CIS 678 - Machine Learning

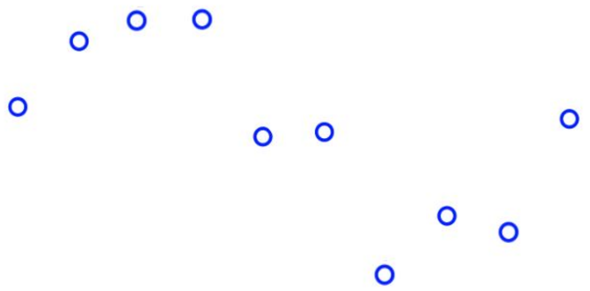
- Linear to Polynomial Regression
- Model Regularization

Plan

- LR to Polynomial Regression
- Regularization
  - Theory
  - Practical - Notebook presentation

Non linear data/function

- Does this data points seem familiar matching a known function?



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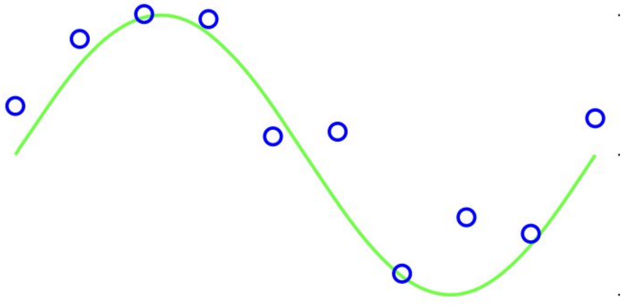
    - A Sinusoidal function

$$y(t) = A\sin(\omega t + \varphi) = A\sin(2\pi f t + \varphi)$$



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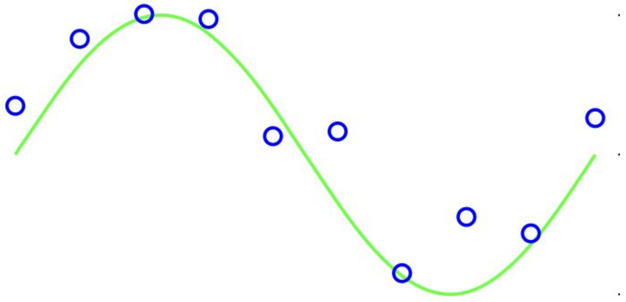
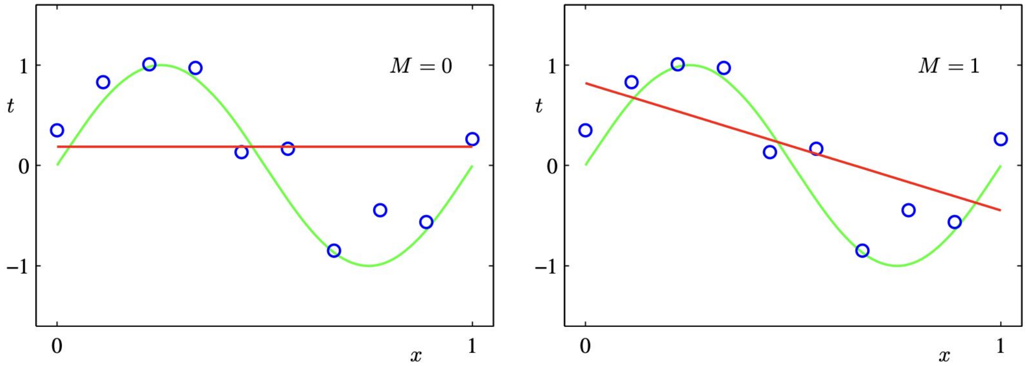


Clearly this is not a linear function; right?

Non linear data/function

- Does this data points seem familiar matching a known function?
- Can we approximate this function using LR ?



$$\hat{y} = \beta_0 + \beta_1 x$$

LR will not work; right?

What no-linear functions we are aware of?

- Can you recall any nonlinear function you learned at your high school/colleges?

