CIS 678 Machine Learning

ML Introduction: Linear Regression (part 3)

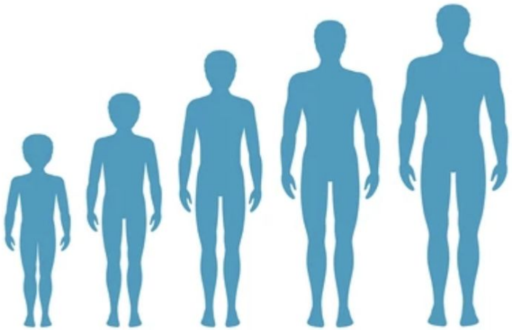
What we'd like to accomplish today

- ~~General Concepts: ***Straight Line** to **Linear Regression***~~
- Gradient Descent Algorithm
  - ➜ A simple two parameter **Linear Regression** model
  - ➜ Hands on **Notebook implementation**
- QA

We will be learning today about Regression

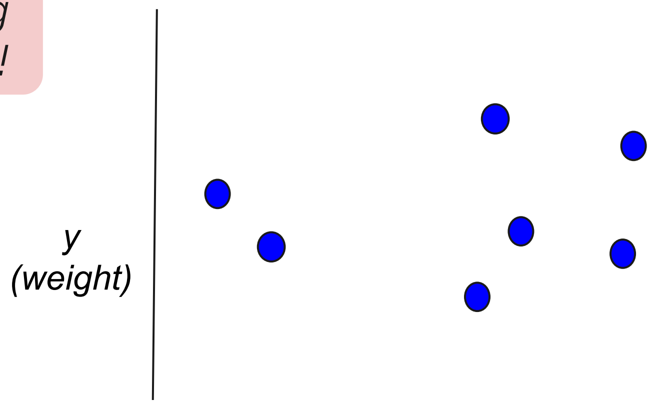
- *Person's weight :* $y \in R$

$$f\,(y|x=height)$$



Select one among a set of Models!

Select one among infinite (∞) Models!



Model, θ: $\{\beta_0, \beta_1\}$

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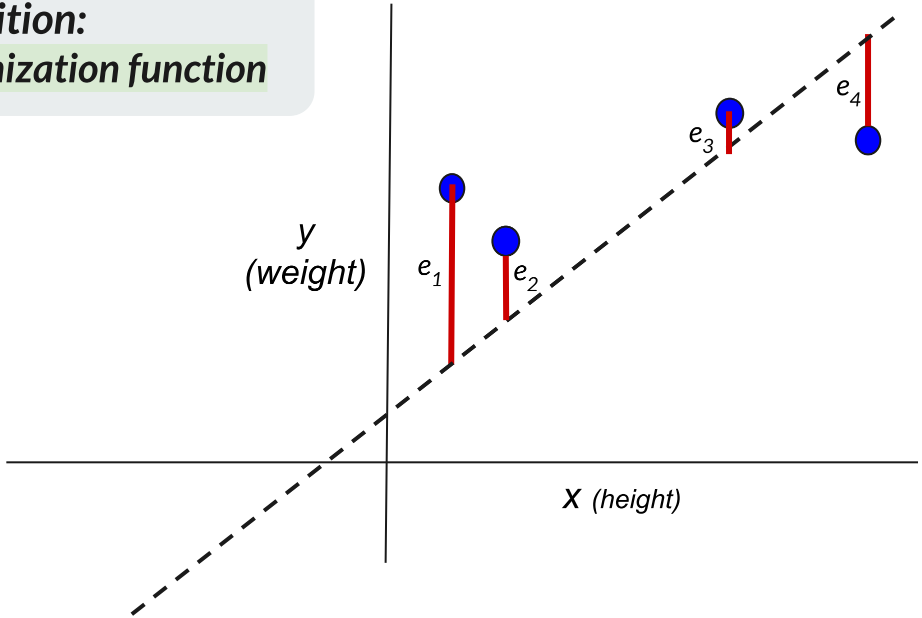
________ ***How to Generalize the Idea for***

- any number of points, and/or

- any models (remember, we have infinite number of possible models)

Linear Regression

Definition:
Optimization function



Prediction function & Model

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\Theta = \{\beta_0, \beta_1\}$$

