TITAS MATH SERIES

Complex Analysis

জটিল বিশ্লেষণ

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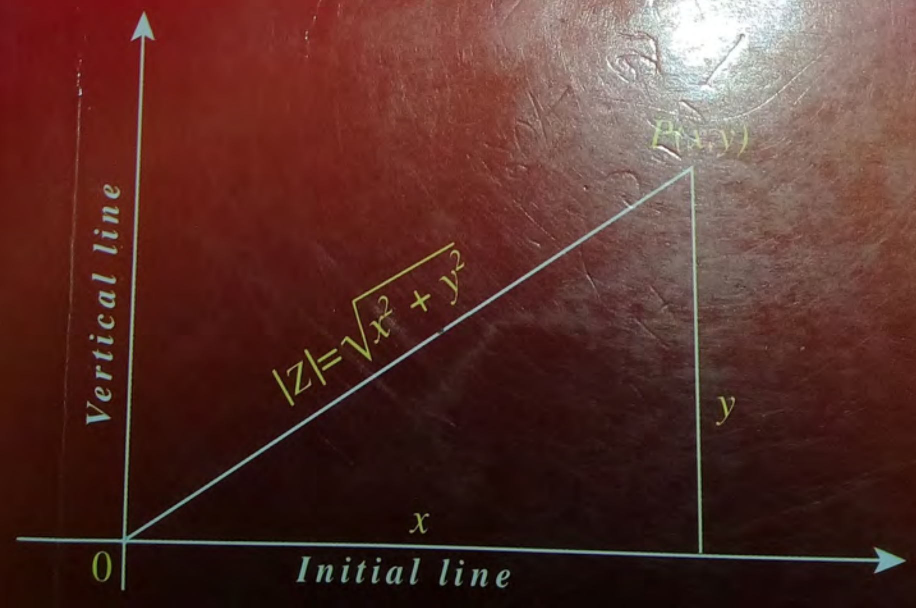

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CHAPTER–1
COMPLEX NUMBER

1.1. Complex Number system :

The equation $x^2 + 3 = 0$ or $x^2 = -3$ has no solution in real number system. This prompted the way to enlarge the real number system. For solving such systems the real number system was enlarged to complex number system. The term complex number was introduced by C. F. Gauss, a German mathematician. Later A. L. Cauchy, B. Riemann, K. Weierstrass and others enriched the subject.

Consider the set of all ordered pair of real numbers (x, y) defined by

$$\mathbf{R}^2 = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}\}.$$

Here 'ordered' means (x, y) and (y, x) are different unless x = y.

We define the operations of addition (+) and multiplication (•) in $\mathbf{R}^2$ as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

Definition–1. A complex number is defined as the ordered pair (x, y) of real numbers, z = (x, y) satisfying the following rules for addition and multiplication [একটি জটিল সংখ্যা z = (x, y) কে বাস্তব সংখ্যার ক্রম জোড় (x, y) হিসাবে বর্ণনা করা হয় যাহা নিম্নের যোগ ও গুণের নিয়ম সিদ্ধ করে]

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$
$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2),$$

where [এখানে] $z_1 = (x_1, y_1)$ and [এবং] $z_2 = (x_2, y_2)$.

It is trivial to prove that for any real numbers a and b, we have

$$(a, 0) + (b, 0) = (a + b, 0)$$
$$(a, 0) \cdot (b, 0) = (ab, 0)$$

This motivates the idea that complex numbers of the form (a, 0) have the same arithmetic properties as the corresponding real numbers a. Thus we can identify the ordered pair (a, 0) by the real number a.

Definition–2. The imaginary unit i (iota) is defined as i = (0, 1).

Proposition–1. $i^2 = -1$.

Proof : $i^2 = i \cdot i = (0, 1) \cdot (0, 1)$
$= (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0)$
$= (-1, 0) \equiv -1$

Proposition–2. Every complex number $z = (x, y)$ can be written as $z = x + iy$ or $x + yi$.

Proof :
$$\begin{aligned} z &= (x, y) = (x, 0) + (0, y) \\ &= (x, 0) + (0, 1)(y, 0) \\ &= x + iy \end{aligned} \qquad \begin{aligned} x + iy &= (x, 0) + (0, 1)(y, 0) \\ &= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y) \\ &= (x, 0) + (0, y) \\ &= (x, y) = z \end{aligned}$$

Similarly,
$$\begin{aligned} z &= (x, y) = (x, 0) + (0, y) \\ &= (x, 0) + (y, 0)(0, 1) \\ &= x + yi \end{aligned}$$

Thus, definition–1 can be modify as, "any number of the form $x + iy$ is called a complex number where $x, y \in \mathbb{R}$." [$x + iy$ আকারের যে কোন সংখ্যাকে জটিল সংখ্যা বলে যেখানে $x, y \in \mathbb{R}$.]

With the introduction of i, Definition–1 for addition and multiplication of complex numbers can be translated as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

The term imaginary number does not mean that such a number does not exist. The letter i is a symbol which denotes imaginary unit just as 1 denotes the unit of real numbers. Thus the imaginary number iy means y unit of imaginary numbers just as x means x units of real numbers. The expression $x + iy$ is not an imaginary number, it is a complex number. If $z = x + iy$, then x is called the real part of z and y is called the imaginary part of z, written as

$$\mathbf{Re}(z) = x, \ \mathrm{Im}(z) = y.$$

[কাল্পনিক সংখ্যা পদটি বুঝায় না যে এরূপ একটি সংখ্যার অস্তিত্ব নাই। i অক্ষরটি একটি প্রতীক যাহা কাল্পনিক একক নির্দেশ করে ঠিক যেমনটা বাস্তব সংখ্যার একক 1 নির্দেশ করে। অতএব কাল্পনিক সংখ্যা iy এর অর্থ কাল্পনিক সংখ্যার y একক ঠিক যেমনটা x অর্থ বাস্তব সংখ্যার x একক। $x + iy$ রাশিটি একটি কাল্পনিক সংখ্যা নয়, ইহা একটি জটিল সংখ্যা। যদি $z = x + iy$ হয় তখন x কে z এর বাস্তব অংশ বলে এবং y কে z এর কাল্পনিক অংশ বলে, যাহা লেখা হয়

$$\mathbf{Re}(z) = x, \ \mathrm{Im}(z) = y.]$$

1.1.1. Compiex number : Any number of the form $x + iy$ is called a complex number, where $x, y \in \mathbb{R}$. If is denoted by Z.

$\therefore z = x + iy$ **[NUH–2013]**

[জটিল সংখ্যা ঃ $x + iy$ আকারের যে কোন সংখ্যাকে জটিল সংখ্যা বলে, যেখানে $x, y \in \mathbb{R}$. ইহাকে z দ্বারা প্রকাশ করা হয়।

$\therefore z = x + iy$

1.1.2. Real and Imaginary parts of Z : [NUH–2013]

If $z = x + iy$ be a complex number, then

Real part of $z = \mathrm{Re}(z) = x$

Imaginary part of $z = \mathrm{Im}(z) = y$

[**z** এর বাস্তব ও কাল্পনিক অংশ ঃ যদি $z = x + iy$ একটি জটিল সংখ্যা হয়, তখন

z এর বাস্তব অংশ $= \mathrm{Re}(z) = x$

z এর কাল্পনিক অংশ $= \mathrm{Im}(z) = y$]

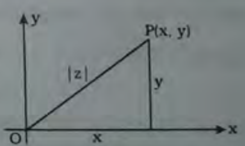
1.2. Equality of complex numbers : Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal $\Leftrightarrow x_1 = x_2$ and $y_1 = y_2$, that is, the real part of the one is equal to the real part of the other, and the imaginary part of the one is equal to the imaginary part of the other.

[জটিল সংখ্যার সমতা ঃ দুইটি জটিল সংখ্যা $z_1 = (x_1, y_1)$ এবং $z_2 = (x_2, y_2)$ সমান $\Leftrightarrow x_1 = x_2$ এবং $y_1 = y_2$ হয়, অর্থাৎ একটির বাস্তব অংশ অপরটির বাস্তব অংশের সমান, এবং একটির কাল্পনিক অংশ অপরটির কাল্পনিক অংশের সমান।]

Note : The phrases "greater than" or "less than" have no meaning in relation between two complex numbers. Inequalities can only occur in relation between the moduli of complex numbers.

[নোট ঃ দুইটি জটিল সংখ্যার সম্পর্কের ক্ষেত্রে 'হতে বৃহত্তর' অথবা 'হতে ক্ষুদ্রতর' প্রবাদের কোন অর্থ নাই। অসমতা শুধুমাত্র জটিল সংখ্যার মানাঙ্ক বা মডুলাসের ক্ষেত্রে হয়।]

1.3. Geometrical representation : Every complex number can be represented geometrically as a point in the xy-plane. We can identify the complex number $z = x + iy$ with the point P(x, y). The set of all real numbers (x, 0) corresponding to the x-axis, called real axis, and the set of all imaginary numbers (0, y) corresponding to the y-axis, called the imaginary axis. The origin identifies complex number $0 = 0 + i0$.



The distance of P from the origin is $\sqrt{x^2 + y^2}$. The nonnegative value of $\sqrt{x^2 + y^2}$ is denoted by $|z|$ and hence $|z| = \sqrt{x^2 + y^2}$. $|z|$ is called the modulas or absolute value of $z = (x, y)$.

[জ্যামিতিক উপস্থাপন ঃ প্রত্যেক জটিল সংখ্যাকে জ্যামিতিকভাবে xy তলে বিন্দু আকারে উপস্থাপন করা যায়। আমরা জটিল সংখ্যা $z = x + iy$ কে বিন্দু $P(x, y)$ দ্বারা চিহ্নিত করতে পারি। x অক্ষের অনুসঙ্গী সকল বাস্তব সংখ্যার সেট $(x, 0)$ কে বাস্তব অক্ষ বলে, এবং y অক্ষের অনুসঙ্গী সকল কাল্পনিক সংখ্যার সেট $(0, y)$ কে কাল্পনিক অক্ষ বলে। মূলবিন্দুটি $0 = 0 + i0$ জটিল সংখ্যা দ্বারা চিহ্নিত হয়।

মূলবিন্দু হতে P বিন্দুর দূরত্ব $\sqrt{x^2 + y^2}$.

$\sqrt{x^2 + y^2}$ এর অঋণাত্মক মানকে $|z|$ দ্বারা প্রকাশ করা হয় এবং অতএব $|z| = \sqrt{x^2 + y^2}$.

$|z|$ কে $z = (x, y)$ এর মডুলাস বা পরম মান বা মানাঙ্ক বলে।]

We shall use the following inequalities [আমরা নিম্নলিখিত অসমতাগুলি ব্যবহার করব]

$$x \le |x| \le \sqrt{x^2 + y^2} \Rightarrow \text{Re}(z) \le |\text{Re}(z)| \le |z|$$

$$y \le |y| \le \sqrt{x^2 + y^2} \Rightarrow \text{Im}(z) \le |\text{Im}(z)| \le |z|$$

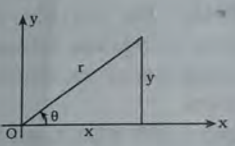
1.4. Polar and exponential form of complex numbers: Let r and θ be polar coordinates of a point $z = (x, y)$. For $z \neq 0$, let [ধরি একটি বিন্দু $z = (x, y)$ এর পোলার স্থানাংক r এবং θ. $z \neq 0$ এর জন্য ধরি]

$x = r\cos\theta,\ y = r\sin\theta$. Then [তখন]

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore r = +\sqrt{x^2 + y^2}$

and [এবং] $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x}$



$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

r is called the modulas or the absolute value and θ is called the amplitude or argument of the complex number z. [r কে জটিল সংখ্যা z এর মডুলাস বা পরমমান বা মানাঙ্ক এবং θ কোণাঙ্ক বা আর্গুমেন্ট বলে]

$$|z| = r = +\sqrt{x^2 + y^2} = 0 \Leftrightarrow x = 0 \text{ and } y = 0 \Leftrightarrow z = x + iy = 0.$$

The value of θ between $-\pi$ and π is called the principal value of the amplitude. We denote it by Arg z. [$-\pi$ এবং π এর মধ্যবর্তী θ এর মানকে কোণাঙ্কের মূখ্য মান বলে। আমরা ইহাকে Arg z দ্বারা প্রকাশ করব।]

Again [আবার] $z = x + iy = r\cos\theta + ir\sin\theta$

$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

which is called the polar or exponential form of the complex number z. [যাহাকে জটিল সংখ্যা z এর পোলার বা সূচক আকার বলে।]

Note : $r = |z|$ is a unique nonnegative number, but $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is a multivalued function.

1.4.1. Modulas and argument of z : **[NUH–2013]**

If $z = x + iy$ be a complex number and $x = r\cos\theta,\ y = r\sin\theta$

$r = |z| = +\sqrt{x^2 + y^2}$ is called the modulas of z and

$\theta = \tan^{-1}\frac{y}{x}$ is called the argument of z.

[যদি $z = x + iy$ একটি জটিল সংখ্যা হয় এবং $x = r\cos\theta,\ y = r\sin\theta$ হয় তখন

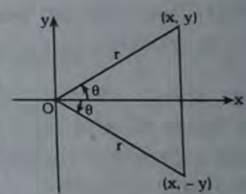
$r = |z| = +\sqrt{x^2 + y^2}$ কে z এর পরম মান বলে এবং

$\theta = \tan^{-1}\frac{y}{x}$ কে z এর আর্গুমেন্ট বা কোণাঙ্ক বলে।]

1.4.2. Argand plane or Argand diagram : When a complex number z is represented by a point $P(x, y)$ in the xy-plane, then this plane is called the Argand plane or Argand diagram or simply a complex plane.

[আর্গান্ড তল বা আর্গান্ড চিত্র ঃ যখন একটি জটিল সংখ্যাকে xy তলে একটি বিন্দু $P(x, y)$ দ্বারা উপস্থাপন করা হয়, তখন এই তলকে আর্গান্ড তল বা আর্গান্ড চিত্র বা সহজে একটি জটিল তল বলে।]

1.5. Conjugate complex number : If $z = x + iy$ is any complex number, then its complex conjugate denoted by $\bar{z}$ is defined as $\bar{z} = x - iy = (x, -y)$.



It is clear that $\bar{z}$ is the mirror image of z into the real axis. This indicates that $z = \bar{z} \Leftrightarrow z$ is purely a real number. Also, $\bar{\bar{z}} = z$.

[অনুবন্ধী জটিল সংখ্যা ঃ যদি $z = x + iy$ যে কোন জটিল সংখ্যা হয়, তখন ইহার জটিল অনুবন্ধীকে $\bar{z}$ দ্বারা প্রকাশ করা হয় যাহা $\bar{z} = x - iy = (x, -y)$ দ্বারা বর্ণিত।

ইহা স্পষ্ট যে $\bar{z}$ হলো x অক্ষের ভিতর z এর আয়না প্রতিবিম্ব। ইহা নির্দেশ করে যে $z = \bar{z} \Leftrightarrow z$ হয় একটি বিশুদ্ধ বাস্তব সংখ্যা। অধিকন্তু, $\bar{\bar{z}} = z$.]

Again [আবার] $\bar{z} = x - iy = r\cos\theta - ir\sin\theta$

$$= r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

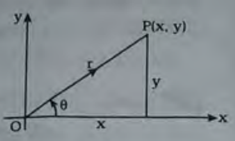
$\therefore |\bar{z}| = r = |z|$ and [এবং] $\text{Amp}\,\bar{z} = -\theta = -\text{Amp}\, z.$

Thus condition for two given numbers z_1 and z_2 to be conjugate [অতএব প্রদত্ত দুইটি সংখ্যা z_1 ও z_2 অনুবন্ধী হওয়ার শর্ত]

(i) $|z_1| = |z_2|$ and (ii) $\text{amp}\, z_1 + \text{amp}\, z_2 = 0$.

1.6. Vector representation of complex number :

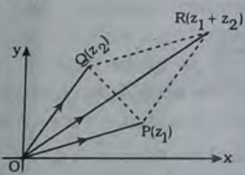
Let P(x, y) be a point in the complex plane corresponding to the complex number $z = x + iy$.



Then the modulas $r = \sqrt{x^2 + y^2}$ is represented by the magnitude of the vector $\overrightarrow{OP}$ and its amplitude θ is represented by the direction of the vector $\overrightarrow{OP}$.

Hence the complex number $z = x + iy$ is completely represents by the vector $\overrightarrow{OP}$. [মনেকরি জটিল তলে $z = x + iy$ জটিল সংখ্যার অনুসঙ্গী P(x, y) একটি বিন্দু। তখন মডুলাস (বা মানাঙ্ক) $r = \sqrt{x^2 + y^2}$ কে ভেক্টর $\overrightarrow{OP}$ এর মান দ্বারা নির্দেশ করা যায় এবং ইহার কোনাঙ্ক θ কে ভেক্টরএর দিকে নির্দেশ করা যায়। অতএব জটিল সংখ্যা $z = x + iy$ কে সম্পূর্ণভাবে ভেক্টর দ্বারা নির্দেশ করা যায়]

(i) Sum : Let the complex numbers $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$ be represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Then the coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively.

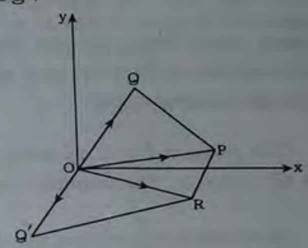


Complete the parallelogram OPRQ. Then the mid point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ which is also the mid point of OR. Hence the coordinate of R is $(x_1 + x_2, y_1 + y_2)$ which represents the point $z_1 + z_2$. Vectorically we have

$$z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

[যোগফল ঃ মনেকরি জটিল সংখ্যা $z_1 = x_1 + iy_1$ এবং $z_2 = x_2 + iy_2$ কে ভেক্টর $\overrightarrow{OP}$ ও $\overrightarrow{OQ}$ দ্বারা নির্দেশ করা যায়। তখন P ও Q এর স্থানাংক যথাক্রমে (x_1, y_1) ও (x_2, y_2), OPRQ সামন্তরিকটি সম্পূর্ণ করি। তখন PQ এর মধ্যবিন্দু $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ যাহা OR এরও মধ্যবিন্দু। অতএব R এর স্থানাংক $(x_1 + x_2, y_1 + y_2)$ যাহা $z_1 + z_2$ বিন্দুকে নির্দেশ করে। ভেক্টর আকারে পাই $z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$]

(ii) Difference : Let the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Produce OQ backwards upto Q′ such that OQ = OQ′.

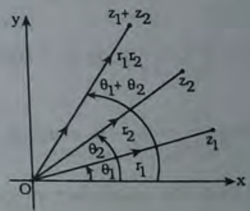


Then Q′ represents the point $-z_2$. Complete the parallelogram OPRQ′. Then R represents the sum of complex numbers represented by points P and Q′, that is $R(z_1 - z_2)$.
Vectorically, $z_1 - z_2 = \overrightarrow{OR} = \overrightarrow{QP}$

[পার্থক্যঃ মনেকরি জটিল সংখ্যা $z_1 = x_1 + iy_1$ এবং $z_2 = x_2 + iy_2$ কে ভেক্টর $\overrightarrow{OP}$ এবং $\overrightarrow{OQ}$ দ্বারা নিদের্শ করা যায়। $\overrightarrow{OQ}$ কে পিছনের দিকে Q′ পর্যন্ত বর্ধিত করি যেন OQ = OQ′ হয়। তখন Q′– z_2 বিন্দুকে নিদের্শ করে। OPRQ সামন্তরিকটি সম্পূর্ণ করি। তখন R. P ও Q′ বিন্দু দ্বারা নির্দেশিত জটিল সংখ্যাদ্বয়ের যোগফল নির্দেশ করে, অর্থাৎ $R(z_1 - z_2)$ ভেক্টর নিয়মে $z_1 - z_2 = \overrightarrow{OR} = \overrightarrow{QP}$]

(iii) Product (গুণফল) : In polar form, let the two complex numbers are [পোলার আকারে ধরি দুইটি জটিল সংখ্যা] $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. $|z_1|$
$= |r_1 e^{i\theta_1}| = r_1 |\cos\theta_1 + i\sin\theta_1| = r_1 \sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1 \cdot 1 = r_1$
Similarly [অনুরূপে] $|z_2| = r_2$.

Then their product is [তখন তাদের গুণফল]



$$\begin{aligned}
z_1 z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\
&= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
\therefore |z_1 z_2| &= |r_1 r_2 e^{i(\theta_1+\theta_2)}| \\
&= |r_1|\,|r_2|\,|e^{i(\theta_1+\theta_2)}|
\end{aligned}$$

$= r_1 r_2$, since [যেহেতু] $|e^{i(\theta_1+\theta_2)}| = |\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)|$
$= \sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)}$
$= 1$

$= |z_1|\,|z_2|$

and [এবং] $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

Thus, we have shown that the modulas of the product of two complex numbers is equal to the product of the moduli of these numbers and the argument of the product of two complex numbers is equal to the sum of the argument of these numbers. [অতএব, আমরা দেখালাম যে দুইটি জটিল সংখ্যার গুণফলের মডুলাস এই সংখ্যাগুলির মডুলাসের গুণফলের সমান এবং দুইটি জটিল সংখ্যার গুণফলের কোণাঙ্ক এই সংখ্যাগুলির কোণাঙ্কের যোগফলের সমান।]

The relation Arg (z_1z_2) = Arg z_1 + Arg z_2 may or may not be true for all z_1 and z_2. For example consider $z_1 = i$ and $z_2 = -1$.

Then Arg $z_1 = \frac{\pi}{2}$ and Arg $z_2 = \pi$.

Now, $z_1z_2 = -i$, Arg $(z_1z_2) = -\frac{\pi}{2}$.

Hence, Arg $(z_1z_2) \neq$ Arg z_1 + Arg z_2 as $\frac{-\pi}{2} \neq \frac{\pi}{2} + \pi$.

(iv) Quotient (ভাগফল) : In polar form let the two complex numbers are [পোলার আকারে ধরি দুইটি জটিল সংখ্যা]

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

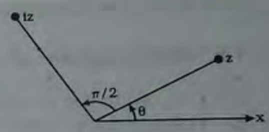
$$\therefore \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2} [\cos(\theta_1-\theta_2) + i \sin(\theta_1-\theta_2)]$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} |\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)| = \frac{r_1}{r_2} \cdot 1 = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

Thus the modulas of quotient of two complex numbers is the quotient of the modulas of the numerator and the denominator, and argument of the quotient of two complex numbers is the difference of the arguments of the numerator and the denominator. [অতএব দুইটি জটিল সংখ্যার ভাগফলের মডুলাস হলো লব ও হরের মডুলাসের ভাগফল, এবং দুইটি জটিল সংখ্যার ভাগফলের কোণাঙ্ক হলো লব ও হরের কোণাঙ্কের পার্থক্য।]

1.7. Geometrical interpretation of multiplication of a complex number by i :



Let z be a complex number whose polar form is

$z = r e^{i\theta}$

$\Rightarrow z = r(\cos\theta + i\sin\theta)$

where r is the modulas and θ is the amplitude (argument) of z.

$$\text{Then } iz = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \cdot r(\cos\theta + i\sin\theta)$$
$$= r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

Thus, the vector iz is one obtained by rotating the vector z through one right angle without changing its length.

[একটি জটিল সংখ্যাকে **i** দ্বারা গুণের জ্যামিতিক ব্যাখ্যা ঃ

মনে করি z একটি জটিল সংখ্যা যার পোলার আকার

$z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$

যেখানে z এর মডুলাস r এবং কোণাঙ্ক θ. তখন

$$iz = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \cdot r(\cos\theta + i\sin\theta)$$
$$= r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

অতএব, ভেক্টর z কে ইহার দৈর্ঘ্য পরিবর্তন না করে এক সমকোণ কোণে ঘুরাইয়া ভেক্টর iz পাওয়া গেল।]

Proposition : A complex number is purely real if the amplitude is O or π and purely imaginary if the amplitude is $\frac{\pi}{2}$ or $\frac{-\pi}{2}$. [একটি জটিল সংখ্যা বিশুদ্ধভাবে বাস্তব হবে যদি ইহার কোণাঙ্ক 0 অথবা π হয় এবং বিশুদ্ধভাবে কাল্পনিক হবে যদি ইহার কোণাঙ্ক $\frac{\pi}{2}$ অথবা $-\frac{\pi}{2}$ হয়।]

Proof : Let z be a complex number. Consider its polar form $z = r(\cos\theta + i\sin\theta)$

When θ = 0 then $z = r(\cos 0 + i\sin 0) = r(1 + i\cdot 0) = r$

When θ = π then $z = r(\cos\pi + i\sin\pi) = r(-1 + i\cdot 0) = -r$

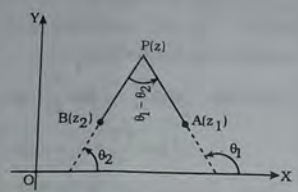
When $\theta = \frac{\pi}{2}$ then $z = r\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = r(0 + i\cdot 1) = ir$.

When $\theta = -\frac{\pi}{2}$ then $z = r\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) = r(0 - i\cdot 1) = -ir$.

Thus, we have if z is purely real then amp $z = 0$ or π
and if z is purely imaginary then amp $z = \frac{\pi}{2}$ or $\frac{-\pi}{2}$.

1.8. Interpretation of $\arg\frac{z - z_1}{z - z_2}$:

Let z, z_1, z_2 be the points P, A, B respectively on the Argands plane so that the complex numbers $z - z_1$ and $z - z_2$ represents the vector $\overrightarrow{AP}$ and $\overrightarrow{BP}$. Here $\arg \overrightarrow{AP} = \theta_1$ and $\arg \overrightarrow{BP} = \theta_2$.



$\therefore \angle BPA = \theta_1 - \theta_2 = \arg(z - z_1) - \arg(z - z_2) = \arg\frac{z - z_1}{z - z_2}$

Thus $\arg\frac{z - z_1}{z - z_2}$ represent the angle between the lines AP and BP in the positive sense. Similarly, $\arg\frac{z - z_2}{z - z_1}$ gives the angle in the negative sense. [$\arg\frac{z - z_1}{z - z_2}$ এর উপস্থাপন ঃ মনেকরি আরগাভতলে z, z_1, z_2 বিন্দু সমূহ যথাক্রমে P, A, B যেন জটিল সংখ্যা $z - z_1$ এবং $z - z_2$ দ্বারা নির্দেশিত ভেক্টর $\overrightarrow{AP}$ এবং $\overrightarrow{BP}$. এখানে $\arg \overrightarrow{AP} = \theta_1$ এবং $\arg \overrightarrow{BP} = \theta_2$.

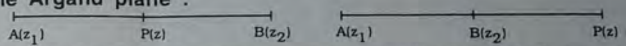
$\therefore \angle BPA = \theta_1 - \theta_2 = \arg(z - z_1) - \arg(z - z_2) = \arg\frac{z - z_1}{z - z_2}$.

অতএব $\arg\frac{z - z_1}{z - z_2}$ ধনাত্মক দিকে AP ও BPরেখার মধ্যবর্তী কোণ নির্দেশ করে। অনুরূপভাবে, arg $\frac{z - z_1}{z - z_2}$ ঋনাত্মক দিকে কোণ দেয়।]

Condition for perpendicularity : If the lines are perpendicular then $\arg\frac{z - z_1}{z - z_2} = \pm\frac{\pi}{2}$. We know that when the argument is $\pm\frac{\pi}{2}$, then the complex number is purely imaginary. Thus, in this case $\frac{z - z_1}{z - z_2}$ is purely imaginary. [লম্বের শর্ত ঃ যদি রেখাদ্বয় লম্ব হয় তখন $\arg\frac{z - z_1}{z - z_2} = \pm\frac{\pi}{2}$ আমরা জানি যে, যখন কোনাঙ্ক $\pm\frac{\pi}{2}$ তখন জটিল সংখ্যাটি বিশুদ্ধ কাল্পনিক। অতএব, এই ক্ষেত্রে $\frac{z - z_1}{z - z_2}$ বিশুদ্ধ কাল্পনিক।]

Condition for parallelism or collinarity : If the lines are parallel or the point A, P, B are collinear, then $\arg\frac{z - z_1}{z - z_2} = 0$ or π. We know that when the argument is 0 or π then the complex number is purely real. Thus, in this case $\frac{z - z_1}{z - z_2}$ is purely real. [সমান্তরাল বা এক রেখীয় শর্ত ঃ যদি রেখাদ্বয় সমান্তরাল হয় অথবা A, P, B বিন্দুগুলি সমরেখীয় হয়, তখন $\arg\frac{z - z_1}{z - z_2} = 0$ অথবা π আমরা জানি যে, যখন কোনাঙ্ক 0 অথবা π তখন জটিল সংখ্যা বিশুদ্ধ বাস্তব। অতএব, এই ক্ষেত্রে $\frac{z - z_1}{z - z_2}$ বিশুদ্ধ বাস্তব]

1.9. Equation of a straight line going the points z_1 and z_2 in the Argand plane :



Let $A(z_1)$ and $B(z_2)$ be any two points on the Argand plane, and P(z) be any point on the line AB.

Then $\arg\frac{z - z_1}{z - z_2} = 0$ or π, according as P(z) lies outside or inside of AB. Therefore, $\frac{z - z_1}{z - z_2}$ is purely real. We know that the complex number z is purely real when $z = \bar{z}$. Thus,

[মনেকরি আরগান্ড সমতলে AB রেখার উপর $A(z_1)$ এবং $B(z_1)$ যে কোন দুইটি বিন্দু। তখন P(z), AB এর বাহিরে বা ভিতরে অনুসারে

$\arg\frac{z - z_1}{z - z_2} = 0$ অথবা π অতএব, $\frac{z - z_1}{z - z_2}$ বিশুদ্ধ বাস্তব। আমরা জানি যে জটিল সংখ্যা z বিশুদ্ধ বাস্তব হয় যখন $z = \bar{z}$ হয়। অতএব]

$$\frac{z - z_1}{z - z_2} = \overline{\left(\frac{z - z_1}{z - z_2}\right)}$$

$$\Rightarrow \frac{z - z_1}{z - z_2} = \frac{\overline{z - z_1}}{\overline{z - z_2}} = \frac{\bar{z} - \overline{z_1}}{\bar{z} - \overline{z_2}}$$

$$\Rightarrow (z - z_1)(\bar{z} - \overline{z_2}) = (z - z_2)(\bar{z} - \overline{z_1})$$

$$\Rightarrow z\bar{z} - z\overline{z_2} - z_1\bar{z} + z_1\overline{z_2} = z\bar{z} - z\overline{z_1} - z_2\bar{z} + \overline{z_1}z_2$$

$$\Rightarrow (\bar{z} - \overline{z_2} - \bar{z} + \overline{z_1})z + (-z_1 + z_2)\bar{z} + (z_1\overline{z_2} - \overline{z_1}z_2) = 0$$

$$\Rightarrow (\overline{z_1} - \overline{z_2})z - (z_1 - z_2)\bar{z} + (z_1\overline{z_2} - z_2\overline{z_1}) = 0$$

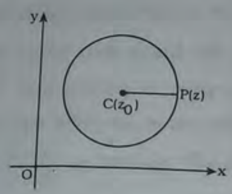
$$\Rightarrow az - \bar{a}\bar{z} + b = 0$$

where $a = \overline{z_1} - \overline{z_2}$, $\bar{a} = z_1 - z_2$ and $b = z_1\overline{z_2} - z_2\overline{z_1}$.

The above equation is the equation of a straight line passing through the points z_1 and z_2.

1.10. Equation of a circle :

Let $C(z_0)$ be the centre and r be the radius of the circle. Let $P(z)$ be any point on its circumference.



Then $\overrightarrow{CP} = z - z_0$

$\therefore |\overrightarrow{CP}| = |z - z_0| = r$ is the equation of the required circle.

The above equation can be written as

$$|z - z_0|^2 = r^2$$

$$\Rightarrow (z - z_0)\overline{(z - z_0)} = r^2$$

$$\Rightarrow (z - z_0)(\bar{z} - \overline{z_0}) = r^2$$

$$\Rightarrow z\bar{z} - z_0\bar{z} - \overline{z_0}z + z_0\overline{z_0} = r^2$$

$$\Rightarrow (\bar{z} - \overline{z_0})z - z_0\bar{z} + (z_0\overline{z_0} - r^2) = 0$$

$$\Rightarrow z\bar{z} - z_0\bar{z} - \overline{z_0}z + (|z_0|^2 - r^2) = 0$$

$$\Rightarrow z\bar{z} + b\bar{z} + \bar{b}z + c = 0,\ \text{Where } b = -z_0,\ \bar{b} = -\overline{z_0} \text{ and } c = |z_0|^2 - r^2$$

which is real. The above equation is the general equation of a circle. Its centre is

$$z_0 = -b \text{ and radius} = \sqrt{|z_0|^2 - c} = \sqrt{|b|^2 - c} = \sqrt{b\bar{b} - c}.$$

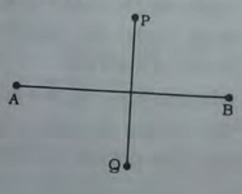
Note : If $z - z_0 = r\,e^{i\theta}$, $0 \le \theta \le 2\pi$, then $|z - z_0| = r|e^{i\theta}| = r \cdot 1 = r$. Thus, $|z - z_0| = r$ gives $z = z_0 + re^{i\theta}$.

Note : If z_0 is at the origin then the equation of the circle is $|z| = r$ and $z = re^{i\theta}$, $0 \le \theta \le 2\pi$.

1.11. Inverse points :

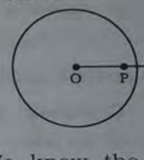
Inverse points with respect to a line.

Let AB be a line on the Argand plane. Then the two points P and Q are said to be inverse points with respect to the line AB if Q is the image of P in AB, that is, if the line AB is the right bisector of PQ.



Inverse point with respect to a circle :

Let O be the centre and r be the radius of a circle. Then two points P and Q are said to be inverse points with respect to the circle if (i) O, P, Q are collinear, and (ii) $OP \cdot OQ = r^2$



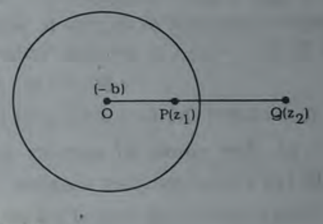
Condition for inverse points : We know the general equation of a circle is

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0$$

where b is a complex constant and c is real. The centre of this circle is $-b$ and radius is $\sqrt{b\bar{b} - c}$.

$$\overrightarrow{OP} = z_1 + b,\ \overrightarrow{OQ} = z_2 + b$$

From the definition of inverse points we have



$$|\overrightarrow{OP}| \cdot |\overrightarrow{OQ}| = r^2$$

$$|z_1 + b| \cdot |z_2 + b| = r^2$$

$$= b\bar{b} - c \cdots (1)$$

Again, O, P, Q, are in collinear, so

$$\arg(z_1 + b) = \arg(z_2 + b) \cdots\cdots (2)$$

Also we know that $\arg z = -\arg \bar{z}$. Using this in (2) we have

$$\arg(z_1 + b) = \arg(z_2 + b) = -\arg\overline{(z_2 + b)} = -\arg(\overline{z_2} + \bar{b})$$

$$\Rightarrow \arg(z_1 + b) + \arg(\overline{z_2} + \bar{b}) = 0$$

$$\Rightarrow \arg((z_1 + b)(\overline{z_2} + \bar{b})) = 0$$

$$\Rightarrow (z_1 + b)(\overline{z_2} + \bar{b}) \text{ is purely real and positive}$$

Again, since $|\overline{z_2 + b}| = |\overline{z_2} + \bar{b}|$ so from (1) we have

$$|z_1 + b|\ |\overline{z_2} + \bar{b}| = b\bar{b} - c$$

$$\Rightarrow z_1\overline{z_2} + b\overline{z_2} + \bar{b}z_1 + b\bar{b} = b\bar{b} - c$$

$$\Rightarrow z_1\overline{z_2} + b\overline{z_2} + \bar{b}z_1 + c = 0$$

which is the required condition. On taking conjugate this condition can be put in the form

$$\overline{z_1}z_2 + \bar{b}z_2 + b\overline{z_1} + c = 0,\ \text{since c is real, so } \bar{c} = c.$$

Particular case : If the circle be a unit circle $|z| = 1$ then $|z|^2 = 1 \Rightarrow z\bar{z} = 1$. Writing the points z_1 and z_2 we have $z_1\overline{z_2} = 1$ $\Rightarrow z_1 = \dfrac{1}{\overline{z_2}}$.

1.12. De moiver's theorem :

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all positive and negative integral or fractional values of n.

1.13. Complex polynomial and equations : Any expression of the form

$$a_0z^n + a_1z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_{n-1}z + a_n$$

is called a complex polynomial of degree n, where

$a_0 \neq 0, a_1, a_2, \cdots, a_n \in \mathbf{C}$ and $n \in \mathbf{N}$.

If $a_0z^n + a_1z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_{n-1}z + a_n = 0$ then it is called a complex equation of degree n, where $a_0 \neq 0, a_1, a_2, \cdots, a_n \in \mathbf{C}$ and $n \in \mathbf{N}$. If $z_1, z_2, \cdots, z_n$ are the roots of the above equation then

$$a_0(z - z_1)(z - z_2)\cdots(z - z_n) = 0$$

It is called the factor form of the above polynomial equation.

1.14. The roots of comlex numbers :

(i) Let z be a complex number.

Then w be an nth root of z if $w^n = z \Rightarrow w = (z)^{1/n}$.

In polar form if $z = r(\cos\theta + i\sin\theta)$, then

$$w = \{r(\cos\theta + i\sin\theta)\}^{1/n}$$
$$= r^{1/n}\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}^{1/n}$$
$$= r^{1/n}\left(\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right)$$

where $k = 0, 1, 2, \cdots (n-1)$.

(ii) The nth roots of unity.

Let $z^n = 1$. Then $z = (1)^{1/n} = (\cos 0 + i\sin 0)^{1/n}$

$$\Rightarrow z = (\cos 2k\pi + i\sin 2k\pi)^{1/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

where $k = 0, 1, 2, \cdots, (n-1)$.

(iii) The nth roots of (-1).

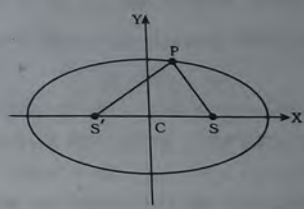
Let $z^n = -1$. Then $z = (-1)^{1/n} = (\cos\pi + i\sin\pi)^{1/n}$

$$\Rightarrow z = \{\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\}^{1/n}$$
$$= \cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}$$

where $k = 0, 1, 2, \cdots, (n-1)$

1.15. Relations in Ellipse and Hyperbola : From the coordinate geometry we know that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the standard form of an ellipse. Also, if S, S′ are the two focus and P be any point on the ellipse then

$$SP + S'P = 2a$$



that is, sum of the length of two focus from a point on the ellipse is equal to the length of the major axis.

In respect of complex variable, let $P(x, y) = P(z)$.

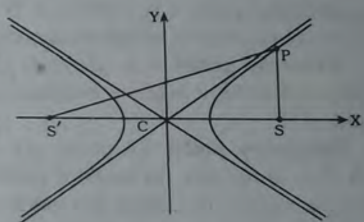
If $S = S(z_0)$, then $S' = S'(-z_0)$

$\therefore SP = |z - z_0|$, $S'P = |z + z_0|$

Therefore, the above relation becomes

$$|z - z_0| + |z + z_0| = 2a = \text{length of the major axis} \quad \cdots\cdots (1)$$

Thus, the equation of the type (1) always hold in the case of an ellipse



Similarly, from any standard geometry book we have

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the standard equation of a hyperbola and

$$SP - S'P = 2a$$

that is, difference of the distance from any point on the hyperbola is equal to the length of the transverse axis. As above in complex variable this relation becomes

$$|z - z_0| + |z + z_0| = 2a = \text{length of the transverse axis.}$$

[উপবৃত্ত ও পরাবৃত্ত সম্পূর্ক সমূহ ঃ স্থানাংক জ্যামিতি হতে আমরা জানি যে, উপবৃত্তের আদর্শ সমীকরণ হল $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ আরো, যদি S, S′ দুইটি ফোকাস এবং উপবৃত্তের উপর P যে কোন বিন্দু হয় তখন

$$SP + S'P = 2a,$$

অর্থাৎ, উপবৃত্তের উপর কোন বিন্দু হতে ফোকসদ্বয়ের দৈর্ঘ্যের সমষ্টি বৃহৎ অক্ষের দৈর্ঘ্যের সমান। জটিল চলক সাপেক্ষে ধরি $P(x, y) = P(z)$

যদি $S = S(z_0)$ হয় তখন $S' = S'(-z_0)$

$\therefore SP = |z - z_0|,\ S'P = |z + z_0|$

অতএব, উপরের সম্পর্কটি দাঁড়ায়

$|z - z_0| + |z + z_0| = 2a =$ বৃহৎ অক্ষের দৈর্ঘ্য ...(1)

অতএব, উপবৃত্তের ক্ষেত্রে (1) এর মত সমীকরণ সব সময় খাটে। অনুরূপে, যে কোন আদর্শ জ্যামিতি পুস্তক হতে পাই পরাবৃত্তের আদর্শ সমীকরণ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ এবং

$$SP - S'P = 2a,$$

অর্থাৎ পরাবৃত্তের উপর অবস্থিত যে কোন বিন্দু হতে a দুরত্বদ্বয়ের পার্থক্য অনুপ্রস্থ (আড়) অক্ষের দৈর্ঘ্যের সমান। উপরের নিয়মে জটির চলকের সাপেক্ষে এই সম্পর্কটি দাঁড়ায়।

$|z - z_0| - |z + z_0| = 2a =$ অনুপ্রস্থ (আড়) অক্ষের দৈর্ঘ্য।]

Regions in the complex plane :

Here we give some definitions only which are necessary for our discussion.

Neighbourhood :

A neighbourhood (or circular neighbourhood) of a point z_0 in the complex plane C is a set of points given by

$$S = \{z \in \mathbf{C} : |z - z_0| < \delta, \delta > 0\}$$

Interior point : A point $z_0 \in S$ is said to be an interior point of S it there exists a neighbourhood of z_0 which is contained in S.

Exterior point : A point $z_0 \in S$ is said to be an exterior point of S if there exists a neighbourhood of z_0 which contains no point of S.

Boundary point : A point $z_0 \in S$ is called a boundary point of S if z_0 is neither an interior point nor a exterior point of S. A boundary point is therefore a point all of whose neighbourhoods contain points in S and points not in S. The totality of all boundary points is called the boundary of S.

Open set : A set S in **C** is said to be open if for each point of S there exists a neighbourhood which is contained in S.

A set is open if it contains none of its boundary points.

Closed set : A set S is said to be closed if it contains all its boundary points.

Some sets are neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Thus, an open set does not have any of its boundary points, while a closed set contains all its boundary points.

Connected set : An open set S is said to be connected if each pair of points z_1 and z_2 in it can be joined by a polygonal path, consisting of a finite number of line segments joined end to end, which lies entirely in S.

The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is open and also connected.

Domain : A non empty open connected set in **C** is called a domain. That is, an open set that is connected is called a domain.

Bounded set : A set S in complex plane is said to be bounded if every point of S lies inside some circle $|z| = R$.

Unbounded set : A set which is not bounded is called unbounded set.

Region : A domain together with some, none or all of its boundary points is referred to as a region.

SOLVED EXAMPLES

Example–1. Express $\frac{(1+2i)^2}{(2+i)^2}$ in the form A + iB. Also find its modulas and argument. **[DUH–1983]**

Solution :
$$\frac{(1+2i)^2}{(2+i)^2} = \frac{1+4i+4i^2}{4+4i+i^2}$$
$$= \frac{1+4i-4}{4+4i-1} \quad [\because i^2 = -1]$$
$$= \frac{-3+4i}{3+4i} \times \frac{3-4i}{3-4i}$$
$$= \frac{-9+12i+12i-16i^2}{3^2-(4i)^2}$$
$$= \frac{-9+24i+16}{9+16} = \frac{7+24i}{25}$$
$$= \frac{7}{25} + i\frac{24}{25} \quad \textbf{(Ans)}$$

Modulas [মডুলাস] $= \left|\frac{(1+2i)^2}{(2+i)^2}\right| = \left|\frac{7}{25} + i\frac{24}{25}\right|$
$$= \sqrt{\left(\frac{7}{25}\right)^2 + \left(\frac{24}{25}\right)^2} = \sqrt{\frac{49+576}{625}}$$
$$= \sqrt{\frac{625}{625}} = 1$$

The principal argument [মুখ্য কোণাঙ্ক] $= \tan^{-1}\left(\frac{24/25}{7/25}\right) = \tan^{-1}\left(\frac{24}{7}\right)$.

The general argument [সাধারণ কোণাঙ্ক] $= 2n\pi +$ principal argument

$$= 2n\pi + \tan^{-1}\left(\frac{24}{7}\right)$$

where [যেখানে] $n = 0, \pm 1, \pm 2, \cdots$ etc.

Example-2. Find the real and imaginary parts of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}$ and also find its modulas. **[RUH-1986]**

Solution : $$\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\phi}{2} + i\,2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\cos\frac{\phi}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\,\frac{e^{i\theta/2}}{e^{i\phi/2}} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\,e^{i(\theta-\phi)/2}$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\left[\cos\left(\frac{\theta-\phi}{2}\right) + i\sin\left(\frac{\theta-\phi}{2}\right)\right]$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,e^{i(\theta-\phi)/2}$$

Real part of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\cos\frac{\theta-\phi}{2}$

Imaginary part of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\sin\frac{\theta-\phi}{2}$

Modulas [মডুলাস] $= \left|\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}\right| = \left|\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\,e^{i(\theta-\phi)/2}\right|$

$$= \left|\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\right|\,\left|e^{i(\theta-\phi)/2}\right|$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot 1 = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \quad \text{Ans.}$$

Example-3. Find the modulas and argument of the following complex numbers :

(i) $\frac{-2}{1+i\sqrt{3}}$, (ii) $\frac{1-i}{1+i}$ **[DUH-2005, JUH-1987]**

Solution : (i) $$\frac{-2}{1+i\sqrt{3}} = \frac{-2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-2+i2\sqrt{3}}{1-i^2 3}$$

$$= \frac{-2+i2\sqrt{3}}{1+3} = \frac{-2}{4} + i\frac{2\sqrt{3}}{4} = -\frac{1}{2}\,i\frac{\sqrt{3}}{2}.$$

$\therefore$ Modulas [মডুলাস] $= \left|\frac{-2}{1+i\sqrt{3}}\right| = \left|-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

The principal argument [মুখ্য কোণাঙ্ক]

$$= \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

The general argument is $2n\pi + \frac{2\pi}{3}$, where $n = 0, \pm 1, \pm 2, \cdots$ etc.

(ii) $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = -i$

Modulas $= \left|\frac{1-i}{1+i}\right| = |-i| = \sqrt{0+(-1)^2} = \sqrt{1} = 1$

The principal argument $= \tan^{-1}\left(\frac{-1}{0}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$

The general argument $= 2n\pi - \frac{\pi}{2}$, where $n = 0, \pm 1, \pm 2 \cdots$ etc.

Example-4. Find the modulas and argument of the complex number $\left(\frac{2+i}{3-i}\right)^2$. **[NUH-2011, DUHT-1991]**

Solution : $\left(\frac{2+i}{3-i}\right)^2 = \frac{4+4i+i^2}{9-6i+i^2} = \frac{4+4i-1}{9-6i-1} = \frac{3+4i}{8-6i}$

$$= \frac{3+4i}{8-6i} \times \frac{8+6i}{8+6i} = \frac{24+18i+32i+24i^2}{8^2-6^2i^2}$$

$$= \frac{24+50i-24}{64+36}$$

$$= \frac{i50}{100} = \frac{i}{2}$$

$\therefore$ Modulas [মডুলাস] $= \left|\left(\frac{2+i}{3-i}\right)^2\right| = \left|\frac{i}{2}\right| = \sqrt{0+\frac{1}{4}} = \frac{1}{2}$

The principal argument [মুখ্য কোণাঙ্ক] $= \tan^{-1}\left(\frac{1/2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

The general argument [সাধারণ কোণাঙ্ক]

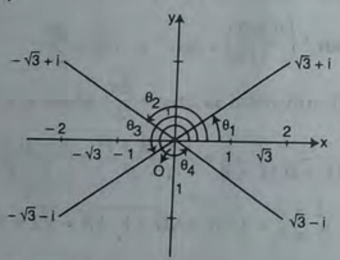
$$= 2n\pi + \frac{\pi}{2}, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

Example-4(i). Plot the four complex numbers $z = \pm\sqrt{3} \pm i$. Find the arguments and the modulas of these z [$z = \pm\sqrt{3} \pm i$ সংখ্যা চতুষ্টয় আরগান্ড ডায়াগ্রাম চিত্রায়িত কর। ইহাদের আর্গুমেন্ট ও পরম মান বাহির কর।] **[NUH-2013]**

Solution : Given [দেওয়া আছে] $z = \pm\sqrt{3} \pm i$. This gives [ইহা দেয়]

(i) $z = \sqrt{3} + i$, (ii) $z = -\sqrt{3} + i$, (iii) $z = \sqrt{3} - i$, (iv) $z = -\sqrt{3} - i$.

First we plot the four complex numbers [আমরা প্রথমে চারটি জটিল সংখ্যাকে চিত্রায়িত করব।]



In case (i) [(i) নং ক্ষেত্রে]

Angument [আর্গুমেন্ট] $\theta_1 = \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\tan 30^0 = 30^0 = \frac{\pi}{6}$

and Modulas of z [এবং z এর পরম মান]

$$|z| = +\sqrt{(\sqrt{3})^2 + 1^2} = +\sqrt{3+1} = +\sqrt{4} = 2$$

In case (ii) [(ii) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta_2 = \tan^{-1}\frac{1}{-\sqrt{3}} = \tan^{-1}(-\tan 30^0)$

$$= \tan^{-1}\tan(180^0 - 30^0) = 150^0 = \frac{5\pi}{6}$$

and Modulas of z [এবং z এর পরম মান]

$$|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$$

In case (iii) [(iii) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \tan^{-1}\tan 30^0$

$$= \tan^{-1}\tan(180^0 + 30^0) = 210^0 = \frac{7\pi}{6}$$

[x ও y ঋণাত্মক বলে z এর অবস্থান তৃতীয় চর্তুভাগে]

এবং Modulas of z [z এর পরমমান], $|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

In case (iv) [(iv) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$$= \tan^{-1}(-\tan 30^0)$$

$$= \tan^{-1}\tan(360^0 - 30^0) = 330^0 = \frac{11\pi}{6}$$

and Modulas [এবং পরমমান] $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2.$

Example-5. Find the square roots of the complex number $5 - 12i$. **[RUH-2002]**

Solution : $5 - 12i = 9 - 12i - 4$

$$= 3^2 - 2 \cdot 3 \cdot 2i + (2i)^2 \quad [\because i^2 = -1]$$

$$= (3-2i)^2$$

$$\Rightarrow \sqrt{5-12i} = \sqrt{(3-2i)^2} = \pm(3-2i)$$

$\therefore$ The square roots of $5 - 12i$ are $3 - 2i$ and $-3 + 2i$. [$5 - 12i$ এর বর্গমূল হলো $3 - 2i$ এবং $-3 + 2i$.]

Example-6. Express $-5 + 5i$ in polar form. **[RUH-1998]**

Solution : Let $-5 = r\cos\theta$ and $5 = r\sin\theta$

Then by squaring and adding we get

$$(-5)^2 + 5^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$\Rightarrow 25 + 25 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow 50 = r^2 \Rightarrow r = \sqrt{50} = 5\sqrt{2}$$

Also $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{5}{-5} = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$\Rightarrow \theta = \frac{3\pi}{4}$

Thus, $-5 + 5i = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

$\Rightarrow -5 + 5i = 5\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

which is the required polar form.

Example–7. Show that the sum of the products of all the nth roots of unity taken 2, 3, 4, ⋯ , (n – 1) at a time is zero. **[RUH–1986]**

Solution : Let $z = (1)^{1/n}$. Then $z^n = 1 \Rightarrow z^n - 1 = 0$

$\Rightarrow z^n + 0 \cdot z^{n-1} + 0 \cdot z^{n-2} + \cdots + 0 \cdot z - 1 = 0$

Let $z_1, z_2, z_3, \cdots, z_n$ be the roots of this equation.

Sum of the products of the roots

taken two at a time is $\sum z_1 z_2 = 0$,

taken three at a time is $\sum z_1 z_2 z_2 = 0$

… … … …

taken (n – 1) at a time is $\sum z_1 z_2 \cdots z_{n-1} = 0$ (Proved)

Example–8. Find the fifth roots of unity. **[JUH–1987]**

Solution : Let [ধরি] $z = (1)^{1/5} = (\cos 0 + i\sin 0)^{1/5}$

$= (\cos 2n\pi + i\sin 2n\pi)^{1/5}$

$= \cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5} = e^{2n\pi i/5}$,

where [যেখানে] n = 0, 1, 2, 3, 4.

Thus, the required roots are [অতএব প্রয়োজনীয় মূলগুলি হল]

$1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$.

Example–9. Find all values of $(1 + i)^{1/4}$.

[NUH–2001, 2005, 2008, 2012(Old)]

Solution : Let [ধরি] $1 = r\cos\theta$ and [এবং] $1 = r\sin\theta$

$\therefore 1^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow r = \sqrt{1+1} = \sqrt{2}$

and [এবং] $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{1}{1} = 1 = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore (1 + i)^{1/4} = (r\cos\theta + ir\sin\theta)^{1/4}$

$= r^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/4}$

$= 2^{1/8}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}^{1/4}$; n = 0, 1, 2, 3.

$= 2^{1/8}\left(\cos\frac{8n\pi + \pi}{16} + i\sin\frac{8n\pi + \pi}{16}\right)$; n = 0, 1, 2, 3.

$= 2^{1/8}\left(\cos\frac{8n+1}{16}\pi + i\sin\frac{8n+1}{16}\pi\right)$; n = 0, 1, 2, 3.

Example–10. Determine all real x and y which satisfy the given relation [সকল বাস্তব x ও y নির্ণয় কর যাহা প্রদত্ত সম্পর্ক সিদ্ধ করে]

(a) $x + iy = |x + iy|$ (b) $x + iy = (x + iy)^2$

(c) Find the real numbers x and y such that $2x - iy + 3ix + y = 4 + i$. **[NUH–2000, 2006]**

Solution : (a) Given that [দেওয়া আছে] $x + iy = |x + iy|$

$\Rightarrow x + iy = \sqrt{x^2 + y^2}$

Equating real and imaginary parts we get, [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$x = \sqrt{x^2 + y^2}$ …… (1)

and $y = 0$ …… (2)

Using (2) in (1) we get, [(1) এ (2) ব্যবহার করে পাই] $x = \sqrt{x^2}$

This equation will be hold if [এই সমীকরণ খাটিবে যদি] $x \geq 0$.

Thus all real [অতএব সকল বাস্তব] $x \geq 0$ and $y = 0$ satisfy the relation

$x + iy = |x + iy|$.

(b) Given that [দেওয়া আছে] $x + iy = (x + iy)^2$

$\Rightarrow x + iy = x^2 + 2ixy + i^2y^2$

$\Rightarrow x + iy = x^2 + i\,2xy - y^2$ $[\because i^2 = -1]$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$x = x^2 - y^2$ …… (1)

and $y = 2xy$ …… (2)

From (2) we have [(2) হতে পাই] $(2x - 1)y = 0$

$\Rightarrow y = 0$ or $2x - 1 = 0$

$\Rightarrow y = 0$ or $x = \frac{1}{2}$

When [যখন] $y = 0$ then (1) gives [তখন (1) দেয়] $x = x^2 \Rightarrow x(x-1) = 0$

$$\therefore x = 0 \text{ or } x = 1$$

When [যখন] $x = \frac{1}{2}$ then (1) gives [তখন (1) দেয়] $\frac{1}{2} = \frac{1}{4} - y^2$

$$\Rightarrow y^2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

$$\Rightarrow y = \pm\sqrt{-\frac{1}{4}} = \pm\frac{1}{2}i$$

Thus the real $x = 0$ or 1 and $y = 0$ satisfy the relation

$$x + iy = |x + iy|.$$

[অতএব বাস্তব $x = 0$ অথবা 1 এবং $y = 0$ প্রদত্ত $x + iy = |x + iy|$ সম্পর্ককে সিদ্ধ করে]

(c) Given that [দেওয়া আছে]

$$2x - iy + 3ix + y = 4 + i$$

$$\Rightarrow 2x + y + i(3x - y) = 4 + i$$

Equating real and imaginary part [বাস্তব কাল্পনিক অংশ সমীকৃত করে পাই]

$$2x + y = 4 \text{ ...... (1)}$$

$$3x - y = 1 \text{ ...... (2)}$$

(1) + (2) gives, $5x = 5 \Rightarrow x = \frac{5}{5} = 1$

From (2) [(2) হতে] $y = 3x - 1 = 3 \times 1 - 1 = 2$

$\therefore x = 1, y = 2$ **Ans.**

Example–11. Find all the roots of the equation $\sinh z = i$. **[NUH–1995]**

Solution : Given that [দেওয়া আছে] $\sinh z = i$

$$\Rightarrow \frac{e^z - e^{-z}}{2} = i$$

$$\Rightarrow w - w^{-1} = 2i \quad \text{where [যেখানে] } w = e^z$$

$$\Rightarrow w^2 - 1 = 2iw$$

$$\Rightarrow w^2 - 2iw - 1 = 0$$

$$\Rightarrow w = \frac{2i \pm \sqrt{(-2i)^2 - 4\cdot1\cdot(-1)}}{2}$$

$$= \frac{2i \pm \sqrt{4i^2 + 4}}{2} = \frac{2i \pm \sqrt{-4+4}}{2}$$

$$\Rightarrow e^z = \frac{2i}{2} = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\Rightarrow e^z = \cos\left(2n\pi + \frac{\pi}{2}\right) \pm i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$\Rightarrow e^z = e^{i(2n\pi + \pi/2)}$$

$$\Rightarrow z = i\left(2n + \frac{1}{2}\right)\pi = \frac{i(4n+1)\pi}{2}$$

where $n = 0, \pm1, \pm2, \cdots$

Example– 11(a). Find all solutions of the equation $\sinh z = 2$.

Solution : Given that $\sinh z = 2$

$$\Rightarrow \frac{e^z - e^{-z}}{2} = 2$$

$$\Rightarrow e^z - \frac{1}{e^z} = 4$$

$$\Rightarrow (e^z)^2 - 1 = 4e^z$$

$$\Rightarrow (e^z)^2 - 4e^z - 1 = 0$$

$$\Rightarrow e^z = \frac{4 \pm \sqrt{(-4)^2 - 4.1.(-1)}}{2}$$

$$= \frac{4 \pm \sqrt{16+4}}{2}$$

$$= \frac{4 \pm 2\sqrt{5}}{2}$$

$$= 2 \pm \sqrt{5}$$

$\therefore z = \ln(2 \pm \sqrt{5})$ **Ans.**

Example–12. Find all solutions of the equation $\cosh z = 2$.

[NUH–2002, 2006, 2010, 2012, DUH–2001]

Solution : Given that [দেওয়া আছে] $\cosh z = 2$

$$\Rightarrow \frac{e^z + e^{-z}}{2} = 2$$

$$\Rightarrow e^z + \frac{1}{e^z} = 4$$

$$\Rightarrow (e^z)^2 + 1 = 4e^z$$

$$\Rightarrow (e^z)^2 - 4e^z + 1 = 0$$

$$\Rightarrow e^z = \frac{4 \pm \sqrt{(-4)^2 - 4}}{2}$$
$$= \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2}$$
$$= \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$
$$\therefore z = ln(2 \pm \sqrt{3}) \quad \text{Ans.}$$

Example–13(a). Prove that $|z|^2 = (Re(z))^2 + (Im(z))^2$, where z is any complex number.

Solution : Let $z = x + iy$. Then $Re(z) = x$, $Im(z) = y$ and
$$|z|^2 = (\sqrt{x^2+y^2})^2 = x^2 + y^2$$
$$= (Re(z))^2 + (Im(z))^2. \quad \text{(Proved)}$$

(b). If z be any complex number, then prove that

(i) $Re(z) = \frac{z+\bar{z}}{2}$, (ii) $Im(z) = \frac{z-\bar{z}}{2i}$

Solution : Let $z = x + iy$. Then $\bar{z} = \overline{x+iy} = x - iy$.

Also $Re(z) = x$, $Im(z) = y$.

(i) $z + \bar{z} = x + iy + x - iy = 2x$
$$\Rightarrow x = \frac{z+\bar{z}}{2} \Rightarrow Re(z) = \frac{z+\bar{z}}{2}$$

(ii) $z - \bar{z} = x + iy - x + iy = 2iy$
$$\Rightarrow y = \frac{z-\bar{z}}{2i} \Rightarrow Im(z) = \frac{z-\bar{z}}{2i} \quad \text{(Proved)}$$

Example–14. If z be a complex number then prove that

(i) z is real if $z = \bar{z}$.

and (ii) z is purely imaginary if $z = -\bar{z}$.

Solution : (i) Let $z = x + iy$ be a complex number.

Then $\bar{z} = x - iy$.

Now $z = \bar{z}$ gives $x + iy = x - iy$
$$\Rightarrow 2iy = 0 \Rightarrow y = 0$$

$\therefore z = x + iy = x + i0 = x$, which is real. Thus, z is real if $z = \bar{z}$

(ii) Let $z = x + iy$ be a complex number. Then $\bar{z} = x - iy$.

Now $z = -\bar{z}$ gives $x + iy = -(x - iy)$
$$\Rightarrow 2x = 0 \Rightarrow x = 0$$

$\therefore z = x + iy = 0 + iy = iy$, which is purely imaginary.

Thus z is purely imaginary if $z = -\bar{z}$.

Example–15. Prove that $|z|^2 = |-z|^2 = |\bar{z}|^2 = |-\bar{z}|^2 = z\bar{z}$.

Solution : Let [ধরি] $z = x + iy$. Then [তখন] $\bar{z} = \overline{x+iy} = x - iy$.
$$|z|^2 = |x+iy|^2 = (\sqrt{x^2+y^2})^2 = x^2+y^2$$
$$|-z|^2 = |-x-iy|^2 = (\sqrt{(-x)^2+(-y)^2})^2 = x^2+y^2$$
$$|\bar{z}|^2 = |x-iy|^2 = (\sqrt{x^2+(-y)^2})^2 = x^2+y^2$$
$$|-\bar{z}|^2 = |-x+iy|^2 = (\sqrt{(-x)^2+y^2})^2 = x^2+y^2$$
$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2$$

Thus we see that [অতএব আমরা দেখি যে]
$$|z|^2 = |-z|^2 = |\bar{z}|^2 = |-\bar{z}|^2 = z\bar{z}.$$

Example–16. Prove that :

(i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$, (ii) $\bar{\bar{z}} = z$, and (iii) $\overline{z+\bar{z}} = z + \bar{z}$.

Solution : (i) Let [ধরি] $z_1 = x_1 + iy_1$ and [এবং] $z_2 = x_2 + iy_2$.

Then [তখন] $\bar{z}_1 = x_1 - iy_1$ and [এবং] $\bar{z}_2 = x_2 - iy_2$.

$\therefore z_1 + z_2 = x_1 + iy_1 + x_2 + iy = (x_1 + x_2) + i(y_1 + y_2)$

$\Rightarrow \overline{z_1+z_2} = x_1 + x_2 - i(y_1 + y_2)$

$\Rightarrow \overline{z_1+z_2} = x_1 - iy_1 + x_2 - iy_2$

$\Rightarrow \overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(ii) Let [ধরি] $z = x + iy$. Then [তখন] $\bar{z} = x - iy \Rightarrow \bar{\bar{z}} = x + iy = z$.

(iii) Let $z = x + iy$. Then $\bar{z} = x - iy$.

$\therefore z + \bar{z} = x + iy + x - iy = 2x$

$\overline{z+\bar{z}} = \overline{2x} = 2x$, since 2x is real.

$\Rightarrow \overline{z+\bar{z}} = 2x = (x+iy) + (x-iy) = z + \bar{z}$

Otherway : $\overline{z+\bar{z}} = \bar{z} + \bar{\bar{z}}$ [by (i)]

$= \bar{z} + z$ [by (ii)]

$\Rightarrow \overline{z+\bar{z}} = z + \bar{z}$

Example–17. Show that

$$Im(iz) = Re(z) \text{ and } Re(iz) = |z|^2\, Im(z^{-1}).$$

Solution : Let $z = x + iy$. Then $Re(z) = x$ and $Im(z) = y$

Now $Im(iz) = Im(ix + i^2y) = Im(ix - y) = x = Re(z)$

and $Re(iz) = Re(ix + i^2y) = Re(ix - y) = -y$

On the other hand, $|z|^2 Im(z^{-1}) = |x + iy|^2 Im\left(\frac{1}{z}\right)$

$$= (x^2 + y^2). Im\left(\frac{1}{x + iy} \cdot \frac{x - iy}{x - iy}\right)$$

$$= (x^2 + y^2). Im\left(\frac{x - iy}{x^2 - i^2y^2}\right)$$

$$= (x^2 + y^2). Im\left(\frac{x - iy}{x^2 + y^2}\right) \quad [\because i^2 = -1]$$

$$= (x^2 + y^2). \frac{-y}{x^2 + y^2} = -y$$

Thus, $Re(iz) = -y = |z|^2 Im(z^{-1})$. (Showed)

Example–18. Show that the modulas of the quotient of two conjugate complex numbers is 1. **[RUH–1985, JUH–1986]**

Solution : Let $z = x + iy$ be a complex number. Then its conjugate number $\bar{z} = \overline{x + iy} = x - iy$.

$$\therefore \left|\frac{z}{\bar{z}}\right| = \left|\frac{x + iy}{x - iy}\right| = \frac{|x + iy|}{|x - iy|} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + (-y)^2}} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = 1$$

Thus, the modulas of the quotient of two conjugate complex numbers is 1.

Other way : Let $z = r\, e^{i\theta}$. Then $\bar{z} = re^{-i\theta}$.

$$\therefore \left|\frac{z}{\bar{z}}\right| = \left|\frac{re^{i\theta}}{re^{-i\theta}}\right| = |e^{i2\theta}| = |\cos 2\theta + i \sin 2\theta|$$

$$= \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1. \quad \text{Hence proved.}$$

Example–19. Show that, $|z|\,\sqrt{2} \geq |Re(z)| + |Im(z)|$, where z is any complex number. **[NUH–94, 13, NU(Pre)–08, DUH–88, 90, 00]**

Solution : Let [ধরি] $z = x + iy$.

Then [তখন] $Re(z) = x$ and [এবং] $Im(z) = y$.

For any two positive real numbers we know that [যে কোন দুইটি ধনাত্মক বাস্তব সংখ্যার জন্য আমরা জানি যে]

Taking $a = |x|$, $b = |y|$ and $m = 2$ we have [$a = |x|$, $b = |y|$ এবং $m = 2$ নিয়ে পাই]

$$\frac{|x|^2 + |y|^2}{2} \geq \left(\frac{|x| + |y|}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 \geq \frac{(|x| + |y|)^2}{2}$$

$$\Rightarrow (\sqrt{x^2 + y^2})^2 \geq \left(\frac{|x| + |y|}{\sqrt{2}}\right)^2$$

$$\Rightarrow \sqrt{x^2 + y^2} \geq \frac{|x| + |y|}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}\,|z| \geq |x| + |y|$$

$$\Rightarrow \sqrt{2}\,|z| \geq |Re(z)| + |Im(z)| \quad \text{(Showed)}$$

Example–20. Prove that $|x| + |y| \leq \sqrt{2}\,|z|$.

(a) If $z = x + iy$ then prove that $|x| + |y| \leq \sqrt{2}\,|x + iy|$

[NUH–1998, 2012(Old)]

Solution :

For any two positive real numbers [যে কোন দুইটি ধনাত্মক বাস্তব সংখ্যার জন্য আমরা জানি যে] $\frac{a^m + b^m}{2} \geq \left(\frac{a + b}{2}\right)^m$

where m is any real number except $o < m < 1$ [যেখানে $o < m < 1$ ব্যতীত m যে কোন বাস্তব সংখ্যা]

Taking $a = |x|$, $b = |y|$ and $m = 2$ we get [$a = |x|$, $b = |y|$ এবং $m = 2$ লইয়া পাই]

$$\frac{|x|^2 + |y|^2}{2} \geq \left(\frac{|x| + |y|}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 \geq \frac{(|x| + |y|)^2}{2}, \qquad \therefore |x|^2 = x^2,\ |y|^2 = y^2$$

$$\Rightarrow (\sqrt{x^2 + y^2})^2 \geq \left(\frac{|x| + |y|}{\sqrt{2}}\right)^2$$

$$\Rightarrow \sqrt{x^2 + y^2} \geq \frac{|x| + |y|}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}\,|z| \geq |x| + |y|$$

$$\Rightarrow |x| + |y| \leq \sqrt{2}\,|z|$$

$$\Rightarrow |x| + |y| \leq \sqrt{2}\,|x + iy| \quad \text{(Proved)}$$

Example-21. If $z_1, z_2, \cdots, z_n$ are complex numbers then show that [যদি $z_1, z_2, \cdots, z_n$ জটিল সংখ্যা হয় তখন দেখাও যে]

(i) $|z_1z_2| = |z_1|\ |z_2|$ **[DUM–1989]**

(ii) $|z_1z_2 \cdots z_n| = |z_1|\ |z_2| \cdots |z_n|$ **[DUH–1985]**

Solution : (i) Let [ধরি] $z_1 = x_1 + iy_1$ and [এবং] $z_2 = x_2 + iy_2$.

Then [তখন] $|z_1| = |x_1 + iy_1| = \sqrt{x_1^2 + y_1^2}$ and [এবং]

$$|z_2| = |x_2 + iy_2| = \sqrt{x_2^2 + y_2^2}.$$

Now [এখন] $|z_1z_2| = |(x_1 + iy_1)(x_2 + iy_2)|$

$$= |x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2|$$
$$= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)|$$
$$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}$$
$$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_1y_2}$$
$$= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}$$
$$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$
$$= \sqrt{x_1^2 + y_1^2}\ \sqrt{x_2^2 + y_2^2}$$
$$= |z_1|\ |z_2|$$

$\therefore |z_1z_2| = |z_1|\ |z_2|$

Other way [অন্যভাবে] **:** We know that [আমরা জানি যে]

$|z|^2 = z\bar{z}$ and [এবং] $\overline{z_1z_2} = \bar{z_1}\ \bar{z_2}$.

$$\therefore |z_1z_2|^2 = (z_1z_2)\ \overline{(z_1z_2)}$$
$$= z_1z_2\ \bar{z_1}\ \bar{z_2} = (z_1\bar{z_1})\ (z_2\bar{z_2}) = |z_1|^2\ |z_2|^2$$
$$= (|z_1|\ |z_2|)^2$$
$$\Rightarrow |z_1z_2| = |z_1|\ |z_2|$$

(ii) $|z_1z_2 \cdots z_n|^2 = (z_1z_2 \cdots z_n)\ \overline{(z_1z_2 \cdots z_n)}$

$$= (z_1z_2 \cdots z_n)\ (\bar{z_1}\ \bar{z_2} \cdots \bar{z_n})$$
$$= (z_1\ \bar{z_1})\ (z_2\bar{z_2}) \cdots (z_n)\ (\bar{z_n})$$
$$= |z_1|^2\ |z_2|^2 \cdots |z_n|^2$$
$$= (|z_1|\ |z_2| \cdots |z_n|)^2$$
$$\Rightarrow |z_1z_2 \cdots z_n| = |z_1|\ |z_2| \cdots |z_n| \quad \text{(proved)}$$

Example-22. For any complex number $z_1, z_2, \cdots, z_n$ prove that [যে কোন জটিল সংখ্যা $z_1, z_2, \cdots, z_n$ এর জন্য প্রমাণ কর যে]

(i) $|z_1 + z_2| \le |z_1| + |z_2|$

[NUH–02 (Old), 03, 04, 05, 07, 10, 12, NU(Pre)–08, DUH–98, 05]

(ii) $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

[RUH–1998, CUH–2004]

(iii) $|z_1 - z_2| \le |z_1| + |z_2|$ **[NUH–2006]**

Solution : We know that [আমরা জানি] $|z|^2 = z\bar{z}$ and $\bar{\bar{z}} = z$.

$$\therefore |z_1 + z_2|^2 = (z_1 + z_2)\ \overline{(z_1 + z_2)}$$
$$= (z_1 + z_2)\ (\bar{z_1} + \bar{z_2})$$
$$= z_1\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_1} + z_2\bar{z_2}$$
$$= |z_1|^2 + z_1\bar{z_2} + \overline{z_1\bar{z_2}} + |z_2|^2$$
$$= |z_1|^2 + 2\text{Re}(z_1\bar{z_2}) + |z_2|^2 \qquad [\because z + \bar{z} = 2\text{Re}(z)]$$
$$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1\bar{z_2}| + |z_2|^2 \qquad [\because \text{Re}(z) \le |z|]$$
$$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1|\ |\bar{z_2}| + |z_2|^2 \qquad [\because |z_1z_2| = |z_1|\ |z_2|]$$
$$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1|\ |z_2| + |z_2|^2 \qquad [\because |z| = |\bar{z}|]$$
$$\Rightarrow |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2|.$$

Other way [অন্যভাবে] **:**

Let $z_1 = r_1\, e^{i\theta_1}$ and $z_2 = r_2\, e^{i\theta_2}$. Then $|z_1| = |r_1\, e^{i\theta_1}|$

$$= |r_1|\ |e^{i\theta_1}| = r_1 \cdot 1 = r_1 \text{ and } |z_2| = r_2$$

$$\therefore z_1 + z_2 = r_1\, e^{i\theta_1} + r_2\, e^{i\theta_2} = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$
$$\Rightarrow |z_1 + z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$$
$$= r_1^2\cos^2\theta_1 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos\theta_2^2$$
$$+ r_1^2\sin^2\theta_1 + 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin\theta_2^2$$
$$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + 2r_1r_2(\cos\theta_1\cos\theta_2$$
$$+ \sin\theta_1\sin\theta_2) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$$
$$= r_1^2 \cdot 1 + 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2 \cdot 1$$

$$\Rightarrow |z_1 + z_2|^2 \le r_1^2 + 2r_1r_2 + r_2^2 \quad [\because \cos(\theta_1 - \theta_2) \le 1]$$

$$\Rightarrow |z_1 + z_2|^2 \le (r_1 + r_2)^2$$

$$\Rightarrow |z_1 + z_2| \le r_1 + r_2$$

$$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2|$$

(ii) $|z_1 + z_2 + \cdots + z_n| = |z_1 + (z_2 + \cdots + z_n)|$

$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2 + \cdots + z_n| \quad \text{[by (i)]}$$

$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2 + (z_3 + \cdots + z_n)|$$

$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + |z_3 + \cdots + z_n|$$

Proceding in the same way we get [একইভাবে অগ্রসর হয়ে আমরা পাই]

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

(iii)
$$\begin{aligned}
|z_1 - z_2|^2 &= |z_1 - z_2)\, \overline{(z_1 - z_2)} && [\because |z|^2 = z\bar{z}] \\
&= (z_1 - z_2)\, (\overline{z_1} - \overline{z_2}) && [\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}] \\
&= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 - (z_1\overline{z_2} + z_2\overline{z_1}) + |z_2)^2 && [\because |z|^2 = z\bar{z}] \\
&= |z_1|^2 - (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + |z_2|^2 && [\because \bar{\bar{z}} = z] \\
&= |z_1|^2 - 2\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2
\end{aligned}$$

$$\left.\begin{aligned}
&\Rightarrow |z_1 - z_2|^2 \le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&\Rightarrow |z_1 - z_2|^2 \le (|z_1| + |z_2|)^2
\end{aligned}\right. \qquad \begin{cases} \because -x \le \sqrt{x^2 + y^2} \\ \Rightarrow -\mathrm{Re}(z) \le |z| \end{cases}$$

$$\Rightarrow |z_1 - z_2| \le |z_1| + |z_2| \quad \text{(Proved)}$$

Example–23. Let **C** be the set of all complex numbers. Consider $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbf{C}$ with $x_1 < x_2$, $y_1 < y_2$. Do you agree that $z_1 < z_2$? What about $|z_1| < |z_2|$? **[NUH–2014]** Prove that

$$\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j| \text{ and } \left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|,$$ where $z_1, z_2, \cdots, z_n$ are complex numbers. **[NUH–1997]**

Solution : Given that $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$. That is, z_1 and z_2 are two points in the Argand (complex) plane. We know that greater than or less than have no meaning in relation between two complex numbers. So $z_1 < z_2$ has no meaning. Thus, I do not agree that $z_1 < z_2$.

[দেওয়া আছে $z_1 = x_1 + iy_1 = (x_1, y_1)$ এবং $z_2 = x_2 + iy_2 = (x_2, y_2)$. অর্থাৎ, আর্গান্ড তলে z_1 ও z_2 দুইটি বিন্দু। আমরা জানি দুইটি জটিল সংখ্যা সর্ম্পকের ক্ষেত্রে হতে বৃহত্তর বা হতে ক্ষুদ্রতর-এর কোন অর্থ নাই। সুতরাং $z_1 < z_2$ ইহার কোন অর্থ নাই। অতএব আমি সম্মত নই যে $z_1 < z_2$.]

2nd Part : Given that [দেওয়া আছে] $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, where [যেখানে] $x_1 < x_2$, $y_1 < y_2$.

$\therefore |z_1| = \sqrt{x_1^2 + y_1^2}$ and [এবং] $|z_2| = \sqrt{x_2^2 + y_2^2}$

Now [এখন] $x_1 < x_2 \Rightarrow x_1^2 < x_2^2$

and [এবং] $y_1 < y_2 \Rightarrow y_1^2 < y_2^2$

$$\Rightarrow x_1^2 + y_1^2 < x_2^2 + y_2^2 \quad \text{[by adding]}$$

$$\Rightarrow \sqrt{x_1^2 + y_1^2} < \sqrt{x_2^2 + y_2^2} \quad \text{[by taking squre root]}$$

$$\Rightarrow |z_1| < |z_2|$$

But this inequality is not always true. We counter this by the following example. [কিন্তু এই অসমতা সব সময় সত্য নয়। আমরা নিম্নের উদাহরণ দ্বারা ইহার বিরোধিতা করি।]

Let [ধরি] $z_1 = x_1 + iy_1 = 1 + i(-7)$

and [এবং] $z_2 = x_2 + iy_2 = 3 + i2$

Here [এখানে] $x_1 = 1, x_2 = 3, y_1 = -7, y_2 = 2$

$$1 < 3 \Rightarrow x_1 < x_2 \text{ and [এবং] } -7 < 2 \Rightarrow y_1 < y_2$$

$$|z_1| = \sqrt{x_1^2 + y_1^2} = \sqrt{1^2 + (-7)^2} = \sqrt{50}$$

$$|z_2| = \sqrt{x_2^2 + y_2^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

It is true that [ইহা সত্য যে] $\sqrt{50} > \sqrt{13} \Rightarrow |z_1| > |z_2|$

Thus, $|z_1| < |z_2|$ is not always true under the given conditions. $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is. [$|z_1| < |z_2|$ সবসময় সত্য নয়। $|z_1| < |z_2|$ এর অর্থ হলো z_1 বিন্দুটি z_2 বিন্দু হতে মূলবিন্দুর অধিক নিকটে।]

3rd Part : Given that [দেওয়া আছে] $\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j|$

$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

Now do as example–22.

4th Part : Given that $\left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$

that is, $|z_1 z_2 \cdots z_n| = |z_1|\ |z_2| \cdots |z_n|$

Now do as example–22.

Example–24. Prove that [প্রমাণ কর]

$$|z_1 - z_2| \geq \big||z_1| - |z_2|\big| \geq |z_1| - |z_2|$$

[NUH–1994, 2002 (Old), 2014, DUH–1998, 2005]

Solution : We know that [আমরা জানি]

$|z|^2 = z\bar{z}$ and [এবং] $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\therefore |z_1 - z_2|^2 = (z_1 - z_2)\,\overline{(z_1 - z_2)}$

$= (z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2)$

$= z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2 \bar{z}_2$

$= |z_1|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2) + |z_2|^2$

$= |z_1|^2 - (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + |z_2|^2$ $\quad [\because \bar{\bar{z}} = z]$

$= |z_1|^2 - 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2$ $\quad [\because z + \bar{z} = 2\mathrm{Re}(z)]$

$\Rightarrow |z_1 - z_2|^2 \geq |z_1|^2 - 2|z_1\bar{z}_2| + |z_2|^2$ $\quad \left[\begin{array}{l}\because x = \mathrm{Re}(z) \leq |z| \\ \Rightarrow -\mathrm{Re}(z) \geq -|z|\end{array}\right]$

$\Rightarrow |z_1 - z_2|^2 \geq |z_1|^2 - 2|z_1|\ |\bar{z}_2| + |z_2|^2$

$\Rightarrow |z_1 - z_2|^2 \geq |z_1|^2 - 2|z_1|\ |z_2| + |z_2|^2$ $\quad [\because |z| = |\bar{z}|]$

$\Rightarrow |z_1 - z_2|^2 \geq (|z_1| - |z|)^2 = (\,\big||z_1| - |z_2|\big|\,)^2$

$\Rightarrow |z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ ...... (1)

Again, we have [আবার, আমাদের আছে]

$\big||z_1| - |z_2|\big| \geq |z_1| - |z_2|$ ...... (2)

From (1) and (2) we have [(1) ও (2) হতে পাই]

$|z_1 - z_2| \geq \big||z_1| - |z_2|\big| \geq |z_1| - |z_2|$ (Proved)

Example–25. If z_1, z_2, z_3, z_4 are complex numbers then show that [যদি z_1, z_2, z_3, z_4 জটিল সংখ্যা হয় তখন দেখাও]

(i) $\left|\dfrac{z_1}{z_2 + z_3}\right| \leq \dfrac{|z_1|}{\big||z_2| - |z_3|\big|}$, where [যেখানে] $|z_2| \neq |z_3|$

[NUH–2011, DUH–98]

(ii) $\left|\dfrac{z_1 + z_2}{z_3 + z_4}\right| \leq \dfrac{|z_1| + |z_2|}{\big||z_3| - |z_4|\big|}$, where [যেখানে] $|z_3| \neq |z_4|$.

[CUH–2000, DUH–2001, 2003, 2006]

Solution : (i) We know that [আমরা জানি] $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$.

Replacing z_2 by $-z_2$ we get [z_2 কে $-z_2$ দ্বারা প্রতিস্থাপন করে পাই]

$|z_1 + z_2| \geq \big||z_1| - |-z_2|\big|$

$\Rightarrow |z_1 + z_2| \geq \big||z_1| - |z_2|\big|$ $\quad [\because |z| = |-z|$ ...... (1)]

$\Rightarrow \dfrac{1}{|z_1 + z_2|} \leq \dfrac{1}{\big||z_1| - |z_2|\big|}$

Multiplying both sides by $|z_1|$ we get [উভয় পক্ষকে $|z_1|$ দ্বারা গুণ করে পাই]

$$\frac{|z_1|}{|z_2 + z_3|} \leq \frac{|z_1|}{\big||z_2| - |z_3|\big|}$$

(ii) We know that [আমরা জানি] $|z_1 + z_2| \leq |z_1| + |z_2|$ ...... (2)

and by (1) we have [এবং (1) হতে পাই]

$|z_3 + z_4| \geq \big||z_3| - |z_4|\big|$

$\Rightarrow \dfrac{1}{|z_3 + z_4|} \leq \dfrac{1}{\big||z_3| - |z_4|\big|}$ ...... (3)

Combining (2) and (3) we get [(1) ও (2) কে একত্র করে পাই]

$\dfrac{|z_1 + z_2|}{|z_3 + z_4|} \leq \dfrac{|z_1| + |z_2|}{\big||z_3| - |z_4|\big|}$ (Proved)

Example–26. Find two complex numbers whose sum is 4 and whose product is 8. [দুইটি জটিল সংখ্যা বাহির কর যাদের যোগফল 4 এবং গুণফল 8]

[NUH–2000, 2006(Old), 2012]

Solution : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. According to the question [ধরি $z_1 = x_1 + iy_1$ এবং $z_2 = x_2 + iy_2$ দুইটি জটিল সংখ্যা। প্রশ্নানুসারে]

$z_1 + z_2 = 4$ ...... (1)

and [এবং] $z_1 z_2 = 8$ ...... (2)

From (1) [(1) হতে] $z_1 + z_2 = 4$

$\Rightarrow x_1 + iy_1 + x_2 + iy_2 = 4$

$\Rightarrow (x_1 + x_2) + i(y_1 + y_2) = 4$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$x_1 + x_2 = 4 \cdots\cdots (3)$

$y_1 + y_2 = 0 \cdots\cdots (4)$

Again, from (2) we get [আবার, (2) হতে পাই]

$(x_1 + iy_1)(x_2 + iy_2) = 8$

$\Rightarrow x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 = 8$

$\Rightarrow (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = 8 \quad [\because i^2 = -1]$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$x_1x_2 - y_1y_2 = 8 \cdots\cdots (5)$

$x_1y_2 + x_2y_1 = 0 \cdots\cdots (6)$

From (4) [(4) হতে] $y_1 = -y_2$

From (6) [(6) হতে] $x_1y_2 = -x_2y_1$

$\Rightarrow \dfrac{x_1}{x_2} = \dfrac{-y_1}{y_2} = \dfrac{-(-y_2)}{y_2} = 1$

$\Rightarrow x_1 = x_2$

From (3) [(3) হতে] $x_1 + x_2 = 4 \Rightarrow 2x_1 = 4 \Rightarrow x_1 = \dfrac{4}{2} = 2$

$\therefore x_1 = x_2 = 2$

From (5) [(5) হতে] $2 \cdot 2 - (-y_2)y_2 = 8$

$\Rightarrow y_2^2 = 8 - 4 = 4$

$\therefore y_2 = \pm 2$

$\therefore y_1 = -y_2 = -(\pm 2) = \mp 2$

That is [অর্থাৎ] $y_1 = -2,\ y_2 = 2$ and [এবং] $y_1 = 2,\ y_2 = -2$.

$\therefore z_1 = x_1 + iy_1 = 2 - 2i$ or $2 + 2i$

$z_2 = x_2 + iy_2 = 2 + 2i$ or $2 - 2i$

Thus, the required two complex numbers are $2 + 2i$ and $2 - 2i$. [অতএব প্রার্থিত (আবশ্যকীয়) দুইটি জটিল সংখ্যা হল $2 + 2i$ এবং $2 - 2i$]

Other way [অন্যভাবে] **:** Let z_1 and z_2 be two complex numbers. Then according to the question [ধরি z_1 ও z_2 দুইটি জটিল সংখ্যা, তখন প্রশ্ন অনুসারে]

$z_1 + z_2 = 4 \cdots\cdots (1)$

and [এবং] $z_1z_2 = 8 \cdots\cdots (2)$

Now [এবং] $(z_1 - z_2)^2 = (z_1 + z_2)^2 - 4z_1z_2$

$= 4^2 - 4 \cdot 8 = 16 - 32 = -16$

$= (4i)^2$

$\therefore z_1 - z_2 = \pm 4i$

$\Rightarrow z_1 - z_2 = 4i \cdots\cdots (3)$

or $z_1 - z_2 = -4i \cdots\cdots (4)$

(1) + (3) gives [দেয়] $2z_1 = 4 + 4i \Rightarrow z_1 = \dfrac{4+4i}{2} = 2 + 2i$

(1) – (3) gives [দেয়] $2z_2 = 4 - 4i \Rightarrow z_2 = \dfrac{4-4i}{2} = 2 - 2i$

(1) + (4) gives [দেয়] $2z_1 = 4 - 4i \Rightarrow z_1 = \dfrac{4-4i}{2} = 2 - 2i$

(1) – (4) gives [দেয়] $2z_2 = 4 + 4i \Rightarrow z_2 = \dfrac{4+4i}{2} = 2 + 2i$

Thus the two complex numbers are $2 + 2i$, $2 - 2i$. [অতএব দুইটি জটিল সংখ্যা হল $2 + 2i$, $2 - 2i$.]

Example–27. Prove that, if sum and product of two complex numbers are both real, then the two numbers must either be real or conjugate. [প্রমাণ কর যে, যদি দুইটি জটিল সংখ্যার যোগফল ও গুণফল উভয়ে বাস্তব হয়, তখন সংখ্যা দুইটি অবশ্যই হয় বাস্তব নতুবা অনুবন্ধী হবে।]

Solution : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. [দুইটি জটিল সংখ্যা] Then their sum [তখন তাদের যোগফল $= z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$ and product [এবং গুণফল] $= z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + i(x_1y_2 + x_2y_1) + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Sum will be real if [যোগফল বাস্তব হবে যদি] $y_1 + y_2 = 0 \Rightarrow y_2 = -y_1 \cdots (1)$

or $y_1 = y_2 = 0 \cdots (2)$

Product will be real if [গুণফল বাস্তব হবে যদি] $x_1y_2 + x_2y_1 = 0$

$$\Rightarrow x_1y_2 = -x_2y_1$$

$$\Rightarrow \frac{x_1}{x_2} = \frac{-y_1}{y_2} \;\ldots\ldots (3)$$

(1) and (3) gives, [(1) এবং (3) দেয়] $\frac{x_1}{x_2} = \frac{-y_1}{-y_1} = 1 \Rightarrow x_1 = x_2$

Now, when [এখন, যখন] $y_1 = y_2 = 0$ then [তখন] $z_1 = x_1 + i0 = x_1$ and $z_2 = x_2 + i0 = x_2$

In this case, z_1 and z_2 are both real. [এই ক্ষেত্রে z_1 ও z_2 উভয়ে বাস্তব]

When [যখন] $x_1 = x_2$ and $y_2 = -y_1$ then [তখন]

$$z_1 = x_1 + iy_1 = x_2 - iy_2 = \overline{x_2 + iy_2} = \overline{z_2}$$

$$z_2 = x_2 + iy_2 = x_1 - iy_1 = \overline{x_1 + iy_1} = \overline{z_1}$$

Thus, if sum and product of two complex numbers are both real, then the two numbers must either be real or conjugate. [অতএব, যদি দুইটি জটিল সংখ্যার যোগফল ও গুণফল উভয়ে বাস্তব হয়, তখন সংখ্যা দুইটি অবশ্যই হয় বাস্তব নতুবা অনুবন্ধী হবে] (Proved)

Example–28. Prove that [প্রমাণ কর]

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$

[NUH–2000, 2006 (Old)]

Interpret the result geometrically and deduce that [ফলটি জ্যামিতিকভাবে ব্যাখ্যা কর এবং প্রতিষ্ঠিত কর]

$$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|.$$

[NUH–1993, 2004 (Old), 2006 (Old), 2012, NU(Pre)–2011]

Solution : $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$$= (z_1 + z_2)\,\overline{(z_1 + z_2)} + (z_1 - z_2)\,\overline{(z_1 - z_2)}$$

$$= (z_1 + z_2)\,(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)\,(\overline{z_1} - \overline{z_2})$$

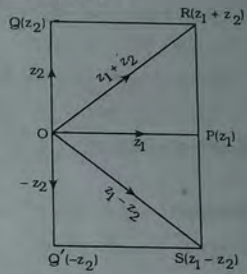
$$= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - \overline{z_1}z_2 + z_2\overline{z_2}$$

$$= |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$$

$$= 2|z_1|^2 + 2|z_2|^2$$

2nd Part (Geometrical interpretation) :

Let z_1 and z_2 be two complex numbers represented by the points P and Q in the Argand diagram. Complete the parallelogram OPRQ. Produce OQ backward upto Q′ so that OQ′ = OQ. Complete the parallelogram OQ′SP. Then the diagonals OR represents $z_1 + z_2$ and OS represents $z_1 - z_2$.



Thus we have $|z_1| = OP$, $|z_2| = OQ = PR$, $|z_1 + z_2| = OR$, $|z_1 - z_2| = OS$. Here P is the middle point of RS. Hence we have [প্রথম এবং দ্বিতীয় সর্ম্পক হতে পাই]

$$OR^2 + OS^2 = 2OP^2 + 2PR^2$$

$$\Rightarrow |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,|z_1|^2 + 2|z_2|^2$$

[২য় অংশ (জ্যামিতিক ব্যাখ্যা) ঃ

মনে করি আর্গান্ড চিত্রে P ও Q বিন্দু দ্বারা নির্দেশিত দুইটি জটিল সংখ্যা z_1 ও z_2. OPQR সামন্তরিকটি সম্পূর্ণ করি। OQ কে পিছনের দিকে Q′ পর্যন্ত বর্ধিত করি যেন OQ′ = OQ হয়। OQ′SP সামন্তরিকটি সম্পূর্ণ করি। তখন কর্ণ OR, $z_1 + z_2$ নির্দেশ করে এবং OS, $z_1 - z_2$ নির্দেশ করে।

অতএব আমরা পাই $|z_1| = OP$, $|z_2| = OQ = PR$, $|z_1 + z_2| = OR$, $|z_1 - z_2| = OS$. এখানে RS এর মধ্যবিন্দু হল P. অতএব আমরা পাই,

$$OR^2 + OS^2 = 2OP^2 + 2PR^2$$

$$\Rightarrow |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,|z_1|^2 + 2|z_2|^2]$$

3rd Part : We know that [আমরা জানি]

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2) \;\ldots\ldots (1)$$

Now [এখন] $(|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}|)^2$

$$= |\alpha + \sqrt{\alpha^2 - \beta^2}|^2 + |\alpha - \sqrt{\alpha^2 - \beta^2}|^2 + 2|\alpha + \sqrt{\alpha^2 - \beta^2}|\;|\alpha - \sqrt{\alpha^2 - \beta^2}|$$

$$= 2|\alpha|^2 + 2|\sqrt{\alpha^2 - \beta^2}|^2 + 2|(\alpha + \sqrt{\alpha^2 - \beta^2})\,(\alpha - \sqrt{\alpha^2 - \beta^2})| \quad \text{[by (1)]}$$

$$= 2|\alpha|^2 + 2|\alpha^2 - \beta^2| + 2|\alpha^2 - \alpha^2 + \beta^2|$$

$$= 2|\alpha|^2 + 2|\alpha+\beta|\ |\alpha-\beta| + 2|\beta|^2$$
$$= |\alpha+\beta|^2 + |\alpha-\beta|^2 + 2|\alpha+\beta|\ |\alpha-\beta|$$
$$= (|\alpha+\beta| + |\alpha-\beta|)^2$$

Taking square root we have [বর্গমূল নিয়ে পাই]

$$|\alpha + \sqrt{\alpha^2-\beta^2}| + |\alpha - \sqrt{\alpha^2-\beta^2}| = |\alpha+\beta| + |\alpha-\beta| \qquad \text{(Proved)}$$

Example–29. If $|z_1| = |z_2|$ and amp z_1 + amp $z_2 = 0$, then $z_2 = \overline{z_1}$.

