CIS 678 - Machine Learning

Basics of Probability

Basic Probability - Data Science

Let us assume an experiment in which one possible outcome is an event denoted by A. The uncertain nature of the experiment is such that event A may – or may not – occur when the experiment is performed. We designate the probability that A will occur by $P(A)$. Let $n_{repetition}$ denote the number of times the experiment is performed and $n_{occurrence}$ denote the number of times A occurs during the $n_{repetition}$ executions of the experiment. With $n_{occurrence}$ and $n_{repetition}$ we can designate the likelihood that event A will occur by the following proportion:

$$\frac{n_{occurrence}}{n_{repetition}} \tag{B.1}$$

To interpret a probability, let us assume the experiments are repeated an infinite number of times. As the number of experiments increases, the ratio $\frac{n_{occurrence}}{n_{repetition}}$ converges to the probability of A. Hence,

$$P(A) = \lim_{n_{repetition} \to \infty} \frac{n_{occurrence}}{n_{repetition}} \tag{B.2}$$

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Example: Let us consider an experiment that involves tossing a coin. The possible outcomes (results) of a coin-toss experiment are a **head** and a **tail**. Since an impartial observer cannot analyze or control the experiment conditions or predict the outcome, the results of tossing the coin are considered random. If we designate to A a **head** outcome for our coin-toss experiment, we assign to $n_{occurrence}$ the number of times A (a **head** outcome) occurs as we toss the coin. As the number of tosses approaches infinity, the proportion of **heads** to the number of repetitions, $\frac{n_{occurrence}}{n_{repetition}}$, approaches 0.5. Thus we conclude that the probability of a **head** occurring, $P(A)$, is 0.5. From Equation (B.2) we get:

$$P(A) = \lim_{n_{repetition} \to \infty} \frac{n_{occurrence}}{n_{repetition}} = 0.5$$

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3. Probability Axioms

The probability $P(A)$ of an event A to occur, as a measure of uncertainty, takes a real number value from the range $[0, 1]$.

The **system of probabilities** assigns probabilities $P(A)$ to events A based on three **probability theory axioms**.

Axiom 1. The probability is greater or equal to zero and less or equal to 1

$$0 \leq P(A) \leq 1 \quad for \quad each\ event\ A \tag{B.3}$$

The probability cannot be negative.

Axiom 2. An event $A = S$, which includes all outcomes from a sample space S, must have a probability equal to 1. An event $A = S$ is called a **certain event**. When $P(A) = 0$, an event cannot occur. $P(A) = 1$ means that an event will always occur.

Axiom 3. Let events $A, B, C, \cdots$ be mutually exclusive (disjoint). The probability of the event that "**A or B or C or** $\cdots$" will happen is

$$\begin{aligned} P(A\ or\ B\ or\ C\ or\ \cdots) &= P(A \cup B \cup C \cup \cdots) \\ &= P(A) + P(B) + P(C) + \cdots \end{aligned} \tag{B.4}$$

The third axiom states that for non-overlapping events (disjoint) the probability of their union is equal to the sum of the probabilities of the individual events.

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Two events are **mutually exclusive** when they cannot occur simultaneously. In other words, the happening of one event excludes the happening of the other.

Two events A and B are considered to be **independent** if the probability of occurrence of one event is not affected by occurrence of the other.

Probability that two independent events A and B will occur simultaneously ($A \cap B$, **A and B**) is

$$P(A \cap B) = P(A \; and \; B) = P(A)P(B) \quad \text{(B.10)}$$

$$P(A \cup B) = P(A \; or \; B) = P(A) + P(B) \quad \text{(B.11)}$$

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6. Multiplicative Rule of Probability

The definition of the conditional probability leads to **multiplicative rule** of probability

$$P(A \ and \ B) = P(A \cap B) = P(A|B)P(B) \tag{B.15}$$

6.1. Independence

Two events A and B are independent if:

$$P(A|B) = P(A) \ \ P(B|A) = P(B) \tag{B.16}$$

and

$$P(A \cap B) = P(A)P(B)$$

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6.2. Bayes Rule (Theorem)

Combining equations

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \; and \; P(B|A) = \frac{P(B \cap A)}{P(A)} \tag{B.17}$$

leads to the **Bayes rule (Bayes theorem)**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{B.18}$$

The Bayes rule is useful to swap events in conditional probability evaluation. The conditional probability $P(A|B)$ can be expressed by the conditional probability $P(B|A)$, $P(A)$ and $P(B)$.