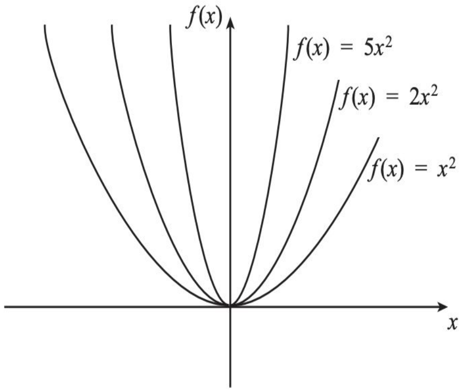
ref

What no-linear functions we are aware of?

- Can you recall any nonlinear function you learned at your high school/colleges?
- **Quadratic (x^2)**

$$f(x) = x^2, \qquad f(x) = 2x^2, \qquad f(x) = 5x^2.$$

What is the impact of changing the coefficient of x^2 as we have done in these examples? One way to find out is to sketch the graphs of the functions.



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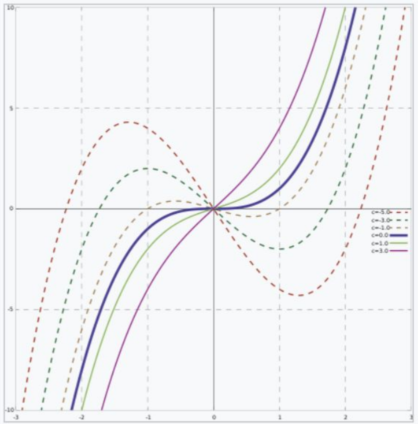
- **Cubic (x^3)**



Cubic functions of the form
$y = x^3 + cx.$
The graph of any cubic function is similar to such a curve.

[ref](ref)

What no-linear functions we are aware of?

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- Quadratic (x^2)
- Cubic (x^3)
-

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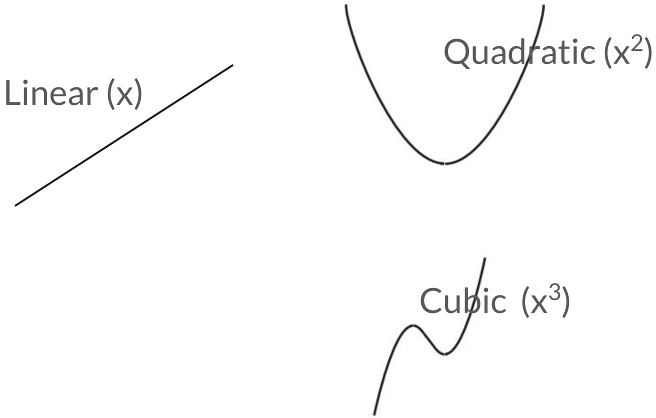

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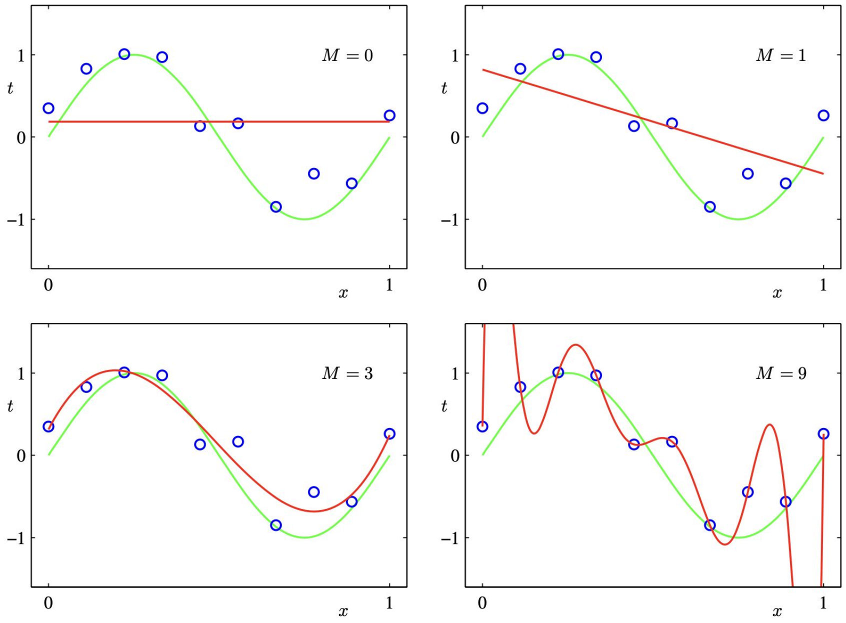
LR to Polynomial Regression