Solution : Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

Then $\overline{z_1} = \overline{r_1 e^{i\theta_1}} = r_1 e^{-i\theta_1}$

Given that $|z_1| = |z_2|$

$$\Rightarrow |r_1 e^{i\theta_1}| = |r_2 e^{i\theta_2}|$$
$$\Rightarrow |r_1|\ |e^{i\theta_1}| = |r_2|\ |e^{i\theta_2}| \qquad [\because |e^{i\theta}| = |\cos\theta + i\sin\theta|$$
$$\Rightarrow r_1 \cdot 1 = r_2 \cdot 1 \Rightarrow r_1 = r_2 \qquad = \sqrt{\cos^2\theta + \sin^2\theta} = 1]$$

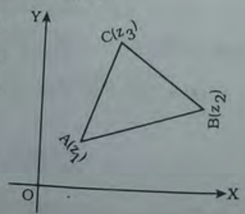
Again, amp z_1 + amp $z_2 = 0$

$$\Rightarrow \text{amp}(z_1 z_2) = 0$$
$$\Rightarrow \theta_1 + \theta_2 = 2n\pi, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$
$$\Rightarrow \theta_2 = 2n\pi - \theta_1$$
$$\therefore z_2 = r_2 e^{i\theta_2} = r_1 e^{i(2n\pi - \theta_1)}$$
$$= r_1 [\cos(2n\pi - \theta_1) + i\sin(2n\pi - \theta_1)]$$
$$= r_1(\cos\theta_1 - i\sin\theta_1)$$
$$= r_1[\cos(-\theta_1) + i\sin(-\theta_1)] \qquad [\because \cos\theta = \cos(-\theta)]$$
$$= r_1 e^{-i\theta_1} = \bar{z}_1.$$

Example–30. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the argand plane, then show that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution : Let the triangle be ΔABC, where the points A, B, C are z_1, z_2, z_3 respectively. The triangle is equilateral, so [মনেকরি ত্রিভুজটি ΔABC, যেখানে A, B, C বিন্দুগুলি যথাক্রমে z_1, z_2, z_3 ত্রিভুজটি সমবাহু, সুতরাং]



$$AB = BC = CA$$
$$\Rightarrow |z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$$
$$\Rightarrow |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \qquad [\because |z| = |-z|]$$
$$\Rightarrow |z_1 - z_2|^2 = |z_2 - z_3|^2 = |z_3 - z_1|^2$$
$$\Rightarrow (z_1 - z_2)\overline{(z_1 - z_2)} = (z_2 - z_3)\overline{(z_2 - z_3)} = (z_3 - z_1)\overline{(z_3 - z_1)}$$
$$\Rightarrow (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = (z_2 - z_3)(\overline{z_2} - \overline{z_3}) = (z_3 - z_1)(\overline{z_3} - \overline{z_1}) \cdots\cdots (1)$$

From the first and second relation we have [প্রথম এবং দ্বিতীয় সম্পর্ক হতে পাই]

$$(z_1 - z_2)(\overline{z_1} - \overline{z_2}) = (z_2 - z_3)(\overline{z_2} - \overline{z_3})$$
$$\Rightarrow \frac{z_1 - z_2}{z_2 - z_3} = \frac{z_2 - z_3}{z_1 - z_2} = \frac{(z_1 - z_2) + (z_2 - z_3)}{(z_2 - z_3) + (z_1 - z_2)} = \frac{z_1 - z_3}{z_1 - z_3}$$
$$\Rightarrow \frac{z_1 - z_2}{z_2 - z_3} = \frac{z_1 - z_3}{z_1 - z_3} \cdots\cdots (2)$$

From the last two relation of (1) we have [(1) এর শেষ দুইটি সম্পর্ক হতে পাই]

$$(z_2 - z_3)(\overline{z_2} - \overline{z_3}) = (z_3 - z_1)(\overline{z_3} - \overline{z_1}) \cdots\cdots (3)$$

Multiplying (2) and (3) we get [(2) ও (3) গুণ করে পাই]

$$\frac{z_1 - z_2}{z_2 - z_3}(z_2 - z_3)(\overline{z_2} - \overline{z_3}) = \frac{z_1 - z_3}{z_1 - z_3}(z_3 - z_1)(\overline{z_3} - \overline{z_1})$$
$$\Rightarrow (z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)(z_1 - z_3)$$
$$\Rightarrow z_1z_2 - z_1z_3 - z_2^2 + z_2z_3 = z_1^2 - z_3z_1 - z_3z_1 + z_3^2$$
$$\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 = z_1^2 + z_2^2 + z_3^2$$
$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \qquad \text{(Showed)}$$

Example–31. If the equation $z^2 + az + b = 0$ has a pair of conjugate complex roots then prove that a, b are both real and $a^2 < 4b$.

Solution : Let the complex conjugate roots of $z^2 + az + b = 0$ are $p + iq$ and $p - iq$ where p and q are real. [মনেকরি $z^2 + az + b = 0$ এর জটিল অনুবন্ধী মূলদ্বয় $p + iq$ এবং $p - iq$ যেখানে p ও q বাস্তব]

sum of the roots [মূলদ্বয়ের যোগফল] $= p + iq + p - iq = -a$

$$\Rightarrow 2p = -a \Rightarrow p = \frac{-a}{2}$$

$\therefore$ a is real, since p is real. [a বাস্তব, যেহেতু p বাস্তব]

Product of the roots [মূলদ্বয়ের গুণফল] = (p + iq) (p − iq) = b

$$\Rightarrow p^2 - i^2q^2 = b$$
$$\Rightarrow p^2 + q^2 = b$$

b is real, since p and q are real. [b বাস্তব, যেহেতু p এবং q বাস্তব]

Also, $p^2 + q^2 = b$ gives, $q^2 = b - p^2 > 0$

$$\Rightarrow b - p^2 > 0 \Rightarrow b - \left(\frac{-a}{2}\right)^2 > 0$$
$$\Rightarrow b - \frac{a^2}{4} > 0 \Rightarrow 4b - a^2 > 0$$
$$\Rightarrow 4b > a^2 \Rightarrow a^2 < 4b$$

Thus, we have a, b are both real and $a^2 < 4b$. [অতএব, a, b উভয়ে বাস্তব এবং $a^2 < 4b$]

Example–32. Solve the equation $|z| - z = 2 + i$.

Solution : Let $z = x + iy$. Then [তখন] $|z| = \sqrt{x^2 + y^2}$

Given that [দেওয়া আছে] $|z| - z = 2 + i$

$$\Rightarrow \sqrt{x^2 + y^2} - (x + iy) = 2 + i$$

Equating real and imaginary parts we get, [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\sqrt{x^2 + y^2} - x = 2 \;\cdots\cdots\; (1)$$
$$-y = 1 \Rightarrow y = -1 \;\cdots\cdots\; (2)$$

Using (2) in (1) we get, [(2) কে (1) এ ব্যবহার করে পাই]

$$\sqrt{x^2 + 1} - x = 2$$
$$\Rightarrow \sqrt{x^2 + 1} = x + 2$$
$$\Rightarrow x^2 + 1 = x^2 + 4x + 4 \quad \text{by squaring}$$
$$\Rightarrow 4x = -3 \Rightarrow x = \frac{-3}{4}$$
$$\therefore z = x + iy = \frac{-3}{4} - i \quad \text{(Ans)}$$

Example–33. Describe geometrically the region of the following :

(i) $|z - 4| > |z|$ **[NUH–1998, DUH–1988, 1998]**

(ii) $|z - i| = |z + i|$ **[NUH–03, 06, 10, 12, DUH–87, RUH–04]**

(iii) $\text{Im}(z) > 1$ **[DUH–1988]**

(iv) $\text{Re}(\bar{z} - 1) = 2$

(v) $|z + 3i| > 4$ **[DUH–1989]**

(vi) $|z| > 4$ **[DUH–1986]**

(vii) $|z - 2 + i| \leq 1$ **[DUH–1988]**

(viii) $|2z + 3| > 4$ **[DUH–1988]**

(ix) $\left|\frac{z-3}{z+3}\right| = 3$

(x) $\left|\frac{z-3}{z+3}\right| > 3$

(xi) $\left|\frac{z-3}{z+3}\right| < 3$ **[DUH–1989]**

(xii) $\text{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$ **[NU(Pre)–08, DUH–1988]**

(xiii) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

[NUH–2000, 06 (Old), 10, 12(Old), 15, DUH– 86, 88, 90, 98, 03]

(xiv) $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$ **[DUH–1987]**

(xv) $1 < |z + i| \leq 2$ **[NUH–2007, 2012(Old), DUH–1989]**

(xvi) $1 < |z + i| < 2$ **[DUH–1990]**

(xvii) $1 < |z - 2i| \leq 2$ **[DUH–1988]**

(xviii) $0 < \text{Re}(iz) < 1$ **[DUH–1990]**

(xix) $\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{2}$

(xx) $-\pi < \arg z < \pi$ **[DUH–1986]**

(xxi) $-\pi < \arg z < \pi,\ z \neq 0$ **[DUH–1988]**

(xxii) $0 < \arg z < 2\pi,\ |z| > 0$ **[DUH–1989]**

(xxiii) $-\pi < \arg z < \pi,\ |z| > 2$ **[DUH–1988]**

(xxiv) $|z - 1| + |z + 1| \leq 3$ **[NUH–2005, CUH–2004]**

(xxv) $|z + 2 - 3i| + |z - 2 + 3i| < 10$ **[DUH–1989]**

(xxvi) $|z - 2| - |z + 2| > 3$ **[DUH–1990]**

(xxvii) $\text{Re}(z^2) > 1$ **[DUH–1989]** (xxviii) $\text{Im}(z^2) > 0$ [DUH–1989]

(xxix) $|z+i| + |z-i| \le 3$ [NUH–1995, 2008]

(xxx) $|z+2i| + |z-2i| = 6$ [NUH–2004 (Old)]

(xxxi) $1 \le |z+1+i| < 2$ [NUH–1994]

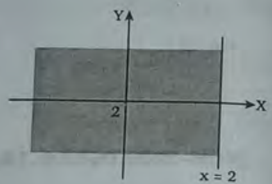
(xxxii) $\text{Re}\left(\frac{1}{z}\right) > \frac{1}{2}$ NUH–1994]

(xxxiii) $\text{Im}(z^2) > 2$ [NUH–1994]

(xxxiv) $|z+1+i| = |z-1+i|$ [NUH–1994

Solution : (i) Let [ধরি] $z = x + iy$.

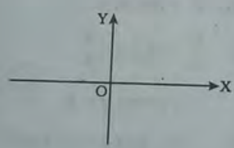
Then [এখন] $|z-4| > |z|$ gives [দেয়]



$$|x + iy - 4| > |x + iy|$$
$$\Rightarrow \sqrt{(x-4)^2 + y^2} > \sqrt{x^2 + y^2}$$
$$\Rightarrow x^2 - 8x + 16 + y^2 > x^2 + y^2$$
$$\Rightarrow -8x + 16 > 0$$
$$\Rightarrow 8x < 16 \Rightarrow x < 2$$

$\therefore$ The region is the set of all points (x, y) such that $x < 2$. That is, the set of all points (x, y) left hand side of the straight line $x = 2$. [এলাকটি হয় সকল বিন্দুসমূহ (x, y) এর সেট যেন $x < 2$ হয়। অর্থাৎ $x = 2$ সরলরেখার বামদিকের সকল (x, y) বিন্দুর সেট।]

(ii) $|z - i| = |z + i|$



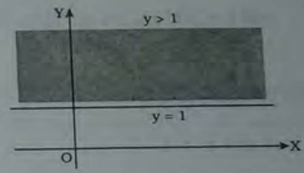
$$\Rightarrow |x + iy - i| = |x + iy + i|$$
$$\Rightarrow |x + i(y-1)| = |x + i(y+1)|$$
$$\Rightarrow \sqrt{x^2 + (y-1)^2} = \sqrt{x^2 + (y+1)^2}$$
$$\Rightarrow x^2 + y^2 - 2y + 1 = x^2 + y^2 + 2y + 1$$
$$\Rightarrow -2y = 2y$$

$\Rightarrow 4y = 0 \Rightarrow y = 0$, which is the equation of real axis (x – axis). Thus the region is the set of all points lie on the real axis.

[যাহা বাস্তব অক্ষ (x অক্ষ) এর সমীকরণ। অতএব এলাকাটি বাস্তব অক্ষের উপর সকল বিন্দুর সেট।]

(iii) $\text{Im}(z) > 1$

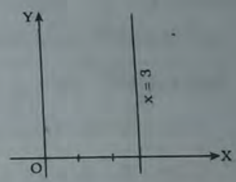
$\Rightarrow \text{Im}(x + iy) > 1 \Rightarrow y > 1$



The region is the set of all points (x, y) such that $y > 1$, that is, the set of all points (x, y) lie in the upper of the line $y = 1$.

[এলাকাটি সকল (x, y) বিন্দুর সেট যেন $y > 1$ হয়, অর্থাৎ $y = 1$ রেখার উপরের সকল (x, y) বিন্দুর সেট।]

(iv) $\text{Re}(\bar{z} - 1) = 2$

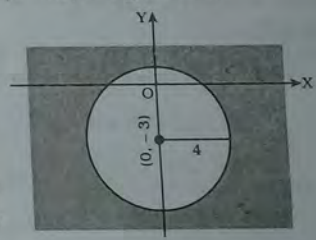


$$\Rightarrow \text{Re}(\overline{x + iy} - 1) = 2, \text{ where } z = x + iy$$
$$\Rightarrow \text{Re}(x - iy - 1) = 2$$
$$\Rightarrow x - 1 = 2$$
$$\Rightarrow x = 3$$

Which is the equation of a straight line parallel to y-axis. Thus the region is the set of all points (3, y), where $y \in \mathbf{R}$.

[যাহা y অক্ষের সমান্তরাল একটি সরলরেখার সমীকরণ। অতএব এলাকাটি সকল (3, y) বিন্দুর সেট যেখানে $y \in \mathbf{R}$.]

(v) Let [ধরি] $z = x + iy$. Then [তখন] $|z + 3i| > 4$ gives [দেয়]



$$|x + iy + 3i| > 4$$
$$\Rightarrow |x + i(y+3)| > 4$$
$$\Rightarrow \sqrt{x^2 + (y+3)^2} > 4$$
$$\Rightarrow x^2 + (y+3)^2 > 4^2$$

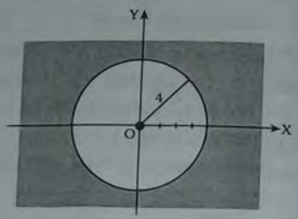
$x^2 + (y+3)^2 = 4^2$ is the equation of a circle whose centre is (0, – 3) and radius is 4.

Thus, the region is the set of all external points of the circle whose centre is (0, – 3) and radius is 4.

[$x^2 + (y+3)^2 = 4^2$ একটি বৃত্তের সমীকরণ যার কেন্দ্র (0, – 3) এবং ব্যাসার্ধ 4. অতএব, এলাকাটি (0, – 3) কেন্দ্র ও 4 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট।]

(vi) Let [ধরি] $z = x + iy$. Then [তখন] $|z| > 4$ gives [দেয়]

$$|x + iy| > 4$$
$$\Rightarrow \sqrt{x^2 + y^2} > 4$$
$$\Rightarrow x^2 + y^2 > 4.$$

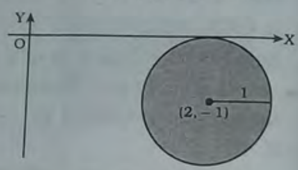


Thus the region is the set of all external points of the circle whose centre is (0, 0) and radius is 4.

[অতএব এলাকাটি (0, 0) কেন্দ্র ও 4 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট ।]

(vii) Let [ধরি] $z = x + iy$. Then [তখন] $|z - 2 + i| \leq |$ gives [দেয়]

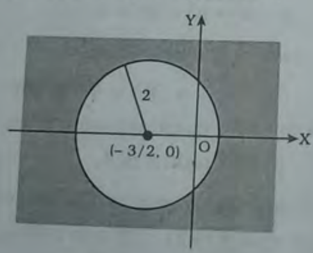
$$|x + iy - 2 + i| \leq 1$$
$$\Rightarrow |(x - 2) + i(y + 1)| \leq 1|$$
$$\Rightarrow \sqrt{(x - 2)^2 + (y + 1)^2} \leq 1$$
$$\Rightarrow (x - 2)^2 + (y + 1)^2 \leq 1$$



Thus, the region is the set of all internal points including the boundary points of a circle whose centre is (2, − 1) and radius is 1. [অতএব, এলাকাটি (2, − 1) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সীমানা বিন্দুসহ সকল অন্তঃস্থ বিন্দুর সেট]

(viii) Let [ধরি] $z = x + iy$. Then [তখন] $|2z + 3| > 4$ gives [দেয়]

$$|2(x + iy) + 3| > 4$$
$$\Rightarrow \sqrt{(2x + 3)^2 + (2y)^2} > 4$$
$$\Rightarrow 4x^2 + 12x + 9 + 4y^2 > 16$$
$$\Rightarrow 4x^2 + 4y^2 + 12x > 7$$
$$\Rightarrow x^2 + y^2 + 3x > \frac{7}{4}$$
$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > \frac{7}{4} + \frac{9}{4}$$

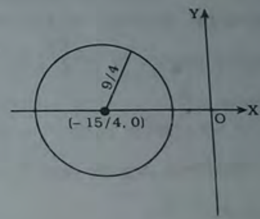


$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > 2^2$$

The region is the set of all external points of the circle whose centre is $\left(\frac{-3}{2}, 0\right)$ and radius is 2. [অতএব, এলাকাটি $\left(\frac{-3}{2}, 0\right)$ কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের বিহঃস্থ বিন্দুর সেট ।]

(ix) Let [ধরি] $z = x + iy$.

Then [তখন] $\left|\frac{z - 3}{z + 3}\right| = 3$ becomes



$$\frac{|z - 3|}{|z + 3|} = 3$$
$$\Rightarrow 3|z + 3| = |z - 3|$$
$$\Rightarrow 3|x + iy + 3| = |x + iy - 3|$$
$$\Rightarrow 3\sqrt{(x + 3)^2 + y^2} = \sqrt{(x - 3)^2 + y^2}$$
$$\Rightarrow 9(x^2 + 6x + 9 + y^2) = x^2 - 6x + 9 + y^2, \text{ by squaring}$$
$$\Rightarrow 8x^2 + 8y^2 + 60x + 72 = 0$$
$$\Rightarrow x^2 + y^2 + \frac{15}{2}x + 9 = 0$$

This is the equation of a circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius $= \sqrt{\left(\frac{15}{4}\right)^2 + 0 - 9} = \sqrt{\frac{81}{16}} = \frac{9}{4}$

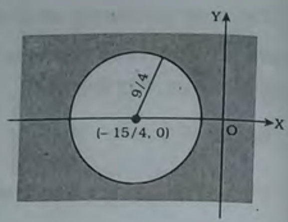
Thus, the region is the set of all boundary points of the circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius $= \frac{9}{4}$.

[ইহা একটি বৃত্তের সমীকরণ যার কেন্দ্র $\left(\frac{-15}{4}, 0\right)$ এবং ব্যাসার্ধ $= \sqrt{\left(\frac{15}{4}\right)^2 + 0 - 9} = \sqrt{\frac{81}{16}} = \frac{9}{4}$. অতএব, এলাকাটি $\left(\frac{-15}{4}, 0\right)$ কেন্দ্র ও $\frac{9}{4}$ ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল সীমানা বিন্দুর সেট ।]

(x) As example–33(ix) for $\left|\frac{z-3}{z+3}\right| > 3$

we will get

$$x^2 + y^2 + \frac{15}{2}x + 9 > 0$$

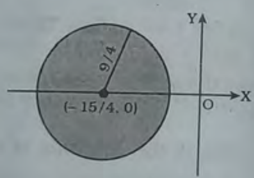


This represents the set of all external points of the circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius is $\frac{9}{4}$.

[ইহা নির্দেশ করে যে এলাকাটি $\left(\frac{-15}{4}, 0\right)$ কেন্দ্র ও $\frac{9}{4}$ ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট।]

(xi) As examples 33(x) for $\left|\frac{z-3}{z+3}\right| < 3$

we will get $x^2 + y^2 + \frac{15}{2}x + 9 < 0$



This represents the set of all internal points of the circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius is $\frac{9}{4}$.

[ইহা নির্দেশ করে যে এলাকাটি $\left(\frac{-15}{4}, 0\right)$ কেন্দ্র ও $\frac{9}{4}$ ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল অন্তঃস্থ বিন্দুর সেট।]

(xii) Let [ধরি] $z = x + iy$. Then [তখন] $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ becomes

$$\text{Re}\left(\frac{1}{x+iy}\right) \le \frac{1}{2}$$

$$\Rightarrow \text{Re}\left(\frac{x-iy}{(x+iy)(x-iy)}\right) \le \frac{1}{2}$$

$$\Rightarrow \text{Re}\left(\frac{x-iy}{x^2+y^2}\right) \le \frac{1}{2}$$

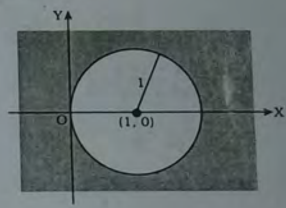
$$\Rightarrow \frac{x}{x^2+y^2} \le \frac{1}{2}$$

$$\Rightarrow 2x \le x^2 + y^2$$

$$\Rightarrow 0 \le x^2 + y^2 - 2x$$

$$\Rightarrow x^2 + y^2 - 2x \ge 0$$

$$\Rightarrow (x-1)^2 + y^2 \ge 1$$



Thus, the region is the set of all external points including the boundary points of the circle whose centre is (1, 0) and radius is 1. [অতএব, এলাকাটি (1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সীমানা বিন্দুসমূহসহ সকল বহিঃস্থ বিন্দুর সেট।]

(xiii) Let [ধরি] $z = x + iy$.

Then [তখন] $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

becomes $\text{Re}\left(\frac{1}{x+iy}\right) < \frac{1}{2}$

$$\Rightarrow \text{Re}\left(\frac{x-iy}{(x+iy)(x-iy)}\right) < \frac{1}{2}$$

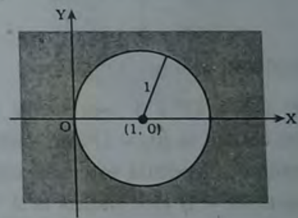
$$\Rightarrow \text{Re}\left(\frac{x-iy}{x^2+y^2}\right) < \frac{1}{2}$$

$$\Rightarrow \frac{x}{x^2+y^2} < \frac{1}{2}$$

$$\Rightarrow 2x < x^2 + y^2$$

$$\Rightarrow x^2 + y^2 - 2x > 0$$

$$\Rightarrow (x-1)^2 + y^2 > 1$$



This represents the set of all external points of the circle whose centre is (1, 0) and radius is 1. [ইহা (1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট নির্দেশ করে।]

(xiv) Let [ধরি] $z = x + iy$.

Then [তখন] $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$ becomes,

$$\text{Im}\left(\frac{1}{x+iy}\right) < \frac{1}{2}$$

$$\Rightarrow \text{Im}\left(\frac{x - iy}{(x + iy)\,(x - iy)}\right) < \frac{1}{2}$$

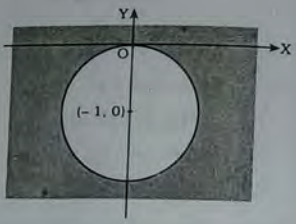
$$\Rightarrow \text{Im}\left(\frac{x - iy}{x^2 + y^2}\right) < \frac{1}{2}$$

$$\Rightarrow \frac{-y}{x^2 + y^2} < \frac{1}{2}$$

$$\Rightarrow -2y < x^2 + y^2$$

$$\Rightarrow x^2 + y^2 + 2y > 0$$

$$\Rightarrow x^2 + (y + 1)^2 > 1$$



This represents the set of all external points of the circle whose centre is (– 1, 0) and radius is 1. [ইহা (– 1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট নির্দেশ করে।]

(xv) Let [ধরি] $z = x + iy$. Given that [দেওয়া আছে] $1 < |z + i| \le 2$

$$\Rightarrow 1 < |x + iy + i| \le 2$$

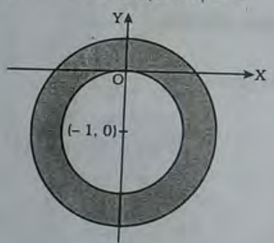
$$\Rightarrow 1 < |x + i(y + 1)| \le 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y + 1)^2} \le 2$$

$$\Rightarrow 1 < x^2 + (y + 1)^2 \le 2^2$$

$$\Rightarrow 1 < x^2 + (y + 1)^2$$

and [এবং] $x^2 + (y + 1)^2 \le 2^2$



$x^2 + (y + 1)^2 > 1$ represents all points external to the circle whose centre is (0, – 1) and radius is 1. Also, $x^2 + (y + 1)^2 \le 2^2$ represents all points internal and boundary of the circle whose centre is (0, – 1) and radius is 2. Thus the region is the set of all common points of external points of the circle $x^2 + (y + 1)^2 = 1$ and internal points of the circle $x^2 + (y + 1)^2 = 2^2$ including its boundary points.

[(0, – 1) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দু $x^2 + (y + 1)^2 > 1$ নির্দেশ করে। $x^2 + (y + 1)^2 \le 2^2$, (0, – 1) কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল অন্তঃস্থ ও সীমানা বিন্দু সমূহ নির্দেশ করে। অতএব এলাকাটি, $x^2 + (y + 1)^2 = 1$ বৃত্তের সকল বিহঃস্থ বিন্দুসমূহ এবং $x^2 + (y + 1)^2 = 2^2$ বৃত্তের সীমানা বিন্দু সমূহসহ অন্তঃস্থ বিন্দু সমূহের সকল সাধারণ বিন্দুর সেট।]

(xvi) $1 < |z + i| < 2$

$$\Rightarrow 1 < |x + iy + i| < 2$$

$$\Rightarrow 1 < |x\, i(y + 1)| < 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y + 1)^2} < 2$$

$$\Rightarrow 1 < x^2 + (y + 1)^2 < 2^2$$

$$\Rightarrow 1 < x^2 + (y + 1)^2 \text{ and [এবং] } x^2 + (y + 1)^2 < 2^2$$

$1 < |z + i| < 2$ represents the common points of the external of the circle $x^2 + (y + 1)^2 = 1$ and internal of the circle $x^2 + (y + 1)^2 = 2^2$. Thus the region is the set of all common points of external points of the circle $x^2 + (y + 1)^2 = 1$ and internal points of the circle $x^2 + (y + 1)^2 = 2^2$. [$1 < |z + i| < 2$ নির্দেশ করে $x^2 + (y + 1)^2 = 1$ বৃত্তের বহিঃস্থ ও $x^2 + (y + 1)^2 = 2^2$ বৃত্তের অন্তঃস্থ বিন্দু সমূহের সাধারণ বিন্দু সমূহ। অতএব, এলাকাটি হল $x^2 + (y + 1)^2 = 1$ বৃত্তের বহিঃস্থ এবং $x^2 + (y + 1)^2 = 2^2$ বৃত্তের অন্তঃস্থ বিন্দুসমূহের সাধারণ বিন্দুগুলির সেট।]

(xvii) Let [ধরি] $z = x + iy$. Given that [দেওয়া আছে] $1 < |z - 2i| \le 2$

$$\Rightarrow 1 < x + iy - 2i| \le 2$$

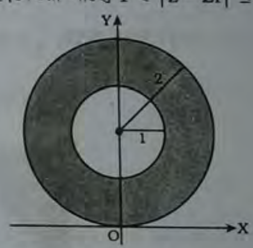
$$\Rightarrow 1 < |x + i(y - 2)| \le 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y - 2)^2} \le 2$$

$$\Rightarrow 1 < x^2 + (y - 2)^2 \le 2^2$$

$$\Rightarrow 1 < x^2 + (y - 2)^2$$

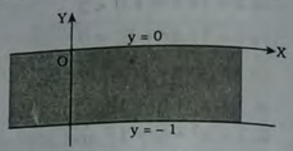
and [এবং] $x^2 + (y - 2)^2 \le 2^2$.



$1 < x^2 + (y - 2)^2$ represents all points external to the circle whose centre is (0, 2) and radius is 1. Also, $x^2 + (y - 2)^2 \le 2^2$ represents all points internal and boundary of the circle whose centre is (0, 2) and radius is 2. Thus the region is the set of all common points of external points of the circle $x^2 + (y - 2)^2 = 1$ and internal points of the circle $x^2 + (y - 2)^2 = 2^2$ including its boundary points.

[$1 < x^2 + (y - 2)^2$ নির্দেশ করে (0, 2) কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুসমূহ। আবার, $x^2 + (y - 2)^2 \le 2^2$ নির্দেশ করে (0, 2) কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের সীমানা বিন্দুসহ সকল অন্তঃস্থ বিন্দু। অতএব, এলাকাটি হল $x^2 + (y - 2)^2 = 1$ বৃত্তের বহিঃস্থ বিন্দু সমূহের এবং $x^2 + (y - 2)^2 = 2^2$ বৃত্তের সীমানা বিন্দুসহ অন্তঃস্থ বিন্দুসমূহের সাধারণ সকল বিন্দুর সেট।]

(xviii) Let [ধরি] $z = x + iy$. Then [তখন] $0 < \text{Re}(iz) < 1$ becomes



$0 < \text{Re}(ix + i^2y) < 1$

$\Rightarrow 0 < \text{Re}(ix - y) < 1$

$\Rightarrow 0 < -y < 1$

$\Rightarrow 0 < -y$ and [এবং] $-y < 1$

$\Rightarrow 0 > y$ and [এবং] $y > -1$

$\Rightarrow y < 0$ and [এবং] $y > -1$

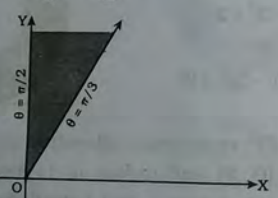
Thus the region lies between the lines $y = 0$ and $y = -1$. [অতএব এলাকাটি $y = 0$ ও $y = -1$ রেখার মধ্যে অবস্থিত।]

(xix) Let [ধরি] $z = x + iy$.

Then [তখন] $\arg z = \theta = \tan^{-1}\dfrac{y}{x}$.

Given that [দেওয়া আছে] $\dfrac{\pi}{3} \le \arg z \le \dfrac{\pi}{2}$

$$\Rightarrow \frac{\pi}{3} \le \theta \le \frac{\pi}{2}$$



Hence the region is the set of all infinite points bounded by the lines $\theta = \arg z = \dfrac{\pi}{3}$ and $\theta = \arg z = \dfrac{\pi}{2}$ including the points on these lines. [অতএব এলাকাটি $\theta = \dfrac{\pi}{3}$ এবং $\theta = \dfrac{\pi}{2}$ রেখাদ্বয়ের উপরস্থ বিন্দুসহ উহাদের দ্বারা সীমাবদ্ধ অসীম সংখ্যক বিন্দুর সেট।]

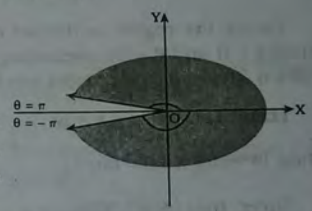
(xx) Let [ধরি] $z = x + iy$.

Then [তখন] $\arg z = \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Given that [দেওয়া আছে]

$-\pi < \arg z < \pi$

$\Rightarrow -\pi < \theta < \pi$



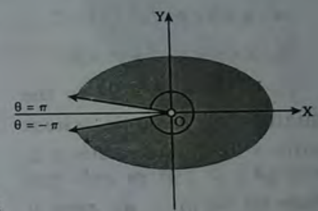
Thus, the region is the set of all infinite points between the lines $\theta = \pi$ and $\theta = -\pi$ as shown in the figure.

[অতএব চিত্রে প্রদর্শন অনুসারে এলাকাটি $\theta = \pi$ এবং $\theta = -\pi$ রেখা দ্বারা সীমাবদ্ধ সকল অসীম সংখ্যক বিন্দুর সেট]

(xxi) Let [ধরি] $z = x + iy$.

Then [তখন] $\arg z = \theta = \tan^{-1}\dfrac{y}{x}$.

Given that [দেওয়া আছে]



$-\pi < \arg\ z < \pi, z \ne 0$

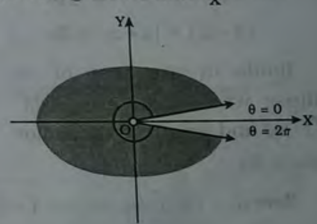
$\Rightarrow -\pi < \theta < \pi, x + iy \ne 0$

$\Rightarrow -\pi < \theta < \pi, x \ne 0, y \ne 0$

Thus, the region is the set of all infinite points between the lines $\theta = \pi$ and $\theta = -\pi$ excluding the origin. [অতএব, এলাকাটি হল মূলবিন্দু ব্যতীত $\theta = \pi$ এবং $\theta = -\pi$ রেখাদ্বয়ের মধ্যবর্তী অসংখ্য বিন্দুর সেট]

(xxii) Let [ধরি] $z = x + iy$. Then [তখন] $\arg z = \theta = \tan^{-1}\dfrac{y}{x}$.

Given that [দেওয়া আছে] $0 < \arg z < 2\pi$, $|z| > 0$



$\Rightarrow 0 < \theta < 2\pi,\ |x + iy| > 0$

$\Rightarrow 0 < \theta < 2\pi,\ \sqrt{x^2 + y^2} > 0$

$\Rightarrow 0 < \theta < 2\pi,\ x^2 + y^2 > 0$

Here $x^2 + y^2 = 0$ is the equation of the point circle which is the origin. [এখানে $x^2 + y^2 = 0$ হল বিন্দু বৃত্তের সমীকরণ যাহা মূলবিন্দু]

Hence the region is the set of all infinite points between the lines $\theta = 0$ and $\theta = 2\pi$ excluding the origin. [অতএব এলাকা হল মূলবিন্দু ব্যতীত $\theta = 0$ এবং $\theta = 2\pi$ রেখাদ্বয়ের মধ্যবর্তী অসংখ্য বিন্দুর সেট]

(xxiii) Let [ধরি] $z = x + iy$.

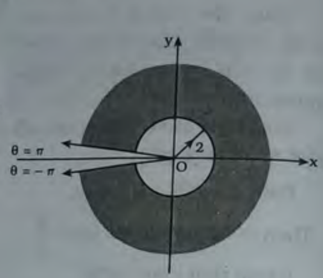
Then [তখন] $\arg z = \theta = \tan^{-1}\frac{y}{x}$.

Given that [দেওয়া আছে] $-\pi < \arg z < \pi$, $|z| > 2$

$\Rightarrow -\pi < \theta < \pi$, $|x + iy| > 2$

$\Rightarrow -\pi < \theta < \pi$, $\sqrt{x^2 + y^2} > 2$

$\Rightarrow -\pi < \theta < \pi$, $x^2 + y^2 > 2^2$



Here $x^2 + y^2 = 2^2$ is the equation of a circle whose centre is (0, 0) and radius is 2. [এখানে $x^2 + y^2 = 2^2$ হল একটি বৃত্তের সমীকরণ যার কেন্দ্র (0, 0) এবং ব্যাসার্ধ 2]

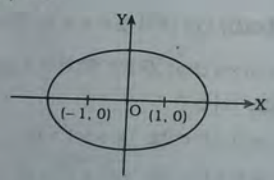
Thus, the region is the set of all infinite common points among the lines $\theta = \pi$, $\theta = -\pi$ and external of the circle $x^2 + y^2 = 2^2$. [অতএব, এলাকাটি হল $\theta = \pi$, $\theta = -\pi$ এবং $x^2 + y^2 = 2^2$ বৃত্তের বহিঃস্থ অসংখ্য সাধারণ বিন্দুর সেট]

(xxiv) Given that $|z - 1| + |z + 1| \leq 3$

The equation of the type

$$|z - z_0| + |z + z_0| = 2a$$

holds in the case of an ellipse whose foci are $S(z_0)$, $S'(-z_0)$ and length of the major axis is 2a.



Here $z_0 = 1 = (1, 0)$, $-z_0 = -1 = (-1, 0)$ and $2a = 3$

Thus, the region is the set of all interior points including the boundary points of the ellipse whose foci are (1, 0), (− 1, 0) and length of the major axis is 3.

[দেওয়া আছে $|z - 1| + |z + 1| \leq 3$

$|z - z_0| + |z + z_0| = 2a$ আকারের সমীকরণ উপবৃত্তের ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 2a.

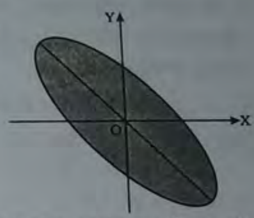
এখানে $z_0 = 1 = (1, 0)$, $-z_0 = -1 = (-1, 0)$ এবং $2a = 3$.

অতএব, এলাকাটি হবে একটি উপবৃত্তের সীমানা বিন্দুসহ অন্তঃস্থ বিন্দু সমূহের সেট যার উপকেন্দ্রদ্বয় S(1, 0), S′(− 1, 0) এবং বৃহৎ অক্ষের দৈঘ্য 3.]

(xxv) Given that $|z + 2 - 3i| + |z - 2 + 3i| < 10$

$\Rightarrow |z + (2 - 3i)| + |z - (2 - 3i)| < 10$

We know that the equation of the type $|z - z_0| + |z + z_0| = 2a$ holds in the case of an ellipse whose foci are $S(z_0)$, $S'(-z_0)$ and length of the major axis is 2a.



Here $z_0 = 2 - 3i = (2, -3)$

$\Rightarrow -z_0 = (-2, 3)$ and $2a = 10$.

Thus, the region is the set of interior points of an ellipse whose foci are S(2, − 3), S′(− 2, 3) and length of the major axis is 10

[দেওয়া আছে $|z + 2 - 3i| + |z - 2 + 3i| < 10$

$\Rightarrow |z + (2 - 3i)| + |z - (2 - 3i)| < 10$

আমরা জানি, $|z - z_0| + |z + z_0| = 2a$ আকারের সমীকরণ একটি উপবৃত্তের ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 2a

এখানে $z_0 = 2 - 3i = (2, -3)$

$\Rightarrow -z_0 = (-2, 3)$ এবং $2a = 10$

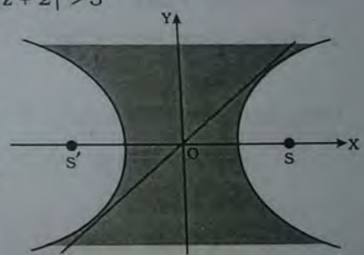
অতএব, এলাকাটি হল একটি উপবৃত্তের অন্তঃস্থ বিন্দু সমূহের সেট যার উপকেন্দ্রদ্বয় S(2, − 3), S′(− 2, 3) এবং বৃহৎ অক্ষের দৈর্ঘ্য 10.]

(xxvi) Given that $|z - 2| - |z + 2| > 3$

We know that the equation of the type

$$|z - z_0| - |z + z_0| = 2a$$

holds in the case of a hyperbola whose foci are $S(z_0)$, $S'(-z_0)$ and length of the transverse axis is 2a.



Here $z_0 = 2 \Rightarrow -z_0 = -2$ and $2a = 3$

Thus the region is the set of external points of a hyperbola whose foci are (2, 0), (– 2, 0) and length of the transverse axis is 3.

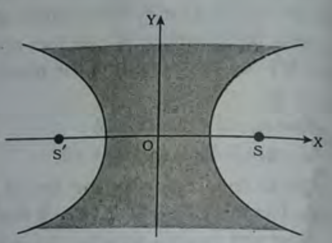
[দেওয়া আছে $|z-2| - |z+2| > 3$

আমরা জানি যে $|z-z_0| - |z+z_0| = 2a$ আকারের সমীকরণ পরাবৃত্তের ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং অনুপ্রস্থ (আড়) অক্ষের দৈর্ঘ্য 2a

এখানে $z_0 = 2 \Rightarrow -z_0 = -2$ এবং $2a = 3$.

অতএব এলাকাটি হল একটি পরাবৃত্তের বহিঃস্থ বিন্দুর সেট যার উপকেন্দ্রদ্বয় (2,0), (–2,0) এবং অনুপ্রস্থ (আড়) অক্ষের দৈর্ঘ্য 3]

(xxvii) Let [ধরি] $z = x + iy$.



Then [তখন] $z^2 = (x + iy)$

$= x^2 + 2ixy + i^2y^2$

$\Rightarrow z^2 = x^2 - y^2 + 2ixy$, where $i^2 = -1$.

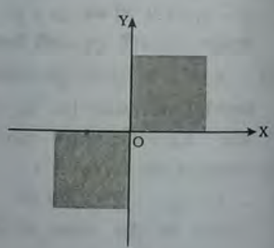
$\Rightarrow Re(z^2) = x^2 - y^2$

Given that [দেওয়া আছে] $Re(z^2) > 1$

$\Rightarrow x^2 - y^2 > 1$

Thus the region is the set of all external points of the rectangular hyperbola $x^2 - y^2 = 1$. [অতএব এলাকাটি হল $x^2 - y^2 = 1$ আয়তাকার পরাবৃত্তের বহিঃস্থ বিন্দুসমূহের সেট]

(xxviii) Let [ধরি] $z = x + iy$.



Then [তখন] $z^2 = (x + iy)^2$

$= x^2 + i^2 y^2 + 2ixy$

$\Rightarrow z^2 = x^2 - y^2 + i2xy \quad [\because i^2 = -1]$

$\Rightarrow Im(z^2) = Im(x^2 - y^2 + i\,2xy)$

$= 2xy$

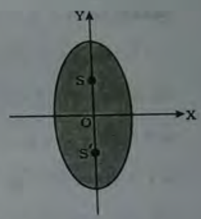
Given that [দেওয়া আছে] $Im(z^2) > 0$

$\Rightarrow 2xy > 0 \Rightarrow xy > 0$

$\Rightarrow (-x)(-y) > 0$

Thus the region is the set of all points in the first and third quadrants. [অতএব, এলাকাটি হল প্রথম ও তৃতীয় চর্তুভাগের বিন্দু সমূহের সেট]

(xxix) Given that $|z+i| + |z-i| \leq 3$



We know that the equation of the type $|z-z_0| + |z+z_0| = 2a$ holds in the case of an ellipse whose foci are $S(z_0)$, $S(-z_0)$ and length of the major axis is 2a.

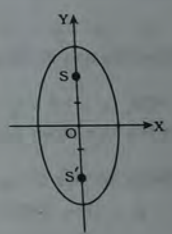
Here $z_0 = i$, $-z_0 = -i$ and $2a = 3$. Thus the region is the set of interior points including boundary points of an ellipse whose foci are S(0, 1), S′(0, – 1) and length of the major axis is z.

[দেওয়া আছে $|z+i| + |z-i| \leq 3$

আমরা জানি যে $|z-z_0| + |z+z_0| = 2a$ উপবৃত্ত এর ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 2a.

এখানে $z_0 = i$, $-z_0 = -i$ এবং $2a = 3$. অতএব এলাকাটি হবে একটি উপবৃত্তের সীমানা বিন্দুসহ অন্তঃস্থ বিন্দুসমূহের সেট যার উপকেন্দ্রদ্বয় S(0, 1), S′(0, – 1) এবং বৃহৎ অক্ষের দৈর্ঘ্য 3.]

(xxx) Given that $|z+2i| + |z-2i| = 6$



We know that the equation of the type $|z-z_0| + |z+z_0| = 2a$ holds in the case of an ellipse whose foci are $S(z_0)$, $S'(-z_0)$ and length of the major axis is 2a.

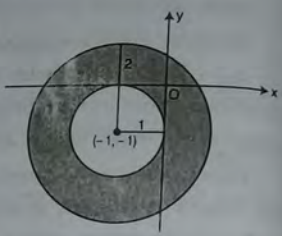
Here $z_0 = 2i$, $-z_0 = -2i$ and $2a = 6$. Thus the region is the set of boundary points of an ellipse whose foci are S(0, 2), S′(0, – 2) and length of the major axis is 6.

[দেওয়া আছে $|z+2i| + |z-2i| = 6$

আমরা জানি $|z-z_0| + |z+z_0| = 2a$ আকারের সমীকরণ একটি উপবৃত্তের ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 2a

এখানে $\dot{z}_0 = 2i$, $-z_0 = -2i$ এবং $2a = 6$ অতএব এলাকাটি হল একটি উপবৃত্তের সীমানা বিন্দু সমূহের সেট যার উপকেন্দ্রদ্বয় S(0, 2), S′(0, – 2) এবং বৃহৎ অক্ষের দৈর্ঘ্য 6]

(xxxi) Let $z = x + iy$. Given that



$$1 \le |z + 1 + i| < 2$$

$$\Rightarrow 1 \le |x + iy + 1 + i| < 2$$

$$\Rightarrow 1 \le |(x + 1) + i(y + 1)| < 2$$

$$\Rightarrow 1 \le \sqrt{(x + 1)^2 + (y + 1)^2} < 2$$

$$\Rightarrow 1 \le (x + 1)^2 + (y + 1)^2 < 2^2$$

$$\Rightarrow 1 \le (x + 1)^2 + (y + 1)^2 \text{ and } (x + 1)^2 + (y + 1)^2 < 2^2$$

$(x + 1)^2 + (y + 1)^2 \ge 1$ represents all points external and boundary of the circle whose centre is $(-1, -1)$ and radius is 1.

Also, $(x + 1)^2 + (y + 1)^2 < 2^2$ represents all internal points of the circle whose centre is $(-1, -1)$ and radius is 2.

Thus the region is the set of all common points of external and boundary points of the circle $(x + 1)^2 + (y + 1)^2 = 1$ and internal points of the circle $(x + 1)^2 + (y + 1)^2 = 2$.

[$(x + 1)^2 + (y + 1)^2 \ge 1$ সমীকরণ $(-1, -1)$ কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের বহিঃস্থ ও সীমানা বিন্দু সমূহ নির্দেশ করে।

আরো, $(x + 1)^2 + (y + 1)^2 < 2^2$ সমীকরণ $(-1, -1)$ কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের অন্তঃস্থ বিন্দু সমূহ নির্দেশ করে।

অতএব এলাকাটি হল $(x + 1)^2 + (y + 1)^2 = 1$ বৃত্তের বহিঃস্থ ও সীমানা বিন্দু এবং $(x + 1)^2 + (y + 1)^2 = 2$ বৃত্তের অন্তঃস্থ বিন্দু সমূহের সাধারণ বিন্দু সমূহের সেট]

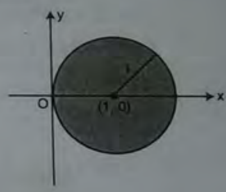
(xxxii) $\text{Re}\left(\frac{1}{z}\right) > \frac{1}{2}$

$$\Rightarrow \text{Re}\left(\frac{1}{\overline{x + iy}}\right) > \frac{1}{2}$$

$$\Rightarrow \text{Re}\left(\frac{x - iy}{x^2 - i^2 y^2}\right) > \frac{1}{2}$$

$$\Rightarrow \text{Re}\left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\right) > \frac{1}{2}$$

$$\Rightarrow \frac{x}{x^2 + y^2} > \frac{1}{2}$$



$$\Rightarrow \frac{x^2 + y^2}{x} < \frac{2}{1}$$

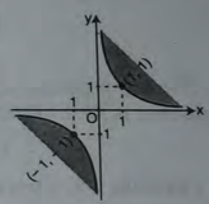
$$\Rightarrow x^2 + y^2 < 2x$$

$$\Rightarrow x^2 - 2x + y^2 < 0$$

$$\Rightarrow (x - 1)^2 + (y - 0)^2 < 1$$

This represents the set of all internal points of the circle whose centre is (1, 0) and radius is 1. [ইহা (1, 0) কেন্দ্র ও1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল অন্তঃস্থ বিন্দু সমূহের সেট নির্দেশ করে।]

(xxxiii) $\text{Im}(z^2) > 2$



$$\Rightarrow \text{Im}\{(x + iy)^2\} > 2$$

$$\Rightarrow \text{Im}(x^2 - y^2 + i2xy) > 2$$

$$\Rightarrow 2ky > 2$$

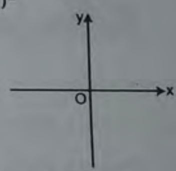
$$\Rightarrow xy > 1 \text{ and } (-x)(-y) > 1$$

(xxxiv) $|z + 1 + i| = |z - 1 + i|$

$$\Rightarrow |x + iy + 1 + i| = |x + iy - 1 + i|$$

$$\Rightarrow |(x + 1) + i(y + 1)| = |(x - 1) + i(y + 1)|$$

$$\Rightarrow \sqrt{(x + 1)^2 + (y + 1)^2} = \sqrt{(x - 1)^2 + (y + 1)^2}$$



$$\Rightarrow (x + 1)^2 + (y + 1)^2 = (x - 1)^2 + (y + 1)^2$$

$$\Rightarrow x^2 + 2x + 1 = x^2 - 2x + 1$$

$$\Rightarrow 4x = 0$$

$$\Rightarrow x = 0,$$

which is the equation of y-axis. [যাহা y অক্ষের সমীকরণ]

Example-34. Determine the set of points in the complex plane which satisfy the inequality $|z + 1 - i| \le |z - 1 + i|$, and sketch it. **[NUH-2002]**

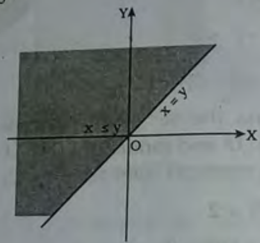
Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $|z + 1 - i| \le |z - 1 + i|$

$$\Rightarrow |x + iy + 1 - i| \le |x + iy - 1 + i|$$

$$\Rightarrow |(x + 1) + i(y - 1)| \le |(x - 1) + i(y + 1)|$$

$$\Rightarrow \sqrt{(x+1)^2+(y-1)^2} \le \sqrt{(x-1)^2+(y+1)^2}$$
$$\Rightarrow x^2+2x+1+y^2-2y+1 \le x^2-2x+1+y^2+2y+1$$
$$\Rightarrow 4x \le 4y$$
$$\Rightarrow x \le y$$



The set of points in the complex plane which satisfy the given inequality is [প্রদত্ত অসমতা সিদ্ধ করে জটিল তলে এমন বিন্দু সমূহের সেট]

$$\{(x, y) : x, y \in \mathbf{R} \text{ and } x \le y\}.$$

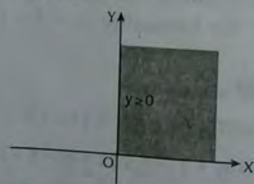
Example–35. Determine the set of points in the complex plane which satisfy the inequality $|z-i| \le |z+i|$, and sketch it.

[NUH–2002(Old), 2014, NU(Pre)–2002]

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $|z-i| \le |z+i|$.

$$\Rightarrow |x+iy-i| \le |x+iy+i|$$
$$\Rightarrow |x+i(y-1)| \le |x+i(y+1)|$$
$$\Rightarrow \sqrt{x^2+(y-1)^2} \le \sqrt{x^2+(y+1)^2}$$
$$\Rightarrow x^2+y^2-2y+1 \le x^2+y^2+2y+1$$
$$\Rightarrow -4y \le 0$$
$$\Rightarrow y \ge 0$$



Thus the set of points in the complex plane which satisfy the given inequality is [অতএব জটিল তলে বিন্দুসমূহের সেট যাহা প্রদত্ত অসমতাকে সিদ্ধ করে তা হল]

$$\{(x, y\} : x, y \in \mathbf{R} \text{ and } y \ge 0\}$$

Example–36. Determine the set of complex numbers such that $\text{Re}\left(\frac{1}{z}\right) < 1$ represent the set in the complex plane.

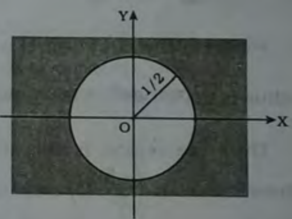
[NUH–2001]

Solution : Let [ধরি] $z = x + iy$.

$$\Rightarrow \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$
$$\Rightarrow \text{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$$

Given that [দেওয়া আছে] $\text{Re}\left(\frac{1}{z}\right) < 1$

$$\Rightarrow \frac{x}{x^2+y^2} < 1$$
$$\Rightarrow x < x^2+y^2$$
$$\Rightarrow x^2+y^2-x > 0$$
$$\Rightarrow \left(x-\frac{1}{2}\right)^2 + y^2 > \left(\frac{1}{2}\right)^2$$



Thus the required set is the external points of the circle whose centre is $\left(\frac{1}{2}, 0\right)$ and radius is $\frac{1}{2}$. [অতএব আবশ্যকীয় সেটটি হল একটি বৃত্তের বহিঃস্থ বিন্দুসমূহ যার কেন্দ্র $\left(\frac{1}{2}, 0\right)$ এবং ব্যাসার্ধ $\frac{1}{2}$.]

Example–37. Describe the region determined by the relations $\left|\frac{z-1}{z+1}\right| = 2$. **[NUH–1995, 2005, 2008]**

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $\left|\frac{z-1}{z+1}\right| = 2$

$$\Rightarrow \frac{|z-1|}{|z+1|} = 2$$

$$\Rightarrow 2|z+1| = |z-1|$$

$$\Rightarrow 2|x+iy+1| = |x+iy-1|$$

$$\Rightarrow 2\sqrt{(x+1)^2+y^2} = \sqrt{(x-1)^2+y^2}$$

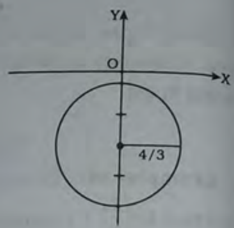
$$\Rightarrow 4(x^2+2x+1+y^2) = x^2-2x+1+y^2$$

$$\Rightarrow 3x^2+3y^2+10x+3=0$$

$$\Rightarrow x^2+y^2+\frac{10}{3}x+1=0$$

$$\Rightarrow x^2+2\cdot\frac{5}{3}\cdot x+\left(\frac{5}{3}\right)^2+y^2=\frac{25}{9}-1$$

$$\Rightarrow \left(x+\frac{5}{3}\right)^2+y^2=\left(\frac{4}{3}\right)^2$$



which is the equation of a circle whose centre is $\left(\frac{-5}{3}, 0\right)$ and radius is $\frac{4}{3}$. [যাহা একটি বৃত্তের সমীকরণ যার কেন্দ্র $\left(\frac{-5}{3}, 0\right)$ এবং ব্যাসার্ধ $\frac{4}{3}$.]

Thus the region is the set of boundary points of the circle whose centre is $\left(\frac{-5}{3}, 0\right)$ and radius $= \frac{4}{3}$. [অতএব এলাকাটি হল একটি বৃত্তের সীমান বিন্দুসমূহের সেট যার কেন্দ্র $\left(\frac{-5}{3}, 0\right)$ এবং ব্যাসার্ধ $= \frac{4}{3}$.]

Example–38. Draw the sketch of the following region :

$$1 < |z-2i| < 2$$

[NUH–1998]

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $1 < |z-2i| < 2$

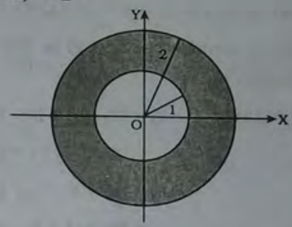
$$\Rightarrow 1 < |x+iy-2i| < 2$$

$$\Rightarrow 1 < \sqrt{x^2+(y-2)^2} < 2$$

$$\Rightarrow 1 < x^2+(y-2)^2 < 2^2$$

$$\Rightarrow 1 < x^2+(y-2)^2$$

and [এবং] $x^2+(y-2)^2 < 2^2$



Thus the region is the set of all common points of external points of the circle $x^2+(y-2)^2 = 1$ and internal points of the circle $x^2+(y-2)^2 = 2^2$. [অতএব এলাকাটি হল $x^2+(y-2)^2 = 1$ বৃত্তের বহিঃস্থ এবং $x^2+(y-2)^2 = 2^2$ বৃত্তের অন্তঃস্থ বিন্দুসমূহের সকল সাধারণ বিন্দু সমূহের সেট ।]

Example–39. Prove that the set of complex numbers form an abelian group.

[RUH–2002]

Solution : Let **C** be the set of complex numbers and

$$\mathbf{C} = \{z : z \text{ is a complex number } (x, y)\}.$$

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3) \in \mathbf{C}$.

(i) Closure law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$$= (x_1 + x_2, y_1 + y_2) \in \mathbf{C}$$

Thus, $z_1, z_2 \in \mathbf{C} \Rightarrow z_1 + z_2 \in \mathbf{C}$

(ii) Associative law : $(z_1 + z_2) + z_3 = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$

$$= z_1 + ((x_2, y_2) + (x_3, y_3))$$

$$= z_1 + (z_2 + z_3)$$

$$\Rightarrow (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) Identity law : $z + 0 = (x, y) + (0, 0) = (x + 0, y + 0) = (x, y) = z$

The complex number $(0, 0)$ is the identity called zero of the set **C**.

(iv) Inverse law : $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$

Thus the complex number $(-x, -y)$ is the additive inverse of the complex number (x, y).

(v) Commutative law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$$= (x_1 + x_2, y_1 + y_2)$$
$$= (x_2 + x_1, y_2 + y_1)$$
$$= (x_2, y_2) + (x_1, y_1)$$
$$= z_2 + z_1$$
$$\Rightarrow z_1 + z_2 = z_2 + z_1$$

The set **C** satisfied all the laws of a an abelian group. Hence the set of all complex numbers form an abelian group.

Example–40. Define addition and multiplication in **C**, the set of complex numbers such that with these definitions **C** is a field. **[RUH–2003]**

Solution : Definition : Let $z_1, z_2 \in$ **C**, where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then their addition and multiplication are defined as

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $z_1 z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

For addition we have the following :

(i) Closure law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in$ **C**

Thus, $z_1, z_2 \in$ **C** $\Rightarrow z_1 + z_2 \in$ **C**

(ii) Associative law : $(z_1 + z_2) + z_3 = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$
$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$
$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$
$$= z_1 + ((x_2, y_2) + (x_3, y_3))$$
$$= z_1 + (z_2 + z_3)$$
$$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) Identity law : $z + 0 = (x, y) + (0, 0) = (x + 0, y + 0) = (x, y) = z$

$\therefore z + 0 = z$

The complex number $0 = (0, 0)$ is the identity called zero of the set **C**.

(iv) Inverse law : $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$

The complex number $(-x, -y)$ is the additive inverse of the complex number (x, y).

(v) Commutative law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$$= (x_1 + x_2, y_1 + y_2)$$
$$= (x_2 + x_1, y_2 + y_1)$$
$$= (x_2, y_2) + (x_1, y_1)$$
$$= z_2 + z_1$$
$$\Rightarrow z_1 + z_2 = z_2 + z_1$$

Hence the set **C** of all complex numbers form an abelian group.

For multiplication we have the following :

(vi) Closure law : $z_1 z_2 = (x_1, y_1) \cdot (x_2, y_2)$

$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \in \mathbf{C}$$

since $x_1 x_2 - y_1 y_2$ and $x_1 y_2 + x_2 y_1 \in$ **R**

(viii) Associative law : $(z_1 z_2) z_3 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)(x_3, y_3)$

$$= ((x_1 x_2 - y_1 y_2) x_3 - (x_1 y_2 + x_2 y_1) y_3, (x_1 x_2 - y_1 y_2) y_3 + (x_1 y_2 + x_2 y_1) y_3)$$
$$= (x_1, y_1)(x_2 x_3 - y_2 y_3, x_2 y_3 + x_3 y_2)$$
$$= z_1 ((x_2, y_2)(x_3, y_3))$$
$$= z_1 (z_2 z_3)$$
$$\Rightarrow (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(viii) Identity law : $(x, y)(1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + 1 \cdot y) = (x, y)$ The complex number $(1, 0)$ is the multiplicative identy of the set **C**.

(ix) Inverse law : z_2 will be the multiplicative inverse of z_1 if

$z_1z_2 = (1, 0)$

$\Rightarrow (x_1, y_1)(x_2, y_2) = (1, 0)$

$\Rightarrow (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = (1, 0)$

$\Rightarrow x_1x_2 - y_1y_2 = 1$ ...... (1)

and $x_1y_2 + x_2y_1 = 0$ ...... (2)

From (2), $y_2 = \dfrac{-x_2y_1}{x_1}$ ...... (3)

By (3), (1) becomes, $x_1x_2 - y_1 \cdot \dfrac{-x_2y_1}{x_1} = 1$

$$\Rightarrow x_1^2x_2 + x_2y_1^2 = x_1$$

$$\Rightarrow x_2 = \frac{x_1}{x_1^2 + y_1^2}$$

From (3), $y_2 = \dfrac{-y_1}{x_1} \cdot \dfrac{x_1}{x_1^2 + y_1^2} = \dfrac{-y_1}{x_1^2 + y_1^2}$

If $(x_1, y_1) \neq (0, 0)$, that is, $x_1^2 + y_1^2 \neq 0$ then (x_2, y_2) exists.

Thus every nonzero complex number (x_1, y_1) has a unique inverse.

(x) Commutative law : $z_1z_2 = (x_1, y_1)(x_2, y_2)$

$$= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$
$$= (x_2x_1 - y_2y_1, x_2y_1 + x_1y_2)$$
$$= (x_2, y_2)(x_1y_1)$$
$$= z_2z_1$$

Thus the set **C** is an abelian multiplicative group.

(xi) Right distributive law :

$z_1(z_2 + z_3) = (z_1, y_1)((x_2, y_2) + (x_3, y_3))$

$$= (x_1, y_1)(x_2 + x_3, y_2 + y_3)$$
$$= (x_1(x_2 + x_3) - y_1(y_2 + y_3), x_1(y_2 + y_3) + (x_2 + x_3)y_1)$$
$$= (x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3, x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1)$$
$$= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) + (x_1x_3 - y_1y_3, x_1y_3 + x_3y_1)$$
$$= (x_1, y_1)(x_2, y_2) + (x_1, y_1)(x_3, y_3)$$
$$= z_1z_2 + z_1z_3$$

Similarly, we can show that left distribution law

$$(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$$

Hence according to the definition of a ring we can say that the set **C** form a ring.

In above we have seen that the multiplication is commutative and unit element (1, 0) exists and in **C** all nonzero elements have multiplicative inverse. Thus the set **C** is a commutative ring with unity such that every nonzero element has a multiplicative inverse. Hence the set **C** of all complex numbers is a field.

Example–41. Describe the locus represented by $\arg\left(\dfrac{z - z_1}{z_2 - z_1}\right) = 0$ where z_1 and z_2 are two given points.

[NUH–1996, RUH–1996]

Solution : Let [ধরি] $z = x + iy$, $z_1 = x_1 + iy_1$ and [এবং] $z_2 = x_2 + iy_2$

$\therefore z - z_1 = (x + iy) - (x_1 + iy_1) = x - x_1 + i(y - y_1)$

$z_2 - z_1 = x_2 + iy_2 - (x_1 + iy_1) = x_2 - x_1 + i(y_2 - y_1)$

Given that [দেওয়া আছে] $\arg\left(\dfrac{z - z_1}{z_2 - z_1}\right) = 0$

$\Rightarrow \arg(z - z_1) - \arg(z_2 - z_1) = 0 \qquad \left[\because \arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\right]$

$\Rightarrow \arg[x - x_1 + i(y - y_1)] - \arg[x_2 - x_1 + i(y_2 - y_1)] = 0$

$$\Rightarrow \tan^{-1}\left(\frac{y - y_1}{x - x_1}\right) - \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = 0$$

$$\Rightarrow \tan^{-1}\frac{\dfrac{y - y_1}{x - x_1} - \dfrac{y_2 - y_1}{x_2 - x_1}}{1 + \dfrac{y - y_1}{x - x_1} \cdot \dfrac{y_2 - y_1}{x_2 - x_1}} = 0$$

$$\Rightarrow \frac{\dfrac{y - y_1}{x - x_1} - \dfrac{y_2 - y_1}{x_2 - x_1}}{1 + \dfrac{y - y_1}{x - x_1}\dfrac{y_2 - y_1}{x_2 - x_1}} = \tan 0 = 0$$

$$\Rightarrow \frac{y-y_1}{x-x_1} - \frac{y_2-y_1}{x_2-x_1} = 0$$

$$\Rightarrow \frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\Rightarrow \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{x}{x_1-x_2} + \frac{y}{y_2-y_1} = \frac{y_1}{y_2-y_1} + \frac{x_1}{x_1-x_2}$$

$$= \frac{x_1 y_1 - x_2 y_1 + x_1 y_2 - x_1 y_1}{(x_1-x_2)(y_2-y_1)}$$

$$= \frac{x_1 y_2 - x_2 y_1}{(x_1-x_2)(y_2-y_1)}$$

$$\Rightarrow \frac{x}{\frac{x_1 y_2 - x_2 y_1}{y_2-y_1}} + \frac{y}{\frac{x_1 y_2 - x_2 y_1}{x_1-x_2}} = 1$$

which is the equation of a straight line. [যাহা একটি সরলরেখার সমীকরণ]

Thus, the locus represented by $\arg\left(\frac{z-z_1}{z_2-z_1}\right) = 0$ is a straight line. [অতএব $\arg\left(\frac{z-z_1}{z_2-z_1}\right) = 0$ দ্বারা নির্দেশিত সঞ্চার পথ একটি সরলরেখা।]

Example–42. Prove that the equation of any circle in the z-plane $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, where α, γ are real constants, while β may be a complex constant. **[NUH–2004]**

Solution : Let [ধরি] $z = x + iy$ and [এবং] $\beta = a + ib$. Then the given equation becomes [তখন প্রদত্ত সমীকরণটি দাঁড়ায়]

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$$

$$\Rightarrow \alpha(x+iy)(x-iy) + (a+ib)(x+iy) + (a-ib)(x-iy) + \gamma = 0$$

$$\Rightarrow \alpha(x^2 - i^2 y^2) + (ax + iay + ibx + i^2 by)$$

$$+ (ax - iay - ibx + i^2 by) + \gamma = 0$$

$$\Rightarrow \alpha(x^2+y^2) + 2ax - 2by + \gamma = 0$$

$$\Rightarrow x^2 + y^2 + 2\frac{a}{\alpha}x + 2\frac{-b}{\alpha} + \frac{\gamma}{\alpha} = 0$$

$$\Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0, \text{ where } g = \frac{a}{\alpha},\ f = \frac{-b}{\alpha},\ c = \frac{\gamma}{\alpha}$$

which is the equation of a circle in the xy plane.

Thus, the equation $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ represents a circle in the z-plane. [যাহা xy তলে একটি বৃত্তের সমীকরণ। অতএব, $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, সমীকরণ z তলে একটি বৃত্ত নির্দেশ করে।]

Example–43. z একটি জটিল সংখ্যা হলে $\left|\frac{z-1}{z+1}\right|$ = ধ্রুব এবং $\text{amp}\left(\frac{z-1}{z+1}\right)$ = ধ্রুব এর সঞ্চার পথসমূহ নির্ণয় কর এবং দেখাও যে, উহারা পরস্পরকে লম্বভাবে ছেদ করে। [If z is a complex number find the locuses of $\left|\frac{z-1}{z+1}\right|$ = constant and $\text{amp}\left(\frac{z-1}{z+1}\right)$ = constant. Show that they cut orthogonally each other.] **[NUH–2011]**

Solution : Given that [দেওয়া আছে]

$$\left|\frac{z-1}{z+1}\right| = \text{constant [ধ্রুবক]}$$

$$\Rightarrow \left|\frac{z-1}{z+1}\right| = c, \text{ say [ধরি]}$$

$$\Rightarrow \frac{|x+iy-1|}{|x+iy+1|} = c$$

$$\Rightarrow \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}} = c$$

$$\Rightarrow (x-1)^2 + y^2 = c^2[(x+1)^2 + y^2]$$

$$\Rightarrow x^2 - 2x + 1 - y^2 + c^2x^2 + 2c^2x + c^2 + c^2y^2$$

$$\Rightarrow (1-c^2)x^2 - 2x - 2c^2x + (1-c^2)y^2 + 1 - c^2 = 0$$

$$\Rightarrow x^2 - \frac{2(1+c^2)}{1-c^2}x + y^2 + 1 = 0$$

$$\Rightarrow x^2 + 2\cdot\frac{c^2+1}{c^2-1}x + y^2 + 1 = 0 \ \cdots\cdots (1)$$

which is the equation of a circle [যাহা একটি বৃত্তের সমীকরণ]

2nd Part [২য় অংশ] ঃ Again, given that [আবার দেওয়া আছে]

$$\text{amp}\left(\frac{z-1}{z+1}\right) = \text{Constant [ধ্রুবক]}$$

$$\Rightarrow \text{amp}(z-1) - \text{amp}(z+1) = k_1, \text{ (ধরি)}$$

$$\Rightarrow \text{amp}(x-1+iy) - \text{amp}(x+1+iy) = k_1$$

$$\Rightarrow \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = k_1$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1+\frac{y}{x-1}\cdot\frac{y}{x+1}}\right) = k_1$$

$$\Rightarrow \tan^{-1}\frac{xy+y-xy+y}{x^2-1+y^2} = k_1$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan k_1 = k, \text{ say [ধরি]}$$

$$\Rightarrow x^2+y^2-1 = \frac{2}{k}y$$

$$\Rightarrow x^2+y^2-2\cdot\frac{1}{k}\cdot y - 1 = 0 \quad \cdots\cdots (2)$$

which is the equation of a circle. [যাহা একটি বৃত্তের সমীকরণ]

3rd part [৩য় অংশ] ঃ Comparing (1) with the circle $x^2 + y^2 + 2g_1 x + 2f_1 + c_1 = 0$ and (2) with the circle $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ we get [(1) কে $x^2 + y^2 + 2g_1 x + 2f_1 + c_1 = 0$ বৃত্তের সাথে এবং (2) নং কে $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ বৃত্তের সাথে তুলনা করে পাই]

$$g_1 = \frac{c^2+1}{c^2-1}, f_1 = 0, \ c_1 = 1$$

and [এবং] $g_2 = 0, \ f_2 = -\frac{1}{k}, c_2 = -1$

Now [এখন] $2g_1 g_2 + 2f_1 f_2 = 2\cdot\frac{c^2+1}{c^2-1}\cdot 0 + 2\cdot 0\cdot\left(\frac{-1}{k}\right) = 0$

and [এবং] $c_1 + c_2 = 1 - 1 = 0$

$\therefore 2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$

Hence the two circles cut orthogonally. [অতএব বৃত্ত দুইটি পরস্পর লম্বভাবে ছেদ করে]

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. Define complex number.
   Ans : Any number of the form x + iy is called a complex number where $x, y \in \mathbb{R}$.
2. If $z = x + iy$, then what is $|z|$?
   Ans : $|z| = \sqrt{x^2+y^2}$.
3. What is geometrical meaning of $|z|$.
   Ans : $|z| = \sqrt{x^2+y^2}$ is the distance of $z = x + iy = (x, y)$ form the origin.
4. What is the argument of the complex number $z = x + iy$.
   Ans : If θ be the argument of z then $\theta = \tan^{-1}\frac{y}{x}$.
5. Is $|z|$ unique?
   Ans : Yes, $|z|$ is a unique non-negative number.
6. What type of function $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is?
   Ans : $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is a multivalued function.
7. What is Argand plane?
   Ans : When a complex number z is represented by a point P(x, y) in the xy plane, then this plane is called the Argand plane.
8. Write the complex conjugate number of $z = x + iy$.
   Ans : The complex conjugate number of z is $\bar{z} = x - iy$.
9. If $z = x + iy$ and $\bar{z} = x - iy$, then do you agree that $z > \bar{z}$?
   Ans : Greater than or less than have no meaning in relation between two complex numbers. So $z > \bar{z}$ has no meaning and there fore I do not agree that $z > \bar{z}$.
10. If $z = x + iy$ and $\bar{z} = x - iy$ then comment about $|z|$ and $|\bar{z}|$.
    Ans : $|z| = \sqrt{x^2+y^2}$ and $|\bar{z}| = \sqrt{x^2+y^2}$. This shows that the two complex numbers $z = (x, y)$ and $\bar{z} = (x, -y)$ are at same (equal) distance from the origin.
11. What is the amplitude when a complex number is purely real?

Ans : The amplitude is 0 or π when a complex number is purely real.

12. What is the amplitude when a complex number is purely imaginary?

Ans : The amplitude is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ when a complex numbers is purely imaginary.

13. Define imaginary unit i (iota).

Ans : The imaginary unit i(iota) is defined as $i = (0, 1)$.

14. Is the expression $x + iy$ is an imaginary number?

Ans : The expression $x + iy$ is not an imaginary number. It is a complex number.

15. Define modulus of a complex number [একটি জটিল সংখ্যার মডুলাসের সংজ্ঞা দাও।] **[NUH–2012]**

Ans : The modulus of a complex number $z = x + iy$ is $\sqrt{x^2 + y^2}$.

16. If $z = x + iy$ then comment about $|z|$ and $|\bar{z}|$. **[NUH–2012]**

[$z = x + iy$ হলে $|z|$ এবং $|\bar{z}|$ সমন্ধে মন্তব্য কর।]

Ans : If $z = x + iy$ then $\bar{z} = x - iy$

$\therefore |z| = |x + iy| = \sqrt{x^2 + y^2}$

and $|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$

$\therefore |z| = |\bar{z}|$, that is, modulus of z and $\bar{z}$ are equal.

17. Show that $|z|^2 = z\bar{z}$. **[NUH–2013]**

Ans : Let $z = x + iy$. Then $\bar{z} = \overline{x + iy} = x - iy$

$\therefore |z|^2 = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$

and $z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$

$\therefore |z|^2 = z\bar{z}$ (Showed)

18. Show that $|z|^2 = |\bar{z}|^2$

Ans : Let $z = x + iy$. Then $\bar{z} = \overline{x + iy} = x - iy$

$\therefore |z|^2 = (\sqrt{x^2 + y^2})^2 = (\sqrt{x^2 + (-y)^2})^2 = |\bar{z}|^2$ (Showed)

19. Find the argument of $-2 - i$. **[NUH–2013]**

Ans : Argument of $-2 - i$ is $\theta = \tan^{-1}\left(\frac{-1}{-2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$.

20. What do you mean by equality of two complex number?

[দুইটি জটিল সংখ্যার সমতা বলতে কি বুঝ?] **[NUH–2013]**

Ans : Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

That is, the real part of the one is equal to the real part of the other, and the imaginary part of the one is equal to the imaginary part of the other.

[দুইটি জটিল সংখ্যা $z_1 = x_1 + iy_1$ এবং $z_2 = x_2 + iy_2$ সমান (সমতা) হবে যদি এবং কেবল যদি $x_1 = x_2$ এবং $y_1 = y_2$ হয়।

অর্থাৎ, একটির বাস্তব অংশ অপরটির বাস্তব অংশের সমান এবং একটির কাল্পনিক অংশ অপরটির কাল্পনিক অংশের সমান হয়]

21. What is the ordered pair of a complex number [একটি জটিল সংখ্যার ক্রম জোড় কি?]

Ans. If $z = x + iy$ be a complex numbers then it can be written as $z = x + iy = (x, y)$. (x, y) is called the ordered pair of the complex number z.

[যদি $z = x + iy$ একটি জটিল সংখ্যা হয় তখন ইহাকে $z = x + iy = (x, y)$ আকারে লেখা যায়। (x, y) কে জটিল সংখ্যা z এর ক্রমজোড় বলে।]

22. What are the real and imaginany parts of a complex number?

[একটি জটিল সংখ্যার বাস্তব ও কাল্পনিক অংশ কি?]

Ans. Let $z = x + iy$ be a complex numbers. Then x is called the real part of z and y is called the imaginany part of z. This is written as $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

[ধরি $z = x + iy$ একটি জটিল সংখ্যা। তখন x কে z এর বাস্তব অংশ এবং y কে z এর কাল্পনিক অংশ বলে। ইহাকে লেখা হয় $\text{Re}(z) = x$ এবং $\text{Im}(z) = y$.

23. What is the vector representation of the complex number?

[জটিল সংখ্যার ভেক্টর উপস্থাপনা কি?]

Ans. Let $z = x + iy$ be a complex number and $P(x,y)$ be a point in the Argand plane. Then $|z| = \sqrt{x^2 + y^2}$ and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

Amplitude θ is represented by the direction of the vector $\overrightarrow{OP}$. Hence the complex number $z = x + iy$ is completely represents by the vector $\overrightarrow{OP}$.

24. Define the product of two complex numbers. [দুইটি জটিল সংখ্যার গুনের সংজ্ঞা দাও]

Ans. The product of two complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ is defined as [দুইটি জটিল সংখ্যা $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ এর গুণফল নিম্নাকারে সংজ্ঞায়িত হয়]

$$\begin{aligned} z_1z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2 \\ &= x_1x_2 + i(x_1y_2 + x_2y_1) - y_1y_2, \because i^2 = -1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

25. Define the quotient (division) of two complex numbers [দুইটি জটিল সংখ্যার ভাগফল সংজ্ঞায়িত কর]

Ans. The quotient (division) of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is defined as [দুইটি জটিল সংখ্যা $z_1 = x_1 + iy_1$ এবং $z_2 = x_2 + iy_2$ নিম্নাকারে সংজ্ঞায়িত হয়]

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{x_1x_2 + ix_2y_1 - ix_1y_2 - i^2 y_1y_2}{x_2^2 - i^2 y_2^2} \\ &= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 - y^2 y_2^2} \quad \because i^2 = -1 \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}, \text{ Provided } x_2^2 + y_2^2 \neq 0. \end{aligned}$$

26. What is the principal argument of $\frac{1-i}{1+i}$? [$\frac{1-i}{1+i}$ এর মূখ্য নতি কত ?] **[NUH–2014]**

Ans. $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = -i$

The principal argument of $\frac{1-i}{1+i}$ [$\frac{1-i}{1+i}$ এর মূখ্যনতি]

$= \tan^{-1}\left(\frac{-1}{0}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$.

27. How many values have the expression $(1+i)^{2/3}$? **[NUH–2015]**

Ans. $(1+i)^{2/3} = (1+2i+i^2)^{1/3} = (2i)^{1/3}$. Thus $(1+i)^{2/3} = (2i)^{1/3}$ have 3 values.

EXERCISE–1

Part–A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. What is the mirror image of a complex number z?
2. Under what condition a complex number z and its conjugate $\bar{z}$ are equal.
3. What are the conditions for two given numbers z_1 and z_2 to be conjugate?
4. What is the effect when a complex number z is multiplied by i.
5. Every real number is a complex number. Is the converse true?

Part–B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. Give the geometrical representation of product of two complex numbers. **[CUH–2002]**

   Ans : See art–1.4

2. Give the geometrical representation of quotient of two complex numbers. **[CUH–2002]**

   Ans : See art–1.6

3. Show that the triangle inequality holds in $\mathbb{C}$. **[RUH–2003, 2006]**

   Ans : See Solved problem–22(i).

4. If z be a complex number, then represent iz geometrically.

   Ans : See art–1.7

5. Prove that a complex number is purely real if the amplitude is 0 or π.

   Ans : See proposition of art–1.7

6. Prove that a complex number is purely imaginary if the amplitude is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

   Ans : See proposition of art–1.7

7. Discuss the locus represented by $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$, where z_1 and z_2 are two given points. **[NUH-1996, RUH-1996]**

Ans : See Solved problem-41.

8. Prove that, if sum and product of two complex numbers are both real, then the two numbers must either be real or conjugate.

Ans : See solved example-27.

9. If z_1 and z_2 are two non-zero complex numbers, then prove that the modulas of their difference is always greater than or equal to the difference of their moduli. **[CUH-2000]**

Ans : See Solved problem-24.

10. Show that $|z|\sqrt{2} \geq |\text{Re}(z)| + |\text{Im}(z)|$, where z is any complex number. **[DUH-1988, 1990]**

Ans : See Solved example-19.

11. If $z = x + iy$, then prove that $|x| + |y| \leq \sqrt{2}\,|x + iy|$. **[NUH-1998, RUH-1997]**

Ans : See Solve example-20.

12. If z is any complex number than show that $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}\,|z|$ **[RUH-2004]**

Ans : See Solved example-19.

13. Find two comple numbers whose sum is 4 and whose product is 8. **[NUH-2000, 2006(Old)]**

Ans : See Solved example-26.

14. If z_1, z_2 are complex numbers then prove that $|z_1 z_2| = |z_1|\,|z_2|$ **[DUH-1989, RUH-1998]**

Ans : See Solved example-21.

15. If $z_1, z_2, \cdots, z_n$ are complex numbers then prove that $|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$. **[DUH-1985]**

Ans : See Solved example-21.

16. For any two complex numbers z_1 and z_2 prove that $|z_1 + z_2| \leq |z_1| + |z_2|$ **[NUH-03, 04, 05, 07, RUH-1997]**

Ans : See Solved example-22(i)

17. For any two complex numbers z_1 and z_2 prove that $|z_1 - z_2| \geq |z_1| - |z_2|$. **[RUH-1998, CUH-2004]**

Ans : See Solved example-24.

18. For any two complex numbers z_1 and z_2 prove that $|z_1 - z_2| \leq |z_1| + |z_2|$ **[NUH-2006]**

Ans : See Solved example-22(iii).

19. If $z_1, z_2, \cdots, z_n$ are complex numbers then prove that $|z_1 + z_2 + .. + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$. **[RUH-1998, CUH-2004]**

Ans : See Solved example-22(ii).

20. If z_1, z_2, z_3 are complex numbers then show that $\left|\frac{z_1}{z_2 + z_3}\right| \leq \frac{|z_1|}{\big||z_2| - |z_3|\big|}$, where $|z_2| \neq |z_3|$. **[NUH-2001]**

Ans : See Solved example-25(i).

21. If z_1, z_2, z_3, z_4 are complex numbers then show that $\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{\big||z_3| - |z_4|\big|}$, where $|z_3| \neq |z_4|$. **[CUH-2000]**

Ans : See solved example-25.

22. Prove that $|z_1 - z_2| \geq \big||z_1| - |z_2|\big| \geq |z_1| - |z_2|$, where z_1, z_2 are complex numbers. **[NUH-1994, 2002(Old), DUH-1998, 2005]**

Ans : See Solved example-24.

23. If z_1 and z_2 are two complex numbers, then prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ **[NUH-2000, 2006(Old), RUH-2002]**

Ans : See Solved example-28.

24. Show that $|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$ **[RUH-2001]**

Ans : See 2nd part of Solved example-28.

25. Show that $|a + \sqrt{a^2 - b^2}| + |a - \sqrt{a^2 - b^2}| = |a + b| + |a - b|$ **[CUH-2004]**

Ans : Write $\alpha = a$ and $\beta = b$ in Solved Example 28(2nd Part).

26. Solve sinhz = i [NUH-1995]

Ans : See Solved example-11.

27. Solve coshz = 2 [NUH-2002, 2006, 2010, DUH-2001]

Ans : See Solved example-12.

28. Prove that $|z| \le |Re(z)| + |Im(z)| \le \sqrt{2}\,|z|$

Ans : See Solved example-19.

29. Prove that $|z|^2 = |-z|^2 = |\bar{z}|^2 = |-\bar{z}|^2 = z\bar{z}$

Ans : See Solved example-15.

30. Show that $Im(iz) = Re(z)$ and $Re(iz) = |z|^2\, Im(z^{-1})$.

Ans : See Solved example-17.

31. If the equation $z^2 + az + b = 0$ has a pair of conjugate complex roots then prove that a, b are both real and $a^2 < 4b$.

Ans : See Solved example-31.

32. Solve the equation $|z| - z = 2 + i$

Ans : See Solved example-32.

33. Determine all real x and y which satify the relation $x + iy = |z + iy|$.

Ans : See Solved example-10(a).

34. Determine all real x and y which satisfy the relation $x + iy = (x + iy)^2$.

Ans : See Solved example-10(b).

35. Find all values of the following :

(i) $(-8i)^{1/3}$, (ii) $(-8-i8\sqrt{3})^{1/4}$ (iii) $(1-i\sqrt{3})^{1/2}$ [DUH-2006]

Ans : (i) $\sqrt{3} - i,\ 2i,\ -\sqrt{3} - i$

(ii) $1 + \sqrt{3}i,\ -\sqrt{3} + i,\ -1 - i\sqrt{3},\ \sqrt{3} - i$

(iii) $\sqrt{3} - i,\ -\sqrt{3} + i$

Part-C (Broad Questions) (বড় প্রশ্ন)

1. Give the geometrical representation of product and quotient of two complex numbers. [CUH-2002]

Ans : See art-1.4 and art-1.6.

2. Prove that the set of complex numbers form an abelian group. [RUH-2002]

Ans : See Solved problem 39 or 40.

3. Define addition and multiplication in $\mathbb{C}$, the set of complex numbers such that with these definitions $\mathbb{C}$ is a field.

[RUH-2003]

Ans : See Definition-1 and Solved problem-40.

4. Prove that a complex number is purely real if the amplitude is 0 or π and purely imaginary if the amplitude is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Ans : See Proposition of art-1.7.

5. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. Interpret the result geometrically and deduce that

$$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|,$$

all the numbers involved being complex. [NUH-2004(Old)]

Ans : See Solved problem-28.

6. Prove that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$ and interpret the identity geometrically. [NUH-2002(Old)]

Ans : See Solved problem-28 [Write $z_1 = z$ and $z_2 = w$].

7. Prove that $|z + w| \le |z| + |w|$ and hence prove that $|z + w| \ge ||z| - |w||$.

Ans : See Solved example-22 and 24.

8. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the argand plane, then show that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

Ans : See Solved example–30.

9. If $z_1, z_2, \cdots, z_n$ are complex numbers then prove that

$$\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j| \text{ and } \left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$$

Ans : See Solved example–23(3rd and 4th part).

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CHAPTER–2
ANALYTIC FUNCTIONS

In this chapter we shall define functions of complex variable and discuss their limit, continuity and differentiability. Later we introduce analytic functions, which have wide range of applications in complex analysis.

2.1. Functions of a complex Variable : Let S be a set of complex numbers. A function f defined on S is a rule which assigns to each $z \in S$ a unique value $f(z) \in C$. We write this as

$$w = f(z),\ z \in S \quad \text{or} \quad f : S \to C.$$

We say f is a complex function of a complex variable. Here w is called the value of f at z. It is customary to locate $w = f(z) \in C$ in another complex plane called w-plane. The set S is called the domain of f and $S_1 = \{f(z) : z \in S\}$, range of f.

When the domain of definition is not specified, we take the largest possible set as domain for f on which f is well defined.

Every functions f(z) can be written as

[জটিল চলকের ফাংশন ঃ মনেকরি S জটিল চলকের একটি সেট। S সেটে বর্ণিত একটি ফাংশন f হল একটি নিয়ম যাহা প্রত্যেক $z \in s$ এর জন্য একটি অনন্য মান $f(z) \in C$ দেয়। আমরা ইহাকে লিখি $w = f(z),\ z \in S$ অথবা $f : S \to C$.

আমরা বলি f হল একটি জটিল চলকের একটি জটিল ফাংশন। এখানে w হল z এ f এর মান। প্রথানুসারে $w = f(z) \in C$ কে w তল নামে অন্য একটি তলে নির্দেশ করানো হয়। S সেটকে f এর ডোমেন বলে এবং f এর রেঞ্জ হল $s_1 = \{f(z) ; z \in S\}$.

যখন সংজ্ঞার ডোমেন সুনির্দিষ্ট থাকবে না, f এর ডোমেনের জন্য সম্ভাব্য বৃহৎ সেট লইব যেখানে f সুবর্ণিত হয়। প্রত্যেক f(z) ফাংশনকে লেখা যায়]

$$f(z) = u(x, y) + iv(x, y) \text{ or } w = f = u + iv \cdots\cdots (1)$$

For an example, let $w = f(z) = z^2$

$\Rightarrow w = f(z) = (x + iy)^2$

$\Rightarrow u + iv = x^2 + i^2y^2 + i2xy$

$\qquad = x^2 - y^2 + i2xy$

Hence $u = x^2 - y^2$ and $v = 2xy$.

On the other hand, if u(x, y) and v(x, y) are two given real-valued functions of the real variables x and y, equation (1) can be used to define a function of the complex variable $z = x + iy$.

[অপর পক্ষে, যদি u (x, y) ও v (x, y) বাস্তব চলক x ও y এর দুইটি বাস্তবমানী ফাংশন হয়, তখন জটিল চলক z = x + iy এর ফাংশন হিসাবে সমীকরণ(1) কে ব্যবহার করা যায়।]

If in equation (1) the number v(x, y) is always zero, then the number f(z) is always real.

[সমীকরণ (1) এ যদি v(x, y) সংখ্যাটি সর্বদা শূণ্য হয়, তখন f(z) সংখ্যাটি সর্বদা বাস্তব হবে।]

Single valued function **[NUH(Phy)–2005]**

The function w = f(z) is called a single valued function if for every value of z there is only one value of w.

[**একমানী ফাংশন ঃ** w = f(z) ফাংশনকে একমানী ফাংশন বলে যদি z এর প্রত্যেক মানের জন্য w এর শুধুমাত্র একটি মান থাকে।]

Example : Let $w = f(z) = z^2$. Then for every value of z, there is only one value of w. Hence $w = f(z) = z^2$ is a single valued function of z.

Multivalued functions : **[NUH–1999]**

Generalization of a concept of a function is the multivalued function. These functions take more than one value at some or all points of the domain of definition.

The function w = f(z) is called a many valued function if for every value of z, there are more value of w.

[**বহুমানী ফাংশন ঃ** w = f(z) ফাংশনকে বহুমানী ফাংশন বলে যদি z এর প্রত্যেক মানের জন্য w এর অনেক (একাধিক) মান থাকে।]

Example : Let $w = f(z) = z^{1/2}$. Here for every nonzero value of z, there are two values of w. Hence $w = f(z) = z^{1/2}$ is a multivalued function of z.

Polynomial : The function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n;\ a_n \neq 0$$

is called a polynomial of degree n, where n is zero or a positive integer and $a_0, a_1, a_2, \cdots, a_n$ are complex constants.

Rational functions : If P(z) and Q(z) are two polynomials then $\frac{P(z)}{Q(z)}$ is called rational functions, which are defined at each point z except where Q(z) = 0.

2.2. Limits :

Definition : The function f(z) defined in some neighbourhood of z_0 is said to have limit w_0 at z_0 if, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

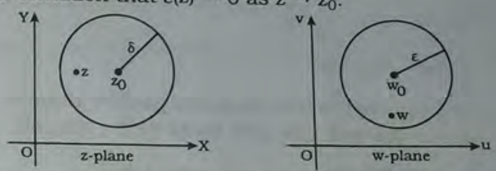
$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \ \cdots\cdots (2)$$

Symbolically, we write $\lim\limits_{z \to z_0} f(z) = w_0$

This is also written as

$$f(z) = w_0 + \varepsilon(z)\ \forall z : 0 < |z - z_0| < \delta$$

with the condition that $\varepsilon(z) \to 0$ as $z \to z_0$.



Since (2) is true for all $z \in \{z : 0 < |z - z_0| < \delta|\}$, $z \to z_0$ means z approaches to z_0 through any path. Hence limit w_0 is independent of path.

If $\lim f(z) \to w_0$ as $z \to z_0$ exists, then it is unique.

Theorem–1. If $\lim\limits_{z \to z_0} f(z)$ exists, then it must be unique.

[যদি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান থাকে, তবে ইহা অবশ্যই অনন্য]

[RUH–1994, CUH–2002]

Proof : Suppose $\lim\limits_{z \to z_0} f(z)$ exists and the limit is not unique. Let there exist two limits w_1 and w_2. Then by definition

$$\lim_{z \to z_0} f(z) = w_1 \text{ and } \lim_{z \to z_0} f(z) = w_2.$$

Also by hypothesis, for any given $\varepsilon > 0$, then exists a $\delta > 0$ such that

[প্রমাণ ঃ ধরি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান এবং সীমামান অনন্য না। ধরি দুইটি সীমা w_1 ও w_2 বিদ্যমান। তখন সংজ্ঞানুসারে

$$\lim_{z \to z_0} f(z) = w_1 \text{ এবং } \lim_{z \to z_0} f(z) = w_2.$$

অধিকন্তু কল্পনা অনুসারে, প্রদত্ত $\varepsilon > 0$ এর জন্য একটি $\delta > 0$ বিদ্যমান থাকবে যেন।

$$|f(z) - w_1| < \frac{\varepsilon}{2} \text{ when } 0 < |z - z_0| < \delta$$

$$|f(z) - w_2| < \frac{\varepsilon}{2} \text{ when } 0 < |z - z_0| < \delta$$

Now [যখন] $|w_1 - w_2| = |w_1 - f(z) + f(z) - w_2|$

$\Rightarrow |w_1 - w_2| \leq |w_1 - f(z)| + |f(z) - w_2|$

$\Rightarrow |w_1 - w_2| \leq |f(z) - w_1| + |f(z) - w_2|$

$\Rightarrow |w_1 - w_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow |w_1 - w_2| < \varepsilon$

$\Rightarrow |w_1 - w_2|$ is less than any positive number ε however small. [যত ক্ষুদ্রই হউক একটি ধনাত্মক সংখ্যা ε হতে ছোট]

This means [এর অর্থ] $|w_1 - w_2| = 0 \Rightarrow w_1 = w_2$

Thus if $\lim\limits_{z \to z_0} f(z)$ exists, it must be unique. [অতএব যদি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান থাকে তবে ইহা অবশ্যই অনন্য।]

Theorem–2. Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

Then $\lim\limits_{z \to z_0} f(z) = w_0 \cdots\cdots (1)$

if and only if

$$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and } \lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0 \cdots\cdots (2)$$

Proof : Necessary condition : First suppose that $\lim\limits_{z \to z_0} f(z) = w_0$. Then, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

$\Rightarrow |u(x, y) + iv(x, y) - (u_0 + iv_0)| < \varepsilon$

whenever $0 < |x + iy - (x_0 + iy_0)| < \delta$

$\Rightarrow |u(x, y) - u_0 + i(v(x, y) - v_0)| < \varepsilon$

whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

Since $|u - u_0| \leq |(u - u_0) + i(v - v_0)|$ $\quad [\because |Re(z)| \leq |z|]$

and $|v - v_0| \leq |(u - u_0) + i(v - v_0)|$ $\quad [\because |Im(z)| \leq |z|]$

it follows that

$$|u(x, y) - u_0| < \varepsilon \text{ and } |v(x, y) - v_0| < \varepsilon$$

whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

$\Rightarrow \lim\limits_{(x, y) \to (x_0, y_0)} u(x, y) = u_0$ and $\lim\limits_{(x, y) \to (x_0, y_0)} v(x, y) = v_0$

Sufficient condition : Let the condition (2) holds. We shall prove that (1) hold. From (2) we can write

$$|u(x, y) - u_0| < \frac{\varepsilon}{2} \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1$$

and $|v(x, y) - v_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2$

Choosing $\delta = \min\{\delta_1, \delta_2\}$ we have

$|f(z) - w_0| = |u(x, y) + iv(x, y) - u_0 - iv_0|$

$\Rightarrow |f(z) - w_0| \leq |u(x, y) - u_0| + |v(x, y) - v_0|$

$\Rightarrow |f(z) - w_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow |f(z) - w_0| < \varepsilon$, wherever $0 < |z - z_0| < \delta$

Thus, $f(z) \to w_0$ as $z \to z_0$

That is, $\lim\limits_{z \to z_0} f(z) = w_0$. Hence proved

2.3. Continuity :

Definition : A complex valued function $f(z)$ is said to be continuous at a point z_0 if for every (given) $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Symbolically, we write $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Thus, in case of continuity at a point, the limiting value and functional value are equal.

[একটি জটিল মান ফাংশন $f(z)$ কে z_0 বিন্দুতে অবিচ্ছিন্ন বলে যদি প্রদত্ত $\varepsilon > 0$ এর জন্য একটি $\delta > 0$ বিদ্যমান থাকে যেন

$|f(z) - f(z_0)| < \varepsilon$ যখন $|z - z_0| < \delta$

প্রতীক আকারে আমরা লিখি $\lim\limits_{z \to z_0} f(z) = f(z_0)$

অতএব, একটি বিন্দুতে অবিচ্ছিন্নতার ক্ষেত্রে, সীমাস্থ মান ও ফাংশনাল মান সমান ।]

A function $f(z)$ is continuous on a set S if it is continuous at every point of S. If a function is not continuous at z_0, then it is discontinuous at z_0.

