CIS 263 Introduction to Data Structures and Algorithms

Designing Algorithms: Introduction to Complexity Analysis

Designing Algorithms

- **Incremental approach:** *Insertion Sort*
- **Divide and conquer:** *Merge Sort*

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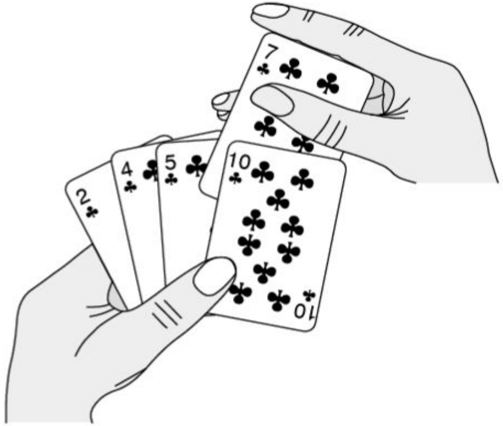
- **Divide and conquer:** Merge Sort



Insertion sort

- Start with the 2nd element
- Find it's position among all those are before it
- If needed to change position (and as identified the position) move others to the right (right shift)

https://www.geeksforgeeks.org/insertion-sort/

Designing Algorithms

- **Incremental approach:** Insertion Sort
- **Divide and conquer:** Merge Sort

6 5 3 1 8 7 2 4

How Efficient an Algorithm is?

Insertion Sort

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Complexity Analysis

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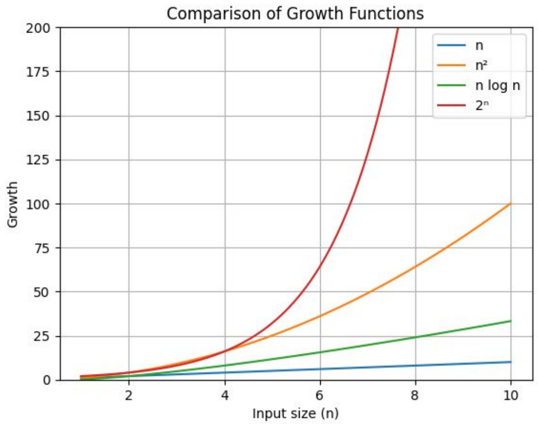
Complexity Analysis *- Concepts & Terminologies*

Growth functions describe how an algorithm’s **resource usage** (usually time or space)

- Increases as the **input size n** grows **asymptotic behavior** (what happens when *n* is large).
- They let us compare algorithms based on *scalability*, not on machine-dependent details.
- They ignore constants and low-order terms that don’t affect scalability



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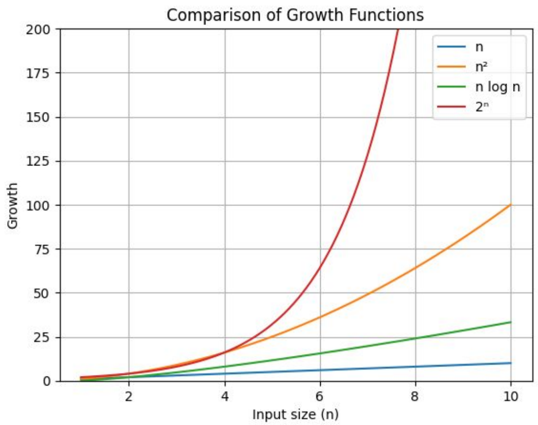
Example:

- $3n^2 + 5n + 20$ grows like n^2
- We say its growth function is **quadratic**



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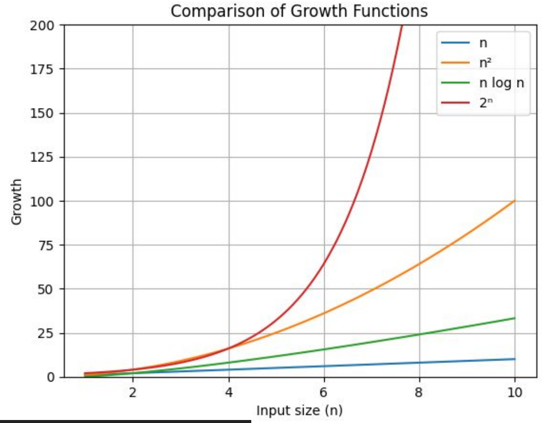