- Polynomial function
  - ***M* is the order/degree of polynomial ..**



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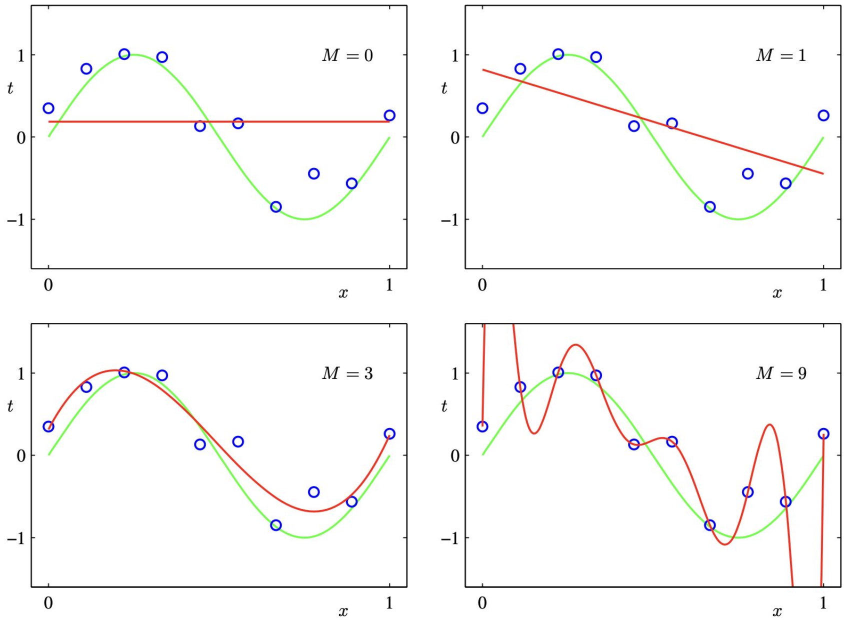
LR to Polynomial Regression

- Polynomial function
  - *M* is the order/degree of polynomial ..
  - **Where to stop? What is the best *M*?**



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Good news is our gradient descent (iterative learning) remains the same!

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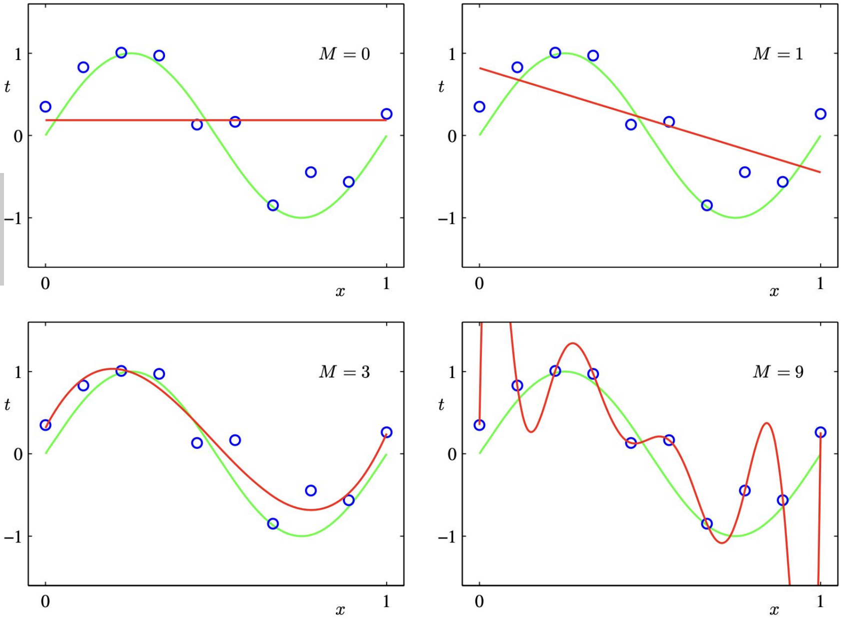