If $f(z_0) = u_0 + iv_0$ then $\lim\limits_{z \to z_0} f(z) = f(z_0)$ gives us

$$\lim_{(x,y) \to (x_0, y_0)} [u(x, y) + iv(x, y)] = u_0 + iv_0$$

$$\Rightarrow \lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0$$

This also give the information that component functions $u(x, y)$ and $v(x, y)$ are also continuous at $z_0 = (x_0, y_0)$.

Proposition : A continuous function of a continuous function is continuous.

OR, A composition of continuous functions is continuous.

Proof : Let $w = f(z)$ be a function which is continuous at z_0. Suppose $g(z)$ is a function which is defined on f-image. The composition $g[f(z)]$ is then defined for all z in the neighbourhood of z_0. Given that $g(z)$ is continuous at $w_0 = f(z_0)$. Then for given $\varepsilon > 0$, there exists a $\gamma > 0$ such that

$$|g(f(z)) - g(f(z_0))| < \varepsilon \text{ whenever } |f(z) - f(z_0)| < \gamma \text{ ...... (1)}$$

Again, since $f(z)$ is continuous at z_0, so for given $\gamma > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \gamma \text{ whenever } |z - z_0| < \delta \text{ ...... (2)}$$

By (2), (1) becomes

$$|(gf)(z) - (gf)(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta.$$

Hence the composition function gf is continuous at z_0.

Proposition : If a function $f(z)$ is continuous on a bounded and closed set $S \subset C$, then minimum and maximum of $|f(z)|$ exist on S.

Proof : Given that $f(z) = u(x, y) + iv(x, y)$ is continuous on S. Then the component functions $u(x, y)$ and $v(x, y)$ are continuous on S.

The function

$$|f(z)| = \sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is then real valued continuous function on the bounded and closed set S. So by real calculas $|f(z)|$ attains its maximum and minimum on S. Thus $f(z)$ is bounded. There exists a non-negative real number M such that $|f(z)| \le M$ for all z in S. The equality holds for at least one such z.

Uniform continuity : A function $f(z)$ is said to be uniformly continuous on a set S if, for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z_1) - f(z_2)| < \varepsilon \text{ whenever } |z_1 - z_2| < \delta;\ \forall z_1, z_2 \in S.$$

Here $\delta = \delta(\varepsilon)$ and δ is independent of z_1 and z_2 in S.

2.4. Differentiability : [NUH-1994, 2002, DUH-1998, 2005]

Definition : Let $f(z)$ be a function defined in some neighbourhood of z_0. Then $f(z)$ is said to be differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists and finite.}$$

This limit is denoted by $f'(z_0)$ and is called the derivative of $f(z)$ at z_0. Thus we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ ...... (1)}$$

[মনে করি z_0 এর কোন নেইবারহুডে $f(z)$ ফাংশনটি বর্ণিত। তখন $f(z)$ কে z_0 এ অন্তরীকরণ যোগ্য বলে যদি

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

বিদ্যমান ও সসীম হয়। এই লিমিটকে $f'(z_0)$ দ্বারা প্রকাশ করা হয় এবং ইহাকে z_0 বিন্দুতে $f(z)$ এর অন্তরক বলে। অতএব আমরা পাই

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.]$$

A function is differentiable on a set S in complex plane if it is differentiable at each point of the set.

Note-1. We observe that f(z) has a derivative $f'(z_0)$ at z_0 if for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon, \text{ whenever } |z - z_0| < \delta.$$

$$\text{If } \eta(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$$

then $\lim\limits_{z \to z_0} \eta(z) = \eta(z_0)$. This shows that $\eta(z)$ is continuous at z_0 and $|\eta(z)| < \varepsilon$ whenever $|z - z_0| < \delta$.

Thus, we get an explicit expression for f(z) in the form

$$f(z) = f(z_0) + (z - z_0) f'(z) + (z - z_0) \eta(z) \quad \text{for } |z - z_0| < \delta.$$

Note-2. Writting $\Delta z = z - z_0$ in (1) we get

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(\Delta z + z_0) - f(z_0)}{\Delta z}$$

Droping the suffix from z_0 we get

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \ldots\ldots (2)$$

Note-3. If w = f(z), $w + \Delta w = f(z + \Delta z)$ then from (2)

$$f'(z) = \lim_{\Delta z \to 0} \frac{w + \Delta w - w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}$$

$$\therefore \frac{dw}{dz} = f'(z).$$

Theorem-3. Every differentiable function is continuous. [প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন]

OR, If f(z) is differentiable at a point then it is continuous there. **[NHU-2014]**

Proof : Let the function f(z) is differentiable at z_0. [মনে করি f(z) ফাংশনটি z_0 এ অন্তরীকরণযোগ্য]

Now [এখন] $\lim\limits_{z \to z_0} [f(z) - f(z_0)] = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} (z - z_0)$

$$\Rightarrow \lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

Hence f(z) is continuous at z_0. [অতএব z_0 এ f(z) অবিচ্ছিন্ন] Thus every differentiable function is continuous. [অতএব প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন ।]

2.5. Analytic (or regular or holomorphic) functions :
[NUH-2000, 03, 05, 05(Old), 06(Old), 06, 12(Old), 12, NU(Pre)-11, (Phy)-06, 10, DUH-05]

The concept of analytic function is the core heart of complex analysis. First we give the definition of an analytic function at a point and in a domain (region).

Definition : A complex function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

[একটি জটিল ফাংশন f(z) কে একটি বিন্দু z_0 এ বৈশ্লেষিক বলে যদি ইহার অন্তরক শুধুমাত্র z_0 বিন্দুতে নয়, z_0 বিন্দুর নেইবারহুডের কিছু বিন্দু z বিন্দুতেও বিদ্যমান থাকে ।]

The above definition follows that if f(z) is analytic at z_0, it is actually analytic at each point in a neighbourhood of z_0.

Definition : A function f(z) is said to be analytic in a domain D (or region R) if it is analytic at each point of D (or R).

[একটি ফাংশন f(z) কে একটি ডোমেন D এ বৈশ্লেষিক বলে যদি ইহা D এর প্রত্যেক বিন্দুতে বৈশ্লেষিক হয় ।]

The terms holomorphic and regular are used with identical meaning in the literature.

Entire function : A complex function $f(z)$ is said to be entire if it is analytic in the whole complex plane.

The functions e^z, $\sin z$, $\cos z$ are entire functions.

Singular point or Singularity : [JUH (Phy)-2000, 03, 05]

If a function $f(z)$ fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of $f(z)$.

[ব্যতিচার বিন্দু ঃ যদি একটি ফাংশন $f(z)$ একটি বিন্দু z_0 এ বৈশ্লেষিক হতে ব্যর্থ হয় কিন্তু z_0 এর প্রত্যেক নেইবারহুডে কমপক্ষে একটি বিন্দু বিদ্যমান থাকে যেখানে ফাংশনটি বৈশ্লেষিক হয়, তখন z_0 কে $f(z)$ এর ব্যতিচার বিন্দু বলে ।]

Cauchy-Riemann Partial Differential Equations :

Theorem-4. Necessary conditions for $f(z)$ to be analytic.

[NHU-2014, JUH (Phy)-2000, 2004]

The necessary condition for $w = f(z) = u(x, y) + iv(x, y)$ to be analytic at a point $z = x + iy$ of its domain is that the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ should exist and satisfy the Cauchy-Riemann partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[উপপাদ্য - 4 : $f(z)$ বৈশ্লেষিক হওয়ার প্রয়োজনীয় শর্তসমূহ ঃ $w = f(z) = u(x, y) + iv(x, y)$ ইহার ডোমেনের একটি বিন্দু $z = x + iy$ এ বৈশ্লেষিক হওয়ার প্রয়োজনীয় শর্ত হল চারটি আংশিক অন্তরীকরণ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ এবং $\frac{\partial v}{\partial y}$ বিদ্যমান থাকবে এবং কচি-রীম্যান আংশিক অন্তরক সমীরকণ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ এবং $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ সিদ্ধ করবে ।]

[NUH-98, 2000, 02, 05 (Old), 06 (Old), Nu()Pre, 07, 10, 12, 13, DUH-03, 05]

Proof : Given that $f(z) = u(x, y) + iv(x, y)$ is analytic (differentiable) at any point $z = x + iy$ of its domain.

[দেওয়া আছে $f(z) = u(x, y) + iv(x, y)$ ইহার ডোমেনের $z = x + iy$ বিন্দুতে বৈশ্লেষিক (অন্তরীকরণযোগ্য ।]

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \quad \dots (1)$$

Since $f(z)$ is analytic, so $f'(z)$ exists, finite and unique independent of the path along which $\Delta z \to 0$.

[যেহেতু $f(z)$ বৈশ্লেষিক, সুতরাং $f'(z)$ বিদ্যমান, সসীম এবং অনন্য, $\Delta z \to 0$ বরাবর পথের উপর অনির্ভরশীল]

Now there are two cases [এখন দুইটি ক্ষেত্র]

Case-1. Along real axis (x-axis) we have $\Delta y = 0$ and $\Delta x \to 0$, so from (1) we have [বাস্তব অক্ষের দিকে $\Delta y = 0$ এবং $\Delta x \to 0$, সুতরাং (1) হতে পাই]

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \dots\dots (2)$$

provided the partial derivatives exist. [আংশিক অন্তরীকরণ বিদ্যমান শর্তে]

Case-2. Along imaginary axis (y-axis) we have $\Delta x = 0$ and $\Delta y \to 0$, so from (1) we have [কাল্পনিক অক্ষের দিকে $\Delta x = 0$ এবং $\Delta y \to 0$ সুতরাং (1) হতে পাই]

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \dots\dots (2)$$

Since f(z) is analytic, so $f'(z)$ exists and the above two limits obtained in (2) and (3) should be unique (identical). [যেহেতু f(z) বৈশ্লেষিক, সুতরাং $f'(z)$ বিদ্যমান এবং উপরের (2) ও (3) প্রাপ্ত লিমিট অনন্য]

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts we get [বাস্তব ও অবাস্তব রাশি সমীকৃত করে পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

That is [অর্থাৎ] $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and [এবং] $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. (Proved)

Theorem-5. Sufficient condition for f(z) to be analytic. **[NUH-1993, 1998, 2001, 2006, 2013]**

The function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if

(i) u, v are differentiable in D and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ are all continuous in D. **[DUH-2006]**

[উপপাদ্য-৫ ঃ f(z) বৈশ্লেষিক হওয়ার জন্য যথেষ্ট শর্ত। একটি ডোমেন D এ $w = f(z) = u(x, y) + iv(x, y)$ বৈশ্লেষিক হবে যদি

(i) D এ u, v অন্তরীকরণযোগ্য হয় এবং $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) আংশিক অন্তরীকরণ $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ এবং $\frac{\partial v}{\partial y}$ সকলেই D এর মধ্যে অবিচ্ছিন্ন হয়।]

Proof : Given that [দেওয়া আছে] $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are continuous and [অবিচ্ছিন্ন এবং]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ ...... (1)}$$

Now [এখন] $\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$

$\Rightarrow \Delta u = [u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)] + [u(x, y + \Delta y) - u(x, y)] \cdots (2)$

From mean value theorem we know that if f(x) is continuous in $a \le x \le b$ and differentiable in $a < x < b$ then

[গড়মান উপপাদ্য হতে আমরা জানি যে, যদি $a \le x \le b$ এ f(x) অবিচ্ছিন্ন হয় এবং $a < x < b$ এ অন্তরীকরণযোগ্য হয় তখন]

$$f(a + h) - f(a) = hf'(a + \theta h) \text{ where [যেখানে] } 0 < \theta < 1$$

Using this result in (2) we get [এইফল (2) এ ব্যবহার করে পাই]

$$\Delta u = \Delta x \cdot \frac{\partial}{\partial x} u(x + \theta\Delta x, y + \Delta y) + \Delta y \cdot \frac{\partial}{\partial y} u(x, y + \theta'\Delta y) \text{ ...... (3)}$$

where [যেখানে] $0 < \theta < 1$ and [এবং] $0 < \theta' < 1$.

Again, since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous, so [আবার যেহেতু $\frac{\partial u}{\partial x}$ এবং $\frac{\partial u}{\partial y}$ অবিচ্ছিন্ন, সুতরাং]

$$\left|\frac{\partial}{\partial x} u(x + \theta\Delta x, y + \Delta y) - \frac{\partial}{\partial x} u(x, y)\right| < \varepsilon$$

and [এবং] $\left|\frac{\partial}{\partial y} u(x, y + \theta'\Delta y) - \frac{\partial}{\partial y} u(x, y)\right| < \eta$

Choosing $\varepsilon_1 < \varepsilon$ and $\eta_1 < \eta$ we get [$\varepsilon_1 < \varepsilon$ এবং $\eta_1 < \eta$ পছন্দ করে পাই]

$$\frac{\partial}{\partial x} u(x + \theta\Delta x, y + \Delta y) - \frac{\partial}{\partial x} u(x, y) = \varepsilon_1$$

$$\Rightarrow \frac{\partial}{\partial x} u(x + \theta\Delta x, y + \Delta y) = \frac{\partial}{\partial x} u(x, y) + \varepsilon_1$$

$$= \frac{\partial u}{\partial x} + \varepsilon_1 \text{ ...... (4)}$$

and [এবং] $\frac{\partial}{\partial y} u(x, y + \theta'\Delta y) - \frac{\partial}{\partial y} u(x, y) = \eta_1$

$$\Rightarrow \frac{\partial}{\partial y} u(x, y + \theta'\Delta y) = \frac{\partial}{\partial y} u(x, y) + \eta_1 = \frac{\partial u}{\partial y} + \eta_1 \text{ ...... (5)}$$

By putting the values of (4) and (5) in (3) we get [(4) ও (5) এর মান (3) এ বসাইয়া পাই]

$$\Delta u = \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y \text{ ...... (6)}$$

where [যেখানে] $\varepsilon_1 \to 0$ and [এবং] $\eta_1 \to 0$ as $\Delta x \to 0$ and [এবং] $\Delta y \to 0$.

Proceeding in the same way we have [একইভাবে অগ্রসর হয়ে পাই]

$$\Delta v = \left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y \text{ ...... (7)}$$

where [যেখানে] $\varepsilon_2 \to 0$ and [এবং] $\eta_2 \to 0$ as $\Delta x \to 0$ and [এবং] $\Delta y \to 0$

Now [এখন] $w = f(z) = u + iv$

$$\Rightarrow \Delta w = \Delta u + i\Delta v$$
$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y + i\left[\left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y\right]$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + (\varepsilon_1 + i\varepsilon_2)\,\Delta x + (\eta_1 + i\eta_2)\,\Delta y$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

where [যেখানে] $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and [এবং] $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and [এবং] $\Delta y \to 0$

$$\Rightarrow \Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z + \varepsilon\Delta x + \eta\Delta y$$
$$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varepsilon\Delta x + \eta\Delta y}{\Delta z}$$

Taking limit $\Delta z \to 0$ on both sides we get [উভয় পক্ষে $\Delta z \to 0$ লিমিট নিয়ে পাই]

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varepsilon\Delta x + \eta\Delta y}{\Delta z}\right)$$
$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + 0 + 0$$
$$\Rightarrow \frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

We see that the derivative exist and unique. Hence $w = f(z)$ is analytic. [আমরা দেখি যে অন্তরীকরণ বিদ্যমান এবং অনন্য। অতএব $w = f(z)$ বৈশ্লেষিক।]

Note-1. A necessary condition for a function f to be analytic in a domain D is the continuity of f throughout D. Satisfaction of the **Cauchy-Riemann (C-R)** equations is also necessary, but note sufficient.

Note-2. The real functions of complex variable are nowhere analytic unless these are constant valued. Hence, Re(z), Im(z), $|z|$, $|z|^2$, etc. are nowhere analytic, since these are real valued but not constant on any domain in complex plane.

2.6. Polar form of C-R equations :

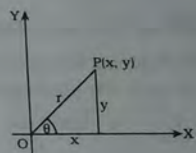
Show that Cauchy-Riemann egrcations in polan form are $= \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

[NUH-97, 03, 04,08, 10, 12(Old), 14, DUH-04]

Or, Statement : The cauchy-Riemann equations in polar co-ordinate are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ [NU(Pre)-2014]

Proof : The relation between cartesian and polar form are [কার্তেসীয় ও পোলার স্থানাংকে সম্পর্ক হল]

$$x = r\cos\theta,\ y = r\sin\theta$$
$$r^2 = x^2 + y^2 \text{ and [এবং] } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$



$$\therefore 2r\frac{\partial r}{\partial x} = 2x$$
$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

Similarly [অনুরূপে]

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

Also [আরো] $\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-y}{x^2 + y^2}$

$$= \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

and [এবং] $\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$

$$= \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Now u, v are functions of x, y; and x, y are functions of r, θ. So, u, v are functions of r, θ. [এখন u, v হল x, y এর ফাংশন এবং x, y হল r, θ এর ফাংশন। সুতরাং u, v হল r, θ এর ফাংশন]

That is [অর্থাৎ] $u = u(r, \theta)$ and [এবং] $v = v(r, \theta)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \quad \text{...... (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \quad \text{...... (2)}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \text{ ...... (3)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \text{ ...... (4)}$$

From Cauchy-Riemann equations we know that [কচি রীম্যান সমীকরণ হতে আমরা জানি]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \text{ ...... (5)}$$

$$\text{and } \sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \text{ ...... (6)}$$

[By equations (1), (2), (3) and (4).]

Multiplying (5) by $\cos\theta$ and (6) by $\sin\theta$, and then adding we get [(5) কে $\cos\theta$ এবং (6) কে $\sin\theta$ দ্বারা গুণ করে এবং অতপর যোগ করে পাই]

$$\cos^2\theta\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta} + \sin^2\theta\frac{\partial u}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$= \cos\theta\sin\theta\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r}\frac{\partial v}{\partial \theta} - \sin\theta\cos\theta\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow (\cos^2\theta + \sin^2\theta)\frac{\partial u}{\partial r} = \frac{1}{r}(\cos^2\theta + \sin^2\theta)\frac{\partial v}{\partial \theta}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ ...... (7)}$$

Again, multiplying (5) by $\sin\theta$ and (6) by $\cos\theta$, and then adding we get [আবার, (5) কে $\sin\theta$ এবং (6) কে $\cos\theta$ দ্বারা গুণ করে এবং অতপর যোগ করে পাই]

$$\cos\theta\sin\theta\frac{\partial u}{\partial r} - \frac{\sin^2\theta}{r}\frac{\partial u}{\partial \theta} - \sin\theta\cos\theta\frac{\partial u}{\partial r} - \frac{\cos^2\theta}{r}\frac{\partial u}{\partial \theta}$$

$$= \sin^2\theta\frac{\partial v}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial v}{\partial \theta} + \cos^2\theta\frac{\partial v}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow -\frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial u}{\partial \theta} = (\sin^2\theta + \cos^2\theta)\frac{\partial v}{\partial r}$$

$$\Rightarrow -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \text{ ...... (8)}$$

Thus [অতএব] $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and [এবং] $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ are the required polar form of C–R equations. [পোলার আকারে আবশ্যকীয় C–R সমীকরণ]

2.7. Harmonic Functions :

[NUH–93, 94, 01, 04, 15, NU(Pre)–08]

Any real valued function of x and y is said to be harmonic in a domain of the xy plane if throughout the domain it has continuous partial derivatives of the first and second order and satisfy the Laplace equation. **[DUH–2005]**

[হারমোনিক ফাংশনঃ xy তলের কোন ডোমেন x ও y এর যে কোন বাস্তব মানী ফাংশনকে হারমোনিক ফাংশন বলে যদি সম্পূর্ণ ডোমেনে ইহার প্রথম ও দ্বিতীয় ক্রমের অবিচ্ছিন্ন আংশিক অন্তরীকরণ থাকে এবং ল্যাপলাস সমীকরণ সিদ্ধ করে ।]

Harmonic conjugate :

[NUH–1993, 1994, 2004, 2014, DUH–2005]

The function v is said to be a harmonic conjugate of u if u and v are harmonic and u, v satisfy the C–R equations.

[হারমোনিক অনুবন্ধীঃ v ফাংশন কে u ফাংশনের হারমোনিক অনুবন্ধী বলে যদি u ও v হারমোনিক হয় এবং u, v কচি-রীম্যান সমীকরণ সমূহ সিদ্ধ করে ।]

If $f(z) = u(x, y) + iv(x, y)$, then u, v are the component functions of f. For component functions we shall prove the following results.

Theorem–6. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D. **[NUH–2001]**

[যদি একটি ফাংশন $f(z) = u(x, y) + iv(x, y)$ কোন ডোমেন D এ বৈশ্লেষিক হয়, তখন ইহার উপাংশ ফাংশন u ও v, D এ হারমোনিক ।]

Or, The real and imaginary parts of an analytic function ore harmonic function. **[NUH–2001, 2015]**

Proof : Since f is analytic in D, so its component functions u and v satisfy the Cauchy-Riemann equations throughout D. [যেহেতু D এ f বৈশ্লেষিক, সুতরাং ইহার উপাংশ ফাংশন u এবং v, D এর সর্বত্র কচি-রীম্যান সমীকরণ সিদ্ধ করবে ।]

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ ...... (1)}$$

Differentiating partially w. r. to x both sides of (1) we get [(1) এর উভয় পক্ষে x এর সাপেক্ষে আংশিক অন্তরীকরণ করে পাই]

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \text{ and [এবং] } \frac{\partial^2 u}{\partial x\,\partial y} = -\frac{\partial^2 v}{\partial x^2} \;\ldots\ldots (2)$$

Again, differentiating partially w. r. to y both sides of (1) we get [আবার, (1) এর উভয় পক্ষে y এর সাপেক্ষে আংশিক অন্তরীকরণ করে পাই]

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial y^2} \text{ and [এবং] } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x} \;\ldots\ldots (3)$$

By calculas, the continuity of the partial derivatives ensures that [ক্যালকুলাস দ্বারা এ আংশিক অন্তরীকরণের অবিচ্ছিন্নতা নিশ্চিত করে যে]

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \text{ and [এবং] } \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} \text{ and [এবং] } \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}; \quad \text{[by (2) and (3)]}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ and [এবং] } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and [এবং] } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus, the both component functions u and v satisfy the Laplace equation. Hence u and v are harmonic in D. [অতএব, উভয় উপাংশ ফাংশন u এবং v ল্যাপলাস সমীকরণ সিদ্ধ করে। অতএব D এ u ও v হারমোনিক।]

That is, $u = \mathrm{Re}\{f(z)\}$, $v = \mathrm{Im}\{f(z)\}$ both are harmonic functions. [অর্থাৎ $u = \mathrm{Re}\{f(z)\}$, $v = \mathrm{Im}\{f(z)\}$ উভয়েই হারমনিক ফাংশন।]

2.8. Laplace equation in polar form :

Theorem-7. The Laplace equation in polar form is [পোলার আকারে ল্যাপলাস সমীকরণ]

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \qquad \textbf{[DUH-1989, 2006, JUH-1991]}$$

Solution : Let [ধরি] $z = x + iy$, $x = r\cos\theta$, $y = r\sin\theta$ and

$$w = f(z) = u + iv.$$

The Cauchy-Riemann equations in polar form are [পোলার আকারে কচি-রীম্যান সমীকরণ হল]

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \;\ldots\ldots (1)$$

$$\text{and [এবং] } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \;\ldots\ldots (2)$$

If the second order partial derivatives of u and v w. r. to r and θ are continuous, then [r ও θ এর সাপেক্ষে u ও v এর যদি দ্বিতীয় ক্রমের আংশিক অন্তরজ থাকে, তখন]

$$\frac{\partial^2 u}{\partial r\,\partial \theta} = \frac{\partial^2 u}{\partial \theta\,\partial r} \;\ldots\ldots (3)$$

$$\text{and [এবং] } \frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial^2 v}{\partial \theta\,\partial r} \;\ldots\ldots (4)$$

From (3) we have [(3) হতে পাই]

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(-r\frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right), \quad \text{[by (1) and (2)]}$$

$$\Rightarrow -\frac{\partial v}{\partial r} - r\frac{\partial^2 v}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}$$

$$\Rightarrow -\frac{1}{r}\frac{\partial v}{\partial r} - \frac{\partial^2 v}{\partial r^2} = \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}, \text{ dividing by r both sides}$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \;\ldots\ldots (5)$$

In the same way from (4) we get [(4) হতে একইভাবে পাই]

$$\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial \theta}\left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right), \quad \text{[by (1) and (2)]}$$

$$\Rightarrow \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

$$\Rightarrow \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \text{ diving both sides by r.}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \;\ldots\ldots (6)$$

If we write $u = v = \psi$ in (5) and (6) we get the same result, which is [(5) ও (6) যদি আমরা $u = v = \psi$ লিখি তাহলে আমরা একই ফল পাই যাহা]

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0.$$

Theorem–8. A function $f(z) = u + iv$ is analytic in a domain D if and only if v is harmonic conjugate of u.

Proof : Let $f(z) = u + iv$ be analytic in D. Then the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations. So u and v are harmonic functions. Hence v is harmonic conjugate of u.

Conversely, suppose that v is harmonic conjugate of u. Then u and v are harmonic functions and the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations. Further, u and v are harmonic, means their second order partial derivatives and hence the first order partial derivatives are continuous. This implies that $f(z) = u + iv$ is analytic.

From the above theorem we see that a necessary and sufficient condition for a function $f(z) = u(x, y) + iv(x, y)$ to be analytic in a domain D is that v be a harmonic conjugate of u in D.

We note that if v is a harmonic conjugate of u in some domain D, then it is not in general true that u is a harmonic conjugate of v there. We show this by an example later.

2.9. Construction of an analytic function :

Method–1 : Let u be a harmonic function and v be its harmonic conjugate function. Then by C–R equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The function $v(x, y)$ is constructed in two steps.

(i) Integrate $\frac{\partial v}{\partial y}\left(\text{which is equal to } \frac{\partial u}{\partial x}\right)$ with respect to y, treating x as constant :

$$v(x, y) = \int \frac{\partial u}{\partial x}\, dy + \phi(x) \;\cdots\cdots (1)$$

where $\phi(x)$ is a function of x only, and hence the partial derivative of $\phi(x)$ with respect to y is zero.

(ii) Differentiating (1) w. r. to x and then replacing $\frac{\partial v}{\partial x}$ by $-\frac{\partial u}{\partial y}$ on the left side we get

$$-\frac{\partial u}{\partial y} = \frac{d}{dx}\int \frac{\partial u}{\partial x}\, dy + \phi'(x) \;\cdots\cdots (2)$$

Since u is harmonic, all terms except those involving x in (2) cancel out, and a formula for $\phi'(x)$ will purely a function of x. Integrating $\phi'(x)$, we will get $\phi(x)$ and put it in (1) so that we get $v(x, y)$. Thus we get our desired analytic function $f(z) = u + iv$.

Method–2. If $v = v(x, y)$ then $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$\Rightarrow dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{by C–R equations} \cdots\cdots (1)$$

$$\Rightarrow dv = M\,dx + N\,dy$$

where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow -\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \left[\because \text{u is harmonic, so u satisfy Laplace equation } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right]$$

This shows that (1) is an exact equation. Therefore, by integrating (1), v can be found. Now both u and v are known and as such $f(z) = u + iv$ can be determined.

Method–3. Milne Thomson method.

Let $z = x + iy$. Then $\bar{z} = x - iy$

$$\therefore z + \bar{z} = x + iy + x - iy = 2x \Rightarrow x = \frac{z + \bar{z}}{2} \cdots\cdots (1)$$

and $z - \bar{z} = x + iy - x + iy = 2iy \Rightarrow y = \frac{z - \bar{z}}{2i} \cdots\cdots (2)$

Now $w = f(z) = u(x, y) + iv(x, y)$

$$= u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iv\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \cdots\cdots (3)$$

The above relation can be treated as a formal identity in two independent variables z and $\bar{z}$. Setting $\bar{z} = z$ in (1), (2) and (3) we get

$$x = \frac{z+z}{2} = z,\ y = \frac{z-z}{2i} = 0 \text{ and}$$

$$w = f(z) = u(z, 0) + iv(z, 0)$$

Now $f(z) = w = u + iv$ gives

$$f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x}\frac{dx}{dz} + i\frac{\partial v}{\partial x}\frac{dx}{dz} \qquad \Bigg| \qquad x = z \Rightarrow \frac{dx}{dz} = 1$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \ \cdots\cdots (4)$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \text{ by C-R equation}$$

If $\frac{\partial u}{\partial x} = \phi_1(x, y) = \phi_1(z, 0)$ and $\frac{\partial u}{\partial y} = \phi_2(x, y) = \phi_2(z, 0)$

then the above relation becomes

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

Integrating this we get

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c \ \cdots\cdots (5)$$

This gives the construction of $f(z)$ when u is given.

Similarly, when v is given and u is unknown then by replacing $\frac{\partial u}{\partial x}$ by $\frac{\partial v}{\partial y}$ in (4) by C–R equation we get

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

If we choose $\frac{\partial v}{\partial y} = \psi_1(x, y) = \psi_1(z, 0)$

and $\frac{\partial v}{\partial x} = \psi_2(x, y) = \psi_2(z, 0)$, then

$$f'(z) = \psi_1(z, 0) + i\psi_2(z, 0)$$

$$\Rightarrow f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\, dz + c.$$

Note : Milne method is very nice and interesting method for getting $f(z)$ rapidly. Some time other methods are very tedious and cumbersome.

2.10. Partial derivative in relation to z and $\bar{z}$:

We have $z = x + iy$ and $\bar{z} = x - iy$

$$\Rightarrow z + \bar{z} = 2x \text{ and } z - \bar{z} = 2iy$$

$$\Rightarrow x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

Every complex function $f(z) = u + iv$ can be written as

$$f(x + iy) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of x and y. This means $f(x + iy)$ may be treated as a function of x and y, that is, $f = f(x, y)$.

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\frac{1}{2i} \qquad \because x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \ \cdots\cdots (1)$$

Again, $f = f(x, y)$ gives

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{-1}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

$$\Rightarrow \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \ \cdots\cdots (2)$$

(1) + (2) gives $\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} + \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$= \frac{1}{2}\cdot 2\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \ \cdots\cdots (3)$$

(2) – (1) gives $\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$

$$=\frac{1}{2}\cdot 2i\frac{\partial}{\partial y}=i\frac{\partial}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial y}=\frac{1}{i}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)$$

$$=i\left(\frac{\partial}{\partial \bar{z}}-\frac{\partial}{\partial z}\right) \text{...... (4)}$$

$$\therefore \frac{\partial}{\partial x}+i\frac{\partial}{\partial y}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}+\frac{\partial}{\partial \bar{z}}-\frac{\partial}{\partial z}=2\frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}-\frac{\partial}{\partial \bar{z}}+\frac{\partial}{\partial z}=2\frac{\partial}{\partial z}$$

Now $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial x^2}-i^2\frac{\partial^2}{\partial y^2}$

$$=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

$$=2\frac{\partial}{\partial \bar{z}}\left(2\frac{\partial}{\partial z}\right)$$

$$=4\frac{\partial^2}{\partial \bar{z}\,\partial z} \text{...... (5)}$$

Again, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial x^2}-i^2\frac{\partial^2}{\partial y^2}$

$$=\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$=2\frac{\partial}{\partial z}\left(2\frac{\partial}{\partial \bar{z}}\right)$$

$$=4\frac{\partial^2}{\partial z\,\partial \bar{z}} \text{...... (6)}$$

From (5) and (6) we have

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial \bar{z}}=4\frac{\partial^2}{\partial \bar{z}\,\partial z}.$$

SOLVED EXAMPLES

Example–1. If $f(z)=\frac{2z-1}{3z+2}$, prove that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=\frac{7}{(3z_0+2)^2} \text{ provided } z_0\neq\frac{-2}{3}.$$

[RUH–2000]

Solution : Given that [দেওয়া আছে] $f(z)=\frac{2z-1}{3z+2}$

$$\therefore \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=\lim_{h\to 0}\frac{1}{h}\left[\frac{2(z_0+h)-1}{3(z_0+h)+2}-\frac{2z_0-1}{3z_0+2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(2z_0+2h-1)(3z_0+2)-(2z_0-1)(3z_0+3h+2)}{(3z_0+3h+2)(3z_0+2)}$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{6z_0^2+4z_0+6hz_0+4h-3z_0-2-6z_0^2-6hz_0-4z_0+3z_0+3h+2}{(3z_0+3h+2)(3z_0+2)}$$

$$=\lim_{h\to 0}\frac{7h}{h(3z_0+3h+2)(3z_0+2)}$$

$$=\lim_{h\to 0}\frac{7}{(3z_0+3h+2)(3z_0+2)}$$

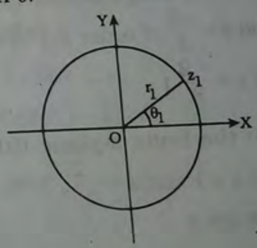
$$=\frac{7}{(3z_0+0+2)(3z_0+2)}=\frac{7}{(3z_0+2)^2}$$

Provided $3z_0+2\neq 0\Rightarrow z_0\neq -\frac{2}{3}$.

Example–2. Prove that $f(z)=\ln z$ has a branch point at $z=0$.

[NUH–2004, 2007]

Solution : Let [ধরি] $x=r\cos\theta$, $y=r\sin\theta$.



$$\therefore f(z)=\ln z=\ln(x+iy)$$

$$=\ln(r\cos\theta+ir\sin\theta)$$

$$=\ln\{r(\cos\theta+i\sin\theta)\}$$

$$=\ln\{re^{i\theta}\}$$

$$=\ln r+\ln e^{i\theta}$$

$$=\ln r+i\theta$$

Suppose we start for [ধরি আমরা শুরু করছি]

$z = z_1 \neq 0$ and let [এবং ধরি] $r = r_1$, $\theta = \theta_1$

$\therefore \ln z_1 = \ln r_1 + i\theta_1$

After making one complete circuit about the origin in the positive or counterclockwise direction, on returning to z_1 we find $r = r_1$, $\theta = \theta_1 + 2\pi$ so that

$$\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi).$$

Thus we have another branch of f(z) and so z = 0 is a branch point.

[ধনাত্মক দিকে বা ঘড়ির কাটার বিপরীত দিকে মূল বিন্দুর চারিদিকে একটি পূর্ণ সার্কিট তৈরীর পর z_1 এর দিকে ফেরত আসতে পাই $r = r_1$, $\theta = \theta_1 + 2\pi$ যেন

$$\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi).$$

অতএব আমরা f(z) এর অন্য একটি ব্রাঞ্চ পদ পাই এবং সেকারণে z = 0 একটি ব্রাঞ্চ বিন্দু।]

Example–3. For the function $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$, locate and name all the singularities in the finite z-plane and also determine where f(z) is analytic.

Solution : Given that [দেওয়া আছে] $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ ...... (1)

In the finite z-plane the singularities will be obtained by solving the equation [সসীম z- চলে ব্যতিচার বিন্দুগুলি নিম্নের সমীকরণ সমাধান করে পাওয়া যাবে]

$$(z-1)^3 (3z+2)^2 = 0$$

$$\Rightarrow (z-1)^3 = 0 \quad \text{or } (3z+2)^2 = 0$$

$$\Rightarrow z = 1, 1, 1 \quad \text{or } z = \frac{-2}{3}, \frac{-2}{3}$$

$\therefore$ The singularities in the finite z-plane are z = 1 of order 3 and $z = \dfrac{-2}{3}$ of order 2. [সসীম z- তলে ব্যতিচার বিন্দু হল 3 ক্রমের z = 1 এবং 2 ক্রমের $z = \dfrac{-2}{3}$]

In the finite z-plane f(z) is analytic everywhere excepts the points z = 1 and $z = \dfrac{-2}{3}$. [সসীম z- তলে z = 1 এবং $z = \dfrac{-2}{3}$ বিন্দু ব্যতীত সর্বত্র বৈশ্লেষিক হবে।]

Example–4. Determine the singular points of the function

$$f(z) = \frac{z^3 + 7}{(z^2 - 2z + 2)(z-3)}.$$

Solution : The singular points are obtained by solving the equation [ব্যতিচার বিন্দুগুলি নিম্নের সমীকরণ সমাধান করে পাওয়া যাবে।]

$$(z^2 - 2z + 2)(z - 3) = 0$$

$$\Rightarrow z - 3 = 0 \text{ or } z^2 - 2z + 2 = 0$$

$$\Rightarrow z = 3 \text{ or } z = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 + i, 1 - i$$

Thus the singular points are [অতএব ব্যতিচার বিন্দুগুলি হল]

z = 3, z = 1 + i, z = 1 – i.

Example–5. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$. [$f(z) = z^3 + z + 1$ ফাংশনকে $f(z) = u(x, y) + iv(x, y)$ আকারে লেখ]

Solution : Let z = x + iy. Then

$$f(z) = z^3 + z + 1$$

$$= (x + iy)^3 + (x + iy) + 1$$

$$= x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3 + x + iy + 1$$

$$= x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1 \quad [\because i^2 = -1]$$

$$= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$$

$$\Rightarrow f(z) = u(x, y) + iv(x, y)$$

where $u(x, y) = x^3 - 3xy^2 + x + 1$ and $v(x, y) = 3x^2y - y^3 + y$.

Example–6. Express the right hand side of

$f(z) = x^2 - y^2 - 2y + i2x + i2xy$ in terms of z.

Solution : Given that $f(z) = x^2 - y^2 - 2y + i2x + i2xy$

$$\Rightarrow f(z) = x^2 + i2xy + i^2y^2 + 2i^2y + i2x \quad [\text{since } i^2 = -1]$$

$$= (x + iy)^2 + i2(x + iy)$$

$$= z^2 + i2z.$$

Example–7. Show that $\frac{d}{dz}(\bar{z})$ does not exist anywhere.

[RUH–198

Solution : By definition we have [সংজ্ঞা অনুসারে পাই]

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta z\to 0} \frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\bar{z}+\overline{\Delta z}-\bar{z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Now along real axis [এখন বাস্তব অক্ষ বরাবর] $\Delta x \to 0$, $\Delta y = 0$ so [সুতরাং]

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta x\to 0} \frac{\Delta x - 0}{\Delta x + 0} = 1$$

Along imaginary axis [কাল্পনিক অক্ষ বরাবর] $\Delta x = 0$, $\Delta y \to 0$, so [সুতরাং]

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta y\to 0} \frac{0 - i\Delta y}{0 + i\Delta y} = -1$$

This shows that limit depends on manner in which $\Delta z \to 0$. Hence $\frac{d}{dz}(\bar{z})$ does not exist any where. Thus $f(z) = \bar{z}$ is non-analytic any where. [ইহা দেখায় যে লিমিট $\Delta z \to 0$ রীতির উপর নির্ভরশীল। অতএব যেকোন স্থানে $\frac{d}{dz}(\bar{z})$ বিদ্যমান নয়। অতএব, যেকোন স্থানে $f(z)$ অবৈশ্লেষিক।]

Example–8. Prove that $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

[NUH–1996, RUH–1998]

Solution : Let [ধরি] $z = x + iy$. Then [তখন] $z \to 0 \Rightarrow x \to 0, y \to 0$

$$\therefore \lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x - iy}{x + iy}$$

Taking limit along the real axis [x অক্ষ বরাবর লিমিট নিয়ে]

$(y = 0, x \to 0)$

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{x\to 0} \frac{x - 0}{x + 0} = 1$$

Again, taking limit along the imaginary axis [আবার, কাল্পনিক অক্ষ বরাবর লিমিট নিয়ে] $(x = 0, y \to 0)$

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{y\to 0} \frac{0 - iy}{0 + iy} = -1$$

The above two limits are not equal, that is, the limit depends on manner in which $z \to 0$. Hence $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

[উপরের লিমিট দুইটি সমান না, অর্থাৎ, লিমিট রীতিতে নির্ভরশীল যেখানে $z \to 0$. অতএব $\lim_{z\to 0} \frac{\bar{z}}{z}$ বিদ্যমান নাই।]

Example–9. Show that the function $f(z) = \bar{z}$ is non-analytic.

[RUH–1997, 1998 , 2002, 2004]

Solution : See example–7.

Example–10. Show that $f(z) = |z|$ is nowhere differentiable but continuous every where. **[DUH–2004]**

Solution : For differentiability we have [অন্তরীকরণ যোগ্যতার জন্য পাই]

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{|z+\Delta z| - |z|}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{|z+\Delta z| - |z|}{\Delta z} \cdot \frac{|z+\Delta z| + |z|}{|z+\Delta z| + |z|}$$

$$= \lim_{\Delta z\to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)\,\overline{(z+\Delta z)} - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z\to 0} \frac{z\bar{z} + z\overline{\Delta z} + \bar{z}\,\Delta z + \Delta z\,\overline{\Delta z} - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{z\,\overline{\Delta z} + \bar{z}\,\Delta z + \Delta z\,\overline{\Delta z}}{\Delta z(|z + \Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \left(\frac{z}{|z + \Delta z| + |z|} \frac{\overline{\Delta z}}{\Delta z} + \frac{\bar{z}}{|z + \Delta z| + |z|} + \frac{\overline{\Delta z}}{|z + \Delta z| + |z|} \right)$$

Along the real axis [বাস্তব অক্ষের বরাবর]

$$\Delta x \to 0,\ \Delta y = 0 \Rightarrow \Delta z = \Delta x + i\Delta y = \Delta x \to 0$$

and [এবং] $\overline{\Delta z} = \overline{\Delta x} = \Delta x \to 0 \Rightarrow \Delta z = \overline{\Delta z}$

$$\therefore f'(z) = \frac{z}{|z + 0| + |z|} \cdot 1 + \frac{\bar{z}}{|z| + |z|} + 0 = \frac{z}{2|z|} + \frac{\bar{z}}{2|z|}$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$\Delta x = 0,\ \Delta y \to 0$$

$$\Rightarrow \Delta z = \Delta x + i\Delta y = i\Delta y \to 0$$

and [এবং] $\overline{\Delta z} = \overline{i\Delta y} = -i\Delta y$. Also [অধিকন্তু] $\Delta z = -\overline{\Delta z}$

$$\therefore f'(z) = \frac{z}{|z| + |z|} \frac{-\Delta z}{\Delta z} + \frac{\bar{z}}{|z| + |z|} + 0 = \frac{-z}{2|z|} + \frac{\bar{z}}{2|z|}$$

Thus [অতএব]

$$f'(z) = \begin{cases} \dfrac{z}{2|z|} + \dfrac{\bar{z}}{2|z|}, & \text{along real axis [বাস্তব অক্ষের দিকে]} \\ \dfrac{-z}{2|z|} + \dfrac{\bar{z}}{2|z|}, & \text{along imaginary axis [কাল্পনিক অক্ষের দিকে]} \end{cases}$$

Hence the given function is nowhere differentiable in the complex plane. [অতএব প্রদত্ত ফাংশনটি জটিল তলে কোথাও অন্তরীকরণযোগ্য নয়]

For continuity, let z_0 be any arbitrary point in the complex plane. Then f(z) will be continuous if [অবিচ্ছিন্নতার জন্য ধরি জটিল তলে ইচ্ছাধীন যে কোন বিন্দু z_0. তখন f(z) অবিচ্ছিন্ন হবে যদি]

$$|f(z) - f(z_0)| < \varepsilon, \text{ whenever [যেখানে] } |z - z_0| < \delta$$

Let ε be given such that [ধরি ε প্রদত্ত যেন] $|f(z) - f(z_0)| < \varepsilon$

$$\Rightarrow \big||z| - |z_0|\big| < \varepsilon$$

$$\Rightarrow \big||z| - |z_0|\big| < |z - z_0| < \varepsilon$$

If we take $\delta = \varepsilon$ then [যদি আমরা $\delta = \varepsilon$ লই তখন]

$$\big||z| - |z_0|\big| < \varepsilon \text{ whenrever } |z - z_0| < \varepsilon = \delta$$

$$\Rightarrow |f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Thus the given function is continuous everywhere. [অতএব প্রদত্ত ফাংশনটি সর্বত্র অবিচ্ছিন্ন ।]

Example–11. For the function, f(z) defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

show that the C–R equations are satisfied at (0, 0) but the function is not differentiable at 0 + i0. **[DUH–2003, 2005]**

Solution : Let [ধরি] $f(z) = u(x, y) + iv(x, y)$

At (0, 0) we have [(0, 0) এ পাই] $f(0) = 0 + i0$

$$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$$

$$\Rightarrow u(0, 0) = 0,\ v(0, 0) = 0$$

Also [অধিকন্তু] $f(z) = \dfrac{(\bar{z})^2}{z} = \dfrac{(\bar{z})^3}{z\bar{z}} = \dfrac{(x - iy)^3}{(x + iy)(x - iy)}$

$$\Rightarrow f(z) = u(x, y) + iv(x, y) = \frac{x^3 - 3ix^2y + 3i^2xy^2 - i^3y^3}{x^2 - i^2y^2}$$

$$= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2 + y^2}$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{y^3 - 3x^2y}{x^2 + y^2}.$$

$\therefore$ At the origin we have [$\therefore$ মূলবিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^3/(\Delta x)^2 - 0}{\Delta x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(\Delta y)^3/(\Delta y)^2 - 0}{\Delta y} = 1$$

Thus at (0, 0) [অতএব (0, 0) এ] $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and [এবং] $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence C–R equations are satisfied at (0, 0). [অতএব (0, 0) এ C–R সমীকরণ সিদ্ধ হয়]

2nd Part : For differentiability we have [অন্তরীকরণ যোগ্যতার জন্য পাই]

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z} = \lim_{z \to 0} \frac{(\bar{z})^2}{z^2}$$

$$= \lim_{(x,\,y)\to(0,\,0)} \left(\frac{x - iy}{x + iy}\right)^2$$

Along the real axis [বাস্তব অক্ষ বরাবর] $f'(0) = \lim_{x\to 0} \left(\frac{x}{x}\right)^2 = 1$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$f'(x) = \lim_{y\to 0} \left(\frac{-iy}{iy}\right)^2 = 1$$

Along the line $y = x$ [$y = x$ রেখা বরাবর]

$$f'(0) = \lim_{x\to 0} \left(\frac{x - ix}{x + ix}\right)^2 = \left(\frac{1 - i}{1 + i}\right)^2$$

$$= \frac{1 - 2i + i^2}{1 + 2i + i^2} = \frac{1 - 2i - 1}{1 + 2i - 1} = \frac{-2i}{2i} = -1$$

Since $f'(0)$ have different values for different paths, $f'(0)$ does not exist, that is, f is not differentiable at $0 + i0$. [যেহেতু ভিন্ন ভিন্ন পথের জন্য $f'(0)$ এর ভিন্ন ভিন্ন মান, সুতরাং $f'(0)$ বিদ্যমান নাই, অর্থাৎ $0 + i0$ এ f অন্তরীকরণযোগ্য না।]

Example–12. Let f denote the function whose values are

$$f(0) = 0 \text{ and } f(z) = \frac{(\bar{z})^2}{z} \text{ when } z \neq 0$$

Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but that the derivative of f fails to exist there.

Solution : See soloved problem–11.

Example–13. Let $f(z) = u + iv = \dfrac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$, when $z \neq 0$ and $f(z) = 0$, when $z = 0$. Show that $f(z)$ is continuous and the Cauchy-Riemann equations are satisfied but $f(z)$ is not differentiable at $z = 0$. **[NUH–2000, 2003, 2005, 2013]**

Solution : We have $f(z) = u + iv = \dfrac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$

$\Rightarrow u(x, y) = \dfrac{x^3 - 3xy^2}{x^2 + y^2}$ and [এবং] $v(x, y) = \dfrac{y^3 - 3x^2y}{x^2 + y^2}$

$\therefore$ At the origin we have [$\therefore$ মূলবিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(\Delta x)^3/(\Delta x)^2 - 0}{\Delta x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y\to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x\to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y\to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{(\Delta y)^3/(\Delta y)^2 - 0}{\Delta y} = 1$$

Thus at (0, 0) [অতএব (0, 0) এ] $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and [এবং] $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence C–R equations are satisfied at (0, 0). [অতএব (0, 0) এ C–R সমীকরণ সিদ্ধ হয়।]

2nd Part : For differentiability we have [অন্তরীকরণ যোগ্যতার জন্য পাই]

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z\to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z} = \lim_{z\to 0} \frac{(\bar{z})^2}{z^2}$$

$$= \lim_{(x,\,y)\to(0,\,0)} \left(\frac{x - iy}{x + iy}\right)^2$$

Along the real axis [বাস্তব অক্ষ বরাবর] $f'(0) = \lim_{x\to 0} \left(\frac{x}{x}\right)^2 = 1$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$f'(x) = \lim_{y\to 0} \left(\frac{-iy}{iy}\right)^2 = 1$$

Along the line $y = x$ [$y = x$ রেখা বরাবর]

$$f'(0) = \lim_{x\to 0} \left(\frac{x - ix}{x + ix}\right)^2 = \left(\frac{1 - i}{1 + i}\right)^2$$

$$= \frac{1 - 2i + i^2}{1 + 2i + i^2} = \frac{1 - 2i - 1}{1 + 2i - 1} = \frac{-2i}{2i} = -1$$

Since $f'(0)$ have different values for different paths, $f'(0)$ does not exist, that is, f is not differentiable at $0 + i0$. [যেহেতু ভিন্ন ভিন্ন পথের জন্য $f'(0)$ এর ভিন্ন ভিন্ন মান, সুতরাং $f'(0)$ বিদ্যমান নাই, অর্থাৎ $0 + i0$ এ f অন্তরীকরণযোগ্য না।]

Example–14. Show that the function

$$f(z) = u + iv = \frac{(1 + i)x^3 - (1 - i)y^3}{x^2 + y^2} \quad \text{if } z \neq 0$$

and $f(0) = 0 \quad$ if $z = 0$,

is continuous and that the Cauchy-Riemann equations are satisfied at the Origin, yet $f'(0)$ does not exist.

[NUH-1993, 2011, NU(Pre)-2014, DUH-1993, 2001, 2006]

Solution : First part (Continuity at origin)

Given [দেওয়া আছে] $f(z) = u + iv = \dfrac{(1+i)x^3 - (1-i)y^3}{x^2+y^2}$, if $z \neq 0$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2}$$

Equating real and imaginary parts [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$u = u(x, y) = \frac{x^3-y^3}{x^2+y^2} \text{ and [এবং] } v = v(x, y) = \frac{x^3+y^3}{x^2+y^3}$$

Along the real axis [বাস্তব অক্ষ বরাবর] $y = 0$ and [এবং] $x \to 0$

$$\therefore \lim_{z\to 0} u = \lim_{\substack{x\to 0\\ y\to 0}} u(x, y) = \lim_{x\to 0} \frac{x^3-0}{x^2} = \lim_{x\to 0} x = 0$$

$$\lim_{z\to 0} v = \lim_{\substack{x\to 0\\ y\to 0}} v = \lim_{x\to 0} \frac{x^3+0}{x^2+0} = \lim_{x\to 0} x = 0$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর] $x = 0,\ y \to 0$

$$\therefore \lim_{z\to 0} u = \lim_{\substack{x\to 0\\ y\to 0}} u = \lim_{y\to 0} \frac{0-y^3}{0+y^2} = \lim_{y\to 0} (-y) = 0$$

$$\lim_{z\to 0} v = \lim_{\substack{x\to 0\\ y\to 0}} v = \lim_{y\to 0} \frac{0+y^3}{0+y^2} = \lim_{y\to 0} y = 0$$

Along the line $y = x$ we have [$y = x$ রেখা বরাবর পাই]

$$\lim_{z\to 0} u = \lim_{\substack{x\to 0\\ y=x}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3-x^3}{x^2+x^2} = 0$$

$$\lim_{z\to 0} v = \lim_{\substack{x\to 0\\ y=x}} \frac{x^3+y^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3+x^3}{x^2+x^2} = \lim_{x\to 0} \frac{2x^3}{2x^2} = 0$$

Thus in all cases [অতএব, সকল ক্ষেত্রে] $\lim_{z\to 0} u = 0,\ \lim_{z\to 0} v = 0$

$$\therefore \lim_{z\to 0} f(z) = \lim_{z\to 0} (u + iv) = \lim_{z\to 0} u + i \lim_{z\to 0} v = 0 + i0 = 0$$

$$\Rightarrow \lim_{z\to 0} f(z) = f(0)$$

Hence $f(z)$ is continuous at the origin. [অতএব মূল বিন্দুতে $f(z)$ অবিচ্ছিন্ন ।]

2nd part (C-R equations) :

Given [দেওয়া আছে] $f(0) = 0 \Rightarrow u + iv = 0$

$$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$$

$$\Rightarrow u(0, 0) = 0 \text{ and [এবং] } v(0, 0) = 0$$

Also, given [আরো দেওয়া আছে]

$$u(x, y) = \frac{x^3-y^3}{x^2+y^2} \text{ and [এবং] } v(x, y) = \frac{x^3+y^3}{x^2+y^2}$$

$$\therefore u(0 + \Delta x, 0) = \frac{(\Delta x)^3 - 0}{(\Delta x)^2 + 0} = \Delta x$$

$$u(0, 0 + \Delta y) = \frac{0 - (\Delta y)^3}{0 + (\Delta y)^2} = -\Delta y$$

$$v(0 + \Delta x, 0) = \frac{(\Delta x)^3 + 0}{(\Delta x)^2 + 0} = \Delta x$$

$$v(0, 0 + \Delta y) = \frac{0 + (\Delta y)^3}{0 + (\Delta y)^2} = \Delta y$$

At the origin we have [মূলবিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0} \frac{u(0+\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y\to 0} \frac{u(0, 0+\Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{-\Delta y - 0}{\Delta y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x\to 0} \frac{v(0+\Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta y\to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y\to 0} \frac{v(0, 0+\Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\Delta y - 0}{\Delta y} = 1$$

Thus [অতএব] $\dfrac{\partial u}{\partial x} = 1 = \dfrac{\partial v}{\partial y}$ and [এবং] $\dfrac{\partial u}{\partial y} = -1 = -\dfrac{\partial v}{\partial x}$

Hence Cauchy-Riemann equations are satisfied at the origin. [অতএব মূল বিন্দুতে কচি-রীম্যান সমীকরণ সিদ্ধ হয় ।]

3rd Part (Differentiability অন্তরীকরণযোগ্যতা) ঃ

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z) - 0}{z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \cdot \frac{1}{x + iy}$$

Now along the real axis [এখন বাস্তব অক্ষ বরাবর] $y = 0$ and [এবং] $x \to 0$

$$\therefore f'(0) = \lim_{x \to 0} \frac{(1+i)x^3 - 0}{(x^2 + 0)(x + 0)} = \lim_{x \to 0} \frac{(1+i)x^3}{x^3} = 1 + i \text{ ...... (1)}$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর] $x = 0$ and [এবং] $y \to 0$

$$\therefore f'(0) = \lim_{y \to 0} \frac{0 - (1-i)y^3}{(0 + y^2)(0 + iy)} = \lim_{y \to 0} \frac{(i-1)y^3}{iy^3} = 1 + i \text{ ...... (2)}$$

Along the line $y = x$ we have [$y = x$ রেখা বরাবর পাই]

$$f'(0) = \lim_{x \to 0} \frac{(1+i)x^3 - (1-i)x^3}{(x^2 + x^2)(x + ix)}$$

$$= \frac{1 + i - 1 + i}{2(1+i)} = \frac{2i}{2(1+i)} = \frac{i}{1+i}$$

$$= \frac{i(1-i)}{1 - i^2} = \frac{i+1}{1+1} = \frac{1}{2}(1+i) \text{ ...... (3)}$$

From (1), (2) and (3) we see that the limit is not unique. Hence $f'(0)$ does not exist. [(1), (2) ও (3) হতে দেখি লিমিট অনন্য নয়। অতএব $f'(0)$ বিদ্যমান নাই।]

Example–15. Prove that $f(z) = |z|^2$ is continuous every where but not differentiable except at the origin.

[NUH-04 (Old), 06 (Old), 07, 15, DUH-03, RUH-94]

Solution : Given that [দেওয়া আছে] $f(z) = |z|^2 = z\bar{z} = x^2 + y^2$

Which shows that f(z) is a function of non zero denominators for all x and y in the z-plane. Thus $f(z) = |z|^2$ is continuous every where. [যাহা দেখায় যে f(z) ফাংশনটি z তলে x এবং y এর যে কোন মানের জন্য অশূন্য হর বিশিষ্ট। অতএব $f(z) = |z|^2$ সর্বত্র অবিচ্ছিন্ন।]

For differentiability we have [অন্তরীকরণযোগ্যতার জন্য পাই]

At $z = 0$ [$z = 0$ এ] $f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z\, \overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$$

At $z \neq 0$, say $z = z_0$ [$z \neq 0$, ধরি $z = z_0$ এ]

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\,\overline{(z_0 + \Delta z)} - z_0 \overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0 \overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{z_0} + z_0 \overline{\Delta z} + \overline{z_0}\,\Delta z + \Delta z\,\overline{\Delta z} - z_0 \overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \overline{z_0}\,\Delta z + \Delta z\,\overline{\Delta z}}{\Delta z}$$

$$= z_0 \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \lim_{\Delta z \to 0} \overline{\Delta z}$$

$$= z_0 \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + 0$$

$$= \overline{z_0} + z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \begin{cases} \overline{z_0} + z_0 & \text{along real axis [বাস্তব অক্ষ বরাবর]} \\ \overline{z_0} - z_0 & \text{along imaginary axis [কাল্পনিক অক্ষ বরাবর} \end{cases}$$

Since the two limits are not equal, so $f'(z_0)$ does not exist. Thus $f(z) = |z|^2$ is continuous every where but not differentiable except at the origin. [যেহেতু দুইটি লিমিট সমান না, সুতরাং $f'(z_0)$ বিদ্যমান না। অতএব $f(z) = |z|^2$ সর্বত্র অবিচ্ছিন্ন কিন্তু মূলবিন্দু ব্যতীত অন্তরীকরণ যোগ্য না।]

Example–16. If f(z) is analytic in a region R, then f(z) is constant if Re f(z) is constant. **[RUH–2001]**

Solution : Let [ধরি] $f(z) = u + iv$. Then [তখন] Re $f(z) = u$

According to the question, Re $f(z)$ = constant $\Rightarrow u$ = constant. [প্রশ্নমতে, Re f(z) = ধ্রুবক ⇒ u = ধ্রুবক]

$$\therefore \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \cdots\cdots (1)$$

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0 \text{ and [এবং] } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0 \cdots\cdots (2)$$

Now [এবং] $f(z) = u + iv$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x}\frac{dx}{dz} + i\frac{\partial v}{\partial x}\frac{dx}{dz} = \frac{\partial u}{\partial x} \cdot 1 + i\frac{\partial v}{\partial x} \cdot 1 = 0 + 0 = 0 \text{ by (1) and (2)}$$

$$z = x + iy, \quad \Rightarrow \frac{dz}{dx} = 1, \quad \Rightarrow \frac{dx}{dz} = 1$$

$\Rightarrow f(z) = c$ = constant, by integrating.

Example–17. If f(z) is analytic in a region R, then f(z) is constant if Im f(z) is constant.

Solution : Let $f(z) = u + iv$. Then Im $f(z) = v$

According to the question Im $f(z) = v$ = constant.

$$\therefore \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0 \cdots\cdots (1)$$

By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \cdots\cdots (2)$$

$f(z) = u + iv$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x}\frac{dx}{dz} + i\frac{\partial v}{\partial x}\frac{dx}{dz} = 0 \cdot 1 + i0 \cdot 1 = 0$$

$$z = x + iy, \quad \Rightarrow \frac{dz}{dx} = 1, \quad \Rightarrow \frac{dx}{dz} = 1$$

$\Rightarrow f(z) = c$, where c is a constant.

Example–18. Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic there.

[NUH–2002, DUH–1977, 1985, 2004, RUH–1984, 1999]

Solution : Given that [দেওয়া আছে] $f(z) = |z|^2$

$$\therefore f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|0 + \Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z\, \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{\Delta z} = \overline{0} = 0$$

$\Rightarrow f(z) = |z|^2$ is differentiable at $z = 0$. [$f(z) = |z|^2$, $z = 0$ এ অন্তরীকরণযোগ্য]

Again [আবার] $f(z) = |z|^2$

$$\Rightarrow f(z) = u + iv = |x + iy|^2$$

$$\Rightarrow u + iv = x^2 + y^2$$

$$\Rightarrow u = x^2 + y^2 \text{ and [এবং] } v = 0$$

$$\therefore \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial x} = 0 \text{ and [এবং] } \frac{\partial v}{\partial y} = 0$$

At $z = 0$ [$z = 0$ এ] $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$ and [এবং] $\frac{\partial v}{\partial y} = 0$

Thus the Cauchy-Riemann equations are satisfied at $z = 0$ but not in the neighbourhood $|z - 0| < \delta$.

Thus $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic there.

[$z = 0$ এ কচি-রীম্যান সমীকরণ সিদ্ধ হয় কিন্তু $|z - 0| < \delta$ নেইবারহুডে না।
অতএব $f(z) = |z|^2$, $z = 0$ এ অন্তরীকরণযোগ্য কিন্তু সেখানে বৈশ্লেষিক না।]

Example–19. Show that $f(z) = 2x + ixy^2$ is no where analytic.

[RUH–1996]

Solution : Let $f(z) = u(x, y) + iv(x, y)$

$$\Rightarrow 2x + ixy^2 = u(x, y) + iv(x, y)$$

$$\Rightarrow u(x, y) = 2x \text{ and } v(x, y) = xy^2$$

$$\therefore \frac{\partial u}{\partial x} = 2, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = y^2 \text{ and } \frac{\partial v}{\partial y} = 2xy$$

Thus we see that [অতএব আমরা দেখি যে] $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

That is, Cauchy-Riemann equations are not satisfied anywhere. Hence f(z) is not analytic at any point, that is, f(z) is no where analytic. [অর্থাৎ, কচি-রীম্যান সমীকরণগুলি কোথাও সিদ্ধ হচ্ছে না। অতএব f(z) কোন বিন্দুতে বৈশ্লেষিক না।]

Example–20. If p and q are functions of x and y satisfying Laplace's equation, then show that (u + iv) is analytic where $u = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}$ and $v = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$. **[RUH–1999]**

Solution : Given that p and q are functions of x and y satisfying Laplace's equation. [দেওয়া আছে x ও y এর ফাংশন p ও q ল্যাপলাস সমীকরণ সিদ্ধ করে]

$$\therefore \quad \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \;\cdots\cdots\; (1)$$

$$\text{and} \quad \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0 \;\cdots\cdots\; (2)$$

Again, given that [আবার, দেওয়া আছে]

$$u = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \text{ and } v = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial^2 p}{\partial x\,\partial y} - \frac{\partial^2 q}{\partial x^2} \;\cdots\cdots\; (3)$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 q}{\partial y\,\partial x} \;\cdots\cdots\; (4)$$

$$\frac{\partial v}{\partial x} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 q}{\partial x\,\partial y} \;\cdots\cdots\; (5)$$

$$\frac{\partial v}{\partial y} = \frac{\partial^2 p}{\partial y\,\partial x} + \frac{\partial^2 q}{\partial y^2} \;\cdots\cdots\; (6)$$

(3) – (6) gives, $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right) = 0$ [by (2)]

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

(4) + (5) gives, $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$ [by (1)]

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, u and v satisfied Cauchy-Riemann equations. Hence (u + iv) is an analytic function. [অতএব, u ও v কচি-রীম্যান সমীকরণ সিদ্ধ করে। অতএব u + iv একটি বৈশ্লেষিক ফাংশন]

Example–21. Prove that, if a function f(z) is differentiable at a point, then f(z) is continuous at that point, but the converse is not nccessarily true. **[NUH–2014]**

Solution : First Part : See art-2.4, Theorem-3.

2nd Part (Converse part) :

The converse of the given statement is not true. We shall prove this by the following counter example.

Let $f(z) = \bar{z} = x - iy$. $\quad \therefore f(0) = \bar{0} = 0$

At $z = 0$ we have

$$|f(z) - f(0)| = |\bar{z} - 0| = |\bar{z}| < \varepsilon \text{ when } |z - 0| = |z| < \varepsilon$$

$\therefore$ f(z) is continuous at $z = 0$

Again, let $h = \Delta z = p + iq$. Then $\bar{h} = p - iq$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\bar{h} - 0}{h} = \lim_{\substack{p \to 0 \\ q \to 0}} \frac{p - iq}{p + iq}$$

Now along the real axis $q = 0$ and $p \to 0$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{p \to 0} \frac{p - 0}{p + 0} = 1$$

Along the imaginary axis, $p = 0$ and $q \to 0$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{q \to 0} \frac{0 - iq}{0 + iq} = -1$$

Since the limit is not unique, so $f(z) = \bar{z}$ is not differentiable at $z = 0$ even though it is continuous.

Example–22. Prove that a function which is analytic at a point, is continuous there but the converse is not necessarily true. **[RUH–1996]**

Solution : See example no 21.

Example–23. Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$, but the function $g(z) = \frac{1}{z}$ is not uniformly continuous in this region.

Solution : Let z_1 and z_2 are two points in $|z| < 1$. [মনেকরি $|z| < 1$ এ z_1 ও z_2 দুইটি বিন্দু]

$$\text{Then } |f(z_1) - f(z_2)| = |z_1^2 - z_2^2|$$
$$= |z_1 + z_2|\,|z_1 - z_2|$$
$$\Rightarrow |f(z_1) - f(z_2)| \le (|z_1| + |z_2|)\,|z_1 - z_2|$$
$$\Rightarrow |f(z_1) - f(z_2)| \le (1 + 1)\,|z_1 - z_2|$$
$$\Rightarrow |f(z_1) - f(z_2)| \le 2|z_1 - z_2|$$

If we choose $|z_1 - z_2| < \delta$ and $\delta = \frac{\varepsilon}{2}$, then $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$ and δ depends only on ε. This ensures that $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

For second case we fix $z_1 = \delta$ where $0 < \delta < 1$ and $z_2 = \frac{\delta}{1+\varepsilon}$

$$\therefore |z_1 - z_2| = \left|\delta - \frac{\delta}{1+\varepsilon}\right| = \frac{\varepsilon}{1+\varepsilon}\delta < \delta$$

$$\text{and } |g(z_1) - g(z_2)| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$$
$$= \left|\frac{1}{\delta} - \frac{1}{\delta/1+\varepsilon}\right| = \left|\frac{1}{\delta} - \frac{1+\varepsilon}{\delta}\right|$$
$$= \left|\frac{1-1-\varepsilon}{\delta}\right| = \frac{\varepsilon}{\delta} > \varepsilon \text{ since } 0 < \delta < 1$$

$\therefore |g(z_1) - g(z_2)| > \varepsilon$ whenever $|z_1 - z_2| < \delta$

Hence $g(z) = \frac{1}{z}$ is not uniformly continuous in the region $|z| < 1$.

Example–24. Show that if a complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. **[DUH–1985]**

Solution : Given that $f(z)$ is differentiable at $z_0 = x_0 + iy_0$. Then by definition we have

[দেওয়া আছে $z_0 = x_0 + iy_0$ এ $f(z)$ অন্তরীকরণযোগ্য। তখন সংজ্ঞানুসারে পাই]

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\Rightarrow f'(z_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{u(x, y) + iv(x, y) - \{u(x_0, y_0) + iv(x_0, y_0)\}}{x + iy - (x_0 + iy_0)} \;\cdots\cdots (1)$$

Now taking the limit through the point (x_0, y_0) and parallel to the real axis (x-axis), we have $y = y_0$ and $x \to x_0$. So from (1) we get

[এখন (x_0, y_0) বিন্দু দিয়ে এবং বাস্তব অক্ষের সমান্তরাল লিমিট নিয়ে পাই $y = y_0$ এবং $x \to x_0$, সুতরাং (1) হতে পাই]

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x + iy_0 - x_0 - iy_0}$$
$$= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}$$
$$= \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0)$$

$$\therefore f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Example–25. If $f'(z) = 0$ in a region R, then the function $f(z)$ must be constant in R. **[DUH–1984]**

Solution : Let $f(z) = u + iv$. Then

$$f'(z) = \frac{\partial}{\partial z}(u + iv)$$
$$\Rightarrow 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + i\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial z} \qquad z = x + iy$$
$$= \frac{\partial u}{\partial x}\cdot 1 + i\frac{\partial v}{\partial x}\cdot 1 \qquad \Rightarrow \frac{\partial z}{\partial x} = 1$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial x} = 0 \;\cdots\cdots (1)$$

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0, \quad \text{[by (1)]}$$

This shows that u and v both are independent of x and y. [ইহা দেখায় যে u ও v উভয়ে x ও y এর অনির্ভরশীল]

Hence $u = \text{constant} = c_1$, say

and $v = \text{constant} = c_2$, say

$\therefore f(z) = u + iv = c_1 + ic_2 = \text{constant}$

Thus, if $f'(z) = 0$ then f(z) must be constant. [অতএব, যদি $f'(z) = 0$ তখন f(z) অবশ্যই ধ্রুবক]

Example–26. Show that an analytic function with constant modulas is constant. **[NUH–2011, 2015, NU(Pre)–2008, DUH–1984, RUH–2006]**

Solution : Let $f(z) = u + iv$ be an analytic function with constant modulas. Than $|f(z)| = \text{constant} = c$, say [মনে করি $f(z) = u + iv$ ধ্রুব মানাংক বিশিষ্ট একটি বৈশ্লেষিক ফাংশন। তখন $|f(z)|$ = ধ্রুবক = c, ধরি]

$$\Rightarrow |u + iv| = c$$
$$\Rightarrow \sqrt{u^2 + v^2} = c$$
$$\Rightarrow u^2 + v^2 = c^2$$

Differentiating this partially w. r. to x and y separately we have [ইহাকে x ও y এর সাপেক্ষে আংশিক অন্তরীকরণ করে পাই]

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and [এবং] $2u\dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y} = 0$

$$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \;\cdots\cdots (1)$$

and [এবং] $u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y} = 0 \;\cdots\cdots (2)$

From Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ হতে পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\cdots\cdots (3)$$

By (3), (2) becomes[(3) দ্বারা (2) দাঁড়ায়]

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \;\cdots\cdots (4)$$

Squaring (1) and (4) and then adding we get [(1) ও (2) কে বর্গ করে এবং অতপর যোগ করে পাই]

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right)^2 = 0$$

$$\Rightarrow u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 - 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u^2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} + v^2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$$

$$\Rightarrow (u^2 + v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$$

$$\Rightarrow c^2 \cdot |f'(z)|^2 = 0$$
$$\Rightarrow |f'(z)|^2 = 0$$
$$\Rightarrow f'(z) = 0$$
$$\Rightarrow f(z) = \text{constant [ধ্রুবক]}$$

$\because f(z) = u + iv$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

Example–27. Let $f(z) = u + iv$ be an analytic function where $u(x, y) = c_1$ = constant and $v(x, y) = c_2$ = constant represent two families of curves. Then these system of families are orthogonal. **[DUH–1975]**

Solution : Given that $f(z) = u + iv$ be an analytic functions, so it must satisfy the Cauchy-Riemann equations [দেওয়া আছে $f(z) = u + iv$ বৈশ্লেষিক ফাংশন, সুতরাং ইহা অবশ্যই কচি-রীম্যান সমীকরণ সিদ্ধ করবে]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow u_x = v_y \text{ and } u_y = -v_x \;\cdots\cdots (1)$$

Now $u(x, y) = c_1$

$$\Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$
$$\Rightarrow u_x\,dx + u_y\,dy = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{u_x}{u_y}$$

$\therefore$ Slope of the first curve [প্রথম বক্ররেখার ঢাল] $m_1 = \dfrac{dy}{dx} = -\dfrac{u_x}{u_y} \;\cdots\cdots (2)$

Similarly, from $v(x, y) = c_2$ we have slope of the second curve [অনুরূপে $v(x, y) = c_2$ হতে দ্বিতীয় বক্ররেখার ঢাল পাই]

$$m_2 = -\frac{v_x}{v_y} \text{ ...... (3)}$$

Product of the slopes, [ঢালদ্বয়ের গুনফল] $m_1 m_2 = \frac{-u_x}{u_y} \cdot \frac{-v_x}{v_y}$.

$$\Rightarrow m_1 m_2 = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1 \text{ by (1)}$$

Hence the given system of families of curves are orthogonal.

Example–28. If $f(z) = u + iv$ is analytic in a region R and if u and v have continuous second order partial derivatives in R, then u and v are harmonic in R.

[NU(Pre)–2008, DUH–1986, 1989, 1991, JUH–1986, 87]

Solution : Given $f(z) = u + iv$ is analytic in the region R. So, by Cauchy-Riemann equations we have [দেওয়া আছে R এলাকায় f(z) বৈশ্লেষিক। সুতরাং কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ ...... (1)}$$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ...... (2)

Again given u and v have continuous second order partial derivatives in R. So we have [আবার দেওয়া আছে R এ u ও v এর দ্বিতীয় ক্রমের অবিচ্ছিন্ন আংশিক অন্তরীকরণ আছে। সুতরাং আমরা পাই]

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x} \text{ ...... (3)}$$

and $\frac{\partial^2 v}{\partial x\, \partial y} = \frac{\partial^2 v}{\partial y\, \partial x}$ ...... (4)

Now from (3) we get [এখন (3) হতে পাই]

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right). \quad \text{[by (1) and (2)]}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus, v satisfy Laplace equation and hence it is harmonic. [অতএব, v ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং ইহা হারমোনিক]

Again, form (4) we get [আবার (4) হতে পাই]

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right). \quad \text{[by (1) and (2)]}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, u satisfy **Laplace equation** and hence it is harmonic. [অতএব, u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং ইহা হারমোনিক]

Example–29. If $f(z) = u + iv$ is a analytic function of $z = x + iy$ and ϕ is any function of x and y with differential coefficient of first order, then show that

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left\{\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right\} |f'(z)|^2 \qquad \textbf{[RUH–2001]}$$

Solution : We have [আমাদের আছে] $\phi = \phi(x, y)$

$$\therefore \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} \text{ ...... (1)}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} \text{ ...... (2)}$$

From Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ হতে পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ ...... (3)}$$

By (3), (2) becomes [(3) দ্বারা (2) দাঁড়ায়]

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\left(-\frac{\partial v}{\partial x}\right) + \frac{\partial \phi}{\partial v} \cdot \frac{\partial u}{\partial x} \text{ ...... (4)}$$

Squaring and adding (1) and (4) we get [(1) ও (4) কে বর্গ ও যোগ করে পাই]

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$
$$+\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2-2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$
$$=\left(\frac{\partial\phi}{\partial u}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}+\left(\frac{\partial\phi}{\partial v}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}$$
$$=\left\{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right\}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\} \quad \ldots\ldots (5)$$

Now $f'(z)=\frac{dw}{dz}=\frac{\partial u}{\partial x}\frac{dx}{dz}+i\frac{\partial v}{\partial x}\frac{dx}{dz}$

$\Rightarrow f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$\therefore |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

$z = x + iy$

$\Rightarrow \frac{dz}{dx}=1 \Rightarrow \frac{dx}{dz}=1$

Putting this value in (5) we get [এই মান (5) মান এ বসাইয়া পাই]

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left\{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right\}|f'(z)|^2 \quad \text{(Showed)}$$

Example–30. If f(z) is analytic at a point z_0, then it must be continuous at z_0. Give an example to show that the converse of this theorem is not necessarily true. **[RUH–2001, 2004]**

Solution : Given f(z) is analytic at z_0. So it is differentiable there. [দেওয়া আছে z_0 এ f(z) বৈশ্লেষিক। সুতরাং ইহা যেখানে অন্তরীকরণ যোগ্য]

$$\therefore f'(z_0)=\lim_{z\to z_0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \quad \text{exist}$$

Now $f(z_0+\Delta z)-f(z_0)=\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\cdot\Delta z$

$$\Rightarrow \lim_{\Delta z\to 0}[f(z_0+\Delta z)-f(z_0)]=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\cdot\lim_{\Delta z\to 0}\Delta z$$

$$\Rightarrow \lim_{\Delta z\to 0} f(z_0+\Delta z)-f(z_0)=f'(z_0)\cdot 0$$

$$\Rightarrow \lim_{\Delta z\to 0} f(z_0+\Delta z)=f(z_0)$$

which shows that f(z) must be continuous at z_0. [যাহা দেখায় যে z_0 এ f(z) অবিচ্ছিন্ন]

2nd Part : See 2nd part of example–21.

Example–31. For what values of z do the function defined by the following equations cease to be analytic.

(i) $z=e^{-v}(\cos u+i\sin u)$, $w=u+iv$

(ii) $z=\sinh u\cos v+i\cosh u\sin v$, $w=u+iv$.

Solution : (i) Given that $w = u + iv$

and $z=e^{-v}(\cos u+i\sin u)$

$$=e^{-v}e^{iu}=e^{iu}e^{i^2v}=e^{i(u+iv)}=e^{iw}$$

$\Rightarrow iw=\ln z$

$\Rightarrow i\frac{dw}{dz}=\frac{1}{z}\Rightarrow\frac{dw}{dz}=\frac{1}{iz}$

when $z=0$ then $\frac{dw}{dz}=\infty$

$\therefore$ w is not analytic at $z = 0$.

(ii) Given that $w = u + iv$

and $z=\sinh u\cos v+i\cosh u\sin v$

$$=\frac{1}{i}\sin iu\cos v+i\cos iu\sin v$$

$\cos iu=\cosh u$

$\sin iu=i\sinh u$

$$=-i\sin iu\cos v+i\cos iu\sin v$$
$$=-i[\sin iu\cos v-\cos iu\sin v]$$
$$=-i[\sin(iu-v)]$$
$$=-i[\sin(iu+i^2v)]$$
$$=-i[\sin i(u+iv)]$$
$$=-i\sin iw=-ii\sinh w=\sinh w$$

$\Rightarrow w=\sinh^{-1}(z)$

$$\therefore \frac{dw}{dz}=\frac{1}{\sqrt{z^2+1}}$$

When $z^2+1=0\Rightarrow z=\pm\sqrt{-1}=\pm\sqrt{i^2}=\pm i$

then $\frac{dw}{dz}=\infty$

Hence the function w is not analytic at $z = i, -i$.

Example–32. Show that the function

$$f(z)=\begin{cases}\dfrac{x^3y^4(x+iy)}{x^6+y^8} & \text{when } z\neq 0\\ 0 & \text{when } z=0\end{cases}$$

the Cauchy-Riemann equations are satisfied at origin, but f(z) is not analytic there. **[RUH–1995]**

Solution : Let $f(z)=u(x,y)+iv(x,y)$

At $z=0$, $f(0)=u(0,0)+iv(0,0)$

$$\Rightarrow 0=u(0,0)+iv(0,0)$$

$$\Rightarrow u(0,0)=0 \text{ and } v(0,0)=0$$

At $z\neq 0$, $f(z)=\dfrac{x^3y^4(x+iy)}{x^6+y^8}$

$$\Rightarrow u(x,y)+iv(x,y)=\frac{x^4y^4}{x^6+y^8}+i\frac{x^3y^5}{x^6+y^8}$$

$$\Rightarrow u(x,y)=\frac{x^4y^4}{x^6+y^8} \text{ and } v(x,y)=\frac{x^3y^5}{x^6+y^8}.$$

(i) For Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণের জন্য পাই]

$$u(0+\Delta x,0)=\frac{0}{(\Delta x)^6+0}=0$$

$$u(0,0+\Delta y)=\frac{0}{0+(\Delta y)^8}=0$$

$$v(0+\Delta x,0)=\frac{0}{(\Delta x)^6+0}=0$$

$$v(0,0+\Delta y)=\frac{0}{0+(\Delta y)^8}=0$$

$\therefore$ At the origin we have [মূল বিন্দুতে পাই]

$$\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{u(0+\Delta x,0)-u(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial u}{\partial y}=\lim_{\Delta y\to 0}\frac{u(0,0+\Delta y)-u(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

$$\frac{\partial v}{\partial x}=\lim_{\Delta x\to 0}\frac{v(0+\Delta x,0)-v(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial v}{\partial y}=\lim_{\Delta y\to 0}\frac{v(0,0+\Delta y)-v(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin. [অতএব মূলবিন্দুতে কচি-রীম্যান সমীকরণগুলি সিদ্ধ হয়]

(ii) For analyticity [বৈশ্লেষিকের জন্য]:

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}, \text{ Choosing } z \text{ in place of } \Delta z$$

$$\Rightarrow f'(0)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y^4(x+iy)}{x^6+y^8}\cdot\frac{1}{x+iy}$$

$$\Rightarrow f'(0)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y^4}{x^6+y^8}$$

Now along the real axis, [এখন বাস্তব অক্ষ বরাবর] $y=0$ and $x\to 0$

$$\therefore f'(0)=\lim_{x\to 0}\frac{0}{x^6+0}=0$$

Along the imaginary axis, [কাল্পনিক অক্ষ বরাবর] $x=0$ and $y\to 0$

$$\therefore f'(0)=\lim_{y\to 0}\frac{0}{0+y^8}=0$$

Along the line $y=x$ we have [$y=x$ রেখা বরাবর পাই]

$$f'(0)=\lim_{\substack{x\to 0\\ y=x}}\frac{x^3y^4}{x^6+y^8}=\lim_{x\to 0}\frac{x^7}{x^6+x^8}=\lim_{x\to 0}\frac{x}{1+x^2}=0$$

Along the curve $y^4=x^3$ we have [$y^4=x^3$ বক্ররেখা বরাবর পাই]

$$f'(0)=\lim_{\substack{x\to 0\\ y^4=x^3}}\frac{x^3y^4}{x^6+y^8}=\lim_{x\to 0}\frac{x^6}{2x^6}=\frac{1}{2}$$

The above limits along different paths are different, so f(z) is not differentiable at $z=0$. Hence f(z) is not analytic at the origin.

[বিভিন্ন পথ বরাবর উপরের লিমিটগুলি ভিন্ন ভিন্ন, সুতরাং $z=0$ তে f(z) অন্তরীকরণ যোগ্য না। অতএব মূলবিন্দুতে f(z) বৈশ্লেষিক না]

Example–33. Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}};\ z \neq 0$$

$$f(0) = 0;\ z = 0$$

in the region including the origin. **[RUH–1998]**

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$\Rightarrow 0 + i0 = u(0, 0) + iv(0, 0)$

$\Rightarrow u(0, 0) = 0$ and $v(0, 0) = 0$.

At $z \neq 0$, $f(z) = \dfrac{x^2 y^5 (x + iy)}{x^4 + y^{10}}$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^3 y^5}{x^4 + y^{10}} + i\frac{x^2 y^6}{x^4 + y^{10}}$$

$$\Rightarrow u(x, y) = \frac{x^3 y^5}{x^4 + y^{10}} \text{ and } v(x, y) = \frac{x^2 y^6}{x^4 + y^{10}}$$

(i) For continuity at the origin [মূলবিন্দুতে অবিচ্ছিন্নতার জন্য]:

Along the real axis we have [বাস্তব অক্ষ বরাবর পাই] $x \to 0$ and $y = 0$

$$\therefore \lim_{z \to 0} u = \lim_{\substack{x \to 0 \\ y = 0}} u = \frac{0}{x^4 + 0} = 0$$

$$\lim_{z \to 0} v = \lim_{x \to 0} v = \frac{0}{x^4 + 0} = 0$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর পাই] $x = 0,\ y \to 0$

$$\therefore \lim_{z \to 0} u = \lim_{\substack{x = 0 \\ y \to 0}} u = \frac{0}{0 + y^{10}} = 0$$

$$\lim_{z \to 0} v = \lim_{\substack{x = 0 \\ y \to 0}} v = \frac{0}{0 + y^{10}} = 0$$

Along the line $y = x$, we have [$y = x$ রেখা বরাবর পাই]

$$\lim_{z \to 0} u = \lim_{\substack{x \to 0 \\ y = 0}} u = \frac{x^3 \cdot x^5}{x^4 + x^{10}} = \lim_{x \to 0} \frac{x^4}{1 + x^6} = \frac{0}{1 + 0} = 0$$

$$\lim_{z \to 0} v = = \lim_{\substack{x \to 0 \\ y = x}} v = \frac{x^2 \cdot x^6}{x^4 + x^{10}} = \lim_{x \to 0} \frac{x^4}{1 + x^6} = \frac{0}{1 + 0} = 0$$

Thus, in all cases [অতএব সকল ক্ষেত্রে] $\lim_{z \to 0} f(z) = \lim_{z \to 0} u + i \lim_{z \to 0} v = 0 + i0 = 0 = f(0)$

Hence $f(z)$ is continuous at the origin. [অতএব, মূলবিন্দুতে $f(z)$ অবিচ্ছিন্ন]

(ii) For Cauchy-Riemann equations [কচি-রীম্যান সমীকরণ সমূহের জন্য] :

$$u(0 + \Delta x, 0) = \frac{0}{(\Delta x)^4 + 0} = 0$$

$$u(0, 0 + \Delta y) = \frac{0}{0 + (\Delta y)^{10}} = 0$$

$$v(0 + \Delta x, 0) = \frac{0}{(\Delta x)^4 + 0} = 0$$

$$v(0, 0 + \Delta y) = \frac{0}{0 + (\Delta y)^{10}} = 0$$

$\therefore$ At the origin we have [মূল বিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.

[অতএব, মূলবিন্দুতে কচি-রীম্যান সমীকরণগুলি সিদ্ধ হয়]

(iii) For differentiability :

$f'(x) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$, Choosing z in place of Δz

$$\Rightarrow f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y^5(x + iy)}{x^4 + y^{10}} \cdot \frac{1}{x + iy}$$

$$\Rightarrow f'(0) = = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Now along the real axis, [এখন বাস্তব অক্ষ বরাবর] $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{0}{x^4 + 0} = 0$$

Along the imaginary axis, [কাল্পনিক অক্ষ বরাবর] $x = 0$ and $y \to 0$

$$\therefore f'(0) = \lim_{y\to 0} \frac{0}{0 + y^{10}} = 0$$

Along the line $y = x$ we have [$y = x$ রেখা বরাবর পাই]

$$f'(0) = \lim_{\substack{x\to 0\\ y=x}} \frac{x^2 y^5}{x^4 + y^{10}} = \lim_{x\to 0} \frac{x^7}{x^4 + x^{10}} = \lim_{x\to 0} \frac{x^3}{1 + x^6} = 0$$

Along the curve $y^5 = x^2$ we have [$y^5 = x^2$ বক্ররেখা বরাবর পাই]

$$f'(0) = \lim_{\substack{x\to 0\\ y^5=x^2}} \frac{x^2 y^5}{x^4 + y^{10}} = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

The above limits along different paths are different, so $f'(0)$ does not exist. Thus the function is not differentiable at $z = 0$. [বিভিন্ন পথ বরাবর উপরের লিমিটগুলি ভিন্ন ভিন্ন, $f'(0)$ বিদ্যমান না। অতএব $z = 0$ তে ফাংশনটি অন্তরীকরণ যোগ্য না]

Example–33(a). Show that the function $f(z) = \sqrt{xy}$ is not regular at the origin although Cauchy–Riemann equations are satisfied at the point (0, 0). [দেখাও যে, $f(z) = \sqrt{xy}$ ফাংশনটি মূলবিন্দুতে regular নয়; যদিও কসি-রীম্যান সমীকরণগুলি (0, 0) বিন্দুতে সিদ্ধ হয়।]

[NU(Pre)–2011]

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$\Rightarrow 0 = u(0, 0) + iv(0, 0)$, $\because f(0) = \sqrt{0 \cdot 0} = 0$

$\Rightarrow u(0, 0) = 0$, $v(0, 0) = 0$

At $z \neq 0$, $f(z) = u(x, y) + iv(x, y) = \sqrt{xy}$

$\Rightarrow u(x, y) = \sqrt{xy}$ and $v(x, y) = 0$

(i) For Cauchy-Riemann equations [কসি-রীম্যান সমীকরণের জন্য]

At the origin we have [মূল বিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y\to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x\to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y\to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy–Riemann equations are satisfied at the origin [অতএব কচি-রীম্যান সমীকরণগুলি মূল বিন্দুতে সিদ্ধ হয়।]

For regular (analytic) [regular এর জন্য] ঃ

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$\Rightarrow f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{xy}}{x + iy}$$

Now along the real axis [এখন বাস্তব অক্ষ বরাবর] $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{0}{x} = 0$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর] $x = 0$ and $y \to 0$

$$\therefore f'(0) = \lim_{y\to 0} \frac{0}{iy} = 0$$

Along the line $y = mx$

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{x \cdot mx}}{x + imx} = \frac{\sqrt{m}}{1 + im}$$

This shows that for different values of m we get different values of $f'(0)$. Thus $f'(0)$ does not exist and so $f(z)$ is not regular (analytic) at (0, 0).

[ইহা দেখায় যে m এর ভিন্ন ভিন্ন মানের জন্য $f'(0)$ এর ভিন্ন ভিন্ন মান পাব। অতএব $f'(0)$ বিদ্যমান নাই এবং সে কারণে $f(z)$, (0, 0) এ regular না] (Showed).

Example-34. Prove that the function $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0$

$$f(0) = 0, z = 0$$

is not analytic at origin although Cauchy-Riemann equations are satisfied there. **[RUH-2004]**

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$$\Rightarrow 0 = u(0, 0) + iv(0, 0)$$

$$\Rightarrow u(0, 0) = 0 \text{ and } v(0, 0) = 0$$

At $z \neq 0$, $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^2 y^2}{x^2 + y^4} + i\frac{xy^3}{x^2 + y^4}$$

$$\Rightarrow u(x, y) = \frac{x^2 y^2}{x^2 + y^4} \text{ and } v(x, y) = \frac{xy^3}{x^2 + y^4}$$

(i) For Cauchy-Riemann equations :

$u(0 + \Delta x, 0) = 0$, $u(0, 0 + \Delta y) = 0$

$v(0 + \Delta x, 0) = 0$ and $v(0, 0 + \Delta y) = 0$

$$\therefore \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.
[অতএব কচি-রীম্যান সমীকরণগুলি মূলবিন্দুতে সিদ্ধ হয়]

(ii) For analyticity :

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}, \text{ Choosing } z \text{ in place of } \Delta z$$

$$\Rightarrow f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2(x + iy)}{x^2 + y^4} \cdot \frac{1}{x + iy}$$

$$\Rightarrow f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{x^2 + y^4}$$

Along the real axis, [বাস্তব অক্ষ বরাবর] $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x \to 0} \frac{0}{x^2 + 0} = 0$$

Along the imaginary axis, [কাল্পনিক অক্ষ বরাবর] $x = 0$ and $y \to 0$

$$f'(0) = \lim_{y \to 0} \frac{0}{0 + y^4} = 0$$

Along the line $y = x$ we have [$y = x$ রেখা বরাবর পাই]

$$f'(0) = \lim_{\substack{x \to 0 \\ y = x}} \frac{xy^2}{x^2 + y^4} = \lim_{x \to 0} \frac{x^3}{x^2 + x^4} = \lim_{x \to 0} \frac{x}{1 + x^2} = 0$$

Along the curve $y^2 = x$ we have [$y^2 = x$ বক্ররেখা বরাবর পাই]

$$f'(0) = \lim_{\substack{x \to 0 \\ y^2 = x}} \frac{xy^2}{x^2 + y^4} = \lim_{x \to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

The above limits along different paths are different, so f(z) is not differentiable at $z = 0$. Hence f(z) is not analytic at the origin.

[বিভিন্ন পথ বরাবর উপরের লিমিটগুলি ভিন্ন ভিন্ন, সুতরাং $z = 0$ তে $f(z)$ অন্তরীকরণ যোগ্য না। অতএব মূলবিন্দুতে $f(z)$ বৈশ্লেষিক না]

Example–35. Show that the function $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$ although Cauchy-Riemann equations are satisfied at the point.

Solution : Given that $f(z) = u + iv = e^{-z^{-4}}$ ...... (1)

$$\text{Now } -z^{-4} = -\frac{1}{z^4} = -\frac{1}{(x+iy)^4} = -\frac{(x-iy)^4}{(x+iy)^4(x-iy)^4}$$

$$= -\frac{x^4 - 4ix^3y + 6x^2 i^2 y^2 - 4xi^3y^3 + i^4y^4}{(x^2+y^2)^4}$$

$$= -\frac{x^4 - 4i\,x^3y - 6x^2y^2 + 4ixy^3 + y^4}{(x^2+y^2)^4}$$

$$= \frac{-(x^4 - 6x^2y^2 + y^4)}{(x^2+y^2)^4} + i\frac{4x^3y - 4xy^3}{(x^2+y^2)^4}$$

$= A + iB$, say

$$\text{Where } A = \frac{6x^2y^2 - x^4 - y^4}{(x^2+y^2)^4} \text{ and } B = \frac{4x^3y - 4xy^3}{(x^2+y^2)^4}$$

$\therefore f(z) = u + iv = e^{A+iB} = e^A[\cos B + i\sin B]$ ...... (2)

$\Rightarrow u = e^A\cos B$ and $v = e^A\sin B$ ...... (3)

Given that [দেওয়া আছে] $f(0) = 0$

$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$

$\Rightarrow u(0, 0) = 0$ and $v(0, 0) = 0$ ...... (4)

Also at $(x, 0)$ we have [$(x, 0)$ এ পাই]

$$A = \frac{-x^4}{x^8} = -\frac{1}{x^4}, B = 0$$

$$\text{At } (0, y),\ A = \frac{-y^4}{y^8} = -\frac{1}{y^4}, B = 0$$

$\therefore u(x, 0) = e^{-1/x^4}\cos 0 = e^{-1/x^4}$

$u(0, y) = e^{-1/y^4}\cos 0 = e^{-1/y^4}$

$v(x, 0) = e^{-1/x^4}\cdot\sin 0 = 0$

$v(0, y) = e^{-1/y^4}\sin 0 = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{e^{-1/x^4}-0}{x}$$

$$= \lim_{x\to 0}\frac{1}{x\left(1 + \frac{1}{x^4} + \frac{1}{2!\,x^8} + \cdots\right)}$$

$$= \lim_{x\to 0}\frac{1}{x + \frac{1}{x^3} + \frac{1}{2!\,x^7} + \cdots} = \frac{1}{\infty} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{e^{-1/y^4}-0}{y} = 0, \text{ as above}$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

The above results show that Cauchy-Riemann equations are satisfied at $z = 0$ [উপরের ফলাফলগুলি দেখায় যে কচি-রীম্যান সমীকরণগুলি $z = 0$ তে সিদ্ধ হয়]

f(z) not analytic at z = 0

$$f'(0) = \lim_{\Delta z\to 0}\frac{f(0+\Delta z)-f(0)}{\Delta z}$$

$$= \lim_{z\to 0}\frac{f(z)-f(0)}{z}, \text{ Choosing } z \text{ for } \Delta z$$

$$= \lim_{z\to 0}\frac{e^{-z^{-4}}-0}{z} \quad \text{...... (5)}$$

Along the path $z = re^{i\frac{\pi}{4}}$ we have $r \to 0$ when $z \to 0$ [$z = re^{i\frac{\pi}{4}}$ পথ বরাবর পাই $r \to 0$ যখন $z \to 0$]

$$\therefore -z^{-4} = \frac{-1}{z^4} = \frac{-1}{(re^{i\pi/4})^4} = \frac{-1}{r^4e^{i\pi}}$$

$$= \frac{-1}{r^4(\cos\pi + i\sin\pi)} = \frac{-1}{r^4(-1+0)} = \frac{1}{r^4} = r^{-4}$$

Thus (5) becomes [অতএব (5) দাঁড়ায়]

$$\therefore f'(0) = \lim_{r\to 0} \frac{e^{r^{-4}} - 0}{re^{i\pi/4}}$$

$$= \lim_{r\to 0} \frac{1}{re^{i\pi/4}}\left[\frac{1}{r^4} + \frac{1}{2!\,r^8} + \cdots\right]$$

$$= \frac{1}{0} = \infty$$

Thus $f'(0)$ does not exist at $z = 0$ and hence $f(z)$ is not analytic at $z = 0$. [অতএব $z = 0$ তে $f'(0)$ বিদ্যমান নয় এবং $z = 0$ তে $f(z)$ বৈশ্লেষিক না]

N. B. For analyticity Cauchy-Riemann equations must be satisfied and also the first order partial derivatives of u and v should be continuous. Here the second condition is not satisfied and hence f(z) is not analytic at z = 0.