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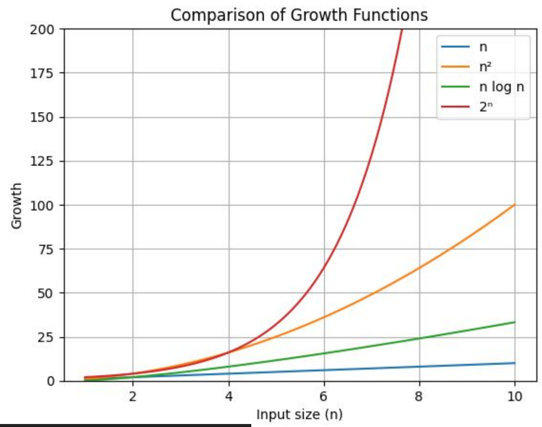

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Big-O: upper bound on growth (e.g., $O(n^2)$)

$$f(n) = 2n^2 + 3n + 1 \Rightarrow O(n^2)$$

Complexity Analysis *- Concepts & Terminologies*

Growth Function	Name	Example
$O(1)$	Constant	Array access
$O(\log n)$	Logarithmic	Binary search
$O(n)$	Linear	Linear search
$O(n \log n)$	Linearithmic	Merge sort
$O(n^2)$	Quadratic	Bubble sort
$O(n^3)$	Cubic	Matrix multiplication (basic)
$O(2^n)$	Exponential	Subset generation
$O(n!)$	Factorial	Traveling Salesman (brute force)

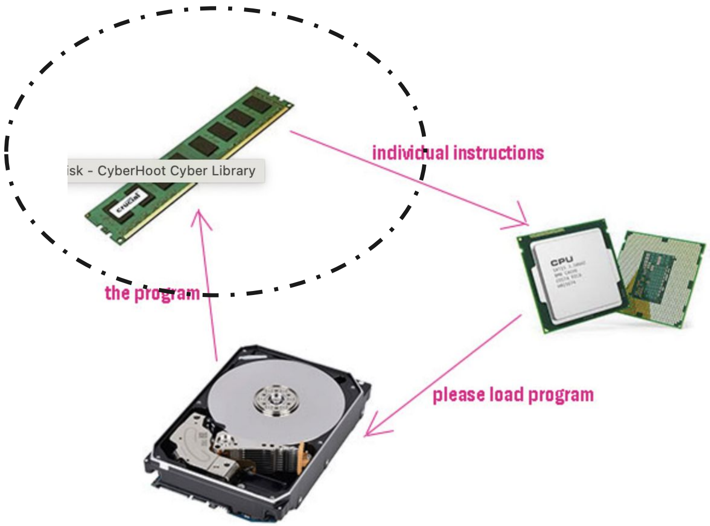
Example growth rates (from best to worst)

Complexity Analysis

- Space complexity
- Time Complexity

Space Complexity

- The **space Complexity of an algorithm** is the **total space taken by the algorithm** with **respect to the input size**.
- Space complexity includes **both Auxiliary space and space used by input**.



Space Complexity

- *Resource independent - Abstractions at the hardware (memory) level*

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Time Complexity or *Complexity (in General)*

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Complexity Analysis – Insertion Sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

Introduction to Algorithms, by Thomas Cormen et al.