The Bayes rule can be extended to a collection of events $A_1, \cdots, A_n$ conditioned on the event B

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n}[P(A_i)P(B|A_i)]} \tag{B.19}$$

where

$$P(B) = \sum_{i=1}^{n}[P(B|A_i)P(A_i)] \tag{B.20}$$

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5. Conditional Probability

The fact that a particular event has already occurred may influence the occurrence of another event, and it influences value of this event's probability. A **conditional probability** is the probability of an event occurring given the knowledge that another event already occurred.

The conditional probability that an event A will happen given that an event B has already happened is denoted by $P(A|B)$.

Example: Consider an experiment of tossing a fair coin twice. Let us define the events A "**head in the first toss**" and B "**head in the second toss**." We may define the event C as "**two heads in the first two tosses**." Before any tossing the probability of tossing "**two heads in the first two tosses**" (probability of happening of the event C) is $P(\textbf{two heads in the first two tosses}) = 0.25$. If in the first toss the outcome was a head (the event A happened), then the conditional probability that "**two head in the first two tosses**" will happen (the event C will happen) is

$$P(\textbf{two heads in the first two tosses} \mid \textbf{a head in the firsttoss}) = 0.5$$

or

$$P(C|A) = 0.5$$

Probability distributions

Distribution: Generally, **a function** showing **all possible values (or intervals)** of the data (**variable**) and how often they occur.

A very important Math/Stat/DS concept

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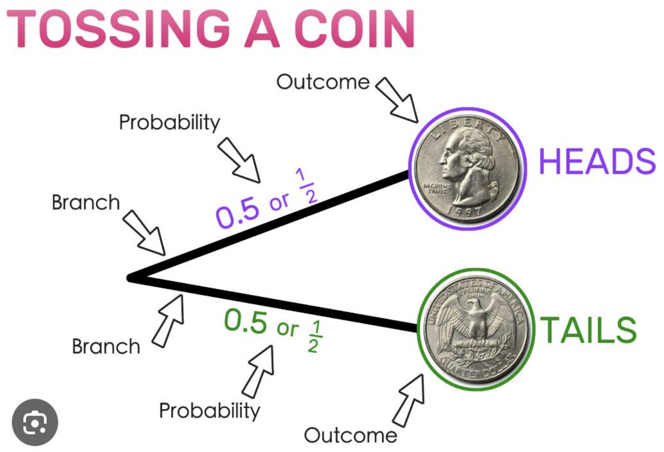

Probability distributions

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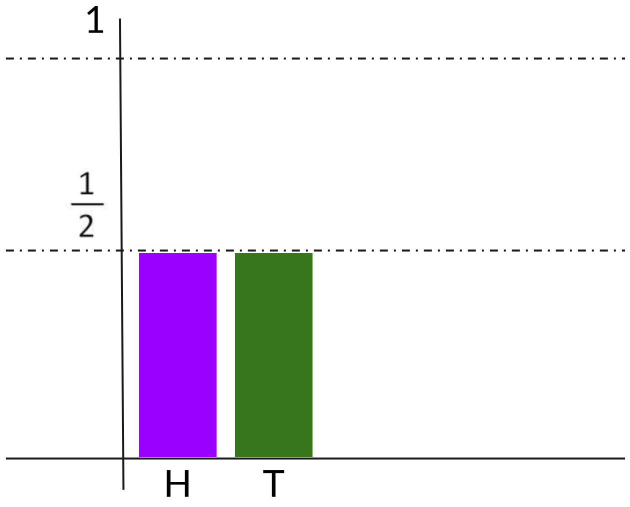


Probability distributions

Distribution: Generally, **a function** showing **all possible values (or intervals)** of the data and how often they occur.