LR to Polynomial Regression

- Polynomial function
  - M is the order ..
  - **Where to stop? What is the best M?**

- Good news is our gradient descent (iterative learning) remains the same!
- You only need to change your objective function (from LR to Polynomial LR)

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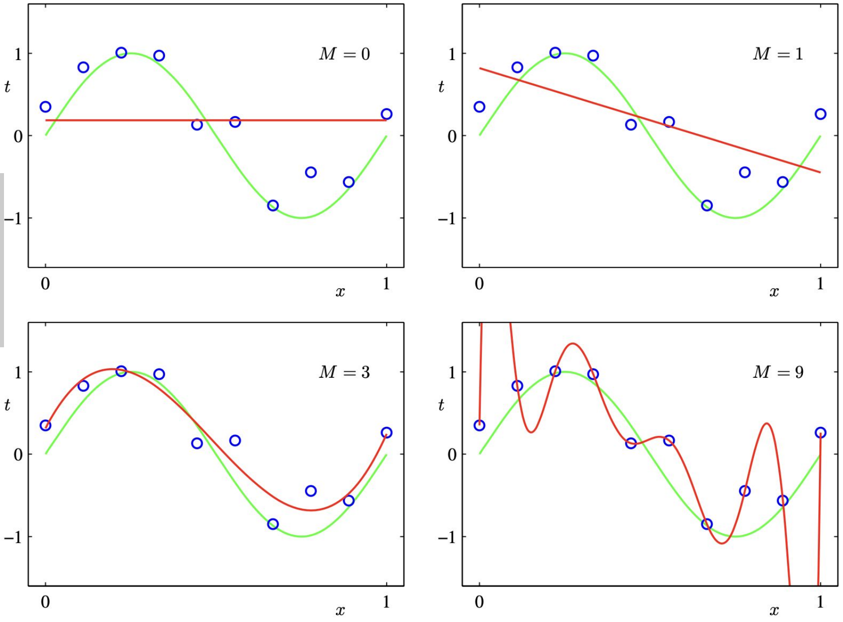

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Model

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\Theta = \{\beta_0, \beta_1\}$$

$$\epsilon = |\hat{y} - y|$$

Optimization function

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

$$\Theta^* = \mathrm{argmin}_\Theta\, E\{(x_i, y_i)\}_{i=1,\cdots,N}$$

Our model got a little bigger: 2 params to *M* param

GPT

I know one of your tricks; get you soon!!

Our model yesterday

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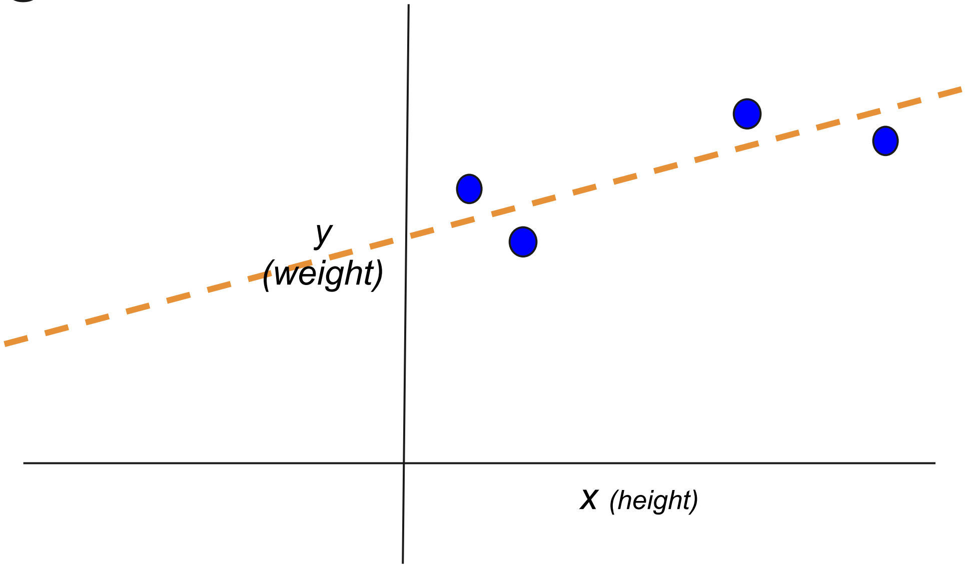
Our model today

Regularization

Regularization



So, essentially we are fitting a function; right?