- **Total cost**

Complexity Analysis – Insertion Sort

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}(t_j - 1) \\
&\quad + c_7\sum_{j=2}^{n}(t_j - 1) + c_8(n-1) . \\
&= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\
&= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) .
\end{aligned}
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- **Best case**

$an + b$ for constants a and b

- **Linear function**

Complexity Analysis – Insertion Sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

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- **Worst case**

  $an^2 + bn + c$ for constants a, b, and c

- **Quadratic function**

Complexity Analysis – Insertion Sort

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$$\boxed{\begin{aligned} \sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\ &= \Theta(n^2) . \end{aligned}}$$

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Complexity Analysis – Insertion Sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1)$$

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&\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\
&= \boxed{\left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2} + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n \\
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Upper Bound of the Algorithm Runtime

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Complexity Analysis – Insertion Sort

Introduction to Algorithms, by Thomas Cormen et al.

- Best case
- Worst case
- **Average case**

Sorting Algorithms – Worst-Case Time Complexity

Algorithm	Worst-Case Time	Notes
Bubble Sort	$O(n^2)$	Educational, very inefficient
Selection Sort	$O(n^2)$	Always $O(n^2)$
Insertion Sort	$O(n^2)$	Fast for small or nearly sorted data
Shell Sort	$O(n^2)$	Depends on gap sequence
Quick Sort	$O(n^2)$	Worst case with poor pivot selection
Tree Sort	$O(n^2)$	Occurs when tree becomes skewed
Bucket Sort	$O(n^2)$	Worst case when all items fall in one bucket

Algorithm	Worst-Case Time	Notes
Merge Sort	$O(n \log n)$	Stable; extra memory required
Heap Sort	$O(n \log n)$	In-place; not stable
Tim Sort	$O(n \log n)$	Python & Java default
Counting Sort	$O(n + k)$	Non-comparison; k = value range
Radix Sort	$O(nk)$	Non-comparison; k = digits
Bucket Sort	$O(n + k)$	Average case (worst can be quadratic)

Designing Algorithms

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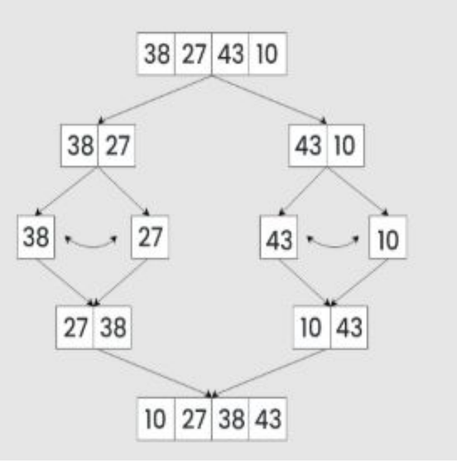

Recursion

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In mathematics, the **factorial** of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. The factorial of n also equals the product of n with the next smaller factorial:

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1 \\ &= n \times (n-1)! \end{aligned}$$

For example,

$$5! = 5 \times 4! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

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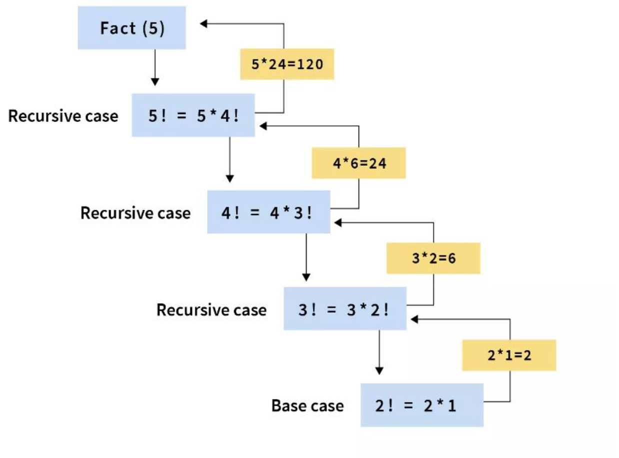