Example–35(a). Prove that the function $f(z) = z^2 + 5iz + 3 - i$ satisfy Cauchy-Riemann equations. **[NUH(Phy)–2005]**

Solution : Given that [দেওয়া আছে]

$$f(z) = z^2 + 5iz + 3 - i$$

$$\Rightarrow u + iv = (x + iy)^2 + 5i(x + iy) + 3 - i$$

$$= x^2 + 2ixy - y^2 + 5ix - 5y + 3 - i$$

$$= (x^2 - y^2 - 5y + 3) + i(2xy + 5x - 1)$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\Rightarrow u = x^2 - y^2 - 5y + 3 \text{ and [এবং] } v = 2xy + 5x - 1$$

$$\therefore \frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y - 5,\ \frac{\partial v}{\partial x} = 2y + 5 \text{ and [এবং] } \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \ \ldots\ldots (1)$$

$$\text{and [এবং] } \frac{\partial u}{\partial y} = -(2y + 5) = -\frac{\partial v}{\partial x} \ \ldots\ldots (2)$$

From (1) and (2) we see that the given equation satisfy the Cauchy-Riemann equations. [সমীকরণ (1) ও (2) হতে দেখি যে প্রদত্ত সমীকরণ কচি-রীম্যান সমীকরণ সিদ্ধ করে।] **(Proved)**

Example–35(b). Test for analyticity of $W_1 = f_1(z) = |z|^2$ and $W_2 = f_2(z) = \frac{1}{2}$. **[NU(Pre)–2006]**

Solution : (i) Given that [দেওয়া আছে]

$$W_1 = f_1(z) = |z|^2$$

$$\Rightarrow u_1 + iv_1 = |x + iy|^2$$

$$\Rightarrow u_1 + iv_1 = x^2 + y^2$$

$$\Rightarrow u_1 = x^2 + y^2 \text{ and [এবং] } v_1 = 0$$

$$\therefore \frac{\partial u_1}{\partial x} = 2x,\ \frac{\partial u_1}{\partial y} = 2y,\ \frac{\partial v_1}{\partial x} = 0,\ \frac{\partial v_1}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial x} = 2x \neq \frac{\partial v_1}{\partial y} \text{ and [এবং] } \frac{\partial u_1}{\partial y} = 2y \neq -\frac{\partial u_1}{\partial x}$$

Therefore, $W_1 = f_1(z) = |z|^2$ does not satisfy Cauchy-Riemann equations and hence not analytic. [অতএব $W_1 = f_1(z) = |z|^2$ কচি-রীম্যান সমীকরণ সিদ্ধ করে না এবং সে কারণে বৈশ্লেষিক না]

(ii) Given that [দেওয়া আছে] $W_2 = f_2(z) = \frac{1}{2}$

$$\Rightarrow u_2 + iv_2 = \frac{1}{2}$$

$$\Rightarrow u_2 = \frac{1}{2} \text{ and [এবং] } v_2 = 0$$

$$\therefore \frac{\partial u_2}{\partial x} = 0,\ \frac{\partial u_2}{\partial y} = 0,\ \frac{\partial v_2}{\partial x} = 0 \text{ and [এবং] } \frac{\partial v_2}{\partial y} = 0$$

$$\therefore \frac{\partial u_2}{\partial x} = 0 = \frac{\partial v_2}{\partial y} \text{ and [এবং] } \frac{\partial u_2}{\partial y} = 0 = -\frac{\partial v_2}{\partial x}$$

Therefore, $W_2 = f_2(z) = \frac{1}{2}$ satisfy Cauchy-Riemann equations and hence analytic. [অতএব $W_2 = f_2(z) = \frac{1}{2}$ কচি-রীম্যান সমীকরণ সিদ্ধ করে এবং সে কারণে বৈশ্লেষিক।]

Example–35(c). Write down the Cauchy-Riemann partial differential equations. How are they related to analytic functions. **[NU(Pre)–2006]**

Solution : If f(z) = u + iv then the Cauchy-Riemann partial differential equations are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Provided the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist.

[যদি f(z) = u + iv হয় তখন কচি-রীম্যান আংশিক অন্তরক সমীকরণগুলি হল $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ এবং $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ এই শর্ত সাপেক্ষে যে চার আংশিক অন্তরীকরণ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ এবং $\frac{\partial v}{\partial y}$ বিদ্যমান]

2nd Part : The Cauchy-Riemann equations related to analytic functions in the following way :

The function f(z) = u + iv is analytic in a domain D if and only if the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

are satisfied in D.

[২য় অংশ ঃ কচি-রীম্যান সমীকরণগুলি নিম্ন লিখিতভাবে বৈশ্লেষিক ফাংশনের সাথে সর্ম্পক যুক্ত ঃ

একটি ডোমেন D এ ফাংশন f(z) = u + iv বৈশ্লেষিক যদি এবং কেবল যদি D এ কচি-রীম্যান সমীকরণ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ এবং $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ সিদ্ধ হয় ।]

Example–35(d). If for all z in the entire complex plane f(z) is analytic and bounded, then prove that f(z) must be a constant.

[NU(Pre)–2006]

Solution : Let [ধরি] f(z) = u + iv. Since f(z) is analytic, so it satisfies the Cauchy-Riemann equations [যেহেতু f(z) বৈশ্লেষিক, সুতরাং ইহা কচি-রীম্যান সমীকরণ সিদ্ধ করে]

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \;\ldots\ldots (1)$$

Again, f(z) is bounded, so its value will reach to a finite constant c, that is [আবার, f(z) সীমাবদ্ধ, সুতরাং ইহার মান একটি সসীম ধ্রুবক c এ পৌছবে; অর্থা]

$$|f(z)| = c$$

$$\Rightarrow |u + iv| = c$$

$$\Rightarrow \sqrt{u^2+v^2} = c$$

$$\Rightarrow u^2 + v^2 = c^2$$

Differentiating this partially w. r. to x and y seperately we have [ইহাকে x ও y এর সাপেক্ষে পৃথকভাবে আংশিক অন্তরীকরণ করে পাই]

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$$\text{and [এবং] } 2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

$$\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \;\ldots\ldots (2)$$

$$\text{and [এবং] } u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0 \;\ldots\ldots (3)$$

By (1), (3) becomes [(1) এর সাহায্যে (3) দাঁড়ায়]

$$-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0 \;\ldots\ldots (4)$$

Squaring (2) and (4), and then adding we get [(2) ও (4) কে বর্গ করে এবং অতপর যোগ করে পাই]

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)^2=0$$

$$\Rightarrow u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2-2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}=0$$

$$\Rightarrow u^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}+v^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow (u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$\Rightarrow c^2\,|f'(z)|^2 = 0$

$\Rightarrow |f'(z)|^2 = 0$

$\Rightarrow f'(z) = 0$

$\Rightarrow f(z) = \text{constant.}$ **(Proved)**

$\because f(z) = u + iv$

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

Example–36. Find an analytic function f(z) such that $\text{Im}\{f'(z)\} = 6xy + 4x$, $f'(0) = 0$ and $f(1 + i) = 0$. **[RUH–1995]**

Solution : Given that [দেওয়া আছে] $\text{Im}\{f'(z)\} = 6xy + 4x$ ...... (1)

Let $w = f'(z) = \frac{df}{dz} = u + iv$

$\therefore \text{Im}\{f'(z)\} = v$

$\Rightarrow v = 6xy + 4x,$ by (1)

$\therefore \frac{\partial v}{\partial x} = 6y + 4$ and $\frac{\partial v}{\partial y} = 6x$ ...... (2)

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6x \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6y - 4 \text{ ...... (3)}$$

Now u = u(x, y), so by calculas we have [সুতরাং ক্যালকুলাসের সাহায্য পাই]

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$\Rightarrow du = 6x\,dx - (6y + 4)\,dy,$ which is an exact equation. [যাহা একটি যথার্থ সমীকরণ]

$\therefore u = \int 6x\,dx - \int (6y + 4)\,dy$

$\Rightarrow u = 3x^2 - 3y^2 - 4y + c$

$\therefore f'(z) = u + iv$

$= 3x^2 - 3y^2 - 4y + c + i(6xy + 4x)$

$= 3(x^2 + 2ixy - y^2) + 4i\left(x - \frac{1}{i}y\right) + c$

$= 3(x^2 + 2ixy + i^2y^2) + 4i(x + iy) + c$

$\Rightarrow f'(z) = 3(x + iy)^2 + 4i(x + iy) + c$ ...... (4)

$\Rightarrow f'(z) = \frac{df}{dz} = 3z^2 + 4iz + c$

$\Rightarrow f = \int (3z^2 + 4iz + c)\,dz$

$\Rightarrow f(z) = z^3 + 2iz^2 + cz + D)$ ...... (5)

Given [দেওয়া আছে] $f'(0) = 0$, so from (4) [সুতরাং (4) হতে]

$f'(0) = 0 + 0 + c \Rightarrow c = 0$

Given [দেওয়া আছে] $f(1 + i) = 0$, so from (5) [সুতরাং (5) হতে]

$f(1 + i) = (1 + i)^3 + 2i(1 + i)^2 + 0 + D \quad [\because c = 0]$

$\Rightarrow 0 = 1 + 3i + 3i^2 + i^3 + 2i + 4i^2 + 2i^3 + D$

$= 1 + 3i - 3 - i + 2i - 4 - 2i + D$

$= -6 + 2i + D$

$\Rightarrow D = 6 - 2i$

By putting the value of c and D in (5) we get [c ও D এর মান (5) এ বসাইয়া পাই]

$f(z) = z^3 + 2iz^2 + 0 + 6 - 2i$

$\Rightarrow f(z) = z^3 + 2iz^2 + 6 - 2i.$ (Ans)

Example–37. Find an analytic function f(z) such that $\text{Re}\{f'(z)\} = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$. **[RUH–1998]**

Solution : Given that [দেওয়া আছে] $\text{Re}\{f'(z)\} = 3x^2 - 4y - 3y^2$

Let $w = f'(z) = \frac{df}{dz} = u + iv$

$\therefore \text{Re}\{f'(z)\} = u$

$\Rightarrow 3x^2 - 4y - 3y^2 = u$

$\therefore \frac{\partial u}{\partial x} = 6x$ and $\frac{\partial u}{\partial y} = -4 - 6y$

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6x \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -4 - 6y$$

Since [যেহেতু] v = v(x, y), so

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$\Rightarrow dv = (6y + 4)\,dx + 6x\,dy$

Here $\frac{\partial}{\partial x}(6y+4)=0$ and $\frac{\partial}{\partial y}(6x)=0$

So the above equation is an exact differential equation. [সুতরাং উপরের সমীকরণটি একটি যথার্থ অন্তরক সমীকরণ]

$\therefore v=\int(6y+4)\,dx+\int 0\,dy$

$=6xy+4x$ omitting the integrating constant

$\therefore f'(z)=u+iv=3x^2-4y-3y^2+i(6xy+4x)$

$=3(x^2+2ixy-y^2)+4i(x+iy)$

$=3(x+iy)^2+4i(x+iy)$

$\Rightarrow \frac{df}{dz}=3z^2+4iz,\ z=x+iy$

$\therefore f(z)=\int(3z^2+4iz)\,dz$

$f(z)=z^3+2iz^2+c,$ where c is the integrating constant.

Given [দেওয়া আছে] $f(1+i)=0$

$\Rightarrow (1+i)^3+2i(1+i)^2+c=0$

$\Rightarrow 1+3i+3i^2+i^3+2i+4i^2+2i^3+c=0$

$\Rightarrow 1+3i-3-i+2i-4-2i+c=0$

$\Rightarrow -6+2i+c=0$

$\Rightarrow c=6-2i$

Thus [অতএব] $f(z)=z^3+2iz^2+6-2i$. (Ans)

Example-38. If $\text{Im}\{f'(z)\}=6x(2y-1)$ and $f(0)=3-2i$, $f(1)=6-5i$ find $f(1+i)$. **[NUH-2000 (Old)]**

Solution : Given [দেওয়া আছে] $\text{Im}\{f'(z)\}=6x(2y-1)$ ...... (1)

Let $w=f'(z)=\frac{df}{dz}=u+iv$

$\therefore \text{Im}\{f'(z)\}=v$

$\Rightarrow 6x(2y-1)=v;$ [by (1)]

$\Rightarrow v=12xy-6x$

$\therefore \frac{\partial v}{\partial x}=12y-6$ and $\frac{\partial v}{\partial y}=12x$ ...... (2)

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=12x$ ...... (3)

and [এবং] $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-12y+6$ ...... (4)

Since $u=u(x,y)$ so by calculas [যেহেতু $u=u(x,y)$, সুতরাং ক্যালকুলাস দ্বারা পাই]

$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$

$\Rightarrow du=12x\,dx+(-12y+6)\,dy$, which is an exact differential equation. [যাহা একটি যথার্থ অন্তরক সমীকরণ]

By integrating we get [যোজিত ফল নিয়ে পাই]

$\int du=\int 12x\,dx+\int(-12y+6)\,dy$

$\Rightarrow u=12\cdot\frac{x^2}{2}+\frac{-12y^2}{2}+6y+c$, c is a integrating constant

$\Rightarrow u=6x^2-6y^2+6y+c$

$\therefore f'(z)=u+iv$

$=6x^2-6y^2+6y+c+i(12xy-6x)$

$=6(x^2-y^2+2ixy)-6i(x+iy)+c$

$=6(x^2+i^2y^2+2ixy)-6i(x+iy)+c$

$=6(x+iy)^2-6i(x+iy)+c$

$\Rightarrow \frac{df}{dz}=6z^2-6iz+c$

$\Rightarrow \int df=\int(6z^2-6iz+c)\,dz$

$\Rightarrow f=6\cdot\frac{z^3}{3}-6i\frac{z^2}{2}+cz+D,$ [D = integrating constant]

$\Rightarrow f(z)=2z^3-3iz^2+cz+D$ ...... (5)

Given [দেওয়া আছে] $f(0)=3-2i$ and [এবং] $f(1)=6-5i$ ...... (6)

Putting $z=0$ in (5) we get [(5) এ $z=0$ বসাইয়া পাই]

$f(0)=0-0+0+D$

$\Rightarrow 3-2i=D$ ...... (7) [by (6)]

Putting $z=1$ in (5) we get [(5) এ $z=1$ বসাইয়া পাই]

$f(1)=2-3i+c+D$

$\Rightarrow 6-5i=2-3i+c+3-2i;$ [by (6) and (7)]

$\Rightarrow 6-5i=5-5i+c$

$\Rightarrow 6-5=c\Rightarrow c=1$

Putting the values of C and D in (5) we get [C ও D এর মান (5) এ বসাইয়া পাই]

$$f(z) = 2z^3 - 3iz^2 + z + 3 - 2i$$

$$\therefore f(1+i) = 2(1+i)^3 - 3i(1+i)^2 + (1+i) + 3 - 2i; \text{ by putting } z = 1+i$$

$$= 2(1 + 3i - 3 - i) - 3i(1 + 2i - 1) + 1 + i + 3 - 2i$$

$$= 2 + 6i - 6 - 2i - 3i + 6 + 3i + 4 - i = 6 + 3i. \quad \text{(Ans)}$$

38 (a). Find an analytic function f(z) such that Im $\{f'(z)\} = 6xy + 4x$, $f'(0) = 0$ and $f(1+i) = 0$. [একটি বৈশ্লেষিক ফাংশন f(z) বাহির কর যেন

Im $\{f'(z)\} = 6xy + 4x$, $f'(0) = 0$ এবং $f(1+i) = 0$ হয়।]

Solution. Given [দেওয়া আছে] Im$\{f'(z)\} = 6xy + 4x$ ...... (1)

$$\text{Let } w = f'(z) = \frac{df}{dz} = u + iv$$

$$\Rightarrow \text{Im}\{f'(z)\} = v$$

$$\Rightarrow 6xy + 4x = v$$

$$\Rightarrow v = 6xy + 4x$$

$$\therefore \frac{\partial v}{\partial x} = 6y + 4 \text{ and } \frac{\partial v}{\partial y} = 6x$$

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6x$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6y - 4$$

Since u = u (x, y) so by Calculas [যেহেতু u = u (x, y), সুতরাং ক্যালকুলাস দ্বারা পাই]

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow du = 6x\,dx - (6y + 4)\,dy$$

$$\Rightarrow \int du = \int 6x\,dx - \int (6y + 4)\,dy$$

$$\Rightarrow u = 6\cdot\frac{x^2}{2} - \left(6\cdot\frac{y^2}{2} + 4y\right) + c$$

$$\Rightarrow u = 3x^2 - 3y^2 - 4y + c$$

$$\therefore f'(z) = u + iv$$

$$= 3x^2 - 3y^2 - 4y + c + i(6xy + 4x)$$

$$= 3(x^2 + i^2y^2 + 2ixy) + 4i(x + iy) + c$$

$$= 3(x + iy)^2 + 4i(x + iy) + c$$

$$= 3z^2 + 4iz + c$$

$$\therefore f(z) = \int (3z^2 + 4iz + c)$$

$$= 3\cdot\frac{z^3}{3} + 4i\cdot\frac{z^2}{2} + cz + D$$

$$= z^3 + 2iz^2 + cz + D \text{ ...... (2)}$$

$$\text{Now } f'(0) = 0 \Rightarrow 3.0 + 4i.0 + c = 0 \Rightarrow c = 0$$

$$f(1+i) = 0 \Rightarrow (1+i)^3 + 2i(1+i)^2 + 0.z + D = 0$$

$$\Rightarrow 1 + 3i + 3i^2 + i^3 + 2i + 4i^2 + 2i^3 + D = 0$$

$$\Rightarrow 1 + 3i - 3 - i + 2i - 4 - 2i + D = 0$$

$$\Rightarrow -6 + 2i + D = 0$$

$$\Rightarrow D = 6 - 2i$$

Putting the value of C and D in (2) [C ও D এর মান (2) এ বসিয়ে]

$$f(z) = z^3 + 2iz^2 + 0.z + 6 - 2i$$

$$\Rightarrow f(z) = z^3 + 2iz^2 + 6 - 2i \quad \text{Ans}$$

Example–39. Prove that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find its harmonic conjugate v and express u + iv as an analytic function of z. **[RUH-1997, 2002, 2004, CUH-1989]**

Solution : Given that [দেওয়া আছে] $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x \text{ ...... (1)}$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y \text{ ...... (2)}$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \text{ ...... (3)}$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \text{ ...... (4)}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$\Rightarrow$ u satisfied Laplace equation. Hence u is harmonic. [u ল্যাপলাস সমীকরণ সিদ্ধ করে। অতএব u হারমোনিক।]

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -3x^2 + 3y^2 + 4y \quad \text{[by (2)]}$$

By integrating this w. r. to x keeping y as constant [y কে স্থির রেখে x এর সাপেক্ষে যোজিত করে পাই]

$$v = \int (-3x^2 + 3y^2 + 4y)\, dx$$

$$\Rightarrow v = -x^3 + 3xy^2 + 4xy + F(y) \text{ ...... (5)}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 6xy + 4x + F'(y); \text{ by C–R equation.}$$

$$\Rightarrow 6xy + 4x = 6xy + 4x + F'(y); \quad \text{[by (1)]}$$

$$\Rightarrow 0 = F'(y)$$

$\therefore F(y) = c_1$ by integrating

Putting this value in (5) we get [এই মান (5) এ বসাইয়া পাই]

$$v = -x^3 + 3xy^2 + 4xy + c_1$$

Let [ধরি] $f(z) = u + iv$

$$= 3x^2y + 2x^2 - y^3 - 2y^2 + i(-x^3 + 3xy^2 + 4xy + c_1)$$

$$= (-ix^3 + 3ixy^2 + 3x^2y - y^3) + 2(x^2 - y^2 + 2ixy) + ic_1$$

$$= -i(x^3 + 3ix^2y + 3i^2xy^2 + i^3y) + 2(x^2 + 2ixy + i^2y^2) + c$$

$$= -i(x + iy)^3 + 2(x + iy)^2 + c, \text{ where } c = ic_1$$

$\Rightarrow f(z) = u + iv = -iz^3 + 2z^2 + c.$ (Ans)

By Milne's methd [মিলনির পদ্ধতি দ্বারা] :

Given that [দেওয়া আছে] $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x = \phi_1(x, y), \text{ say ...... (1)}$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \text{ ...... (2)}$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y = \phi_2(x, y), \text{ say ...... (3)}$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \text{ ...... (4)}$$

(2) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$

$\Rightarrow$ u satisfy Laplace equation [u ল্যাপলাস সমীকরণ সিদ্ধ করে]

$\Rightarrow$ u is harmonic. [u হারমোনিক]

Putting x = z and y = 0 in (1) and (3) we get [(1) ও (3) এ x = z এবং y = 0 বসাইয়া পাই]

$$\phi_1(z, 0) = 0 + 4z = 4z$$

and $\phi_2(z, 0) = 3z^2 - 0 - 0 = 3z^2$

By Milne's theorem we have [মিলনির উপপাদ্য দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 4z - i3z^2$$

$\Rightarrow f(z) = 2z^2 - iz^3 + c$, by integrating

$$\Rightarrow u + iv = 2(x + iy)^2 - i(x + iy)^3 + c$$

$$= 2(x^2 + 2ixy + i^2y^2) - i(x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3) + c$$

$$= 2x^2 + 4ixy - 2y^2 - ix^3 + 3x^2y - 3ixy^2 - y^3 + c_1 + ic_2$$

where $c = c_1 + ic_2$

Equating imaginary parts we get, [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = 4xy - x^3 - 3xy^2 + c_2$$

$$\left.\begin{array}{l} \text{Thus, } v = -x^3 - 3xy^2 + 4xy + c_2 \\ \text{and } u + iv = -iz^3 + 2z^2 + c \end{array}\right\} \text{(Ans)}$$

Example–40. Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic and also find the harmonic conjugate if $f(z) = u + iv$ is analytic. **[DUH–1991, 2003, CUH–1985, JUH–1987, 1989]**

Solution : Given that [দেওয়া আছে] $u = 2x - x^3 + 3xy^2$

$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 = \phi_1(x, y)$, say ...... (1)

$\frac{\partial u}{\partial y} = 6xy = \phi_2(x, y)$, say ...... (2)

$\frac{\partial^2 u}{\partial x^2} = -6x$ ...... (3)

$\frac{\partial^2 u}{\partial y^2} = 6x$ ...... (4)

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$

This shows that u satisfy Laplace equation and hence u is harmonic. [ইহা দেখায় যে u ল্যাপলাস সমীকরণ সিদ্ধ করে এবং সে কারণে u হারমোনিক।]

To find the harmonic conjugate v, we have by putting $x = z$ and $y = 0$ in (1) and (2) we get [হারমোনিক অনুবন্ধী v বাহির করতে আমরা (1) ও (2) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$\phi_1(z, 0) = 2 - 3z^2$

and [এবং] $\phi_2(z, 0) = 0$

By Milne's theorem we have [মিলনীর উপপাদ্য দ্বারা পাই]

$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

$= 2 - 3z^2 - 0$

$\therefore f(z) = \int (2 - 3z^2)\, dz$

$= 2z - z^3 + c$

$\Rightarrow u + iv = 2(x + iy) - (x + iy)^3 + c$

$= 2(x + iy) - (x^3 + 3ix^2y + 3i^2 xy^2 + i^3 y^3) + c$

$= 2(x + iy) - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$

$= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3 + c_1)$; where $ic_1 = c$

Equating imaginary parts we have [কাল্পনিক রাশি সমীকৃত করে পাই]

$v = 2y - 3x^2y + y^3 + c_1$ (Ans)

Example–41. Find the harmonic conjugate of the function $u = x^3 + 6x^2y - 3xy^2 - 2y^3$. **[DUH–1987, 1991, JUH–1991]**

Solution : Given [দেওয়া আছে] $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 + 6x^2y - 3xy^2 - 2y^3)$

$= 3x^2 + 12xy - 3y^2 = \phi_1(x, y)$, say ...... (1)

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 + 6x^2y - 3xy^2 - 2y^3)$

$= 6x^2 - 6xy - 6y^2 = \phi_2(x, y)$, say ...... (2)

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 12xy - 3y^2)$

$= 6x + 12y$ ...... (3)

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(6x^2 - 6xy - 6y^2)$

$= -6x - 12y$ ...... (4)

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y$

Since u satisfies Laplace equation, so u is harmonic.

Let v be the harmonic conjugate of u, so that

[যেহেতু u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি u এর হারমোনিক অনুবন্ধী v.]

$f(z) = u + iv$ is analytic. [বৈশ্লেষিক]

Now from (1), $\phi_1(z, 0) = 3z^2 + 0 - 0 = 3z^2$

and from (2), $\phi_2(z, 0) = 6z^2 - 0 - 0 = 6z^2$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা]

$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

$= 3z^2 - 6iz^2$

$\Rightarrow f(z) = \int (3z^2 - 6iz^2)\, dz$

$\Rightarrow u + iv = \int (3 - 6i)\, z^2\, dz$

$= (3 - 6i)\frac{z^3}{3} + c_1 + ic_2$, where $c_1 + ic_2$ = complex constant

$= (1 - 2i)\, z^3 + c_1 + ic_2$

$= (1 - 2i)\, (x + iy)^3 + c_1 + ic_2$

$$= (1-2i)(x^3 + 3ix^2y - 3xy^2 - iy^3) + c_1 + ic_2$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3 - i2x^3 + 6x^2y + 6ixy^2 - 2y^3 + c_1 + ic_2$$
$$= (x^3 - 3xy^2 + 6x^2y - 2y^3 + c_1) + i(3x^2y - y^3 - 2x^3 + 6xy^2 + c_2)$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = 3x^2y - y^3 - 2x^3 + 6xy^2 + c_2. \quad \text{(Ans)}$$

Example–42. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic fuunction. Find v such that $u + iv$ is analytic.

[NUH–98, 06, 10, 12(Old) NU(Pre)–13, RUH–1986, CUH–1986, JUH (Phy)–1996]

Solution : Given [দেওয়া আছে] $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$
$$= 3x^2 - 3y^2 + 6x = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$
$$= -6xy - 6y = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 6x)$$
$$= 6x + 6 \cdots\cdots (3)$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-6xy - 6y)$$
$$= -6x - 6 \cdots\cdots (4)$$

(3) + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$

$$\Rightarrow \nabla^2 u = 0$$

$\therefore$ u satisfies Laplace equation, so u is a harmonic function. Let v is the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic. [$\therefore$ u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u ফাংশন হারমোনিক। v, u এর হারমোনিক অনুবন্ধী হবে যেন $f(z) = u + iv$.]

2nd Part : Putting $x = z, y = 0$ in (1) and (2) we get [(1) ও (2) এ $x = z, y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = 3z^2 + 6z$$

and [এবং] $\phi_2(z, 0) = -0 - 0 = 0$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 3z^2 + 6z - 0i$$
$$\Rightarrow f(z) = \int (3z^2 + 6z)\, dz$$
$$\Rightarrow u + iv = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c_1 + ic_2, \text{ where } c_1 + ic_2 \text{ is complex constant.}$$
$$= z^3 + 3z^2 + c_1 + ic_2$$
$$= (x + iy)^3 + 3(x + iy)^2 + c_1 + ic_2$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3 + 3x^2 + 6ixy - 3y^2 + c_1 + ic_2$$
$$= x^3 - 3xy^2 + 3x^2 - 3y^2 + c_1 + i(3x^2y - y^3 + 6xy + c_2)$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = 3x^2y - y^3 + 6xy + c_2. \quad \text{(Ans)}$$

Example–43. Show that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic and hence find its harmonic conjugate v if $f(z) = u + iv$ is analytic.

[NUH–2002]

Solution : Given [দেওয়া আছে] $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2y + 2x^2 - y^3 - 2y^2)$$
$$= 6xy + 4x = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2y + 2x^2 - y^3 - 2y^2)$$
$$= 3x^2 - 3y^2 - 4y = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(6xy + 4x)$$
$$= 6y + 4 \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3x^2 - 3y^2 - 4y)$$
$$= -6y - 4 \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4$
$$\Rightarrow \nabla^2 u = 0$$

Since u satisfies Laplace equation, so u is harmonic. Let v is the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic. [যেহেতু u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি u এর হারমোনিক অনুবন্ধী v যেন $f(z) = u + iv$ বৈশ্লেষিক হয়।]

Putting $x = z$, $y = 0$ in (1) and (2) we get [(1) ও (2) এ $x = z$, $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = 0 + 4z = 4z$$

and [এবং] $\phi_2(z, 0) = 3z^2 - 0 - 0 = 3z^2$

$\therefore$ By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$\Rightarrow f'(z) = 4z - i3z^2$$
$$\Rightarrow f(z) = \int (4z - i3z^2)\, dz$$
$$\Rightarrow u + iv = 4 \cdot \frac{z^2}{2} - i3\frac{z^3}{3} + c_1 + ic_2,$$ where $c_1 + ic_2$ is complex constant.
$$= 2z^2 - iz^3 + c_1 + ic_2$$
$$= 2(x+iy)^2 - i(x+iy)^3 + c_1 + ic_2$$
$$= 2x^2 + 4ixy - 2y^2 - ix^3 + 3x^2y + i3xy^2 - y^3 + c_1 + ic_2$$
$$= (2x^2 - 2y^2 + 3x^2y - y^3 + c_1) + i(4xy - x^3 + 3xy^2 + c_2)$$

Equating imaginary parts we get [কাল্পনিক রাশি সমীকৃত করে পাই]

$$v = 4xy - x^3 + 3xy^2 + c_2. \text{ (Ans)}$$

Example–44. Show that $\psi(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is a harmonic in the region $C - \{(0, 0)\}$. Find the harmonic conjugate of this function such that $f(z) = \psi + i\phi$ is analytic and also find f(z) interms of z. **[NUH–2004, DUH–1986]**

Solution : Given [দেওয়া আছে] $\psi = \psi(x, y) = \frac{1}{2}\log(x^2 + y^2)$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2} = \phi_1(x, y), \text{ say} \cdots\cdots (1)$$

$$\frac{\partial \psi}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2} = \phi_2(x, y), \text{ say} \cdots\cdots (2)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \cdots\cdots (3)$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2} \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}$
$$= \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2}$$
$$= 0$$

$\therefore \psi = \psi(x, y)$ is harmonic.

Given, $f(z) = \psi + i\phi$ will be analytic

$\therefore \phi$ is the harmonic conjugate of ψ

Putting $x = z$ and $y = 0$ in (1) and (2) we get

[$\therefore \psi = \psi(x, y)$ হারমোনিক। দেওয়া আছে $f(z) = \psi + i\phi$ বৈশ্লেষিক হবে।

$\therefore \psi$ এর হারমোনিক অনুবন্ধী ϕ। (1) ও (2) এ $x = z$, $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = \frac{z}{z^2 + 0} = \frac{1}{z}$$

and [এবং] $\phi_2(z, 0) = \frac{0}{z^2 + 0} = 0$

$\therefore$ By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= \frac{1}{z} + i0$$
$$\Rightarrow f(z) = \int \frac{1}{z}\, dz$$
$$\Rightarrow \psi + i\phi = \log z + (c_1 + ic_2);$$ $c_1 + ic_2$ is the imaginary constant.
$$= \log(x + iy) + c_1 + ic_2$$
$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x} + c_1 + ic_2$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\phi = \tan^{-1}\frac{y}{x} + c_2. \text{ (Ans)}$$

Example–45. Show that $u = \frac{1}{2}\log(x^2 + y^2)$ satisfies the laplace's equation and find v if $f(z) = u + iv$ is analytic.

[DUH–2006, RUH–1974]

Solution : Do as example 4 by writting $\psi = u$ and $\phi = v$

Example–46. Show that $u = \sin x \cosh y + 2\cos x \sin hy + x^2 - y^2 + 4xy$ is harmonic in some domain and determine the corresponding analytic function.

[DUH–1984, 1986, 1987, 1988, 1990, JUH–1988]

Solution : Given $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$$\therefore \frac{\partial u}{\partial x} = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y - 2\cos x \sinh y + 2 \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y + 2\cos x \sinh y - 2 \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y - 2\cos x \sinh y + 2$

$$+ \sin x \cosh y + 2\cos x \sinh y - 2$$

$$= 0$$

$\therefore$ u satisfies Laplace equation, so u is harmonic [u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক]

2nd part : Let v is the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic. [মনেকরি v হল u এর হারমোনিক অনুবন্ধী যেন $f(z) = u + iv$ বৈশ্লেষিক হয়]

Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) ও (2) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\cos z \cdot \cos 0 - 2\sin z \cdot \sin 0 + 2z + 4 \cdot 0 = \phi_1(z, 0)$$

$$\Rightarrow \cos z + 2z = \phi_1(z, 0) \cdots\cdots (5)$$

and $\sin z \cdot \sin 0 + 2\cos z \cdot \cos 0 - 0 + 4z = \phi_2(z, 0)$

$$\Rightarrow 2\cos z + 4z = \phi_2(z, 0) \cdots\cdots (6)$$

By Milne's theorem we get [মিলনির উপপাদ্য দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= \cos z + 2z - i(2\cos z + 4z)$$

$$\Rightarrow f(z) = \int (\cos z + 2z - 2i\cos z - 4iz)\, dz$$

$$\Rightarrow u + iv = \sin z + 2\frac{z^2}{2} - 2i\sin z - 4i \cdot \frac{z^2}{2} + c$$

$$= \sin z + z^2 - 2i\sin z - 2iz^2 + c$$

$$= (1 - 2i)z^2 + (1 - 2i)\sin z + c. \text{ (Ans)}$$

Example–47. Show that $\phi = \log\{(x-1)^2 + (y-2)^2\}$ is harmonic in every region which does not include the point (1, 2). Find a function ψ such that $\phi + i\psi$ is analytic and express $\phi + i\psi$ as a function of z.

[DUH–2004]

Solution : Given [দেওয়া আছে] $\phi = \log\{(x-1)^2 + (y-2)^2\}$

$$\therefore \frac{\partial \phi}{\partial x} = \frac{1}{(x-1)^2 + (y-2)^2} \cdot 2(x-1) = \phi_1(x, y), \text{ say [ধরি] } \cdots\cdots (1)$$

$$\frac{\partial \phi}{\partial y} = \frac{2(y-2)}{(x-1)^2 + (y-2)^2} = \phi_2(x, y), \text{ say [ধরি] } \cdots\cdots (2)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\{(x-1)^2 + (y-2)^2\} \cdot 2 - 2(x-1) \cdot 2(x-1)}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= \frac{2(y-2)^2 - 2(x-1)^2}{\{(x-1)^2 + (y-2)^2\}^2} \cdots\cdots (3)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\{(x-1)^2 + (y-2)^2\} \cdot 2 - 2(y-2) \cdot 2(y-2)}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= \frac{2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2} \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{2(y-2)^2 - 2(x-1)^2}{\{(x-1)^2 + (y-2)^2\}^2} + \frac{2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2}$

$$= \frac{2(y-2)^2 - 2(x-1)^2 + 2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= 0$$

Since ϕ satisfy Laplace equation, so ϕ is harmonic. Let ψ is the harmonic conjugate ϕ, so that $\phi + i\psi$ is analytic. [যেহেতু ϕ ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং ϕ হারমোনিক। ধরি ϕ এর হারমোনিক অনুবন্ধী ψ যেন $\phi + i\psi$ বৈশ্লেষিক হয়।]

Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) এবং (2) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = \frac{2(z-1)}{(z-1)^2 + 2^2}$$

and [এবং] $\phi_2(z, 0) = \dfrac{-4}{(z-1)^2 + 2^2}$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= \frac{2(z-1)}{(z-1)^2 + 2^2} - i\frac{-4}{(z-1)^2 + 2^2}$$

$$\Rightarrow f(z) = \int\left[\frac{2(z-1)}{(z-1)^2 + 2^2} + i4\frac{1}{(z-1)^2 + 2^2}\right]dz$$

$$\Rightarrow \phi + i\psi = \int\frac{2(z-1)}{(z-1)^2 + 2^2}\,d(z-1) + i4\int\frac{1}{(z-1)^2 + 2^2}\,dz$$

$$= \log\{(z-1)^2 + 2^2\} + i4 \cdot \frac{1}{2}\tan^{-1}\frac{z-1}{2} + c$$

$$= \log\{(z-1)^2 + 4\} + i2\tan^{-1}\left(\frac{z-1}{2}\right) + c$$

$$= 2\left[\frac{1}{2}\log\{(z-1)^2 + 2^2\} + i\tan^{-1}\left(\frac{z-1}{2}\right)\right] + c$$

$$= 2\log\{2 + i(z-1)\} + c$$

$$= 2\log\{2 + i(x + iy - 1)\} + c$$

$$= 2\log\{2 + ix - y - i\} + c$$

$$= 2\log\{2 - y + i(x-1)\} + c$$

$$= 2\left[\frac{1}{2}\log\{(2-y)^2 + (x-1)^2\} + i\tan^{-1}\frac{x-1}{2-y}\right] + c$$

$$= \log\{(x-1)^2 + (y-2)^2\} + 2i\tan^{-1}\frac{x-1}{2-y} + c_1 + ic_2,$$

where $c = c_1 + ic_2$

Equating imaginary parts [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\psi = 2\tan^{-1}\left(\frac{x-1}{2-y}\right) + c_2. \text{ Ans.}$$

Example-48. Show that the function $u = 2x(1-y)$ is harmonic and find a function v such that $f(z) = u + iv$ is analytic. Also find $f(z)$ in terms of z.

[NUH-1996, 2001, NU(Pre)-2008, DUH-1983]

OR, Find the harmaonic conjugate of $u = 2x(1-y)$. **[NUH-2014]**

Solution : Given [দেওয়া আছে] $u = 2x(1-y)$

$\therefore \dfrac{\partial u}{\partial x} = 2(1-y) = \phi_1(x, y)$, say ...... (1)

$\dfrac{\partial^2 u}{\partial x^2} = 0$ ...... (2)

$\dfrac{\partial u}{\partial y} = -2x = \phi_2(x, y)$, say ...... (3)

$\dfrac{\partial^2 u}{\partial y^2} = 0$ ...... (4)

(2) + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 + 0 = 0$

Since u satisfies Laplace equation, so u is harmonic. Let v is the harmonic conjugate of u such that $f(z) = u + iv$ is analytic. [যেহেতু u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি v হল u এর হারমোনিক অনুবন্ধী যেন $f(z) = u + iv$ বৈশ্লেষিক হয়]

Putting $x = z$ and $y = 0$ in (1) and (3) we get [(1) ও (3) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = 2(1-0) = 2$$

and $\phi_2(z, 0) = -2z$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2 + i2z$$

$$\Rightarrow f(z) = \int(2 + 2iz)\,dz$$

$$\Rightarrow f(z) = 2z + 2i\frac{z^2}{2} + c$$

$$\Rightarrow f(z) = iz^2 + 2z + c \text{ ...... (5)}$$

$$\Rightarrow u + iv = i(x + iy)^2 + 2(x + iy) + c$$

$$= i(x^2 - y^2 + 2ixy) + 2x + 2iy + c$$

$$= ix^2 - iy^2 - 2xy + 2x + 2iy + c$$

$$= 2x - 2xy + i(x^2 - y^2 + 2y) + c_1 + ic_2, \text{ where } c = c_1 + ic_2$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = x^2 - y^2 + 2y + c_2$$

$$\left.\begin{array}{l} \text{Thus, } v = x^2 - y^2 + 2y + c_2 \\ \text{and } f(z) = iz^2 + 2z + c \end{array}\right\} \text{(Ans)}$$

Example–49. Show that the function $u = e^x(x \cos y - y \sin y)$ is a harmonic function and find the corresponding analytic function $f(z) = u + iv$. From it find v.

[NUH–1994, DUH–1988, 2005, CUH–85, 88, JUH–91]

Solution : We have [আমাদের আছে] $u = e^x (x \cos y - y \sin y)$

$$\therefore \frac{\partial u}{\partial x} = e^x (x \cos y - y \sin y) + e^x \cos y = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x (x \cos y - y \sin y) + e^x \cos y + e^x \cos y \cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = e^x (-x \sin y - \sin y - y \cos y) = \phi_2(x, y), \text{ say } \cdots\cdots (3)$$

$$\text{and } \frac{\partial^2 u}{\partial y^2} = e^x (-x \cos y - \cos y - \cos y + y \sin y) \cdots\cdots (4)$$

(2) + (4) gives,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x (x \cos y - y \sin y) + e^x \cos y + e^x \cos y + e^x(-x \cos y - \cos y - \cos y + y \sin y)$$
$$= e^x(x \cos y - y \sin y + \cos y + \cos y - x \cos y - \cos y - \cos y + y \sin y)$$
$$= e^x \times 0 = 0$$

$\therefore$ u satisfies Laplace equation. [u ল্যাপলাস সমীকরণ সিদ্ধ করে]

$\Rightarrow$ u is a harmonic function. [u হারমোনিক]

Putting $x = z$, $y = 0$ in (1) and (3) we get [(1) ও (3) এ $x = z$, $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = e^z (z \cdot 1 - 0) + e^z \cdot 1$$
$$= ze^z + e^z$$
$$\text{and } \phi_2(z, 0) = e^z (-0 - 0 - 0) = 0$$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$\Rightarrow f(z) = \int (ze^z + e^z - i0)\, dz$$

$$\Rightarrow u + iv = ze^z - \int 1 \cdot e^z + e^z$$
$$= ze^z - e^z + e^z + c, \text{ c is a complex constant.}$$
$$= ze^z + c$$
$$= (x + iy)\, e^{x+iy} + c$$
$$= (x + iy)\, e^x \cdot e^{iy} + c$$
$$= e^x(x + iy)(\cos y + i \sin y) + c$$
$$= e^x(x \cos y + ix \sin y + iy \cos y - y \sin y) + c_1 + ic_2$$
$$\text{where } c = c_1 + ic_2$$
$$= e^x(x \cos y - y \sin y) + ie^x(x \sin y + y \cos y) + c_1 + ic_2$$

Equating imaginary parts we get, [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = e^x (x \sin y + y \cos y) + c_2. \text{ (Ans)}$$

Example–50. Show that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic. Find v such that $f(z) = u + iv$ is analytic. **[NUH–2011, NUH (Phy)–2003, 2006, DUHT–1989, 2001, DUH–1981, 1982, DUMPT–1990, CUH–1982, RUMP–1985]**

Solution : We have [আমাদের আছে] $u = e^{-x} (x \sin y - y \cos y)$.

$$\therefore \frac{\partial u}{\partial x} = -e^{-x} (x \sin y - y \cos y) + e^{-x} \sin y = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x} (x \sin y - y \cos y) - e^{-x} \sin y - e^{-x} \sin y$$
$$= e^{-x} (x \sin y - y \cos y) - 2e^{-x} \sin y \cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y - \cos y + y \sin y) = \phi_2(x, y), \text{ say } \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x \sin y + \sin y + \sin y + y \cos y) \cdots\cdots (4)$$

(2) + (4) gives,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}(x \sin y - y \cos y) - 2e^{-x} \sin y + e^{-x} (-x \sin y + \sin y + \sin y + y \cos y)$$
$$= e^{-x} (x \cancel{\sin y} - y \cancel{\cos y} - 2 \sin y - x \cancel{\sin y} + 2 \sin y + y \cancel{\cos y})$$
$$= e^{-x} \times 0 = 0$$

Since u satisfies Laplace equation, so u is harmonic. Let v is the harmonic conjugate of u such that $f(z) = u + iv$ is analytic. [যেহেতু u ল্যাপলাসের সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি u এর হারমোনিক অনুবর্ত্মী v যেন $f(z) = u + iv$ বৈশ্লেষিক হয়।]

Putting $x = z$ and $y = 0$ in (1) and (3) we get [(1) ও (3) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = -e^{-z}(0-0) + 0 = 0$$

and [এবং] $\phi_2(z, 0) = e^{-z}(z \cdot 1 - 1 + 0) = e^{-z}(z-1) = ze^{-z} - e^{-z}$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$\Rightarrow f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz$$

$$\Rightarrow u + iv = \int [0 - i(ze^{-z} - e^{-z})]\, dz$$

$$= -i\int ze^{-z}\, dz + i\int e^{-z}\, dz$$

$$= -i\left[-ze^{-z} - \int -e^{-z}\, dz\right] + i(-1)\, e^{-z}$$

$$= ize^{-z} + ie^{-z} - ie^{-z} + c$$

$$= ize^{-z} + c$$

$$= i(x + iy)e^{-(x+iy)} + c$$

$$= (ix - y)\, e^{-x} \cdot e^{-iy} + c$$

$$= e^{-x}(ix - y)(\cos y - i \sin y) + c$$

$$= e^{-x}(ix \cos y + x \sin y - y \cos y + iy \sin y) + c_1 + ic_2$$

where $c = c_1 + ic_2$

$$= e^{-x}(x \sin y - y \cos y) + ie^{-x}(x \cos y + y \sin y) + c_1 + ic_2$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = e^{-x}(x \cos y + y \sin y) + c_2. \text{ (Ans)}$$

Example–50(i). Show that $u(x, y)$ is harmonic. Find its harmonic conjugate $v(x, y)$ and the corresponding analytic function $f(z) = u + iv$ where $u(x, y) = x^2 - y^2 + 2e^{-x} \sin y$. [দেখাও যে, $u(x, y)$ একটি হারমনিক ফাংশন এর হারমনিক কনজুগেট $v(x, y)$ এবং এদের অ্যানালেটিক ফাংশন $f(z)$ নির্ণয় কর, যেখানে $u(x, y) = x^2 - y^2 + 2e^{-x} \sin y$.] **[NUH–2012]**

Solution : Given that [দেওয়া আছে] $u(x, y) = x^2 - y^2 + 2e^{-x} \sin y$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2e^{-x} \sin y = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + 2e^{-x} \sin y$$

$$\frac{\partial u}{\partial y} = -2y + 2e^{-x} \cos y = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 - 2e^{-x} \sin y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2e^{-x} \sin y - 2 - 2e^{-x} \sin y = 0$$

$\Rightarrow$ u satisfy Laplace equation [u ল্যাপলাস সমীকরণ সিদ্ধ করে]

$\therefore$ u harmonic [u হারমনিক] 1st Part proved.

2nd Part : Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) ও (3) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = 2z - 2e^{-z} \sin 0 = 2z$$

and $\phi_2(z, 0) = 0 + 2e^{-z} \cos 0 = 2e^{-z}$

By Milne's theorem we have [মিলনির উপপাদ্য দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2z - i2e^{-z}$$

$$\Rightarrow f(z) = \int (2z - 2ie^{-z})\, dz$$

$$= 2 \cdot \frac{z^2}{2} - 2i \cdot \frac{e^{-z}}{-1} + c$$

$$= z^2 + 2ie^{-z} + c$$

which is the corresponding analytic function. [যাহা অনুসঙ্গী বৈশ্লেষিক ফাংশন]

$$\Rightarrow f(z) = (x + iy)^2 + 2ie^{-(x+iy)} + c, \text{ where } z = x + iy$$

$$= x^2 + i^2 y^2 + 2ixy + 2ie^{-x} e^{-iy} + c$$

$$= x^2 - y^2 + 2ixy + 2ie^{-x}[\cos(-y) + i \sin(-y)] + c$$

$$\Rightarrow u + iv = x^2 - y^2 + 2ixy + 2ie^{-x} \cos y + 2e^{-x} \sin y + c_1 + ic_2$$

where $c = c_1 + ic_2$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$u = x^2 - y^2 + 2e^{-x}\sin y + c_1$$

$$v = 2xy + 2e^{-x}\cos y + c_2. \text{ (Ans)}$$

Example–51. If $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2+y^2}$, then show that both u and v satisfy the Laplace's equation but u + iv is not an analytic function of z. **[DUH–1987, DUHT–1991]**

Solution : Given [দেওয়া আছে] $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = 2x \;\cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \;\cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = -2y \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \;\cdots\cdots (4)$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 0 + y\cdot 2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \;\cdots\cdots (5)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{2xy}{(x^2+y^2)^2}\right]$$

$$= \frac{(x^2+y^2)^2\cdot 2y - 2xy\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\,[(x^2+y^2)2y - 8x^2y]}{(x^2+y^2)^4} = \frac{2y(x^2+y^2-4x^2)}{(x^2+y^2)^3}$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} \;\cdots\cdots (6)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot(-1) + y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \;\cdots\cdots (7)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

$$= \frac{(x^2+y^2)^2\cdot 2y - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\,\{(x^2+y^2)\cdot 2y - (y^2-x^2)\cdot 4y\}}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2-2y^2+2x^2)}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3} \;\cdots\cdots (8)$$

(2 + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

and (6) + (8) gives, $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{2y(y^2-3x^2)}{(x^2+y^2)^3} + \dfrac{2y(3x^2-y^2)}{(x^2+y^2)^3}$

$$= \frac{2y^3 - 6x^2y + 6x^2y - 2y^3}{(x^2+y^2)^3} = 0$$

$\therefore$ u and v both satisfy Laplace's equation. [u ও v উভয়ে ল্যাপলাস সমীকরণ সিদ্ধ করে]

From (1) and (7) we have [(1) ও (7) হতে পাই] $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$

and from (3) and (5) we have [এবং (3) ও (5) হতে পাই] $\dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

$\therefore$ u + iv is not an analytic function of z. [u + iv, z এর বৈশ্লেষিক ফাংশন না] (Showe)

Example–52. Show that $u = \dfrac{y}{x^2+y^2}$ is harmonic and finds harmonic conjugate v and f(z) = u + iv if f(z) is analytic. **[DUH–1976]**

Solution : Given that [দেওয়া আছে] $u = \dfrac{y}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)^2\cdot(-2y) - (-2xy)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-2y(x^4 + 2x^2y^2 + y^4) + 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{-2x^4y - 4x^2y^3 - 2y^5 + 8x^4y + 8x^2y^3}{(x^2+y^2)^4}$$

$$= \frac{6x^4y + 4x^2y^3 - 2y^5}{(x^2+y^2)^4} \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)^2 \cdot (-2y) - (x^2-y^2) \cdot 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{-2x^4y - 4x^2y^3 - 2y^5 - 4x^4y + 4y^5}{(x^2+y^2)^4}$$

$$= \frac{-6x^4y - 4x^2y^3 + 2y^5}{(x^2+y^2)^4} \;\cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{6x^4y + 4x^2y^3 - 2y^5}{(x^2+y^2)^4} - \frac{6x^4y + 4x^2y^3 - 2y^5}{(x^2+y^2)^4}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ u satisfied Laplace equation and hence u is harmonic.

[u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক]

To find v, we put x = z and y = 0 in (1) and (2) [v পাওয়ার জন্য (1) ও (2) এ x = z এবং y = 0 বসাই]

$$\phi_1(z, 0) = 0 \text{ and } \phi_2(z, 0) = \frac{z^2 - 0}{(z^2+0)^2} = \frac{1}{z^2}$$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 0 - i\frac{1}{z^2}$$

$$\Rightarrow f(z) = \frac{i}{z} + c$$

$$\Rightarrow u + iv = \frac{i}{x+iy} + c$$

$$= \frac{i(x-iy)}{(x+iy)(x-iy)} + c$$

$$= \frac{ix + y}{x^2+y^2} + c_1 + ic_2, \text{ where } c = c_1 + ic_2$$

Equating imaginary part we get, [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = \frac{x}{x^2+y^2} + c_2. \text{ (Ans)}$$

Example–53. Find the harmonic conjugate of the function $u = e^{x^2-y^2} \cos 2xy$ and the corresponding analytic function $f(z) = u + iv$. **[DUH–1978, CUH–2004, JUH–1987]**

Solution : Given that [দেওয়া আছে] $u = e^{x^2-y^2} \cos 2xy$

$$\therefore \frac{\partial u}{\partial x} = 2xe^{x^2-y^2} \cos 2xy + e^{x^2-y^2}(-\sin 2xy) \cdot 2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy) = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = -2ye^{x^2-y^2} \cos 2xy + e^{x^2-y^2} \cdot (-\sin 2xy) \cdot 2x$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy) = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4xe^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$
$$+ 2e^{x^2-y^2}(\cos 2xy - 2xy \sin 2xy - 2y^2 \cos 2xy)$$
$$= 2e^{x^2-y^2}(2x^2 \cos 2xy - 2xy \sin 2xy$$
$$+ \cos 2xy - 2xy \sin 2xy - 2y^2 \cos 2xy)$$

$$\frac{\partial^2 u}{\partial y^2} = 4ye^{x^2-y^2}(y \cos 2xy + x \sin 2xy)$$
$$- 2e^{x^2-y^2}(\cos 2xy - 2xy \sin 2xy + 2x^2 \cos 2xy)$$
$$= 2e^{x^2-y^2}(2y^2 \cos 2xy + 2xy \sin 2xy - \cos 2xy$$
$$+ 2xy \sin 2xy - 2x^2 \cos 2xy)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2e^{x^2-y^2}(2x^2 \cos 2xy - 2xy \sin 2xy$$
$$+ \cos 2xy - 2xy \sin 2xy - 2y^2 \cos 2xy + 2y^2 \cos 2xy$$
$$+ 2xy \sin 2xy - \cos 2xy + 2xy \sin 2xy - 2x^2 \cos 2xy)$$
$$= 2e^{x^2-y^2} \times 0 = 0$$

$\Rightarrow$ u is harmonic. [u হারমোনিক]

Now putting x = z and y = 0 in (1) and (2) we get [(1) ও (2) এ x = z এবং y = 0 বসাইয়া পাই]

$$\phi_1(z, 0) = 2e^{z^2}(z \cdot 1 - 0) = 2ze^{z^2}$$

and $\phi_2(z, 0) = -2z^2(0 + 0) = 0$

By Milne's theorem we have [মিলনির উপপাদ্য দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0) = 2ze^{z^2}$$

$$\Rightarrow f(z) = \int 2ze^{z^2}\, dz$$

$$= \int e^{z^2} \cdot d(z^2) = e^{z^2} + c$$

$$\Rightarrow u + iv = e^{(x+iy)^2} + c = e^{x^2-y^2} \cdot e^{i2xy} + c$$

$$= e^{x^2-y^2}(\cos 2xy + i \sin 2xy) + c_1 + ic_2,\ c = c_1 + ic_2$$

$$\left.\begin{aligned} &\Rightarrow v = e^{x^2-y^2} \sin 2xy + c_2 \\ &\text{and } f(z) = u + iv = e^{z^2} + c \end{aligned}\right\}. \text{ (Ans)}$$

Example–54. Show that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic and find the harmonic conjugate v. Also f(z) = u + iv if f(z) is analytic. **[DUMP–1991]**

Solution : Given that [দেওয়া আছে] $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2y - 2 = \phi_1(x, y), \text{ say} \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 = \phi_2(x, y), \text{ say} \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \cdots\cdots (4)$$

(3) + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

$\Rightarrow$ u is harmonic. [u হারমোনিক]

For finding v [v নির্ণয়], 1st method [১ম পদ্ধতি] :

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 2y - 2 \cdots\cdots (5)$$

and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -2y - 2x + 3 \cdots\cdots (6)$

From (5) [(5) হতে] $v = \int (2x - 2y - 2)\, dy$

$$v = 2xy - y^2 - 2y + F(x) \cdots\cdots (7)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 2y + F'(x)$$

$$\Rightarrow 2y + 2x - 3 = 2y + F'(x)$$

$$\Rightarrow F'(x) = 2x - 3$$

$$\Rightarrow F(x) = \int (2x - 3)\, dx = x^2 - 3x + c$$

Putting this value in (7) we get, [এই মান (7) এ বসাইয়া পাই]

$$v = 2xy - y^2 - 2y + x^2 - 3x + c$$

$$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c. \text{ (Ans)}$$

2nd method [২য় পদ্ধতি] :

$$v = v(x, y)$$

$$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy; \text{ by C–R equations}$$

$$\Rightarrow dv = (2y + 2x - 3)\, dx + (2x - 2y - 2)\, dy \cdots\cdots (8)$$

$$= M\, dx + N\, dy$$

where $M = 2y + 2x - 3$ and $N = 2x - 2y - 2$

$$\therefore \frac{\partial M}{\partial y} = 2 \text{ and } \frac{\partial N}{\partial x} = 2 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence equation (8) is an exact differential equation. [অতএব সমীকরণ (8) একটি যথার্থ অন্তরক সমীকরণ]

$\therefore v = \int (2y + 2x - 3)\,dx + \int (-2y - 2)\,dy + c,$

[In 1st integral y is constant]

$\Rightarrow v = 2xy + x^2 - 3x - y^2 - 2y + c$

$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c.$ (Ans)

3rd method [৩য় পদ্ধতি] :

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

$= 2z - 0 - 2 - i(-0 - 2z + 3)$

$= 2z - 2 + 2iz - 3i$

$= (2 + 2i)\,z - 2 - 3i$

$\Rightarrow f(z) = \int \{(2 + 2i)\,z - 2 - 3i\}\,dz$

$= (2 + 2i)\dfrac{z^2}{2} - 2z - 3iz + c$ ...... (9)

$= (1 + i)\,(x + iy)^2 - 2(x + iy) - 3i(x + iy) + c$

$\Rightarrow u + iv = (1 + i)\,(x^2 - y^2 + 2ixy) - 2(x + iy) - 3i(x + iy) + c_1 + ic_2$

where $c = c_1 + ic_2$

Equating imaginary parts [কাল্পনিক অংশ সমীকৃত করে]

$v = 2xy + x^2 - y^2 - 2y - 3x + c_2$

$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c_2.$ (Ans)

Third Part : Form (9), $f(z) = (1 + i)z^2 - 2z - 3iz + c$

$= (1 + i)z^2 - (2 + 3i)\,z + c$

OR, $f(z) = u + iv$

$= x^2 - y^2 - 2xy - 2x + 3y + i(x^2 - y^2 + 2xy - 3x - 2y + c)$

$= (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) - 2(x + iy) - 3i(x + iy) + c$

$= (x + iy)^2 + i(x + iy)^2 - 2(x + iy) - 3i(x + iy) + c$

$= z^2 + iz^2 - 2z - 3iz + c$, where $z = x + iy$

$= (1 + i)z^2 - (2 + 3i)z + c.$ (Ans)

Example–55. Prove that $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic and find its harmonic conjugate. **[DUH–2000, RUH–2000]**

Solution : Give that [দেওয়া আছে] $u = e^{-2xy}\sin(x^2 - y^2)$

$\therefore \dfrac{\partial u}{\partial x} = -2ye^{-2xy}\sin(x^2 - y^2) + e^{-2xy} \cdot 2x \cdot \cos(x^2 - y^2)$

$\Rightarrow \dfrac{\partial u}{\partial x} = 2e^{-2xy}\,[-y\sin(x^2 - y^2) + x\cos(x^2 - y^2)] = \phi_1(x, y)$, say .... (1)

$\dfrac{\partial u}{\partial y} = -2xe^{-2xy}\sin(x^2 - y^2) - 2ye^{-2xy}\cos(x^2 - y^2)$

$\Rightarrow \dfrac{\partial u}{\partial y} = -2e^{-2xy}\,[x\sin(x^2 - y^2) + y\cos(x^2 - y^2)] = \phi_2(x, y)$, say .... (2)

$\dfrac{\partial^2 u}{\partial x^2} = -4ye^{-2xy}\,[-y\sin(x^2 - y^2) + x\cos(x^2 - y^2)]$

$+ 2e^{-2xy}[-2xy\sin(x^2 - y^2) + \cos(x^2 - y^2) - 2x^2\sin(x^2 - y^2)]$

$= 4e^{-2xy}\,[y^2\sin(x^2 - y^2) - xy\cos(x^2 - y^2) - xy\cos(x^2 - y^2)$

$+ \dfrac{1}{2}\cos(x^2 - y^2) - x^2\sin(x^2 - y^2)]$ ...... (3)

$\dfrac{\partial^2 u}{\partial y^2} = 4xe^{-2xy}\,[x\sin(x^2 - y^2) + y\cos(x^2 - y^2)$

$- 2e^{-2xy}\,[-2xy\cos(x^2 - y^2) + \cos(x^2 - y^2) + 2y^2\sin(x^2 - y^2)]$

$= 4e^{-2xy}\,[x^2\sin(x^2 - y^2) + xy\cos(x^2 - y^2) + xy\cos(x^2 - y^2)$

$- \dfrac{1}{2}\cos(x^2 - y^2) - y^2\sin(x^2 - y^2)]$ ...... (4)

(3) + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic. [u হারমোনিক]

To find v, the harmonic conjugate of u, we put $x = z$ and $y = 0$ in (1) and (2) [u এর হারমোনিক অনুবন্ধী পাওয়ার জন্য আমরা (1) ও (2) এ $x = z$ এবং $y = 0$ বসাব]

$\therefore \phi_1(z, 0) = 2(0 + z\cos z^2) = 2z\cos z^2$

and $\phi_2(z, 0) = -2(z\sin z^2 + 0) = -2z\sin z^2$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$f'(z) = \phi_1(z, 0) - i\phi_2\,(z, 0)$

$= 2z\cos z^2 + i2z\sin z^2$

$= 2z(\cos z^2 + i\sin z^2) = 2ze^{iz^2}$

$\Rightarrow f(z) = \int 2ze^{iz^2}\,dz$ Putting $iz^2 = t$

$= \int e^t \cdot \dfrac{1}{i}\,dt$ $\Rightarrow 2iz\,dz = dt$

$= \dfrac{1}{i}e^t + c$

$= -ie^{iz^2} + c$

$$= -ie^{i(x+iy)^2} + c$$
$$= -ie^{-2xy} \cdot e^{i(x^2-y^2)} + c$$
$$= -ie^{-2xy} [\cos (x^2 - y^2) + i \sin (x^2 - y^2)] + c$$
$$\Rightarrow u + iv = e^{-2xy} [-i \cos (x^2 - y^2) + \sin (x^2 - y^2)] + c_1 + ic_2$$
where $c = c_1 + ic_2$

Equating imaginary parts [কাল্পনিক অংশ সমীকৃত করে পাই]
$$v = -e^{-2xy} \cos(x^2 - y^2) + c_2. \quad \text{(Ans)}$$

56. If $w = f(z) = u + iv$ is analytic and $u - v = e^x$ (cosy − siny), find f(z) interms of z. [যদি $w = f(z) = u + iv$ বৈশ্লেষিক এবং $u - v = e^x$ (cosy − siny) হয় তবে f(z) কে z–এর মাধ্যমে প্রকাশ কর। **[NUH–2015]**

Solution : Given that [দেওয়া আছে]

$u + iv = f(z)$ ...... (1)

and $u - v = e^x (\cos y - \sin y)$ ...... (2)

$$\therefore (u - v) + i (u + v) = v + i^2v + iu + iv$$
$$= (u + iv) + i (u + iv)$$
$$= f(z) + i f(z), \text{ by (1)}$$
$$= (1 + i) f(z) \text{ ...... (3)}$$

Let, $u - v = U$, $u + v = V$ and $(1 + i) f(z) = F(z)$.

Then from (3) we have [তখন (3) হতে পাই]

$$U + iV = F(z) \text{ ...... (4)}$$

is an analytic function [একটি বৈশ্লেষিক ফাংশন]

and [এবং] $U = u - v = e^x (\cos y - \sin y)$

$$\therefore \frac{\partial U}{\partial x} = e^x (\cos y - \sin y) = \varphi_1 (x, y) \text{, say ..... (5)}$$

$$\frac{\partial U}{\partial y} = e^x (-\sin y - \cos y) = \varphi_2 (x, y)\text{, say ...... (6)}$$

Putting $x = z$ and $y = 0$ in (5) and (6) [(5) ও (6) এ $x = z$ এবং $y = 0$ বসাইয়া]

$$\varphi_1 (z, 0) = e^z (\cos 0 - \sin 0) = e^z (1 - 0) = e^z$$

and $\varphi_2 (z, 0) = e^z (-\sin 0 - \cos 0) = e^z\, 0 - 1) = -e^z$

By Milne's method we have [মিলনীর পদ্ধতি দ্বারা পাই]

$$F'(z) = \varphi_1 (z, 0) - i\, \phi_2 (z, 0)$$
$$= e^z - i (-e^z)$$
$$= (1 + i) e^z$$
$$\Rightarrow F(z) = (1 + i) \int e^z dz$$
$$\Rightarrow (1 + i) f(z) = (1 + i) e^z + A, \text{ A is amintegrating constant}$$
$$\Rightarrow f(z) = e^z + \frac{A}{1 + i}$$
$$\Rightarrow f(z) = e^z + C, \text{ where } C = \frac{A}{1 + i}$$
$$\therefore f(z) = e^z + C \quad \text{Ans}$$

Example–57. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find f(z) subject to the condition $f\left(\dfrac{\pi}{2}\right) = \dfrac{1}{2} (3 - i)$. **[CUH–2003]**

Solution : Give that $u + iv = f(z)$ ...... (1)

$$\text{and } u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x} \text{ ...... (2)}$$

$$\therefore (u - v) + i(u + v) = u + i^2v + iu + iv$$
$$= (u + iv) + i(u + iv)$$
$$= f(z) + if(z), \quad \text{[by (1)]}$$
$$= (1 + i) f(z) \text{ ...... (3)}$$

Let $u - v = U$, $u + v = V$ and $(1 + i) f(z) = F(z)$. Then from (3) we have

$U + iV = F(z)$ ...... (4) is an analytic function.

$$\text{and } U = u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{(\cosh y - \cos x)(\sin x + \cos x) - (e^y - \cos x + \sin x) \cdot \sin x}{(\cosh y - \cos x)^2}$$

$$= \frac{\sin x \cosh y - \sin x \cos x + \cos x \cosh y - \cos^2 x - e^y \sin x + \sin x \cos x - \sin^2 x}{(\cosh y - \cos x)^2}$$

$$= \frac{-1 + \sin x \cosh y + \cos x \cosh y - e^y \sin x}{(\cosh y - \cos x)^2} = \phi_1(x, y), \text{ say} \cdots (5)$$

$$\frac{\partial U}{\partial y} = \frac{(\cosh y - \cos x)(e^y) - (e^y - \cos x + \sin x)\sinh y}{(\cosh y - \cos x)^2}$$

$$= \frac{e^y \cosh y - e^y \cos x - e^y \sinh y + \cos x \sinh y - \sin x \sinh y}{(\cosh y - \cos x)^2}$$

$$= \phi_2(x, y), \text{ say} \cdots\cdots (6)$$

Putting $x = z$ and $y = 0$ in (5) and (6)

$$\phi_1(z, 0) = \frac{-1 + \sin z + \cos z - \sin z}{(1 - \cos z)^2} = \frac{-1}{1 - \cos z}$$

and $\phi_2(z, 0) = \dfrac{1 - \cos z - 0 + 0 - 0}{(1 - \cos z)^2} = \dfrac{1}{1 - \cos z}$

By Milne's method we have

$$F'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= \frac{-1}{1 - \cos z} - i\frac{1}{1 - \cos z} = -(1 + i)\frac{1}{1 - \cos z} = \frac{-(1 + i)}{2\sin^2\frac{z}{2}}$$

$$= \frac{-(1 + i)}{2}\operatorname{cosec}^2\frac{z}{2}$$

$$\Rightarrow F(z) = -\frac{1}{2}(1 + i)\int \operatorname{cosec}^2\frac{z}{2}\,dz$$

$$= (1 + i)\cot\frac{z}{2} + c$$

$$\Rightarrow (1 + i) f(z) = (1 + i)\cot\frac{z}{2} + c$$

$$\Rightarrow f(z) = \cot\frac{z}{2} + A, \text{ where } A = \frac{c}{1 + i} \cdots\cdots (7)$$

By the given condition $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$

$$\Rightarrow \cot\frac{\pi}{4} + A = \frac{3 - i}{2}$$

$$\Rightarrow 1 + A = \frac{3 - i}{2}$$

$$\Rightarrow A = \frac{3 - i}{2} - 1 = \frac{3 - i - 2}{2} = \frac{1 - i}{2}$$

Putting this value in (7) we get,

$$f(z) = \cot\frac{z}{2} + \frac{1}{2}(1 - i). \quad \text{(Ans)}$$

Example–58. If $f(z)$ is analytic function of z, then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f(z)|^{p-2}\,|f'(z)|^2$$ **[RUH–1988]**

Also, from it show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

[NUH–1997, 2004(Old), 2012, DUH–1984, 1986, 1988, 1990, RUH–1973, 1988]

and $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^3 = 9|f(z)|\;|f'(z)|^2$

Solution : $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}\{|f(z)|^2\}^{p/2}$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\{f(z)\,f(\bar{z})\}^{p/2}$$

$$= 4\frac{\partial}{\partial z}\{f(z)\}^{p/2}\cdot\frac{\partial}{\partial \bar{z}}\{f(\bar{z})\}^{p/2}$$

$$= 4\cdot\frac{p}{2}\cdot\{f(z)\}^{\frac{p}{2}-1}\cdot f'(z)\cdot\frac{p}{2}\{f(\bar{z})\}^{\frac{p}{2}-1}\cdot f'(\bar{z})$$

$$= p^2\{f(z)\,f(\bar{z})|^{\frac{p}{2}-1}\cdot f'(z)\,f'(\bar{z})$$

$$= p^2\{|f(z)|^2|^{\frac{p}{2}-1}\cdot|f'(z)|^2$$

$$= p^2|f(z)|^{p-2}\cdot|f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}\,|f'(z)|^2 \cdots\cdots (1)$$

2nd Part : Putting $p = 2$ in (1) we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2^2|f(z)|^{2-2}\,|f'(z)|^2$$

$$= 4|f'(z)|^2. \quad \text{(Showed)}$$

3rd Part : Putting $p = 3$ in (1) we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^3 = 3^2|f(z)|^{3-2}\,|f'(z)|^2$$

$$= 9|f(z)|\;|f'(z)|^2 \quad \text{(Showed)}$$

Example–59. If $f(z) = u + iv$ is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^7 = 49\ |f(z)|^5\ |f'(z)|^2 \qquad \text{[RUH–1996]}$$

Solution : $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^7 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\{|f(z)|^2\}^{7/2}$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\{f(z)\ f(\bar{z})\}^{7/2}$$

$$= 4\frac{\partial}{\partial z}\{f(z)\}^{7/2}\cdot\frac{\partial}{\partial \bar{z}}\{f(\bar{z})\}^{7/2}$$

$$= 4\cdot\frac{7}{2}\{f(z)\}^{5/2}\cdot f'(z)\cdot\frac{7}{2}\{f(\bar{z})\}^{5/2}\cdot f'(\bar{z})$$

$$= 49\{f(z)\ f(\bar{z})\}^{5/2}\cdot f'(z)\ f'(\bar{z})$$

$$= 49\{|f(z)|^2\}^{5/2}\cdot |f'(z)|^2$$

$$= 49\ |f(z)|^5\ |f'(z)|^2. \quad \text{(Proved)}$$

Example–60. If $f(z)$ is an analytic function of z, then show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2. \qquad \text{[NUH–2006 (Old),}$$

DUH–1986, 1990, 1998, 2004, RUH–1973, 1984, 1998]

Solution : $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}f(z)\ f(\bar{z})$

$$= 4\frac{\partial}{\partial z}\{f(z)\cdot f'(\bar{z})$$

$$= 4f'(z)\cdot f'(\bar{z})$$

$$= 4|f'(z)|^2 \quad \text{(Showed)}$$

Example–61. Show that $\psi = \log|f(z)|$ is harmonic in a region R if $f(z)$ is analytic in R and $f(z)\ f'(z) \neq 0$ in R.

Solution : We have $\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f(z)|$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\frac{1}{2}\log|f(z)|^2$$

$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log\{f(z)\cdot f(\bar{z})\}$$

$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\{\log f(z) + \log f(\bar{z})\}$$

$$= 2\frac{\partial}{\partial z}\left\{0+\frac{f'(\bar{z})}{f(\bar{z})}\right\}$$

$$= 2\times 0 = 0$$

Here ψ satisfies Laplace equation.

$\therefore\ \psi = \log|f(z)|$ is harmonic.

Example–62. If $f(z)$ is analytic function such that $f'(z) \neq 0$, then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

If $|f'(z)|$ is the product of a function of x and a function of y, then show that $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$, where α is real and β and γ are complex constants. **[JUH–1989]**

Solution : 1st Part :

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\frac{1}{2}\log|f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\log\{f'(z)\cdot f'(\bar{z})\}$$

$$= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\{\log f'(z) + \log f'(\bar{z})\}$$

$$= 2\frac{\partial}{\partial z}\left\{0+\frac{f''(\bar{z})}{f'(\bar{z})}\right\} = 2\times 0 = 0 \quad \text{(Proved)}$$

2nd Part : Given $|f'(z)|$ is the product of a function x and a function of y. Let $|f'(z)| = P(x)\cdot Q(y)$. Then from the first part,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{P(x)\cdot Q(y)\}=0$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\{\log P(x)+\log Q(y)\}=0$$

$$\Rightarrow\frac{\partial^2}{\partial x^2}\log P(x)+\frac{\partial^2}{\partial y^2}\log Q(y)=0$$

$\Rightarrow \dfrac{d^2}{dx^2}\log P(x) + \dfrac{d^2}{dy^2}\log Q(y) = 0$, since $P(x)$ and $Q(y)$ are functions of x and y alone respectively.

Let $\dfrac{d^2}{dx^2}\log P(x) = c$, then $\dfrac{d^2}{dy^2}\log Q(y) = -c$

$\Rightarrow\dfrac{d}{dx}\log P(x)=cx+d$	$\Rightarrow\dfrac{d}{dy}\log Q(y)=-cy+d'$
$\Rightarrow\log P(x)=\dfrac{1}{2}cx^2+dx+e$	$\Rightarrow\log Q(y)=-\dfrac{1}{2}cy^2+d'y+e'$
$\Rightarrow P(x)=\exp\left(\dfrac{1}{2}cx^2+dx+e\right)$	$\Rightarrow Q(y)=\exp\left(-\dfrac{1}{2}cy^2+d'y+e'\right)$

where d, e, d', e' are integrating real constants.

Now, $|f'(z)| = P(x)\cdot Q(y)$

$$=\exp\left(\frac{1}{2}cx^2+dx+e\right)\cdot\exp\left(-\frac{1}{2}cy^2+d'y+e'\right)$$

$$=e^{\frac{1}{2}cx^2+dx+e}\cdot e^{-\frac{1}{2}cy^2+d'y+e'}$$

$$=e^{\frac{1}{2}c(x^2-y^2)+dx+d'y+(e+e')}\ \ldots\ldots(1)$$

Now putting $\beta=a_1+ib_1,\ \gamma=c_1+id_1,\ z=x+iy$ we have

$$|\exp(\alpha z^2+\beta z+\gamma)|=|\exp\{\alpha(x+iy)^2+(a_1+ib_1)(x+iy)+c_1+id_1\}|$$

$$=\exp\{\alpha(x^2-y^2+2ixy)+(a_1x-b_1y)+i(b_1x+a_1y)+c_1+id_1\}|$$

$$=\left|e^{\alpha(x^2-y^2)+(a_1x-b_1y)+c_1+i(2\alpha xy+b_1x+a_1y+d_1)}\right|$$

$$=\left|e^{\alpha(x^2-y^2)+(a_1x-b_1y)+c_1}\right|\left|e^{i(2\alpha xy+b_1x+a_1y+d_1)}\right|$$

$$=\left|e^{\alpha(x^2-y^2)+a_1x-b_1y+c_1}\right|\cdot 1;\quad \because |e^{i\theta}|=1$$

$$=\left|e^{\alpha(x^2-y^2)+a_1x-b_1y+c_1}\right|\ \ldots\ldots(1)$$

From (1) and (2) we have if $f'(z)=\exp(\alpha z^2+\beta z+\gamma)$, then

$$e^{\frac{1}{2}c(x^2-y^2)+dx+d'y+(e+e')}=e^{\alpha(x^2-y^2)+a_1x-b_1y+c_1}$$

$$\Rightarrow\frac{1}{2}c(x^2-y^2)+dx+d'y+(e+e')=\alpha(x^2-y^2)+a_1x-b_1y+c_1$$

Equating the coefficients of like terms

$$\frac{1}{2}c=\alpha,\ d=a_1,\ d'=-b_1,\ e+e'=c_1$$

$\Rightarrow\alpha=\dfrac{1}{2}c=$ real, since c is real.

Thus, $f'(z)=\exp(\alpha z^2+\beta z+\gamma)$ (Showed)

Example–63. Show that the function $\psi = ln[(x-1)^2+(y-2)^2]$ is a harmonic other than the point (1, 2) and find a function ϕ such that $\phi+i\psi$ is analytic. [দেখাও যে (1, 2) বিন্দু ব্যতীত $\psi = ln[(x-1)^2+(y-2)^2]$ ফাংশনটি হারমোনিক এবং $\phi+i\psi$ বৈশ্লেষিক হলে ϕ ফাংশনটি নির্ণয় কর।]

[NU Pre–2013]

Solution : Given [দেওয়া আছে] $\psi = ln[(x-1)^2+(y-2)^2$

$$\therefore\frac{\partial\psi}{\partial x}=\frac{2(x-1)}{(x-1)^2+(y-2)^2}=\psi_1(x,y),\text{ say [ধরি] }\ldots\ldots(1)$$

$$\frac{\partial\psi}{\partial y}=\frac{2(y-2)}{(x-1)^2+(y-2)^2}=\psi_2(x,y),\text{ say [ধরি] }\ldots\ldots(2)$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\{(x-1)^2+(y-2)^2\}\cdot2-2(x-1)\cdot2(x-1)}{\{(x-1)^2+(y-2)^2\}^2}$$

$$=\frac{2(y-2)^2-2(x-1)^2}{\{(x-1)^2+(y-2)^2\}^2}\ \ldots\ldots(3)$$

$$\frac{\partial^2\psi}{\partial y^2}=\frac{\{(x-1)^2+(y-2)^2\}\cdot2-2(y-2)\cdot2(y-2)}{\{(x-1)^2+(y-2)^2\}^2}$$

$$=\frac{2(x-1)^2-2(y-2)^2}{\{(x-1)^2+(y-2)^2\}^2}\ \ldots\ldots(4)$$

(3) + (4) gives [(3) + (4) করে পাই]

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{2(y-2)^2-2(x-1)^2}{\{(x-1)^2+(y-2)^2\}^2}+\frac{2(x-1)^2-2(y-2)^2}{\{(x-1)^2+(y-2)^2\}^2}$$

$$=\frac{2(y-2)^2-2(x-1)^2+2(x-1)^2-2(y-2)^2}{\{(x-1)^2+(y-2)^2\}^2}$$

$$=0$$

Since ψ satisfy Laplace equation, so ψ is harmonic. Let ϕ is the harmonic conjugate of ψ, so that $\psi + i\phi$ is analytic. [যেহেতু ψ ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং ψ হারমোনিক। ধরি ϕ হলো ψ এর হারমোনিক অনুবন্ধী, সুতরাং $\psi + i\phi$ বৈশ্লেষিক।]

Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) ও (2) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\psi_1(z, 0) = \frac{2(z-1)}{(z-1)^2 + 2^2} \text{ and [এবং] } \psi_2(z, 0) = \frac{-4}{(z-1)^2 + 2^2}$$

By Milne's method we get [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \psi_1(z - 0) - i\psi_2(z, 0)$$

$$= \frac{2(z-1)}{(z-1)^2 + 2^2} - i\frac{-4}{(z-1)^2 + 2^2}$$

$$\Rightarrow f(z) = \int\left[\frac{2(z-1)}{(z-1)^2 + 2^2} + i4\frac{1}{(z-1)^2 + 2^2}\right]dz$$

$$\Rightarrow \psi + i\phi = \int\frac{2(z-1)}{(z-1)^2 + 2^2}\,d(z-1) + i4\int\frac{1}{(z-1)^2 + 2^2}\,dz$$

$$= ln\{(z-1)^2 + 2^2\} + i4\cdot\frac{1}{2}\tan^{-1}\left(\frac{z-1}{2}\right) + c$$

$$= ln\{(z-1)^2 + 2^2\} + i2\tan^{-1}\left(\frac{z-1}{2}\right) + c$$

$$= 2\left[\frac{1}{2}ln\{(z-1)^2 + 2^2\} + i\tan^{-1}\left(\frac{z-1}{2}\right)\right] + c$$

$$= 2\,ln\{2 + i(z-1)\} + c$$

$$= 2\,ln\{2 + i(x + iy - 1)\} + c$$

$$= 2\,ln\{2 - y + i(x-1)\} + c$$

$$= 2\left[\frac{1}{2}ln\{(2-y)^2 + (x-1)^2\} + i\tan^{-1}\left(\frac{x-1}{2-y}\right)\right] + c$$

$$= ln\{(x-1)^2 + (y-2)^2\} + 2i\tan^{-1}\left(\frac{x-1}{2-y}\right) + c_1 + ic_2$$

[where [যেখানে] $c = c_1 + ic_2$]

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\phi = 2\tan^{-1}\left(\frac{x-1}{2-y}\right) + c_2. \text{ Ans}$$

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. Define analytic function at a point. **[NUH-2012, 2015]**

Ans : A complex function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

[একটি জটিল ফাংশন f(z)কে একটি বিন্দু z_0 এ বৈশ্লেষিক বলে যদি ইহার অন্তরক শুধুমাত্র z_0 বিন্দুতে নয়, z_0 বিন্দুর প্রতিবেশের কিছু বিন্দু z বিন্দুতেও বিদ্যমান থাকে।]

2. Define entire function. **[NUH-2014]**

Ans : A complex function f(z) is said to be entire if it is analytic in the whole complex plane. [একটি জটিল ফাংশন f(z)কে entire বলে যদি ইহা সম্পূর্ণ জটিল তলে বৈশ্লেষিক হয়।]

3. What are Cauchy-Riemann partial differential equations?

Ans : If $w = f(z) = u(x, y) + iv(x, y)$ then the Cauchy-Riemann partial differential equations are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

[যদি $w = f(z) = u(x, y) + iv(x, y)$ হয় তখন কচি-রীম্যান সমীকরণগুলি হলো $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ এবং $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$]

4. Write the polar form of Cauchy-Riemann equations– **[NUH-2012]**

Ans : The polar form of Cauchy-Rienann equations are [কচি-রীম্যান সমীকরণগুলির পোলার আকার হল]

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

5. Write the polar form of Laplace equation.

OR, Write down the laplace equation in polar form of complex number. **[NUH-2013]**

Ans : The polar form of Laplace equation of complex number is [জটিল সংখ্যার ল্যাপলাস সমীকরণের পোলার আকার হল]

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

6. Can every complex function f(z) be written as
$$f(z) = u(x, y) + iv(x, y)?$$
Ans : Yes, every complex function f(z) be always written as
$$f(z) = u(x, y) + iv(x, y).$$
[হ্যাঁ, প্রত্যেক জটিল ফাংশন f(z)কে সর্বদা $f(z) = u(x, y) + iv(x, y)$ আকারে লেখা যায়]

7. When a complex valued function is continuous at a point?
Ans : A complex valued function f(z) is said to be continuous at a point z_0 if for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$, whenever $|z - z_0| < \delta$.
[একটি জটিল মানী ফাংশন f(z) কে একটি z_0 বিন্দুতে অবিচ্ছিন্ন বলে যদি প্রত্যেক প্রদত্ত $\varepsilon > 0$ এর জন্য একটি $\delta > 0$ বিদ্যমান থাকবে যেন $|f(z) - f(z_0)| < \varepsilon$ যখন $|z - z_0| < \delta$ হয়।]

8. When a maximum and minimum value of $|f(z)|$ exist?
Ans : If a function f(z) is continuous on a bounded and closed set $S \subset C$, then maximum and minimum value of $|f(z)|$ exist on S. [একটি সীমায়িত ও বদ্ধ সেট $S \subset C$ এ যদি f(z) ফাংশন অবিচ্ছিন্ন হয় তখন $|f(z)|$ এর সর্বোচ্চ ও সর্বনিম্ন মান বিদ্যমান।]

9. When a function f(z) is differentiable at a point?
OR, When a complex function f(z) is said to be differentiable at a point? **[NUH-2014]**
Ans : A function f(z) is said to be differentiable at a point z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and finite.
[একটি ফাংশন f(z) কে z_0 বিন্দুতে অন্তরীকরণযোগ্য বলে যদি $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ বিদ্যমান থাকে ও সসীম হয়।]

10. When a function is differentiable on a set?
Ans : A function is differentiable on a set if it is differentiable at each point of the set. [একটি ফাংশন একটি সেটে অন্তরীকরণযোগ্য যদি ইহা সেটের প্রত্যেক বিন্দুতে অন্তরীকরণযোগ্য হয়।]

11. Does every differentiable function is continuous?
Ans : Yes, every differentiable function is continuous.
[হ্যাঁ, প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন।]

12. Does every continuous function is differentiable?
Ans : No, every continuous function need not be differentiable. [না, প্রত্যেক অবিচ্ছিন্ন ফাংশন অন্তরীকরণযোগ্য না ও হতেপারে।]

13. Define harmonic function? **[NUH-2014]**
Ans : Any real valued function of x and y is said to be harmonic if it has continuous partial derivatives of the first and second order and satisfy the Laplace equation.
[x ও y এর যে কোন বাস্তব মানী ফাংশনকে হারমোনিক ফাংশন বলে যদি ইহার প্রথম ও দ্বিতীয় ক্রমের অবিচ্ছিন্ন আংশিক অন্তরীকরণ থাকে এবং ল্যাপলাস সমীকরণ সিদ্ধ করে।]

14. Define harmonic conjugate. **[NUH-2012]**
Ans : The function v is said to be a harmonic conjugate of u if u and v are harmonic and u, v satisfy the Cauchy-Riemann equations. [v ফাংশনকে u এর হারমোনিক অনুবন্ধী বলে যদি u ও v হারমোনিক হয় এবং u,v কচি-রীম্যান সমীকরণ সমূহ সিদ্ধ করে।]

15. When is a function analytic? **[NUH-2013]**
Ans : When the derivative of the complex function f(z) exists not only at some point z_0 but also at each point z in some neighbourhood of z_0, then the function f(z) is said to be analytic. [যখন জটিল ফাংশন f(z) এর অন্তরক শুধুমাত্র কোন বিন্দু z_0 এ বিদ্যমান থাকে না, z_0 এর কোন প্রতিবেশের প্রত্যেক বিন্দুতে বিদ্যমান থাকে, তখন f(z) ফাংশনকে বৈশ্লেষিক বলে।]

16. Give an example of a harmonic function. **[NUH-2013]**
Ans : $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is an example of a harmonic function. [হারমোনিক ফাংশনের একটি উদাহরণ]

17. What do you mean by orthogonal system [লম্বিক জোট বলিতে কি বুঝ?]
Ans : Let $u(x, y) = C_1$ and $v(x, y) = C_2$ be two families of curves. They will form orthogonal system if they intersect at right angles at each point of their intersections.

18. Why $|z|$, $|z|^2$ are not analytic?
OR. Are $|z|$, $|z|^2$ analytic?
Ans : The real functions of complex variable are now here analytic unless these are constant valued. Here $|z|$, $|z|^2$ are real valued but not constant on any domain in complex plane and so are not analytic.

EXERCISE-2

Part-A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. Define a multivalued function with example.
2. What is the meaning of the symbol $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$?
3. When a real function of complex variable is analytic?
4. Why the real valued function Re(z), Im(z), $|z|$ and $|z|^2$ are not analytic?
5. Express the right hand side of $f(z) = x^2 - y^2 - 2y + i2x + i2xy$ interms of z.'

Part-B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. State and prove the sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic. [NUH-1993, 1998, 2001, 2006, DUH-1999, 2001, 2003, CUH-2002, 2004]

   Ans : See theorem-5.
2. Obtain a set of conditions sufficient for a complex function to be analytic in a domain. [CUH-2001]

   Ans : See theorem-5.
3. State and derive the necessary condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R. [CUH-2003]

   Ans : See theorem-4.
4. Define limit, continuity and uniform continuity. [CUH-2004]

   Ans : See the definitions of art-2.2 and 2.3.
5. If f(z) is analytic in a region R, then f(z) is constant if Re f(z) is constant. [RUH-2001]

   Ans : See solved example-16.
6. State and prove the necessary condition for a function to be analytic. [NUH-2005(Old), 2007, DUH-1998, 2003, 2005, RUH-1997, 2002, 2004]

   Ans : See theorem-4.

7. Let $W = f(z) = u(x, y) + iv(x, y)$ be defined in a region R. If in R the Cauchy Riemann equations are satisfied and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous, then f(z) is analytic in R. [RUH-2001]

   Ans : See theorem-5.
8. State and prove the sufficient conditions for a function to be analytic in a region. [NUH-2006, RUH-1995]

   Ans : See theorem-5.
9. If $\lim_{z \to z_0} f(z)$ exists, then it must be unique, prove it. [RUH-1994, CUH-2002]

   Ans : See theorem-1.
10. Show that if the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = x + iy$ then the four partial derivatives u_x, v_x, u_y, v_y should exist and satisfy the equations $u_x = v_y$, $u_y = -v_x$. [NUH-2004(Old)]

    Ans : See theorem-4.
11. Does $\lim_{z \to 0} \frac{\bar{z}}{z}$ exist? [NUH-1996]

    Ans : See solved example-8.
12. Prove that $f(z) = \ln z$ has a branch point at $z = 0$. [NUH-2007]

    Ans : See solved example-2.

Part-C (Broad Questions) (বড় প্রশ্ন)

1. State and prove the necessary and sufficient conditions for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic. [CUH-2004]

   Ans : See theorem-4 and 5.
2. Prove that differentiability of f(z) implies continuity but the converse is not true in general.

   Ans : See theorem-3 and solved example-21.
3. Prove that an analytic function with constant modulas is constant. [RUH-2006]

   Ans : See solve example-26.
4. If $f(z) = u + iv$ is analytic in a region R, prove that u and v are harmonic in R.

   Ans : See Theorem of art-2.7.

5. If f(z) is analytic at a point z_0, then it must be continuous at z_0. Give an example to show that the converse of this theorem is not necessarily true. **[RUH-2001, 2004]**

Ans : See theorem-3 and Solved example-21.

6. Prove that a function which is analytic at a point is continuous there but the converse is not necessarily true. **[RUH-1996]**

Ans : See theorem-3 and solve example-21.

7. If w = f(z) = u + iv is an analytic function, show in polar form the Cauchy Riemann equations are

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ **[NUH-1997, 2003, 2004, 2008, 2010, DUH-2004, RUH-2006]**

Ans : See art-2.6.

8. If w = f(z) = u + iv is an analytic function, show in polar form the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

[NUH-1997, 2003, DUH-2004, RUH-2006]

Ans : See art 2.6

9. Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation

$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}=0$$ **[RUH-2000, 2006]**

Ans : See theorem of art 2.8

10. If f(z) = u + iv is an analytic function of z = x + iy and ϕ is any function of x and y with differential coefficient of first order, then show that

$$\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2=\left\{\left(\frac{\partial \phi}{\partial u}\right)^2+\left(\frac{\partial \phi}{\partial v}\right)^2\right\}|f'(z)|^2.$$ **[RUH-2001]**

Ans : See solved example-29

11. If p and q are functions of x and y satisfying Laplace's equation, then show that (u + iv) is analytic where $u=\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}$ and $v=\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}$. **[RUH-1999]**

Ans : See solved example-20

12. What is mean by saying that a function of a complex variable f(z) is analytic at z_0? Show that validity of the Cauchy-Riemann equation is necessary condition for analyticity, but not a sufficient condition. State a set of conditions sufficient for analyticity. **[NUH-1995]**

Ans : See definition, theorem-4, 5 and Notes of art 2.5

13. What is meant by the analyticity of a complex function at a point? Prove that the analyticity of a function at a point implies the continuity of the function at that points. Give an example to show that the converse is not necessarily true. **[NUH-1996]**

Ans : See definition of art 2.5, theorem-3 of art 2.4 and solved example-21

14. Find the necessary and sufficient conditions for f(z) = u + iv to be regular, where u and v both real. **[NUH-1998]**

Ans : See theorem-4 and 5 of art 2.5.

15. Show that the real and imaginary parts of an analytic function are harmonic functions. **[NUH-2001]**

Ans : See theorem-6 of art 2.7

16. Find the polar form of Cauchy-Riemann equations. **[CUH-2003]**

Ans : See art 2.6

17. What do you mean by saying that a complex function (i) is differentiable at z_0, (ii) is analytic at z_0? State and prove necessary conditions for f(z) to be analytic at z_0. **[NUH-2002]**

Ans : See definition in art 2.4, definition in art 2.5 and theorem-4 of art 2.5

18. State and prove the Cauchy-Riemann equations in polar form. [NUH–2004]

Ans : art–2.6

19. Show that the function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ for $z \neq 0$, $f(0) = 0$ is not differentiable at the origin although Cauchy-Riemann's equations are satisfied at that point. [DUH–1994]

Ans : See solved example–14

20. Define differentiability of a complex function. If f(z) is differentiable at z_0, show that it must be continuous at z_0. Give an example to show that the converse is not true. [DUH–1998, 2004]

Ans : See definition in art 2.4; theorem–3 and solved example–21.

21. Show that $f(z) = |z|^2$ is differentiable at the origin, but nowhere else. [DUH–2003]

Ans : See solved example–15.

22. Let $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & \text{, where } z \neq 0 \\ 0 & \text{, where } z = 0 \end{cases}$

Show that f(z) satisfies Cauchy-Riemann equations at the origin, but $f'(0)$ does not exist. [DUH–2003]

Ans : See solved example–11.

23. Ascertain whether the function f(z) defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

is differentiable at z = 0. [DUH–2005]

Ans : See solved example–11.

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CHAPTER–3
COMPLEX INTEGRATION AND RELATED THEOREMS

3.1. If the complex function f(z) is analytic, then its indefinite integral is the same as the process of real variable. Thus if f(z) is analytic function of z, and if $\int f(z)\,dz = F(z)$ then $F'(z) = f(z)$.

But in the case of definite integral, the concept of real variable can not be applied in the case of complex variable, because for the definite integral $\int_a^b f(x)\,dx$ the path of integration is always along the x-axis from x = a to x = b, or may be along the y-axis of integral $\int_a^b f(y)\,dy$, but in the case of a complex function f(z) the path of the definite integral $\int_a^b f(z)\,dz$ can be along any curve from z = a to z = b. So the value depends upon the path (curve) of integration. This variation in values can be made to disappear if the different curve (paths) from a to b are regular curves.