Fitting Error

$$\epsilon = |\hat{y} - y|$$

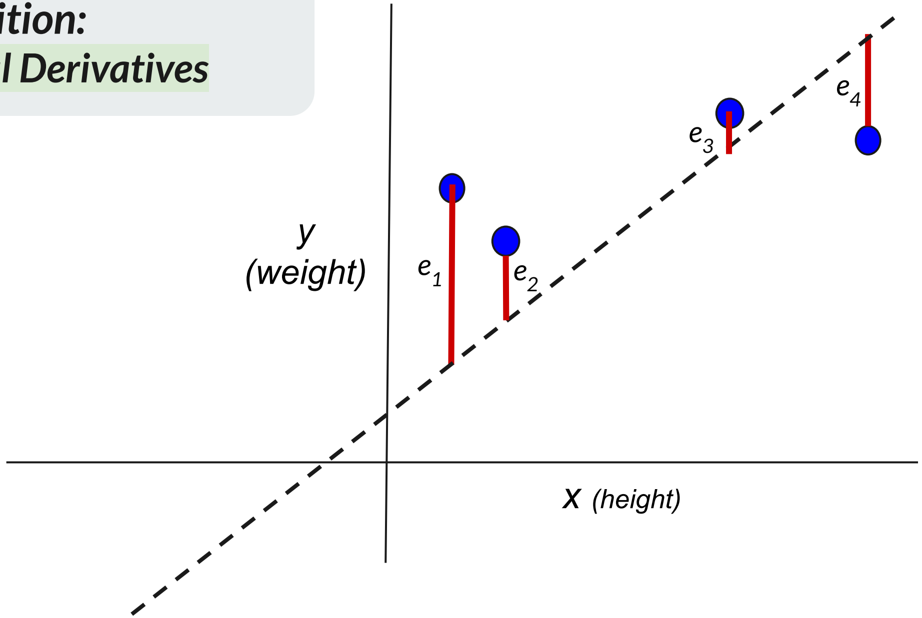
Optimization/loss/error function

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

$$\Theta^* = \text{argmin}_\Theta\, E\{(x_i, y_i)\}_{i=1,\cdots,N}$$

Linear Regression

Definition:
Partial Derivatives



- **Optimization/Loss Function:**

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\beta_0 + \beta_1 x_i - y_i)^2$$

- **Gradient w.r.t.** β_0**:**

$$\frac{\partial E_\Theta}{\partial \beta_0} = \sum_{i=1}^{N}(\beta_0 + \beta_1 x_i - y_i)$$

- **Gradient w.r.t.** β_1**:**

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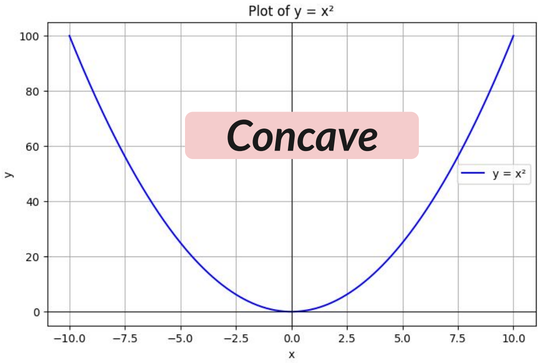
Linear Regression

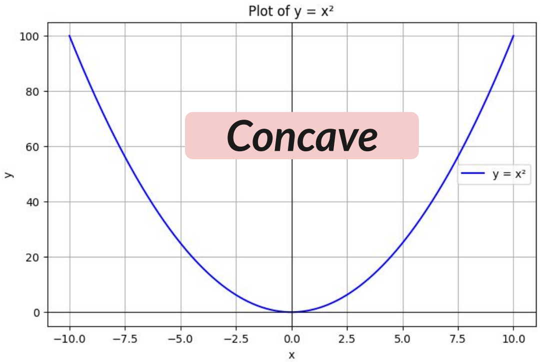
Quadratic functions



$$y = x^2$$

Linear Regression

Quadratic functions



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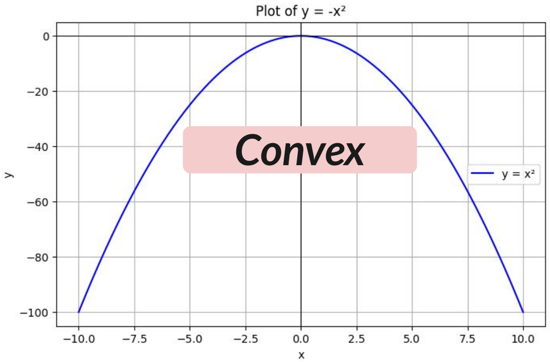
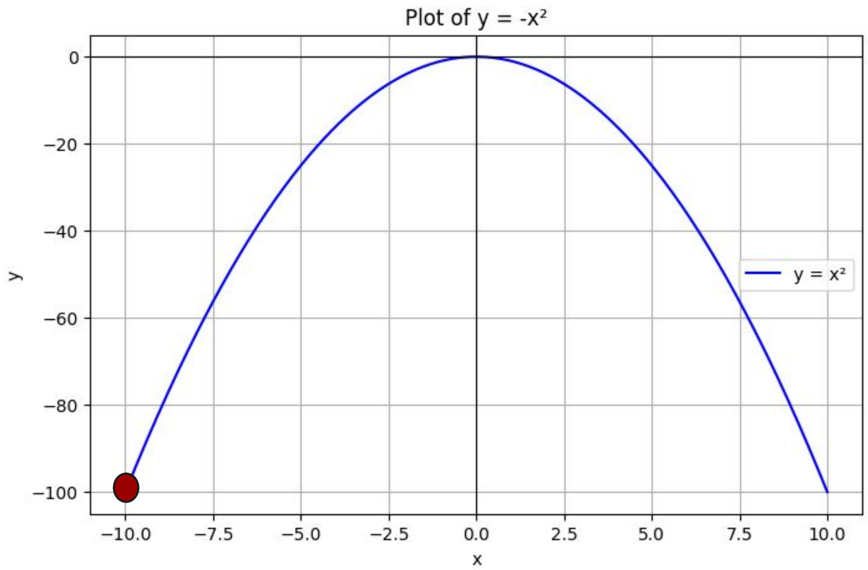


$$y = -x^2$$

Linear Regression

Gradient ascent - General idea:

- *One wants to reach the top starting from the red point*



$y = -x^2$ (Quadratic function)