Model

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\Theta = \{\beta_0, \beta_1\}$$

Fitting Error

$$\epsilon = |\hat{y} - y|$$

Optimization function

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

$$\Theta^* = \text{argmin}_\Theta\, E\{(x_i, y_i)\}_{i=1,\cdots,N}$$

Regularization

Model

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\Theta = \{\beta_0, \beta_1\}$$

$$E(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{y(x_n, \mathbf{w}) - t_n\}^2$$

$$\epsilon = |\hat{y} - y|$$

Same model, two different notations

Optimization function

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

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Regularization

$$E(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{y(x_n, \mathbf{w}) - t_n\}^2$$

Essentially, the same formulation

Generally **ML** vs **Math** conventions

Model

$$\hat{y} = \beta_0 + \beta_1 x$$

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$$\epsilon = |\hat{y} - y|$$

Optimization function

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

$$\Theta^* = \operatorname{argmin}_\Theta E\{(x_i, y_i)\}_{i=1,\cdots,N}$$

Regularization

$$E(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{y(x_n, \mathbf{w}) - t_n\}^2$$

x : scalar
$\boldsymbol{x}$, $\mathbf{x}$: vector
$\mathbf{X}$: Matrix

Model

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\Theta = \{\beta_0, \beta_1\}$$

$$\epsilon = |\hat{y} - y|$$

Optimization function

$$E_\Theta = \frac{1}{2}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

Essentially, the same formulation

Generally ML vs Math conventions

$$W^* = \text{argmin}_W\, E\{(x_i, t_i)\}_{i=1,\cdots,N}$$

$$\Theta^* = \text{argmin}_\Theta\, E\{(x_i, y_i)\}_{i=1,\cdots,N}$$

Regularization

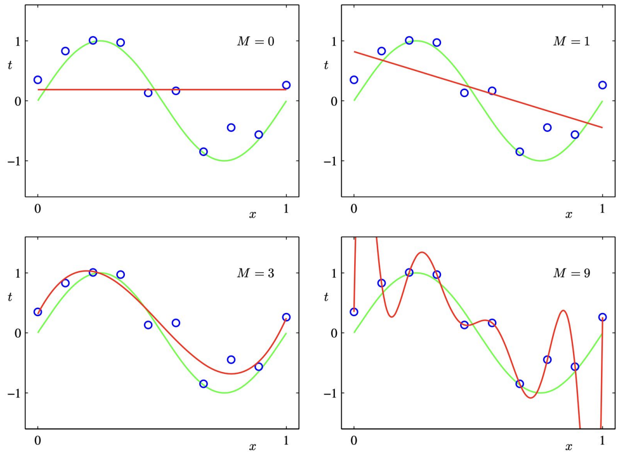

Table 1.1 Table of the coefficients $\mathbf{w}^\star$ for polynomials of various order. Observe how the typical magnitude of the coefficients increases dramatically as the order of the polynomial increases.

	$M = 0$	$M = 1$	$M = 6$	$M = 9$
$w_0^\star$	0.19	0.82	0.31	0.35
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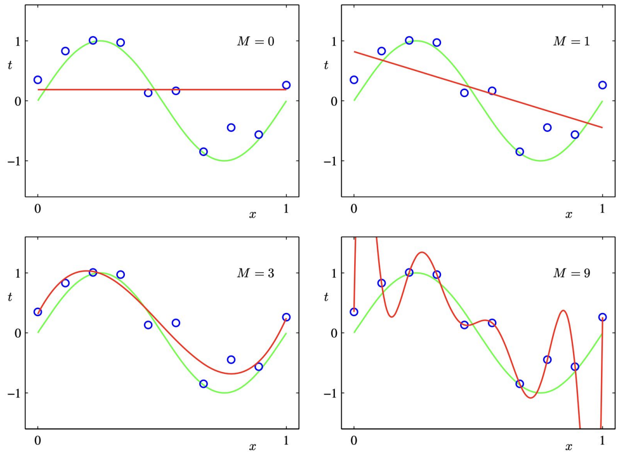
Regularization