3.2. Some definitions :

Continuous arc : Let z be a point on an arc such that

$$z = z(t) = \phi(t) + i\psi(t) \cdots\cdots (1)$$

$$\Rightarrow x = \phi(t) \text{ and } y = \psi(t) \cdots\cdots (2)$$

If $\phi(t)$ and $\psi(t)$ are both real continuous functions of the real variable t defined in the range $\alpha \le t \le \beta$, then the arc is said to be a continuous arc.

Multiple point of the arc : If the above equation (1) and (2) are satisfied by more than one value of t, then we say that the point z = (x, y) is a multiple point of the arc.

Jordan arc : A continuous arc with out multiple points is called a Jordan arc.

Regular arc : An arc of a Jordan curve is said to be regular if $\phi'(t)$ and $\psi'(t)$ are also continuous in the given range.

Continuous Jordan curve : If a Jordan curve consists of a chain of finite number of continuous arcs then it is called a continuous Jordan curve.

Differentiable : An arc is said to be differentiable if $z'(t)$ exists and continuous.

Closed curve : If the starting and ending points of a curve coincide then the curve is called a closed curve.

Simple closed curve : A closed curve which does not intersect itself anywhere is called a simple closed curve.

Simply connected region : A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R.

Multiply connected region : A region R which is not simply connected is called multiply connected.

Contour : A Jordan curve consisting of continuous chain of a finite number of regular arcs is called a contour.

or A contour is a piece wise smoth path, that is a curve consisting of finite number of smoth arcs joined end to end.

Let A be the starting point of the first arc and B be the end point of the last arc. Then the integral along such a curve is written as $\int_{AB} f(z)\,dz$.

Closed contour : If the starting point A of the first arc and the ending point B of the last arc coincide, then the contour is said to be the closed contour. In this case the integral is written as

$$\int_C f(z)\,dz \text{ or } \oint_C f(z)\,dz.$$

Partitions : Let the closed interval [a, b] is divided into n sub-intervals

$$[a = t_0, t_1], [t_1, t_2], [t_2, t_3], \cdots, [t_{n-1}, t_n = b]$$

such that $a = t_0 < t_1 < t_2 < \cdots < t_n = b$

Then the set $P = \{t_0, t_1, \cdots, t_n\}$ is called the partition of the interval [a, b].

The greatest of the numbers

$$t_1 - t_0, t_2 - t_1, \cdots, t_n - t_{n-1}$$

is called the norm of P which is denoted by $|P|$.

Rectifiable arc : An arc is said to be rectifiable if it has a finite arc length.

Length of arc : Let $\gamma(t)$ be a smoth arc. Then the real valued function

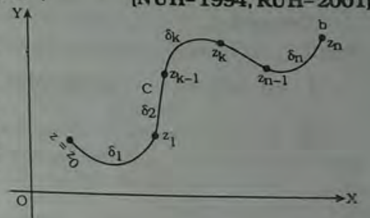
$$|\gamma'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over [a, b]. The length of the arc γ is defined as

$$L = \int_a^b |\gamma'(t)|\,dt.$$

3.3. Complex line integral : **[NUH–1994, RUH–2001]**

Let the complex function f(z) be continuous at all points of a rectifiable curve C whose starting and ending points are a and b respectively. Let us divide C into n parts at the points $z_1, z_2, \cdots, z_n$. Let



[মনেকরি দৈর্ঘ্য নির্ণয় যোগ্য একটি বক্ররেখা C যার শুরু এবং শেষ বিন্দু যথাক্রমে a ও b এর সকল বিন্দুতে জটিল ফাংশন f(z) অবিচ্ছিন্ন। $z_1, z_2, \ldots, z_n$ বিন্দুতে C কে n সংখ্যাক অংশে বিভক্ত করি। ধরি]

$$S_n = (z_1 - z_0) f(\delta_1) + (z_2 - z_1) f(\delta_2) + \cdots + (z_n - z_{n-1}) f(\delta_n)$$

$$\Rightarrow S_n = \sum_{k=1}^{n} (z_k - z_{k-1}) f(\delta_k), \text{ where } z_{k-1} \le \delta_k \le z_k \cdots\cdots (1)$$

If the divisions n increases in such a way that the chord length $|\delta_k| \to 0$ then S_n tends to a limit uniquely which does not depend on the path for any mode of partition and we call this the integral of f(z) over C from a to b and write as [যদি ভাগ দৈর্ঘ্য n এমনভাবে বৃদ্ধি পায় যেন জ্যা দৈর্ঘ্য $|\delta_k| \to 0$ হয় তখন S_n অনন্যভাবে একটি লিমিটে ধাবিত হবে যাহা যে কোন প্রকৃতির বিভাজনের পথের উপর নির্ভর করে না এবং আমরা ইহা C এর উপর a হতে b এ f(z) এর ইন্টিগ্রাল বলব এবং লিখব]

$$\int_a^b f(z)\,dz \text{ or } \int_C f(z)\,dz$$

$$\therefore \int_a^b f(z)\,dz = \int_C f(z)\,dz = \sum_{k=1}^{\infty} (z_k - z_{k-1}) f(\delta_k) \cdots\cdots (2)$$

This is called the complex line integral or line integral of f(z) along the curve C. This is also called the definite integral of f(z) from a to b along the curve C. [ইহাকে C বক্ররেখা বরাবর f(z) এর জটিল রৈখিক ইন্টিগ্রাল বা লাইন ইন্টিগ্রাল বলা হয়। ইহাকে C বক্ররেখা বরাবর a হতে b এ f(z) এর নির্দিষ্ট ইন্টিগ্রাল ও বলে ।]

Note : If f(z) is analytic at all points of a region R and C is any curve lying in R, then f(z) must be integrable along C.

3.4. An inequality for complex integrals (ML inequality) :

Theorem–1. If a function f(z) is continuous on a contour C of length L, and if M be the upper bound of $|f(z)|$ on C then

$$\left|\int_C f(z)\,dz\right| \le ML \qquad \textbf{[NUH–2011, DUH–2005, RUH–2000, 2006]}$$

OR, If f(z) is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \le M$ on C, then

$$\left|\int_C f(z)\,dz\right| \le ML \qquad \textbf{[NUH–2004, 2014, RUH–2000, 2006]}$$

Proof : From the definition of complex integral we have [জটিল যোগজের সংজ্ঞা হতে পাই]

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1})\; f(\delta_k) \;\cdots\cdots (1)$$

where [যেখানে] $z_{k-1} \le \delta_k \le z_k$

Let $x = \phi(t)$, $y = \psi(t)$ be the equation of the curve C. Then its length is [ধরি C বক্ররেখার সমীকরণ $x = \phi(t)$, $y = \psi(t)$. তখন ইহার দৈর্ঘ্য]

$$L = \int ds = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt \cdots\cdots (2)$$

Also $z = x + iy$ gives $dz = dx + i\,dy$ [আরো $z = x + iy$ দেয় $dz = dx + i\,dy$]

$$\Rightarrow |dz| = |dx + i\,dy| = \sqrt{dx^2 + dy^2}$$

$$\therefore \int_C |dz| = \int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= L, \text{ by}(2) \cdots\cdots (3)$$

$$\text{Now [এখন]} \left|\sum_{k=1}^{n} (z_k - z_{k-1})\, f(\delta_k)\right| \le \sum_{k=1}^{n} |(z_k - z_{k-1})\, f(\delta_k)|$$

$$\Rightarrow \left|\sum_{k=1}^{n} (z_k - z_{k-1})\, f(\delta_k)\right| \le \sum_{k=1}^{n} |z_k - z_{k-1}|\; |f(\delta_k)|$$

$$\Rightarrow \lim_{n\to\infty} \left|\sum_{k=1}^{n} (z_k - z_{k-1})\, f(\delta_k)\right| \le \lim_{k\to\infty} \sum_{k=1}^{n} |z_k - z_{k-1}|\; |f(\delta_k)|$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\; |dz|$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \le M \int_C |dz|, \quad \text{by hypothesis}$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \le ML, \quad \text{by(3).} \quad \text{(Proved)}$$

3.5. Cauchy's Fundamental theorem :

Cauchy-Goursat theorem (Proof by triangle method) :

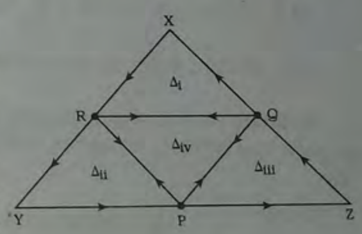
[NUH(Pre)–2006, 2014]

Theorem–2. Let f(z) be analytic in a region D bounded by a triangle C and on its boundary (i. e. on the sides of the triangle).

Then $\oint_C f(z)\,dz = 0$.

Proof : Let C be the triangle XYZ in the Argand plane and we denote it by Δ. Let P, Q, R be the mid points of the sides YZ, ZX and XY respectively.

Join these points so that the triangle is divided into four congruent triangles. Let they are Δ_i, Δ_{ii}, Δ_{iii} and Δ_{iv} respectively. Given that f(z) is analytic inside and on the boundary of the triangle XYZ.



[মনেকরি আরগান্ড তলে C ত্রিভুজটি XYZ এবং আমরা ইহাকে Δ দ্বারা চিহ্নিত করব। ধরি P, Q, R হল যথাক্রমে YZ, ZX ও XY বাহুর মধ্যবিন্দু। বিন্দুগুলি যোগকরি যেন ত্রিভুজটি চারটি সর্বসম ত্রিভুজে বিভক্ত হয়। ধরি তারা যথাক্রমে Δ_i, Δ_{ii}, Δ_{iii} ও Δ_{iv}. দেওয়া আছে XYZ ত্রিভুজের ভিতরে ও সীমানায় f(z) বৈশ্লেষিক ।]

$$\therefore \oint_C f(z)\,dz = \oint_{XYZX} f(z)\,dz$$
$$= \int_{QXR} f(z)\,dz + \int_{RYP} f(z)\,dz + \int_{PZQ} f(z)\,dz$$
$$= \left\{\int_{QXR} f(z)\,dz + \int_{RQ} f(z)\,dz\right\} + \left\{\int_{RYP} f(z)\,dz + \int_{PR} f(z)\,dz\right\}$$
$$+ \left\{\int_{PZQ} f(z)\,dz + \int_{QP} f(z)\,dz\right\} + \int_{QR} f(z)\,dz + \int_{RP} f(z)\,dz + \int_{PQ} f(z)\,dz$$
$$= \int_{QXRQ} f(z)\,dz + \int_{RYPR} f(z)\,dz + \int_{PZQP} f(z)dz + \int_{QRPQ} f(z)\,dz \;\cdots\cdots (1)$$

where $\int_{RQ} f(z)\,dz = -\int_{QR} f(z)\,dz,\ \int_{PR} f(z)\,dz = -\int_{RP} f(z)\,dz$

and $\int_{QP} f(z)\,dz = -\int_{PQ} f(z)\,dz$

Thus (1) becomes [অতএব (1) দাঁড়ায়]

$$\oint_{\Delta} f(z)\,dz = \oint_{\Delta_i} f(z)\,dz + \oint_{\Delta_{ii}} f(z)\,dz + \oint_{\Delta_{iii}} f(z)\,dz + \oint_{\Delta_{iv}} f(z)\,dz \;\cdots\cdots (2)$$

Among the four terms let the first terms has the largest value (if there are two or more such terms we also consider the first term). Thus (2) becomes by taking modulas [চারটি পদের মধ্যে ধরি প্রথম পদটির মান বৃহৎ। অতএব পরমমান নিয়ে (2) দাঁড়ায়]

$$\left|\oint_{\Delta} f(z)\,dz\right| \le \left|\oint_{\Delta_i} f(z)\,dz\right| + \left|\oint_{\Delta_{ii}} f(z)\,dz\right| + \left|\oint_{\Delta_{iii}} f(z)\,dz\right| + \left|\oint_{\Delta_{iv}} f(z)\,dz\right|$$
$$\Rightarrow \left|\oint_{\Delta} f(z)\,dz\right| \le \left|\oint_{\Delta_i} f(z)\,dz\right| + \left|\oint_{\Delta_i} f(z)\,dz\right| + \left|\oint_{\Delta_i} f(z)\,dz\right| + \left|\oint_{\Delta_i} f(z)\,dz\right|$$
$$\Rightarrow \left|\oint_{\Delta} f(z)\,dz\right| \le 4\left|\oint_{\Delta_1} f(z)\,dz\right| \;\cdots\cdots (3)$$

Again by joining the midpoints of sides of Δ_1 we get a triangle Δ_2 such that [আবার Δ_1 এর বাহুগুলির মধ্যবিন্দু যোগকরে আমরা একটি ত্রিভুজ Δ_2 পাই যেন]

$$\left|\oint_{\Delta_1} f(z)\,dz\right| \le 4\left|\oint_{\Delta_2} f(z)\,dz\right| \;\cdots\cdots (4)$$

By (4), (3) becomes [(4) দ্বারা (3) দাঁড়ায়]

$$\left|\oint_{\Delta} f(z)\,dz\right| \le 4^2\left|\oint_{\Delta_2} f(z)\,dz\right| \;\cdots\cdots (5)$$

Continuing the above process upto n times we get [উপরের পদ্ধতিটি n বার করে পাই]

$$\left|\oint_{\Delta} f(z)\,dz\right| \le 4^n\left|\oint_{\Delta_n} f(z)\,dz\right| \;\cdots\cdots (6)$$

When $n \to \infty$ then the triangle Δ_n shirnks to a point, say z_0 which lies with in every triangles $\Delta_1, \Delta_2, \cdots$ Now f(z) is analytic at z_0 since z_0 lies inside or on the boundary. Thus we have [যখন $n \to \infty$ তখন Δ_n ত্রিভুজটি একটি বিন্দুতে পরিণত হয়, ধরি z_0 যাহা $\Delta_1, \Delta_2, \cdots$ প্রত্যেক ত্রিভুজের মধ্যে অবস্থিত। এখন z_0 এ f(z) বৈশ্লেষিক, যেহেতু z_0 ভিতরে অথবা সীমানায় অবস্থিত। অতএব আমরা পাই।]

$$f(z) = f(z_0) + (z - z_0)\, f'(z_0) + (z - z_0)\eta \;\cdots\cdots (7)$$

where for given any $\varepsilon > 0$, we can find a $\delta > 0$ such that [যেখানে প্রদত্ত $\varepsilon > 0$ এর জন্য আমরা একটি $\delta > 0$ পাই যেন]

$$|\eta| < \varepsilon \text{ whenever } |z - z_0| < \delta \;\cdots\cdots (8)$$

Integrating (7) over the boundary Δ_n we get [Δ_n সীমানার উপর (7) কে যোজিত করে পাই]

$$\int_{\Delta_n} f(z)\,dz = \int_{\Delta_n} f(z_0)\,dz + \int_{\Delta_n} (z - z_0)\, f'(z_0)\,dz + \int_{\Delta_n} (z - z_0)\,\eta\,dz$$
$$= 0 + 0 + \int_{\Delta_n} (z - z_0)\,\eta\,dz$$
$$\Rightarrow \left|\int_{\Delta_n} f(z)\,dz\right| = \left|\int_{\Delta_n} (z - z_0)\,\eta\,dz\right| \;\cdots\cdots (9)$$

Let S and S_n are the perimeters of the triangle Δ and Δ_n respectively. Then $S_1 = \frac{S}{2}, S_2 = \frac{S_1}{2} = \frac{S}{2^2}$ and so on. [ধরি S ও S_n হল যথাক্রমে ত্রিভুজ Δ ও Δ_n এর পরিসীমা। তখন $S_1 = \frac{S}{2}, S_2 = \frac{S_1}{2} = \frac{S}{2^2}$ এবং এইরূপ।]

Thus [অতএব], $S_n = \frac{S}{2^n}$. Let z be any point on Δ_n, then [:ধরি Δ_n এর উপর z যে কোন বিন্দু, তখন]

$$|z - z_0| < \frac{S}{2^n} < \delta \;\cdots\cdots (10)$$

Now by (8) and (10), equation (9) becomes [এখন (8) ও (10) দ্বারা (9) দাঁড়ায়]

$$\left|\int_{\Delta_n} f(z)\, dz\right| \leq \frac{S}{2^n}\cdot\varepsilon\cdot\frac{S}{2^n} = \frac{\varepsilon S^2}{4^n}$$

Putting this results in (6) we get, [এই ফলাফল (6) এ বসাইয়া পাই]

$$\left|\oint_{\Delta} f(z)\, dz\right| \leq 4^n \cdot \frac{\varepsilon S^2}{4^n} = \varepsilon S^2$$

Since ε is arbitrary so we have [যেহেতু ε ইচ্ছামূলক সুতরাং আমরা পাই]

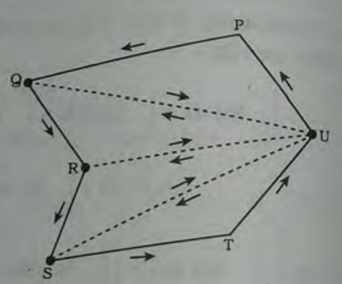
$$\oint_{\Delta} f(z)\, dz = 0$$

That is, $\oint_C f(z)\, dz = 0$ (Proved)

Cauchy-Goursat theorem for any closed polygon. **[RUH–2004]**

Theorem–3. Let f(z) be analytic in a region D bounded by a closed polygon C and on its boundary, then $\oint_C f(z)\, dz = 0$.

Proof : Let us consider a closed polygon PQRSTUP as shown in the adjacent figure. By joining the lines QU, RU and SU the polygon is subdivided into triangles. Then by Cauchy's theorem for triangles and the fact that the integrals along QU and UQ, RU and UR, SU and US cancel, we have



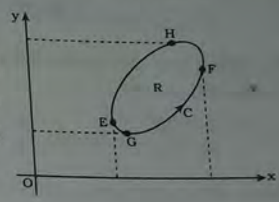
[পাশের চিত্রে দেখানো মত একিট বন্ধ বহুভুজ PQRSTUP বিবেচনা করি। QU, RU এবং SU রেখাগুলি যোগ করায় বহুভুজটি ত্রিভুজ সমূহে উপবিভক্ত হয়েছে। তখন ত্রিভুজের জন্য কচির উপপাদ্য দ্বারা এবং QU ও UQ, RU ও UR, SU ও US বরাবর ইন্টিগ্রাল বর্জিত হয়েছে ঘটনায় আমরা পাই]

$$\int_{\text{PQRSTUP}} f(z)\, dz = \int_{\text{PQUP}} f(z)\, dz + \int_{\text{QRUQ}} f(z)\, dz + \int_{\text{RSUR}} f(z)\, dz + \int_{\text{STUS}} f(z)\, dz$$
$$= 0 + 0 + 0 + 0$$

$$\Rightarrow \oint_C f(z)\, dz = 0 \quad \text{(Proved)}$$

3.6. Green's Theorem–4 :

If P(x, y) and Q(x, y) be continuous and have continuous partial derivatives in a region R and on its boundary C, then



$$\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

✪ The Theorem is valid for both simply and multiply connected regions.

Cauchy's Theorem or Cauchy's Fundamental Theorem or Cauchy's Integral Theorem.

Theorem–5. If f(z) is analytic in a region R and on its closed boundary C with derivative f′(z) which is continuous at all points inside R and on C, then [একটি এলাকা R এর ভিতর এবং ইহার বদ্ধ সীমা C এর উপর যদি একটি ফাংশন f(z), R এর ভিতর সকল বিন্দুতে ও C এর উপর অবিচ্ছিন্ন অন্তরক f′(z) সহ বৈশ্লেষিক হয় তবে]

$$\oint_C f(z)\, dz = 0$$

[NUH–95, 97, 04 (Old), 06, 11, NU(Phy)–04, RUH–99]

Proof : Let [ধরি] $z = x + iy$.

Then [তখন] $\frac{\partial z}{\partial x} = 1$ and [এবং] $\frac{\partial z}{\partial y} = i$.

Given, f(z) is analytic and f′(z) is continuous. [দেওয়া আছে f(z) বৈশ্লেষিক এবং f′(z) অবিচ্ছিন্ন।]

$$\frac{\partial}{\partial z}[f(z)] = \frac{\partial}{\partial z}(u + iv), \quad \text{where [যেখানে] } f(z) = u + iv = u(x, y) + iv(x, y)$$

$$\Rightarrow \frac{\partial}{\partial x}(u + iv)\cdot\frac{\partial x}{\partial z} = \frac{\partial}{\partial y}(u + iv)\cdot\frac{\partial y}{\partial z};$$

Since f′(z) exist so f′(z) will be independent of manner.

$$\Rightarrow \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot 1 = \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\cdot\frac{1}{i}$$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ldots\ldots (1) \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ldots\ldots (2)$$

which are continuous inside R and on C. [যাহা R এর ভিতর এবং C এর উপর অবিচ্ছিন্ন।]

$\therefore$ By applying Green's theorem we get [গ্রীনের উপপাদ্য প্রয়োগ করে পাই]

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

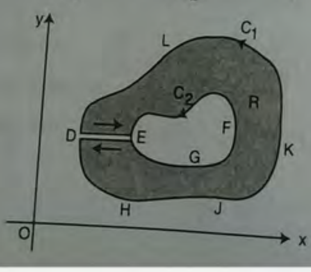
$$= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy, \quad \text{[by (1) and (2)]}$$

$$= 0 + i0 = 0$$

$$\therefore \oint_C f(z)\,dz = 0 \qquad \text{(Proved).}$$

Theorem–6. Let f(z) be analytic in a region bounded by two simple closed curves C_1 and C_2(C_2 lies inside C_1) and on these curves then [দুই সরল বদ্ধ বক্ররেখা C_1 এবং C_2 (C_1 এর ভিতর C_2 অবস্থিত) দ্বারা সীমাবদ্ধ একটি এলাকার ভিতর এবং উহাদের উপর যদি f(z) বৈশ্লেষিক হয় তখন]

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz. \text{ **[NU(Pre)–2006, 2014]**}$$



Proof : We construct a cros-cut DE. Since f(z) is analytic in the region R, so by Cauchy's theorem we have [আমরা একটি ক্রশ কাট DE গঠন করি। যেহেতু R এর মধ্যে f(z) বৈশ্লেষিক, সুতরাং কচির উপপাদ্য দ্বারা পাই]

$$\int_{DEFGEDHJKLD} f(z)\,dz = 0$$

$$\text{or, } \int_{DE} f(z)\,dz + \int_{EFGE} f(z)\,dz + \int_{ED} f(z)\,dz + \int_{DHJKLD} f(z)\,dz = 0 \ldots\ldots (1)$$

The integrals $\int_{DE} f(z)\,dz$ and $\int_{ED} f(z)\,dz$ are same by in opposite direction.

$$\therefore \int_{DE} f(z)\,dz = -\int_{ED} f(z)\,dz$$

$$\Rightarrow \int_{DE} f(z)\,dz + \int_{ED} f(z)\,dz = 0$$

Thus, from (1) we get [অতএব (1) হতে পাই]

$$\int_{EFGE} f(z)\,dz + \int_{DHJKLD} f(z)\,dz = 0$$

$$\Rightarrow \int_{DHJKLD} f(z)\,dz = -\int_{EFGE} f(z)\,dz = \int_{EGFE} f(z)\,dz$$

$$\Rightarrow \oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz. \text{ **(Proved)**}$$

Cauchy's integral formula :

Theorem–7. Let f(z) be analytic inside and on a simple closed curve C. If a is any point inside C, then [মনে করি একটি সরল বদ্ধ বক্ররেখা C এর উপর ও ভিতর f(z) বৈশ্লেষিক। যদি C এর ভিতর a যে কোন বিন্দু হয় তবে]

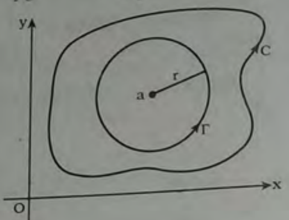
$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - a}\,dz$$

[NUH–95, 97, 99, 04, 05, 06 (Old), 07, 12(Old), 14,NU(Pre)–08, 13 DUH–75, 83, 85, 88, 91, 2000, 04, 06, RUH–73, 75, 77, 02, 06, CUH–81, 86, 87, JUH–1983, 85, 87]

where C is traversed in the positive sense (i. e. in the counter-clock wise direction). [যেখানে C ধনাত্মক দিকে পরিভ্রমণ করে।]

Proof : We know that if f(z) is analytic in a region bounded by two simple closed curves C and C_1 (C_1 lies inside C) and on these curves then [আমরা জানি যে, যদি f(z) দুইটি সরল বদ্ধ বক্ররেখা C এবং C_1 (C_1, C এর ভিতর) দ্বারা সীমাবদ্ধ এলাকায় এবং এই বক্ররেখাগুলির উপর বৈশ্লেষিক হয় তখন]

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz. \;\cdots\cdots (1)$$



Here the function $\frac{f(z)}{z-a}$ is analytic inside and on C except at the point z = a. [এখানে $\frac{f(z)}{z-a}$ ফাংশনটি z = a বিন্দু ব্যতীত C এর উপর এবং ইহার ভিতর বৈশ্লেষিক।]

$$\therefore \oint_C \frac{f(z)}{z-a}\, dz = \oint_\Gamma \frac{f(z)}{z-a}\, dz \;\cdots\cdots (2); \quad \text{By (1).}$$

where Γ is a circle with centre a and radius r. [যেখানে Γ একটি বৃত্ত যার কেন্দ্র a এবং ব্যাসার্ধ r]

$$\therefore |z-a| = r \Rightarrow z - a = re^{i\theta},\ 0 \le \theta \le 2\pi$$

$$\Rightarrow z = a + re^{i\theta} \Rightarrow dz = 0 + ire^{i\theta}\, d\theta = ire^{i\theta}\, d\theta$$

Putting these in values (2) we get [এই মানগুলি (2) এ বসাইয়া পাই]

$$\oint_C \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a + r\, e^{i\theta})}{r\, e^{i\theta}}\, ire^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

Taking limit $r \to 0$ on both sides and making use of the continuity of f(z), we get [উভয় পক্ষে লিমিট $r \to 0$ নিয়ে এবং f(z) এর অবিচ্ছিন্নতা বিবেচনায় রেখে পাই]

$$\oint_C \frac{f(z)}{z-a}\, dz = \lim_{r \to 0} i \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

$$= i\, f(a) \int_0^{2\pi} d\theta = i\, f(a).\ [\theta]_0^{2\pi} = i\, 2\pi\, f(a)$$

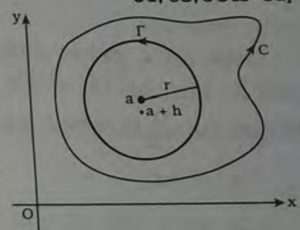
$$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \qquad \text{(Proved)}$$

3.7. Cauchy's integral formula for the first derivative.

Theorem–8. Let f(z) be analytic inside and on a simple closed curve C and a is a point inside C. Then [মনে করি একটি সরল বদ্ধ বক্ররেখা C এর ভিতর ও উপর f(z) বৈশ্লেষিক এবং C এর অভ্যন্তরে a একটি বিন্দু। তখন]

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz.$$

[NUH–99, 03, 12, 15, NU(Pre)– 11 DUH–1987, 88, 89, 90, 04, 05, RUH–73, 80, 82, 84, 85, 95, 97, 01, 03, JUH–91]



Proof : Let a be a point inside C and a + h be a neighbouring point of a inside C From Cauchy's integral formula we know that

[মনে করি C এর ভিতর a একটি বিন্দু এবং C এর ভিতরে এর প্রতিবেশী বিন্দু a + h. কচির যোজিত সূত্র হতে আমরা জানি]

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \;\cdots\cdots (1)$$

Applying (1) we can write [(1) প্রয়োগ করে আমরা লিখতে পারি]

$$\frac{f(a+h) - f(a)}{h}$$

$$= \frac{1}{2\pi i} \oint_C \frac{1}{h} \left(\frac{1}{z - (a+h)} - \frac{1}{z-a} \right) f(z)\, dz$$

$$= \frac{1}{2\pi i} \oint_C \left(\frac{\cancel{z} - a - \cancel{z} + a + h}{h(z-a)\,(z-a-h)} \right) f(z)\, dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{1}{(z-a)\,(z-a-h)}\, f(z)\, dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{z - a}{(z - a)^2 (z - a - h)} f(z)\, dz$$

$$= \frac{1}{2\pi i} \oint_C \left[\frac{1}{(z - a)^2} + \frac{h}{(z - a)^2 (z - a - h)}\right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^2} dz + \frac{1}{2\pi i} \oint_C \frac{hf(z)\, dz}{(z - a)^2 (z - a - h)}$$

$$\Rightarrow \left|\frac{f(a + h) - f(a)}{h} - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^2} dz\right|$$

$$= \left|\frac{1}{2\pi i} \oint_C \frac{hf(z)\, dz}{(z - a)^2 (z - a - h)}\right|$$

$$= \frac{|h|\ |f(z)| \oint_C |dz|}{|2\pi i|\ |z - a - h|\ |z - a|^2} \text{ ...... (2)}$$

Here f(z) is continuous on C, so it is bounded. Then there exist a constant M such that $|f(z)| \leq M$. We choose h so small in absolute value that $a + h$ lies in Γ and $|h| < \frac{r}{2}$, where the equation of the circle Γ is

[এখানে, f(z), C এর উপর অবিচ্ছিন্ন, সুতরাং ইহা সীমাবদ্ধ। তখন একটি ধ্রুবক M থাকবে যেন $|f(z)| \leq M$ হয়। আমরা h কে এত ছোট পছন্দ করব যেন $a + h$ থাকে Γ এর ভিতর এবং $|h| < \frac{r}{2}$ হয় যেখানে Γ বৃত্তের সমীকরণ $|z - a| = r$.]

$$|z - a| = r.$$

$$\therefore |z - a - h| \geq |z - a| - |h| > r - \frac{r}{2} = \frac{r}{2}$$

The length of Γ [Γ এর দৈর্ঘ্য] $= \oint_C |dz| = 2\pi r$

Now $\dfrac{|h|\,|f(z)|\,|dz|}{|2\pi i|\,|z - a - h|\,|z - a|^2} \leq \dfrac{|h| M 2\pi r}{2\pi \frac{r}{2} \cdot r^2} = \dfrac{2|h|M}{r^2} \to 0$ as $h \to 0$.

$\therefore$ Taking limit $h \to 0$ in (2) we get [লিমিট $h \to 0$ নিয়ে (2) হতে পাই]

$$\left|\frac{f(a + h) - f(a)}{h} - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^2} dz\right| = 0, \text{ when [যখন] } h \to 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^2} dz$$

$$\Rightarrow f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^2} dz \qquad \text{(Proved)}$$

3.8. Higher derivatives of an analytic function :

Theorem–9. Let f(z) be analytic inside and on the boundary C of a simply-connected region R. Then f(z) has, at every point $z = a$ of R, derivatives of all orders and their values are given by [মনে করি একটি সরল সংযুক্ত অঞ্চল R এর অভ্যন্তরে এবং সীমানা C এর উপরে f(z) বৈশ্লেষিক হয়। তখন R এর প্রত্যেক বিন্দু $z = a$ এ f(z) এর সকল ক্রমের অন্তরক আছে এবং তাদের মান প্রদত্ত হয়]

$$f^n(a) = \frac{\underline{|n}}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}}, \text{ where [যেখানে] } n = 0, 1, 2, 3, \cdots$$

[DUH–89, 90, 01, 03, JUH–86, 88, 89, 90, RUH–85]

Proof : Given f(z) is analytic inside and on the boundary C. From Cauchy's integral formula, and also from Cauchy's integral formula for first derivates we have [দেওয়া আছে C এর অভ্যন্তরে এবং সীমানার উপর f(z) বৈশ্লেষিক, কচির যোজিত সূত্র হতে এবং প্রথম অন্তরকের জন্য কচির যোজিত সূত্র হতে পাই]

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz \text{ ...... (1)}$$

$$\text{and [এবং] } f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^2} dz \text{ ...... (2)}$$

where $\frac{f(z)}{z - a}$ is not analytic at $z = a$ inside C. [যেখানে C এর ভিতর $z = a$ এ $\frac{f(z)}{z - a}$ বৈশ্লেষিক না]

From (1) and (2) we see that the given problem is true for $n = 0$ and 1.

We shall now prove the theorem by induction method. Let the theorem is true for $n = m$. Then

[(1) ও (2) হতে দেখি যে প্রদত্ত সমস্যাটি $n = 0$ ও 1 এর জন্য সত্য। এখন আমরা আরোহী পদ্ধতিতে উপপাদ্যটি প্রমাণ করব। ধরি উপপাদ্যটি $n = m$ এর জন্য সত্য। তখন]

$$f^{(m)}(a) = \frac{\underline{|m}}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{m+1}} dz \text{ ...... (3)}$$

$$\therefore \frac{f^{(m)}(a + h) - f^{(m)}(a)}{h}$$

$$= \frac{\underline{|m}}{2\pi i\, h}\left[\oint_C \frac{f(z)dz}{(z - a - h)^{m+1}} - \oint_C \frac{f(z)}{(z - a)^{m+1}} dz\right], \text{ by (3)}$$

$$= \frac{|\underline{m}}{2\pi i h} \oint_C \frac{1}{(z-a)^{m+1}} \left\{ \frac{1}{\left(\frac{z-a-h}{z-a}\right)^{m+1}} - 1 \right\} f(z)\, dz$$

$$= \frac{|\underline{m}}{2\pi i\, h} \oint_C \frac{1}{(z-a)^{m+1}} \left\{ \left(1 - \frac{h}{z-a}\right)^{-(m+1)} - 1 \right\} f(z)\, dz$$

$$= \frac{|\underline{m}}{2\pi i\, h} \oint_C \frac{1}{(z-a)^{m+1}} \left[\left\{ 1 + (m+1)\frac{h}{z-a} + \frac{(m+1)(m+2)}{|\underline{2}} \left(\frac{h}{z-a}\right)^2 + \cdots \right\} - 1 \right] f(z)\, dz$$

$$= \frac{|\underline{m}}{2\pi i\, h} \oint_C \frac{1}{(z-a)^{m+1}} \left[(m+1)\frac{h}{z-a} + \frac{(m+1)(m+2)}{|\underline{2}} \cdot \frac{h^2}{(z-a)^2} + \cdots \right] f(z)\, dz$$

Now taking limits $h \to 0$ on both sides

$$\underset{h \to 0}{\text{Lt}} \frac{f^{(m)}(a+h) - f^{(m)}(a)}{h} = \frac{|\underline{m}}{2\pi i} \oint_C \frac{1}{(z-a)^{m+1}} \cdot \frac{m+1}{z-a} f(z)\, dz$$

$$\Rightarrow f^{(m+1)}(a) = \frac{|\underline{m+1}}{2\pi i} \oint_C \frac{1}{(z-a)^{m+2}} f(z)\, dz$$

Thus, the theorem is true for $n = m + 1$. Since the theorem is true for $n = 1$, so it is true for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, and so on. Thus, for any integral value of n (0 or + ve) we have [অতএব, উপপাদ্যটি $n = m + 1$ এর জন্য সত্য। যেহেতু উপপাদ্যটি $n = 1$ এর জন্য সত্য, সুতরাং ইহা $n = 1 + 1 = 2$, $n = 2 + 1 = 3$ এর জন্য সত্য এবং এভাবে আরো। অতএব, n এর যেকোন পূর্ণ মানের জন্য (0 বা + ve) আমরা পাই]

$$f^{(n)}(a) = \frac{|\underline{n}}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz. \qquad \text{(Proved)}$$

3.9. Morera's Theorem (The converse of Cauchy's theorem) [মরিরার উপপাদ্য] ঃ [NUH–1999, 2003]

Theorem–10. If f(z) is continuous in a simply-connected region R and if $\oint_C f(z)\, dz = 0$ around every simple closed curve C in R, then f(z) is analytic in R. [একটি সরল সংযুক্ত এলাকা R এ যদি f(z) অবিচ্ছিন্ন হয় এবং R এর মধ্যে প্রত্যেক সরল বদ্ধ বক্ররেখা C এর চারিদিকে যদি $\oint_C f(z)\, dz = 0$ হয়, তখন f(z), R এ বৈশ্লেষিক।] [NUH–1999, RUH–1998, 2001, 2006]

Proof : Let a be a fixed point and z any variable point in the region R, then the value of the integral

$$\int_a^z f(z)\, dz \;\cdots\cdots (1)$$

along any curve in R joining a to z is the same, since $\oint_C f(z)\, dz = 0$.

Since the integral (1) is independent of the curve joining a to z and depends only upon these points only, therefore taking ξ as the variable of integration, we can write

[প্রমাণ ঃ মনে করি R এলাকায় a একটি নির্দিষ্ট বিন্দু এবং z যেকোন চলমান বিন্দু। তখন R এলাকায় a হতে z এ সংযোগকারী যে কোন বক্ররেখা বরাবর

$$\int_a^z f(z)\, dz \;\cdots\cdots (1)$$

যোগজের মান একই হবে, যেহেতু $\oint_C f(z)\, dz = 0$.

সেহেতু (1) নং যোগজটি a হতে z এ সংযোগকারী বক্ররেখার উপর অনির্ভরশীল এবং শুধুমাত্র এই বিন্দুগুলির উপর নির্ভরশীল, সুতরাং ξ কে যোগজীকরণের চলক নিয়ে আমরা লিখতে পারি]

$$F(z) = \int_a^z f(\xi)\, d\xi \;\cdots\cdots (2)$$

If $z + h$ is a point in R near the point z, then we have [যদি R এ z এর নিকটস্থ একটি বিন্দু $z + h$ হয় তখন]

$$F(z+h) = \int_a^{z+h} f(\xi)\, d\xi \;\cdots\cdots (3)$$

$$\therefore F(z+h) - F(z) = \int_a^{z+h} f(\xi)\, d\xi - \int_a^z f(\xi)\, d\xi$$

$$= \int_a^z f(\xi)\, d\xi + \int_z^{z+h} f(\xi)\, d\xi - \int_a^z f(\xi)\, d\xi$$

$$= \int_z^{z+h} f(\xi)\, d\xi \;\cdots\cdots (4)$$

Since the integral (4) is independent of the curve joining z to $z + h$, hence taking the integral along a straight line joining z to $z + h$, we have [যেহেতু (4) যোগজটি z হতে $z + h$ সংযোগকারী বক্ররেখার উপর অনির্ভরশীল, সুতরাং যোগজটি z হতে $z + h$ সংযোগকারী একটি সরলরেখা বরাবর নিয়ে পাই]

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_z^{z+h} f(\xi)\,d\xi - f(z)$$

$$=\frac{1}{h}\left[\int_z^{z+h} f(\xi)\,d\xi - hf(z)\right]$$

$$=\frac{1}{h}\left[\int_z^{z+h} f(\xi)\,d\xi - \int_z^{z+h} f(z)\,d\xi\right]$$

$$=\frac{1}{h}\int_z^{z+h} [f(\xi)-f(z)]\,d\xi \cdots\cdots (5)$$

Since $f(\xi)$ is continuous at the point z of the domain R, therefore for a given $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|f(\xi)-f(z)| < \varepsilon \text{ when ever } |\xi - z| < \delta \cdots\cdots (6)$$

Choosing h such that $h < \delta$, (6) is satisfied for every ξ on the straight line joining z to z + h. Thus, from (5) we have

[যেহেতু ডোমেন R এর z বিন্দুতে $f(\xi)$ অবিচ্ছিন্ন, সুতরাং প্রদত্ত $\varepsilon > 0$ এর জন্য একটি $\delta > 0$ বিন্দুমান থাকবে যেন

$$|f(\xi)-f(z)| < \varepsilon \text{ যেখানে } |\xi - z| < \delta \cdots\cdots (6)$$

h কে পছন্দ করি যেন $h < \delta$ হয়। z হতে z + h সংযোগকারী সরলরেখার উপর প্রত্যেক ξ এর জন্য (6) সিদ্ধ হয়। অতএব, (5) হতে পাই]

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| = \left|\frac{1}{h}\int_z^{z+h} [f(\xi)-f(z)]\,d\xi\right|$$

$$\le \frac{1}{|h|}\left|f(\xi)-f(z)\right| \left|\int_z^{z+h} d\xi\right|$$

$$< \frac{1}{|h|}\,\varepsilon\cdot\left|\int_z^{z+h} d\xi\right| = \frac{1}{|h|}\cdot\varepsilon|h| = \varepsilon$$

Hence $\lim\limits_{h\to 0} \dfrac{F(z+h)-F(z)}{h} = f(z)$.

Thus $F'(z)$ exist and $F'(z) = f(z)$

$$\int_z^{z+h} d\xi = [\xi]_z^{z+h} = z+h-z = h$$

Since the derivative $F'(z)$ of $F(z)$ exist at every point $z \in R$ hence the function F(z) is analytic in R. We know the derivative of an analytic function is also analytic, therefore $F'(z)$ is also analytic at every point in R.

Hence the function f(z) is analytic in R.

[অতএব $F'(z)$ বিদ্যমান এবং $F'(z) = f(z)$.

যেহেতু প্রত্যেক বিন্দু $z \in R$ এ F(z) এর অন্তরক $F'(z)$ বিদ্যমান, অতএব F(z) ফাংশন R এর মধ্যে বৈশ্লেষিক। আমরা জানি বৈশ্লেষিক ফাংশনের অন্তরকও বৈশ্লেষিক। অতএব, R এর মধ্যে প্রত্যেক বিন্দুতে $F'(z)$ বৈশ্লেষিক। অতএব, R এর মধ্যে f(z) বৈশ্লেষিক।]

(Proved)

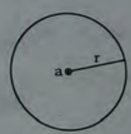
Cauchy's Inequality :

If f(z) is analytic inside and on a circle C of radius r and centre at z = a, then

$$|f^{(n)}(a)| \le \frac{M\underline{|n}}{r^n},\ n = 0, 1, 2, \ldots$$

where M is a constant such that $|f(z)| \le M$. **[DUH-2006]**

Proof : Given f(z) is analytic inside and on a circle $|z-a| = r$. Then by Cauchy's integral formula we have



$$f^{(n)}(a) = \frac{\underline{|n}}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$\Rightarrow |f^{(n)}(a)| = \left|\frac{\underline{|n}}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right|$$

$$\le \frac{\underline{|n}}{2\pi}\,\frac{|f(z)|}{|z-a|^{n+1}}\left|\oint_C dz\right|$$

$$\le \frac{\underline{|n}}{2\pi}\,\frac{M\cdot 2\pi r}{r^{n+1}};\qquad \left|\oint_C dz\right| = \text{Length of the circle} = 2\pi r$$

$$= \frac{M\underline{|n}}{r^n}$$

$$\Rightarrow |f^{(n)}(a)| \le \frac{M\underline{|n}}{r^n} \qquad \text{(Proved)}$$

Liouville's Theorem

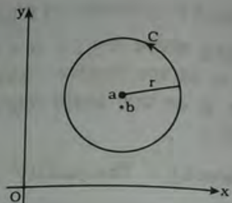
Theorem–11. If for all z in the entire complex plane,

(i) f(z) is analytic and (ii) f(z) is bounded,

then f(z) must be a constant

[NUH-95, 2000, 01, 03, 04(Old), 11, 14

NU(Pre)-06, 08, 13, RUH-94, 97, 2000, 04]



Proof : Let a and b be any two points in the z plane. Suppose that C is a circle of radius r having centre at a and enclosing point b. Then by Cauchy's integral formula, we have

[প্রমাণ ঃ মনে করি z তলে a ও b যেকোন দুইটি বিন্দু। ধরি b বিন্দুকে ধারণকারী a কেন্দ্র বিশিষ্ট C একটি বৃত্ত। তখন কচির যোগজ সূত্র দ্বারা পাই]

$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-b} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

$$= \frac{i}{2\pi i}\oint_C \left(\frac{1}{z-b} - \frac{1}{z-a}\right) f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{z-a-z+b}{(z-b)\,(z-a)}\, f(z)\,dz$$

$$= \frac{b-a}{2\pi i}\oint_C \frac{f(z)}{(z-b)\,(z-a)}\,dz$$

Now, f(z) is bounded, so there exist a constant M such that [এখন, f(z) সীমাবদ্ধ, সুতরাং একটি ধ্রুবক M থাকবে যেন]

$$|f(z)| \le M.$$

$$|z-a| = r,\ |z-b| = |z-a+a-b| \ge |z-a| - |a-b|$$

$$\Rightarrow |z-b| \ge r - |a-b| \ge r - \frac{r}{2} = \frac{r}{2}$$

where, if we choose [যেখানে, যদি আমরা পছন্দ করি] $|a-b| < \frac{r}{2}$

and the length of the circle C [এবং C বৃত্তের দৈর্ঘ্য] $= 2\pi r$

$$\therefore\ |f(b) - f(a)| = \left|\frac{b-a}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-b)\,(z-a)}\right|$$

$$= \frac{|b-a|}{2\pi}\ \frac{|f(z)|\ \left|\oint_C dz\right|}{|z-b|\ |z-a|}$$

$$\le \frac{|b-a|}{2\pi}\ \frac{M\cdot 2\pi r}{\frac{r}{2}\cdot r} = \frac{2|b-a|M}{r}$$

When [যখন] $r \to \infty$ then [তখন] $|f(b) - f(a)| = 0$

$\Rightarrow f(b) - f(a) = 0$

$\Rightarrow f(b) = f(a)$, for any arbitrary points a and b in the complex plane. [জটিল তলে ইচ্ছাধীন যে কোন বিন্দু a এবং b এর জন্য]

This shows that f(z) must be a constant. [ইহা দেখায় যে f(z) অবশ্যই ধ্রুবক।] (Proved)

3.10. Fundamental Theorem of Algebra :

Theorem–12. Every polynomial equation $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ with degree $n \ge 1$ and $a_n \ne 0$ has at least one root. [প্রত্যেক বহুপদী সমীকরণ $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ মাত্রা $n \ge 1$ এবং $a_n \ne 0$ সহ- এর কমপক্ষে একটি মূল আছে।]

[NUH-95, DUH-95, 01, 03, RUH-94, 02, 06]

Proof : If f(z) = 0 has no root, then $F(z) = \frac{1}{f(z)}$ is finite and analytic for all values of z.

Therefore, $|F(z)| = \frac{1}{|f(z)|}$ is bounded. Then by Liouville's theorem F(z) and thus f(z) must be constant, which is a contradiction to our hypothesis. Hence the polynomial equation f(z) = 0 has at least one root. (Proved)

[প্রমাণঃ যদি f(z) = 0 এর কোন মূল না থাকে, তখন z এর সকল মানের জন্য $F(z) = \frac{1}{f(z)}$ সসীম এবং বৈশ্লেষিক হবে। অতএব, $|F(z)| = \frac{1}{|f(z)|}$ সীমায়িত। তখন লিউভলির উপপাদ্য দ্বারা F(z) এবং অতএব f(z) অবশ্যই ধ্রুবক হবে, যাহা আমাদের কল্পনার বিরোধী। অতএব বহুপদী সমীকরণ f(z) = 0 এর কমপক্ষে একটি মূল আছে।]

Polynomial Equation :

Theorem–13. Every polynomial equation $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, where the degree $n \ge 1$ and $a_n \ne 0$, has exactly n roots. [উপপাদ্য–13 ঃ প্রত্যেক বহুপদী সমীকরণ $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, যেখানে মাত্রা $n \ge 1$ এবং $a_n \ne 0$, এর ঠিক n সংখ্যক মূল আছে।]

[DUH-2004, RUH-2002]

Proof : By fundamental theorem of algebra, $f(z)$ has at least one root. Let z_1 be that root. Then $f(z_1) = 0$

$$\Rightarrow f(z) - f(z_1) = a_n(z^n - z_1^n) + a_{n-1}(z^{n-1} - z_1^{n-1}) + \cdots + a_1(z - z_1)$$

$$= (z - z_1)\, f_1(z)$$

where $f_1(z)$ is a polynomial of degree $(n - 1)$.

Again, by fundamental theorem of algebra, $f_1(z) = 0$ has at least one root and let that root is z_2.

Then $f_1(z) = (z - z_2)\, f_2(z)$

where $f_2(z)$ is a polynomial of degree $n - 2$.

Thus, $f(z) = (z - z_1)(z - z_2)\, f_2(z)$

Continuing the above process, we can show that $f(z)$ has exactly n roots. (Proved)

[প্রমাণ ঃ বীজগণিতের মৌলিক উপপাদ্য দ্বারা, $f(z)$ এর কমপক্ষে একটি মূল আছে। ধরি ঐ মূলটি z_1. অতএব $f(z_1) = 0$

$$\Rightarrow f(z) - f(z_1) = a_n(z^n - z_1^n) + a_{n-1}(z^{n-1} - z_1^{n-1}) + \cdots + a_1(z - z_1)$$

$$= (z - z_1)\, f_1(z)$$

যেখানে $f_1(z)$ হল $(n - 1)$ মাত্রার বহুপদী।

আবার, বীজগণিতের মৌলিক উপপাদ্য দ্বারা $f_1(z) = 0$ এর কমপক্ষে একটি মূল আছে এবং ধরি মূলটি z_2 তখন

$f_1(z) = (z - z_2)\, f_2(z)$

যেখানে $f_2(z)$ হল $(n - 2)$ মাত্রার বহুপদী।

অতএব, $f(z) = (z - z_1)(z - z_2)\, f_2(z)$

উপরের পদ্ধতি অনবরত চালিয়ে আমরা দেখাতে পারি যে $f(z)$ এর ঠিক n সংখ্যক মূল আছে]

3.11. Winding Number : In this section we introduce the concept of winding number (or index) of a closed path with respect to a point. **[NUH-2004, 2008, 2012(Old), 2015]**

Definition : Let γ be a closed path (contour) and z be a point not on the path γ. Then the index or winding number of γ with respect to z is given by the integral $\frac{1}{2\pi i}\int_\gamma \frac{1}{s - z}\,ds$. It is denoted by $n(\gamma, z)$. Therefore, $n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{s - z}\,ds$.

The winding number counts the number of rounds of a path around the point.

[**Winding** সংখ্যা ঃ মনে করি γ একটি বদ্ধ পথ (কন্টুর) এবং γ পথের উপর z একটি বিন্দু। তখন z এর সাপেক্ষে γ এর সূচক বা winding সংখ্যা $\frac{1}{2\pi i}\int_\gamma \frac{1}{s - z}\,ds$ যোগজ দ্বারা প্রদত্ত হয়। ইহাকে $n(\gamma, z)$ দ্বারা প্রকাশ করা হয়। অতএব,

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{s - z}\,dx$$

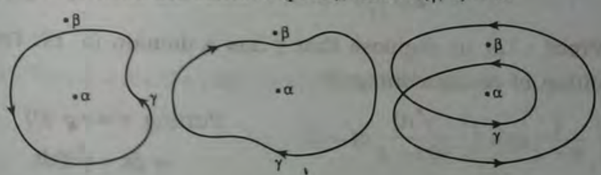
winding সংখ্যা দ্বারা একটি বিন্দুর চারদিকে একটি পথের ঘূর্ণনের সংখ্যাকে গণনা করা হয়।]

Note : (i) $n(\gamma, z)$ will be a positive integer if the contour γ winds counterclockwise direction around z.

(ii) $n(\gamma, z)$ will be a negative integer if the contour γ winds clockwise direction around z.

(iii) If the point z is enclosed by a simple closed positive oriented contour, then $n(\gamma, z) = 1$.

(iv) If z is outside the contour (path), then $n(\gamma, z) = 0$.



$n(\gamma, \alpha) = 1$	$n(\gamma, \alpha) = -1$	$n(\gamma, \alpha) = 2$
$n(\gamma, \beta) = 0$	$n(\gamma, \beta) = 0$	$n(\gamma, \beta) = 1$

Propertise of the winding Number :

[NUH-04, 08, 12(Old), 13]

From the definition we have the following properties.

(a) If γ is the sum of two closed contours γ_1 and γ_2 in the complex plane having the same initial points, then for every $z \notin \gamma$.

(i) $n(\gamma, z) = n(\gamma_1, z) + n(\gamma_2, z)$

(ii) $n(-\gamma, z) = -n(\gamma, z)$

(b) If the complex plane is C and γ_1, γ_2 be homotopic closed paths in $C - \{z\}$, then $n(\gamma_1, z) = n(\gamma_2, z)$, $\forall z \in \gamma_1 \cup \gamma_2$.

[**Winding** সংখ্যার ধর্মসমূহ ঃ

সংজ্ঞা হতে আমরা নিম্নলিখিত ধর্মসমূহ পাই

(a) কাল্পনিক তলে একই আদি বিন্দু নিয়ে γ_1 ও γ_2 দুইটি বদ্ধ কন্টুরের সমষ্টি γ হলে, প্রত্যেক $z \notin \gamma$ এর জন্য

(i) $n(\gamma, z) = n(\gamma_1, z) + n(\gamma_2, z)$

(ii) $n(-\gamma, z) = -n(\gamma, z)$

(b) যদি C কাল্পনিক তল হয় এবং $C - \{z\}$ এ γ_1 ও γ_2 হোমোটোপিক (homotopic) বদ্ধ পথ হয়, তখন

$$n(\gamma_1, z) = n(\gamma_2, z), \forall z \in \gamma_1 \cup \gamma_2.]$$

Theorem–14. Let γ be a closed path (contour) and z be a point not on γ. Then $n(\gamma, z)$ is an integer (depending on γ and on z) and

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{s-z}\,ds \text{ is an integer.} \qquad \textbf{[DUH–2006]}$$

Proof : Let us suppose that γ has a domain [a, b]. Then by definition of contour integral

$$\int_\gamma \frac{1}{s-z}\,ds = \int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt \cdots (1) \qquad \text{Putting } s = \gamma = \gamma(t)$$
$$\Rightarrow ds = \gamma'(t)dt$$

We define the complex valued function F(x) on [a, b] as

$$F(x) = \int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt,\ a \le x \le b \cdots (2)$$

Thus (1) becomes, $\int_\gamma \frac{1}{s-z}\,ds = F(x)$ we must show that $F(b) = 2\pi i n$ for some integer n.

Now F is continuous on [a, b] and has a derivative

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt$$
$$\Rightarrow F'(x) = \frac{\gamma'(x)}{\gamma(x)-z} \cdots (3)$$

at each point of continuity of γ'.

Let us definite a function G(t) by

$$G(t) = e^{-F(t)}[\gamma(t) - z],\ a \le t \le b \cdots (4)$$

which is also continuous on [a, b].

Differentiating (4) w. r. to t we get,

$$G'(t) = e^{-F(t)}\gamma'(t) - F'(t)\,e^{-F(t)}[\gamma(t) - z]$$
$$= e^{-F(t)}[\gamma'(t) - F'(t)\{\gamma(t) - z\}]$$
$$= e^{-F(t)}\left[\gamma'(t) - \frac{\gamma'(t)}{\gamma(t)-z}\{\gamma(t) - z\}\right];\ \text{by (3)}$$
$$= e^{-F(t)}[\gamma'(t) - \gamma'(t)] = 0$$

By integrating, G(t) = constant, throughout [a, b]

Hence, G(a) = G(b)

$\Rightarrow e^{-F(a)}[\gamma(a) - z] = e^{-F(b)}[\gamma(b) - z]$

$\Rightarrow \gamma(a) - z = e^{-F(b)}[\gamma(a) - z] \quad [\because \gamma(a) = \gamma(b) \ne z \text{ and From (2) } F(a) = 0]$

$\Rightarrow 1 = e^{-F(b)}$

$\Rightarrow e^{-F(b)} = 1 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi,\ k = 0, \pm 1, \pm 2 \cdots$

$= e^{i2k\pi}$

$\Rightarrow -F(b) = i2k\pi$

$\Rightarrow F(b) = i2(-k)\pi = i2n\pi$

where n = – k and n is any integer.

$\therefore n(\gamma, z) = \frac{1}{2\pi i}\cdot F(b) = \frac{1}{2\pi i}2\pi i n = n$ which is an integer.

SOLVED EXAMPLES

Example–1. Show that $\int_{(0,1)}^{(2,5)} (3x + y)\, dx + (2y - x)\, dy = 32$, along the straight line joining the points (0, 1) and (2, 5).

[DUMP–1990]

Solution : The equation of the straight line joining the points (0, 1) and (2, 5) is

$$\frac{y-1}{1-5} = \frac{x-0}{0-2}$$

$$\Rightarrow \frac{y-1}{-4} = \frac{x}{-2}$$

$$\Rightarrow \frac{y-1}{2} = \frac{x}{1}$$

$$\Rightarrow y - 1 = 2x$$

$$\Rightarrow y = 2x + 1$$

$$\therefore dy = d(2x+1) = 2dx + 0 = 2dx$$

$$\therefore \int_{(0,1)}^{(2,5)} (3x + y)\, dx + (2y - x)dy$$

$$= \int_0^2 (3x + 2x + 1)dx + \{2(2x+1) - x\} \cdot 2dx$$

$$= \int_0^2 (5x + 1 + 8x + 4 - 2x)\, dx$$

$$= \int_0^2 (11x + 5)\, dx$$

$$= \left[\frac{11x^2}{2} + 5x\right]_0^2$$

$$= \left[\frac{11}{2} \times 2^2 + 5 \times 2\right]$$

$$= 22 + 10$$

$$= 32. \quad \text{(Showed)}$$

Example–2. Evaluate the following :

(a) $\int_C z\, dz$ (b) $\int_C dz$ (c) $\int_C |dz|$.

Solution : From the definition of complex line integral we have

$$\int_C f(z)\, dz = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1})\, f(\delta_k),\ z_{k-1} \le \delta_k \le z_k \cdots\cdots (1)$$

(a) Here f(z) = z. since δ_k is arbitrary and hence putting $\delta_k = z_k$ and z_{k-1} successively in (1) we get,

$$\int_C z\, dz = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1})\, z_k \cdots\cdots (2)$$

and $$\int_C z\, dz = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1})\, z_{k-1} \cdots\cdots (3)$$

(2) + (3) gives, $$2\int_C z\, dz = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1})(z_k + z_{k-1})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (z_k^2 - z_{k-1}^2)$$

$$= \lim_{n\to\infty} [(z_1^2 - z_0^2) + (z_2^2 - z_1^2) + (z_3^2 - z_2^2) + \cdots + (z_n^2 - z_{n-1}^2)]$$

$$= \lim_{n\to\infty} [z_n^2 - z_0^2] = b^2 - a^2$$

$\int_C z\, dz = \frac{1}{2}(b^2 - a^2)$, if $z_0 = a$ and $z_n = b$ where a and b are the starting and ending points of C.

If C is a closed curve then a = b.

$$\therefore \int_C z\, dz = \frac{1}{2}(b^2 - b^2) = 0$$

(b) In this case f(z) = 1

$$\therefore \int_C 1 \cdot dz = \lim_{n\to\infty} \left[\sum_{k=1}^{n} (z_k - z_{k-1}) \cdot 1\right]$$

$$\Rightarrow \int_C dz = \lim_{n\to\infty} [(z_1 - z_0) + (z_2 - z_1) + \cdots + (z_n - z_{n-1})]$$

$$= \lim_{n\to\infty} (z_n - z_0) = b - a$$

= Chord AB, where A and B are end points of C.

If C is a closed curve then A = B, so chord AB = 0

$$\therefore \int_C dz = 0.$$

(c) In this case f(z) = 1 and in place of dz we have $|dz|$.

$$\therefore \oint_C |dz| = \lim_{n\to\infty} \sum_{k=1}^{n} |z_k - z_{k-1}| \cdot 1$$

$$= \lim_{n\to\infty} \{|z_1 - z_0| + |z_2 - z_1| + \cdots + |z_n - z_{n-1}|\}$$

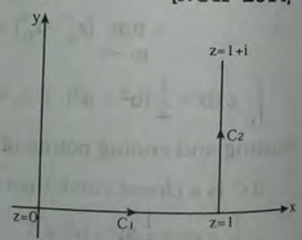
$$= \lim_{n\to\infty} \{\text{chord } z_0z_1 + \text{chord } z_1z_2 + \cdots + \text{chord } z_{n-1}\, z_n\}$$

$$= \text{Arc } z_0z_1 + \text{Arc } z_1z_2 + \cdots + \text{Arc } z_{n-1}\, z_n$$

= Arc length of C between the points z_0 and z_n.

Example – 2(i) Evaluate the integral $\int_C f(z)\, dz$ where $f(z) = z^2$ and c is the line segment from z = 0 to z = 1 and then z = 1 to z = 1 + i **[NUH-2014]**

Solution :



We have $\int_C f(z)\, dz = \int_C z^2 dz.$

Let the line segment are C_1 and C_2.

On C_1 we have $z = x \Rightarrow dz = dx$

and limits of x are 0, 1.

On C_2 we have $z = 1 + iy \Rightarrow dz = idy$

and limits of y are 0, 1.

$$\therefore \int_C f(z)\, dz = \int_C z^2\, dz$$

$$= \int_{C_1} z^2 dz + \int_{C_2} z^2 dz$$

$$= \int_0^1 x^2 dx + \int_0^1 (1 + iy)^2\, idy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\int_0^1 (1 + 2iy + i^2y^2)\, dy$$

$$= \left(\frac{1}{3} - 0\right) + i\left[y + iy^2 - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{3} + i\left(1 + i - \frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{2}{3}i + i^2$$

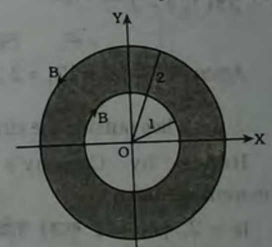
$$= \frac{1}{3} + \frac{2}{3}i - 1$$

$$= \frac{2}{3}i - \frac{2}{3}$$

$$= -\frac{2}{3}(1 - i) \textbf{ Ans.}$$

Example–3. Evaluate $\int_B \frac{dz}{z^2(z^2+9)}$ where B is the circle $|z| = 2$, described in the positive direction, together with the circle $|z| = 1$ described in the negative direction. **[RUH-1999]**

Solution :



Let [ধরি] $\int_B \frac{dz}{z^2(z^2+9)} = \int_B f(z)\, dz$

where [যেখানে] $f(z) = \frac{1}{z^2(z^2+9)}$

f(z) is fails to be analytic at $z^2 = 0$ and $z^2 + 9 = 0$

That is, at z = 0 and z = ± 3i

all of which lie outside the annular region with boundary B. Hence by Cauchy's integral theorem

[$z^2 = 0$ এবং $z^2 + 9 = 0$ এ f(z) বৈশ্লেষিক হতে ব্যর্থ হয়।

অর্থাৎ, $z = 0$ এবং $z = \pm 3i$, যার সকলে B দ্বারা সীমাবদ্ধ বলয়াকার বাহিরে অবস্থিত। অতএব কচির ইনটিগরাল উপপাদ্য দ্বারা পাই]

$$\int_B f(z)\,dz = 0$$

$$\Rightarrow \int_B \frac{dz}{z^2(z^2+9)} = 0$$

Example–4. Show that

$$\frac{1}{2\pi i}\oint \frac{e^z}{z-2}\,dz = \begin{cases} e^2, \text{ if C is the circle } |z| = 3 \\ 0, \text{ if C is the circle } |z| = 1 \end{cases}$$ **[DUH–1984]**

Solution : From Cauchy's integral formula we have [কচির ইনটিগরাল সূত্র হতে পাই]

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \;\cdots\cdots (1)$$

Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$. [এখানে $|z| = 3$ বৃত্তের ভিতর ও উপরে $f(z) = e^z$ বৈশ্লেষিক]

Again [আবার] $z = a = 2$

$\Rightarrow |z| = |2| = 2 < 3$

$\therefore$ z = 2 lies inside the circle $|z| = 3$

[$z = 2$, $|z| = 3$ বৃত্তের ভিতরে অবস্থিত]

Hence by (1) [অতএব (1) দ্বারা]

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = f(2)$$

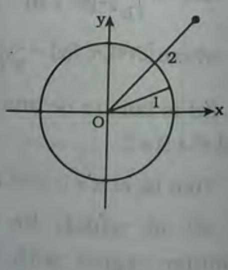
$$= e^2. \qquad \text{(Showed)}$$

Again [আবার] $|z| = |2| = 2 > 1$

$\therefore$ z = 2 lies outside the circle $|z| = 1$.

Hence by Cauchy's integral theorem we have

[$z = 2$, $|z| = 1$ বৃত্তের বাহিরে অবস্থিত। অতএব কচির ইনটিগরাল উপপাদ্য দ্বারা পাই]



$$\oint_C f(z)\,dz = 0$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0. \qquad \text{(Showed)}$$

Example–5. Show that $\oint_C \frac{\sin 3z}{z + \frac{\pi}{2}}\,dz = 2\pi i$, where C is the circle $|z| = 5$. **[NUH–2014, RUH–1981]**

Solution : Let [ধরি] $f(z) = \sin 3z$

Then [তখন] $f(z) = \sin 3z$ is analytic inside and on the circle $|z| = 5$. [$|z| = 5$ বৃত্তের ভিতর ও উপর বৈশ্লেষিক]

Also [আরো] here $z = -\frac{\pi}{2}$

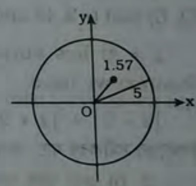
$$\Rightarrow |z| = \left|-\frac{\pi}{2}\right| = \frac{\pi}{2} = \frac{3{\cdot}14}{2} = 1{\cdot}57 < 5$$

$\therefore z = -\frac{\pi}{2}$ lies in the circle $|z| = 5$.

Hence by Cauchy's integral formula we have

[$z = -\frac{\pi}{2}$, $|z| = 5$ বৃত্তের মধ্যে অবস্থিত। অতএব কচির ইনটিগরাল সূত্র দ্বারা পাই]

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = f(a)$$



$$\Rightarrow \oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\Rightarrow \oint_C \frac{\sin 3z}{z + \frac{\pi}{2}}\,dz = 2\pi i\, f\left(\frac{-\pi}{2}\right)$$

$$= 2\pi i \sin\left(\frac{-3\pi}{2}\right)$$

$$= 2\pi i \cos 0^0$$

$$= 2\pi i \times 1 = 2\pi i. \qquad \text{(Showed)}$$

Example–6. Show that

$$\oint_C \frac{e^{3z}}{z - \pi i}\,dz = \begin{cases} -2\pi i, \text{ if C is the circle } |z-1| = 4 \\ 0 \;\;, \text{ if C is the ellipse } |z-2| + |z+2| = 6 \end{cases}$$

[NUH–1999, RUH–1981, 1986]

Solution : $f(z) = e^{3z}$ is analytic inside and on the circle $|z - 1| = 4$ [$|z - 1| = 4$ বৃত্তের ভিতর ও উপর $f(z) = e^{3z}$ বৈশ্লেষিক]

Also [আরো] $z = a = \pi i$

$$\Rightarrow |z| = |\pi i| = \pi = 3{\cdot}14 < 4$$

$\therefore$ $z = \pi i$ lies inside the given circle. [$\therefore$ $z = \pi i$ প্রদত্ত বৃত্তের ভিতরে অবস্থিত]

Hence by Cauchy's integral formula we have [অতএব কচির যোজিত সূত্র দ্বারা পাই]

$$\oint_C \frac{e^{3z}}{z - \pi i}\, dz = 2\pi i\, f(\pi i)$$

$$= 2\pi i \cdot e^{3i\pi}$$

$$= 2\pi i\,(\cos 3\pi + i \sin 3\pi)$$

$$= 2\pi i\,(-\cos 0 + i \sin 0)$$

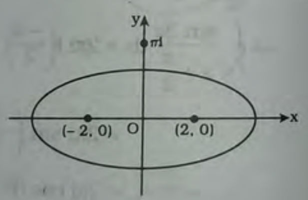
$$= 2\pi i\,(-1 + 0)$$

$$= -2\pi i. \quad \text{(Showed)}$$

2nd part : Here $f(z) = \dfrac{e^{3z}}{z - \pi i}$.

$|z - 2| + |z + 2| = 6$ is the equation of an ellipse whose foci are (2, 0) and (– 2, 0) and length of the major axis is 6.

$z = \pi i$ lies outside the ellipse. Hence by Cauchy's integral theorem we have

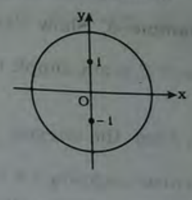


[$|z - 2| + |z + 2| = 6$ একটি উপবৃত্তের সমীকরণ যার ফোকাস হল (2, 0) ও (– 2, 0) এবং বৃহৎ অক্ষের দৈর্ঘ্য 6. $z = \pi i$ উপবৃত্তের বাহিরে অবস্থিত। অতএব কচির যোজিত উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 0$$

$$\Rightarrow \oint_C \frac{e^{3z}}{z - \pi i}\, dz = 0. \quad \text{(Showed)}$$

Example–7. Show that $\oint_C \dfrac{e^{tz}}{z^2 + 1}\, dz = 2\pi i \sin t$, where C is the circle $|z| = 3$ and $t > 0$. **[NUH–2004(Old), 04, 06(Old), 07, 11, 13 NU(Pre)–11, DUH–85, RUH–1975, 1997, CUMP–1986, JUH–1988, 2007, RUMP–1984]**



Solution : Here $f(z) = e^{tz}$ is analytic inside and on the given circle $|z| = 3$. [এখানে f(z) ফাংশনটি $|z| = 3$ প্রদত্ত বৃত্তের ভিতর ও উপর বৈশ্লেষিক।]

Again [আবার] $z^2 + 1 = z^2 - (-1)$

$$= z^2 - i^2$$

$$= (z + i)(z - i)$$

$$\therefore \oint_C \frac{e^{tz}}{z^2 + 1}\, dz = \oint_C \frac{e^{tz}}{(z + i)(z - i)}\, dz$$

$$= \oint_C \frac{1}{2i}\left(\frac{1}{z - i} - \frac{1}{z + i}\right) e^{tz}\, dz$$

$$= \frac{1}{2i}\oint_C \frac{e^{tz}}{z - i}\, dz - \frac{1}{2i}\oint_C \frac{e^{tz}}{z + i}\, dz \;\ldots\ldots (1)$$

Here [এখানে] $z = i \Rightarrow |z| = |i| = 1 < 3$

and [এবং] $z = -i \Rightarrow |z| = |-i| = 1 < 3$

$\therefore$ $z = i$ and $z = -i$ lie inside the circle $|z| = 3$

Hence by Cauchy's integral formula we get

[$\therefore$ $z = i$ এবং $z = -i$, $|z| = 3$ বৃত্তের ভিতর। অতএব কচির ইন্‌টিগরাল সূত্র দ্বারা পাই]

$$\oint_C \frac{e^{tz}}{z - i} = 2\pi i \cdot f(i) = 2\pi i\, e^{it}$$

and $$\oint_C \frac{e^{tz}}{z + i} = 2\pi i \cdot f(-i) = 2\pi i \cdot e^{-it}$$

Putting these values in (1) we get [এই মানগুলি (1) এর বসাইয়া পাই]

$$\oint_C \frac{e^{tz}}{z^2 + 1}\, dz = \frac{1}{2i}\,(2\pi i\, e^{it}) - \frac{1}{2i}\,(2\pi i\, e^{-it})$$

$$= \pi e^{it} - \pi e^{-it}$$

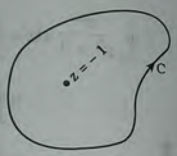
$$= 2\pi i\, \frac{e^{it} - e^{-it}}{2i}$$

$$= 2\pi i \sin t. \quad \text{(Showed)}$$

Example–8. Show that $\frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^3}dz = \left(t - \frac{1}{2}t^2\right)e^{-t}$ where C is any simple closed curve enclosing $z = -1$ and $t > 0$. **[RUH–1974, CUMP–1987]**

Or, Find the integral $\frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^3}dz$, where C is any simple closed curve enclosing $z = -1$ and $t > 0$. **[NUH–2015]**

Solution : From Cauchy's integral formula for higher derivatives we know that [উচ্চ অন্তরকের জন্য কচির ইনটিগরাল সূত্র হতে আমরা জানি]



$$f^{(n)}(a) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{1}{\lfloor n} f^{(n)}(a)$$

$$= \frac{1}{\lfloor n}\left[\frac{d^n}{dz^n}f(z)\right]_{z=a} \quad \cdots\cdots (1)$$

Let [ধরি] $f(z) = ze^{tz}$ and [এবং] $a = -1, n = 2$

Then $f(z) = ze^{tz}$ is analytic inside and on C.

Thus, from (1) we get [তখন C এর ভিতর ও উপর $f(z) = ze^{tz}$ বৈশ্লেষিক। অতএব, (1) হতে পাই]

$$\frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^{2+1}}dz = \frac{1}{\lfloor 2}\left[\frac{d^2}{dz^2}(ze^{tz})\right]_{z=-1}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^3}dz = \frac{1}{2}\left[\frac{d}{dz}(e^{tz} + t\,ze^{tz})\right]_{z=-1}$$

$$= \frac{1}{2}[te^{tz} + te^{tz} + t^2ze^{tz}]_{z=-1}$$

$$= \frac{1}{2}[te^{-t} + te^{-t} + t^2 \cdot (-1)\, e^{-t}]$$

$$= \frac{1}{2}(2te^{-t} - t^2 \cdot e^{-t})$$

$$= \frac{1}{2}(2t - t^2)\, e^{-t}$$

$$= \left(t - \frac{1}{2}t^2\right)e^{-t}. \qquad \text{(Showed)}$$

Example–9. Show that $\oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i e^{-2}}{3}$, where C is the circle $|z| = 3$. **[DUH–1991]**

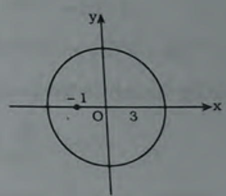
Solution : From Cauchy's integral formula for higher derivative we know that [উচ্চতর অন্তরকের জন্য কচির যোজিত সূত্র হতে জানি যে]

$$f^{(n)}(a) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a)$$

$$= \frac{2\pi i}{\lfloor n}\left[\frac{d^n}{dz^n}f(z)\right]_{z=a} \quad \cdots\cdots (1)$$

Let $f(z) = e^{2z}$ and $a = -1, n = 3$



Then $f(z) = e^{2z}$ is analytic inside and on the circle $|z| = 3$ [তখন $|z| = 3$ বৃত্তের ভিতর ও উপর $f(z) = e^{2z}$ বৈশ্লেষিক]

Also, $z = a = -1 \Rightarrow |z| = |-1| = 1 < 3$

$\therefore z = -1$ lies inside the circle $|z| = 3$

Hence from (1) we get, [অতএব (1) হতে পাই]

$$\oint_C \frac{e^{2z}}{(z+1)^{3+1}}dz = \frac{2\pi i}{\lfloor 3}\left[\frac{d^3}{dz^3}e^{2z}\right]_{z=-1}$$

$$\Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{6}[8e^{2z}]_{z=-1}$$

$$= \frac{\pi i}{3} \cdot 8e^{-2} = \frac{8\pi i e^{-2}}{3}. \qquad \text{(Showed)}$$

Example–10. Show that

(i) $\left(\frac{a^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{\lfloor n\, z^{n+1}}dz$

[DUH–2006, RUH–1973, 2006, CUH–1990]

(ii) $\sum_{n=0}^{\infty}\left(\frac{a^n}{\lfloor n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta$ **[CUH–1990]**

Solution : Let [ধরি] $f(z) = e^{az}$

Differentiating this successively w. r. to z we get [ইহাকে z এর সাপেক্ষে পর্যায়ক্রমে অন্তরীকরণ করে পাই]

$$f'(z) = ae^{az}$$

$$f''(z) = a^2e^{az}$$

$$f'''(z) = a^3e^{az}$$

$$\dots \quad \dots \quad \dots$$

$$f^{(n)}(z) = a^ne^{az}$$

When [যখন] $z = 0$ then [তখন] $f^{(n)}(0) = a^n e^0 = a^n \cdot 1 = a^n \;\dots\dots (1)$

By Cauchy's integral formula we have [কচির যোগজ সূত্র দ্বারা পাই]

$$f^{(n)}(b) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-b)^{n+1}}\,dz$$

Putting $b = 0$ we get [b = 0 বসাইয়া পাই]

$$f^{(n)}(0) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz \;\dots\dots (z)$$

From (1) and (2) we have [(1) ও (2) হতে পাই]

$$a^n = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz$$

$$\Rightarrow \frac{a^n}{\lfloor n} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz$$

Multiplying both sides by $\frac{a^n}{\lfloor n}$ we get [উভয় পক্ষকে $\frac{a^n}{\lfloor n}$ দ্বারা গুণ করে পাই]

$$\left(\frac{a^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{\lfloor n \; z^{n+1}}\,dz \qquad \textbf{[1st Part Proved]}$$

Taking summation over n from 0 to ∞ we get [0 হতে ∞ তে n এর উপর যোগফল নিয়ে পাই]

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{\lfloor n}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{\lfloor n \; z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\oint_C e^{az}\cdot\sum_{n=0}^{\infty}\frac{\left(\frac{a}{z}\right)^n}{\lfloor n}\cdot\frac{1}{z}\,dz$$

$$= \frac{1}{2\pi i}\oint_C e^{az}\cdot e^{a/z}\cdot\frac{1}{z}\,dz$$

$$\left[\because e^x = 1 + \frac{x}{\lfloor 1} + \frac{n^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots = \sum_{n=0}^{\infty}\frac{x^n}{\lfloor n}\right]$$

$$= \frac{1}{2\pi i}\oint_C e^{a(z+1/z)}\cdot\frac{1}{z}\,dz \;\dots\dots (3)$$

Let C be the circle of radius 1 and centre at (0, 0). [ধরি (0, 0) তে কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্ত C.]

Then [তখন] $x = 1\cos\theta = \cos\theta,\ y = \sin\theta$

$$z = x + iy = \cos\theta + i\sin\theta = e^{i\theta}$$

$$dz = i\,e^{i\theta}\,d\theta$$

$$z + \frac{1}{z} = z + z^{-1}$$

$$= \cos\theta + i\sin\theta + (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

Limit of θ is 0 to 2π. [θ এর সীমার হতে 2π]

Putting these values in (3) we get [এইমানগুলি (3) এ বসাইয়া পাই]

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\int_0^{2\pi} e^{a.2\cos\theta}\cdot\frac{1}{e^{i\theta}}\cdot ie^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta \qquad \textbf{[2nd Part Proved]}$$

Example–11. What is Cauchy's integral formula? Using this evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$ describe in the positive sense. **[NUH–1997, 2001, 2013, NU(Pre)–2008]**

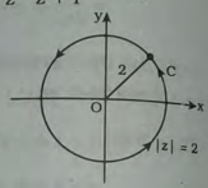
Solution : Let f(z) be analytic inside and on a simple closed curve C. If a is any point inside C, then cauchy's integral formula is

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

where C is traversed in the positive sense

2nd Part : Here [এখানে] $\frac{z}{(9-z^2)(z+i)}=\frac{z}{(3+z)(3-z)(z+i)}$

Let [ধরি] $\frac{z}{(3+z)(3-z)(z+i)}=\frac{A}{3+z}+\frac{B}{3-z}+\frac{c}{z+i}$ ...... (1)



$\Rightarrow z = A(3-z)(z+i)+B(3+z)(z+i)$
$+\,c(3-z)(3+z)$ ...... (2)

Putting z = 3 in (2) [(2) এ z = 3 বসাইয়া]

$$3 = B(3+3)(3+i)$$
$$= 6(3+i)B$$
$$\therefore B=\frac{3}{6(3+i)}=\frac{1}{2(3+i)}$$

Putting z = − 3 in (2) [(2) এ z = − 3 বসাইয়া]

$$-3 = 6(-3+i)A$$
$$\Rightarrow A=\frac{-3}{6(-3+i)}=\frac{-1}{2(-3+i)}$$

Putting z = − i in (2) [(2) এ z = − i বসাইয়া]

$$-i = c(3-i)(3+i) = c(9-i^2) = c(9+1) = 10c$$
$$\therefore c=\frac{-i}{10}$$

Putting the values of A, B, C in (1) we get [A, B, C এর মান (1) এ বসাইয়া পাই]

$$\frac{z}{(3+z)(3-z)(z+i)}=\frac{-1}{2(-3+i)}\cdot\frac{1}{3+z}+\frac{1}{2(3+i)}\cdot\frac{1}{3-z}-\frac{i}{10}\cdot\frac{1}{z+i}$$

$$\Rightarrow \int_C\frac{z}{(9-z^2)(z+i)}\,dz=-\frac{1}{2(-3+i)}\int_C\frac{dz}{3+z}+\frac{1}{2(3+i)}\cdot\int_C\frac{dz}{3-z}$$
$$-\frac{i}{10}\int_C\frac{dz}{z+i}\ \text{......}(3)$$

Now [এখন] $|3| = 3,\ |-3| = 3,\ |-i| = 1$

$\therefore z = 3, -3$ lie outside the circle $|z| = 2$

and $z = -i$ lie inside the circle $|z| = 2$.

∴ By Cauchy's integral theorem we have

[∴ z = 3, − 3, |z| = 2 বৃত্তের বাহিরে এবং z = − i, |z| = 2 বৃত্তের ভিতরে অবস্থিত।

∴ কচির যোজিত উপপাদ্য দ্বারা পাই]

$$\int_C\frac{dz}{3+z}=0$$

and [এবং] $\int_C\frac{dz}{3-z}=0$

By Cauchy's integral formula [কচির যোজিত সূত্র দ্বারা পাই]

$$\int_C\frac{dz}{z+i}=2\pi i\cdot f(a);\qquad \text{Here } z=a=-i$$
$$=2\pi i\cdot f(-i)\qquad f(z)=1$$
$$=2\pi i\cdot 1\qquad \therefore f(-i)=1$$
$$=2\pi i$$

Putting these values in (3) we get [এইমানগুলি (3) এ বসাইয়া পাই]

$$\int_C\frac{z}{(9-z^2)(z+i)}\,dz=-\frac{1}{(2-3+i)}\times 0+\frac{1}{2(3+i)}\times 0-\frac{i}{10}\times 2\pi i$$
$$=\frac{-\pi}{5}i^2=\frac{-\pi}{5}(-1)$$
$$=\frac{\pi}{5}\quad \text{Ans.}$$

Example–12. Evaluate $\oint_C \frac{\sin \pi z^2 \cos \pi z^2}{(z-1)(z-2)}\,dz$, where C is the circle $|z| = 3$. **[RUH–1997]**

Solution : We have $\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$

$$\therefore \oint_C\frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}\,dz=\oint_C\frac{\sin \pi z^2+\cos \pi z^2}{z-2}\,dz$$
$$-\oint_C\frac{\sin \pi z^2+\cos \pi z^2}{z-1}\,dz\ \text{......}(1)$$

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$. Then f(z) is analytic inside C and $z = 1$, $z = 2$ lie in the circle $|z| = 3$. Therefore by Cauchy's integral formula we have from (1)

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i \{\sin \pi(2)^2 + \cos \pi(2)^2\}$$
$$- 2\pi i \{\sin \pi (1)^2 + \cos \pi (1)^2\}$$
$$= 2\pi i\{0 + 1 - 0 + 1\} = 2\pi i \cdot (2) = 4\pi i \quad \text{(Ans)}$$

Example–13. Prove that $\int F(z)\, G'(z)\, dz = F(z)\, G(z) - \int F'(z)\, G(z)\, dz$. **[RUH–1998]**

Solution : We have $\frac{d}{dz}\{F(z)\, G(z)\} = F(z)\, G'(z) + F'(z)\, G(z)$

Integrating both sides w. r. to z we get

$$\int \frac{d}{dz}\{F(z)\, G(z)\}\, dz = \int F(z)\, G'(z)\, dz + \int F'(z)\, G(z)\, dz$$
$$\Rightarrow F(z)\, G(z) = \int F(z)\, G'(z)\, dz + \int F'(z)\, G(z)\, dz$$
$$\Rightarrow \int F(z)\, G'(z)\, dz = F(z)\, G(z) - \int F'(z)\, G(z)\, dz \quad \text{(Proved)}$$

Example–14. Let C denotes the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$, described in the positive sense. Determine the following integrals :

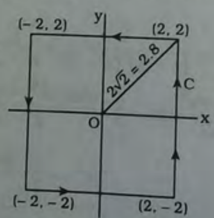
(i) $\int_c \frac{1}{z^2 + 9} dz$ **[NUH–96, 12]** (ii) $\int_c \frac{1}{z(z^2 + 9)} dz$ **[NUH–1996]**

(iii) $\int_c \frac{1}{(z^2 + 1)(z^2 + 9)} dz$ **[NUH–1996]**

Solution : The square is bounded by the lines

$$x = 2, x = -2, y = 2, y = -2$$

[বর্গক্ষেত্রটি $x = 2, x = -2, y = 2, y = -2$ রেখাগুলি দ্বারা সীমাবদ্ধ]



(i) $z^2 + 9 = (z + 3i)(z - 3i)$

$|3i| = 3$ and [এবং] $|-3i| = 3$

$\therefore z = 3i$ and $z = -3i$ each lies outside of C. [$\therefore z = 3i$ এবং $z = -3i$ প্রত্যেকেই C এর বাহিরে অবস্থিত।]

Hence by Cauchy's integral formula we have [অতএব কচির যোজিত সূত্র দ্বারা পাই]

$$\int_c \frac{1}{z^2 + 9} dz = \int_c \frac{1}{(z + 3i)(z - 3i)} dz$$
$$= \int_c \frac{1}{6i}\left(\frac{1}{z - 3i} - \frac{1}{z + 3i}\right) dz$$
$$= \frac{1}{6i}\int_c \frac{1}{z - 3i} dz - \frac{1}{6i}\int_c \frac{1}{z - (-3i)} dz$$
$$= \frac{1}{6i} \times 0 - \frac{1}{6i} \times 0 \quad \Big| \quad \text{Here } f(z) = 1$$
$$= 0 \text{ Ans.}$$

(ii) $\frac{1}{z(z^2 + 9)} = \frac{1}{z(z + 3i)(z - 3i)}$

Let $\frac{1}{z(z^2 + 9)} = \frac{A}{z} + \frac{B}{z + 3i} + \frac{D}{z - 3i}$

$$\Rightarrow 1 = A(z + 3i)(z - 3i) + Bz(z - 3i) + Dz(z + 3i) \cdots (1)$$

Putting $z = 0, 3i, -3i$ successively we get [পর্যায়ক্রমে $z = 0, 3i, -3i$ বসাইয়া পাই]

$$1 = A(3i)(-3i) \Rightarrow A = \frac{1}{-9i^2} = \frac{1}{9}$$
$$1 = 3i\, D(3i + 3i) \Rightarrow D = \frac{1}{18i^2} = \frac{-1}{18}$$
and [এবং] $1 = (-3i)\, B(-6i) \Rightarrow B = \frac{1}{18i^2} = \frac{-1}{18}$

$|0| = 0$, $|3i| = 3$ and [এবং] $|-3i| = 3$

Thus only z = 0 lie in C. [অতএব মাত্র z = 0, C এর মধ্যে অবস্থিত।]

$$\therefore \int_c \frac{1}{z(z^2+9)}\,dz = \int_c \left(\frac{1/9}{z} + \frac{-1/18}{z+3i} + \frac{-1/18}{z-3i}\right) dz$$

$$= \frac{1}{9}\int_c \frac{1}{z-0}\,dz - \frac{1}{18}\int_c \frac{1}{z+3i}\,dz - \frac{1}{18}\int_c \frac{1}{z-3i}\,dz$$

$$= \frac{1}{9}\cdot 2\pi i\, f(0) - \frac{1}{18}\times 0 - \frac{1}{18}\times 0$$

$$= \frac{1}{9}\cdot 2\pi i\cdot 1 = \frac{2}{9}\pi i \quad \text{Ans.}$$

Here $f(z) = 1$, $\therefore f(0) = 1$

Other way : Let $f(z) = \dfrac{1}{z^2+9}$ which is analytic inside and on C.

Since $z^2 + 9 = 0$ gives $z = \pm 3i$ and $|z| = 3$ $\therefore z = \pm 3i$ lie outside C

$$\therefore \int_c \frac{1}{z(z^2+9)}\,dz = \int_c \frac{f(z)}{z-0}\,dz$$

$$= 2\pi i\cdot f(0)$$

$$= 2\pi i\cdot\frac{1}{9} = \frac{2\pi i}{9} \quad \text{Ans.}$$

$f(z) = \dfrac{1}{z^2+9}$

$\Rightarrow f(0) = \dfrac{1}{0+9} = \dfrac{1}{9}$

(iii) $\dfrac{1}{(z^2+1)(z^2+9)} = \dfrac{1}{(z+i)(z-i)(z+3i)(z-3i)}$

We have $\dfrac{1}{(z+i)(z-i)(z+3i)(z-3i)}$

$$= \frac{1}{(-2i)(2i)(-4i)(z+i)} + \frac{1}{2i(4i)(-2i)(z-i)} + \frac{1}{(-2i)(-4i)(-6i)(z+3i)} + \frac{1}{(4i)(2i)(6i)(z-3i)}$$

$$= \frac{1}{16i^3(z+i)} + \frac{1}{-16i^3(z-i)} + \frac{1}{-48i^3(z+3i)} + \frac{1}{48i^3(z-3i)}$$

$$\therefore \int_c \frac{1}{(z^2+1)(z^2+9)}\,dz = \frac{1}{16i^3}\int_c \frac{1}{z-(-i)}\,dz - \frac{1}{16i^3}\int_c \frac{1}{z-i}\,dz - \frac{1}{48i^3}\int_c \frac{1}{z+3i}\,dz + \frac{1}{48i^3}\int_c \frac{1}{z-3i}\,dz$$

$$= \frac{1}{16i^3}\cdot 2\pi i\, f(-i) - \frac{1}{16i^3}\cdot 2\pi i\cdot f(i) - \frac{1}{48i^3}\times 0 + \frac{1}{48i^3}\times 0$$

[$\because$ only $z = \pm i$ lie inside C]

$$= \frac{-\pi}{8}\cdot 1 + \frac{\pi}{8}\cdot 1 - 0 + 0 = 0 \quad \text{Ans.}$$

Other way : Let $f(z) = \dfrac{1}{z^2+9}$ which is analytic inside and on C.

$$\text{Then } \int_c \frac{1}{(z^2+1)(z^2+9)}\,dz$$

$$= \int_c \frac{f(z)}{z^2+1}\,dz$$

$$= \int_c \frac{f(z)}{(z+i)(z-i)}\,dz$$

$$= \frac{1}{2i}\int_c \frac{f(z)}{z-i}\,dz - \frac{1}{2i}\int \frac{f(z)}{z+i}\,dz$$

$$= \frac{1}{2i}\cdot f(i) - \frac{1}{2i} f(-i)$$

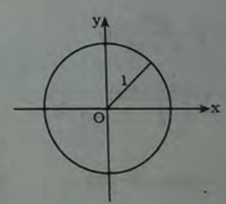
$$= \frac{1}{2i}\cdot\frac{1}{8} - \frac{1}{2i}\cdot\frac{1}{8} = 0 \quad \text{Ans.}$$

$f(z) = \dfrac{1}{z^2+9}$

$\Rightarrow f(i) = \dfrac{1}{i^2+9} = \dfrac{1}{8}$

$f(-i) = \dfrac{1}{i^2+9} = \dfrac{1}{8}$

Example–15. Using Cauchy's integral formula evaluate $\int_c \dfrac{dz}{z(z-4)^2}$, where C is the circle $|z| = 1$, described counter clockwise. **[NUH–2002]**



Solution : Here $|0| = 0$ and $|4| = 4$, $z = 0$ lie inside C and $z = 4$ lie outside C. So let $f(z) = \dfrac{1}{(z-4)^2}$ which is analytic inside and on C. Then [এখানে $|0| = 0$ এবং $|4| = 4$, $z = 0$, C এর ভিতরে অবস্থিত এবং $z = 4$, C এর বাহিরে অবস্থিত। সুতরাং ধরি $f(z) = \dfrac{1}{(z-4)^2}$ যাহা C এর ভিতর ও উপর বৈশ্লেষিক। তখন]

$$\int_c \frac{dz}{z(z-4)^2}\,dz = \int_c \frac{f(z)}{z-0}\,dz$$

$$= 2\pi i\cdot f(0)$$

$$= 2\pi i\cdot\frac{1}{16} = \frac{\pi i}{8} \quad \text{Ans.}$$

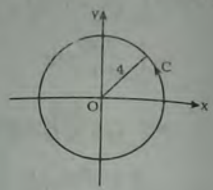
$f(z) = \dfrac{1}{(z-4)^2}$

$\Rightarrow f(0) = \dfrac{1}{(0-4)^2} = \dfrac{1}{16}$

Example–16. Evaluate $\int_c \frac{e^{3z}}{z + \pi i} dz$ where the circle $|z + 1| = 4$.

[NUH-1994, 2014, DUH-1994]

Solution : Here the centre of the circle is (– 1, 0) and radius is 4



Here $|z| = |-\pi i| = \pi = 3{\cdot}14 < 4$

$\therefore z = -\pi i$ lies inside C.

Let $f(z) = c^{3z}$. Then f(z) is analytic inside and on C.

Hence by cauchy's integral formula we have

[এখানে বৃত্তের কেন্দ্র (– 1, 0) এবং ব্যাসার্ধ 4.

এখানে $|z| = |-\pi i| = \pi = 3{\cdot}14 < 4$

$\therefore z = -\pi i$, C এর ভিতর অবস্থিত। ধরি $f(z) = c^{3z}$. তখন C এর ভিতর ও উপর f(z) বৈশ্লেষিক। অতএব কচির যোজিত সূত্র দ্বারা পাই]

$$\int_c \frac{e^{3z}}{z + \pi i} dz$$

$$= \int_c \frac{f(z)}{z - (-\pi i)} dz$$

$$= 2\pi i \cdot f(-\pi i) \qquad f(z) = e^{3z}$$

$$= 2\pi i \cdot e^{-i3\pi} \qquad f(-\pi i) = e^{-3\pi i}$$

$$= 2\pi i (\cos 3\pi - i \sin 3\pi)$$

$$= 2\pi i (-1 - 0)$$

$$= -2\pi i \quad \text{Ans.}$$

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. When the indefinite integral of f(z) is the same as the process of real variable?

   Ans : If the complex function f(z) is analytic, then its indefinite integral is the same as the process of real variable.

2. Define Jordan arc.

   Ans : A continuous arc without multiple points is called a Jordan arc.

3. Define continuous Jordan curve.

   Ans : If a Jordan curve consists of a chain of finite number of continuous arcs then it is called a continuous Jordan curve.

4. When a curve is said to be closed?

   Ans : If the starting and ending points of a curve coincide then the curve is called a closed curve.

5. What is a contour?

   Ans : A Jordan curve consisting of continuous chain of a finite number of regular arcs is called a contour.

   OR, A contour is a curve consisting of finite number of smoth arcs joined end to end.

6. What is a closed contour? **[NUH–2015]**

   Ans : If the starting point A of the first arc and the ending point B of the last arc coincide, then the contour is said to be a closed contour.

7. Define rectifiable arc.

   Ans : An arc is said to be a rectifiable if it has a finite arc length.

8. Write the complex line integral along the curve C.

   OR, Define line integral of f(z) along a curve. **[RUH–2001]**

   Ans : The line integral $\int_C f(z)\, dz$ is called the complex line integral along the curve C. [রেখা সমাকলন $\int_C f(z)\, dz$ কে C বক্ররেখা বরাবর জটিল রেখা সমাকলন বলে।]

9. Under what condition a complex function $f(z)$ be integrable along a curve C?

Ans : If $f(z)$ is analytic at all points of a region R and C is any curve lying in R, then $f(z)$ must be integrable along C.