Linear Regression

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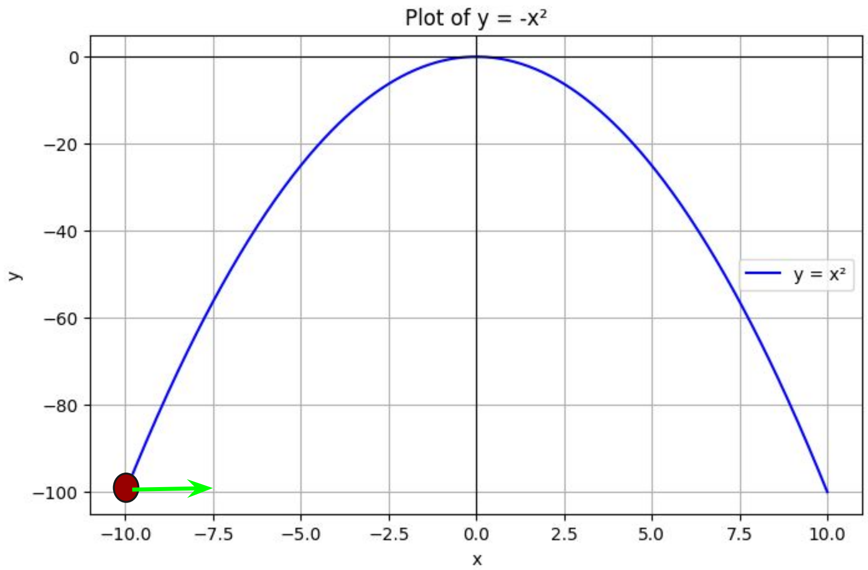

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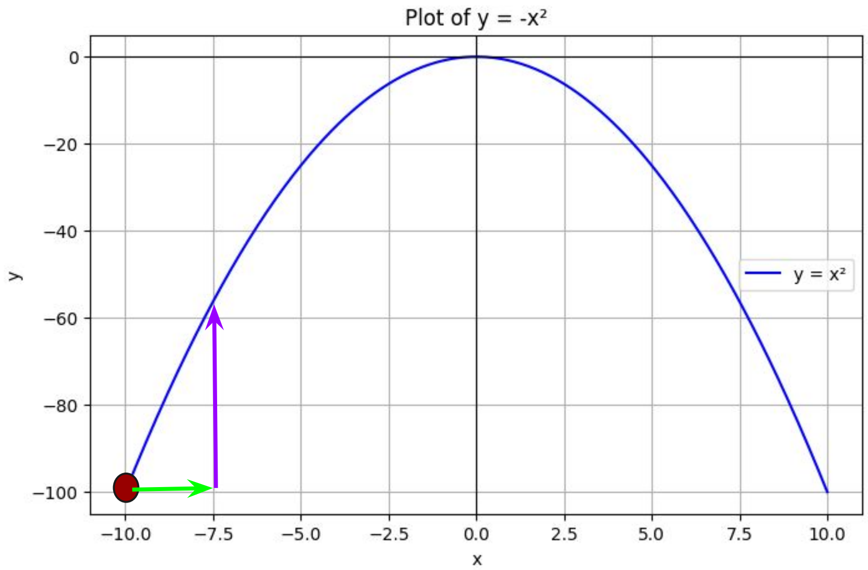

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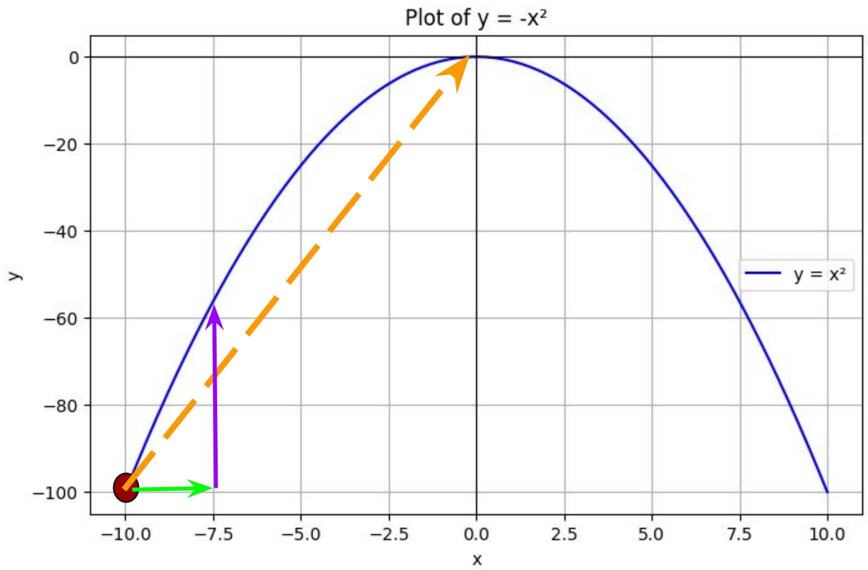