- Uniform distribution (discrete and continuous)
- Binomial distribution (discrete, binary)
- Multinomial distribution (discrete, general)
- Normal/Gaussian distribution (continuous)

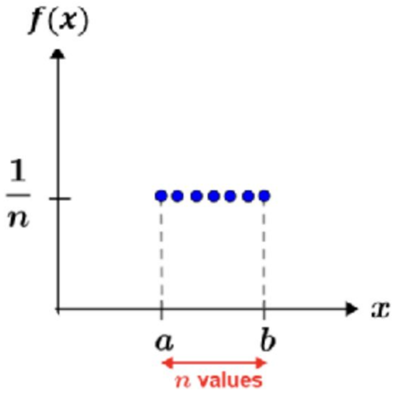
A very important Math/Stat/DS concept

Discrete Uniform Distribution

- Uniform/Equal probability
- ***X*** is a random variable
- ***n*** is the number of different choices ***X*** has

$$f(x) = \frac{1}{n}\ ,\ \mathrm{x} = 1, 2, 3, \ldots, \mathrm{n}$$

Tossing a (**fair**) coin, throwing a (**fair**) dice

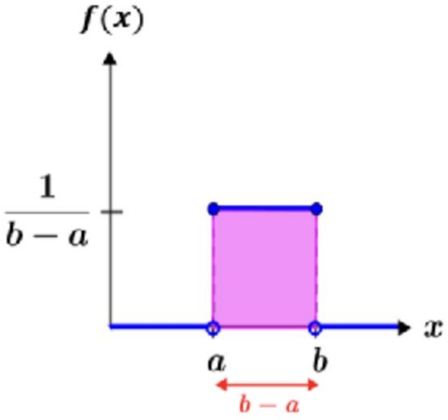


Continuous Uniform Distribution

- Uniform/Equal probability

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

Throwing a (**fair**) dirt



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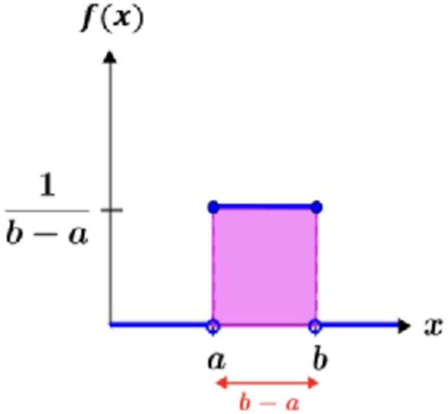
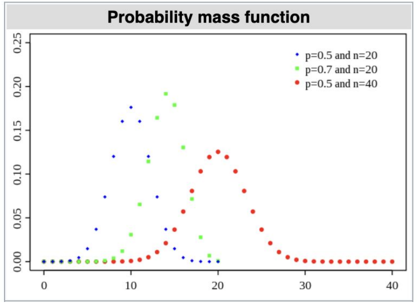


Binomial Distribution

Doing a repeated independent trials/experiments

- If you throw a coin n time, what will be the probability of getting 1 - n (denoted as *k*) heads



In general, if the random variable X follows the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, we write $X \sim B(n, p)$. The probability of getting exactly k successes in n independent Bernoulli trials is given by the probability mass function:

$$f(k,n,p) = \Pr(k;n,p) = \Pr(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

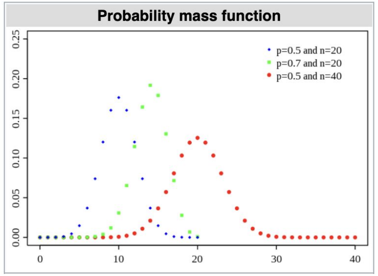
for $k = 0, 1, 2, ..., n$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Notation	$B(n,p)$
Parameters	$n \in \{0,1,2,\ldots\}$ – number of trials $p \in [0,1]$ – success probability for each trial $q = 1-p$

Binomial Distribution

- Binary variable (two outcomes): tossing a coin, answering a yes/no question, etc.
- Doing a repeated independent trials/experiments
- If you throw a coin n time, what will be the probability of getting 1 - n (denoted as *k*) heads



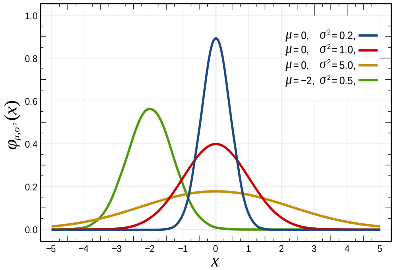
Probability mass function: The probabilities associated with all (hypothetical) values must be non-negative and sum up to 1,

$$\boxed{\sum_x p_X(x) = 1}$$

$$\boxed{p_X(x) \geq 0.}$$

Normal (Gaussian) Distribution

- Continuous (real/measurement) variable (infinite outcomes): height or weight of students, salary of a group of professional with similar background etc.



