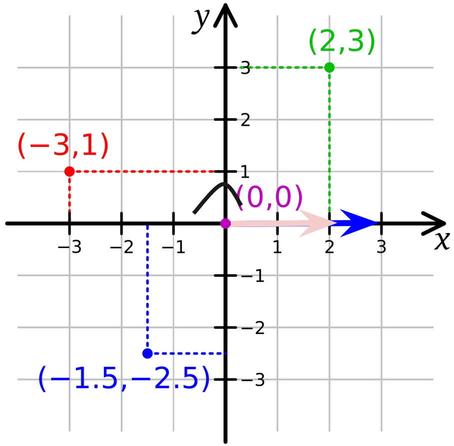
- Negative distance, **which is unique**

What's your interpretation of negative distances?

Cosine distance (angular)

- Distances: L1, L2, Cosine

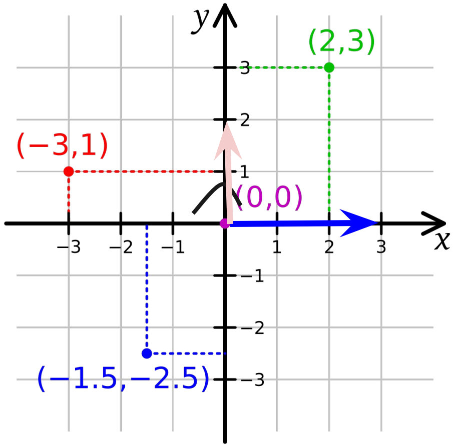
- Cosine distance range (-1, +1)
- Two proportional vectors (same direction) have a cosine similarity of 1; [3, 0], [2, 0]



Cosine distance (angular)

- Distances: L1, L2, Cosine

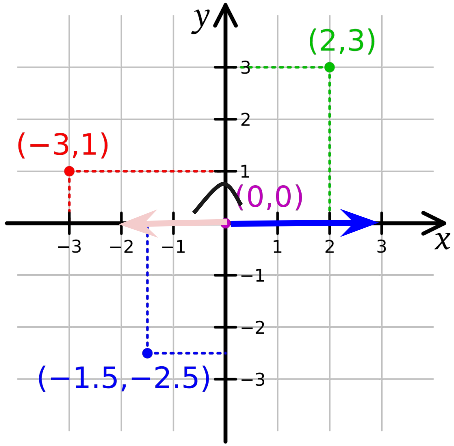
- Cosine distance range (-1, +1)
- Two orthogonal vectors have a cosine similarity of 0; [3, 0], [0, 2]



Cosine distance (angular)

- Distances: L1, L2, Cosine

- Cosine distance range (-1, +1)
- Two opposite vectors have a cosine similarity of -1; [3, 0], [-2, 0]



Break!

Basic Math - Concept of Vectors

We are aware of Scalars: A person’s

Height (1.72m)

Basic Math - Concept of Vectors

We are aware of Scalars: A person's

Height (1.72m)
Weight (72kg)

Basic Math - Concept of Vectors

We are aware of Scalars: A person’s

Height (1.72m)
Weight (72kg)
Salary (100K)

Basic Math - Concept of Vectors

We are aware of Scalars: A person's

Height (1.72m)
Weight (72kg)
Salary (100K)
....

Basic Math - Concept of Vectors

A closed form definition of a person through some features

[Height (1.72m), Weight (72kg), Salary (100K)]

Basic Math - Concept of Vectors

A closed form definition of a person through some features

- no explicit unit mentions

[1.72, 72, 100]

Basic Math - Concept of Vectors

A closed form definition of a person through some features

- no explicit unit mentions

[1.72, 72, 100]

Is a vectoried representation of the person through some attributes: height, weight,salary

Basic Math - Concept of Vectors

We are aware of Scalars: A person's height, weight, salary

1. Vectors

We begin by defining a mathematical abstraction known as a **vector space**. In linear algebra the fundamental concepts relate to the n-**tuples** and their algebraic properties.

Definition: An ordered n-**tuple** is considered as a sequence of n **terms** $(a_1, a_2, \cdots, a_n)$, where n is a positive integer.