[যদি একটি এলাকা R এর সকল বিন্দুতে $f(z)$ বৈশ্লেষিক হয় এবং R এর মধ্যে C বক্ররেখার অবস্থান হয়, তখন C বরাবর $f(z)$ অবশ্যই যোগজীকরণ যোগ্য ।]

10. Under what conditions $\oint_C f(z)\, dz = 0$? [NUH–2012]

OR, State couchy's integral theorem. [NUH–2014]

Ans : If $f(z)$ is analytic in a region R and on its closed boundary C with derivatives $f'(z)$ which is continuous at all points inside R and on C, then $\oint_C f(z) = 0$. [যদি একটি এলাকা R এবং ইহার বদ্ধ সীমানা C এর উপর $f(z)$ অন্তরক $f'(z)$ যাহা R এর ভিতর এবং C এর উপর অবিচ্ছিন্ন সহ, বৈশ্লেষিক হয় তখন $\oint_C f(z)\, dz = 0$]

11. Write a formula for finding the value of $f(z)$ at a point $z = a$.

Ans : Let $f(z)$ be analytic inside and on a simple closed curve C. If a is any point inside C, then the value of $f(z)$ at $z = a$ is $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz$. [মনেকরি একটি সরল বদ্ধ বক্ররেখা C এর ভিতর ও উপর $f(z)$ বৈশ্লেষিক। যদি C এর ভিতর যে কোন বিন্দু a হয়, তখন $z = a$ এ $f(z)$ এর মান

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\, dz.$$

12. Write a formula of $f'(a)$ for the derivatives of $f(z)$ at $z = a$.

Ans : If $f(z)$ be analytic inside and on a simple closed curve C and a is a point inside C, then $f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\, dz$.

[যদি একটি সরল বদ্ধ বক্ররেখা C এর ভিতর ও উপর $f(z)$ বৈশ্লেষিক হয় এবং C এর ভিতর a যে কোন বিন্দু হয়, তখন $f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\, dz$.

13. What is the formula of n-th derivatives of $f(z)$ at $z = a$?

Ans : Let $f(z)$ be analytic inside and on the boundary C of a simply connected region R. Then at every point $z = a$ of R, $f(z)$ has derivatives of all orders given by the formula

$$f^n(a) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz, \text{ where } n = 0, 1, 2, \cdots$$

[মনেকরি সরলভাবে সংযুক্ত একটি এলাকা R এর ভিতর এবং সীমানা C এর উপর $f(z)$ বৈশ্লেষিক। তখন R এর প্রত্যেক বিন্দু $z = a$ এ প্রদত্ত সূত্র

$$f^n(a) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}, \text{ যেখানে } n = 0, 1, 2, \ldots$$

দ্বারা $f(z)$ এর সকল ক্রমের অন্তরক আছে ।]

14. Which theorem is known as the converse of Cauchy's theorem?

Ans : Morera's theorem is known as the converse of Cauchy's theorem. [মরিরার উপপাদ্যটি কচির উপপাদ্যের বিপরীত বলে পরিচিত ।]

15. What is Morera's theorem?

OR, State Morera's theorem. [NUH–2012, 2014]

Ans : If $f(z)$ is continuous in a simply connected region R and if $\oint_C f(z)\, dz = 0$ around every simple closed curve C in R, then $f(z)$ is analytic.

This is Known as Morera's theorem. [সরলভাবে সংযুক্ত একটি এলাকা R এ যদি $f(z)$ অবিচ্ছিন্ন হয় এবং R এর ভিতর প্রত্যেক সরল বদ্ধ বক্ররেখা C এর চারিদিকে $\oint_C f(z)\, dz = 0$ হয়, তখন $f(z)$ বৈশ্লেষিক। ইহাই মরিরার উপপাদ্য নামে অভিহিত ।]

16. What is Cauchy's inequality?

Ans : If $f(z)$ is analytic inside and on C of radius r and centre at $z = a$ then the Cauchy's inequality is

$$|f^n(a)| \le \frac{M\lfloor n}{r^n}, \; n = 0, 1, 2, \cdots$$

where M is a constant such that $|f(z)| \le M$.

17. Under what conditions a complex function f(z) be constant?

Ans : A complex function (z) be constant if for all z in the entrire complex plane (i) f(z) is analytic and (ii) f(z) is bounded. [একটি জটিল ফাংশন f(z) ধ্রুবক হবে যদি সমগ্র জটিল তলে সকল z এর জন্য(i) f(z) বৈশ্লেষিক হয় এবং(ii) f(z) সীমায়িত হয়।]

18. Write fundamental theorem of Algebra.

OR, State fundamental theorem of Algebra. **[NUH–2015]**

Ans : Every polynomial equation $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has at least one root.

19. Define winding number.

Ans : Let γ be a closed path and z be a point not on the path γ. Then the winding number of γ with respect to z is given by the integral $\frac{1}{2\pi i}\int_\gamma \frac{1}{s-z}\,ds$. [মনেকরি γ একটি বদ্ধ পথ এবং z একটি বিন্দু যাহা γ এর পথের উপর না। তখন z এর সাপেক্ষে γ এর winding সংখ্যা $\frac{1}{2\pi i}\int_\gamma \frac{1}{s-z}\,ds$ সমাকলন দ্বারা দেওয়া (প্রদত্ত) হয়।]

20. What is the significant of a winding number?

Ans : The winding number counts the number of rounds of a path around the point. [Winding সংখ্যা একটি বিন্দুর চারিদিকে একটি পথের ঘূর্ণনের সংখ্যা গণনা করে।]

21. When the winding number is zero? **[NUH–2012]**

Ans : If the point z is outside the contour (closed path) then the winding number is zero. [যদি z বিন্দুটি কন্টুরের (বদ্ধপথ) বাহিরে হয় তখন Winding সংখ্যা শূন্য।]

22. When the winding number is one?

Ans : If the point z is enclosed by a simple closed positive oriented contour, then winding number is one. [যদি z বিন্দুটি একটি ধনাত্মক দিকে নির্দেশিত সরল বদ্ধ কন্টুর দ্বারা আবদ্ধ হয়, তখন Winding সংখ্যা এক।]

23. Write down the two properties of Winding number. **[NUH–2013]**

Ans : The two properties of Winding number are

(a) If γ is the sum of two closed contours γ_1 and γ_2 in the complex plane having the same initial points, then for every $z \notin \gamma$,

(i) $n(\gamma, z) = n(\gamma_1, z) + n(\gamma_2, z)$

(ii) $n(-\gamma, z) = -n(\gamma, z)$

(b) If the complex plane is C and γ_1, γ_2 be homotopic closed paths in C – {z}, then $n(\gamma_1, z) = n(\gamma_2, z)$, $\forall z \in \gamma_1 \cup \gamma_2$.

[(a) কাল্পনিক তলে একই আদি বিন্দু নিয়ে γ_1 ও γ_2 দুইটি বদ্ধ কন্টুরের সমষ্টি γ হলে, প্রত্যেক $z \notin \gamma$ এর জন্য

(i) $n(\gamma, z) = n(\gamma_1, z) + n(\gamma_2, z)$

(ii) $n(-\gamma, z) = -n(\gamma, z)$

(b) যদি C কাল্পনিক তল হয় এবং C – {z} এ γ_1 ও γ_2 হোমোটোপিক (homotopic) বদ্ধ পথ হয়, তখন

$$n(\gamma_1, z) = n(\gamma_2, z),\ \forall z \in \gamma_1 \cup \gamma_2.]$$

24. What is simply connected region ? [সরল আবদ্ধ এলাকা কি?] **[NUH–2014]**

Ans. A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R.

EXERCISE–3

Part–A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. Why the definite integral $\int_a^b f(z)\,dz$ can not be integrated as the process of real variable?
2. Define the norm of a partition.
   OR, What is the norm of a partition.
3. Mention an inequality for complex integrals.
4. If $\left|\int_c f(z)\,dz\right| \leq ML$, then write the meaning of C, M and L.

Part–B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. Prove that if f(z) is integrable along a curve C having length L and $|f(z)| \le M$ on C, them $\left|\int_c f(z)\, dz\right| \le ML$. **[NUH–04, DUH–98]**

   Ans : See theorem–1.

2. Prove that if f(z) is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \le M$ on C, then $\left|\int_c f(z)\, dz\right| \le ML$. **[RUH–1994, 2000, 2006]**

   Ans : See theorem–1.

3. State and prove Cauchy's Fundamental theorem for any closed polygon. **[RUH–2004]**

   Ans : See theorem–3.

4. Define a simple closed curve and a simple connected region. **[RUH–2003]**

   Ans : See art–3.2.

5. For any simple closed curve C, show that (i) $\oint_c dz = 0$; (ii) $\oint_c z d z = 0$. **[RUH–2003]**

   Ans : See solved problem–2.

6. Using Lioville's theorem prove the Fundamental theorem of algebra. **[RUH–1996]**

   Ans : See theorem–12.

7. State and prove the fundamental theorem of algebra. **[DUH–2001, 2003, 2004, RUH–1999]**

   Ans : See theorem–12.

8. What do you mean by winding number? Discuss the properties of winding number. **[NUH–2004]**

   Ans : See art 3.11.

Part–C (Broad Questions) (বড় প্রশ্ন)

1. State and Prove Cauchy's integral formula. **[NUH–1995, 2004 (Old), 2006, 2007, DUH–1998, 2004]**

   Ans : See theorem–5.

2. State Liouville's theorem. Apply it to prove the fundamental theorem of Algebra. **[NUH–1995]**

   Ans : See theorem–11 and theorem–12.

3. Suppose the complex valued function f(z) is analytic and $f'(z)$ is continuous with in and on a closed contour C. Prove that $\int_c f(z)\, dz = 0$. Can the restriction about continuity be removed? **[NUH–1997, DUH–1999]**

   Ans : See theorem–5.

4. State and prove the converse theorem of Cauchy. **[NUH–1999, RUH–1998]**

   Ans : See theorem–10.

5. State and prove Morera's theorem. **[NUH–99, 03, RUH–01]**

   Ans : See theorem–10.

6. State and prove Cauchy's integral formula and hence derive the expression for the derivative for an analytic function. **[NUH–1999]**

   Ans : See theorem–7 and 8.

7. If for all z in the entire complex plane f(z) is analytic and bounded, then prove that f(z) must be a constant. **[NUH–2000]**

   Ans : See theorem–11.

8. State and prove Liouville's theorem. **[NUH–2001**

   Ans : See theorem–11.

9. If f(z) is analytic inside and on the boundary C of a simply connected region R, then prove that $f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}\, dz$. **[NUH–2003, DUH–2004]**

   Ans : See theorem–8.

10. State and prove the Cauchy's theorem.

    Ans : See theorem–5.

11. State and prove Cauchy's fundamental theorem for a triangle. **[RUH–2006]**

    Ans : See theorem–2.

12. If $f(z)$ is continuous in a simply connected region R and if $\oint_c f(z)\, dz = 0$ around every simple closed curve C in R, then prove that $f(z)$ is analytic in R. **[RUH-2006]**
Ans : See theorem-10.

13. State the Fundamental theorem of Algebra. Prove that every polynomial equation $P(z) = a_0 + a_1 z + a_1 z^2 + \cdots + a_n z^n = 0$, $n \geq 1$, $a_n \neq 0$ has exactly n roots. **[DUH-2004, RUH-2002, 2004]**
Ans : See theorem-12 and 13.

14. A function which is analytic and bounded in the entire complex plane is constant. **[DUH-03, RUH-94, 97, 00, 04]**
Ans : See theorem-11.

15. If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C, then $f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^2}\, dz$.
[RUH-1995, 1997, 2001, 2003]
Ans : See theorem-8.

16. State and prove Cauchy's fundamental theorem and hence prove that $\oint_c dz = 0$ and $\oint_c z\, dz = 0$, where C is a simple closed curve. **[RUH-1999]**
Ans : See theorem-5 and solved problem-2.

17. If $f(z)$ is analytic and bounded for all z in the complex plane, then show that $f(z)$ is constant throughout the z-plane.
[DUH-98, 99]
Ans : See theorem-11.

18. If $f(z)$ is analytic within and on a closed contour C and z_0 is any point with in C, prove that
$$f'(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^2}\, dz$$
where C is taken in the positive direction. **[DUH-1999, 2005]**
Ans : Se theorem-8.

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CHAPTER-4
SINGULARITIES, RESIDUE AND SOME THEOREMS

4.1. Zero or root of an analytic function : **[NUH-1993]**

A value of z for which the analytic function $f(z) = 0$ is called a zero of $f(z)$.

If $f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic, $g(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a zero of order n of the function $f(z)$.

Simple zero : If $f(z)$ has a zero of order one at $z = z_0$, then $f(z)$ is said to have a simple zero at $z = z_0$.

Singular point or critical point of an analytic function : **[NUH-1994, 2006 (Old), 2008, NUH(Pre)-2013]**

A point at which an analytic function $f(z)$ fails or ceases to be analytic is called a singular point. [যে বিন্দুতে একটি ফাংশন $f(z)$ বৈশ্লেষিক হতে ব্যর্থ হয় বা বৈশ্লেষিক হতে বিরত হয় বা থেমে যায়, সেই বিন্দুকে ব্যতিচার বিন্দু বলে।

4.2. Types of singularities : **[NUH-93, 96, 11, DUH-05]**

Isolated and non-isolated singularity : Let $z = z_0$ be a singularity of $f(z)$. If there is no other singularity with in a small circle surrounding the point $z = z_0$, then this point is called an isolated singularity. If the singularity $z = z_0$ is not an isolated singularity then it is called a non-isolated singularity. [**বিচ্ছিন্ন ও অবিচ্ছিন্ন ব্যতিচার বিন্দু :** মনেকরি $f(z)$ এর একটি ব্যতিচার বিন্দু $z = z_0$ যদি $z = z_0$ কে একটি ছোট বৃত্ত ঘিরে তার ভিতর আর কোন ব্যতিচারিতা না থাকে তখন এই বিন্দুকে বিচ্ছিন্ন ব্যতিচার বিন্দু বলে। যদি $z = z_0$ বিচ্ছিন্ন ব্যতিচার বিন্দু না হয় তখন ইহাকে অবিচ্ছিন্ন ব্যতিচার বিন্দু বলে।]

Ordinary point : If $z = z_0$ is not a singular point of $f(z)$ and there can be found a small circle surrounding the point $z = z_0$ which encloses no singular point, then $z = z_0$ is called an ordinary point of $f(z)$. [সাধারণ বিন্দু : যদি $z = z_0$ বিন্দু $f(z)$ এর ব্যতিচার বিন্দু না হয় এবং $z = z_0$ বিন্দুকে ঘিরে একটি ছোট বৃত্ত পাওয়া যায় যাহা আর কোন ব্যতিচার বিন্দুকে ধারণ করে না, তখন $z = z_0$ কে $f(z)$ এর সাধারণ বিন্দু বলে।]

Pole : If there exists a positive integer n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$$

then $z = z_0$ is called a pole of order n. [NUH-02, 03, 06, 08]

If $n = 1$ then z_0 is called a simple pole.

If $n = 2$ then z_0 is called a double pole and so on.

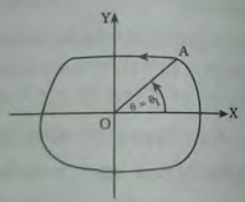
[পোল ঃ যদি একটি ধনাত্মক পূর্ণসংখ্যা বিদ্যমান থাকে যে

$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$ তখন $z = z_0$ কে n ক্রমের পোল বলে। যদি $n = 1$ হয় তখন z_0 কে সরল পোল বলে। যদি $n = 2$ হয় তখন z_0 কে দ্বিপোল বলে।]

Branch points and branch lines [NUH-2004, 2007]

To define branch point and branch line we discuss the situation by the following example.

Let $w = z^{1/2}$. Suppose that z is allowed to make a complete circuit around the origin $(z = 0)$ in the counterclockwise direction.



In polar form let $z = re^{i\theta}$.

Then $w = (re^{i\theta})^{1/2} = \sqrt{r}\, e^{i\theta/2}$.

If in this case $\theta = \theta_1$ then $w = \sqrt{r}\, e^{i\theta_1/2}$.

After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and

$$w = \sqrt{r}\, e^{i(\theta_1 + 2\pi)/2} = \sqrt{r}\, e^{i(\pi + \theta_1/2)}$$

$$= \sqrt{r}\left[\cos\left(\pi + \frac{\theta_1}{2}\right) + i \sin\left(\pi + \frac{\theta_1}{2}\right)\right]$$

$$= \sqrt{r}\left[-\cos\frac{\theta_1}{2} - i \sin\frac{\theta_1}{2}\right]$$

$$= -\sqrt{r}\, e^{i\theta_1/2}$$

Again, after a second complete circuit back to A, we have $\theta = \theta_1 + 4\pi$ and

$$w = \sqrt{r}\, e^{i(\theta_1 + 4\pi)/2} = \sqrt{r}\, e^{i(2\pi + \theta_1/2)}$$

$$= \sqrt{r}\left[\cos\left(2\pi + \frac{\theta_1}{2}\right) + i \sin\left(2\pi + \frac{\theta_1}{2}\right)\right]$$

$$= \sqrt{r}\left[\cos\frac{\theta_1}{2} + i \sin\frac{\theta_1}{2}\right]$$

$$= \sqrt{r}\, e^{i\theta_1/2}$$

Thus we see that in the first complete circuit we do not obtain the same value of w but after second complete circuit we obtain the same value of w with which we started.

The above situation can be stated as, if $0 \leq \theta \leq 2\pi$ we are one branch of the multiple valued function $w = z^{1/2}$, while if $2\pi \leq \theta \leq 4\pi$ we are on the other branch of the function.

Again, let $w = f(z) = z^{1/2}$

$$\Rightarrow u(r, \theta) + iv(r, \theta) = \sqrt{r}\, e^{i\theta/2} = \sqrt{r}\left(\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right) \text{ ...... (1)}$$

$$\Rightarrow u(r, \theta) = \sqrt{r} \cos\frac{\theta}{2} \text{ and } v(r, \theta) = \sqrt{r} \sin\frac{\theta}{2} \text{ ...... (2)}$$

$$\Rightarrow \text{In short, } u = \sqrt{r} \cos\frac{\theta}{2} \text{ and } v = \sqrt{r} \sin\frac{\theta}{2}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos\frac{\theta}{2} \text{ and } \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}} \sin\frac{\theta}{2}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2} \sin\frac{\theta}{2} \text{ and } \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2} \cos\frac{\theta}{2}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos\frac{\theta}{2} = \frac{1}{r} \cdot \frac{\sqrt{r}}{2} \cos\frac{\theta}{2} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

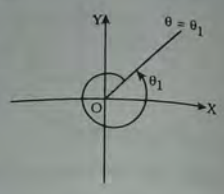
and $$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}} \sin\frac{\theta}{2} = \frac{1}{r} \cdot \frac{\sqrt{r}}{2} \sin\frac{\theta}{2} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Thus, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ which are polar form of Cauchy-Riemann equations.

Also, $\frac{d}{dz}(z^{1/2}) = \frac{1}{2\sqrt{z}}$

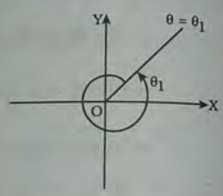
The function $z^{1/2} = \sqrt{r}\, e^{i\theta/2}$, $|z| > 0$, $-\pi < \theta < \pi$ is called the principal branch of the multivaled function $z^{1/2}$.

The function defined in (1) is single valued and continuous on $|z| > 0, \theta_1 < \arg z < \theta_1 + 2\pi$.



The function in (1) is not continuous on the line $\theta = \theta_1$ as there are points arbitrarily close to z at which the values of $v(r, \theta)$ are nearer to $\sqrt{r}\sin\frac{\theta_1}{2}$ and also points such that the values of $v(r, \theta)$ are nearer to $-\sqrt{r}\sin\frac{\theta_1}{2}$. The above discussion also shows that the function in (1) is not only continuous but also analytic. Now we define the following.

Branch line (Branch cut) : A branch line (cut) is a portion of a line or curve which is introduced in order to define the branch of a multivalued function.



Example : The origin and $\theta = \theta_1$ form branch cut for the function

$$w = f(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2}, r > 0, \theta_1 < \theta < \theta_1 + 2\pi.$$

Branch point : [NUH-2013]

A multivalued function f(z) defined in some domain S is said to have a branch point at z_0 if, when z describes an arbitrary small circle about z_0, then for every branch F of f, F(z) does not return to its original value.

[ব্রাঞ্চ পয়েন্ট ঃ একটি ডোমেন S এ বর্ণিত একটি বহুমানী ফাংশন f(z) এর z_0 বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে যদি, f এর প্রত্যেক ব্রাঞ্চF এর জন্য z_0 বিন্দুর চারিদিকে z একটি ইচ্ছাধীন ক্ষুদ্র বৃত্ত বর্ণনা করে যেখানে F(z) তার আদি মান দিতে পারে না।]

Example : Let $w = f(z) = z^{1/2}$.

In polar form $z = re^{i\theta}$, so $w = \sqrt{r}\, e^{i\theta/2}$

After complete circuit, θ becomes $\theta + 2\pi$ and w becomes,

$$w = \sqrt{r}\, e^{i(\theta+2\pi)/2} = -\sqrt{r}\, e^{i\theta/2}.$$

which shows that w has not returned to its original value. Since a complete circuit about $z = 0$ altered the branch of the function $w = f(z) = z^{1/2}$, so $z = 0$ is a branch point.

Removable singularity : If $\lim_{z \to z_0} f(z)$ exists then z_0 is called a removable singularity of f(z).

Essential singularity : A singular point which is not a pole, branch point or removable singularity is called an essential singularity.

Singularities at infinity : The function f(z) has a singularity at $z = \infty$ if $w = 0$ is a singularity of $f\left(\frac{1}{w}\right)$.

N. B. If a function f(z) is single-valued and has a singularity, then this singularity is either a pole or an essential singularity. For this reason a pole is sometimes called a non-essential singularity.

Meromorphic function : A complex function f(z) which has poles as its only singularities in the finite part of the plane is called a meromorphic function.

Entire function : A complex function f(z) which has no singularities in the finite part of the plane is called an entire function. The functions e^z, sinz, cosz are entire functions.

4.3. Working rule for poles and singularies :

1. (i) If $\lim_{z \to z_0} f(z) = \infty$ then $z = z_0$ is a pole of f(z).

   (ii) If there are only m terms in the negative powers of $z - z_0$ then $z = z_0$ is a pole of order m.
2. If $\lim_{z \to z_0} f(z)$ exists finitely then $z = z_0$ is a removable singularity
3. If $\lim_{z \to z_0} f(z)$ does not exist then $z = z_0$ is an essential singularity.
4. If the principal part of f(z) contains infinite number of terms then $z = z_0$ is an isolated essential singularity

[In the expansion $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$, $0 < |z-z_0| < R$, the second term $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ is called the principal part of f(z).]

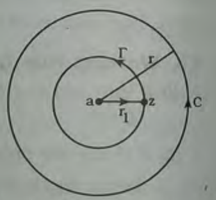
4.4. Taylor's theorem :

Theorem–1. If f(z) is analytic for all values of z inside a circle C with centre at a, then [যদি a কেন্দ্র বিশিষ্ট বৃত্ত C এর ভিতরে z এর সকল মানের জন্য f(z) বৈশ্লেষিক হয় তবে]

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \frac{(z-a)^3}{\lfloor 3} f'''(a) + \cdots$$

[NUH–2005, 2013(Old), 2008, 2012(Old), 2013, 2015 NU(Pre)–2011, DUH–1975, 1983, 1988, 2005, DUM–1988, RUH–1973, 1982, 1984, 1988, CUH–1981]

Proof : Let a be the centre and r be the radius of the circle C. Let z be any point inside C such that $|z-a| = r_1 < r$. Then by Cauchy's integral formula we have [মনে করি C বৃত্তের কেন্দ্র a এবং ব্যাসার্ধ r, ধরি C এর ভিতর z যে কোন বিন্দু যেন $|z-a| = r_1 < r$. তখন কচির যোজিত সূত্র দ্বারা পাই]



$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)-(z-a)}\,dw$$

$$= \frac{1}{2\pi i}\oint_C \left(1-\frac{z-a}{w-a}\right)^{-1} \frac{f(w)}{w-a}\,dw$$

$$= \frac{1}{2\pi i}\oint_C \left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^n + \left(\frac{z-a}{w-a}\right)^{n+1}+\left(\frac{z-a}{w-a}\right)^{n+2}+\cdots\infty\right]\frac{f(w)}{w-a}\,dw$$

$$= \frac{1}{2\pi i}\oint_C \left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n\left\{1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots\right\}\right]\frac{f(w)\,dw}{w-a}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}\,dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n} + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)dw}{(w-a)^{n+1}}\cdot\frac{1}{1-\frac{z-a}{w-a}}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}\,dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n} + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n\,(w-z)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}\,dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n} + U_n \;\cdots\cdots (1)$$

Where [যেখানে] $U_n = \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n\,(w-z)}$

Now [এখন] $|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| \geq r - r_1$

where $|w-a| = r$ for all points on C. [যেখানে C এর উপর সকল বিন্দুর জন্য $|w-a| = r$]

Since f(z) is analytic inside C, so there exist a constant M such that $|f(w)| \leq M$ and $\oint_C |dw| = 2\pi r$ = length of the circle. [যেহেতু C এর ভিতর f(z) বৈশ্লেষিক, সুতরাং একটি ধ্রুবক M থাকবে যেন $|f(w)| \leq M$ এবং $\oint_C |dw| = 2\pi r$ = বৃত্তের দৈর্ঘ্য ।]

$$\therefore |U_n| = \left|\frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n\,(w-z)}\right| \leq \frac{1}{2\pi}\frac{|(z-a)^n|\;|f(w)|\oint_C |dw|}{|(w-a)^n|\;|w-z|}$$

$$\leq \frac{1}{2\pi}\frac{r_1^n\, M\, 2\pi r}{r^n\,(r-r_1)} = \frac{M}{1-\frac{r_1}{r}}\left(\frac{r_1}{r}\right)^n \to 0 \text{ when } n\to\infty$$

$$\left[\left|\frac{r_1}{r}\right| < 1 \Rightarrow \left|\frac{r_1}{r}\right|^n \to 0 \text{ as } n\to\infty\right]$$

Thus, when $n\to\infty$ we have [অতএব যখন $n\to\infty$ তখন পাই]

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \frac{(z-a)^3}{\lfloor 3} f'''(a) + \cdots\infty$$

LAURENT'S THEOREM

Theorem–2. If f(z) is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with centre at a and respective radii r_1 and r_2 $(r_2 < r_1)$, then for all z in R. [r_1 ও r_2 $(r_2 < r_1)$ ব্যাসার্ধ বিশিষ্ট a এক কেন্দ্রিক দুইটি বৃত্ত C_1 ও C_2 দ্বারা তৈরী রিং আকারের R এলাকার ভিতর ও সীমানার উপর যদি f(z) বৈশ্লেষিক হয়, তখন R এর সকল z এর জন্য]

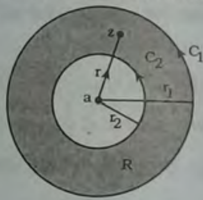
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where $a_n = \dfrac{1}{2\pi i}\oint_{C_1} \dfrac{f(w)}{(w-a)^{n+1}}\, dw,\ n = 0, 1, 2, \cdots$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{-n+1}}\, dw,\ n = 1, 2, 3, \cdots$$

[DUH–1982, 1985, 1988, 2003, RUH–1976, 1979, 1983, RUMP–1985, 1988, CUH–1982]

Proof : The ring shaped region [রিং আকারের এলাকা $R = \{|z-a| = r,\ r_2 < r < r_1\}$. Then by Cauchy's integral formula we have [তখন কচির যোজিত সূত্র দ্বার পাই]



$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\, dw \cdots\cdots (1)$$

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a}\left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n + \cdots\right]$$

$$\Rightarrow \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)\, dw}{w-z} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a}\, dw + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2}\, dw + \cdots$$

$$+ \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^n}\, dw + U_n$$

$$= a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + U_n \cdots\cdots (2)$$

where $a_0 = \dfrac{1}{2\pi i}\oint_{C_1} \dfrac{f(w)}{w-a}\, dw,\ a_1 = \dfrac{1}{2\pi i}\oint_{C_1} \dfrac{f(w)}{(w-a)^2}\, dw, \cdots$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\, dw \text{ and } U_n = \frac{1}{2\pi i}\oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\, dw$$

Again, $-\dfrac{1}{w-z} = \dfrac{1}{z-w} = \dfrac{1}{z-a-(w-a)} = \dfrac{1}{z-a}\left(1 - \dfrac{w-a}{z-a}\right)^{-1}$

$$= \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots + \left(\frac{w-a}{z-a}\right)^{n-1} + \left(\frac{w-a}{z-a}\right)^n\right]$$

$$\Rightarrow -\frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{z-a}\, dw + \frac{1}{2\pi i}\oint_{C_2} \frac{w-a}{(z-a)^2} f(w)\, dw$$

$$+ \cdots + \frac{1}{2\pi i}\oint_{C_2} \frac{(w-a)^{n-1}}{(z-a)^n} f(w)\, dw + V_n$$

$$= \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + V_n \cdots\cdots (3)$$

where $a_{-1} = \dfrac{1}{2\pi i}\oint_{C_2} f(w)\, dw,\ a_{-2} = \dfrac{1}{2\pi i}\oint_{C_2} (w-a) f(w)\, dw, \cdots,$

$a_{-n} = \oint_{C_2} (w-a)^{n-1} f(w)\, dw$ and $V_n = \dfrac{1}{2\pi i}\oint_{C_2} \left(\dfrac{w-a}{z-a}\right)^n \dfrac{f(w)}{z-w}\, dw$

From (1), (2) and (3) we have [(1), (2) ও (3) হতে পাই]

$$f(z) = a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1}$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + U_n + V_n \cdots\cdots (4)$$

Now, for all points w on C_1 we have [এখন C_1 এর উপর সকল বিন্দু w এর জন্য পাই]

$$|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| \geq r_1 - r$$

and for all points w on C_2 we have [এবং C_2 এর উপর সকল বিন্দু w এর জন্য পাই]

$$|z-w| = |(z-a)-(w-a)| \geq |z-a| - |w-a| \geq r - r_2$$

Length of the circle C_1 [C_1 বৃত্তের দৈর্ঘ্য] $= \oint_{C_1} |dw| = 2\pi r_1$

and Length of the circle C_2 [এবং C_2 বৃত্তের দৈর্ঘ্য] $= \oint_{C_2} |dw| = 2\pi r_2$

f(z) is continuous, so it is bounded. Then there exists a constant M_1 such that $|f(z)| \leq M_1$ on C_1 and a constant M_2 such that $|f(z)| \leq M_2$ on C_2. [f(z) অবিচ্ছিন্ন, সুতরাং ইহা সীমায়িত। তখন একটি ধ্রুবক M_1 বিদ্যমান থাকবে যেন C_1 এর উপর $|f(z)| \leq M_1$ এবং একটি ধ্রুবক M_2 থাকবে যেন C_2 এর উপর $|f(z)| \leq M_2$ হয়।

Also, $r < r_1$ and $r_2 < r$

$$\Rightarrow \frac{r}{r_1} < 1 \text{ and } \frac{r_2}{r} < 1$$

$$\Rightarrow \left(\frac{r}{r_1}\right)^n < 1 \text{ and } \left(\frac{r_2}{r}\right)^n < 1$$

$$\Rightarrow \left(\frac{r}{r_1}\right)^n \to 0 \text{ and } \left(\frac{r_2}{r}\right)^n \to 0 \text{ as } n \to \infty$$

i. e. $\lim\limits_{n\to\infty} \left(\frac{r}{r_1}\right)^n = 0$ and $\lim\limits_{n\to\infty} \left(\frac{r_2}{r}\right)^n = 0$

$$\therefore |U_n| = \left|\frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw\right| \leq \frac{1}{2\pi}\left|\frac{z-a}{w-a}\right|^n \frac{|f(w)|\oint_{C_1}|dw|}{|w-z|}$$

$$\Rightarrow |U_n| \leq \frac{1}{2\pi}\left(\frac{r}{r_1}\right)^n \frac{M_1}{r_1 - r} 2\pi r_1 = \frac{M_1}{1 - \frac{r}{r_1}}\left(\frac{r}{r_1}\right)^n = 0 \text{ when } n \to \infty$$

and $$|V_n| = \left|\frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right| \leq \frac{1}{2\pi}\left|\frac{w-a}{z-a}\right|^n \frac{|f(w)|\oint_{C_2}|dw|}{|z-w|}$$

$$\Rightarrow |V_n| \leq \frac{1}{2\pi}\left(\frac{r_2}{r}\right)^n M_2 \frac{2\pi r_2}{r - r_2} = \frac{M_2}{\frac{r}{r_2} - 1}\left(\frac{r_2}{r}\right)^n = 0 \text{ when } n \to \infty$$

Thus, from (4) we have [অতএব (4) হতে পাই]

$$f(z) = \{a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + \cdots\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + \cdots\right\}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}. \qquad \text{(Proved)}$$

Theorem–3. If f(z) has a pole at z = a, then $|f(z)| \to \infty$ as $z \to a$.

[NUH–1996]

Proof : Let z = a is a pole of order m. Then by Laurent's Theorem [z = a, m ক্রমের একটি পোল। তখন লরেন্টের উপপাদ্য দ্বারা পাই]

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{1}{(z-a)^m}[b_m + b_1(z-a)^{m-1} + \cdots + b_{m-1}(z-a)]$$

When $z \to a$ then the square brakets term tends to b_m and the whole right hand expression tend to infinity. [যখন $z \to a$ তখন বর্গ বন্ধনীর পদ b_m হয় এবং ডানদিকের সম্পূর্ণ রাশি অসীম হয়।]

Hence [অতএব] $|f(z)| \to \infty$ as $z \to a$.

4.5. Residues and Residues theorem :

[NUH–2000, 02, 06 (Old), NU(Pre)–06, 07, 13, 14 DUH–83, 89, 90] [NUH–2011, NU(Phy)–2004]

Definition : If the function f(z) is analytic within a circle C of radius r and centre a, except at z = a, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laruent's expansion [সংজ্ঞা ঃ r ব্যাসার্ধ ও a কেন্দ্র বিশিষ্ট C বৃত্তের ভিতর z = a ব্যতীত যদি f(z) ফাংশন বৈশ্লেষিক হয়, তখন লরেন্টের বিস্তৃতিতে $\frac{1}{z-a}$ এর সহগ a_{-1}]

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-1}}{(z-a)^n}$$

$$= \{a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\right\}$$

around z = a is called the residue of f(z) at z = a. It is denoted by Res (a) or a_{-1}. [z = a এর চারিদিকে z = a এ f(z) এর অবশেষ বলে। ইহাকে Res (a) বা a_{-1} দ্বারা নির্দেশ করা হয়]

Theorem–4. Let f(z) be analytic inside and on a simple closed curve C except at point a inside C, then

If z = a is a simple pole then $a_{-1} = \lim\limits_{z\to a} (z-a) f(z)$

Proof : The corresponding Laurent series is [অনুসঙ্গী লরেন্ট ধারা]

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{a_{-1}}{z-a}, \text{ where } a_{-1} \neq 0$$

Multiplying this by (z − a) and then taking limit z → a we get [ইহাকে (z − a) দ্বারা গুণ করে এবং অতপর z → a লিমিট নিয়ে পাই]

$$\lim_{z\to a} (z-a) f(z) = \lim_{z\to a}\left[\sum_{n=0}^{\infty} a_n (z-a)^{n+1} + a_{-1}\right]$$

$$= 0 + a_{-1}$$

$$\Rightarrow a_{-1} = \lim_{z\to a} (z-a) f(z) \qquad \text{(Proved)}$$

Residue at a multiple point.

Theorem–5. Let f(z) be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of f(z) at a is given by

$$a_{-1} = \lim_{z\to a} \frac{1}{\lfloor\underline{m-1}} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

[NUH–93, 2000, 02, 05, 06(Old), 11, NU(Pre)–08, 13, 14, DUH–04]

Proof : If f(z) has a pole a of order m, then Laurent series of f(z) is [যদি f(z) এর m ক্রমের একটি পোল a থাকে, তখন f(z) এর লরেন্ট ধারা।]

$$f(z) = \{a_0 + a_1 (z-a) + a_2 (z-a)^2 + \cdots\} + \frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-m+1}}{(z-a)^{m-1}} + \frac{a_{-m}}{(z-a)^m}$$

$$\Rightarrow (z-a)^m f(z) = \{a_0(z-a)^m + a_1(z-a)^{m+1} + a_2(z-a)^{m+2} + \cdots\} + a_{-1}(z-a)^{m-1} + a_{-2}(z-a)^{m-2} + \cdots + a_{-m+1}(z-a) + a_{-m};$$

[Multiplying by $(z-a)^m$]

Differentiating this (m − 1) times w. r. to z we get [ইহাকে z এর সাথে (m − 1) বার অন্তরীকরণ করে পাই]

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = \left\{a_0 \lfloor\underline{m}\,(z-a) + \frac{a_1\lfloor\underline{m-1}}{\lfloor\underline{2}}(z-a)^2 + \cdots\right\} + a_{-1}\lfloor\underline{m-1}$$

Now taking limit z → a we get [এখন z → a লিমিট নিয়ে পাই]

$$\lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = a_{-1}\lfloor\underline{m-1}$$

$$\therefore a_{-1} = \lim_{z\to a} \frac{1}{\lfloor\underline{m-1}} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}. \qquad \text{(Proved)}$$

CAUCHY'S RESIDUE THEOREM

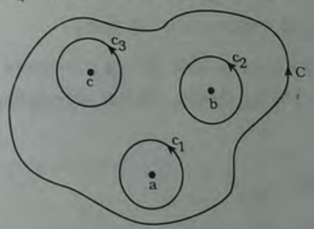
Theorem–6. If f(z) is analytic inside and on a simple closed curve C except at a finite number of points a, b, c, ··· inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \cdots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

$$= 2\pi i \text{ (sum of the residues)}.$$

[NUH–98, 02, 06, 11, 15, NU(Phy)–03, 04, DUH–03]

Proof : Let us construct circles $c_1, c_2, c_3, \cdots$ with centre a, b, c, ··· which lie entirely within C. Then[a, b, c, ··· কেন্দ্র বিশিষ্ট c_1, c_2, c_3 বৃত্তগুলি অংকন করি যারা সম্পূর্ণভাবে C এর মধ্যে অবস্থিত। তখন]



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots$$

$$= 2\pi i\, a_{-1} + 2\pi i\, b_{-1} + 2\pi i\, c_{-1} + \cdots$$

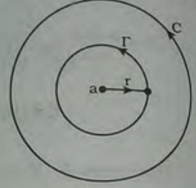
$$= 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots$$

$$= 2\pi i \text{ (sum of the residues)} \qquad \text{(Proved)}$$

MAXIMUM MODULUS THEOREM

Theorem–7. Let f(z) be analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

[DUH–75, 86, 89, 90, 02, 04, JUH–89, 91]



Proof : Given f(z) is analytic inside and on C, so f(z) is continuous inside and on C. Hence f(z) reaches its maximum value M (upper bound) with in or on C.

If possible let $|f(z)|$ have its maximum value M (say) at a point $z = a$ lying with in C, and not on C.

i. e. $|f(a)| = M$, where a lies with in C.

If Γ be a small circle of radius r and centre at $z = a$, lying entriety with in C, then f(z) is analytic with in and on Γ. Hence by Cauchy's integral formula, we have [প্রমাণ ঃ দেওয়া আছে C এর ভিতর ও উপর f(z) বৈশ্লেষিক, সুতরাং C এর ভিতর ও উপর f(z) অবিচ্ছিন্ন। অতএব C এর ভিতর বা উপর f(z) ইহার সর্বোচ্চ মান M এ পৌছঁবে।

যদি সম্ভব হয় তবে ধরি $z = a$ বিন্দুতে $|f(z)|$ এর সর্বোচ্চ মান M, C এর ভিতর অবস্থিত এবং C এর উপর না।

অর্থাৎ $|f(a)| = M$, যেখানে a হল C এর ভিতরে।

যদি r ব্যাসার্ধ ও $z = a$ কেন্দ্র বিশিষ্ট ছোট বৃত্ত Γ সম্পূর্ণভাবে C এর ভিতরে অবস্থিত হয়, তখন Γ এর ভিতর ও উপর f(z) বৈশ্লেষিক। অতএব কচির যোজিত সূত্র দ্বারা পাই]

$$f(a) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)\,dz}{z-a}$$

$$\Rightarrow |f(a)| = \left|\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)\,dz}{z-a}\right| \le \frac{1}{|2\pi i}\frac{|f(z)|\oint_{\Gamma}|dz|}{|z-a|} \quad \text{...... (1)}$$

Since f(z) is continuous at $z = a$, then there exist $\varepsilon > 0$, such that $|f(z)| \le M - \varepsilon$, $\forall$ z on the circle Γ. Also, $|z - a| = r$. Thus, from (1) we get,

$$|f(a)| \le \frac{1}{2\pi}\frac{M-\varepsilon}{r}\oint_{\Gamma}|dz|$$

$$\Rightarrow M \le \frac{1}{2\pi}\cdot\frac{M-\varepsilon}{r}\cdot 2\pi r$$

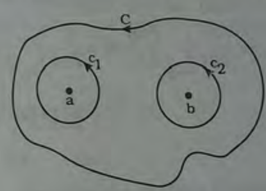
$\Rightarrow M \le M - \varepsilon$, which is impossible, since M can not less than $M - \varepsilon$.

Hence the maxim of $|f(z)|$ occurs on C. (Proved)

THE ARGUMENT THEOREM

Theorem–8. Let f(z) be analytic inside and on a simple closed curve C except for a pole $z = a$ of order p inside C. Suppose also that inside C f(z) has only one zero $z = b$ of order n and no zeros on C. Prove that

$$\frac{1}{2\pi i}\oint_{C}\frac{f'(z)}{f(z)}\,dz = n - p$$ **[DUH–2002]**



Proof : Let c_1 and c_2 be two non overlapping circles lying inside C enclosing $z = a$ and $z = b$ respectively. Then

$$\frac{1}{2\pi i}\oint_{C}\frac{f'(z)}{f(z)}\,dz$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{C_2}\frac{f'(z)}{f(z)}\,dz \quad \text{...... (1)}$$

Since f(z) has a pole of order p at $z = a$ we have

$$f(z) = \frac{F(z)}{(z-a)^p} \quad \text{...... (2)}$$

where F(z) is analytic and different from zero inside and on c_1.

Taking logarithm of (2) of both sides we get

$$\log f(z) = \log F(z) - p\log(z-a)$$

Differentiating this w. r. to z we get,

$$\frac{1}{f(z)} \cdot f'(z) = \frac{F'(z)}{F(z)} - p \cdot \frac{1}{z-a}$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{C_1} \frac{f'(z)}{f(z)} dz$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{F'(z)}{F(z)} dz - \frac{p}{2\pi i} \oint_{C_1} \frac{dz}{z-a}$$

$$= \frac{1}{2\pi i} \cdot 0 - \frac{p}{2\pi i} \cdot 2\pi i \cdot 1$$

$$= -p \;\cdots\cdots (3)$$

[$F(z)$ is analytic,
$\therefore F'(z)$ and $\frac{F'(z)}{F(z)}$ are also analytic on c_1
$\therefore \oint_C \frac{F'(z)}{F(z)} dz = 0$]

Again, since f(z) has a zero of order n, at z = b, we have

$$f(z) = (z-b)^n G(z)$$

where G(z) is analytic and different from zero inside and on c_2.

$$\Rightarrow \log f(z) = n \log (z-b) + \log G(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{G'(z)}{G(z)}; \text{ diff. w. r. to z.}$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{C_2} \frac{f'(z)}{f(z)} dz = \frac{n}{2\pi i} \oint_{C_2} \frac{1}{z-b} dz + \frac{1}{2\pi i} \oint_{C_2} \frac{G'(z)}{G(z)} dz$$

$$= \frac{n}{2\pi i} \cdot 2\pi i + \frac{1}{2\pi i} \times 0$$

$$= n \;\cdots\cdots (4)$$

By (3) and (4), (1) becomes

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = n - p \qquad \text{(Proved)}$$

THE GENERAL ARGUMENT THEOREM

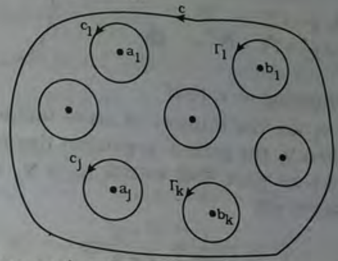
Theorem–9. Let f(z) be analytic inside and on a simple closed curve C apart from a finite number of poles inside C and no zeros on C. Then $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P$

Where N is the number of zeros

and P is the number of poles inside C counting multiplicities.

[NUH–1989, 1991, 1985, 2006, RUH–1978, 1986, CHU–1981]

Proof : Let $a_1, a_2, \cdots a_j$ and $b_1, b_2, \cdots, b_k$ be the respective poles and zeros of f(z) lying inside C and suppose their multiplicities are $p_1, p_2, \cdots, p_j$ and $n_1, n_2, \cdots, n_k$.



We now enclose each pole and zero by non-overlapping circles $c_1, c_2, \cdots, c_j$ and $\Gamma_1, \Gamma_2; \cdots, \Gamma_k$.

This can always be done since the poles and zeros are isolated. Then we have

[প্রমাণ ঃ মনে করি $a_1, a_2, \cdots a_j$ এবং $b_1, b_2, \cdots, b_k$, C এর ভিতর অবস্থিত f(z) এর স্ব-স্ব পোল ও শূণ্যক সমূহ এবং তাদের সংখ্যাধিক্য $p_1, p_2, \cdots, p_j$ ও $n_1, n_2, \cdots, n_k$. আমরা এখন প্রত্যেক পোল ও শূন্যককে একসাথে মিলে না যায় এমন বৃত্ত $c_1, c_2, \cdots, c_j$ ও $\Gamma_1, \Gamma_2; \cdots, \Gamma_k$ দ্বারা আবৃত করি। যেহেতু পোল ও শুণ্যক বিচ্ছিন্ন তাই এরূপ সব সময় করা যায়। তখন আমরা পাই]

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \sum_{r=1}^{j} \frac{1}{2\pi i} \oint_{\Gamma_r} \frac{f'(z)}{f(z)} dz + \sum_{r=1}^{k} \frac{1}{2\pi i} \oint_{C_r} \frac{f'(z)}{f(z)} dz$$

$$= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r$$

$$= N - P. \qquad \text{(Proved)}$$

ROUCHE'S THEOREM

Theorem–10. If f(z) and g(z) are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then f(z) + g(z) and f(z) have the same number of zeros inside C. [যদি f(z) ও g(z) একটি সাধারণ বদ্ধ বক্ররেখা C এর ভিতর ও উপর বৈশ্লেষিক হয় এবং C এর উপর যদি $|g(z)| < |f(z)|$ হয়, তবে C এর ভিতর f(z) + g(z) ও f(z) এর একই সংখ্যক শূন্যক (zeros) থাকবে।] **[NUH–1995, 2004, 2007, 2012, 2014, DUH–1975, 1978, 1984, 1987, 1989, 1990, 1995, 2002, 2003, RUH–1972, 1975, 1982, 1984, 1986, 1988, CUH–1987, 1989, 1993, JUH–1986, 1989.]**

Proof : Let [ধরি] $F(z) = \frac{g(z)}{f(z)}$

$$\Rightarrow g(z) = f(z)\,F(z)$$

or in short, $g = f\,F$ ...... (1)

$$\Rightarrow g' = f'F + fF' \text{ ...... (2)}$$

[By differentiating w. r. to z]

If N_1 and N_2 are the number of zeros inside C of f + g and f respectively, then by the general argument theorem we have [যদি C এর ভিতর f + g ও f এর শূন্যের সংখ্যা যথাক্রমে N_1 ও N_2 হয়, তবে সাধারণ যুক্তি (আরগুমেন্ট) উপপাদ্য দ্বারা পাই]

$$N_1 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz \text{ ...... (3)}$$

and [এবং] $N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$ ...... (4)

using the fact that these two functions have no poles inside C. [C এর ভিতর এই দুইটি ফাংশনের কোন পোল নাই এই ঘটনা প্রয়োগ করে]

Now [এখন] $N_1 - N_2 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$

$$= \frac{1}{2\pi i}\oint_C \frac{f' + f'F + fF'}{f + fF}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz; \quad \text{By (1) and (2)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f'(1 + F) + fF'}{f(1 + F)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \left(\frac{f'}{f} + \frac{F'}{1 + F}\right)dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz + \frac{1}{2\pi i}\oint_C \frac{F'}{1 + F}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C F'(1 + F)^{-1}\,dz$$

$$= \frac{1}{2\pi i}\oint_C F'(1 - F + F^2 - F^3 + \cdots)\,dz$$

= 0, using the given fact $|F| < 1$ one so that the given series is uniformly convergent on C and term by term integration gives the value zero. [$|F| < 1$ ঘটনাটি আরেকবার ব্যবহার করি যেন প্রদত্ত ধারাটি C এর উপর সুষমভাবে অভিসারি হয় এবং পদ ক্রম পদ যোজিত ফল শূন্য দেয় ।]

Thus [অতএব] $N_1 = N_2$ and hence Proved [এবং প্রমাণিত] ।

SOLVED PROBLEMS

Problem-1. $f(z) = \frac{1}{(z-1)^2}$ has a singularity at z = 1, because

$$f(1) = \frac{1}{(1-1)^2} = \infty$$

That is, f(z) fails to be analytic at z = 1.

Also, $\lim_{z\to 1} (z-1)^2 f(z) = \lim_{z\to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1 \neq 0.$

Thus z = 1 is a pole of order 2.

Problem-2. $f(z) = \frac{\sin z}{z}$ has a singular point at z = 0.

But $\lim_{z\to 0} \frac{\sin z}{z} = 1$, so the singular point z = 0 is a removal singularity of f(z).

Problem-2(a). Let $f(z) = e^{1/z-1}$. Then $f(1) = e^{\infty} = \infty$

$\therefore$ z = 1 is a singular point of f(z).

But $\lim_{z\to 1} f(z) = \lim_{z\to 1} e^{1/z-1} = \infty$

This shows that z = 1 is not a removal singularity.

Also z = 1 is not a branch point or pole of f(z).

Hence z = 1 is an essential singularity.

Problem-3. Classify singularitics. Give an example of each kind. [ব্যতিক্রমী বিন্দুগুলির শ্রেণীবিভাগ কর। প্রতিটি ধরণের একটি করে উদাহরণ দাও।]

[NUH-2008]

OR, Discuss the different kinds of singularities [বিভিন্ন প্রকার ব্যতিক্রমী বিন্দুগুলির বর্ণনা দাও।] **[NUH-2011]**

OR, Classify the different kinds of singularities of a complex function and illustrate. **[NUH-2015]**

Solution : A point at which an analytic function f(z) fails or ceases to be analytic is called a singular point.]

The singular points are of two types-isolated and non-isolated singular points.

Isolated singular point : Let $z = z_0$ be a singularity of f(z). If there is no other singularity with in a small circle surrounding the point $z = z_0$, that is, there exists a deleted neighbourhood of z_0 $\{z : 0 < |z - z_0| < \delta\}$ in which f(z) is analytic, then the point is called an isolated singularity.

Example : $f(z) = \frac{1}{z-1}$ has an isolated singularity at $z = 1$, since $f(z)$ is analytic in $0 < |z-1| < \delta, \delta > 0$.

Isolated singular points are of three types :

1. **Removable singularity**, 2. **Pole** and 3. **Essential singularity. Removable singularity**. If $\lim_{z \to z_0} f(z)$ exists then z_0 is called a removable singularity of $f(z)$.

Example : $f(z) = \frac{\sin z}{z}$ has a singular point at $z = 0$.

But $\lim_{z \to 0} \frac{\sin z}{z} = 1$, so the singular point $z = 0$ is a removable singularity of $f(z)$.

Pole : If there exists a positive integer n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$$

then $z = z_0$ is called a pole of order n.

Example : $f(z) = \frac{1}{(z-1)^2}$ has a singularity at $z = 1$, because $f(1) = \frac{1}{(1-1)^2} = \infty$. That is, $f(z)$ fails to be analytic at $z = 1$.

Also, $\lim_{z \to 1} (z-1)^2 f(z) = \lim_{z \to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1 \neq 0$.

Thus $z = 1$ is a pole of order 2.

Essential singularity : A singular point which is not a pole, branch point or removable singularity is called an essential singularity.

Example : Let $f(z) = e^{\frac{1}{z-1}}$. Then $f(1) = e^{\infty} = \infty$.

$\therefore z = 1$ is a singular point of $f(z)$. But $\lim_{z \to 1} f(z) = \lim_{z \to 1} e^{\frac{1}{z-1}} = \infty$.

This shows that $z = 1$ is not a removal singularity.

Also $z = 1$ is not a branch point or pole of $f(z)$.

Hence $z = 1$ is an essential singularity.

Singularity at infinity : The function $f(z)$ has a singularity at $z = \infty$ if $w = 0$ is a singularity of $f\left(\frac{1}{w}\right)$.

Example : Let $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$.

Then putting $z = \frac{1}{w}$ we get

$$f\left(\frac{1}{w}\right) = \frac{\frac{1}{w^8} + \frac{1}{w^4} + 2}{\left(\frac{1}{w} - 1\right)^3 \left(\frac{3}{w} + 2\right)^2} = \frac{1 + w^4 + 2w^8}{w^3(1-w)^3 (3+2w)^2}$$

when $w = 0$ then $f\left(\frac{1}{w}\right) = \infty$

$\therefore w = 0 \Rightarrow z = \frac{1}{w} = \frac{1}{0} = \infty$ is a singularity of $f(z)$.

Branch point is also a singular point.

A multivalued function $f(z)$ defined in some domain S is said to have a branch point at z_0 if, when z describes an arbitrary small circle about z_0, then for every branch F of f, $F(z)$ does not return to its original value.

Example : 1. $f(z) = z^{1/2}$ has a branch point at $z = 0$.

2. $f(z) = \ln(z^2 + z - 2)$ has branch point $z^2 + z - 2 = 0$
$\Rightarrow z = 1$ and $z = -2$.

Non-isolated singular point : If the singularity $z = z_0$ is not an isolated singularity then it is called a non-isolated singularity.

Example : Let $f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)}$. Then $f(z) = \infty$ when

$$\tan\left(\frac{\pi}{z}\right) = 0 \Rightarrow \frac{\pi}{z} = n\pi \Rightarrow \frac{1}{z} = n \Rightarrow z = \frac{1}{n}$$

$$\Rightarrow z = 1, \frac{1}{2}, \frac{1}{3}, \cdots, 0$$

Here $z = 0$ is a non isolated singular point and all other are isolated singular points.

[সমাধান ঃ যে বিন্দুতে একটি ফাংশন $f(z)$ বৈশ্লেষিক হতে ব্যর্থ হয় বা বৈশ্লেষিক হতে বিরত হয় বা থেমে যায়, সেই বিন্দুকে ব্যতিচার বিন্দু বলে। ব্যতিচার বিন্দু দুই প্রকার বিচ্ছিন্ন ও অবিচ্ছিন্ন ব্যতিচার বিন্দু।

Isolated [বিচ্ছিন্ন] ব্যতিচার বিন্দু ঃ মনে করি $f(z)$ এর একটি ব্যতিচার বিন্দু $z = z_0$. যদি $z = z_0$ কে একটি ছোট বৃত্ত ঘিরে তার ভিতর আর কোন ব্যতিচারিতা না থাকে তখন এই বিন্দুকে বিচ্ছিন্ন ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ $f(z) = \frac{1}{z-1}$ এর $z = 1$ এ একটি বিচ্ছিন্ন ব্যতিচার বিন্দু আছে, কারণ $0 < |z-1| < \delta, \delta > 0$ এ $f(z)$ বৈশ্লেষিক।

বিচ্ছিন্ন ব্যতিচার বিন্দু তিন প্রকার ঃ

১। অপসারণযোগ্য ব্যতিচার বিন্দু, ২। পোল, ৩। অপরিহার্য ব্যতিচার বিন্দু।

অপসারণযোগ্য ব্যতিচার বিন্দু ঃ যদি $\lim_{z \to z_0} f(z)$ বিদ্যমান থাকে তখন z_0 কে $f(z)$ এর অপসারণযোগ্য ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ $f(z) = \frac{\sin z}{z}$ এর $z = 0$ একটি ব্যতিচার বিন্দু।

কিন্তু $\lim_{z \to 0} \frac{\sin z}{z} = 1$, সুতরাং $z = 0$ হল $f(z)$ এর অপসারণযোগ্য ব্যতিচার বিন্দু।

পোল ঃ যদি একটি ধনাত্মক পূর্ণসংখ্যা n বিদ্যমান থাকে যেন

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$$

তখন $z = z_0$ কে n মাত্রার পোল বলে।

উদাহরণ ঃ $f(z) = \frac{1}{(z-1)^2}$ এর $z = 1$ একটি ব্যতিচার বিন্দু, কারণ $f(1) = \frac{1}{(1-1)^2} = \infty$. অর্থাৎ $z = 1$ এ $f(z)$ বৈশ্লেষিক হতে ব্যর্থ হয়।

কিন্তু $\lim_{z \to 1} (z-1)^2 f(z) = \lim_{z \to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1 \neq 0.$

অতএব $z = 1$ হল 2 মাত্রার একটি পোল।

অপরিহার্য ব্যতিচার বিন্দু ঃ একটি ব্যতিচার বিন্দু যা পোল, ব্রাঞ্চ বিন্দু বা অপসারণযোগ্য ব্যতিচার বিন্দু না তাকে অপরিহার্য ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ ধরি $f(z) = e^{\frac{1}{z-1}}$. তখন $f(1) = e^{\infty} = \infty$

$\therefore z = 1$ হল $f(z)$ এর একটি ব্যতিচার বিন্দু।

কিন্তু $\lim_{z \to 1} f(z) = \lim_{z \to 1} e^{\frac{1}{z-1}} = \infty$

ইহা দেখায় যে $z = 1$ অপসারণযোগ্য ব্যতিচার বিন্দু না।

আরো, $z = 1$, $f(z)$ এর ব্রাঞ্চ বিন্দু বা পোল না।

অতএব $z = 1$ একটি অপরিহার্য ব্যতিচার বিন্দু।

ব্রাঞ্চ বিন্দু ও ব্যতিচার বিন্দু। কোন ডোমেন S এ সংজ্ঞায়িত একটি বহুমানী ফাংশন $f(z)$ এর z_0 বিন্দুতে ব্রাঞ্চ বিন্দু আছে যদি z_0 এর চারিদিকে একটি ইচ্ছাধীন ছোট বৃত্ত অংকিত হলে তখন f এর প্রত্যেক ব্রাঞ্চ F এর জন্য $F(z)$ ইহার আদি অবস্থায় ফিরে আসতে না পারে।

উদাহরণ ঃ ১। $f(z) = z^{1/2}$ ফাংশনের $z = 0$ একটি ব্রাঞ্চ বিন্দু।

২। $f(z) = ln(z^2 + z - 2)$ ফাংশনের $z^2 + z - 2 = 0$

অর্থাৎ $z = 1$ এবং $z = -2$ ব্রাঞ্চ বিন্দু।

অবিচ্ছিন্ন ব্যতিচার বিন্দু ঃ যে ব্যতিচার বিন্দু বিচ্ছিন্ন নয় তাকে অবিচ্ছিন্ন (non-isolated) ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ ধরি $f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)}$

$$f(z) = \infty \text{ যখন } \tan\frac{\pi}{z} = 0$$

$$\Rightarrow \frac{\pi}{z} = n\pi \Rightarrow z = \frac{1}{n}$$

$$\Rightarrow z = 1, \frac{1}{2}, \frac{1}{3}, \cdots, 0.$$

এখানে $z = 0$ অবিচ্ছিন্ন ব্যতিচার বিন্দু এবং অন্য সকল বিচ্ছিন্ন ব্যতিচার বিন্দু।]

Problem–4. $f(z) = ln\,(z^2 - 5z + 6)$ has branch points where

$$z^2 - 5z + 6 = 0$$

$$\Rightarrow (z-2)(z-3) = 0$$

$$\Rightarrow z = 2 \text{ and } z = 3.$$

Thus, $z = 2$ and $z = 3$ are branch points of $f(z) = ln\,(z^2 - 5z + 6)$.

Problem–5. Show that $f(z) = \frac{(z+5i)^3}{(z^2-2z+5)^2}$ has double poles at $z = 1 \pm 2i$ and a simple pole at infinity.

Solution : Given that $f(z) = \frac{(z+5i)^3}{(z^2-2z+5)^2}$

f(z) has singularity when $(z^2-2z+5)^2=0$

$$\Rightarrow z^2-2z+5=0$$

$$\Rightarrow z=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$$

$$\therefore f(z)=\frac{(z+5i)^3}{\{(z-1-2i)(z-1+2i)\}^2}$$

$$=\frac{(z+5i)^3}{\{z-(1+2i)\}^2\{z-(1-2i)\}^2}$$

Now $\lim\limits_{z\to 1+2i}\{z-(1+2i)\}^2 f(z)$

$$=\lim_{z\to 1+2i}\frac{(z+5i)^3}{\{z-(1-2i)\}^2}$$

$$=\frac{(1+2i+5i)^3}{(1+2i-1+2i)^2}=\frac{(1+7i)^3}{-4}\neq 0$$

and $\lim\limits_{z\to 1-2i}\{z-(1-2i)\}^2 f(z)$

$$=\lim_{z\to 1-2i}\frac{(z+5i)^3}{\{z-(1+2i)\}^2}$$

$$=\frac{(1-2i+5i)^3}{(1-2i-1-2i)^2}=\frac{(1+3i)^3}{-4}\neq 0$$

$\therefore$ f(z) has double poles at $z=1\pm 2i$.

Again, $$f\left(\frac{1}{w}\right)=\frac{\left(\frac{1}{w}+5i\right)^3}{\left(\frac{1}{w^2}-\frac{2}{w}+5\right)^2}=\frac{(1+5iw)^3}{w(1-2w+5w^2)}$$

$$\therefore \lim_{w\to 0} wf\left(\frac{1}{w}\right)=\lim_{w\to 0}\frac{(1+5iw)^3}{1-2w+5w^2}=\frac{1}{1}=1\neq 0$$

$\therefore w=0$ is a simple pole of $f\left(\frac{1}{w}\right)$.

Hence $z=\infty$ is a simple pole of f(z).

Problem-6. Prove that $f(z)=e^{-1/z^2}$ has no singularities.

Solution : $f(z)=e^{-1/z^2}=\dfrac{1}{e^{1/z^2}}$

Poles are obtained when $e^{1/z^2}=0$. This is not possible for any real or complex value of z. Hence f(z) has no poles.

Again $f(z)=e^{-1/z^2}=0$ gives $e^{-1/z^2}=0=e^{-\infty}$

$$\Rightarrow \frac{1}{z^2}=\infty\Rightarrow z^2=0\Rightarrow z=0,0.$$

Thus $z=0$ is a zero of f(z) and hence no singularity.

Therefore, $f(z)=e^{-1/z^2}$ has no singularities.

Problem-7. Discuss the nature of the singularities of $f(z)=\dfrac{z-2}{z^2}\sin\dfrac{1}{z-2}$.

Solution : When $z^2=0\Rightarrow z=0,0$ then $f(z)=\infty$.

$$\lim_{z\to 0} z^2\cdot f(z)=\lim_{z\to 0} z^2\cdot\frac{z-2}{z^2}\sin\frac{1}{z-2}$$

$$=\lim_{z\to 0}(z-2)\sin\frac{1}{z-2}$$

$$=-2\sin\left(\frac{-1}{2}\right)=2\sin\frac{1}{2}\neq 0.$$

$\therefore z=0$ is a pole of order 2.

Problem-8. Find the singular points of the function $\dfrac{z^2}{(z+1)^2}\sin\left(\dfrac{1}{z-1}\right)$ and determine their nature.

[NUH-1995, 2000, 2015, NU(Pre)-2006, 2014]

Solution : Let [ধরি] $f(z)=\dfrac{z^2}{(z+1)^2}\sin\left(\dfrac{1}{z-1}\right)$

when [যখন] $z=-1$ then [তখন] $f(z)=\infty$

$\therefore z=-1$, is the singular point of the given function. [প্রদত্ত ফাংশনের একটি ব্যতিচার বিন্দু ।]

Now at $z=-1$ we have [এখন $z=-1$ এ পাই]

$$\lim_{z\to -1}(z+1)^2\cdot f(z)=\lim_{z\to -1}(z+1)^2\cdot\frac{z^2}{(z+1)^2}\sin\left(\frac{1}{z-1}\right)$$

$$=\lim_{z\to -1} z^2\sin\left(\frac{1}{z-1}\right)$$

$$= (-1)^2 \sin\left(\frac{1}{-1-1}\right)$$

$$= -\sin\frac{1}{2}, \text{ which is finite. [যাহা সসীম]}$$

$\therefore z = -1$ is a pole of order 2. [$z = -1$, 2 ক্রমের একটি পোল]

Again, We have [আবার আমরা পাই]

$$\sin\frac{1}{z-1} = \frac{1}{z-1} - \frac{1}{\lfloor 3\,(z-1)^3} + \frac{1}{\lfloor 5\,(z-1)^5} + \cdots$$

In the above expansion, there are infinite number of terms in the negative powers of $(z-1)$. Hence $z = 1$ is an isolated essential singularity of $\sin\frac{1}{z-1}$ and hence of $f(z)$. [উপরের বিস্তৃতিতে $(z-1)$ এর ঋণাত্মক ঘাতের অসীম সংখ্যক পদ আছে। অতএব $z = 1$, $\sin\frac{1}{z-1}$ এর একটি বিচ্ছিন্ন অপরিহার্য ব্যতিচার বিন্দু এবং অতএব $f(z)$ এর]

Problem–8(i). Identify the singularities of $f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2-1)^2}$.

[$f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2-1)^2}$ এর সিংগুলারিটি নির্ণয় কর।] **[NUH–2012]**

Solution : Given $f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2-1)^2} = \dfrac{\sin\left(\frac{1}{z}\right)}{(z-1)^2(z+1)^2}$

When [যখন] $z = 1, -1$ তখন $f(z) = \infty$

$\therefore z = 1, -1$ are the singularities of $f(z)$ [$f(z)$ এর সিংগুলারিটি হল $z = 1$, $z = -1$]

$$\text{Now } \lim_{z\to 1} (z-1)^2 \cdot f(z) = \lim_{z\to 1} (z-1)^2 \cdot \frac{\sin\left(\frac{1}{z}\right)}{(z-1)^2(z+1)^2}$$

$$= \lim_{z\to 1} \frac{\sin\left(\frac{1}{z}\right)}{(z+1)^2} = \frac{1}{4}\sin(1)$$

Which is finite [যাহা সসীম]

$\therefore z = 1$ is a pole of order 2 [$z = 1$, 2 ক্রমের একটি পোল]

$$\text{Again [আবার]}, \lim_{z\to -1} (z+1)^2 f(z) = \lim_{z\to -1} (z+1)^2 \cdot \frac{\sin\left(\frac{1}{z}\right)}{(z-1)^2(z+1)^2}$$

$$= \lim_{z\to -1} \frac{\sin\left(\frac{1}{z}\right)}{(z-1)^2} = \frac{1}{4}\sin(-1)$$

Which is finite [যাহা সসীম]

$\therefore z = -1$ is a pole of order 2 [$z = -1$ হল 2 ক্রমের একটি পোল]

Moreove [অধিকন্তু] $\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{\lfloor 3\; z^3} + \frac{1.}{\lfloor 5\; z^5} - \cdots$

In the above expansion, there are infinite number of terms in the negative powers of z. Hence $z = 0$ is an isolated essential singularity of $\sin\frac{1}{z}$ and hence of $f(z)$.

[উপরের বিস্তৃতিতে z এর ঋণাত্মক ঘাতের অসীম সংখ্যক পদ আছে। অতএব $z = 0$, $\sin\frac{1}{z}$ এর একটি বিচ্ছিন্ন অপরিহার্য ব্যতিচার বিন্দু (সিনগুলারিটি) এবং অতএব $f(z)$ এর]

Problem–9. Find the nature and location of the singularities of the function $f(z) = \dfrac{1}{z(e^z-1)}$. **[NUH–2005(Old)]**

OR, Determine and classify all the singularities of the function $f(z) = \dfrac{1}{z(e^z-1)}$. **[NUH–2013]**

Solution : Given that [দেওয়া আছে] $f(z) = \dfrac{1}{z(e^z-1)}$

In the finite z-plane the singularities will be obtained by solving the equation [সসীম z তলে নিম্নের সমীকরণ সমাধান করে ব্যতিচার বিন্দু পাওয়া যাবে]

$$z(e^z-1) = 0$$

$$\Rightarrow z = 0 \text{ or, } e^z - 1 = 0$$

Now [এখন] $e^z - 1 = 0$ gives [দেয়] $e^z = 1 = \cos 0 + i\sin 0$

$$= \cos 2n\pi + i\sin 2n\pi$$

$$= e^{i2n\pi}$$

$\Rightarrow z = i2n\pi$, where [যেখানে] $n = 0, \pm 1, \pm 2, \cdots$

$\therefore$ The singularities are [ব্যতিচার বিন্দুগুলি হল] $z = 0$ and [এবং] $z = i2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$

Problem–10. For the function $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$, locate and name all the singularities in the finite z-plane and also determine where f(z) is analytic.

[NUH–2003, 2006(Old), 2008, NU(Pre)–2011, 2013, DUH–1984]

Solution : Given [দেওয়া আছে] $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$ ...... (1)

In the finite z-plane the singularities will be obtained by solving the equation $(z-1)^3(3z+2)^2 = 0$ [সসীম z তলে ব্যতিচার বিন্দুগুলি পাওয়া যাবে $(z-1)^3(3z+2)^2 = 0$ সমীকরণ সমাধান করে]

$$\Rightarrow (z-1)^3 = 0 \quad \text{or} \quad (3z+2)^2 = 0$$

$$\Rightarrow z = 1, 1, 1 \quad \text{or} \quad z = -\frac{2}{3}, -\frac{2}{3}$$

$\therefore$ The singularities in the finite z-plane are $z = 1$ and $z = -\frac{2}{3}$.

[সসীম z তলে ব্যতিচার বিন্দুগুলি হল $z = 1$ ও $z = -\frac{2}{3}$.]

To determine whether there is a singularity at $z = \infty$, let $z = \frac{1}{w}$.

Then from (1) we get [$z = \infty$ তে ব্যতিচার বিন্দু নির্ণয়ের জন্য ধরি $z = \frac{1}{w}$ তখন (1) হতে পাই]

$$f\left(\frac{1}{w}\right) = \frac{\frac{1}{w^8} + \frac{1}{w^4} + 2}{\left(\frac{1}{w} - 1\right)^3 \left(\frac{3}{w} + 2\right)^2}$$

$$= \frac{1 + w^4 + 2w^8}{w^8 (1-w)^3 (3+2w)^2 \cdot \frac{1}{w^5}}$$

$$= \frac{1 + w^4 + 2w^8}{w^3 (1-w)^3 (3+2w)^2}$$

when [যখন] $w = 0$, then [তখন] $f\left(\frac{1}{w}\right) = \infty$

$\therefore w = 0 \Rightarrow \frac{1}{z} = 0 \Rightarrow z = \frac{1}{0} = \infty$ is a singularity of f(z) [f(z) এর একটি ব্যতিচার বিন্দু]

Now [এখন] $\lim\limits_{z \to 1} (z-1)^3 f(z) = \lim\limits_{z \to 1} (z-1)^3 \cdot \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$

$$= \lim_{z \to 1} \frac{z^8 + z^4 + 2}{(3z+2)^2} = \frac{1 + 1 + 2}{(3+2)^2} = \frac{4}{25} \neq 0$$

$$\lim_{z \to -2/3} \left(z + \frac{2}{3}\right)^2 f(z) = \lim_{z \to -2/3} \frac{(3z+2)^2}{9} \cdot \frac{z^8 + z^4 + 1}{(z-1)^3(3z+2)^2}$$

$$= \lim_{z \to -2/3} \frac{z^8 + z^4 + 1}{9(z-1)^3} = \frac{\left(\frac{2}{3}\right)^8 + \left(\frac{2}{3}\right)^4 + 1}{9\left(-\frac{2}{3} - 1\right)^3} \neq 0$$

$$\lim_{w \to 0} (w-0)^3 f\left(\frac{1}{w}\right) = \lim_{w \to 0} w^3 \cdot \frac{1 + w^4 + 2w^8}{w^3(1-w)^3(3+2w)^2}$$

$$= \lim_{w \to 0} \frac{1 + w^4 + 2w^8}{(1-w)^3(3+2w)^2}$$

$$= \frac{1 + 0 + 0}{(1-0)^3(3+0)^2}$$

$$= \frac{1}{1 \times 9} = \frac{1}{9} \neq 0$$

$\therefore z = 1$ is a pole of order 3 [3 ক্রমের একটি পোল]

$z = -\frac{2}{3}$ is a pole of order 2 [2 ক্রমের একটি পোল]

and $z = \infty$ is a pole of order 3 for the function f(z). [এবং $z = \infty$, f(z) এর 3 ক্রমের একটি পোল]

2nd Part : $z = 1$ and $z = -\frac{2}{3}$ are the singularities for the f(z) in the finite z-plane. Thus, in the finite z-plane f(z) is analytic every where excepts the points $z = 1$ and $z = -\frac{2}{3}$. [সসীম z তলে f(z) এর ব্যতিচার বিন্দু $z = 1$ এবং $z = -\frac{2}{3}$. অতএব, সসীম z তলে $z = 1$ ও $z = -\frac{2}{3}$ বিন্দুদ্বয় ব্যতীত f(z) সর্বত্র বৈশ্লেষিক ।] Ans.

Problem–10(a). Calculate the Residue of the function $\dfrac{z^2}{z^2 + a^2}$ (take only the positive value of z). **[NUH–2012, NU(Phy)–2003]**

Solution : Given that [দেওয়া আছে] $f(z) = \dfrac{z^2}{z^2 + a^2}$

The poles of f(z) are obtained by solving the equations

$$z^2 + a^2 = 0$$

[$z^2 + a^2 = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$$\Rightarrow z = \pm\sqrt{-a^2} = \pm ai$$

The positive value of $z = ai$ which is a simple pole. [ধনাত্মক মান $z = ai$ যাহা একটি সরল পোল]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim_{z \to ai} \left\{(z - ai)\frac{z^2}{z^2 + a^2}\right\}$

$$= \lim_{z \to ai} \frac{z^2(z - ai)}{(z + ai)(z - ai)}$$

$$= \lim_{z \to ai} \frac{z^2}{z + ai}$$

$$= \frac{(ai)^2}{ai + ai} = \frac{-a^2}{2ai} = \frac{ai}{2}. \quad \textbf{(Ans)}$$

Problem–11. Find the residues of $f(z) = \frac{z^2 - 2z}{(z + 1)^2 (z + 4)}$ at all its poles in a finite plane. **[NUH–2013]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2 - 2z}{(z + 1)^2 (z + 4)}$

The poles of f(z) are obtained by solving the equation

$$(z + 1)^2 (z + 4) = 0$$

[$(z + 1)^2 (z + 4) = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$\Rightarrow (z + 1)^2 = 0$ and $z + 4 = 0$

$\Rightarrow z = -1, -1$ and $z = -4$

$\therefore z = -1$ is a double pole and $z = -4$ is a simple pole.

[$z = -1$ একটি দ্বিপোল এবং $z = -4$ একটি সরল পোল]

Residue at $z = -1$ is [$z = -1$ এ অবশেষ] $\lim_{z \to -1} \frac{1}{1!}\frac{d}{dz}\{(z + 1)^2 f(z)\}$

$$= \lim_{z \to -1} \frac{d}{dz}\left\{(z+1)^2 \cdot \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)}\right\}$$

$$= \lim_{z \to -1} \frac{d}{dz}\left(\frac{z^2 - 2z}{z^2 + 4}\right)$$

$$= \lim_{z \to -1} \frac{(z^2 + 4)(2z - 2) - (z^2 - 2z) \cdot 2z}{(z^2 + 4)^2}$$

$$= \lim_{z \to -1} \frac{2z^3 + 8z - 2z^2 - 8 - 2z^3 + 4z^2}{(z^2 + 4)^2}$$

$$= \lim_{z \to -1} \frac{2z^2 + 8z - 8}{(z^2 + 4)^2}$$

$$= \frac{2 - 8 - 8}{(1 + 4)^2} = \frac{-14}{25}$$

Residue at $z = -4$ [$z = -4$ এ অবশেষ] $\lim_{z \to -4} (z + 4) \cdot f(z)$

$$= \lim_{z \to -4} \left\{(z + 4) \cdot \frac{z^2 - 2z}{(z + 1)^2 (z + 4)}\right\}$$

$$= \lim_{z \to -4} \frac{z^2 - 2z}{(z + 1)^2} = \frac{16 + 8}{(-4 + 1)^2} = \frac{24}{9}$$

The residues at $z = -1$ is $\frac{-14}{25}$ and residue at $z = -4$ is $\frac{24}{9}$. **(Ans)**

Problem–11(a). Find the residues of the function

$$f(z) = \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)}.$$ **[DUH–1988, 1989]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)}$

The poles of f(z) are obtained by solving the equation

$$(z + 1)^2 (z^2 + 4) = 0$$

[$(z + 1)^2 (z^2 + 4) = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$\Rightarrow (z + 1)^2 = 0$ and $z^2 + 4 = 0$

$\Rightarrow z = -1, -1$ and $z = \pm 2i$

$\therefore z = -1$ is a double pole and $z = 2i$, $z = -2i$ are simple poles.

[$z = -1$ একটি দ্বিপোল এবং $z = 2i$, $z = -2i$ সরল পোল]

Residue at $z = -1$ is [$z = -1$ এ অবশেষ] $\lim_{z \to -1} \frac{1}{1!}\frac{d}{dz}\{(z + 1)^2 f(z)\}$

$$= \lim_{z \to -1} \frac{d}{dz}\left\{(z + 1)^2 \cdot \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)}\right\}$$

$$= \lim_{z \to -1} \frac{d}{dz}\left(\frac{z^2 - 2z}{z^2 + 4}\right)$$

$$= \lim_{z \to -1} \frac{(z^2 + 4)(2z - 2) - (z^2 - 2z) \cdot 2z}{(z^2 + 4)^2}$$

$$= \lim_{z \to -1} \frac{2z^3 + 8z - 2z^2 - 8 - 2z^3 + 4z^2}{(z^2 + 4)^2}$$

$$= \lim_{z \to -1} \frac{2z^2 + 8z - 8}{(z^2 + 4)^2}$$

$$= \frac{2 - 8 - 8}{(1 + 4)^2} = \frac{-14}{25}$$

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z\to 2i} \left\{(z-2i)\cdot\dfrac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$

$$= \lim_{z\to 2i}\left\{(z-2i)\cdot\frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}\right\}$$

$$= \lim_{z\to 2i}\frac{z^2-2z}{(z+1)^2(z+2i)}$$

$$= \frac{4i^2-4i}{(2i+1)^2(4i)} = \frac{-4-4i}{(4i^2+4i+1)\,4i}$$

$$= \frac{-4-4i}{(4i-3)\,4i} = \frac{-1-i}{4i^2-3i} = \frac{-1-i}{-4-3i}$$

$$= \frac{1+i}{4+3i}\times\frac{4-3i}{4-3i} = \frac{4-3i+4i-3i^2}{16-9i^2}$$

$$= \frac{4+i+3}{16+9} = \frac{7+i}{25}$$

Residue at $z = -2i$ is [$z = -2i$ এ অবশেষ]

$$\lim_{z\to -2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$$

$$= \lim_{z\to -2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}\right\}$$

$$= \lim_{z\to -2i}\left\{\frac{z^2-2z}{(z+1)^2(z-2i)}\right\}$$

$$= \frac{4i^2+4i}{(-2i+1)^2(-4i)}$$

$$= \frac{-4+4i}{(4i^2-4i+1)(-4i)}$$

$$= \frac{1-i}{(-4-4i+1)i}$$

$$= \frac{1-i}{-4i+4+i}$$

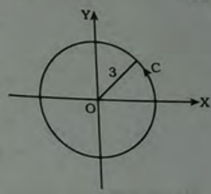
$$= \frac{1-i}{4-3i}$$

$$= \frac{1-i}{4-3i}\times\frac{4+3i}{4+3i}$$

$$= \frac{4-4i+3i-3i^2}{16-9i^2}$$

$$= \frac{7-i}{25}$$

Problem–12. Show that $\oint_C \dfrac{e^{tz}}{(z^2+1)^2}\,dz = \pi i(\sin t - t\cos t)$, where C is the circle $|z| = 3$ and $t > 0$ **[RUH–1980, 1982, 1996]**



Solution : Here the circle is $|z| = 3$. The poles of $\dfrac{e^{tz}}{(z^2+1)^2}$ are obtained by solving the equation $(z^2+1)^2 = 0$

$$\Rightarrow \{(z+i)(z-i)\}^2 = 0$$

$\Rightarrow z = i, -i$ both of are double poles and lie inside C, since $|i| = |-i| = 1 < 3$.

[সমাধান ঃ এখানে বৃত্তটি $|z| = 3 \cdot \dfrac{e^{tz}}{(z^2+1)^2}$ এর পোলগুলি $(z^2+1)^2 = 0$ সমীকরণ হতে পাওয়া যাবে।

$\Rightarrow \{(z+i)(z-i)\}^2 = 0$

$\Rightarrow z = i, -i$ উভয়ে দ্বিপোল এবং C এর ভিতরে, যেহেতু $|i| = |-i| = | < 3$.]

Residue of $z = i$ is [$z = -i$ এ অবশেষ]

$$\lim_{z\to i}\frac{1}{1!}\frac{d}{dz}\left\{(z-i)^2\cdot\frac{e^{tz}}{(z-i)^2(z+i)^2}\right\}$$

$$= \lim_{z\to i}\frac{d}{dz}\left\{\frac{e^{tz}}{(z+i)^2}\right\}$$

$$= \lim_{z\to i}\frac{(z+i)^2\cdot te^{tz} - e^{tz}\cdot 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{(z+i)\,te^{tz} - 2e^{tz}}{(z+i)^3}$$

$$= \frac{2i+e^{it}-2e^{it}}{8i^3} = \frac{(it-1)e^{it}}{-4i}$$

$$= \frac{-(t+i)\,e^{it}}{4}$$

Similarly residue at $z = -i$ [একইভাবে $z = -i$ এ অবশেষ]

$$\text{is } \frac{-(t-i)\,e^{-it}}{4}$$

[Replacing i by $-i$ in the above result]

Therefore by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{e^{tz}}{(z^2+1)^2}\,dz = 2\pi i\ [\text{sum of the residues}]$$

$$= -2\pi i\left[\frac{(t+i)\,e^{it}}{4} + \frac{(t-i)\,e^{-it}}{4}\right]$$

$$= -2\pi i\left[\frac{t(e^{it}+e^{-it})}{4}+\frac{i}{4}(e^{it}-e^{-it})\right]$$
$$= -2\pi i\left[\frac{2t\cos t}{4}+\frac{i}{4}\cdot 2i\sin t\right]$$
$$= -\pi i\,[t\cos t-\sin t]$$
$$= \pi i\,(\sin t - t\cos t) \qquad \text{(Showed)}$$

Problem–13. Show that $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{1}{\pi}$, where C is the circle $|z| = 4$. **[CUH–1987]**

Solution : Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

Poles of f(z) are obtained from the equation [f(z) এর পোল পাওয়ার সমীকরণ]

$(z+\pi i)^2(z-\pi i)^2 = 0$

$\Rightarrow z = \pi i, \pi i$ and $z = -\pi i, -\pi i$

$\Rightarrow z = \pi i$ and $-\pi i$ are two poles of f(z) each of double poles.

[$z = \pi i$ এবং $-\pi i$ হল এর পোল, প্রত্যেকে দ্বিপোল]

Residue at $z = \pi i$ is [$z = \pi i$ এ অবশেষ]

$$\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z-i\pi)^2\cdot\frac{e^z}{(z+\pi i)^2(z-\pi i)^2}\right\}$$
$$= \lim_{z\to\pi i}\cdot\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}$$
$$= \lim_{z\to\pi i}\frac{(z+\pi i)^2\cdot e^z - e^z\cdot 2(z+\pi i)}{(z+\pi i)^4}$$
$$= \lim_{z\to\pi i}\frac{e^z(z+\pi i-2)}{(z+\pi i)^3} = \frac{e^{i\pi}(2i\pi-2)}{(2i\pi)^3}$$
$$= \frac{2(i\pi-1)\,e^{i\pi}}{-8i\pi^3} = \frac{(1-i\pi)\,e^{i\pi}}{4i\pi^3} = \frac{-(\pi+i)}{4\pi^3}\,e^{i\pi}$$

Similarly, residue at $z = -\pi i$ is [একইভাবে $z = \pi i$ এ অবশেষ]

$$\frac{-(\pi+i)}{4\pi^3}\,e^{-i\pi}.$$

Therefore, by Cauchy's residue theorem we have [অতএব, কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\ \text{[Sum of the residues]}$$

$$= 2\pi i\left[\frac{-(\pi+i)\,e^{i\pi}}{4\pi^3}-\frac{(\pi-i)}{4\pi^3}\,e^{-i\pi}\right]$$
$$= \frac{i}{2\pi^2}\left[-\pi(e^{i\pi}+e^{-i\pi}) - i\,(e^{i\pi}-e^{-i\pi})\right]$$
$$= \frac{i}{2\pi^2}\left[-2\pi\cos\pi - i\cdot 2i\sin\pi\right]$$
$$= \frac{i}{2\pi^2}\left[-2\pi\cdot(-1) - 2i^2\times 0\right]$$
$$= \frac{i}{\pi} \qquad \text{(Showed)}$$

Problem–14. Show that

$$I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{1}{2}(t-1)+\frac{1}{2}e^{-t}\cos t,$$

where C is the circle with equation $|z| = 3$.

[NU(Phy)–2004, CUH–1988, RUH–1984]

Solution : Poles of $\frac{e^{zt}}{z^2(z^2+2z+2)}$ are obtained by solving

$z^2(z^2+2z+2) = 0$

[$z^2(z^2+2z+2) = 0$ সমীকরণ সমাধান করে $\frac{e^{zt}}{z^2(z^2+2z+2)}$ এর পোল পাওয়া যাবে]

$\Rightarrow z^2 = 0$ and [এবং] $z^2+2z+2 = 0$

$\Rightarrow z = 0, 0$ and [এবং] $(z+1)^2 = -1 = i^2$

$\Rightarrow z+1 = \pm i$

$\Rightarrow z = -1 \pm i$

$|-1+i| = \sqrt{1+1} = \sqrt{2} < 3$

$|-1-i| = \sqrt{1+1} = \sqrt{2} < 3$

$\therefore$ $z = 0$ is a pole of order 2 inside C. [C এর ভিতর 2 ক্রমের পোল $z = 0$]

$z = -1+i$ is a pole of order 1 inside C. [C এর ভিতর এক ক্রমের পোল $z = -1+i$]

$z = -1-i$ is a pole of order 1 inside C. [C এর ভিতর এক ক্রমের পোল $z = -1-i$]

Now residue at $z = 0$ is [এখন $z = 0$ এ অবশেষ]

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{z^2\cdot\frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to 0} \frac{d}{dz}\left\{\frac{e^{zt}}{z^2+2z+2}\right\}$$

$$= \lim_{z\to 0} \frac{(z^2+2z+2)\cdot te^{zt} - e^{zt}(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{(0+0+2)\,te^0 - e^0(0+2)}{(0+0+2)^2} = \frac{2t-2}{4} = \frac{t-1}{2}$$

Residue at $z = -1+i$ is [$z = -1+i$ এ অবশেষ]

$$\lim_{z\to -1+i}\left\{(z+1-i)\cdot\frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to -1+i}\left\{(z+1-i)\cdot\frac{e^{zt}}{z^2(z+1-i)\,(z+1+i)}\right\}$$

$$= \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2(z+1+i)}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2\,(-1+i+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i+i^2)\,(2i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i-1)\,(2i)}$$

$$= \frac{e^{(-1+i)t}}{-4i^2} = \frac{e^{-t}}{4}e^{it}$$

Similarly, residue at $z = -1-i$ is [অনুরূপে, $z = -1-i$ এ অবশেষ]

$\frac{e^{-t}}{4}e^{-it}$ [Replacing i by – i in the above result]

Therefore by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{1}{2}(t-1) + \frac{e^{-t}}{4}e^{it} + \frac{e^{-t}}{4}e^{-it}$$

$$= \frac{1}{2}(t-1) + \frac{e^{-t}}{4}(e^{it}+e^{-it})$$

$$= \frac{1}{2}(t-1) + \frac{e^{-t}}{4}\cdot 2\cos t$$

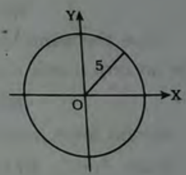
$$= \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t \qquad \text{(Showed)}$$

Problem–15. Show that $\oint_C \frac{zf'(z)}{f(z)}\,dz = 4\pi i$, where $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ and C is the circle $|z| = 5$. **[JUH–1989]**

Solution : Given that $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$

$$\Rightarrow f(z) = z^3(z-2) + z(z-2) - 10(z-2)$$
$$= (z-2)(z^3+z-10)$$
$$= (z-2)\{z^2(z-2) + 2z(z-2) + 5(z-2)\}$$
$$= (z-2)^2(z^2+2z+5)$$
$$= (z-2)^2\{z-(-1+2i)\}\{z-(-1-2i)\}$$

The zeros of f(z) are obtained from



$f(z) = 0$

$\Rightarrow (z-2)^2\{z-(-1+2i)\}\{z-(-1-2i)\} = 0$

$\Rightarrow z = 2, 2, -1+2i, -1-2i$

$|2| = 2 < 5$

$|-1+2i| = \sqrt{1+4} = \sqrt{5} < 5$

$|-1-2i| = \sqrt{1+4} = \sqrt{5} < 5$

There are 2 zeros of order 2 and two other simple zeros. All zeros are inside C.

Again, f(z) has no poles in C. Also here $g(z) = z$

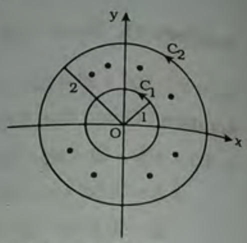
$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)}\,dz = [2\cdot g(2) + 1\cdot g(-1+2i) + 1\cdot g(-1-2i)] - 0$$

$$\Rightarrow \oint_C \frac{z\,f'(z)}{f(z)}\,dz = 2\pi i\,[2\cdot 2 + (-1+2i) + (-1-2i)]$$
$$= 2\pi i\,[4 - 1 + 2i - 1 - 2i]$$
$$= 2\pi i\,(2) = 4\pi i \qquad \text{(Ans)}$$

Problem–16. Show that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

[NUH–2014, DUH–2006, DUH–1989, RUH–1984, CUH–1987]

Solution : Let the circle $|z| = 1$ and $|z| = 2$ are C_1 and C_2 respectively.



Let $f(z) = 12$ and $g(z) = z^7 - 5z^3$.

Then on C_1 we have

$$|g(z)| = |z^7 - 5z^3| \le |z^7| + 5|z^3|$$

$$\Rightarrow |g(z)| \le 1 + 5$$

$$\Rightarrow |g(z)| \le 6 < 12 < f(z)$$

$$\Rightarrow |g(z)| < f(z).$$

Also, both $f(z)$ and $g(z)$ are analytic inside and on C_1.

Therefore, by Rouche's theorem

$$f(z) + g(z) = z^7 - 5z^3 + 12 \text{ and } f(z) = 12$$

have the same number of zeros inside C_1. Evidently, $f(z) = 12$ has no zero inside C_1 and so the given equation

$$f(z) + g(z) = z^7 - 5z^3 + 12 = 0 \cdots\cdots (1)$$

has no zeros inside C_1.

Again, For C_2, let $f(z) = z^7$ and $g(z) = -5z^3 + 12$.

$$\therefore |g(z)| = |-5z^3 + 12| \le |-5z^3| + |12|$$

$$\Rightarrow g(z) \le 40 + 12 = 52 < 2^7 = |f(z)|$$

$$\Rightarrow g(z) < |f(z)|$$

Both $f(z)$ and $g(z)$ are analytic inside and on C_2.

Therefore, by Rouche's theorem

$$f(z) + g(z) = z^7 - 5z^3 + 12 \text{ and } f(z) = z^7$$

have the same number of zeros inside C_2. Here $f(z) = z^7$ have seven zeros all lie inside C_2.

$$\therefore f(z) + g(z) = z^7 - 5z^3 + 12 = 0 \cdots\cdots (2)$$

has all zeros inside C_2

From (1) and (2) we have that the roots of the given equation lie between the circles C_1 and C_2, that is, $|z| = 1$ and $|z| = 2$.

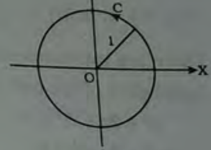
(Showed)

Problem–17. If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$.

[NUH–2004, 2007, 2012(Old), DUH–1986, RUMP–1988]

Solution : Let C be circle [মনে করি C বৃত্তটি] $|z| = 1$ and [এবং] $f(z) = az^n$ and [এবং] $g(z) = -e^z$.

On the circle C we have [C বৃত্তের উপর পাই]



$$|f(z)| = |az^n|$$

$$\Rightarrow |f(z)| = |a|\ |z^n| = a|z|^n$$

$$= a \cdot 1 = a > e \cdots\cdots (1)$$

$$|g(z)| = |-e^z| = |e^z|$$

$$= \left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right|$$

$$\Rightarrow |g(z)| \le 1 + |z| + \frac{1}{2!}|z|^2 + \frac{1}{3!}|z|^3 + \cdots$$

$$\Rightarrow |g(z)| \le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$$

$$\Rightarrow |g(z)| \le e < a = |f(z)|$$

$$\Rightarrow |g(z)| < f(z) \quad \text{by (1)}$$

Therefore, by the Rouche's theorem, $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C. Here $f(z) = az^n$ gives n zeros inside C. Thus, the given equation has n zeros inside the circle $|z| = 1$. [অতএব, রচির উপপাদ্য দ্বারা C এর ভিতর $f(z) + g(z)$ ও $f(z)$ এর একই সংখ্যক শূন্য থাকবে। এখানে C এর ভিতর $f(z) = az^n$, n সংখ্যক শূন্য দেয়। অতএব প্রদত্ত সমীকরণের $|z| = 1$ বৃত্তের ভিতর n সংখ্যক মূল আছে।]

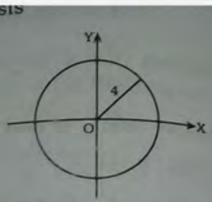
Problem–18. If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the region $|z| < \frac{1}{2}$. **[NUH–2006, DUH–1989]**

Solution : Do as 17.

Problem–19. Show that $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = -2$, where $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$ and C is the circle $|z| = 4$.

[RUH–1988, CUH–1983]

Solution : Here C is the circle $|z| = 4$ and $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$ which is analytic inside and on C.



Zeros of f(z) are obtained from

$$(z^2+1)^2=0$$

$$\Rightarrow (z+i)^2(z-i)^2=0$$

$$\Rightarrow z=-i,-i,i,i$$

$$|z|=|-i| \text{ or } |i|=1<4$$

There are two zeros each of order 2 lying in C.