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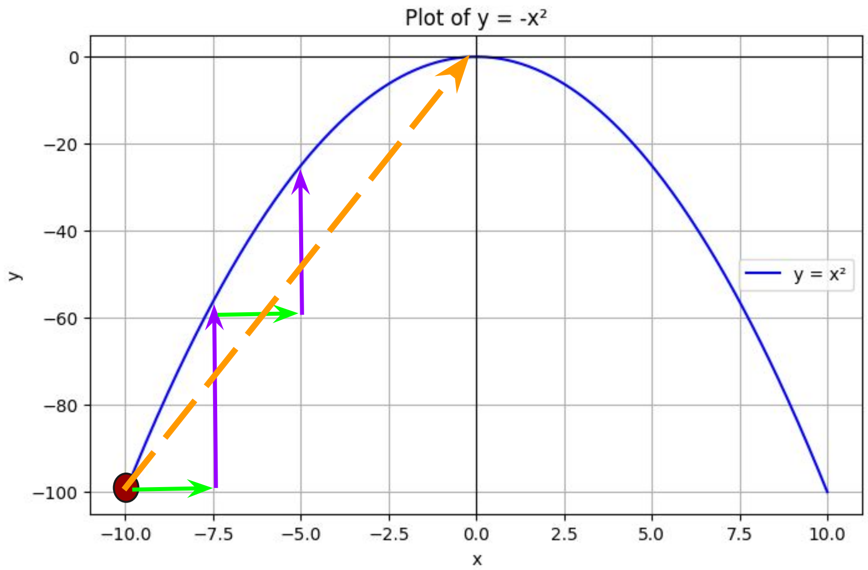

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- *Usually until f(x) ~ 0*
- *The width of each stair is called 'step-size'/'learning-rate in many ML algorithms.*
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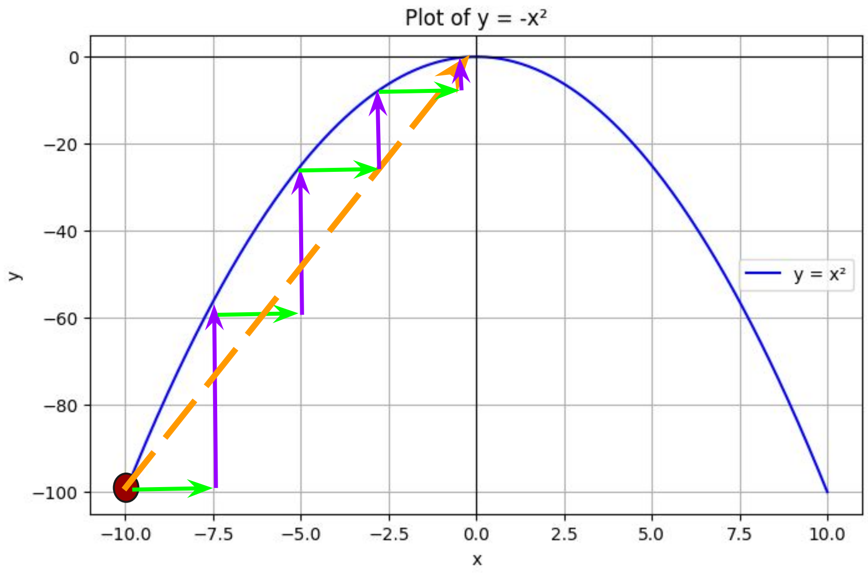

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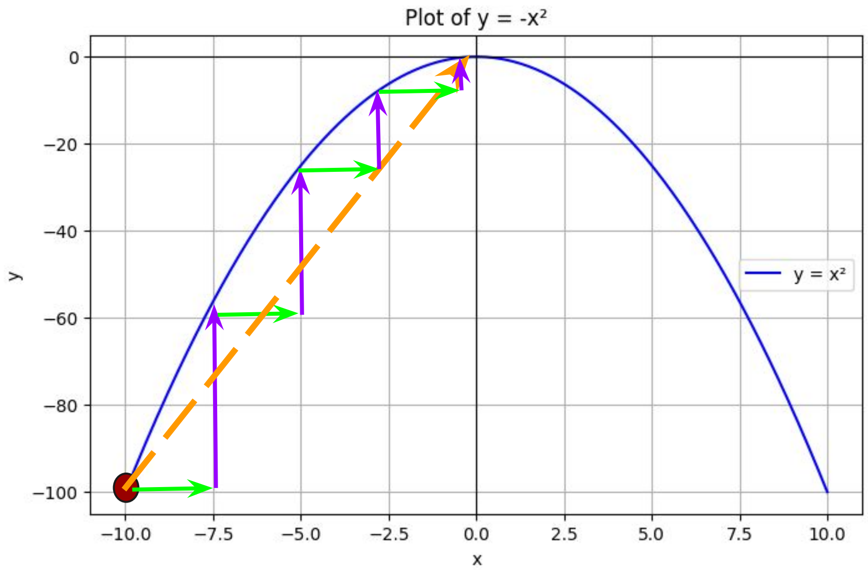

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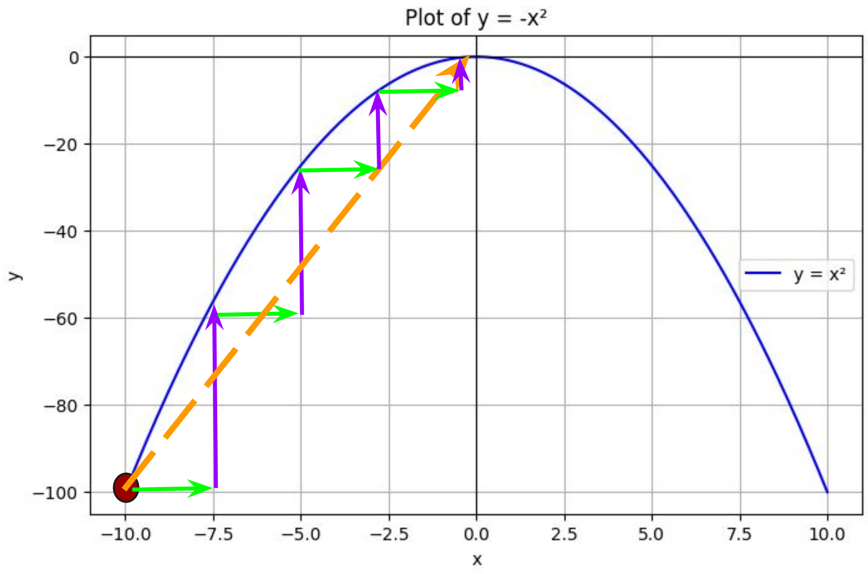