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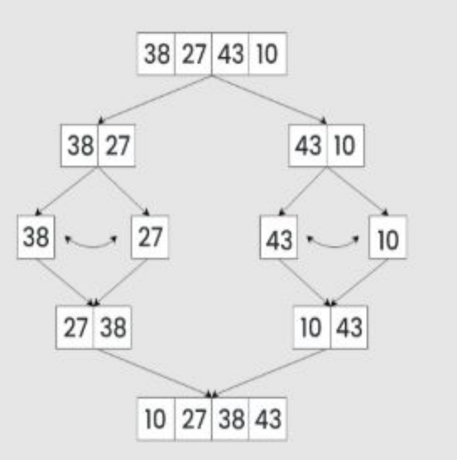

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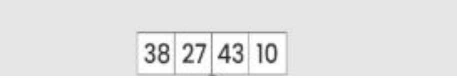

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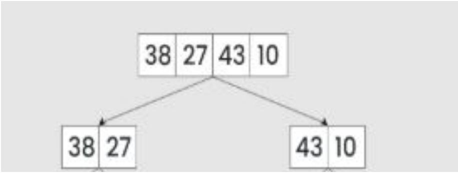

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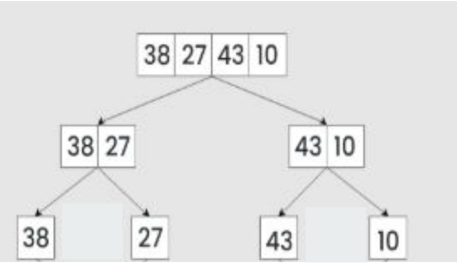

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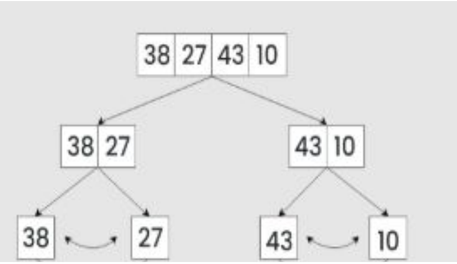

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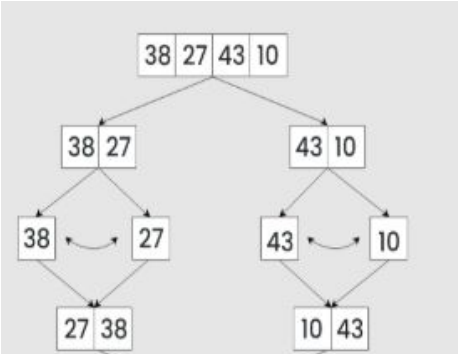

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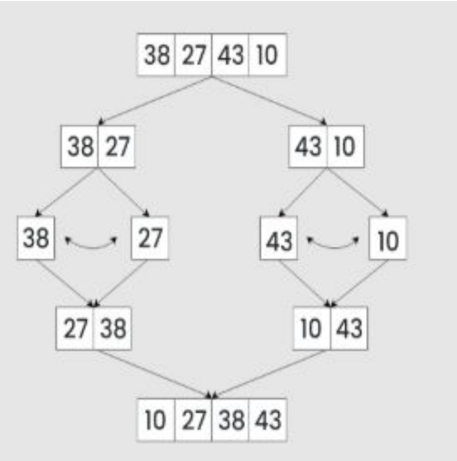

Merge Sort

Complexity

Worst Case Time Complexity [Big-O]: **O(n*log n)**

Best Case Time Complexity [Big-omega]: **O(n*log n)**

Average Time Complexity [Big-theta]: **O(n*log n)**

Comparing complexities

1K data points	Insertion sort: O(1K**2) ~ 1M Bubble sort: O(1K**2) ~ 1M Merge sort: O(1K * log (1K)) ~ 7K

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1K data points	Insertion sort: O(1K**2) ~ 1M Bubble sort: O(1K**2) ~ 1M Merge sort: O(1K * log (1K)) ~ 7K
1000K (1M) data points	Insertion sort: O(1M**2) ~ 1 Trillion Bubble sort: O(1M**2) ~ 1 Trillion Merge sort: O(1M * log (1M)) ~ 14M