Again pole of f(z) can be obtained from

$$(z^2+2z+2)^3=0$$

$$\Rightarrow z^2+2z+2=0$$

$$\Rightarrow (z+1)^2=-1=i^2$$

$$\Rightarrow z+1=\pm i$$

$$\Rightarrow z=-1\pm i$$

$$|z|=|-1+i|=\sqrt{1+1}=\sqrt{2}<4$$

and $|z|=|-1-i|=\sqrt{1+1}=\sqrt{2}<4$

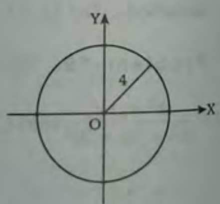
There are two poles each of order 3 lying in C. Thus by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N-P$$

$$=(2+2)-(3+3)=-2 \qquad \text{(Showed)}$$

Problem-20. For the function $f(z)=\dfrac{(z^2+1)^2(z+5)}{(z^2+2z+2)^3}$, find $\oint_C \dfrac{f'(z)}{f(z)}\,dz$ Where C is the circle $|z|=4$.

Solution : Here C is the circle $|z|=4$ and $f(z)=\dfrac{(z^2+1)^2(z+5)}{(z^2+2z+2)^3}$ which is analytic inside and on C.



Zeros of f(z) are obtained from the equation f(z) = 0

$$\Rightarrow (z^2+1)^2(z+5)=0$$

$$\Rightarrow (z+i)^2(z-i)^2(z+5)=0$$

$$\Rightarrow z=i,-i,-5$$

$z=i,-i$ lie in C but $z=-5$ lie outside C.

$z=i$ and $-i$ are zeros of order 2.

Poles of f(z) are obtained from the equation

$$(z^2+2z+2)^3=0$$

$$\Rightarrow z^2+2z+2=0$$

$$\Rightarrow (z+1)^2=-1=i^2$$

$$\Rightarrow z+1=\pm i$$

$$\Rightarrow z=-1\pm i$$

$$|z|=|-1+i|=\sqrt{1+1}=\sqrt{2}<4$$

and $|z|=|-1-i|=\sqrt{1+1}=\sqrt{2}<4$

$\therefore z=-1+i$ and $z=-1-i$ are poles each of order 3 and lie in C.

Thus by general argument theorem we have

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N-P$$

$$=(2+2)-(3+3)=-2$$

$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz=-4\pi i$$

Problem-21. If C is the circle $|z|=\pi$, then show that

$$\oint_C \frac{f'(z)}{f(z)}\,dz=\begin{cases}14\pi i & \text{if } f(z)=\sin \pi z\\ 12\pi i & \text{if } f(z)=\cos \pi z\\ 2\pi i & \text{if } f(z)=\tan \pi z\end{cases}$$ **[RUH-1985, 2000]**

Solution : First Part : Here C is the circle $|z|=\pi=3{\cdot}14$. $f(z)=\sin \pi z$ is analytic inside and on C. It has no poles inside C. The zeros of f(z) can be obtained from

$$f(z)=\sin \pi z=0$$

$\Rightarrow \pi z=n\pi \Rightarrow z=n$, where $n=0,\pm1,\pm2,\pm3,\pm4,\cdots$ etc. Among these $0,1,-1,2,-2,3,-3$, lie with in C and each of them are simple zero. Hence by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N-P$$

$$=(1+1+1+1+1+1+1)-0$$

$$=7-0=7$$

$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz=14\pi i \qquad \text{(Showed)}$$

Second part : Here $f(z) = \cos \pi z$. There are no poles of $f(z)$ inside C. The zeros of f(z) are obtained from

$$f(z) = \cos \pi z = 0$$

$$\Rightarrow \pi z = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = n + \frac{1}{2}, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

Only the zeros $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{-3}{2}, \frac{5}{2}, \frac{-5}{2}$ lie with in C and the order of each is one. Hence by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$

$$= (1+1+1+1+1+1) - 0 = 6 - 0 = 6$$

$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz = 12\pi i \qquad \text{(Showed)}$$

Third Part : Here $f(z) = \tan \pi z = \frac{\sin \pi z}{\cos \pi z}$.

The zeros of f(z) can be obtained from

$$f(z) = \tan \pi z = 0$$

$$\Rightarrow \sin \pi z = 0$$

$$\Rightarrow \pi z = n\pi$$

$$\Rightarrow z = n,\ n = 0, \pm 1, \pm 2, \pm 3, \cdots \text{ etc.}$$

The zeros with in C are 0, 1, – 1, 2, – 2, 3, – 3, each of order 1.

The poles are obtained from

$$\cos \pi z = 0 \Rightarrow \pi z = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = n + \frac{1}{2},\ n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

The poles inside C are $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{-3}{2}, \frac{5}{2}, \frac{-5}{2}$ each of order 1. Hence by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$

$$= (1+1+1+1+1+1+1) - (1+1+1+1+1+1)$$

$$= 7 - 6$$

$$= 1$$

$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \qquad \text{(Showed)}$$

Problem–22. Evaluate $\oint_C \frac{e^{3z}}{z + \pi i}\,dz$, where C is the circle $|z+1| = 4$.

[DUH–1994]

Solution : The given circle is C : $|z+1| = 4$

$$\oint_C \frac{e^{3z}}{z+\pi i}\,dz = \oint_C f(z)\,dz$$

$$\text{where } f(z) = \frac{e^{3z}}{z+\pi i}$$

The poles of f(z) is given by the equation $z + \pi i = 0 \Rightarrow z = -\pi i$

$$|z| = |-\pi i| = \pi < 4.$$

$\therefore$ The pole $z = -\pi i$ lie inside C.

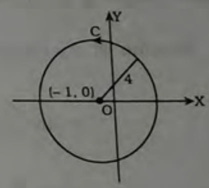
Residue at $z = -\pi i$ is $\lim_{z \to -\pi i} (z + i\pi)\, f(z)$

$$= \lim_{z \to -\pi i} (z + i\pi)\frac{e^{3z}}{z+\pi i}$$

$$= \lim_{z \to -\pi i} e^{3z} = e^{-i3\pi}$$

$$= \cos 3\pi - i \sin 3\pi$$

$$= (-1) - 0 = -1$$



Therefore, by Cauchy's residue theorem

$$\oint_C \frac{e^{3z}}{z+\pi i}\,dz = 2\pi i(-1) = -2\pi i \qquad \text{Ans.}$$

Problem–23. Evaluate the integral $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2}\,dz$, $C = \{z : z = 1 + 2e^{i\theta}, 0 \le \theta \le 2\pi\}$ using Cauchy's residue theorem.

[NUH–2005, 2008, 2010]

Solution : Equation of the given curve is [প্রদত্ত বক্ররেখার সমীকরণ]

$$z = 1 + 2e^{i\theta}$$

$$\Rightarrow z - 1 = 2e^{i\theta}$$

$$\Rightarrow |z-1| = |2e^{i\theta}|$$

$$\Rightarrow |z-1| = 2 \qquad \because |e^{i\theta}| = 1$$

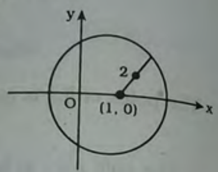
$\therefore$ The given cure is a circle whose centre is (1, 0) and radius is 2. [প্রদত্ত বক্ররেখা একটি বৃত্ত যার কেন্দ্র (1, 0) এবং ব্যাসার্ধ 2]

Now [এখন] $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} = \oint_C f(z)\, dz$, say

where [যেখানে] $f(z) = \frac{e^{-iz}}{(z+3)(z-i)^2}$

The poles of f(z) are given by $(z+3)(z-i)^2 = 0$

[$(z+3)(z-i)^2 = 0$ দ্বারা f(z) এর পোল পাওয়া যাবে]



$$\Rightarrow z = -3 \text{ and } z = i, i$$

$$|-3| = 3 > 2 \text{ and } |i| = 1 < 2$$

∴ The pole z = i lies inside the circle which is a double pole (pole of order 2). [z = i পোলটি বৃত্তের ভিতর অবস্থিত যাহা দ্বিপোল।]

Residue at z = i is [z = i এ অবশেষ] $\lim_{z \to i} \frac{1}{|\underline{2-1}} \frac{d}{dz}[(z-i)^2 \cdot f(z)]$

$$= \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{e^{-iz}}{(z+3)(z-i)^2}\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{e^{-iz}}{z+3}\right]$$

$$= \lim_{z \to i} \frac{(z+3)(-ie^{-iz}) - e^{-iz} \cdot 1}{(z+3)^2}$$

$$= \frac{(i+3)(-ie) - e}{(i+3)^2}$$

$$= \frac{(1 - 3i - 1)\, e}{i^2 + 6i + 9}$$

$$= \frac{-3ie}{8 + 6i}$$

$$= \frac{-3ie}{2(4+3i)} \times \frac{4-3i}{4-3i}$$

$$= \frac{-12ie - 9e}{2(16+9)}$$

$$= \frac{-12ie - 9e}{50}$$

∴ By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} = 2\pi i \cdot (\text{Residue at } z = i)$$

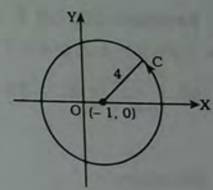
$$= 2\pi i \cdot \frac{-12ie - 9e}{50}$$

$$= \frac{(12 - i9)\,\pi e}{25}. \quad \textbf{(Ans)}$$

Problem–24. Evaluate the value of $\oint_C \frac{e^{3z}}{z - \pi i}\, dz$, where C is a curve (a) $|z-1| = 4$ and (b) $|z-2| + |z+2| = 6$ **[NUH–1999]**

Solution : (a) Given [দেওয়া আছে] $|z-1| = 4$

Let $I = \oint_C \frac{e^{3z}}{z - \pi i}\, dz = \oint_C f(z)\, dz$



where [যেখানে] $f(z) = \frac{e^{3z}}{z - \pi i}$

Pole of f(z) is $z = \pi i$ [f(z) এর পোল $z = \pi i$]

$$|z| = |\pi i| = \pi < 4$$

∴ The pole $z = \pi i$ lie inside C. [$z = \pi i$ পোলটি C এর ভিতর অবস্থিত]

Residue at $z = \pi i$ is [$z = \pi i$ এ অবশেষ]

$$\lim_{z \to \pi i} (z - \pi i)\, f(z)$$

$$= \lim_{z \to \pi i} (z - \pi i) \frac{e^{3z}}{z - \pi i} = e^{i3\pi}$$

$$= \cos 3\pi + i \sin 3\pi$$

$$= -1 + i(0)$$

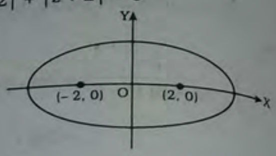
$$= -1$$

Therefore, by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{e^{3z}}{z - \pi i}\, dz = 2\pi i\,(-1)$$

$$= -2\pi i \qquad \text{Ans.}$$

(b) In this case the curve is $|z-2| + |z+2| = 6$ which is the equation of an ellipse whose focii are (2, 0) and (– 2, 0) and length of the major axis is 6. Here the pole is $z = \pi i$.



$\therefore |z| = |\pi i| = \pi = 3{\cdot}14 > 3$

$\therefore$ The pole lies out side the ellipse.

Hence by Cauchy's integral theorem we have

$$\oint_C f(z)\, dz = 0$$

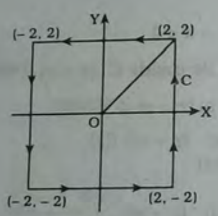
$$\Rightarrow \oint_C \frac{e^{3z}}{z - \pi i}\, dz = 0$$

Problem–25. Let C denotes the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$ described in the positive sense. Determine

(a) $\oint_C \frac{1}{z^2+9}\, dz$, (b) $\oint_C \frac{1}{z(z^2+9)}\, dz$, (c) $\oint_C \frac{1}{(z^2+1)(z^2+9)}\, dz$.

[NUH–1996]

Solution :



(a) $\oint_C \frac{1}{z^2+9}\, dz$

Poles are obtained from the equation $z^2 + 9 = 0$ [$z^2 + 9 = 0$ সমীকরণ হতে পোল পাওয়া যাবে]

$$\Rightarrow z^2 = -9 = (3i)^2$$

$$\Rightarrow z = \pm 3i$$

Each pole lie outside the square C. [প্রত্যেক পোল C বর্গের বাহিরে অবস্থিত]

$\therefore \oint_C \frac{1}{z^2+9}\, dz = 0$ Ans.

(b) $\oint_C \frac{1}{z(z^2+9)}\, dz$

Here the poles are [এখানে পোলগুলি হল] $z = 0$, $z = 3i$, $z = -3i$

Only the pole $z = 0$ lie inside C.

Residue at $z = 0$ is [$z = 0$ এ অবশেষ]

$$\lim_{z\to 0} z \cdot \frac{1}{z(z^2+9)} = \lim_{z\to 0} \frac{1}{z^2+9} = \frac{1}{9}$$

Therefore, by cauchy's residue theorem [অতএব, কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{1}{z(z^2+9)}\, dz = 2\pi i \left(\frac{1}{9}\right) = \frac{2\pi i}{9} \quad \text{Ans.}$$

(c) $\oint_C \frac{1}{(z^2+1)(z^2+9)}\, dz$

Poles are obtained from the equation $(z^2+1)(z^2+9) = 0$ [পোলগুলি $(z^2+1)(z^2+9) = 0$ সমীকরণ হতে পাওয়া যায়।]

$$\Rightarrow (z+i)(z-i)(z+3i)(z-3i) = 0$$

$$\Rightarrow z = i, -i, 3i, -3i$$

Only the poles $i, -i$ lie inside C which are simple poles. [শুধুমাত্র $i, -i$ পোলগুলি C এর ভিতর অবস্থিত যারা সরল পোল]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim_{z\to i} \left\{ (z-i) \cdot \frac{1}{(z^2+1)(z^2+9)} \right\}$

$$= \lim_{z\to i} \frac{1}{(z+i)(z^2+9)}$$

$$= \frac{1}{2i(i^2+9)} = \frac{1}{16i} = \frac{-i}{16}$$

Similarly, residue at $z = -i$ is [অনুরূপে, $z = -i$ এ অবশেষ] $\frac{i}{16}$

[Replacing i by – i in the above result]

Therefore, by Cauchy's residue theorem we have [অতএব, কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\, dz = 2\pi i \text{ (sum of the residues)}$$

$$= 2\pi i \left(\frac{-i}{16} + \frac{i}{16}\right) = 0 \text{ Ans.}$$

Problem–26. Expand $\log\left(\frac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$

[DUH–1988, 1990, DUMP–1991]

Solution : When $|z| < 1$ then we have [যখন $|z| < 1$ তখন পাই]

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \cdots\cdots \quad (1)$$

and $$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \frac{z^5}{5} \cdots\cdots \quad (2)$$

(1) – (2) gives, $\log(1+z) - \log(1-z)$

$$= 2z + \frac{2z^3}{3} + \frac{2}{5}z^5 + \cdots$$

$$\Rightarrow \log\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

This series converses for $|z| < 1$. This series also converges for $|z| = 1$ except $z = -1$ [ধারাটি $|z| < 1$ এর জন্য অভিসারী।ধারাটি $z = -1$ ব্যতীত $|z| = 1$ এর জন্যও অভিসারী]

Problem–26(a). Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$ **[NUH–2007, 2010, 2012(Old)]**

Solution : Given that [দেওয়া আছে]

$f(z) = \sin z \qquad \therefore f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore f'(z) = \cos z \qquad \therefore f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \qquad \therefore f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z \qquad \therefore f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f^{iv}(z) = \sin z \qquad \therefore f^{iv}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

and so on.

Therefore, the Taylor series about $z = \frac{\pi}{4}$ is [অতএব $z = \frac{\pi}{4}$ এ টেইলর ধারা]

$$f(z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{\left(z - \frac{\pi}{4}\right)^2}{2!}\frac{1}{\sqrt{2}} - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!}\frac{1}{\sqrt{2}} + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \cdots\right]. \textbf{ (Ans)}$$

Problem–26(b). $z = 0$ বিন্দুর চারিদিকে $f(z) = ln(1 + z)$ ফাংশনটিকে Taylor এর ধারায় বিস্তার কর এবং ধারাটির জন্য convergence রিজিয়ন নির্ণয় কর। [State Taylor's theorem for the complex function $f(z)$. Expand $f(z) = ln(1 + z)$ in Taylor's series about $z = 0$ and determine the region of convergence for the series.]

Solution : দেওয়া আছে [Given] $f(z) = ln(1 + z) \quad \therefore f(0) = ln\, 1 = 0$

$\therefore f'(z) = \frac{1}{1+z} \qquad f'(0) = \frac{1}{1+0} = 1$

$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$

$f'''(z) = (-1)(-2)\cdot\frac{1}{(1+z)^3} \qquad f'''(0) = (-1)(-2) = (-1)^2\, \underline{|2}$

$f'''(z) = (-1)(-2)(-3)\cdot\frac{1}{(1+z)^4}, \qquad f^{iv}(0) = (-1)(-2)(-3) = (-1)^3\, \underline{|3}$

$\cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots \quad \cdots$

$f^n(z) = \frac{(-1)^{n-1}\, \underline{|n-1}}{(1+z)^n} \qquad f^n(0) = (-1)^{n-1}\, \underline{|n-1}$

অতএব, $z = 0$ বিন্দুর চারিদিকে টেইলরের উপপাদ্য [Therefore, the Taylor's series about the point $z = 0$ is]

$$f(z) = f(0) + (z-0) f'(0) + \frac{(z-0)^2}{\underline{|2}} f''(0) + \frac{(z-0)^3}{\underline{|3}} f'''(0) + \frac{(z-0)^4}{\underline{|4}} f^{iv}(0) + \cdots$$

$$= 0 + z \cdot 1 + \frac{z^2}{\lfloor 2} \cdot (-1) + \frac{z^3}{\lfloor 3} (-1)^2 \lfloor 2 + \frac{z^4}{\lfloor 4} (-1)^3 \lfloor 3 + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

অভিসারী এলাকা (Convergence region) : আমরা জানি $ln(1+z)$ অভিসারী হয় যখন $|z| < 1$ হয়। [We know that $ln(1+z)$ is convergent when $|z| < 1$]

Problem–27. Expand the function $f(z) = \frac{1}{z-3}$ in a Laurent series for the region : (i) $|z| < 3$, (ii) $|z| > 3$ **[DUH–1983]**

Solution : (i) We express f(z) in a manner so that the binomial expansion is valid for $|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$.

$$\therefore f(z) = \frac{1}{z-3} = \frac{1}{-3\left(1-\frac{z}{3}\right)} = \frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= -\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\right)$$

$$= -\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} - \frac{z^3}{81} - \cdots \quad \text{Ans.}$$

(ii) In this case $|z| > 3 \Rightarrow \frac{|z|}{3} > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

$$\therefore f(z) = \frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \cdots\right)$$

$$= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \cdots \quad \text{Ans.}$$

Problem–28. Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in a laurent series for the region $1 < |z| < 2$ **[DUH–1987]**

Solution : $1 < |z| < 2$

$$\Rightarrow 1 < |z| \text{ and } |z| < 2$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1$$

We expand f(z) in a way such that the bionomial expansion is valid for $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$.

$$\therefore f(z) = \frac{z}{(z+1)(z+2)} = \frac{2}{z+2} - \frac{1}{z+1}$$

$$= \frac{2}{2\left(1+\frac{z}{2}\right)} - \frac{1}{z\left(1+\frac{1}{z}\right)}$$

$$= \left(1+\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right) - \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\right) \quad \text{Ans.}$$

Problem–29. Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series for the region (i) $0 < |z| < 2$; (ii) $|z| > 2$. **[DUMP–1991]**

Solution : (i) In this case $0 < |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$.

We expand f(z) in a manner so that bionomial expansion is valid in $0 < |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$.

$$\therefore f(z) = \frac{1}{z(z-2)} = \frac{1}{2}\left(\frac{1}{z-2} - \frac{1}{z}\right)$$

$$= \frac{-1}{4\left(1-\frac{z}{2}\right)} - \frac{1}{2z}$$

$$= -\frac{1}{4}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{2z}$$

$$= -\frac{1}{2z} - \frac{1}{4}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$= -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots \quad \text{Ans.}$$

(ii) In $|z| > 2$ we have $\left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore f(z) = \frac{1}{z(z-2)}$$
$$= \frac{1}{2}\left(\frac{1}{z-2} - \frac{1}{z}\right)$$
$$= \frac{1}{2z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{2z}$$
$$= -\frac{1}{2z} + \frac{1}{2z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \frac{2^4}{z^4} + \cdots\right)$$
$$= -\frac{1}{2z} + \frac{1}{2z} + \frac{1}{z^2} + \frac{2}{z^3} + \frac{4}{z^4} + \frac{8}{z^5} + \cdots$$
$$= \frac{1}{z^2} + \frac{2}{z^3} + \frac{4}{z^4} + \frac{8}{z^5} + \cdots \quad \text{Ans.}$$

Problem–30. Expand $f(z) = \frac{z^2}{(z-1)(z-2)}$ in a laurent series for the region $1 < |z| < 2$ **[NUH–04 (Old) 08, DUH–1971, 85, 88, 94]** and $0 < |z| < 1$. **[NUH–2004, 2008]**

Solution : Here [এখানে] $f(z) = \frac{z^2}{(z-1)(z-2)} = 1 + \frac{A}{z-1} + \frac{B}{z-2}$, say

Then [তখন] $z^2 = (z-1)(z-2) + A(z-2) + B(z-1)$

Putting $z = 1$ [$z = 1$ বসাইয়া] $1 = -A \Rightarrow A = -1$

Putting $z = 2$ [$z = 2$ বসাইয়া] $4 = B$

$$\therefore f(z) = 1 - \frac{1}{z-1} + \frac{4}{z-2} \quad \cdots\cdots (1)$$

We expand f(z) in a manner so that the binomial expansion is valid in

$$1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1 \text{ and } \left|\frac{z}{2}\right| < 1.$$

[আমরা f(z) কে এমন রীতিতে বিস্তৃত করব যেন দ্বিপদী বিস্তর $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1$ এবং $\left|\frac{z}{2}\right| < 1$ এ বৈধ হয়।]

$$\therefore f(z) = 1 - \frac{1}{z-1} + \frac{4}{z-2}$$

$$= 1 - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{4}{2\left(1-\frac{z}{2}\right)}$$
$$= 1 - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - 2\left(1-\frac{z}{2}\right)^{-1}$$
$$= 1 - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) - 2\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$
$$= \left(1 - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right) - 2\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$
$$= \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - 1 - z - \frac{z^2}{2} - \frac{z^3}{2^2} - \cdots \quad \text{Ans.}$$

For 2nd part we have [দ্বিতীয় অংশের জন্য পাই] $0 < |z| < 1$

$\Rightarrow |z| < 1$ and [এবং] $|z| < 2$

$\Rightarrow |z| < 1$ and [এবং] $\left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = 1 - \frac{1}{z-1} + \frac{4}{z-2}$$
$$= 1 + \frac{1}{1-z} - \frac{4}{2\left(1-\frac{z}{2}\right)}$$
$$= 1 + (1-z)^{-1} - 2\left(1-\frac{z}{2}\right)^{-1}$$
$$= 1 + (1 + z + z^2 + z^3 + \cdots) - 2\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$
$$= \frac{z^2}{2} + \frac{3}{4}z^3 + \cdots \quad \textbf{(Ans)}$$

Problem–30 (i). Expand the function $f(z) = \frac{1}{z^2 - 3z + 2}$ in a Laurent's series for the region $1<|z|<2$.

Solution. Here $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)}$
$$= \frac{-1}{z-1} + \frac{1}{z-2}$$

We expand f(z) in a manner so that the Binomial expansion valid in $1<|z|2 \Rightarrow \frac{1}{|z|} < 1$ and $\left|\frac{z}{2}\right| < 1$

[আমরা f(z) কে এমন রীতিতে বিস্তৃত করব যেন দ্বিপদী বিস্তার $1<|z|<2 \Rightarrow \frac{1}{|z|} < 1$ এবং $\left|\frac{z}{2}\right| < 1$ এ বৈধ হয়।]

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right)$$

$$= \left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right) - \left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots\right)$$

$$= \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} - \frac{z^3}{2^4} - \cdots \textbf{ Ans.}$$

Problem–31. Expand the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ for the following regions :

(i) $2 < |z| < 3$ (ii) $|z| < 2$ (iii) $|z| > 3$.

[NUH–06, 2010, 2013, DUH–90, DUMP–88, 89, RUH–80, 85]

Solution : Let [ধরি] $\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{A}{z+2} + \frac{B}{z+3}$

$$\Rightarrow z^2 - 1 = (z+2)(z+3) + A(z+3) + B(z+2)$$

When [যখন] $z = -2$ then [তখন] $3 = A \Rightarrow A = 3$

When [যখন] $z = -3$ then [তখন] $8 = -B \Rightarrow B = -8$

$$\therefore f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3} \;\cdots\cdots (1)$$

(i) In this case we expand f(z) in a manner so that the bionomial expansion valid in $2 < |z| < 3$ [এইক্ষেত্রে f(z) কে এমন রীতিতে বিস্তৃত করব যেন দ্বিপদী বিস্তার $2 < |z| < 3$ বৈধ হয়।]

$$\Rightarrow 2 < |z| \text{ and [এবং] } |z| < 3$$

$$\Rightarrow \frac{2}{|z|} < 1 \text{ and [এবং] } \frac{|z|}{3} < 1$$

$$\Rightarrow \left|\frac{2}{z}\right| < 1 \text{ and [এবং] } \left|\frac{z}{3}\right| < 1.$$

Thus from (1) we get [অতএব (1) হতে পাই]

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= \cdots + \frac{12}{z^3} - \frac{6}{z^2} + \frac{3}{z} - \frac{5}{3} + \frac{8z}{3^2} - \frac{8z^2}{3^3} + \frac{8z^3}{3^4} - \cdots \quad \text{Ans.}$$

(ii) In this case [এইক্ষেত্রে] $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1.$

Also [আরো] $|z| < 2 < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

Thus from (1) we get [অতএব (1) হতে পাই]

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= -\frac{1}{6} - \left(\frac{3}{2^2}-\frac{8}{3^2}\right)z + \left(\frac{3}{2^3}-\frac{8}{3^3}\right)z^2 - \left(\frac{3}{2^4}-\frac{8}{3^4}\right)z^3 + \cdots \text{ Ans.}$$

(iii) In this case [এইক্ষেত্রে] $|z| > 3 \Rightarrow \frac{|z|}{3} > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

Also [আরো] $|z| > 3 \Rightarrow |z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$. Thus from (1) [অতএব (1) হতে পাই]

$$\therefore f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) - \frac{8}{z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\cdots\right)$$

$$= 1 + (3-8)\frac{1}{z} + (-6+24)\frac{1}{z^2} + (12-72)\frac{1}{z^3} + \cdots$$

$$= 1 - \frac{5}{z} + \frac{18}{z^2} - \frac{60}{z^3} + \cdots \quad \text{Ans.}$$

Problem-32. Expand the function $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ for the following regions : (i) $1 < |z| < 4$; (ii) $|z| < 1$; (iii) $|z| > 4$.

[DUMP-1988, RUMP-1986]

Solution : Let $\dfrac{(z-2)(z+2)}{(z+1)(z+4)} = 1 + \dfrac{A}{z+1} + \dfrac{B}{z+4}$

$\Rightarrow (z-2)(z+2) = (z+1)(z+4) + A(z+4) + B(z+1)$

When $z = -1$ then $(-3)(1) = 3A \Rightarrow A = -1$

When $z = -4$ then $(-6)(-2) = -3B \Rightarrow B = -4$.

$\therefore f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)} = 1 - \dfrac{1}{z+1} - \dfrac{4}{z+4}$ ...... (1)

(i) We expand f(z) in such a way that the bicnomial expansion valid in $1 < |z| < 4$

$\Rightarrow 1 < |z|$ and $|z| < 4$

$\Rightarrow \left|\dfrac{1}{z}\right| < 1$ and $\left|\dfrac{z}{4}\right| < 1$

$$\therefore f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \left(1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \cdots\right)$$

$$= \cdots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + \frac{z}{4} - \frac{z^2}{4^2} + \frac{z^3}{4^3} - \cdots \text{ Ans.}$$

(ii) In this case $|z| < 1$, so $|z| < 4 \Rightarrow \left|\dfrac{z}{4}\right| < 1$.

Thus from (1) we vet,

$$f(z) = 1 - \frac{1}{(1+z)} - \frac{4}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - (1+z)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - (1 - z + z^2 - z^3 + \cdots) - \left(1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \cdots\right)$$

$$= -1 + \left(1+\frac{1}{4}\right)z - \left(1+\frac{1}{4^2}\right)z^2 + \left(1+\frac{1}{4^3}\right)z^3 + \cdots \quad \text{Ans.}$$

(iii) In this case $|z| > 4 \Rightarrow \dfrac{|z|}{4} > 1 \Rightarrow \left|\dfrac{4}{z}\right| < 1$

Also $|z| > 4 \Rightarrow |z| > 1 \Rightarrow \left|\dfrac{1}{z}\right| < 1$. Thus from (1),

$$f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{z\left(1+\frac{4}{z}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{4}{z}\left(1 - \frac{4}{z} + \frac{4^2}{z^2} - \frac{4^3}{z^3} + \cdots\right)$$

$$= 1 - (1+4)\frac{1}{z} + (1+4^2)\frac{1}{z^2} - (1+4^3)\frac{1}{z^3} + \cdots \quad \text{Ans.}$$

Problem-33. Expand $f(z) = \dfrac{z^3}{(z+1)(z-2)}$ in a laurent series in the powers of $(z+1)$ in the region $0 < |z+1| < 3$. **[DUH-1976]**

Solution : Given that $0 < |z+1| < 3 \Rightarrow \left|\dfrac{z+1}{3}\right| < 1$.

$$f(z) = \frac{z^3}{(z+1)(z-2)}$$

$$= \frac{z^3+1-1}{(z+1)(z-2)} = \frac{z^3+1}{(z+1)(z-2)} - \frac{1}{(z+1)(z-2)}$$

$$= \frac{z^2-z+1}{z-2} - \frac{1}{3}\left(\frac{1}{z-2} - \frac{1}{z+1}\right)$$

$$= \frac{z(z-2)+z+1}{z-2} - \frac{\frac{1}{3}}{z-2} + \frac{\frac{1}{3}}{z+1}$$

$$= z + \frac{z-2+3}{z-2} - \frac{\frac{1}{3}}{z-2} + \frac{\frac{1}{3}}{z+1}$$

$$= z + 1 + \frac{3}{z-2} - \frac{\frac{1}{3}}{z-2} + \frac{\frac{1}{3}}{z+1}$$

$$= (z+1) + \frac{\frac{1}{3}}{z+1} + \frac{\frac{8}{3}}{z-2}$$

$$= (z+1) + \frac{1}{3(z+1)} + \frac{8}{3}\cdot\frac{1}{z+1-3}$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left(1 - \frac{z+1}{3}\right)^{-1}$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left\{1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right\}$$

$$= \frac{1}{3(z+1)} - \frac{8}{9} + \frac{19}{27}(z+1) - \frac{8}{9}\left\{\frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right\} \text{ Ans.}$$

Problem–34. Expand $f(z) = \frac{1}{(z^2+1)(z+2)}$ in a laurent series for the region (i) $1 < |z| < 2$, (ii) $|z| > 1$.

[NUH–2005(Old), 2011, DUH–1978]

Solution : Given that [দেওয়া আছে] $1 < |z| < 2$

$$\Rightarrow 1 < |z| \text{ and } |z| < 2$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1$$

$$\Rightarrow \left|\frac{1}{z^2}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1.$$

Let [ধরি] $\frac{1}{(z^2+1)(z+2)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+1}$

$$\Rightarrow 1 = A(z^2+1) + (Bz+C)(z+2)$$

Putting $z = -2$ [$z = -2$ বসাইয়া] $1 = 5A \Rightarrow A = \frac{1}{5}$

Equating the coefficients of z^2 [z^2 এর সহগ সমীকৃত করে]

$$0 = A + B \Rightarrow B = -A = -\frac{1}{5}$$

Equating the coefficients of z [z এর সহগ সমীকৃত করে]

$$0 = 2B + C \Rightarrow C = -2B = \frac{2}{5}$$

$$\therefore f(z) = \frac{1}{(z^2+1)(z+2)}$$

$$\Rightarrow f(z) = \frac{\frac{1}{5}}{z+2} + \frac{-\frac{1}{5}z + \frac{2}{5}}{z^2+1}$$

$$= \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$$

We expand this so that the binomial expansions valid for $\left|\frac{1}{z^2}\right|$ < and $\left|\frac{z}{2}\right| < 1$. [আমরা ইহাকে বিস্তৃত করব যেন দ্বিপদী বিস্তার $\left|\frac{1}{z^2}\right|$ এবং $\left|\frac{z}{2}\right| < 1$ এ বৈধ হয়]

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{2\left(1+\frac{z}{2}\right)} - (z-2)\cdot\frac{1}{z^2\left(1+\frac{1}{z^2}\right)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \left(\frac{z-2}{z^2}\right)\left(1+\frac{1}{z^2}\right)^{-1}\right]$$

$$= \frac{1}{10}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) - \frac{z-2}{5z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right). \text{ Ans.}$$

(ii) $|z| > 2 \Rightarrow 2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1.$

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{z\left(1+\frac{2}{z}\right)} - (z-2)\cdot\frac{1}{z^2+1}\right]$$

$$= \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1} - \frac{z-2}{z^2+1}$$

$$= \frac{1}{5z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right) - \frac{z-2}{z^2+1}$$

Problem–35. Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ in a laurent series for the following region :

(i) $1 < |z| < 3$, (ii) $|z| > 3$, (iii) $0 < |z+1| < 2$, (iv) $|z| < 1$

[NUH–1998, 2003, 2005, DUH–2004]

Solution : Given [দেওয়া আছে] $f(z) = \frac{1}{(z+1)(z+3)}$

$$= \frac{1}{(z+1)(-1+3)} + \frac{1}{(-3+1)(z+3)}$$

$$= \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

(i) $1 < |z| < 3$

$$\Rightarrow 1 < |z| \text{ and } |z| < 3$$

$$\Rightarrow |z| > 1 \text{ and } |z| < 3$$

$$\Rightarrow \frac{1}{|z|} < 1 \text{ and } \frac{|z|}{3} < 1.$$

We write f(z) in a manner so that the binomial expansion is valid for $1 < |z| < 3 \Rightarrow \frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$. [আমরা f(z) কে এমন রীতিতে লিখব যেন দ্বিপদী বিস্তার $1 < |z| < 3 \Rightarrow \frac{1}{|z|} < 1$ এবং $\frac{|z|}{3} < 1$ এ বৈধ হয়]

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$= \frac{1}{2z}\cdot\frac{1}{1+\frac{1}{z}} - \frac{1}{6}\cdot\frac{1}{1+\frac{z}{3}}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\cdots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right) - \left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\cdots\right) \quad \text{Ans.}$$

(ii) $|z| > 3$

$\Rightarrow |z| > 1$ and [এবং] $|z| > 3$

$\Rightarrow \frac{1}{|z|} < 1$ and [এবং] $\frac{3}{|z|} < 1$.

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$= \frac{1}{2z}\cdot\frac{1}{\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{2z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\cdots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right) - \left(\frac{1}{2z}-\frac{3}{2z^2}+\frac{9}{2z^3}-\frac{27}{2z^4}+\cdots\right)$$

$$= \left(\frac{3}{2z^2}-\frac{1}{2z^2}\right) + \left(\frac{1}{2z^3}-\frac{9}{2z^3}\right) + \left(-\frac{1}{2z^4}+\frac{27}{2z^4}\right) + \cdots$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots \quad \text{Ans.}$$

(iii) $0 < |z+1| < 2$

$\Rightarrow |z_1| < 2$, where [যেখানে] $z_1 = z + 1$

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$= \frac{1}{2z_1} - \frac{1}{2(z_1+2)}$$

$$= \frac{1}{2z_1} - \frac{1}{4\left(1+\frac{z_1}{2}\right)}$$

$$= \frac{1}{2z_1} - \frac{1}{4}\left(1+\frac{z_1}{2}\right)^{-1}$$

$$= \frac{1}{2z_1} - \frac{1}{4}\left(1-\frac{z_1}{2}+\frac{z_1^2}{2^2}-\frac{z_1^3}{2^3}+\cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \frac{1}{32}(z+1)^3 + \cdots \quad \text{Ans.}$$

(iv) $|z| < 1 \Rightarrow |z| < 1$ and [এবং] $|z| < 3 \Rightarrow |z| < 1$ and [এবং] $\frac{|z|}{3} < 1$

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}(1-z+z^2-z^3+\cdots) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= \left(\frac{1}{2}-\frac{1}{6}\right) + \left(\frac{-z}{2}+\frac{z}{18}\right) + \left(\frac{z^2}{2}-\frac{z^2}{54}\right) + \left(-\frac{z^3}{2}+\frac{1}{162}z^3\right) + \cdots$$

$$= \frac{2}{6} - \frac{8}{18}z + \frac{26}{54}z^2 - \frac{80}{162}z^3 + \cdots$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots \quad \text{Ans.}$$

Problem–36. Find the Laurent expansion of

$$f(z) = \frac{z-1}{(z+2)(z+3)}$$ in each of the following region :

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$. **[NUH–2002]**

Solution : We have [দেওয়া আছে] $f(z) = \frac{z-1}{(z+2)(z+3)}$

$$= \frac{-2-1}{(z+2)(-2+3)} + \frac{-3-1}{(-3+2)(z+3)}$$

$$= \frac{-3}{z+2} + \frac{4}{z+3}$$

(i) In this case [এইক্ষেত্রে] $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

Also [আরো] $|z| < 2 < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$.

$$\therefore f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$

$$= \frac{-3}{2\left(1+\frac{z}{2}\right)} + \frac{4}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{-3}{2}\left(1+\frac{z}{2}\right)^{-1} + \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{-3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right) + \frac{4}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= \left(\frac{-3}{2}+\frac{4}{3}\right) + \left(\frac{3}{4}-\frac{4}{9}\right)z + \left(\frac{-3}{8}+\frac{4}{27}\right)z^2 + \left(\frac{3}{16}-\frac{4}{81}\right)z^3 + \cdots$$

$$= -\frac{1}{6} + \frac{11}{36}z - \frac{49}{216}z^2 + \frac{179}{1296}z^3 - \cdots$$

(ii) In this case [এইক্ষেত্রে] $2 < |z| < 3$

$\Rightarrow 2 < |z|$ and [এবং] $|z| < 3$

$\Rightarrow \left|\frac{2}{z}\right| < 1$ and [এবং] $\left|\frac{z}{3}\right| < 1$

$$\therefore f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$

$$= \frac{-3}{z\left(1+\frac{2}{z}\right)} + \frac{4}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{-3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) + \frac{4}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= \cdots - \frac{12}{z^3} + \frac{6}{z^2} - \frac{3}{z} + \frac{4}{3} - \frac{4z}{9} + \frac{4z^2}{27} - \frac{4z^3}{81} + \cdots$$

(iii) Here [এখানে] $|z| > 3 \Rightarrow \left|\frac{z}{3}\right| > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

Also [আরো] $|z| > 3 \Rightarrow |z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$

$$= \frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{4}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{-3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) + \frac{4}{z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\cdots\right)$$

$$= (-3+4)\frac{1}{z} + (6-12)\frac{1}{z^2} + (-12+36)\frac{1}{z^3} + \cdots$$

$$= \frac{1}{z} - \frac{6}{z^2} + \frac{24}{z^3} - \cdots \quad \text{Ans.}$$

Problem-37. Find the Laurent expansion of the function $f(z) = \frac{z^2+1}{(z+1)(z-2)}$ in each of the regions

(i) $1 < |z| < 2$ **[NUH-01, 12]**

(ii) $0 < |z| < 1$ **[NUH-01, 12]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2+1}{(z+1)(z-2)}$

Let [এবং] $\frac{z^2+1}{(z+1)(z-2)} = 1 + \frac{A}{z+1} + \frac{B}{z-2}$

$\Rightarrow z^2 + 1 = (z+1)(z-2) + A(z-2) + B(z+1)$

When [যখন] $z = -1$, then [তখন] $2 = 0 + A(-3) + 0 \Rightarrow A = \frac{-2}{3}$

When [যখন] $z = 2$, then [তখন] $5 = 0 + 0 + 3B \Rightarrow B = \frac{5}{3}$

$$\therefore f(z) = \frac{z^2+1}{(z+1)(z-2)} = 1 - \frac{2/3}{z+1} + \frac{5/3}{z-2} \cdots (1)$$

(i) In this case [এইক্ষেত্রে] $1 < |z| < 2$

$\Rightarrow 1 < |z|$ and [এবং] $|z| < 2$

$\Rightarrow \left|\frac{1}{z}\right| < 1$ and [এবং] $\left|\frac{z}{2}\right| < 1$

Therefore, from (1) we have [অতএব (1) হতে পাই]

$$f(z) = 1 - \frac{2/3}{z\left(1+\frac{1}{z}\right)} + \frac{5/3}{-2\left(1-\frac{z}{2}\right)}$$

$$= 1 - \frac{2}{3z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{5}{6}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{2}{3z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{5}{6}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$= \cdots - \frac{2}{3z^3} + \frac{2}{3z^2} - \frac{2}{3z} + \frac{1}{6} - \frac{5z}{12} - \frac{5z^2}{24} - \frac{5z^3}{48} - \cdots$$

(ii) Here [এখানে] $0 < |z| < 1 \Rightarrow |z| < 1$

Also [আরো] $|z| < 1 \Rightarrow |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = 1 - \frac{2/3}{z+1} + \frac{5/3}{-2\left(1 - \frac{z}{2}\right)}$$

$$= 1 - \frac{2}{3}(1+z)^{-1} - \frac{5}{6}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{2}{3}(1 - z + z^2 - z^3 + \cdots) - \frac{5}{6}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$= \left(1 - \frac{2}{3} - \frac{5}{6}\right) + \left(\frac{2}{3} - \frac{5}{12}\right)z + \left(\frac{-2}{3} - \frac{5}{24}\right)z^2 + \left(\frac{2}{3} - \frac{5}{48}\right)z^3 + \cdots$$

$$= -\frac{1}{2} + \frac{1}{4}z - \frac{7}{8}z^2 + \frac{9}{16}z^3 + \cdots$$

Problem-38. Obtain Laurent expansion of $\frac{1}{z^2(z-3)^2}$ about the point $z = 3$. **[NUH-1995]**

Solution : Let [ধরি] $z - 3 = u \Rightarrow z = u + 3$.

$$\therefore \frac{1}{z^2(z-3)^2} = \frac{1}{u^2(u+3)^2}$$

$$= \frac{1}{9u^2\left(1 + \frac{u}{3}\right)^2}$$

$$= \frac{1}{9u^2}\left(1 + \frac{u}{3}\right)^{-2}$$

$$= \frac{1}{9u^2}\left[1 - (2)\frac{u}{3} + \frac{(-2)(-3)}{\lfloor 2}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{\lfloor 3}\left(\frac{u}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \cdots$$

$[\because u = z - 3]$ Ans.

Problem-39. Find the laurent series expansion of $f(z) = \frac{3}{z(2-z-z^2)}$ in Powers of z, valid in the region (i) $0 < |z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$ **[NUH-1996]**

Solution : $f(z) = \frac{3}{z(2-z-z^2)}$

$$2 - z - z^2 = 2 - 2z + z - z^2 = 2(1-z) + z(1-z) = (1-z)(2+z)$$

$$= \frac{3}{z(1-z)(2+z)}$$

$$= 3\left[\frac{1}{z(1-0)(2+0)} + \frac{1}{1\cdot(1-z)(2+1)} + \frac{1}{-2(1+2)(2+z)}\right]$$

$$= 3\left[\frac{1}{2z} + \frac{1}{3(1-z)} - \frac{1}{6(2+z)}\right]$$

$$= \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$

(i) $0 < |z| < 1$

$\Rightarrow |z| < 1$ and [এবং] $|z| < 2$

$\Rightarrow |z| < 1$ and [এবং] $\frac{|z|}{2} < 1$

$$\therefore f(z) = \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$

$$= \frac{3}{2z} + \frac{1}{1-z} - \frac{1}{4\left(1 + \frac{z}{2}\right)}$$

$$= \frac{3}{2z} + (1-z)^{-1} - \frac{1}{4}\left(1 + \frac{z}{2}\right)^{-1}$$

$$= \frac{3}{2z} + \left(1 + z + z^2 + z^3 + \cdots\right) - \frac{1}{4}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right)$$

$$= \frac{3}{2z} + \left(1 - \frac{1}{4}\right) + \left(1 + \frac{1}{8}\right)z + \left(1 - \frac{1}{16}\right)z^2 + \cdots$$

$$= \frac{3}{2z} + \frac{3}{4} + \frac{9}{8}z + \frac{15}{16}z^2 + \cdots \quad \text{Ans.}$$

(ii) $1 < |z| < 2$

$\Rightarrow 1 < |z|$ and [এবং] $|z| < 2$

$\Rightarrow |z| > 1$ and [এবং] $|z| < 2$

$\Rightarrow \frac{1}{|z|} < 1$ and [এবং] $\frac{|z|}{2} < 1$

$$\therefore f(z) = \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$
$$= \frac{3}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{4\left(1+\frac{z}{2}\right)}$$
$$= \frac{3}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}$$
$$= \frac{3}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{4}\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\right)$$
$$= \cdots - \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{2z} - \left(\frac{1}{4}-\frac{z}{8}+\frac{z^2}{16}-\cdots\right) \quad \text{Ans.}$$

(iii) $|z| > 2 \Rightarrow \frac{|z|}{2} > 1 \Rightarrow \frac{2}{|z|} < 1$ and $|z| > 2 \Rightarrow |z| > 1 \Rightarrow \frac{1}{|z|} < 1$

$$\therefore f(z) = \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$
$$= \frac{3}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{2}{z}\right)}$$
$$= \frac{3}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{2}{z}\right)^{-1}$$
$$= \frac{3}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{2z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\frac{2^4}{z^4}+\cdots\right)$$
$$= \frac{-3}{z^3} + \frac{3}{z^4} - \frac{9}{z^5} + \cdots \quad \text{Ans.}$$

Problem-40. Expand $f(z) = \frac{3z-3}{(2z-1)(z-2)}$ in a laurent series for the region $|z| < 1$ and $|z| > 1$. **[NUH-2000, NU(Pre)-2006, 2014]**

Solution : Given [দেওয়া আছে] $f(z) = \frac{3z-3}{(2z-1)(z-2)}$

$$= \frac{\frac{3}{2}-3}{(2z-1)\left(\frac{1}{2}-2\right)} + \frac{3\times 2-3}{(2\times 2-1)(z-2)}$$
$$= \frac{-3}{-3(2z-1)} + \frac{3}{3(z-2)}$$
$$= \frac{1}{2z-1} + \frac{1}{z-2}$$

Now for the region $|z| < 1$ we have [এখন $|z| < 1$ এলাকার জন্য পাই]

$$f(z) = \frac{1}{2z-1} + \frac{1}{z-2}$$
$$= \frac{1}{2}\cdot\frac{1}{z-\frac{1}{2}} + \frac{1}{z-2}$$
$$= \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}(1-2z)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$
$$= -(1-2z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$
$$= -[1+(2z)+(2z)^2+\cdots] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$
$$= -(1+2z+4z^2+\cdots) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$
$$= \frac{-3}{2} - \frac{9}{4}z - \frac{33}{8}z^2 - \cdots \quad \text{Ans.}$$

For the region $|z| > 1$ [$|z| > 1$ এলাকার জন্য] $\Rightarrow \frac{1}{|z|} < 1$

$$f(z) = \frac{1}{2z-1} + \frac{1}{z-2}$$
$$= \frac{1}{2z\left(1-\frac{1}{2z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$
$$= \frac{1}{2z}\left(1-\frac{1}{2z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$
$$= \frac{1}{2z}\left(1+\frac{1}{2z}+\frac{1}{2^2z^2}+\frac{1}{2^3z^3}+\cdots\right) + \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right)$$
$$= \left(\frac{1}{2}+1\right)\frac{1}{z} + \left(\frac{1}{4}+2\right)\frac{1}{z^2} + \left(\frac{1}{8}+4\right)\frac{1}{z^3} + \cdots$$
$$= \frac{3}{2z} + \frac{9}{4z^2} + \frac{33}{8z^3} + \cdots \quad \text{Ans.}$$

Problem-41. Show that $\cos h\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$,

Where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cos h(2\cos\theta)\, d\theta$.

[DUMP-1980, RUH-1973]

Solution : Let $f(z) = \cos h\left(z+\frac{1}{z}\right)$ which is analytic everywhere in the finite z-plane except at $z = 0$. Thus, f(z) is analytic in the annular region $r \le |z| \le R$, where r is very small and R is very large. By Laurent's theorem we have

$$f(z) = \cos h\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \text{ ...... (1)}$$

Where $a_n = \frac{1}{2\pi i}\oint_C \cos h\left(z+\frac{1}{z}\right)\frac{dz}{z^{n+1}}$ ...... (2)

$b_n = \frac{1}{2\pi i}\oint_C \cos h\left(z+\frac{1}{z}\right) z^{n-1}\, dz$ ...... (3)

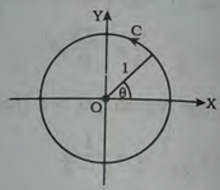
and C is any circle with centre at the origin. Let us consider the circle C of unit radius. Then

$|z| = 1 \Rightarrow z = e^{i\theta}$

$\Rightarrow dz = ie^{i\theta}\, d\theta.$

and $z + \frac{1}{z} = e^{i\theta} + e^{-i\theta}$

$= 2\cos\theta,\ 0 \le \theta \le 2\pi.$



Putting these values in (2) we get,

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \cosh(2\cos\theta)\cdot\frac{ie^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\,(\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cdot\cos n\theta\, d\theta$$

$$- \frac{i}{2\pi}\int_0^{2\pi} \cos h\,(2\cos\theta)\cdot\sin n\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cdot\cos n\theta\, d\theta - \frac{i}{2\pi}\times 0$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta\cdot\cos h(2\cos\theta)\, d\theta$$

$\because \int_0^{2\pi} F(\theta)\, d\theta = 0$ if $F(2\pi - \theta) = -F(\theta)$ which is true for $F(\theta) = \cosh(2\cos\theta)\sin n\theta$

From the values of a_n and b_n we see that if we replace n by – n in the value of a_n we get the value of b_n.

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cos(-n\theta)\cdot\cos h\,(2\cos\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta\cdot\cos h(2\cos\theta)\, d\theta = a_n$$

Thus from (1), $f(z) = \cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$

$$\Rightarrow \cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta\cdot\cos h\,(2\cos\theta)\, d\theta$ (Showed)

Problem–42. Show that $e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n$, where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta.$$

Solution : Let $f(z) = e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)}$. Then f(z) is analytic in the finite z-plane except the point $z = 0$. So f(z) is analytic in the annulas region $r \le |z| \le R$ where r is small and R is large. Hence by Laurent's theorem

$$f(z) = e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n \text{ ...... (1)}$$

Where $a_n = \frac{1}{2\pi i}\oint_C e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)}\frac{dz}{z^{n+1}}$ ...... (2)

and C is any circle. Consider the circle C as $|z| = 1$.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta,\ 0 \le \theta \le 2\pi$

$\therefore z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2\,i\sin\theta.$

$$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{1}{2}c\cdot 2i\sin\theta}\cdot\frac{ie^{i\theta}\,d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ic\sin\theta}\cdot e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - c\sin\theta)}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [\cos(n\theta - c\sin\theta) - i\sin(n\theta - c\sin\theta)]\,d\theta$$

$= \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(n\theta - c\sin\theta)\,d\theta$, since the other term vanishes, as $\displaystyle\int_0^{2\pi}\sin(n\theta - c\sin\theta)\,d\theta$ denoted it by $\displaystyle\int_0^{2\pi} F(\theta)\,d\theta$ which satisfied $F(2\pi - \theta) = -F(\theta)$.

Thus, $e^{\frac{1}{2}c(z-1/z)} = \displaystyle\sum_{n=-\infty}^{\infty} a_n z^n$

where $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\cos(n\theta - c\sin\theta)\,d\theta$. (Showed)

Problem–43. Prove that $z = \pm i$ are branch points of the function $f(z) = (z^2+1)^{1/3}$.

Solution : We have $w = f(z) = (z^2+1)^{1/3}$

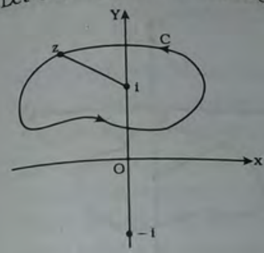
$$\Rightarrow w = [(z+i)(z-i)]^{1/3} = (z+i)^{1/3}(z-i)^{1/3}$$

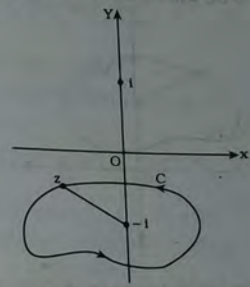
$\therefore \arg w = \dfrac{1}{3}\arg(z+i) + \dfrac{1}{3}\arg(z-i)$ $[\because \arg(z_1z_2) = \arg z_1 + \arg z_2]$

$\Rightarrow$ Change in $\arg w = \dfrac{1}{3}$ [Change in $\arg(z+i)$]

$+\dfrac{1}{3}$ [Change in $\arg(z-i)$]

Let C be a circuit enclosing the point i but not the point – i.





Let the point z goes once counterclockwise around C.

$\therefore$ Change in $\arg(z-i) = 2\pi$, Change in $\arg(z+i) = 0$

Thus, Change in $\arg w = \dfrac{1}{3}\times 0 + \dfrac{1}{3}\times 2\pi = \dfrac{2\pi}{3}$

Hence w does not return to its original value, so a change in branch has occured. Since $z = i$ altered the branch of the given function, so $z = i$ is a branch point of the given function.

In the same way considering as a circuit enclosing $z = -i$ but not $z = i$ we can show that $z = -i$ is another branch point.

Hence $z = \pm i$ are the branch points of the given function.

Problem–44. For the function $f(z) = (z^2+1)^{1/2}$ find branch points, branch line and show that a complete circuit around these points produces no change in the branches of the function.

Solution : Let $w = f(z) = (z^2+1)^{1/2}$

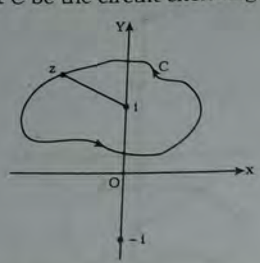
$\Rightarrow w = (z+i)^{1/2}(z-i)^{1/2}$

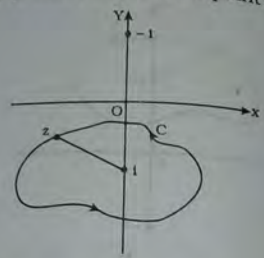
Then $\arg w = \dfrac{1}{2}\arg(z+i) + \dfrac{1}{2}\arg(z-i)$

$\Rightarrow$ Change in $\arg w = \dfrac{1}{2}$ [Change in $\arg(z+i)$]

$+\dfrac{1}{2}$ [Change in $\arg(z-i)$]

Let C be the circuit enclosing the points i but not the point $-i$.





Let the point z goes once counter clockwise around C.

$\therefore$ Change in $\arg(z-i) = 2\pi$ and Change in $\arg(z+i) = 0$

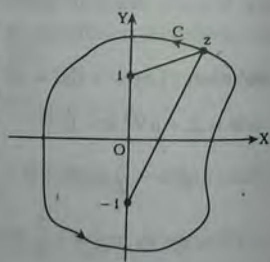
Thus Change in $\arg w = \frac{1}{2}\times 0 + \frac{1}{2}\times 2\pi = \pi$

Hence w does not return to its original values, so a change in branch has occurred. Since $z = i$ altered the branch of the given function, so $z = i$ is a branch point of the given function.

In the same way consider C as a circuit enclosing $z = -i$ but not $z = i$ we can show that $z = -i$ is another branch point. Hence $z = \pm i$ are the branch points of the given function.

2nd Part (Branch line) : The line segment $-1 \le y \le 1$ $(x = 0)$ is the branch line.

3rd Part : If C encloses both branch point $z = \pm i$ and z moves counterclockwise around C, then Change in $\arg(z-i) = 2\pi$ and Change in $\arg(z+i) = 2\pi$.



$\therefore$ Change in arg

$$w = \frac{1}{2}\times 2\pi + \frac{1}{2}\times 2\pi = 2\pi.$$

Hence a complete circuit around both the branch point does not change the branch.

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. What is a zero of an analytic function?

   OR, What is meant by zero of an analytic functoin?
   [বৈশ্লেষিক ফাংশনের শূণ্য বলতে কি বুঝায়?] **[NUH–2014]**

   Ans : A value of z for which the analytic function $f(z) = 0$ is called a zero of $f(z)$.

2. What is a zero of order n of an analytic function?

   Ans : If $f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic, $g(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a zero of order n of the function $f(z)$.

3. What is a simple zero of an analytic function?

   Ans : If $f(z)$ has a zero of order one at $z = z_0$, then $f(z)$ is said to have a simple zero at $z = z_0$.

4. Define singular point of a complex function $f(z)$. **[NUH–2012]**

   Ans : A point at which an analytic function $f(z)$ fails to be analytic is called a singular point.

5. Define isolated singularity of a complex function $f(z)$.

   Ans : A singular point $z = z_0$ is called an isolated singularity of $f(z)$ if there is no other singularity with in a small circle surrounding the point $z = z_0$.

6. Define ordinary point of a complex function $f(z)$.

   Ans : If $z = z_0$ is not a singular point of $f(z)$ and there can be found a small circle surrounding the point $z = z_0$ which encloses no singular point, then $z = z_0$ is called an ordinary point of $f(z)$.

7. What is pole? **[NUH–2013]**

   OR, Define pole of a complex function. **[NUH–2015]**

   Ans : If there exists a positive integer n such that
   $$\lim_{z\to z_0} (z - z_0)^n f(z) = A \neq 0.$$
   then $z = z_0$ is called a pole of order n.

8. Why a pole is called a non-essential singularity?

   Ans : If a function $f(z)$ is single valued and has a singularity, then this singularity is either a pole or an essential singularity. For this reason a pole is sometimes called a non-essential singularity.

9. Define removable singularity.
Ans : A point z_0 is called a removable singularity of a complex function f(z) if $\lim_{z \to z_0} f(z)$ exists.

10. Define essential singularity.
Ans : A singular point which is not a pole, branch point or removable singularity is called an essential singularity.

11. Define meromorphic function. **[NUH-2014]**
Ans : A complex function f(z) which has poles as its only singularities in the finite part of the plane is called a meromorphic function.

12. Define entire function with an example. **[NUH-2014]**
OR, What do you mean by entire function?
Ans : A complex function f(z) which has no singularities in the finite part of the plane is called an entire function.
The functions e^z, sinz, cosz are entire functions.

13. Write a rule to determine a pole.
Ans : If $\lim_{z \to z_0} f(z) = \infty$ then $z = z_0$ is a pole of f(z).

14. Write a rule to determine a pole of order m.
Ans : If there are only m terms in the negative powers of $z - z_0$ of f(z) then $z = z_0$ is a pole of order m.

15. What is the pole of $f(z) = \frac{z^2 - 3z}{z - 2}$? [$f(z) = \frac{z^2 - 3z}{z - 2}$ এর পোল কত?] **[NUH-2012]**
Ans : The pole of $f(z) = \frac{z^2 - 3z}{z - 2}$ is 2, since

$$\lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} (z-2)\frac{z^2 - 3z}{z-2} = 2^2 - 6 = -2 \neq 0.$$

16. What is branch point of complex number? **[NUH-2013]**
Ans : A multivalued function f(z) defined in some domain S is said to have a branch point at z_0 if, when z describes an arbitrary small circle about z_0, then for every branch F of f, F(z) does not return to its original value.
[একটি ডোমেন S এ বর্ণিত একটি বহুমানী ফাংশন f(z) এর z_0 বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে যদি, f এর প্রত্যেক ব্রাঞ্চF এর জন্য z_0 বিন্দুর চারিদিকে z একটি ইচ্ছাধীন ক্ষুদ্র বৃত্ত বর্ণনা করে যেখানে F(z) তার আদি মান দিতে পারে না।]

17. What is meant by residue of complex number? **[NUH-2013]**
OR, Define residue at a pole of a complex function. **[NUH-15]**
Ans : If the function f(z) is analytic within a circle C of radius r and centre a, except at z = a, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laruent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$= \{a_0 + a_1 (z-a) + a_2(z-a)^2 + \cdots\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\right\}$$

around z = a is called the residue of f(z) at z = a. It is denoted by Res (a) or a_{-1}.

18. Write the formula to find the residue of f(z) at the pole z = a of order m. [f(z) ফাংশনের z = a বিন্দুতে অবস্থিত m ক্রমের পোলে রেসিডিউ নির্ণয়ের সূত্রটি লিখ।] **[NUH-2014]**
Ans : The formula to find the residue of f(z) at the pole z = a of order m is [f(z) ফাংশনের z = a বিন্দুতে অবস্থিত m ক্রমের পোলে রেসিডিউ নির্ণয়ের সূত্রটি হল।]

$$\lim_{z \to a} \frac{1}{\underline{|m-1}} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}.$$

19. Write a function which has two isolated singularity.
[এমন একটি ফাংশন লিখ যার দুইটি বিচ্ছিন্ন (সতন্ত্র) বিন্দু আছে।] **[NUH-2014]**
Ans : The function $f(z) = \frac{z^2}{(z-1)(z-2)}$ has two isolated singularity z = 1 and z = 2.

20. Write a complex function which has a zero at z = 1 but has no singularty. **[NUH-2015]**
Ans : The complex function f(z) = z − 1 has a Zero at z = 1 but has no singularity.

EXERCISE-4

Part-A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. Define branch line.
2. Define branch point of a multivalued function f(z).
3. State maximum modulus theorem.
4. Write the statement of Rouche's theorem.
5. Why the function $f(z) = e^{-1/z^2}$ has no pole?

Part–B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. Derive the formula to find residue of a function at a pole of order m.
[NUH–1993, 2000, 2002, 2005(Old), 2006(Old), DUH–1993]
Ans : See theorem – 5 of art 4.5.
2. If f(z) is analytic inside and on a simple closed curve C except for a finite number of poles P and zeros N of f(z), then
$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = N - P.$ **[DUH–1994]**
Ans : See theorem – 9.
3. State and prove Cauchy's residue theorem. **[NUH–1998, 2002, 2006]**
Ans : See theorem – 6.
4. Let f(z) be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of f(z) at a is given by
$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$
Ans : See theorem – 5.
5. Show that if z_0 is a pole f(z), then $|f(z)| \to \infty$ as $z \to z_0$. **[NUH–1996]**
Ans : See theorem – 3 (Write z_0 instead of a).

Part–C (Broad Questions) (বড় প্রশ্ন)

1. State and prove the theorem on Laurent's series. **[DUH–1993]**
Ans : See theorem – 2.
2. State and prove Rouche's theorem.
[NUH–1995, 2004, 2007, DUH–2000, 2003, RUH–1997, 2001]
Ans : See theorem – 10.
3. State and prove the Argument theorem. **[DUH–2002, RUH–1996, 1998]**
Ans : See theorem – 8.
4. State and prove Taylor's theorem. **[NUH–2005 (Old), DUH–1998, 2001]**
Ans : See theorem–1.
5. Establish Taylor series for a complex function. **[NUH–2004, 2008, DUH–2005]**
Ans : See theorem–1.
6. State and prove maximum modulas principle. **[DUH–2000, 2002]**
Ans : See theorem–7

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CHAPTER–5
CALCULUS OF RESIDUES CONTOUR INTEGRATION

Evaluation of real definite integrals by contour integration.

Contour means closed path. We choose a closed curve C is usually called a contour. The contour may be a circle, semi-circle or a quadrant of a circle. The process of integration along a contour is called contour integration. To evaluate a definite integral we first determine the poles of f(z) and then calculate the residues at the poles which are inside the contour. Finally we use the Cauchy's residue theorem

$$\oint_C f(z)\,dz = 2\pi i \times \text{(sum of the residues of f(z) at the poles with in C)} \quad \cdots\cdots (1)$$

There are so many types of problems. According to the problems we classify some important problems in the following way.

5.1. Integration round the unit circle.

In this case the integrals are of the type

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta \text{ or } \int_{-\pi}^{\pi} f(\sin\theta, \cos\theta)\,d\theta$$

We consider the unit circle $|z| = 1$ as the closed contour C so that

$$z = e^{i\theta} \Rightarrow d\theta = \frac{1}{iz}dz$$

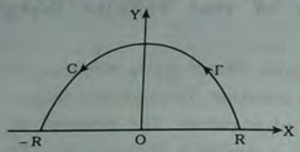
$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \text{ and } \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

The integrals then become $\oint_C f(z)\,dz$ which can be calculated by using (1).

5.2. Evaluation of $\int_{-\infty}^{\infty} f(x)\,dx$ or $\int_0^{\infty} f(x)\,dx$

Improper integrals where no poles on the real axis.

We can evaluate $\int_{-\infty}^{\infty} f(x)\,dx$ if f(z) has no poles on the real axis and probably some poles on the upper half plane. We consider the circle $|z| = R$ and its upper half as Γ as shown in the figure.



All poles can be found by putting denominator of f(z) equal to zero.

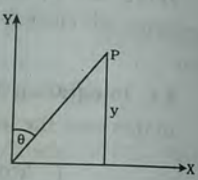
Jordan's inequality.

The inequality $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$, where $0 \le \theta \le \frac{\pi}{2}$ is called the Jordan's inequality.

Consider $y = \cos\theta$.

When θ increases then $\cos\theta$ decreases

$\Rightarrow$ When θ increases then y decreases.



The mean ordinate between $\theta = 0$ to θ is

$$\frac{1}{\theta}\int_0^\theta y\,d\theta = \frac{1}{\theta}\int_0^\theta \cos\theta\,d\theta = \frac{1}{\theta}[\sin\theta]_0^\theta = \frac{\sin\theta}{\theta}$$

When $\theta = 0$ then $y = \cos 0 = 1$

When $\theta = \frac{\pi}{2}$ then the mean ordinate $= \frac{\sin\frac{\pi}{2}}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi}$

Thus when $0 < \theta < \frac{\pi}{2}$ then the mean ordinate lies between 1 and $\frac{2}{\pi}$, that is, $\frac{2}{\pi} < \frac{\sin\theta}{\theta} < 1$.

Hence $\frac{\pi}{2} \le \frac{\sin\theta}{\theta} \le 1$, when $0 \le \theta \le \frac{\pi}{2}$

Jordan's Lemma :

If f(z) is analytic except a finite number of singularities and $f(z) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to\infty}\int_\Gamma e^{imz} f(z)\,dz = 0,\ m > 0$$

where Γ is the semi-circle $|z| = R$, $\text{Im}(z) \ge 0$.

Proof : Here R is large, so we can ensure that all singularities with in Γ and none on its boundary.

Since $f(z) \to 0$ uniformly as $z \to \infty$, so for given $\varepsilon > 0$, there exists $R > 0$ such that $|f(z)| < \varepsilon$ for all z on Γ.

Also, $z = Re^{i\theta} \Rightarrow |dz| = |iRe^{i\theta}\,d\theta| = R\,d\theta$

and $|e^{imz}| = |e^{imR(\cos\theta + i\sin\theta)}| = |e^{imR\cos\theta}|\ |e^{-mR\sin\theta}|$

$= 1 \cdot |e^{-mR\sin\theta}| = e^{-mR\sin\theta}$

Now, $\left|\int_\Gamma e^{imz} f(z)\,dz\right| \le \int_\Gamma |e^{imz}|\ |f(z)|\ |dz|$

$\Rightarrow \left|\int_\Gamma e^{imz} f(z)\,dz\right| \le \int_0^\pi e^{-mR\sin\theta} \cdot \varepsilon \cdot R\,d\theta$

$\Rightarrow \left|\int_\Gamma e^{imz} f(z)\,dz\right| \le 2\varepsilon R\int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta$; By Jordan's inequality

$\Rightarrow \left|\int_\Gamma e^{imz} f(z)\,dz\right| \le 2\varepsilon R\left[\frac{e^{-2mR\theta/\pi}}{-2mR/\pi}\right]_0^{\pi/2}$

$\Rightarrow \left|\int_\Gamma e^{imz} f(z)\,dz\right| \le \frac{2\varepsilon R\pi}{-2mR}[e^{-mR} - 1]$

$\Rightarrow \left|\int_\Gamma e^{imz} f(z)\,dz\right| \le \frac{\varepsilon\pi}{m}(i - e^{-mR}) \to 0$ when $R \to \infty$ and $\varepsilon \to 0$.

$\Rightarrow \int_\Gamma e^{imz} f(z)\,dz \to 0$ when $R \to \infty$

Hence, $\lim_{R\to\infty}\int_\Gamma e^{imz} f(z)\,dz = 0$. (Proved)

5.3. Improper integrals involving sines and cosines :

In this case the integrals are of the form

$$\int_{-\infty}^{\infty} f(x)\sin mx\,dx \ \text{ or } \int_{-\infty}^{\infty} f(x)\cos mx\,dx$$

To evaluate the integrals we use the proces discussed in art-5.2.

5.4. Integration along indented contours :

When the poles lie on the real axis and the integration is not possible at that poles, then we excluded those poles on the real axis by enclosing them with semi-circles of small radi i. This procedure is known as indenting at a point.

Now f(z) is regular along the modified contour C, and the integral $\oint_C f(z)\,dz$ can be evaluated by the process and theorems discussed in art-5.2 and 5.3, and also by using the following theorems.

Theorem : If $\lim_{z\to a} (z-a) f(z) = k$ and γ be the arc of the circles $|z-a| = r$ given by $\theta_1 \le \theta \le \theta_2$, then

$$\lim_{r\to 0} \int_\gamma f(z)\,dz = ik(\theta_2 - \theta_1)$$

where k is the residue of f(z) at the simple pole $z = \alpha$ and the integration is taken in anti-clock direction.

Proof : Given that $\lim_{z\to a} (z-a) f(z) = k$. So for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|(z-a) f(z) - k| < \varepsilon \text{ whenever } 0 < |z-a| < \delta$$

or, $(z-a) f(z) = k + \varepsilon$, where $\varepsilon \to 0$ as $z \to a$

$$\therefore \int_\gamma f(z)\,dz = \int_\gamma \frac{k+\varepsilon}{z-a}\,dz$$

$$= \int_{\theta_1}^{\theta_2} \frac{k+\varepsilon}{re^{i\theta}} \cdot ire^{i\theta}\,d\theta$$

$$\left|\begin{array}{l} |z-a| = r \\ \Rightarrow z-a = re^{i\theta} \\ \Rightarrow dz = ire^{i\theta}\,d\theta \end{array}\right.$$

$$= \int_{\theta_1}^{\theta_2} ik\,d\theta + \int_{\theta_1}^{\theta_2} i\varepsilon\,d\theta$$

$$= ik\,[\theta]_{\theta_1}^{\theta_2} + \int_{\theta_1}^{\theta_2} i\varepsilon\,d\theta$$

$$= ik(\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} i\varepsilon\,d\theta$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz - ik(\theta_2 - \theta_1)\right| = \left|\int_{\theta_1}^{\theta_2} i\varepsilon\,d\theta\right|$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz - ik(\theta_2 - \theta_1)\right| \le \int_{\theta_1}^{\theta_2} |i\varepsilon|\,d\theta$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz - ik(\theta_2 - \theta_1)\right| \le \varepsilon\,(\theta_2 - \theta_1)$$

When $r \to 0$ then $z - a \to 0 \Rightarrow z \to a$ and so $\varepsilon \to 0$

$$\therefore \lim_{r\to 0} \left[\int_\gamma f(z)\,dz - ik\,(\theta_2 - \theta_1)\right] = 0$$

$$\Rightarrow \lim_{r\to 0} \int_\gamma f(z)\,dz = ik(\theta_2 - \theta_1)$$

5.5. Integration through a branch cut

Or, Integration involving many valued functions.

Integrals of the form $\int_0^\infty x^{n-1} Q(x)\,dx$, where n is not an integer, then x^{n-1} is a many-valued function. This type of integral can be evaluated by the method of contour integration by taking the integral round a large circle $|z| = R$ a small circle $|z| = r$ enclosing the branch point $z = 0$ and a cut along the real axis joining the ends of the two circles.

5.6. Other types of contours :

Some integrals may not be evaluated by the contours considering above discussion. Solely, the choice of contour depends on the nature of the function to be integrated. We consider such contour as rectangle, triangle, etc.

Now we shall do all types of problems which are helpful and important for the learner.

SOLVED PROBLEMS

Evaluate the following by using the method of contour integration.

GROUP–A

1. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ **[NUH–93, 03, 15 (Phy), DUH–93, RUH–2000]**

2. $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ **[NUH–02, 04, 06 (Old), 07, 12(Old), 12]**

3. $\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta}$ **[NUH–1995]**

3(a). $\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$ **[NUH–2010]**

4. $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ **[NUH–2000,05 (Old), NU(Pre)–06, 13, 14]**

5. $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$ **[RUH–2001]**

6. $\int_0^{2\pi} \frac{1}{1+a\sin x}\,dx,\ 0<a<1$ **[NUH–1996]**

7. $\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$ **[RUH–1996]**

8. $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$ **[RUH–1998, 2002]**

9. $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$ **[NUH–1997, DUH–2006]**

10. $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$

[NUH–99, 03, 06, 08, NU(Pre)–08, DUH–90, 94]

11. $\int_0^{2\pi} \frac{\sin 2\theta}{5-3\cos\theta}\,d\theta$ **[NUH–2001]**

12. $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta},\ a>|b|$ **[DUH–1975, 1977, 1987]**

13. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta},\ a>|b|$ **[DUH–2005, RUH–1995, 1997]**

14. (i) $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ **[DUH–1984, CUH–2000, 2002]**

(ii) $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2}$

15. $\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta},\ 0<a<1$

16. (i) $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta,\ a^2<1$

[DUH–1984, RUH–1973, 1975, 1981]

(ii) $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta,\ a^2<1$

[DUH–1976, 1986, CUH–1990]

17. $\int_0^{\pi} \frac{a\,d\theta}{a^2+\cos^2\theta},\ a>0$ **[NUH–1998]**

18. $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta},\ a>0$ **[DUH–86, 03, 04, CUH–01, 04]**

18(i). $\int_0^{\pi} \frac{dx}{1+\sin^2 x}$ **[NUH–2013]**

19. (i) $\int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{a+b\cos\theta},\ a>b>0$ **[NU(Pre)–2011]**

(ii) $\int_0^{\pi} \frac{\sin^2\theta\,d\theta}{a+b\cos\theta},\ a>b>0$

[DUH–1977, 1983, 1987, 1990, CUH–1982]

20. $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta$ **[NUH–04(Old), 08, 11, CUH–03]**

21. By calculas of residue prove that

(i) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$

[CUH–2001, 2003, 2004]

(ii) $\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta + \sin\theta)\,d\theta = \frac{2\pi}{n!}(-1)^n$

GROUP–B

22. $\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$

23. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx$ **[NUH–1995]**

23(i). $\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)}\,dx$ **[NUH–2013]**

24. $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)^2} dx$
[NUH-2000, RUH-1996, NU(Pre)-2006, 2014, CUH-2001]

25. $\int_0^\infty \frac{2x^2}{(x^2+9)(x^2+4)^2} dx$ **[NUH-1993]**

26. $\int_{-\infty}^\infty \frac{1}{(x^2+b^2)(x^2+c^2)^2} dx$
[DUH-1980, 1982, 1988, DUH-1984, RUH-1979]

27. $\int_{-\infty}^\infty \frac{x^2}{(x^2+1)^2(x^2+2x+2)} dx$ **[RUH-1984]**

28. $\int_0^\infty \frac{dx}{x^4+1}$
[NUH-02, 05, 12, DUH-86, 88, RUH-98, 01]

29. $\int_0^\infty \frac{dx}{x^4+a^4}, a>0$ **[NUH-01, 04, 06, 08, 12(Old), DUH-88]**

30. $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx, a>0$ **[NUH-1996]**

30. (i) $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2}$ **[NUH-2014]**

31. $\int_0^\infty \frac{1}{x^6+1} dx$ **[NU(Pre)-2011, CUH-2004]**

32. $\int_0^\infty \frac{x^6}{(a^4+x^4)^2} dx$ **[DUH-1977, 1985, DUH-1980]**

33. $\int_0^\infty \frac{x^6}{(x^4+1)^2} dx$ **[NUH-1998]**

34. $\int_0^\infty \frac{\log(x^2+1)}{x^2+1} dx$ **[NUH-97, 03, 06 (Old), NU(Pre)-08, 11, 13, DUH-05, RUH-82, 97, 99, CUH-81, 89, 03]**

34. (a) $\int_0^\infty \frac{z^a}{(1+z^2)^2} dz, 0<a<1$ **[NUH-1996]**

(b) $\int_0^\infty \frac{dz}{1+z^2}$ **[NU(Phy)-2005]**

GROUP-C

35. $\int_{-\infty}^\infty \frac{\cos x}{(x^2+1)(x^2+9)} dx$ **[RUH-1977]**

36. $\int_{-\infty}^\infty \frac{\cos x}{(x^2+8)(x^2+12)} dx$ **[DUH-1977]**

37. $\int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$ **[DUH-1982, 1987, 1989]**

38. $\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1} dx$ **[NUH-1998]**

39. (i) $\int_0^\infty \frac{\cos mx}{x^2+a^2} dx$ **[DUH-1978, 1988, 2006, RUH-1976]**

(ii) $\int_{-\infty}^\infty \frac{\sin mx}{x^2+a^2} dx$

40. (i) $\int_0^\infty \frac{\cos ax}{x^2+1} dx, a>0$
[NUH-96, 05 (Old), 06(Old), 11, 15, NU(Pre)-05, 13]

(ii) $\int_{-\infty}^\infty \frac{\sin ax}{x^2+1} dx$

41. $\int_{-\infty}^\infty \frac{\cos 2x}{x^2+1} dx$ **[NUH-1995, 2004(Old), 2012]**

42. $\int_0^\infty \frac{\cos 3x}{x^2+4} dx$ **[RUH-1999]**

43. (i) $\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2} dx$ **[DUH-1986]**

(ii) $\int_{-\infty}^\infty \frac{\sin mx}{(x^2+a^2)^2} dx$

44. $\int_0^\infty \frac{\cos mx}{(x^2+1)^2} dx, m>0$ **[RUH-1995, 1997, CUH-2002]**

45. $\int_0^\infty \frac{\cos mx}{x^4+a^4} dx$ **[DUH-1975, 1986]**

46. $\int_0^\infty \frac{x \sin mx}{x^4+a^4} dx$ **[DUH-1976, 1978, 1986]**

47. (i) $\int_0^\infty \frac{x \cos mx}{x^2+a^2} dx$

(ii) $\int_0^\infty \frac{x \sin mx}{x^2+a^2} dx, a>0, m>0$
[NUH-97, 05 (Old), NU(Phy)-05, DUH-76, 78, 03]

48. $\int_0^\infty \frac{x \sin x}{a^2+x^2} dx$ **[NUH-2003, CUH-1989]**

49. $\int_0^\infty \frac{x \sin x}{x^2+4} dx$ **[NUH-94, 05,(Old) NU(Pre)-05, DUH-94, RUH-75, CUH-01]**

GROUP–D

50. $\int_0^\infty \frac{\sin x}{x}\,dx$ [NUH–94, 12(Old), DUH–85, 88, 94, 04, 05, RUH–1995, 2002, 2003]

51. $\int_0^\infty \frac{\sin mx}{x}\,dx,\ m > 0$ [NUH–93, 03, 04, 06, 07, 15, DUH–93, 02, 04, 06, 07]

52. $\int_0^\infty \frac{\cos x}{a^2 - x^2}\,dx$ [NUH–2000, NU(Pre)–2006, 2014]

53. $\int_0^\infty \frac{\sin \pi x}{x(1 - x^2)}\,dx$ [NUH–2001, 2005 (Old), 2006 (Old), 2014, NUP–05, DUH–03, RUH–73, 74, 81, CUH–04]

54. (i) $\int_0^\infty \frac{(\log x)^2}{1 + x^2}\,dx$ [RUH–2001, CUH–2004]

(ii) $\int_0^\infty \frac{\log x}{1 + x^2}\,dx$

55. $\int_0^\infty \frac{\log x}{(1 + x^2)^2}\,dx$ [RUH–2002]

GROUP–E

56. If $0 < p < 1$, then show that

(i) $\int_0^\infty \frac{x^{p-1}}{1 + x}\,dx = \frac{\pi}{\sin p\pi}$

[NUH–98, 05, 08, 11, NU(Phy)–06, DUH–86, RUH–96, 01]

(ii) $\Gamma(p)\,\Gamma(1 - p) = \frac{\pi}{\sin p\pi}$

57. Show that $\int_{-\infty}^\infty \frac{e^{px}}{1 + e^x}\,dx = \frac{\pi}{\sin p\pi}$, where $0 < p < 1$ [NUH–97, NU(Pre)–08, 11, DUH–76, 78, 87, RUH–98, 01, 02, CUH–01]

58. Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

[NUH–2005, 2007, RUH–1981, 1999]

59. Show that $\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{2\cos\left(\frac{\pi a}{2}\right)}$ [DUH–1975, 1989]

60. By contour integration prove that

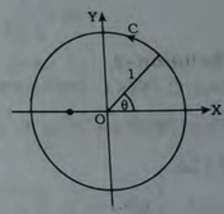
$$\int_{-\infty}^\infty \frac{\cos x^2 + \sin x^2 - 1}{x^2}\,dx = 0$$

SOLUTIONS
GROUP–A

Solution–1. Let us consider the unit circle $|z| = 1$ as the contour C. [কন্টুর C কে আমরা একক বৃত্ত $|z| = 1$ বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$ where [যেখানে] $0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$



$$\therefore \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{2 + \frac{z^2 + 1}{2z}}$$

$$= \frac{1}{i}\oint_C \frac{2z/z}{4z + z^2 + 1}\,dz$$

$$= \frac{2}{i}\oint_C \frac{dz}{z^2 + 4z + 1}$$

$$= \frac{2}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$

where [যেখানে] $f(z) = \frac{1}{z^2 + 4z + 1}$

The poles of f(z) are obtained by solving the equation

$z^2 + 4z + 1 = 0$

[f(z) এর পোল $z^2 + 4z + 1 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$\Rightarrow z = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

$$|z| = |-2 + \sqrt{3}| = |-2 + 1.73| = |-0.27| = 0.27 < 1$$

$$|z| = |-2 - \sqrt{3}| = |-2 - 1.73| = |-3.73| = 3.73 > 1$$

$\therefore$ The pole $z = -2 + \sqrt{3}$ lies inside the contour C which is a simple pole. [$z = -2 + \sqrt{3}$ পোলটি C কন্টুরের ভিতরে অবস্থিত যাহা একটি সরল পোল।]

Residue at $z = -2 + \sqrt{3}$ is [$z = -2 + \sqrt{3}$ এ অবশেষ]

$$\lim_{z \to -2+\sqrt{3}} (z + 2 - \sqrt{3})\,f(z)$$

$$= \lim_{z \to -2+\sqrt{3}} (z + 2 - \sqrt{3}) \cdot \frac{1}{z^2 + 4z + 1}$$

$$= \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})\cdot \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$$= \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\cdot 2\pi i \cdot (\text{Residue at } z = -2+\sqrt{3})$$

$$= 4\pi \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}. \text{ (Ans)}$$

Solution–2. Let us consider the unit circle $|z| = 1$ as the contour C. [কন্টুর C হিসাবে একক বৃত্ত $|z| = 1$ কে বিবেচনা করি]

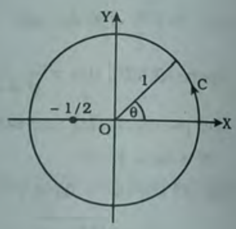
Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$; where [যেখানে] $0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{5+4\cdot\frac{z^2+1}{2z}}$$

$$= \frac{1}{i}\oint_C \frac{dz}{5z+2z^2+2}$$

$$= \frac{1}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$



where [যেখানে] $f(z) = \frac{1}{2z^2+5z+2}$

The poles of f(z) are obtained by solving the equation

$$2z^2 + 5z + 2 = 0$$

[f(z) এর পোল $2z^2 + 5z + 2 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$\Rightarrow z = \frac{-5 \pm \sqrt{25-16}}{4} = \frac{-5\pm 3}{4} = -\frac{1}{2}, -2$$

$|z| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$ and $|z| = |-2| = 2 > 1$.

Only the pole $z = -\frac{1}{2}$ lies inside the contour C which is a simple pole. [একমাত্র পোল $z = -\frac{1}{2}$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরলপোল]

Residue at $z = -\frac{1}{2}$ is [$z = -\frac{1}{2}$ এ অবশেষ]

$$\lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot f(z) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot \frac{1}{2z^2+5z+2}$$

$$= \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot \frac{1}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1}{2}\cdot \lim_{z\to -\frac{1}{2}} \frac{1}{z+2}$$

$$= \frac{1}{2}\cdot \frac{1}{-\frac{1}{2}+2} = \frac{1}{2\cdot\frac{3}{2}} = \frac{1}{3}$$

Now by Cauchy's residue theorem we have from (1) [এখন কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{i}\cdot 2\pi i \left(\text{Residue at } z = -\frac{1}{2}\right)$$

$$= 2\pi\cdot\frac{1}{3} = \frac{2\pi}{3} \quad \text{(Ans)}$$

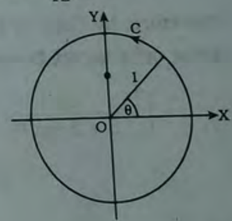
Solution–3. Let us consider the unit circle $|z| = 1$ as the contour C. [কন্টুর C হিসাবে একক বৃত্ত $|z| = 1$ কে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$

where [যেখানে] $0 \le \theta \le 2\pi$.

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$= \frac{z^2-1}{2iz}$$



$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{3+2\cdot\frac{z^2-1}{2iz}}$$

$$= \oint_C \frac{\frac{1}{iz}\,dz}{\frac{3iz+z^2-1}{2iz}} = 2\oint_C \frac{dz}{z^2+3iz-1}$$

$$= 2\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$

where [যেখানে] $f(z) = \dfrac{1}{z^2 + 3iz - 1}$

The poles of f(z) are obtained from the equation [f(z) এর পোল $z^2 + 3iz - 1 = 0$ সমীকরণ হতে পাওয়া যাবে]

$$z^2 + 3iz - 1 = 0$$

$$\Rightarrow z = \frac{-3i \pm \sqrt{9i^2 + 4}}{2} = \frac{-3i \pm \sqrt{-5}}{2} = \frac{(-3 \pm \sqrt{5})i}{2}$$

$$|z| = |(-3 + \sqrt{5})i| = |-3 + \sqrt{5}| < 1$$

and $|z| = |(-3 - \sqrt{5})i| = |-3 - \sqrt{5}| > 1$

$\therefore$ The only pole [এক মাত্র পোল] $z = (-3 + \sqrt{5})i$ lies inside the contour C which is a simple pole. [C এর ভিতর অবস্থিত যাহা সরল পোল]

Residue at $z = (-3 + \sqrt{5})i$ is [$z = (-3 + \sqrt{5})i$ এ অবশেষ]

$$\lim_{z \to (-3+\sqrt{5})i} \{z - (-3 + \sqrt{5})i\} \cdot \frac{1}{z^2 + 3iz - 1}$$

$$= \lim_{z \to (-3+\sqrt{5})i} \{z - (-3 + \sqrt{5})i\} \cdot \frac{1}{\{z - (-3 + \sqrt{5})i\}\{z - (-3 - \sqrt{5})i\}}$$

$$= \frac{1}{(-3 + \sqrt{5})i - (-3 - \sqrt{5})i} = \frac{1}{(-3 + \sqrt{5} + 3 + \sqrt{5})i} = \frac{1}{2i\sqrt{5}}$$

Therefore, by Cauchy's residue theorem we have from (1) [অতএব, কচির অবশেষে উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{3 + 2\sin\theta} = 2 \cdot 2\pi i(\text{Residue at the pole})$$

$$= 4\pi i \cdot \frac{1}{2i\sqrt{5}} = \frac{2\pi}{\sqrt{5}} \quad \text{(Ans)}$$

Solution-3(a) : Consider the unit circle $|z| = 1$ as the contour C. [$|z| = 1$ একক বৃত্তকে কন্টুর C হিসাবে বিবেচনা করি]

তখন [Then] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$

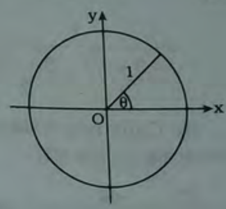
$$\Rightarrow d\theta = \frac{dz}{iz}, \text{ where [যেখানে] } 0 \le \theta \le 2\pi.$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = \oint_C \frac{\frac{dz}{iz}}{3 + 2 \cdot \frac{z^2 + 1}{2z}}$$

$$= \frac{1}{i} \oint_C \frac{dz}{3z + z^2 + 1}$$

$$= \frac{1}{i} \oint_C f(z)\, dz, \text{ say (ধরি) ...... (1)}$$



where [যেখানে] $f(z) = \dfrac{1}{z^2 + 3z + 1}$

The poles of f(z) can be obtained by solving the equation $z^2 + 3z + 1 = 0$ [f(z) এর পোল $z^2 + 3z + 1 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$\Rightarrow z = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2} = \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$$

$$|z| = \left|\frac{-3 + \sqrt{5}}{2}\right| < 1 \text{ and [এবং] } |z| = \left|\frac{-3 - \sqrt{5}}{2}\right| > 1$$

The pole $z = \dfrac{-3 + \sqrt{5}}{2}$ lies inside the contour which is a simple pole. [$z = \dfrac{-3 + \sqrt{5}}{2}$ পোলটি কন্টুরের ভিতরে অবস্থিত যাহা একটি সরল পোল]

$z = \dfrac{-3 + \sqrt{5}}{2}$ [Residue at [এ অবশেষ] $z = \dfrac{-3 + \sqrt{5}}{2}$]

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) f(z)$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) \cdot \frac{1}{z^2 + 3z + 1}$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) \cdot \frac{1}{\left(z - \frac{-3 + \sqrt{5}}{2}\right)\left(z - \frac{-3 - \sqrt{5}}{2}\right)}$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \frac{1}{\left(z + \frac{3 + \sqrt{5}}{2}\right)}$$

$$= \frac{1}{\frac{-3+\sqrt{5}}{2}+\frac{3+\sqrt{5}}{2}} = \frac{1}{\sqrt{5}}$$

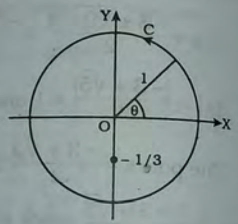
By Cauchy's residue theorem we have form (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi}\frac{d\theta}{3+2\cos\theta} = \frac{1}{i}\cdot 2\pi i\cdot\left[\frac{1}{\sqrt{5}}\right] = \frac{2\pi}{\sqrt{5}}. \quad \text{(Ans)}$$

Solution–4. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$

where [যেখানে] $0 \le \theta \le 2\pi$.



$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$= \frac{z^2-1}{2iz}.$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{5+3\sin\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{5+3\cdot\frac{z^2-1}{2iz}}$$

$$= \oint_C \frac{2}{10iz+3z^2-3}\,dz$$

$$= 2\oint_C \frac{1}{3z^2+10iz-3}\,dz$$

$$= 2\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$

where [যেখানে] $f(z) = \frac{1}{3z^2+10iz-3}$.

The poles of f(z) are obtained by solving the equation

$$3z^2 + 10iz - 3 = 0$$

[$3z^2 + 10iz - 3 = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$$\Rightarrow z = \frac{-10i \pm \sqrt{100i^2+36}}{6}$$

$$= \frac{-10i \pm 8i}{6} = \frac{-i}{3}, -3i$$

$\left|\frac{-i}{3}\right| = \frac{1}{3} < 1$ and [এবং] $|-3i| = 3 > 1$

Thus the only pole $z = -\frac{i}{3}$ lies inside the contour C. [অতএব কন্টুর C এর মধ্যে একমাত্র পোল $z = -\frac{i}{3}$.]