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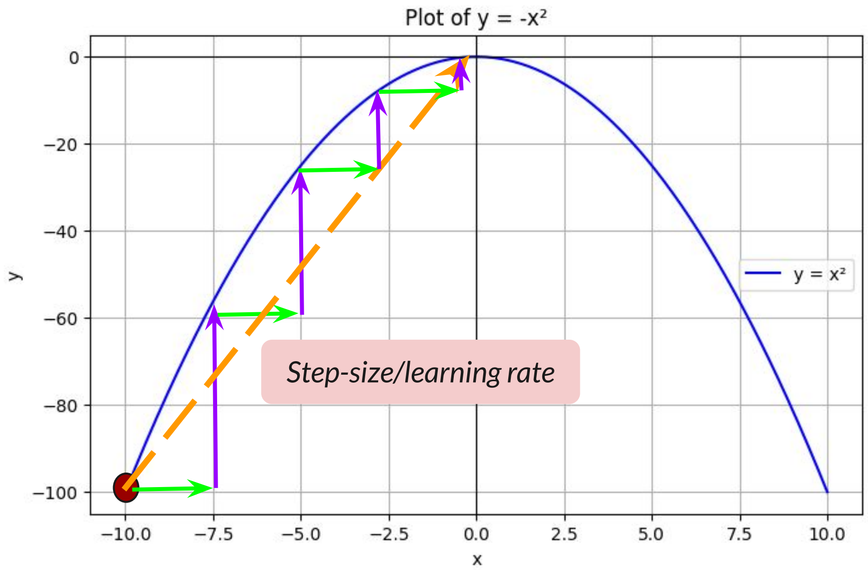

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Questions

Linear Regression

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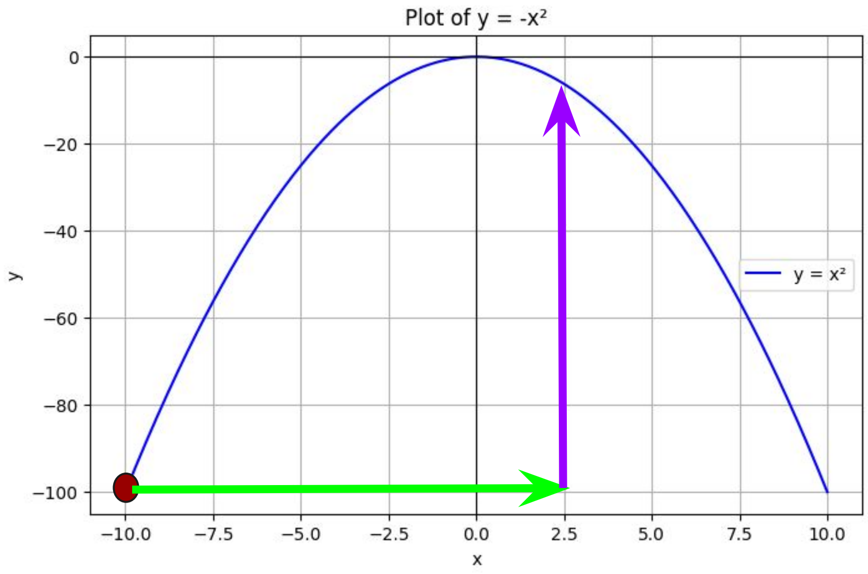

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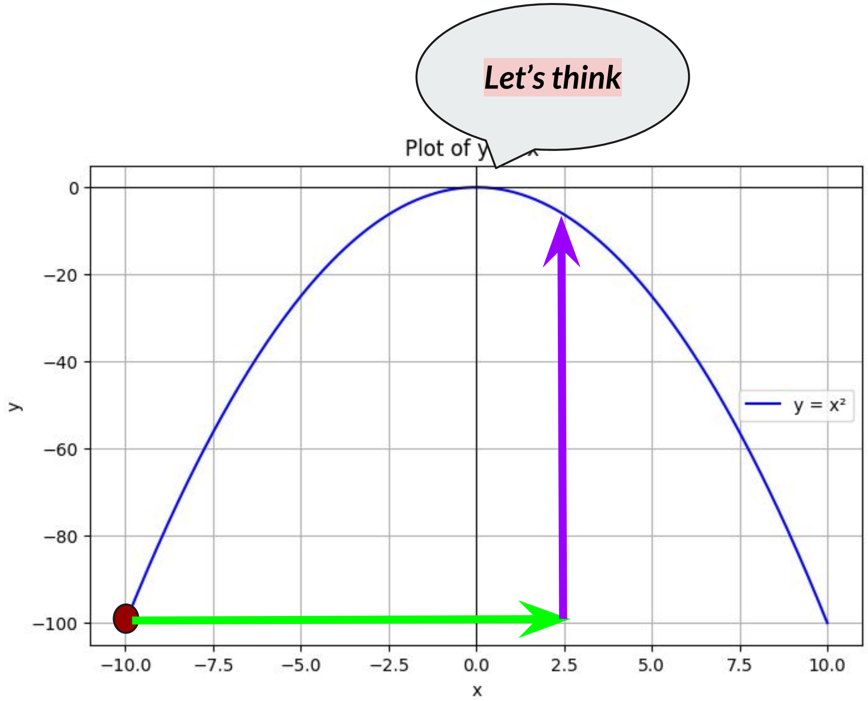