In statistics, a **normal distribution** or **Gaussian distribution** is a type of continuous probability distribution for a real-valued random variable. The general form of its probability density function is

Normal Distribution Formula

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$\mu =$ mean of x

$\sigma =$ standard deviation of x

$\pi \approx$ 3.14159 ...

$e \approx$ 2.71828 ...

Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbb{R}$ = mean (location) $\sigma^2 \in \mathbb{R}_{>0}$ = variance (squared scale)

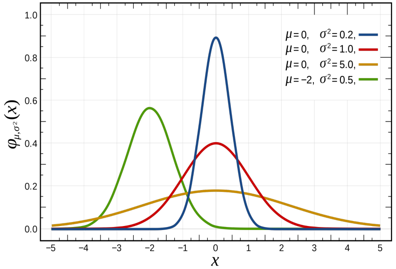
Normal (Gaussian) Distribution

- Continuous (real/measurement) variable (infinite outcomes): your blood pressure, expected salary.

Probability density function: The probabilities associated with all (hypothetical) values must be non-negative and sum up to 1,

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

$$P[a \le X \le b] = \int_a^b f(x)\ dx$$

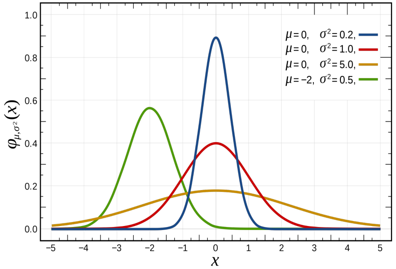


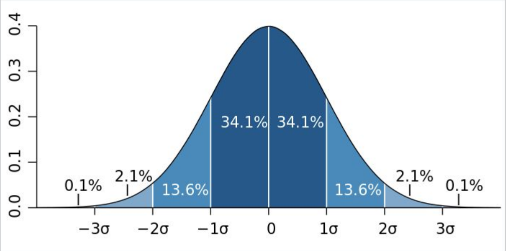
Normal (Gaussian) Distribution

- Continuous (real/measurement) variable (infinite outcomes): your blood pressure, expected salary.

Carl Friedrich Gauss

Portrait by Christian Albrecht Jensen, 1840 (copy from Gottlieb Biermann, 1887)[1]



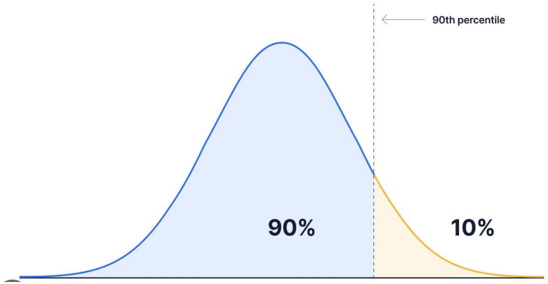


For the normal distribution, the values less than one standard deviation away from the mean account for 68.27% of the set; while two standard deviations from the mean account for 95.45%; and three standard deviations account for 99.73%.

Percentile

Percentile: The **percentile** reported for a given score is the percentage of values in the data set that fall below that certain score. For example,

- If **your score** was reported to be at the **90th percentile**, that means 90% of the other people who took the test scored lower than you did.



Z-score/Z-distribution

Standard-score/Z-score/Z-distribution: Z-score is a statistical measurement that describes a value's relationship to the mean of a group of values. Z-score is measured in terms of standard deviations from the mean. If a Z-score is 0, it indicates that the data point's score is identical to the mean score.

$$Z = \frac{x - \mu}{\sigma}$$

If a data set has a normal distribution, and you standardize all the data to obtain standard scores, those standard scores are called *z*-values. All *z*-values have what is known as a standard normal distribution (or *Z*-distribution). The *standard normal distribution* is a special normal distribution with a mean equal to 0 and a standard deviation equal to 1.

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