We see that an ordered n-**tuple** has **terms** whereas a set has members.

Example: A sequence (5) is called an ordered 1-**tuple**. A 2-**tuple**, for example (3, 6) (where 6 appears after 3) is called an ordered pair, and 3-**tuple** is called an ordered triple. A sequence (9, 3, 4, 4, 1) is called an ordered 5-**tuple**.

Let us denote the set of all ordered 1-**tuples** of real numbers by $\mathbb{R}$. We will write for example $(3.5) \in \mathbb{R}$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Basic Math - Concept of Vectors

We are aware of Scalars: A person’s height, weight, salary

Physics vector: velocity (scalar value + direction)

Algebraic vector (in general): Common representation of an entity (1 to n dimension):

- A person’s (height, weight, salary), say [1.78, 72, 100] : once defined, we have to follow it.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

One hot encoding: Important DS/ML concept (an example application)

Basic Math - Vector Operations

Vector operation rules

1. $\mathbf{x}+\mathbf{y} \in \mathbb{R}^n$
2. $\alpha \cdot \mathbf{x} \in \mathbb{R}^n$
3. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x} \in \mathbb{R}^n$ (*commutativity*)
4. $\alpha \cdot (\mathbf{x}+\mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y}$ (*distributivity*)
5. $(\alpha+\beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$ (*distributivity*)
6. $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ (*associativity*)
7. $(\alpha\beta) \cdot \mathbf{x} = \alpha \cdot (\beta \cdot \mathbf{x})$ (*associativity*)

Basic Math - Vector Operations

Vector Operation

1.1.2. Vector Addition

Addition of vectors is defined:

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}=\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

Example:

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix} 2 \\ 6 \\ -5 \end{bmatrix}+\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}=\begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}$$

Basic Math - Vector Operations

Vector Operation

1.1.4. Zero Vector

The **zero** vector **sometimes denoted 0** is a vector having all elements equal to zero, e.g., the 2-dimensional **0** vector:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A.7}$$

Basic Math - Vector Operations

Vector Operation

1.1.9. Inner Product

The **inner** or **dot** product of two vectors **x** and **y of the same dimension** is a **scalar** defined by:

$$\mathbf{x}^T \cdot \mathbf{y} = (\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \tag{A.11}$$

Note that the inner product of vector **x** and **y** requires that a transposed vector **x** be multiplied by the **y** vector. Sometimes the inner product is denoted simply by juxtaposition of the vectors x and y, for example, as $< \mathbf{x}, \mathbf{y} >$ or $(\mathbf{x}, \mathbf{y})$.

Example: The inner product of two vectors $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

$$\mathbf{x}^T \mathbf{y} = [4\,1\,7]^T \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = 4 \cdot 0 + 1 \cdot 2 + 7 \cdot (-3) = 19$$

Basic Math - Vector Operations

Vector Operation

1.1.10. Orthogonal Vectors

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal** if their inner product is equal to zero

$$\mathbf{x}^T\mathbf{y} = 0 \tag{A.12}$$

here 0 is a scalar.

Example: Two vectors $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and are orthogonal, since their inner product is equal to zero

$$\mathbf{x}^T \cdot \mathbf{y} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}^T = \begin{bmatrix} 0\, 2 \end{bmatrix} = 4 \cdot 0 + 0 \cdot 2 = 0$$

Basic Math - Vector Operations

Vector Operation

1.1.11. Vector Norm

The magnitude of a vector may be measure in different ways. One method, called the vector **norm**, is a function from $\mathbb{R}^n$ into $\mathbb{R}$ for $\mathbf{x}$ an element of $\mathbb{R}^n$. It is denoted $||\mathbf{x}||$ and satisfies the following conditions:

1. $||\mathbf{x}|| \geq 0$, and the equality holds if and only if $\mathbf{x} = \mathbf{0}$
2. $||\alpha\mathbf{x}|| = |\alpha| \cdot ||\mathbf{x}||$, where $|\alpha|$ is the absolute value of scalar α

and is defined as:

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{A.13}$$

Example: For the vector $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ the norm is

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{4^2 + 3^2} = 5$$