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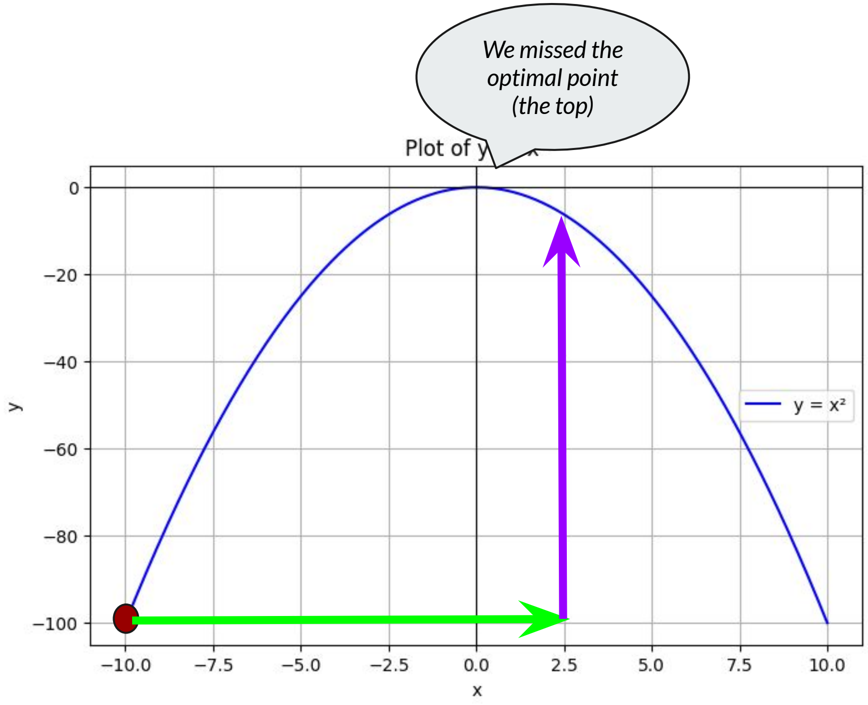

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Another Question!

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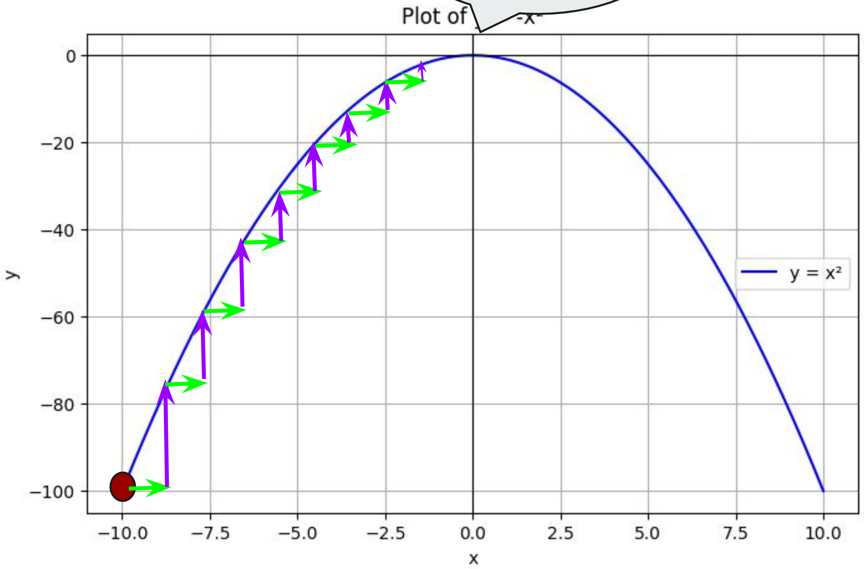
Let's think



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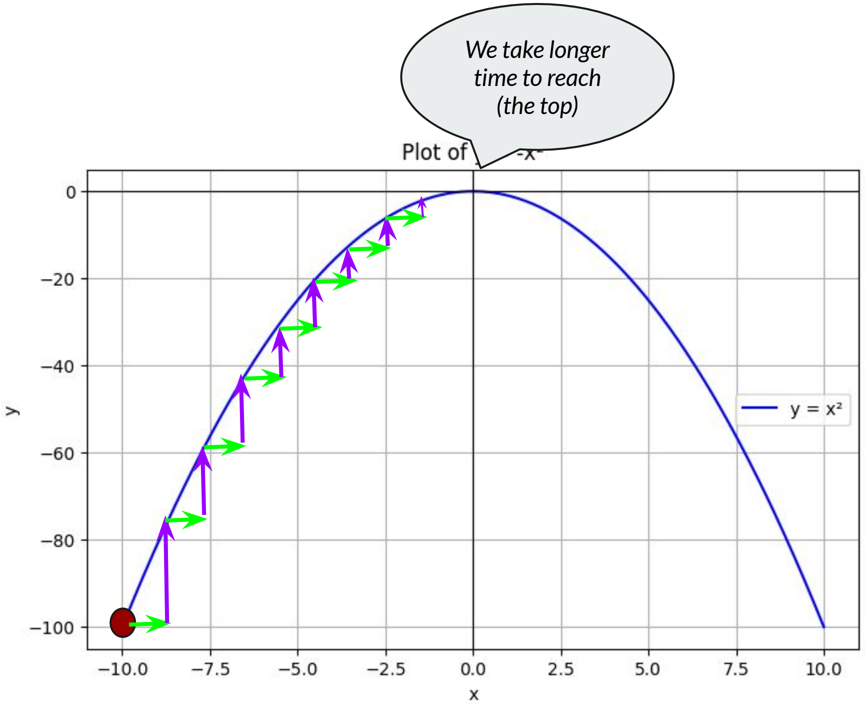