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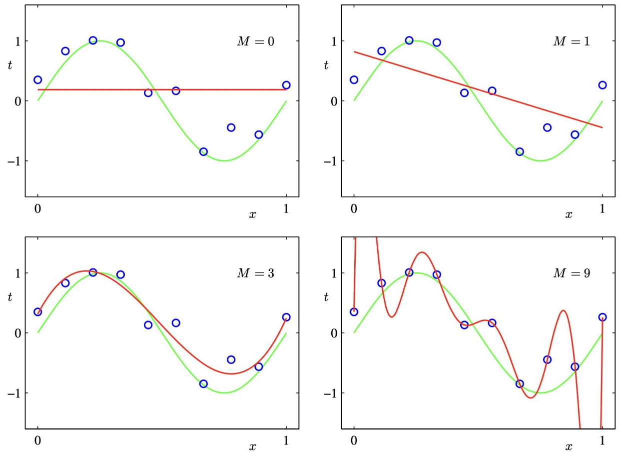
Absolute values are increasing

Regularization



$$E(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{y(x_n,\mathbf{w}) - t_n\}^2$$

Regularizer

$$\widetilde{E}(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{y(x_n,\mathbf{w}) - t_n\}^2 + \frac{\lambda}{2}\|\mathbf{w}\|^2$$

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Abs values
Are increasing

Regularization

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$||\mathbf{W}||^2$

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decreasing

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Abs values
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||W||^2

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How to control this?

Regularization

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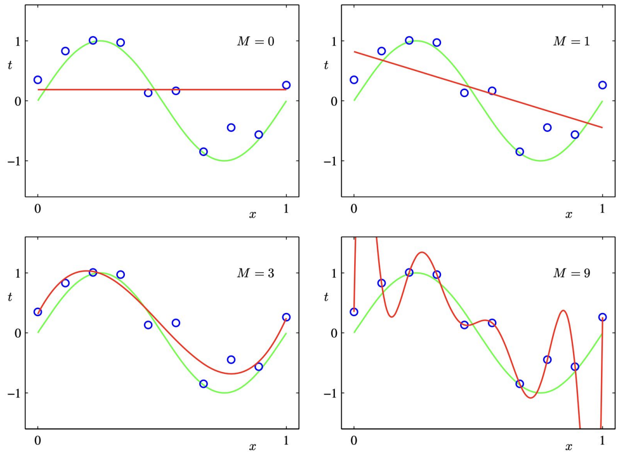
Who to control this?

Lamda is called the
Hyper Parameter of this model

Polynomial Regression with Regularization



Learned function is **nonlinear**

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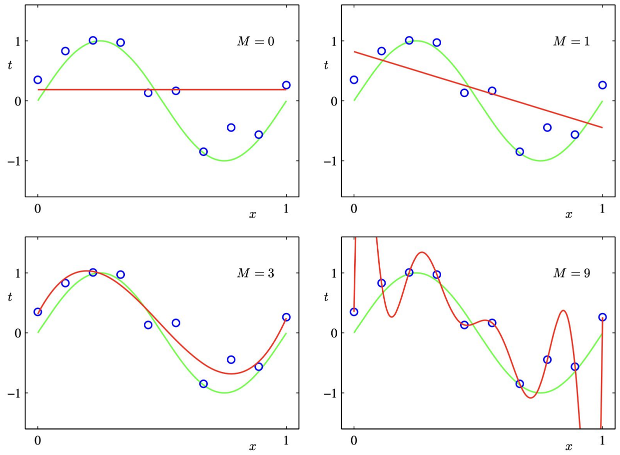
$$\widetilde{E}(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^{N} \{y(x_n, \mathbf{w}) - t_n\}^2 + \frac{\lambda}{2} \|\mathbf{w}\|^2$$

Polynomial Regression with Regularization



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Model (still) **linear**

Classification

- General Idea (two steps process)
  - LR (Bias Only)
  - LR (general)

Notebook presentation

- Without regularizer
- With regularizer

Predictive modeling: Regression (diabetes)

Predictive modeling: Classification