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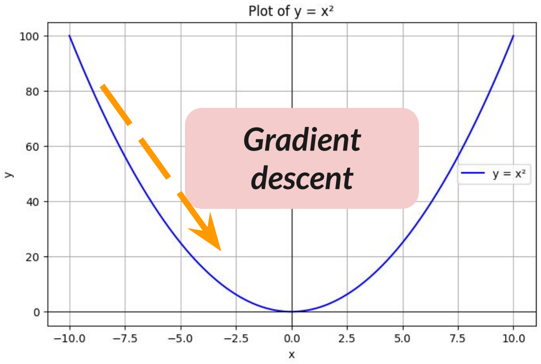
Now we are ready for the



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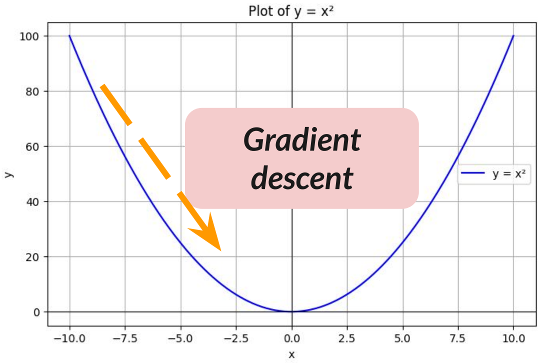

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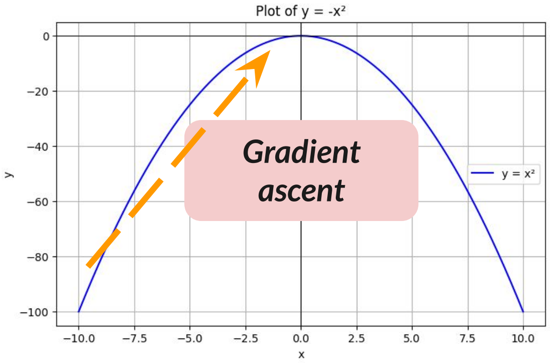

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Gradient Descent Algorithm

Gradient descent (- ascent)

1. Start with an initial ***(β_0, β_1)*** and a ***learning rate (L)*** , a scalar, which controls the gradient step.
2. For ***N*** training data points, estimate the **model loss**
3. Estimate the gradient (vector of partial derivatives): $\nabla E_\Theta = \left[\frac{\delta}{\delta \beta_0}(E_\Theta), \frac{\delta}{\delta \beta_1}(E_\Theta)\right]$
4. Update parameters

$$\beta_0 \leftarrow \beta_0 - L * \frac{\delta}{\delta \beta_0}(E_\Theta)$$

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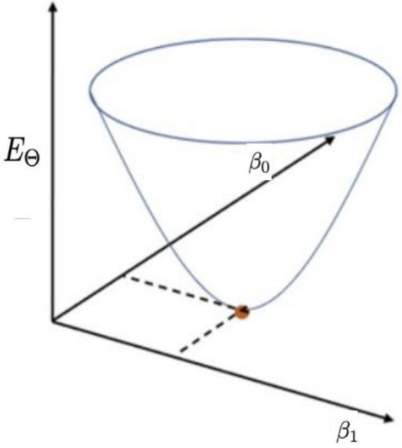
4. Go to step 2) and iterate until the model loss reaches a predefined **threshold** or a certain **number of iterations** are executed.



- **Optimization/Loss Function:**

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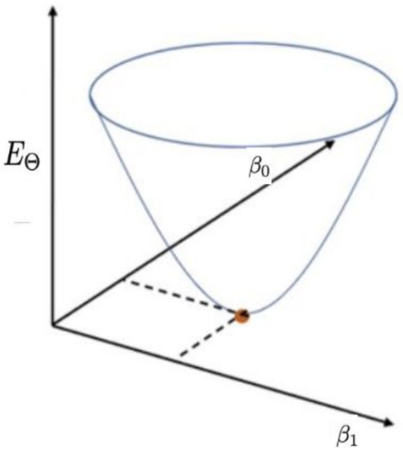

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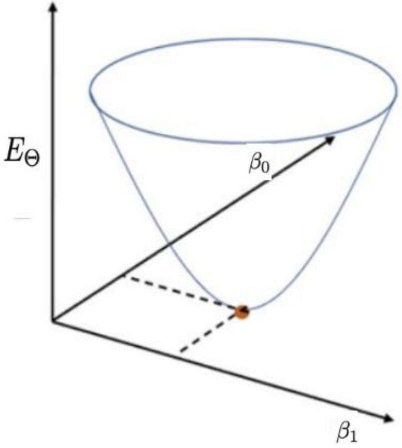

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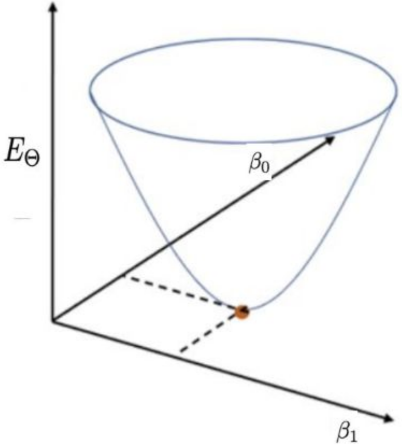

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Batch Gradient descent (- ascent)

Gradient descent (- ascent) Variants

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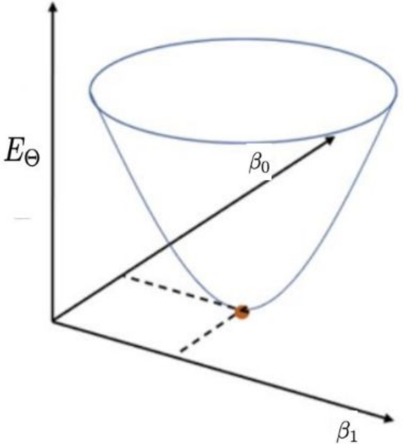

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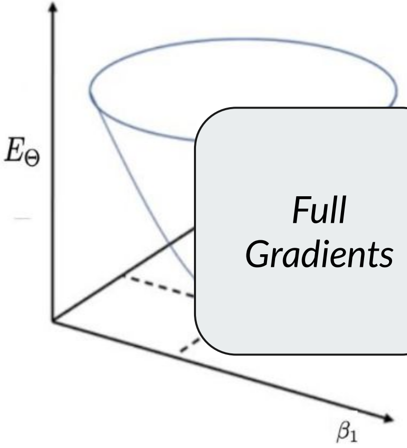

Full Gradients

- **Optimization/Loss Function:**

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\beta_0 + \beta_1 x_i - y_i)^2$$

- **Gradient w.r.t. β_0:**

$$\frac{\partial E_\Theta}{\partial \beta_0} = \sum_{i=1}^{N}(\beta_0 + \beta_1 x_i - y_i)$$

- **Gradient w.r.t. β_1:**

$$\frac{\partial E_\Theta}{\partial \beta_1} = \sum_{i=1}^{N}(\beta_0 + \beta_1 x_i - y_i)\, x_i$$

Full Gradients

Gradient descent (- ascent) Variants

1. Start with an initial **_(β_0, β_1)_** and a **_learning rate (L)_** , a scalar, which controls the gradient step.
2. For ***N*** training data points, estimate the **model loss**
3. Estimate the gradient (vector of partial derivatives): $\nabla E_\Theta = \left[\frac{\delta}{\delta\beta_0}(E_\Theta), \frac{\delta}{\delta\beta_1}(E_\Theta)\right]$

4. Update parameters

$$\beta_0 \leftarrow \beta_0 - L * \frac{\delta}{\delta\beta_0}(E_\Theta)$$

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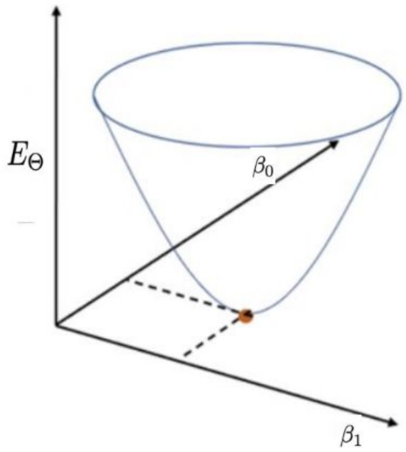

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$$\frac{\partial E_\Theta}{\partial \beta_1} = \sum_{i=1}^{N}(\beta_0 + \beta_1 x_i - y_i)\, x_i$$

Batch Gradient descent: *N=n (random) and n< N*

Gradient descent (- ascent) Variants

1. Start with an initial ***(β_0, β_1)*** and a ***learning rate (L)*** , a scalar, which controls the gradient step.
2. For ***N*** training data points, estimate the **model loss**
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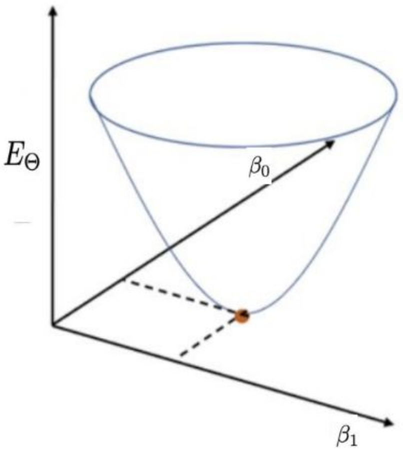

- **Optimization/Loss Function:**

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$$\frac{\partial E_\Theta}{\partial \beta_0} = \sum_{i=1}^{\boxed{N}}(\beta_0 + \beta_1 x_i - y_i)$$

- **Gradient w.r.t.** β_1**:**

$$\frac{\partial E_\Theta}{\partial \beta_1} = \sum_{i=1}^{\boxed{N}}(\beta_0 + \beta_1 x_i - y_i)\, x_i$$

Stochastic Gradient descent: *N=1 (random)*

Notebook presentation

- Notebook github
  - ➜ [Gradient descent training (Linear Regression)]

GPT

Our model today

What we have discussed today

- ~~General Concepts: ***Straight Line** to **Linear Regression***~~
- Gradient Descent Algorithm
  - ➜ A simple two parameter **Linear Regression** model
  - ➜ Hands on **Notebook implementation**
- **QA …….**