Residue at $z = \frac{-i}{3}$ is [$z = \frac{-i}{3}$ এ অবশেষ] $\lim\limits_{z\to\frac{-i}{3}}\left(z+\frac{i}{3}\right)f(z)$

$$= \lim_{z\to\frac{-i}{3}}\left(z+\frac{i}{3}\right)\cdot\frac{1}{3z^2+10iz-3}$$

$$= \lim_{z\to\frac{-i}{3}}\left(z+\frac{i}{3}\right)\cdot\frac{1}{3\left(z+\frac{i}{3}\right)(z+3i)}$$

$$= \lim_{z\to\frac{-i}{3}}\frac{1}{3(z+3i)} = \frac{1}{3\left(\frac{-i}{3}+3i\right)}$$

$$= \frac{1}{-i+9i} = \frac{1}{8i}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi}\frac{d\theta}{5+3\sin\theta} = 2\cdot 2\pi i\left(\text{Residue at } z = \frac{-i}{3}\right)$$

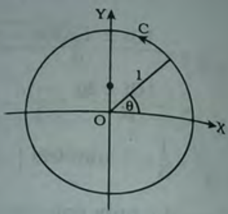
$$= 4\pi i\cdot\frac{1}{8i} = \frac{\pi}{2} \quad \text{(Ans)}$$

Solution–5. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$, where [যেখানে] $0 \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{2+\sin\theta}=\oint_C\frac{\frac{1}{iz}\,dz}{2+\frac{z^2-1}{2iz}}$$



$$=2\oint_C\frac{dz}{4iz+z^2-1}$$

$$=2\oint_C f(z)\,dz,\ \text{say} \cdots\cdots (1)$$

where [যেখানে] $f(z)=\dfrac{1}{z^2+4iz-1}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল $z^2+4iz-1=0$ সমীকরণ হতে পাওয়া যাবে]

$$z^2+4iz-1=0$$

$$\Rightarrow z=\frac{-4i\pm\sqrt{16i^2+4}}{2}=\frac{-4i\pm 2\sqrt{3}i}{2}$$

$$\Rightarrow z=-2i\pm\sqrt{3}i=(-2+\sqrt{3})i,\ (-2-\sqrt{3})i$$

$$|(-2+\sqrt{3})i|=|-2+\sqrt{3}|<1$$

and $|(-2-\sqrt{3})i|=|2+\sqrt{3}|>1$

$\therefore$ The only pole [একমাত্র পোল] $z=(-2+\sqrt{3})i$ lies inside the contour C which is a simple pole. [কন্টুর C এর ভিতরে অবস্থিত যাহা একটি সরল পোল]

Residue at $z=(-2+\sqrt{3})i$ is

$$\lim_{z\to(-2+\sqrt{3})i}\{z-(-2+\sqrt{3})i\}\cdot f(z)$$

$$=\lim_{z\to(-2+\sqrt{3})i}\{z-(-2+\sqrt{3})i\}\cdot\frac{1}{z^2+4iz-1}$$

$$=\lim_{z\to(-2+\sqrt{3})i}\{z-(-2+\sqrt{3})i\}\cdot\frac{1}{\{z-(-2+\sqrt{3})i\}\{z-(-2-\sqrt{3})i\}}$$

$$=\lim_{z\to(-2+\sqrt{3})i}\frac{1}{z-(-2-\sqrt{3})i}$$

$$=\frac{1}{(-2+\sqrt{3})i-(-2-\sqrt{3})i}=\frac{1}{(-2+\sqrt{3}+2+\sqrt{3})i}=\frac{1}{2\sqrt{3}i}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}=2\cdot 2\pi i\ \{\text{Residue at } z=(-2+\sqrt{3})i\}$$

$$=4\pi i\cdot\frac{1}{2\sqrt{3}i}=\frac{2\pi}{\sqrt{3}}\ \text{(Ans)}$$

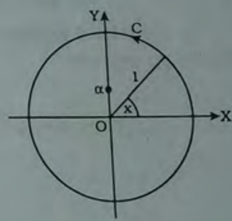
Solution–6. Let us consider the unit circle $|z|=1$ as the contour C. [একক বৃত্ত $|z|=1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then [তখন] $z=e^{ix}\Rightarrow dz=ie^{ix}dx=iz\,dx\Rightarrow dx=\frac{1}{iz}dz$, where [যেখানে] $0\le x\le 2\pi$

$$\sin x=\frac{1}{2i}(e^{ix}+e^{-ix})=\frac{1}{2i}\left(z-\frac{1}{z}\right)=\frac{z^2-1}{2iz}$$

$$\therefore\int_0^{2\pi}\frac{1}{1+a\sin x}\,dx$$

$$=\oint_C\frac{\frac{1}{iz}\,dz}{1+a\cdot\frac{z^2-1}{2iz}}$$



$$=2\oint_C\frac{dz}{2iz+az^2-a}$$

$$=2\oint_C f(z)\,dz,\ \text{say}\cdots\cdots (1)$$

where [যেখানে] $f(z)=\dfrac{1}{az^2+2iz-a}$

The poles of f(z) are obtained from the equation [f(z) এর পোল $az^2+2iz-a=0$ সমীকরণ হতে পাওয়া যাবে]

$$az^2+2iz-a=0$$

$$\Rightarrow z=\frac{-2i\pm\sqrt{4i^2+4a^2}}{2a}=\frac{-2i\pm\sqrt{4a^2-4}}{2a}$$

$$=\frac{-2i\pm 2\sqrt{a^2-1}}{2a}=\frac{-i\pm\sqrt{a^2-1}}{a}=\frac{-i\pm i\sqrt{1-a^2}}{a},\quad\because a<1$$

Let $\alpha=\dfrac{-i+i\sqrt{1-a^2}}{a}$ and $\beta=\dfrac{-i-i\sqrt{1-a^2}}{a}$

Since $0 < a < 1$, so only the simple pole $z = \alpha$ lies inside the contour. [যেহেতু $0 < a < 1$, সুতরাং একমাত্র সরল পোল $z = \alpha$ কনটুরের ভিতরে অবস্থিত]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z \to \alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{az^2 + 2iz - a}$$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{a(z - \alpha)(z - \beta)}$$

$$= \lim_{z \to \alpha} \frac{1}{a(z - \beta)} = \frac{1}{a(\alpha - \beta)}$$

$$= \frac{1}{a \cdot \frac{1}{a}(-i + i\sqrt{1 - a^2} + i + i\sqrt{1 - a^2})}$$

$$= \frac{1}{2i\sqrt{1 - a^2}}$$

$\therefore$ By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{1}{1 + a \sin x} dx = 2 \oint_C f(z)\, dz$$

$$= 2 \cdot 2\pi i. \text{ (Residue at } z = \alpha)$$

$$= 4\pi i \cdot \frac{1}{2i\sqrt{1 - a^2}}$$

$$= \frac{2\pi}{\sqrt{1 - a^2}} \quad \text{(Ans)}$$

Solution–7. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$

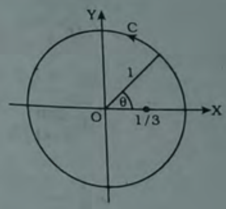
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \oint_C \frac{\frac{1}{iz} dz}{\left(5 - 3 \cdot \frac{z^2 + 1}{2z}\right)^2}$$

$$= \frac{1}{i} \oint_C \frac{\frac{1}{z} dz}{\left(\frac{10z - 3z^2 - 3}{2z}\right)^2}$$

$$= \frac{4}{i} \oint_C \frac{z\, dz}{(-3 + 10z - 3z^2)^2}$$

$$= \frac{4}{i} \oint_C f(z)\, dz, \text{ say } \cdots\cdots (1)$$



where [যেখানে] $f(z) = \frac{z}{(-3 + 10z - 3z^2)^2} = \frac{z}{(3z^2 - 10z + 3)^2}$

The poles of $f(z)$ are obtained by solving the equation [$f(z)$ এর পোল $(3z^2 - 10z + 3)^2 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$(3z^2 - 10z + 3)^2 = 0$$

$$\Rightarrow 3z^2 - 10z + 3 = 0$$

$$\Rightarrow 3z^2 - z - 9z + 3 = 0$$

$$\Rightarrow 3z\left(z - \frac{1}{3}\right) - 9\left(z - \frac{1}{3}\right) = 0$$

$$\Rightarrow \left(z - \frac{1}{3}\right)(3z - 9) = 0$$

$$\Rightarrow 3\left(z - \frac{1}{3}\right)(z - 3) = 0$$

$$\Rightarrow z = \frac{1}{3}, 3$$

$$\left|\frac{1}{3}\right| = \frac{1}{3} < 1 \text{ and } |3| = 3 > 1.$$

$\therefore z = \frac{1}{3}$ lies inside the contour C which is a double pole. [$z = \frac{1}{3}$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি দ্বিপোল]

Residue at $z = \frac{1}{3}$ is [$z = \frac{1}{3}$ এ অবশেষ] $\lim\limits_{z \to 1/3} \frac{d}{dz}\left\{\left(z - \frac{1}{3}\right)^2 f(z)\right\}$

$$= \lim_{z \to 1/3} \frac{d}{dz}\left[\left(z - \frac{1}{3}\right)^2 \cdot \frac{z}{\left\{3\left(z - \frac{1}{3}\right)(z - 3)\right\}^2}\right]$$

$$= \lim_{z \to 1/3} \frac{d}{dz}\left\{\frac{z}{9(z - 3)^2}\right\}$$

$$=\frac{1}{9}\cdot\lim_{z\to 1/3}\frac{(z-3)^2\cdot 1 - z\cdot 2(z-3)}{(z-3)^4}$$

$$=\frac{1}{9}\lim_{z\to 1/3}\frac{z-3-2z}{(z-3)^3}=\frac{1}{9}\cdot\frac{\frac{1}{3}-3-\frac{2}{3}}{\left(\frac{1}{3}-3\right)^3}$$

$$=\frac{1}{9}\cdot\frac{-10/3}{-512/27}=\frac{10}{512}=\frac{5}{256}$$

Now by Cauchy's residue theorem we have from (1) [এখন কচি অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi}\frac{d\theta}{(5-3\cos\theta)^2}=\frac{4}{i}\cdot 2\pi i\cdot\left(\text{Residue at } z=\frac{1}{3}\right)$$

$$=8\pi\cdot\frac{5}{256}=\frac{5\pi}{32}\quad\text{(Ans)}$$

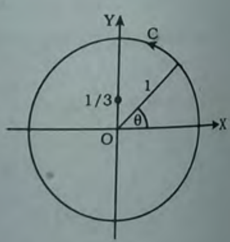
Solution–8. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি।]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$, where [যেখানে] $0 \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)=\frac{1}{2i}\left(z-\frac{1}{z}\right)=\frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2}=\oint_C\frac{\frac{1}{iz}\,dz}{\left(5-3\cdot\frac{z^2-1}{2iz}\right)^2}$$

$$=\frac{1}{i}\oint_C\frac{\frac{1}{z}\,dz}{\left(\frac{10iz-3z^2+3}{i2z}\right)^2}$$



$$=\frac{-4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2}$$

$$=\frac{-4}{i}\oint_C f(z)\,dz,\ \text{say} \ \cdots\cdots (1)$$

where [যেখানে] $f(z)=\dfrac{z}{(3z^2-10iz-3)^2}$

The poles of f(z) are obtained from the equation

$(3z^2 - 10iz - 3)^2 = 0$

[$(3z^2 - 10iz - 3)^2 = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাব]

$\Rightarrow 3z^2 - 10iz - 3 = 0$

$$\Rightarrow z=\frac{10i\pm\sqrt{100i^2+36}}{6}=\frac{10i\pm 8i}{6}=3i,\ \frac{i}{3}$$

$|3i| = 3 > 1$ and $\left|\frac{i}{3}\right| = \frac{1}{3} < 1$

Only the pole $z = \frac{i}{3}$ lies inside the contour C which is a double pole (pole of order 2). [শুধু $z = \frac{i}{3}$ পোলটি কন্টুর C এর ভিতর অবস্থিত যাহা একটি দ্বিপোল]

$\therefore$ Residue at $z = \frac{i}{3}$ is [$z = \frac{i}{3}$ অবশেষ] $\displaystyle\lim_{z\to i/3}\cdot\frac{1}{1!}\frac{d}{dz}\left\{\left(z-\frac{i}{3}\right)^2 f(z)\right\}$

$$=\lim_{z\to i/3}\frac{d}{dz}\left\{\left(z-\frac{i}{3}\right)^2\cdot\frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$=\lim_{z\to i/3}\frac{d}{dz}\left\{\left(z-\frac{i}{3}\right)^2\cdot\frac{z}{9\left(z-\frac{i}{3}\right)^2(z-3i)^2}\right\}$$

$$=\lim_{z\to i/3}\frac{d}{dz}\left\{\frac{z}{9(z-3i)^2}\right\}$$

$$=\frac{1}{9}\lim_{z\to i/3}\frac{(z-3i)^2\cdot 1-z\cdot 2(z-3i)}{(z-3i)^4}$$

$$=\frac{1}{9}\lim_{z\to i/3}\frac{z-3i-2z}{(z-3i)^3}$$

$$=\frac{1}{9}\lim_{z\to i/3}\frac{-z-3i}{(z-3i)^3}$$

$$=\frac{1}{9}\frac{\frac{i}{3}-3i-\frac{2i}{3}}{\left(\frac{i}{3}-3i\right)^3}=\frac{-10i/3}{9\cdot\left(\frac{-8i}{3}\right)^3}=\frac{-10i}{-512i^3}=\frac{-5}{256}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{-4}{i}\cdot 2\pi i. \left(\text{Residue at } z = \frac{i}{3}\right)$$

$$= -8\pi\left(\frac{-5}{256}\right) = \frac{5\pi}{32} \quad \text{(Ans)}$$

Solution–9. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর হিসাবে বিবেচনা করি]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}dz$, where $0 \le \theta \le 2\pi$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

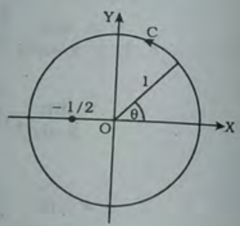
$$\therefore \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \text{Real part of} \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta}\,d\theta$$

$= \text{R. P.} \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta}\,d\theta$, where R. P. means real part of

$$= \text{R. P.} \oint_C \frac{z^2}{5+4\,\frac{z^2+1}{2z}}\cdot\frac{1}{iz}\,dz$$

$$= \text{R. P.} \frac{1}{i}\oint_C \frac{z^2}{5z+2z^2+2}\,dz$$

$$= \text{R. P.} \frac{1}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$



where $f(z) = \frac{z^2}{2z^2+5z+2}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল $2z^2 + 5z + 2 = 0$ সমীকরণ সমাধান করে পাই]

$$2z^2 + 5z + 2 = 0$$

$$\Rightarrow z = \frac{-5 \pm \sqrt{25-16}}{4} = \frac{-5 \pm 3}{4} = \frac{-1}{2}, -2$$

$$|-2| = 2 > 1 \text{ and } \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$$

The only pole $z = -\frac{1}{2}$ lies inside the contour C which is a simple pole. [একমাত্র পোল $z = -\frac{1}{2}$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = -\frac{1}{2}$ is [$z = -\frac{1}{2}$ এ অবশেষ] $\lim\limits_{z\to -1/2}\left(z+\frac{1}{2}\right)\cdot f(z)$

$$= \lim_{z\to -1/2}\left(z+\frac{1}{2}\right)\cdot\frac{z^2}{2z^2+5z+2}$$

$$= \lim_{z\to -1/2}\left(z+\frac{1}{2}\right)\cdot\frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \lim_{z\to -1/2}\frac{z^2}{2(z+2)} = \frac{\left(-\frac{1}{2}\right)^2}{2\left(-\frac{1}{2}+2\right)}$$

$$= \frac{1}{8\times\frac{3}{2}} = \frac{1}{12}$$

Therefore, by Cauchy's residue theorem we have from (1) [অতএব, কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \text{R. P.}\ \frac{1}{i}\cdot 2\pi i\left[\text{Residue at } z = -\frac{1}{2}\right]$$

$$= \text{R. P.}\left(2\pi\times\frac{1}{12}\right) = \frac{\pi}{6} \quad \text{(Ans)}$$

Solution–10. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি ।]

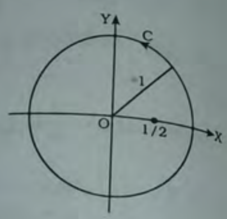
Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}dz$, where [যেখানে] $\theta \le \theta \le 2\pi$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \text{Real part of} \int_0^{2\pi} \frac{\cos 3\theta + i\sin 3\theta}{5-4\cos\theta}\,d\theta$$

$= \text{R. P.} \int_0^{2\pi} \frac{e^{i3\theta}\,d\theta}{5-4\cos\theta}$, where R. P. means real part of [যেখানে R. P. অর্থ বাস্তব অংশ]

$$= \text{R. P.} \oint_C \frac{z^3}{5 - 4 \cdot \frac{z^2+1}{2z}} \cdot \frac{1}{iz}\, dz$$



$$= \text{R. P.} \frac{1}{i} \oint_C \frac{z^3\, dz}{5z - 2z^2 - 2}$$

$$= \text{R. P.} \frac{-1}{i} \oint_C \frac{z^3}{2z^2 - 5z + 2}\, dz$$

$$= \text{R. P.} \frac{-1}{i} \oint_C f(z)\, dz, \text{ say } \cdots\cdots (1)$$

where [যেখানে] $f(z) = \dfrac{z^3}{2z^2 - 5z + 2}$

The poles of f(z) are obtained by solving the equation

$2z^2 - 5z + 2 = 0$

[$2z^2 - 5z + 2 = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$$\Rightarrow z = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm 3}{4} = 2, \frac{1}{2}$$

Only the pole $z = \frac{1}{2}$ lie inside the contour C which is a simple pole. [শুধুমাত্র $z = \frac{1}{2}$ পোলটি কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = \frac{1}{2}$ is [$z = \frac{1}{2}$ এ অবশেষ] $\displaystyle\lim_{z \to 1/2} \left(z - \frac{1}{2}\right) \cdot f(z)$

$$= \lim_{z \to 1/2} \left(z - \frac{1}{2}\right) \cdot \frac{z^3}{2z^2 - 5z + 2}$$

$$= \lim_{z \to 1/2} \left(z - \frac{1}{2}\right) \cdot \frac{z^3}{2\left(z - \frac{1}{2}\right)(z - 2)}$$

$$= \lim_{z \to 1/2} \frac{z^3}{2(z-2)} = \frac{(1/2)^3}{2\left(\frac{1}{2} - 2\right)}$$

$$= \frac{1}{8 \times 2 \times \frac{-3}{2}} = \frac{-1}{24}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \text{R. P.} \frac{-1}{i} \cdot 2\pi i \left[\text{Residue at } z = \frac{1}{2}\right]$$

$$= \text{R. P.} (-2\pi)\left(\frac{-1}{24}\right) = \frac{\pi}{12} \quad \text{(Ans)}$$

Solution–11. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর হিসাবে বিবেচনা করি।]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$.

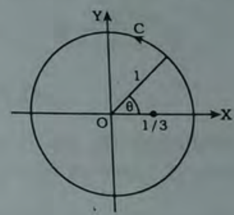
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{\sin 2\theta}{5 - 3\cos\theta}\, d\theta$$

$$= \text{Imaginary part of } \int_0^{2\pi} \frac{\cos 2\theta + i \sin 2\theta}{5 - 3\cos\theta}\, d\theta$$

$$= \text{I. P.} \int_0^{2\pi} \frac{e^{i2\theta}}{5 - 3\cos\theta}\, d\theta, \text{ where I.P. stands for imaginary part of}$$

$$= \text{I. P.} \oint_C \frac{z^2}{5 - 3 \cdot \frac{z^2+1}{2z}} \cdot \frac{1}{iz}\, dz$$



$$= \text{I. P.} \frac{2}{i} \oint_C \frac{z^2}{10z - 3z^2 - 3}\, dz$$

$$= \text{I. P.} \frac{-2}{i} \oint_C \frac{z^2}{3z^2 - 10z + 3}\, dz$$

$$= \text{I. P.} \frac{-2}{i} \oint_C f(z)\, dz \cdots\cdots (1)$$

where $f(z) = \dfrac{z^2}{3z^2 - 10z + 3}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল পাওয়া যাবে $3z^2 - 10z + 3 = 0$ সমীকরণ সমাধান করে।]

$3z^2 - 10z + 3 = 0$

$\Rightarrow 3z^2 - 9z - z + 3 = 0$

$\Rightarrow 3z(z - 3) - 1(z - 3) = 0$

$\Rightarrow (3z - 1)(z - 3) = 0$

$\Rightarrow z = \frac{1}{3}, 3$

The only pole $z = \frac{1}{3}$ lies inside the contour C which is a simple pole. [একমাত্র পোল $z = \frac{1}{3}$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = \frac{1}{3}$ is [$z = \frac{1}{3}$ এ অবশেষ] $\lim_{z \to 1/3} \left(z - \frac{1}{3}\right) \cdot f(z)$

$$= \lim_{z \to 1/3} \left(z - \frac{1}{3}\right) \cdot \frac{z^2}{3z^2 - 10z + 3}$$

$$= \lim_{z \to 1/3} \left(z - \frac{1}{3}\right) \cdot \frac{z^2}{3\left(z - \frac{1}{3}\right)(z - 3)}$$

$$= \lim_{z \to 1/3} \frac{z^2}{3(z-3)} = \frac{(1/3)^2}{3\left(\frac{1}{3} - 3\right)} = \frac{-1}{72}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{\sin 2\theta}{5 - 3\cos\theta} d\theta = \text{I. P.} \frac{-2}{i} \cdot 2\pi i \left[\text{Residue at } z = \frac{1}{3}\right]$$

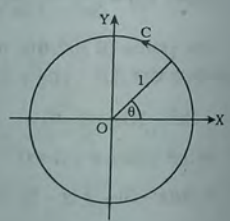
$$= \text{I. P.} \left(-4\pi \times \frac{-1}{72}\right) = 0 \text{ (Ans)}$$

Solution–12. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর হিসাবে বিবেচনা করি।]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$, where $0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \oint_C \frac{\frac{dz}{iz}}{a + b \cdot \frac{z^2+1}{2z}}$$



$$= \frac{2}{i} \oint_C \frac{1}{2az + bz^2 + b} dz$$

$$= \frac{2}{i} \oint_C f(z)\, dz, \text{ say} \cdots\cdots (1)$$

where $f(z) = \frac{1}{bz^2 + 2az + b}$

The poles of $f(z)$ are obtained by solving the equation

$bz^2 + 2az + b = 0 \cdots\cdots (2)$

[$bz^2 + 2az + b = 0$ সমীকরণ সমাধান করে $f(z)$ এর পোল সমূহ পাওয়া যাবে।

$$\Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-2a \pm 2\sqrt{a^2 - b^2}}{2b}$$

$$\Rightarrow z = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $z = \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$ and $z = \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$

Here $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} = \frac{-a + \sqrt{a^2 - b^2}}{b} \times \frac{-a - \sqrt{a^2 - b^2}}{-a - \sqrt{a^2 - b^2}}$

$$= \frac{(-a)^2 - (\sqrt{a^2 - b^2})^2}{-b(a + \sqrt{a^2 - b^2})} = \frac{a^2 - a^2 + b^2}{-b(a + \sqrt{a^2 - b^2})}$$

$$= \frac{b^2}{-b(a + \sqrt{a^2 - b^2})}$$

$$= \frac{-b}{a + \sqrt{a^2 - b^2}}$$

Given that [দেওয়া আছে] $a > |b| \Rightarrow a^2 > b^2$

$$\Rightarrow (a^2 - b^2) > 0$$

$$\Rightarrow \sqrt{a^2 - b^2} > 0$$

$\Rightarrow a + \sqrt{a^2 - b^2} > a > |b|$; adding a both sides

$$\Rightarrow \frac{1}{a + \sqrt{a^2 - b^2}} < \frac{1}{|b|}$$

$$\Rightarrow \frac{|b|}{a + \sqrt{a^2 - b^2}} < 1$$

$$\therefore |\alpha| = \left|\frac{-b}{a + \sqrt{a^2 - b^2}}\right| = \frac{|b|}{a + \sqrt{a^2 - b^2}} < 1$$

Again, $\alpha\beta = \frac{-a + \sqrt{a^2 - b^2}}{b} \cdot \frac{-a - \sqrt{a^2 - b^2}}{b}$

$$= \frac{(-a)^2 - (\sqrt{a^2 - b^2})^2}{b^2} = \frac{a^2 - a^2 + b^2}{b^2} = \frac{b^2}{b^2} = 1$$

$\left[\text{Or, from (2) product of the roots } \alpha\beta = \frac{b}{b} = 1\right]$

$$\Rightarrow \beta = \frac{1}{\alpha} \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1, \text{ since } |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

Thus only the pole $z = \alpha$ lies inside the contour C which is a simple pole. [অতএব, একমাত্র পোল $z = \alpha$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha} (z-\alpha)\cdot f(z)$

$$= \lim_{z\to\alpha} (z-\alpha)\cdot\frac{1}{bz^2+2az+b}$$

$$= \lim_{z\to\alpha} (z-\alpha)\cdot\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{b(z-\beta)} = \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{b}\cdot\frac{1}{\dfrac{-a+\sqrt{a^2-b^2}}{b} - \dfrac{-a-\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1}{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}} = \frac{1}{2\sqrt{a^2-b^2}}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\cdot 2\pi i \quad [\text{Residue at } z=\alpha]$$

$$= 4\pi\cdot\frac{1}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad \text{(Ans)}$$

Solution–13. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর হিসাবে বিবেচনা করি ।]

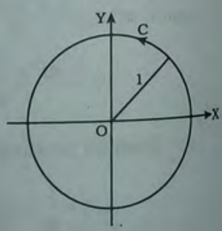
Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$, where $0 \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{a+b\cdot\frac{z^2-1}{2iz}}$$

$$= 2\oint_C \frac{dz}{2iaz+bz^2-b}$$

$$= 2\oint_C f(z)\,dz, \text{ say} \;\cdots\cdots (1)$$



where $f(z) = \dfrac{1}{bz^2+2iaz-b}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল $bz^2 + 2iaz - b = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$bz^2 + 2iaz - b = 0 \;\cdots\cdots (2)$$

$$\Rightarrow z = \frac{-2ia \pm \sqrt{4a^2 i^2 + 4b^2}}{2b} = \frac{-2ia \pm 2\sqrt{b^2-a^2}}{2b}$$

$$= \frac{-ia \pm i\sqrt{a^2-b^2}}{b}$$

Let $z = \alpha = \dfrac{i(-a+\sqrt{a^2-b^2})}{b}$ and $z = \beta = \dfrac{i(-a-\sqrt{a^2-b^2})}{b}$

Given that [দেওয়া আছে] $a > |b| \Rightarrow a^2 > b^2$

$$\Rightarrow a^2 - b^2 > 0$$

$$\Rightarrow \sqrt{a^2-b^2} > 0$$

$$\Rightarrow a + \sqrt{a^2-b^2} > a > |b|$$

$$\Rightarrow \frac{1}{a+\sqrt{a^2-b^2}} < \frac{1}{|b|}$$

$$\Rightarrow \frac{|b|}{a+\sqrt{a^2-b^2}} < 1$$

$$\therefore |\alpha| = \left|\frac{i(-a+\sqrt{a^2-b^2})}{b}\right| = \left|\frac{-a+\sqrt{a^2-b^2}}{b}\right| \quad \because |i| = 1$$

$$= \left|\frac{-a+\sqrt{a^2-b^2}}{b} \times \frac{-a-\sqrt{a^2-b^2}}{-(a+\sqrt{a^2-b^2})}\right|$$

$$= \left|\frac{a^2-(a^2-b^2)}{b(a+\sqrt{a^2-b^2})}\right| = \left|\frac{a^2-a^2+b^2}{b(a+\sqrt{a^2-b^2})}\right|$$

$$= \left|\frac{b^2}{b(a+\sqrt{a^2-b^2})}\right| = \frac{|b|}{a+\sqrt{a^2-b^2}} < 1$$

From equation (2), Product of the roots [সমীকরণ (2) হতে পাই মূলগয়ের গুণফল] $\alpha\beta = \dfrac{-b}{b} = -1$

$$\Rightarrow |\alpha\beta| = |-1| \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1 \quad \because |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

Thus the only pole $z = \alpha$ lies inside the contour C which is a simple pole. [অতএব শুধুমাত্র $z = \alpha$ পোল কন্টুর C এর ভিতর অবস্থিত, যাহা একটি সরল পোল]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z \to \alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{bz^2 + 2iaz - b}$$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{b(z - \alpha)(z - \beta)}$$

$$= \frac{1}{b} \cdot \lim_{z \to \alpha} \frac{1}{z - \beta} = \frac{1}{b} \cdot \frac{1}{\alpha - \beta}$$

$$= \frac{1}{b} \cdot \frac{1}{\frac{i}{b}\left(-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}\right)}$$

$$= \frac{1}{2i\sqrt{a^2 - b^2}}$$

Therefore, by Cauchy's residue theorem we have from (1) [অতএব, কচির অবশেষ উপপাদ্যে দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = 2 \cdot 2\pi i \cdot (\text{Residue at } z = \alpha)$$

$$= 4\pi i \cdot \frac{1}{2i\sqrt{a^2 - b^2}}$$

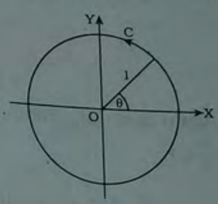
$$= \frac{2\pi}{\sqrt{a^2 - b^2}} \quad \text{(Ans)}$$

Solution–14. (i) & (ii) : Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর হিসাবে বিবেচনা করি।]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} d\theta$$



$$= \oint_C \frac{\frac{1}{iz} dz}{\left(a + b \cdot \frac{z^2 + 1}{2z}\right)^2}$$

$$= \frac{4}{i} \oint_C \frac{z\, dz}{(2az + bz^2 + b)^2}$$

$$= \frac{4}{i} \oint_C f(z)\, dz, \text{ say } \ldots\ldots (1)$$

where $f(z) = \dfrac{z}{(bz^2 + 2az + b)^2}$

The poles of $f(z)$ are obtained by solving the equation [$f(z)$ এর পোল সমূহ নিম্নের সমীকরণ সমাধান করে পাওয়া যাবে]

$$(bz^2 + 2az + b)^2 = 0$$

$$\Rightarrow bz^2 + 2az + b = 0 \ldots\ldots (2)$$

$$\Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$z = \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \text{ and } z = \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

$$|z| = |\alpha| = \left|\frac{-a + \sqrt{a^2 - b^2}}{b}\right|$$

$$= \left|\frac{-a + \sqrt{a^2 - b^2}}{b} \times \frac{-a - \sqrt{a^2 - b^2}}{-a - \sqrt{a^2 - b^2}}\right|$$

$$= \left|\frac{a^2 - (a^2 - b^2)}{-b(a + \sqrt{a^2 - b^2})}\right| = \left|\frac{b}{a + \sqrt{a^2 - b^2}}\right| \ldots\ldots (3)$$

Given that [দেওয়া আছে] $a > b > 0 \Rightarrow a^2 > b^2$

$$\Rightarrow a^2 - b^2 > 0$$

$$\Rightarrow \sqrt{a^2 - b^2} > 0$$

$$\Rightarrow a + \sqrt{a^2 - b^2} > a > b$$

$$\Rightarrow \frac{1}{a + \sqrt{a^2 - b^2}} < \frac{1}{b}$$

$$\Rightarrow \frac{b}{a + \sqrt{a^2 - b^2}} < 1$$

$$\Rightarrow \left|\frac{b}{a+\sqrt{a^2-b^2}}\right| < 1, \text{ since } a > b > 0$$

$$\Rightarrow |\alpha| < 1, \text{ by (3)}$$

Again from (2), product of the roots [আবার (2) হতে, মূলদ্বয়ের গুণফল]

$$\alpha\beta = \frac{b}{b} = 1$$

$$\Rightarrow |\alpha\beta| = 1 \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1, \text{ since } |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

Therefore, the only pole $z = \alpha$ lies inside the contour C which is a double pole (pole of order 2). [অতএব একমাত্র পোল $z = \alpha$ কন্টুর C এর ভিতর অবস্থিত, যাহা একটি দ্বিপোল]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha} \frac{1}{1!}\frac{d}{dz}\{(z-\alpha)^2 \cdot f(z)\}$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left\{(z-\alpha)^2 \cdot \frac{z}{(bz^2+2az+b)^2}\right\}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left\{(z-\alpha)^2 \cdot \frac{z}{b^2(z-\alpha)^2\,(z-\beta)^2}\right\}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left\{\frac{z}{b^2(z-\beta)^2}\right\}$$

$$= \frac{1}{b^2}\lim_{z\to\alpha} \frac{(z-\beta)^2 \cdot 1 - z \cdot 2(z-\beta)}{(z-\beta)^4}$$

$$= \frac{1}{b^2}\lim_{z\to\alpha} \frac{z-\beta-2z}{(z-\beta)^3}$$

$$= \frac{1}{b^2}\lim_{z\to\alpha} \frac{-z-\beta}{(z-\beta)^3} = \frac{1}{b^2}\cdot\frac{-\alpha-\beta}{(\alpha-\beta)^3}$$

$$= \frac{-1}{b^2}\cdot\frac{\frac{1}{b}\left(-a+\sqrt{a^2-b^2}-a-\sqrt{a^2-b^2}\right)}{\left\{\frac{1}{b}\left(-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}\right)\right\}^3}$$

$$= -\frac{-2a}{(2\sqrt{a^2-b^2})^3} = \frac{9}{4(\sqrt{a^2-b^2})^3} = \frac{a}{4(a^2-b^2)^{3/2}}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{4}{i}\cdot 2\pi i \cdot (\text{Residue at } z=\alpha)$$

$$= 8\pi \cdot \frac{a}{4(a^2-b^2)^{3/2}} = \frac{2\pi a}{(a^2-b^2)^{3/2}} \quad \text{(Ans)}$$

(ii) $$\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$$

$$= \frac{1}{2}\cdot\frac{2\pi a}{(a^2-b^2)^{3/2}} = \frac{\pi a}{(a^2-b^2)^{3/2}} \quad \text{(Ans)}$$

Solution-15. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি।]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}dz$, where $0 \le \theta \le 2\pi$.

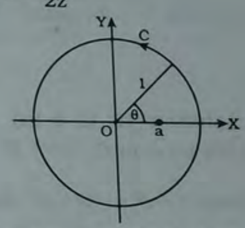
$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \oint_C \frac{\frac{1}{iz}dz}{1+a^2-2a\cdot\frac{z^2+1}{2z}}$$

$$= \frac{1}{i}\oint_C \frac{dz}{(1+a^2)z - az^2 - a}$$

$$= \frac{-1}{i}\oint_C \frac{dz}{az^2-(1+a^2)z+a}$$

$$= \frac{1}{i}\oint_C f(z)\,dz, \text{ say } \quad \cdots\cdots (1)$$



where $f(z) = \frac{1}{az^2-(1+a^2)z+a}$

The poles of f(z) will be obtained from the equation [f(z) এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যায়]

$$az^2-(1+a^2)z+a = 0$$

$$\Rightarrow az^2 - z - a^2z + a = 0$$

$$\Rightarrow az\left(z-\frac{1}{a}\right) - a^2\left(z-\frac{1}{a}\right) = 0$$

$$\Rightarrow a\left(z-\frac{1}{a}\right)(z-a) = 0$$

$$\Rightarrow z = \frac{1}{a}, a$$

Since [যেহেতু] $0 < a < 1$. so [সুতরাং] $\left|\frac{1}{a}\right| > 1$ and $|a| < 1$.

$\therefore$ The pole $z = a$ lies inside the contour C which is a simple pole. [$z = a$ পোলটি কন্টুরC এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = a$ is [$z = a$ এ অবশেষ] $\lim\limits_{z \to a} (z - a) \cdot f(z)$

$$= \lim_{z \to a} (z - a) \cdot \frac{1}{az^2 - (1 + a^2) z + a}$$

$$= \lim_{z \to a} (z - a) \cdot \frac{1}{a (z - a) \left(z - \frac{1}{a}\right)}$$

$$= \lim_{z \to a} \frac{1}{a\left(z - \frac{1}{a}\right)} = \frac{1}{a\left(a - \frac{1}{a}\right)} = \frac{1}{a^2 - 1}$$

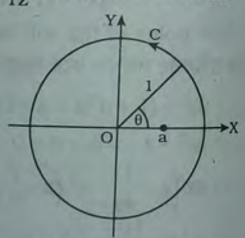
By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্যে দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{-1}{i} \cdot 2\pi i \cdot (\text{Residue at } z = a)$$

$$= -2\pi \cdot \frac{1}{a^2 - 1} = \frac{2\pi}{1 - a^2} \quad \text{(Ans)}$$

Solution–16. (i) & (ii) : Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি।]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$



$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta$$

$$= \text{Real part of} \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{1 - 2a\cos\theta + a^2} d\theta$$

$= \text{R. P.} \int_0^{2\pi} \frac{e^{i2\theta}}{1 - 2a\cos\theta + a^2} d\theta$, where R. P. means real part of

$$= \text{R. P.} \oint_C \frac{z^2}{1 - 2a \cdot \frac{z^2 + 1}{2z} + a^2} \cdot \frac{1}{iz} dz$$

$$= \text{R. P.} \frac{1}{i} \oint_C \frac{z^2}{z - az^2 - a + a^2 z} dz$$

$$= \text{R. P.} \frac{-1}{i} \oint_C f(z)\, dz, \text{ say } \cdots\cdots (1)$$

where $f(z) = \frac{z^2}{az^2 - z - a^2 z + a}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল নিম্নের সমীকরণ হতে পাওয়া যায়]

$$az^2 - z - a^2 z + a = 0$$

$$\Rightarrow z(az - 1) - a(az - 1) = 0$$

$$\Rightarrow (z - a)(az - 1) = 0$$

$$\Rightarrow z = a, \frac{1}{a}$$

Given that [দেওয়া আছে] $a^2 < 1$, so $\frac{1}{a^2} > 1$

Thus the only pole $z = a$ lies inside the contour C which is a simple pole. [একমাত্র পোল $z = a$ কন্টুর C এর ভিতর অবস্থিত, যাহা সরল পোল]

Residue at $z = a$ is [$z = a$ এ অবশেষ] $\lim\limits_{z \to a} (z - a) \cdot f(z)$

$$= \lim_{z \to a} (z - a) \cdot \frac{z^2}{az^2 - z - a^2 z + a}$$

$$= \lim_{z \to a} (z - a) \cdot \frac{z^2}{a(z - a)\left(z - \frac{1}{a}\right)}$$

$$= \lim_{z \to a} \frac{z^2}{a\left(z - \frac{1}{a}\right)} = \frac{a^2}{a\left(a - \frac{1}{a}\right)} = \frac{a^2}{a^2 - 1}$$

Therefore, by Cauchy's residue theorem we have from (1) [অতএব, কচির অবশেষ উপপাদ্যে দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = \text{R. P.} \frac{-1}{i} \cdot 2\pi i \, [\text{Residue at } z = a]$$

$$= \text{R. P.} \left[-2\pi \times \frac{a^2}{a^2 - 1}\right] = \frac{\pi a^2}{1 - a^2} \quad \text{(Ans)}$$

(ii) $\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta$

$$= \frac{\pi a^2}{2(1 - a^2)} \quad \text{(Ans)}$$

Solution–17. Let $I = \int_0^{\pi} \frac{a\, d\theta}{a^2 + \cos^2\theta} = \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 2\cos^2\theta}$

$$= \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 1 + \cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a\, d\phi}{2a^2 + 1 + \cos\phi}$$

Let $2\theta = \phi \Rightarrow 2\, d\theta = d\phi$

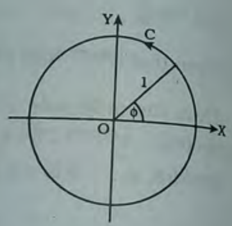
θ	0	π
ϕ	0	2π

Now let us consider the unit circle $|z| = 1$ as the contour C [এখন একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then $z = e^{i\phi} \Rightarrow dz = ie^{i\phi}\, d\phi = iz\, d\phi \Rightarrow d\phi = \frac{1}{iz}\, dz$, where $0 \leq \phi \leq 2\pi$

$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}) = \left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$

$$\therefore I = \int_0^{2\pi} \frac{a\, d\phi}{1 + 2a^2 + \cos\phi}$$



$$= \oint_C \frac{a}{1 + 2a^2 + \frac{z^2 + 1}{2z}} \cdot \frac{1}{iz}\, dz$$

$$= \frac{2a}{i} \oint_C \frac{1}{(1 + 2a^2)\, 2z + z^2 + 1}\, dz$$

$$= \frac{2a}{i} \oint_C f(z)\, dz \;\ldots\ldots (1)$$

where $f(z) = \frac{1}{z^2 + (1 + 2a^2)\, 2z + 1}$

The poles of $f(z)$ are obtained by solving the equation [$f(z)$ এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যায়]

$$z^2 + (2 + 4a^2)\, z + 1 = 0 \;\ldots\ldots (2)$$

$$\Rightarrow z = \frac{-(2 + 4a^2) \pm \sqrt{(2 + 4a^2)^2 - 4}}{2}$$

$$= \frac{-(2 + 4a^2) \pm \sqrt{4 + 16a^2 + 16a^4 - 4}}{2}$$

$$= \frac{-2(1 + 2a^2) \pm 4a\sqrt{1 + a^2}}{2}$$

$$= -(1 + 2a^2) \pm 2a\sqrt{1 + a^2}.$$

Let $z = \alpha = -(1 + 2a^2) + 2a\sqrt{1 + a^2}$ and $z = \beta = -(1 + 2a^2) - 2a\sqrt{1 + a^2}$

$|\alpha| = |-(1 + 2a^2) + 2a\sqrt{1 + a^2}| = |1 + 2a^2 - 2a\sqrt{1 + a^2}|$

$= |a^2 + a^2 + 1 - 2a\sqrt{1 + a^2}|$

$= |(\sqrt{a^2 + 1} - a)^2| < 1$, since $a > 0$

Again from (2), product of the roots is [আবার (2) হতে, মূলদ্বয়ের গুণফল] $\alpha\beta = 1$

$\Rightarrow |\alpha\beta| = 1 \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1$, since $|\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$

Thus the only pole $z = \alpha$ lies inside the contour C which is a simple pole. [অতএব একমাত্র পোল $z = \alpha$ কন্টুর C এর ভিতর অবস্থিত, যাহা সরল পোল]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z \to \alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{z^2 + (1 + 2a^2)\, 2z + 1}$$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{(z - \alpha)(z - \beta)}$$

$$= \lim_{z \to \alpha} \frac{1}{z - \beta}$$

$$= \frac{1}{\alpha - \beta} = \frac{1}{-(1 + 2a^2) + 2a\sqrt{1 + a^2} + (1 + 2a^2) + 2a\sqrt{1 + a^2}}$$

$$= \frac{1}{4a\sqrt{1 + a^2}}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$I = \frac{2a}{i} \cdot 2\pi i$ [Residue at $z = \alpha$]

$$\Rightarrow \int_0^{\pi} \frac{a\, d\theta}{a^2 + \cos^2\theta} = 4\pi a \cdot \frac{1}{4a\sqrt{1 + a^2}} = \frac{\pi}{\sqrt{1 + a^2}} \quad \text{(Ans)}$$

Solution–18. Let $I = \int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 2\sin^2\theta}$

$$= \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 1 - \cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a\, d\phi}{2a^2 + 1 - \cos\phi}$$

Putting $2\theta = \phi \Rightarrow 2\, d\theta = d\phi$

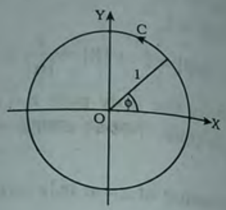
θ	0	π
ϕ	0	2π

Now let us consider the unit circle $|z| = 1$ as the contour C.
[এখন একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then $z = e^{i\phi} \Rightarrow dz = ie^{i\phi}\, d\phi = iz\, d\phi \Rightarrow d\phi = \frac{1}{iz}\, dz$, where $0 \le \phi \le 2\pi$.

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore I = \int_0^{2\pi} \frac{a\, d\phi}{1 + 2a^2 - \cos\phi}$$



$$= \oint_C \frac{a}{1 + 2a^2 - \frac{z^2+1}{2z}} \cdot \frac{1}{iz}\, dz$$

$$= \frac{2a}{i} \oint_C \frac{1}{(1 + 2a^2)\, 2z - z^2 - 1}$$

$$= \frac{-2a}{i} \oint_C \frac{dz}{z^2 - (2 + 4a^2)\, z + 1}$$

$$= \frac{-2a}{i} \oint_C f(z)\, dz \cdots\cdots (1)$$

where $f(z) = \frac{1}{z^2 - (2 + 4a^2)\, z + 1}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যাবে]

$$z^2 - (2 + 4a^2)z + 1 = 0 \cdots\cdots (2)$$

$$\Rightarrow z = \frac{(2 + 4a^2) \pm \sqrt{(2 + 4a^2)^2 - 4}}{2}$$

$$= \frac{(2 + 4a^2) \pm \sqrt{4 + 16a^2 + 16a^4 - 4}}{2}$$

$$= \frac{2(1 + 2a^2) \pm \sqrt{16a^2(1 + a^2)}}{2}$$

$$= \frac{2(1 + 2a^2) \pm 2a\sqrt{1 + a^2}}{2}$$

$$= (1 + 2a^2) \pm 2a\sqrt{1 + a^2}$$

Let $z = \alpha = (1 + 2a^2) + 2a\sqrt{1 + a^2}$

and $z = \beta = (1 + 2a^2) - 2a\sqrt{1 + a^2}$

$$|z| = |\beta| = |1 + 2a^2 - 2a\sqrt{1 + a^2}|$$

$$= |a^2 + 1 - 2a\sqrt{1 + a^2} + a^2|$$

$$= |(\sqrt{1 + a^2} - a)^2| < 1, \text{ since } a > 0$$

From (2), product of the roots [(2) হতে, মূলদ্বয়ের গুণফল], $\alpha\beta = 1$

$$|\alpha\beta| = 1 \Rightarrow |\alpha|\,|\beta| = 1$$

$$\Rightarrow |\alpha| = \frac{1}{|\beta|} > 1, \text{ since } |\beta| < 1 \Rightarrow \frac{1}{|\beta|} > 1$$

Thus, the only pole $z = \beta$ lies inside the contour C which is a simple pole. [অতএব, একমাত্র পোল $z = \beta$ কন্টুর C এর ভিতর আছে যাহা একটি সরল পোল]

Residue at $z = \beta$ is $\lim_{z \to \beta} (z - \beta) \cdot f(z)$

$$= \lim_{z\to\beta} (z - \beta) \cdot \frac{1}{z^2 - (2 + 4a^2)\, z + 1}$$

$$= \lim_{z\to\beta} (z - \beta) \cdot \frac{1}{(z - \alpha)\,(z - \beta)}$$

$$= \lim_{z\to\beta} \frac{1}{z - \alpha} = \frac{1}{\beta - \alpha}$$

$$= \frac{1}{(1 + 2a^2) - 2a\sqrt{1 + a^2} - (1 + 2a^2) - 2a\sqrt{1 + a^2}}$$

$$= \frac{-1}{4a\sqrt{1 + a^2}}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$I = -\frac{2a}{i} \cdot 2\pi i \quad \text{[Residue at } z = \beta]$$

$$\Rightarrow \int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = -4\pi a \cdot \frac{-1}{4a\sqrt{1 + a^2}} = \frac{\pi}{\sqrt{1 + a^2}} \quad \text{(Ans)}$$

Solution–18(i). Let $I = \int_0^{\pi} \frac{dx}{1 + \sin^2 x}$

$$\Rightarrow I = \int_0^{\pi} \frac{2\, dx}{2 + 2\sin^2 x}$$

$$= \int_0^{\pi} \frac{2\, dx}{2 + 1 - \cos 2x}$$

$$= \int_0^{2\pi} \frac{d\theta}{3 - \cos\theta}$$

Putting $2x = \theta \Rightarrow 2\, dx = d\theta$

x	0	π
θ	0	2π

Now let us consider the unit circle $|z| = 1$ as the contour C.
[এখন একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$

$$\Rightarrow d\theta = \frac{1}{iz} dz, \text{ where } 0 \le \theta \le 2\pi.$$

$$\therefore \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

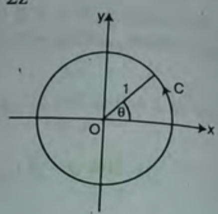
$$\therefore I = \int_0^{2\pi} \frac{d\theta}{3 - \cos\theta}$$

$$= \oint_C \frac{1}{3 - \frac{z^2+1}{2z}} \cdot \frac{1}{iz} dz$$

$$= \frac{1}{i} \oint_C \frac{2z}{z(6z - z^2 - 1)} dz$$

$$= \frac{-2}{i} \oint_C \frac{1}{z^2 - 6z + 1} dz$$

$$\Rightarrow I = \frac{-2}{i} \oint_C f(z)\, dz \;\cdots\cdots (1)$$



where [যেখানে] $f(z) = \dfrac{1}{z^2 - 6z + 1}$

The poles of f(z) are obtained by solving the equation $z^2 - 6z + 1 = 0$. [$z^2 - 6z + 1 = 0$ সমীকরণ সমাধান করে f(z) এর পোলসমূহ পাওয়া যাবে।]

$$\Rightarrow z = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

$$|z| = |3 + 2\sqrt{2}| > 1 \text{ and } |z| = |3 - 2\sqrt{2}| < 1$$

The only pole $z = 3 - 2\sqrt{2}$ lies inside the contour [একমাত্র $z = 3 - 2\sqrt{2}$ পোলটি কন্টুরের ভিতরে অবস্থিত।]

Residue at $z = 3 - 2\sqrt{2}$ is

$$\lim_{z \to (3-2\sqrt{2})} \{z - (3 - 2\sqrt{2})\} \cdot f(z)$$

$$= \lim_{z \to (3-2\sqrt{2})} \{z - (3 - 2\sqrt{2})\} \cdot \frac{1}{z^2 - 6z + 1}$$

$$= \lim_{z \to (3-2\sqrt{2})} \{z - (3 - 2\sqrt{2})\} \cdot \frac{1}{\{z - (3 - 2\sqrt{2})\}\{z - (3 + 2\sqrt{2})\}}$$

$$= \lim_{z \to (3-2\sqrt{2})} \frac{1}{z - 3 - 2\sqrt{2}}$$

$$= \frac{1}{3 - 2\sqrt{2} - 3 - 2\sqrt{2}} = \frac{1}{-4\sqrt{2}}$$

$\therefore$ By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$I = \frac{-2}{i} \cdot 2\pi i \text{ [Residue at } z = 3 - 2\sqrt{2}]$$

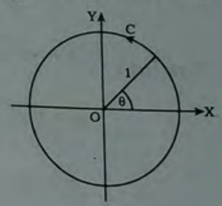
$$\Rightarrow \int_0^{\pi} \frac{dx}{1 + \sin^2 x} = -4\pi \cdot \frac{1}{-4\sqrt{2}} = \frac{\pi}{\sqrt{2}}. \quad \textbf{(Ans)}$$

Solution–19. (i) & (ii) : Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$



$$\therefore \int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \int_0^{2\pi} \frac{2\sin^2\theta\, d\theta}{2a + 2b\cos\theta}$$

$$= \int_0^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b\cos\theta} d\theta$$

$$= \text{Real part of} \int_0^{2\pi} \frac{1 - (\cos 2\theta + i\sin 2\theta)}{2a + 2b\cos\theta} d\theta$$

$$= \text{R. P.} \int_0^{2\pi} \frac{1 - e^{i2\theta}}{2a + 2b\cos\theta} d\theta, \text{ R. P. means real part of}$$

$$= \text{R. P.} \oint_C \frac{1 - z^2}{2a + 2b \cdot \frac{z^2+1}{2z}} \cdot \frac{1}{iz} dz$$

$$= \text{R. P.} \frac{1}{i} \oint_C \frac{1 - z^2}{2az + bz^2 + b} dz$$

$$= \text{R. P.} \frac{1}{i} \oint_C f(z)\, dz \;\cdots\cdots (1)$$

where $f(z) = \dfrac{1 - z^2}{bz^2 + 2az + b}$

The poles of f(z) are obtained by solving the equation [f(z) এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যাবে]

$$bz^2 + 2az + b = 0 \;\cdots\cdots (2)$$

$$\Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$= \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Let $z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $z = \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

Given that [দেওয়া আছে] $a > b > 0 \Rightarrow a^2 > b^2$

$$\Rightarrow a^2 - b^2 > 0$$
$$\Rightarrow \sqrt{a^2-b^2} > 0$$
$$\Rightarrow a + \sqrt{a^2-b^2} > a > b$$
$$\Rightarrow \frac{1}{a+\sqrt{a^2-b^2}} < \frac{1}{b}$$
$$\Rightarrow \frac{b}{a+\sqrt{a^2-b^2}} < 1$$

Now $$|\alpha| = \left|\frac{-a+\sqrt{a^2-b^2}}{b}\right|$$
$$= \left|\frac{-a+\sqrt{a^2-b^2}}{b} \times \frac{-a-\sqrt{a^2-b^2}}{-a-\sqrt{a^2-b^2}}\right|$$
$$= \left|\frac{a^2-(a^2-b^2)}{-b(a+\sqrt{a^2-b^2})}\right|$$
$$= \left|\frac{b}{a+\sqrt{a^2-b^2}}\right| = \frac{b}{a+\sqrt{a^2-b^2}} < 1, \text{ since } a > b > 0$$
$$\Rightarrow |\alpha| < 1$$

From (2), product of the roots is [(2) হতে, মূলদ্বয়ের গুণফল] $\alpha\beta = \frac{b}{b} = 1$

$$|\alpha\beta| = 1 \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1, \text{ since } |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

∴ The only pole $z = \alpha$ lies inside the contour C which is a simple pole. [একমাত্র পোল $z = \alpha$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = \alpha$ is $\lim_{z\to\alpha} (z-\alpha)\cdot f(z)$

$$= \lim_{z\to\alpha} (z-\alpha)\cdot\frac{1-z^2}{bz^2+2az+b}$$
$$= \lim_{z\to\alpha} (z-\alpha)\cdot\frac{1-z^2}{b(z-\alpha)(z-\beta)}$$
$$= \lim_{z\to\alpha} \frac{1-z^2}{b(z-\beta)}$$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)}$$
$$= \frac{\alpha\beta-\alpha^2}{b(\alpha-\beta)} = \frac{-\alpha(\alpha-\beta)}{b(\alpha-\beta)} = \frac{-\alpha}{b}, \quad \because \alpha\beta = 1$$
$$= \frac{a-\sqrt{a^2-b^2}}{b^2}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \text{R. P. } \frac{1}{i}\cdot 2\pi i \text{ (Residue at } z=\alpha)$$
$$= \text{R. P. } 2\pi\cdot\frac{a-\sqrt{a^2-b^2}}{b^2}$$
$$= \frac{2\pi}{b^2}(a-\sqrt{a^2-b^2}) \quad \text{(Ans)}$$

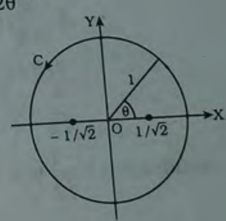
(ii) $$\int_0^{\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta}$$
$$= \frac{1}{2}\cdot\frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$$
$$= \frac{\pi}{b^2}(a-\sqrt{a^2-b^2}) \quad \text{(Ans)}$$

Solution–20. Let us consider the unit circle $|z| = 1$ as the closed contour C. [$|z| = 1$ একক বৃত্তকে বদ্ধ কন্টুর C বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz}dz$, where [যেখানে] $0 \le \theta \le 2\pi$.

$$\cos 2\theta = \frac{1}{2}(e^{i2\theta}+e^{-i2\theta}) = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right) = \frac{z^4+1}{2z^2}$$

$$\therefore \int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta}{5-4\cos 2\theta}d\theta$$



$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1+e^{i6\theta}}{5-4\cos 2\theta}d\theta$$
$$= \text{R. P. of } \frac{1}{2}\oint_C \frac{1+z^6}{5-4\cdot\frac{z^4+1}{2z^2}}\cdot\frac{1}{iz}dz$$
$$= \text{R. P. of } \frac{-1}{2i}\oint_C \frac{z(1+z^6)}{2z^4-5z^2+2}dz$$
$$= \text{R. P. of } \frac{-1}{2i}\oint_C f(z)\, dz$$

where [যেখানে] $f(z) = \frac{z(1+z^6)}{2z^4-5z^2+2}$

The poles of f(z) can be obtained from the equation

$2z^4 - 5z^2 + 2 = 0$

[$2z^4 - 5z^2 + 2 = 0$ সমীকরণ হতে f(z) এর পোল পাওয়া যাবে]

$\Rightarrow 2z^4 - 4z^2 - z^2 + 2 = 0$

$\Rightarrow 2z^2(z^2 - 2) - 1(z^2 - 2) = 0$

$\Rightarrow (2z^2 - 1)(z^2 - 2) = 0$

$\therefore 2z^2 - 1 = 0$ gives [দেয়] $z = \pm\frac{1}{\sqrt{2}}$

$z^2 - 2 = 0$ gives [দেয়] $z = \pm\sqrt{2}$

The simple poles $z = \frac{1}{\sqrt{2}}$ and $z = \frac{-1}{\sqrt{2}}$ lies inside the contour C.

[সরল পোল $z = \frac{1}{\sqrt{2}}$ এবং $z = \frac{-1}{\sqrt{2}}$ কন্টুর C এর ভিতর অবস্থিত]

Residue at $z = \frac{1}{\sqrt{2}}$ is [$z = \frac{1}{\sqrt{2}}$ এ অবশেষ] $\lim\limits_{z\to 1/\sqrt{2}} \left(z - \frac{1}{\sqrt{2}}\right)\cdot f(z)$

$$= \lim_{z\to 1/\sqrt{2}} \left(z - \frac{1}{\sqrt{2}}\right)\cdot\frac{z(1+z^6)}{2z^4 - 5z^2 + 2}$$

$$= \lim_{z\to 1/\sqrt{2}} \left(z - \frac{1}{\sqrt{2}}\right)\cdot\frac{z(1+z^6)}{2\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)(z^2 - 2)}$$

$$= \lim_{z\to 1/\sqrt{2}} \frac{z(1+z^6)}{2\left(z + \frac{1}{\sqrt{2}}\right)(z^2 - 2)}$$

$$= \frac{\frac{1}{\sqrt{2}}\left(1 + \frac{1}{8}\right)}{2\left(\frac{2}{\sqrt{2}}\right)\left(\frac{1}{2} - 2\right)}$$

$$= \frac{9/8\sqrt{2}}{\frac{4}{\sqrt{2}}\cdot\frac{-3}{2}} = \frac{9}{8\sqrt{2}}\cdot\frac{\sqrt{2}}{-6} = \frac{-9}{48}$$

Residue at $z = \frac{-1}{\sqrt{2}}$ is [$z = \frac{-1}{\sqrt{2}}$ অবশেষ] $\lim\limits_{z\to -1/\sqrt{2}} \left(z + \frac{1}{\sqrt{2}}\right)\cdot f(z)$

$$= \lim_{z\to -1/\sqrt{2}} \left(z + \frac{1}{\sqrt{2}}\right)\cdot\frac{z(1+z^6)}{\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)(z^2 - 2)}$$

$$= \frac{\frac{-1}{\sqrt{2}}\left(1 + \frac{1}{8}\right)}{2\cdot\frac{-2}{\sqrt{2}}\cdot\left(\frac{1}{2} - 2\right)}$$

$$= \frac{-9}{8\sqrt{2}}\cdot\frac{1}{\frac{-4}{\sqrt{2}}\cdot\frac{-3}{2}} = \frac{-9}{48}$$

Therefore, by Cauchy's residue theorem we have [অতএব, কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta = \text{R. P. of} \frac{-1}{2i}\oint_C f(z)\, dz$$

$$= \text{R. P. of} \cdot \frac{-1}{2i}\cdot 2\pi i \quad \text{[Sum of the residues]}$$

$$= \text{R. P. of} \cdot (-\pi)\left(\frac{-9}{48} - \frac{9}{48}\right)$$

$$= \frac{18\pi}{48} = \frac{3\pi}{8} \quad \text{(Ans)}$$

Solution–21. (i) : Let $I = \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta$

$$= \text{R. P. of} \int_0^{2\pi} e^{\cos\theta}\cdot e^{i(\sin\theta - n\theta)}\, d\theta$$

$$= \text{R. P. of} \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\cdot e^{-in\theta}\, d\theta$$

$$= \text{R. P. of} \int_0^{2\pi} e^{e^{i\theta}}\cdot e^{-in\theta}\, d\theta$$

Now let us consider the closed contour C as the unit circle $|z| = 1$. [এখন একক বৃত্ত $|z| = 1$ কে বদ্ধ কন্টুর C হিসাবে বিবেচনা করি]

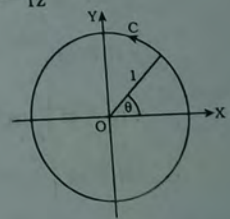
Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\therefore I = \text{R. P. of} \oint_C e^z\cdot z^{-n}\cdot\frac{dz}{iz}$$

$$= \text{R. P. of} \frac{1}{i}\oint_C \frac{e^z}{z^{n+1}}\, dz$$

$$= \text{R. P. of} \frac{1}{i}\oint_C f(z)\, dz \quad \text{...... (1)}$$

where $f(z) = \frac{e^z}{z^{n+1}}$



Poles of f(z) are obtained from $z^{n+1} = 0 \Rightarrow z = 0$

Thus $z = 0$ is a pole inside the contour C of order $n + 1$. [$z^{n+1} = 0$ হতে f(z) এর পোল $z = 0$ পাওয়া যায়। অতএব C কন্টুরের ভিতর $n + 1$ ক্রমের পোল $z = 0$]

Residue at $z = 0$ is [$z = 0$ এ অবশেষ] $\lim\limits_{z \to 0} \frac{1}{n!} \frac{d^n}{dz^n} \{(z - 0)^{n+1} \cdot f(z)\}$

$$= \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} \left(z^{n+1} \cdot \frac{e^z}{z^{n+1}} \right)$$

$$= \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} (e^z)$$

$$= \frac{1}{n!} \lim_{z \to 0} e^z = \frac{1}{n!} e^0 = \frac{1}{n!}$$

Hence by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ [Residue at } z = 0]$$

$$= 2\pi i \cdot \frac{1}{n!} = \frac{2\pi i}{n!}$$

Putting this value in (1) we get, [এই মান (1) এ বসাইয়া পাই]

$$I = \text{R. P. of} \frac{1}{i} \cdot \frac{2\pi i}{n!}$$

$$\Rightarrow I = \frac{2\pi}{n!}$$

$$\therefore \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!} \quad \text{(Proved)}$$

(ii) : Let $I = \int_0^{2\pi} e^{-\cos\theta} \cdot \cos(n\theta + \sin\theta)\, d\theta$

$$= \text{R. P. of} \int_0^{2\pi} e^{-\cos\theta} \cdot e^{-i(n\theta + \sin\theta)}\, d\theta$$

$$= \text{R. P. of} \int_0^{2\pi} e^{-\cos\theta - i\sin\theta} \cdot e^{-in\theta}\, d\theta$$

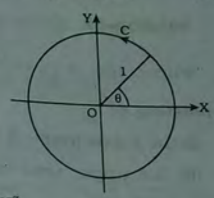
$$= \text{R. P. of} \int_0^{2\pi} e^{-(\cos\theta + i\sin\theta)} \cdot e^{-in\theta}\, d\theta$$

$$= \text{R. P. of} \int_0^{2\pi} e^{-e^{i\theta}} \cdot e^{-in\theta}\, d\theta$$

Now let us consider the closed contour C as the unit circle $|z| = 1$. [এখন একক বৃত্ত $|z| = 1$ কে বদ্ধ কন্টুর C হিসাবে বিবেচনা করি]

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$

$$\Rightarrow d\theta = \frac{1}{iz} dz$$



$$\therefore I = \text{R. P. of} \oint_C e^{-z} \cdot z^{-n} \cdot \frac{1}{iz} dz$$

$$= \text{R. P. of} \frac{1}{i} \oint_C \frac{e^{-z}}{z^{n+1}} dz$$

$$= \text{R. P. of} \frac{1}{i} \oint_C f(z)\, dz, \text{ where } f(z) = \frac{e^{-z}}{z^{n+1}} \quad \cdots\cdots (1)$$

Poles of f(z) are obtained from the equation $z^{n+1} = 0 \Rightarrow z = 0$.

Thus, $z = 0$ is a pole inside the contour C of order $n + 1$. [$z^{n+1} = 0$ হতে f(z) এর পোল $z = 0$ পাওয়া যায়। অতএব C কন্টুরের ভিতর $n + 1$ ক্রমের পোল $z = 0$]

Residue at $z = 0$ is [$z = 0$ এ অবশেষ] $\lim\limits_{z \to 0} \frac{1}{n!} \frac{d^n}{dz^n} \{(z - 0)^{n+1} \cdot f(z)\}$

$$= \lim_{z \to 0} \frac{1}{n!} \frac{d^n}{dz^n} \left\{ z^{n+1} \cdot \frac{e^{-z}}{z^{n+1}} \right\}$$

$$= \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} (e^{-z})$$

$$= \frac{1}{n!} \lim_{z \to 0} (-1)^n \cdot e^{-z}$$

$$= \frac{(-1)^n}{n!} e^0$$

$$= \frac{(-1)^n}{n!}$$

Hence by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \cdot \text{[Residue at } z = 0]$$

$$= 2\pi i \cdot \frac{(-1)^n}{n!}$$

Putting this value in (1) we get

$$I = \text{R. P. of} \frac{1}{i} \cdot 2\pi i \cdot \frac{(-1)^n}{n!}$$

$$\Rightarrow \int_0^{2\pi} e^{-\cos\theta} \cdot \cos(n\theta + \sin\theta)\, d\theta = \frac{2\pi(-1)^n}{n!} \quad \text{(Proved)}$$

GROUP-B

Solution–22. Let us consider the integral $\oint_C f(z)\,dz$,

where $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large,

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$ are obtained from the equation.

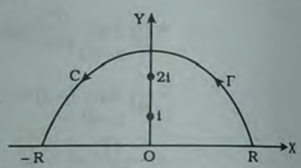
[$\oint_C f(z)\,dz$ সমাকলন যোগজ'টি বিবেচনা করি, যেখানে $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R খুব বড়

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

$f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$ এর পোল সমূহ নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$(z^2+1)(z^2+4) = 0$$
$$\Rightarrow z^2+1 = 0,\ z^2+4 = 0$$
$$\Rightarrow z = \pm i,\ z = \pm 2i$$



The simple poles $z = i$ and $z = 2i$ lie inside the contour C. [সরল পোল $z = i$ এবং $z = 2i$ কন্টুর C এর ভিতরে অবস্থিত]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim\limits_{z\to i}(z-i)\cdot f(z)$

$$= \lim_{z\to i}(z-i)\cdot\frac{1}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to i}(z-i)\cdot\frac{1}{(z+i)(z-i)(z^2+4)}$$

$$= \lim_{z\to i}\frac{1}{(z+i)(z^2+4)}$$

$$= \frac{1}{2i(i^2+4)} = \frac{1}{6i}$$

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z\to 2i}(z-2i)\cdot f(z)$

$$= \lim_{z\to 2i}(z-2i)\cdot\frac{1}{(z^2+1)(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i}\frac{1}{(z^2+1)(z+2i)} = \frac{1}{(4i^2+1)4i} = \frac{-1}{12i}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i\ \text{[sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\left[\frac{1}{6i} - \frac{1}{12i}\right] \quad \ldots\ldots (1)$$

when $R\to\infty$ then [যখন $R\to\infty$ তখন] $\displaystyle\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)}\,dx$$

Here $\displaystyle\lim_{z\to\infty} z\cdot f(z) = \lim_{z\to\infty}\frac{z}{(z^2+1)(z^2+4)}$

$$= \lim_{z\to\infty}\frac{z}{z^4\left(1+\frac{1}{z^2}\right)\left(1+\frac{4}{z^2}\right)}$$

$$= \lim_{z\to\infty}\frac{1}{z^3\left(1+\frac{1}{z^2}\right)\left(1+\frac{4}{z^2}\right)} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

Now taking limit $R\to\infty$ in (1) and then using the above results we get, [এখন (1) এ লিমিট $R\to\infty$ নিয়ে এবং উপরের ফলাফল ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} f(x)\,dx + 0 = 2\pi i\times\frac{1}{12i}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$$

$$\Rightarrow 2\int_0^{\infty}\frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$$

$$\Rightarrow \int_0^{\infty}\frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{12} \quad \text{(Ans)}$$

Solution–23. Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{z^2}{(z^2+4)(z^2+9)}$ and C is the closed contour consisting of :

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \dfrac{z^2}{(z^2+4)(z^2+9)}$ will be obtained by solving the equation

[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{z^2}{(z^2+4)(z^2+9)}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R খুব বড়

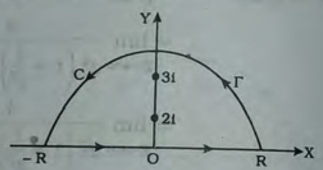
(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

$f(z) = \dfrac{z^2}{(z^2+4)(z^2+9)}$ এর পোল সমূহ নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$(z^2+4)(z^2+9) = 0$$

$$z^2 + 4 = 0 \Rightarrow z = \pm 2i$$

$$z^2 + 9 = 0 \Rightarrow z = \pm 3i$$



The poles $z = 2i$ and $z = 3i$ lie inside the contour which are simple poles. [$z = 2i$ এবং $z = 3i$ পোলগুলি কন্টুরের ভিতরে অবস্থিত যাহা সরল পোল]

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z\to 2i} (z-2i)\cdot f(z)$

$$= \lim_{z\to 2i} (z-2i)\cdot\frac{z^2}{(z^2+4)(z^2+9)}$$

$$= \lim_{z\to 2i} (z-2i)\cdot\frac{z^2}{(z+2i)(z-2i)(z^2+9)}$$

$$= \lim_{z\to 2i} \frac{z^2}{(z+2i)(z^2+9)}$$

$$= \frac{4i^2}{4i(4i^2+9)} = \frac{i}{(-4+9)} = \frac{i}{5}$$

Residue at $z = 3i$ is [$z = 3i$ এ অবশেষ] $\lim\limits_{z\to 3i} (z-3i)\cdot f(z)$

$$= \lim_{z\to 3i} (z-3i)\cdot\frac{z^2}{(z^2+4)(z^2+9)}$$

$$= \lim_{z\to 3i} (z-3i)\cdot\frac{z^2}{(z^2+4)(z+3i)(z-3i)}$$

$$= \lim_{z\to 3i} \frac{z^2}{(z^2+4)(z+3i)}$$

$$= \frac{9i^2}{(9i^2+4)\,6i} = \frac{3i}{2(-9+4)} = \frac{-3i}{10}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z) = 2\pi i \text{ (Sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\left(\frac{i}{5} - \frac{3i}{10}\right) \cdots\cdots (1)$$

Now when $R \to \infty$ then [যখন $R \to \infty$ তখন] $\displaystyle\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

Also, $\displaystyle\lim_{z\to\infty} z\,f(z) = \lim_{z\to\infty} \frac{z^3}{(z^2+4)(z^2+9)} = 0$

Thus $\displaystyle\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$

Therefore, from (1) by taking limit $R \to \infty$ we get [অতএব, $R \to \infty$ লিমিট নিয়ে (1) হতে পাই]

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx + 0 = 2\pi i\left(\frac{-i}{10}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{\pi}{5} \quad \text{(Ans for 23)}$$

$\Rightarrow 2\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{\pi}{5}$, since $f(x) = \dfrac{x^2}{(x^2+4)(x^2+9)}$ is an even function.

$$\Rightarrow \int_0^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{\pi}{10} \quad \text{[Ans of 23(i)]}$$

Solution–24. Consider $\oint_C f(z)\, dz$, where $f(z) = \dfrac{1}{(z^2+1)(z^2+4)^2}$

and C is the closed contour consisting of :

(i) the x-axis from $-R$ to R, where R is large,

(ii) the upper semi-circle Γ of the circle $|z| = R$,

which lies above the x-axis.

The poles of $f(z) = \dfrac{1}{(z^2+1)(z^2+4)^2}$ will be obtained by solving the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{1}{(z^2+1)(z^2+4)^2}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

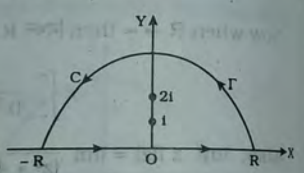
(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

$f(z) = \dfrac{1}{(z^2+1)(z^2+4)^2}$ এর পোল সমূহ নিম্নের সমীকরণ সমাধাণ করে পাওয়া যায় ।]

$(z^2+1)(z^2+4)^2 = 0$

$z^2 + 1 = 0$ gives $z = \pm\sqrt{-1} = \pm i$

$z^2 + 4 = 0$ gives $z = \pm\sqrt{-4} = \pm 2i$



The pole $z = i$ and $z = 2i$ lie inside the contour. $z = i$ is a simple pole and $z = 2i$ is a double pole. [$z = i$ এবং $z = 2i$ পোল কন্টুরের ভিতরে অবস্থিত । $z = i$ সরল পোল এবং $z = 2i$ দ্বিপোল]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim\limits_{z\to i}(z-i)\, f(z)$

$$= \lim_{z\to i}(z-i)\cdot\frac{1}{(z^2+1)(z^2+4)^2}$$

$$= \lim_{z\to i}(z-i)\cdot\frac{1}{(z+i)(z-i)(z^2+4)^2}$$

$$= \lim_{z\to i}\frac{1}{(z+i)(z^2+4)^2}$$

$$= \frac{1}{2i(i^2+4)^2} = \frac{i}{-2(9)} = \frac{-i}{18}$$

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z\to 2i}\dfrac{d}{dz}\{(z-2i)^2 \cdot f(z)\}$

$$= \lim_{z\to 2i}\frac{d}{dz}\left\{(z-2i)^2\cdot\frac{1}{(z^2+1)(z+2i)^2(z-2i)^2}\right\}$$

$$= \lim_{z\to 2i}\frac{d}{dz}\left\{\frac{1}{(z^2+1)(z+2i)^2}\right\}$$

$$= \lim_{z\to 2i}\frac{1}{(z^2+1)(z+2i)^2}\left[0 - \frac{2z}{z^2+1} - \frac{2}{z+2i}\right]$$

$$= \frac{-1}{(4i^2+1)\,16i^2}\left[\frac{4i}{4i^2+1} + \frac{2}{4i}\right]$$

$$= \frac{-1}{-3(-16)}\left[\frac{4i}{-3} - \frac{i}{2}\right]$$

$$= \frac{-1}{48}\cdot\frac{-8i-3i}{6} = \frac{11i}{288}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i\ (\text{sum of the residues})$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i\left(\frac{-i}{18} + \frac{5i}{288}\right) \ \ldots\ldots (1)$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন] $\displaystyle\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx$

$$= \int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)^2}\, dx$$

$$= 2\int_0^{\infty}\frac{1}{(x^2+1)(x^2+4)^2}\, dx, \text{ since } f(x) \text{ is an even function.}$$

Also, $\displaystyle\lim_{z\to\infty} z f(z) = \lim_{z\to\infty}\frac{z}{(z^2+1)(z^2+4)^2}$

$$= \lim_{z\to\infty}\frac{1}{z^5\left(1+\frac{1}{z^2}\right)\left(1+\frac{4}{z^2}\right)^2}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore \lim_{R\to\infty}\int_\Gamma f(z)\, dz = 0$$

Now taking the limit $R \to \infty$ in (1) and putting these values we get [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং এই মানগুলি বসিয়ে পাই]

$$2\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\,dx + 0 = 2\pi i\left(\frac{-16i+11i}{288}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\,dx = \pi i \times \frac{-5i}{288}$$

$$= \frac{5\pi}{288} \text{ (Ans)}$$

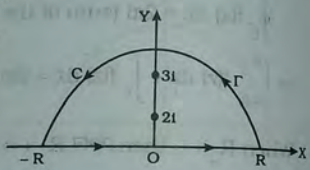
Solution–25. Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{2z^2}{(z^2+9)(z^2+4)^2}$

and C is the closed contour consisting of :

(i) the x-axis from $-R$ to R, where R is very large,

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \dfrac{2z^2}{(z^2+9)(z^2+4)^2}$



will be obtained by solving the equation. [$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{2z^2}{(z^2+9)(z^2+4)^2}$ এবং C কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

$f(z) = \dfrac{2z^2}{(z^2+9)(z^2+4)^2}$ এর পোল সমূহ নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$(z^2+9)(z^2+4)^2 = 0$$

$z^2 + 9 = 0$ gives $z = \pm\sqrt{-9} = \pm 3i$

$z^2 + 4 = 0$ gives $z = \pm\sqrt{-4} = \pm 2i$

Only the poles $z = 3i$ and $z = 2i$ lie inside the contour. $z = 3i$ is a simple pole and $z = 2i$ is a double pole. [একমাত্র $z = 3i$ এবং $z = 2i$ পোলগুলি কন্টুরের ভিতর অবস্থিত। $z = 3i$ সরল পোল এবং $z = 2i$ দ্বিপোল]

Residue at $z = 3i$ is [$z = 3i$ এ অবশেষ] $\lim\limits_{z\to 3i} (z-3i)\,f(z)$

$$= \lim_{z\to 3i} (z-3i)\cdot\frac{2z^2}{(z^2+9)(z^2+4)^2}$$

$$= \lim_{z\to 3i} (z-3i)\cdot\frac{2z^2}{(z+3i)(z-3i)(z^2+4)^2}$$

$$= \lim_{z\to 3i} \frac{2z^2}{(z+3i)(z^2+4)^2}$$

$$= \frac{18i^2}{6i(9i^2+4)^2} = \frac{3i}{25}$$

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z\to 2i} \dfrac{1}{1!}\dfrac{d}{dz}\{(z-2i)^2\cdot f(z)\}$

$$= \lim_{z\to 2i} \frac{d}{dz}\left\{(z-2i)^2\cdot\frac{2z^2}{(z^2+9)(z^2+4)^2}\right\}$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left\{(z-2i)^2\cdot\frac{2z^2}{(z^2+9)(z+2i)^2(z-2i)^2}\right\}$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left\{\frac{2z^2}{(z^2+9)(z+2i)^2}\right\}$$

$$= 2\lim_{z\to 2i} \frac{z^2}{(z^2+9)(z+2i)^2}\left[\frac{2}{z} - \frac{2z}{z^2+9} - \frac{2}{z+2i}\right]$$

$$= 2\,\frac{4i^2}{(4i^2+9)\,16i^2}\left[\frac{2}{2i} - \frac{4i}{4i^2+9} - \frac{2}{4i}\right]$$

$$= \frac{1}{10}\left[-i - \frac{4i}{5} + \frac{i}{2}\right] = \frac{1}{10}\times\frac{-13i}{10} = \frac{-13i}{100}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \text{ (Sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz = 2\pi i\left(\frac{3i}{25} - \frac{13i}{100}\right) \text{...... (1)}$$

When $R \to \infty$ then $\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{2x^2}{(x^2+9)(x^2+4)^2}\, dx$$

$$= 2\int_{0}^{\infty} \frac{2x^2}{(x^2+9)(x^2+4)^2}\, dx, \text{ since } f(x) \text{ is an even function}$$

Also, $\lim\limits_{z\to\infty} z\, f(z) = \lim\limits_{z\to\infty} \frac{2z^3}{(z^2+9)(z^2+4)^2}$

$$= \lim_{z\to\infty} \frac{2}{z^3\left(1+\frac{9}{z^2}\right)\left(1+\frac{4}{z^2}\right)^2} = \frac{2}{\infty} = 0$$

$$\therefore \lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0$$

Taking limit $R \to \infty$ in (1) and then putting these values we get [(1) এ $R \to \infty$ লিমিট নিয়ে এবং তারপর এই মানগুলি বসাইয়া পাই]

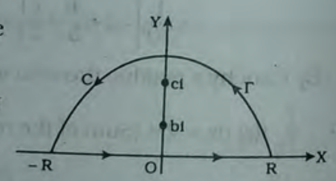
$$2\int_{0}^{\infty} \frac{2x^2}{(x^2+9)(x^2+4)^2}\, dx + 0 = 2\pi i \times \frac{-i}{100}$$

$$\Rightarrow \int_{0}^{\infty} \frac{2x^2}{(x^2+9)(x^2+4)^2}\, dx = \frac{\pi}{100} \text{ (Ans)}$$

Solution–26. Consider $\oint_C f(z)\, dz$ where $f(z) = \frac{1}{(z^2+b^2)(z^2+c^2)^2}$ and C is the contour consisting of :

(i) the x-axis from $-R$ to R, where R is very large,

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of f(z) are obtained from the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{1}{(z^2+b^2)(z^2+c^2)^2}$ এবং C কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

f(z) এর পোল নিম্নের সমীকরণ হতে পাওয়া যায় ।]

$$(z^2+b^2)(z^2+c^2)^2 = 0$$

$z^2 + b^2 = 0$ gives $z = \pm bi$

$z^2 + c^2 = 0$ gives $z = \pm ci$

The simple pole $z = bi$ and the double pole $z = ci$ lie inside the contour. [সরল পোল $z = bi$ এবং দ্বিপোল $z = ci$ কন্টুরের ভিতর অবস্থিত]

Residue at $z = bi$ is [$z = bi$ এ অবশেষ] $\lim\limits_{z\to bi} (z - bi)\, f(z)$

$$= \lim_{z\to bi} (z-bi) \cdot \frac{1}{(z^2+b^2)(z^2+c^2)^2}$$

$$= \lim_{z\to bi} (z-bi) \cdot \frac{1}{(z-bi)(z+bi)(z^2+c^2)^2}$$

$$= \lim_{z\to bi} \frac{1}{(z+bi)(z^2+c^2)^2}$$

$$= \frac{1}{2bi(-b^2+c^2)^2} = \frac{-i}{2b(b^2-c^2)^2}$$

Residue at $z = ci$ is [$z = ci$ এ অবশেষ] $\lim\limits_{z\to ci} (z - ci)^2 \,.\, f(z)$

$$= \lim_{z\to ci} \frac{d}{dz}\left\{(z-ci)^2 \cdot \frac{1}{(z^2+b^2)(z^2+c^2)^2}\right\}$$

$$= \lim_{z\to ci} \frac{d}{dz}\left\{(z-ci)^2 \cdot \frac{1}{(z^2+b^2)(z+ci)^2(z-ci)^2}\right\}$$

$$= \lim_{z\to ci} \frac{d}{dz}\left\{\frac{1}{(z^2+b^2)(z+ci)^2}\right\}$$

$$= \lim_{z\to ci} \frac{1}{(z^2+b^2)(z+ci)^2}\left[0 - \frac{2z}{z^2+b^2} - \frac{2}{z+ci}\right]$$

$$= \frac{-1}{(-c^2+b^2)\,4c^2}\left[\frac{2ci}{-c^2+b^2} + \frac{2}{2ci}\right]$$

$$= \frac{1}{4c^2(b^2-c^2)}\left[\frac{2ci}{b^2-c^2} - \frac{i}{c}\right]$$

$$= \frac{1}{4c^2(b^2-c^2)} \cdot \frac{(2c^2-b^2+c^2)i}{c(b^2-c^2)}$$

$$= \frac{(3c^2-b^2)i}{4c^3(b^2-c^2)^2}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ (sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz = 2\pi i\left[\frac{-i}{2b(b^2-c^2)^2}+\frac{(3c^2-b^2)i}{4c^3(b^2-c^2)^2}\right] \cdots\cdots (1)$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন] $\displaystyle\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)(x^2+c^2)^2}\,dx$$

$$\text{and } \lim_{z\to\infty} z\,f(z) = \lim_{z\to\infty} \frac{z}{(z^2+b^2)(z^2+c^2)^2}$$

$$= \lim_{z\to\infty} \frac{1}{z^5\left(1+\frac{b^2}{z^2}\right)\left(1+\frac{c^2}{z^2}\right)^2} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

Now taking the limit $R \to \infty$ in (1) and then putting these values we get [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর এই মানগুলি বসাইয়া পাই]

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)(x^2+c^2)^2}\,dx + 0$$

$$= -2\pi \cdot \frac{-2c^3+3bc^2-b^3}{4bc^3(b^2-c^2)^2}$$

$$= \frac{\pi(b^3-3bc^2+2c^3)}{2bc^3(b^2-c^2)^2}$$

$$= \frac{\pi(b-c)^2(b+2c)}{2bc^3(b^2-c^2)^2}$$

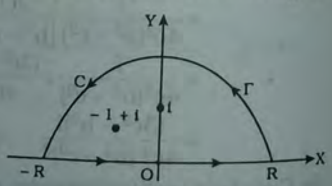
$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)(x^2+c^2)^2}\,dx = \frac{\pi(b+2c)}{2bc^3(b+c)^2} \text{ (Ans)}$$

Solution–27. Consider $\oint_C f(z)\,dz$, where

$f(z) = \dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$ and C is the contour consisting of

(i) the x-axis from $-R$ to R, where R is large,

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of f(z) are obtained from the equation

[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$ এবং C কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

f(z) এর পোল সমূহ নিম্নের সমীকরণ হতে পাওয়া যায় ।]

$$(z^2+1)^2(z^2+2z+2) = 0$$

$$z^2+1 = 0 \text{ gives } z = \pm\sqrt{-1} = \pm i$$

$$z^2+2z+2 = 0 \text{ gives } z = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm 2i}{2} = -1\pm i$$

The poles $z = i$ and $z = -1+i$ lie inside the contour.

$z = i$ is a double pole and $z = -1+i$ is a simple pole. [$z = i$ এবং $z = -1+i$ কন্টুরের ভিতরে অবস্থিত $z = i$ দ্বিপোল এবং $z = -1+i$ সরল পোল ।]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\displaystyle\lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\{(z-i)^2 \cdot f(z)\}$

$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{z^2}{(z^2+1)^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{z^2}{(z-i)^2(z+i)^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{z^2}{(z+i)^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to i} \frac{z^2}{(z+i)^2(z^2+2z+2)}\left[\frac{2}{z} - \frac{2}{z+i} - \frac{2z+2}{z^2+2z+2}\right]$$

$$= \frac{i^2}{4i^2(i^2+2i+2)}\left[\frac{2}{i} - \frac{2}{2i} - \frac{2i+2}{i^2+2i+2}\right]$$

$$= \frac{1}{4(1+2i)}\left[\frac{1}{i} - \frac{2i+2}{1+2i}\right]$$

$$= \frac{1}{4(1+2i)} \times \frac{1+2i-2i^2-2i}{i(1+2i)}$$

$$= \frac{3}{4i(1+4i+4i^2)} = \frac{3}{4i(-3+4i)} = \frac{-3}{4(4+3i)}$$

$$= \frac{-3(4-3i)}{4(16-9i^2)} = \frac{9i-12}{100}$$

Residue at $z = -1 + i$ is [$z = 1 + i$ এ অবশেষ] $\lim_{z \to -1+i} \{(z + 1 - i) \cdot f(z)\}$

$$= \lim_{z \to -1+i} \left\{(z + 1 - i) \cdot \frac{z^2}{(z^2 + 1)^2 (z + 1 - i)(z + 1 + i)}\right\}$$

$$= \lim_{z \to -1+i} \frac{z^2}{(z^2 + 1)^2 (z + 1 + i)}$$

$$= \frac{(-1 + i)^2}{\{(-1 + i)^2 + 1\}^2 (-1 + i + 1 + i)}$$

$$= \frac{1 - 2i + i^2}{(1 - 2i + i^2 + 1)^2 \, 2i} = \frac{-2i}{2i(1 - 2i)^2}$$

$$= \frac{-1}{1 - 4i + 4i^2} = \frac{-1}{-3 - 4i}$$

$$= \frac{3 - 4i}{9 - 16i^2}$$

$$= \frac{3 - 4i}{25}$$

Now by Cauchy's residue theorem we have [এখন কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ (sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \left(\frac{9i - 12}{100} + \frac{3 - 4i}{25}\right) \ldots\ldots (1)$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন] $\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx$

$$= \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2 (x^2 + 2x + 2)}\, dx$$

Also, $\lim_{z \to \infty} z\, f(z) = \lim_{z \to \infty} \frac{z^3}{(z^2 + 1)^2 (z^2 + 2z + 2)}$

$$= \lim_{z \to \infty} \frac{1}{z^3 \left(1 + \frac{1}{z^2}\right)^2 \left(1 + \frac{2}{z} + \frac{2}{z^2}\right)} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0.$$

Now taking the limits $R \to \infty$ in (1) and putting these values we get, [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং এই মানগুলি বসাইয়া পাই]

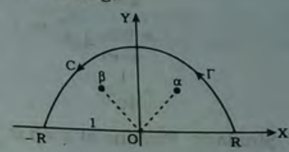
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2 (x^2 + 2x + 2)}\, dx + 0 = 2\pi i \times \frac{9i - 12 + 12 - 16i}{100}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2 (x^2 + 2x + 2)}\, dx = \frac{-14i^2\pi}{100} = \frac{7\pi}{50} \quad \text{(Ans)}$$

Solution-28. Consider $\oint_C f(z)\, dz$, where $f(z) = \frac{1}{z^4 + 1}$ and C is the contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \frac{1}{z^4 + 1}$ will be ohtained by solving the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{1}{z^4 + 1}$ এবং C কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত

$f(z) = \frac{1}{z^4 + 1}$ এর পোল সমূহ $z^4 + 1 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে।]

$$z^4 + 1 = 0$$

$$\Rightarrow z^4 = -1 = \cos\pi + i\sin\pi$$

$$= \cos(2n\pi + \pi) + i\sin(2n\pi + \pi)$$

$$= \cos(2n + 1)\pi + i\sin(2n + 1)\pi$$

$$\Rightarrow z = \cos\frac{2n + 1}{4} + i\sin\left(\frac{2n + 1}{4}\right)\pi$$

$$= e^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

The poles are [পোলগুলি হল] $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

The amplitudes of the first two poles lies between 0 and π.

Therefore, the poles, $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$ lie inside the contour, both of them are simple poles. Let $z = \alpha = e^{i\pi/4}$ and $z = \beta = e^{i3\pi/4}$.

[প্রথম দুইটি পোলের কোনাঙ্ক 0 ও π এর মধ্যে অবস্থিত। অতএব, $z = e^{i\pi/4}$ এবং $z = e^{i3\pi/4}$ পোল সমূহ কন্টুরের ভিতরে অবস্থিত, তাদের উভয়েই সরল পোল। ধরি $z = \alpha = e^{i\pi/4}$ এবং $z = \beta = e^{i3\pi/4}$]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z \to \alpha} (z - \alpha)\, f(z)$

$$= \lim_{z \to \alpha} \frac{z - \alpha}{z^4 + 1} ; \frac{0}{0} \text{ form}$$

$$= \lim_{z \to \alpha} \frac{1}{4z^3} ; \text{ by L. Hospital rule}$$

$$= \frac{1}{4\alpha^3}$$

$$z^4 = -1 \Rightarrow \alpha^4 = -1 \Rightarrow \alpha^4 + 1 = 0$$

Similarly, Residue at $z = \beta$ is $\frac{1}{4\beta^3}$ [অনুরূপে $z = \beta$ এ অবশেষ $\frac{1}{4\beta^3}$]

Now by Cauchy's residue theorem we have [এখন কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ [Sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz = 2\pi i \left(\frac{1}{4\alpha^3} + \frac{1}{4\beta^3} \right) \text{ ...... (1)}$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$$

$$= 2 \int_0^{\infty} \frac{dx}{x^4 + 1}, \text{ since } f(x) \text{ is an even function.}$$

Also, $\lim\limits_{z \to \infty} z f(z) = \lim\limits_{z \to \infty} \frac{z}{z^4 + 1} ; \frac{\infty}{\infty}$ form

$$= \lim_{z \to \infty} \frac{1}{4z^3}, \text{ by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore \lim_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0$$

$$\frac{1}{4\alpha^3} + \frac{1}{4\beta^3} = \frac{1}{4} \left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} \right)$$

$$= \frac{1}{4} \left(\frac{\alpha}{\alpha^4} + \frac{\beta}{\beta^4} \right)$$

$$= \frac{1}{4} (-\alpha - \beta); \quad \because z^4 = -1 \Rightarrow \alpha^4 = -1 \text{ and } \beta^4 = -1$$

$$= -\frac{1}{4} \left(e^{i\pi/4} + e^{i3\pi/4} \right)$$

$$= -\frac{1}{4} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} + \cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4} \right)$$

$$= -\frac{1}{4} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} - \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)$$

$$= \frac{-2}{4}\, i \sin\frac{\pi}{4} = \frac{-i}{2\sqrt{2}}$$

Now taking limit $R \to \infty$ in (1) and then putting above all results we get [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের মানগুলি বসিয়ে পাই]

$$2 \int_0^{\infty} \frac{dx}{x^4 + 1} + 0 = 2\pi i \left(\frac{-i}{2\sqrt{2}} \right)$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \pi \quad \text{(Ans)}$$

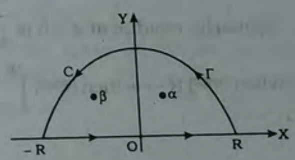
Solution–29. Consider $\oint_C f(z)\, dz$, where $f(z) = \frac{1}{z^4 + a^4}$ and C is the contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \frac{1}{z^4 + a^4}$ will be obtained from the equation

$$z^4 + a^4 = 0$$



[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{1}{z^4 + a^4}$ এবং C কন্টুরটি গঠিত হচ্ছে

(i) x অক্ষ, $-R$ হতে R, যেখানে R অনেক বৃহৎ।

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধবৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবিস্থত।

$f(z) = \frac{1}{z^4 + a^4}$ এর পোল পাওয়া যাবে $z^4 + a^4 = 0$ সমীকরণ হতে।]

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = a(-1)^{1/4}$$

$$= a(\cos \pi + i \sin \pi)^{1/4}$$

$$= a[\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)]^{1/4}$$

$$= a\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$$

$$= ae^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

The poles are [পোলগুলি হলো] $ae^{i\pi/4}$, $ae^{i3\pi/4}$, $ae^{i5\pi/4}$, $ae^{i7\pi/4}$.

The amplitudes of the first two poles are lie between 0 and π. So they are inside the contour and each of them are simple pole. [প্রথম দুইটি পোলের কোণাঙ্ক 0 ও π এর মধ্যে অবস্থিত। সুতরাং তারা কন্টুরের ভিতরে অবস্থিত এবং প্রত্যেকে সরল পোল।]

Let [ধরি] $z = ae^{i\pi/4} = \alpha$ and [এবং] $z = ae^{i3\pi/4} = \beta$.

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha} (z-\alpha)\cdot f(z)$

$$= \lim_{z\to\alpha} \frac{z-\alpha}{z^4 + a^4}; \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{z\to\alpha} \frac{1}{4z^3}, \text{ by L. Hospital rule}$$

$$= \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4}$$

$$= \frac{\alpha}{4(-a^4)} = \frac{-\alpha}{4a^4} \qquad \because z^4 = -a^4 \Rightarrow \alpha^4 = -a^4$$

Similarly, residue at $z = \beta$ is $\frac{-\beta}{4a^4}$ [একইভাবে $z = \beta$ এ অবশেষ $\frac{-\beta}{4a^4}$]

When [যখন] $R \to \infty$ then [তখন] $\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$$

$$= 2\int_0^{\infty} \frac{dx}{x^4 + a^4}$$

Also [আরো] $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \frac{z}{z^4 + a^4}; \frac{\infty}{\infty}$ form

$$= \lim_{z\to\infty} \frac{1}{4z^3}; \text{ by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

Sum of the residues [অবশেষগুলির যোগফল]

$$= \frac{-\alpha}{4a^4} - \frac{\beta}{4a^4}$$

$$= \frac{-1}{4a^4}(\alpha + \beta)$$

$$= \frac{-a}{4a^4}(e^{i\pi/4} + e^{i3\pi/4})$$

$$= \frac{-1}{4a^3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= \frac{-1}{4a^3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \frac{-1}{4a^3}\left(2i\sin\frac{\pi}{4}\right)$$

$$= \frac{-i}{2\sqrt{2}a^3}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz = 2\pi i \times \frac{-i}{2\sqrt{2}a^3}$$

Taking limit $R \to \infty$ and using the above results we get [$R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} f(x)\,dx + 0 = \frac{\pi}{\sqrt{2}a^3}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{\sqrt{2}a^3}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}a^3}$$

$$= \frac{\sqrt{2}\pi}{4a^3}. \text{ (Ans)}$$

Solution–30. Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \frac{1}{(z^2 + a^2)^2}$ and C is the closed contour consisting of

(i) the x-axis from – R to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \frac{1}{(z^2 + a^2)^2}$ are obtained from the equation

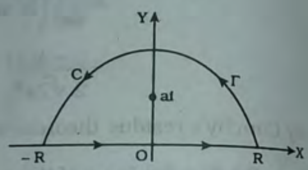
[$\oint_C f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{1}{(z^2 + a^2)^2}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, – R হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{1}{(z^2 + a^2)^2}$ এর পোল সমূহ নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$(z^2 + a^2)^2 = 0$$
$$\Rightarrow z^2 + a^2 = 0$$
$$\Rightarrow z^2 = - a^2$$
$$\Rightarrow z = \pm ai$$



Only the pole $z = ai$ lies inside the contour which is a double pole. [একমাত্র পোল $z = ai$ কন্টুরের ভিতর অবস্থিত যাহা একটি দ্বিপোল]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim\limits_{z\to ai} \frac{1}{1!}\frac{d}{dz}\{(z - ai)^2 \cdot f(z)\}$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{(z - ai)^2 \cdot \frac{1}{(z^2 + a^2)^2}\right\}$$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{(z - ai)^2 \cdot \frac{1}{(z + ai)^2\,(z - ai)^2}\right\}$$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{\frac{1}{(z + ai)^2}\right\}$$

$$= \lim_{z\to ai} \frac{-2}{(z + ai)^3} = \frac{-2}{(2ai)^3}$$

$$= \frac{-1}{4a^3 i^3} = \frac{-i}{4a^3}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i\,[\text{Residue at } z = ai]$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i\left(\frac{-i}{4a^3}\right) \text{...... (1)}$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2}\, dx$$

Also, $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \frac{z}{(z^2 + a^2)^2}$; $\frac{\infty}{\infty}$ form

$$= \lim_{z\to\infty} \frac{1}{2(z^2 + a^2) \cdot 2z} \text{ ; by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

So, $\lim\limits_{R\to\infty} \int_\Gamma f(z)\, dz = 0$

Taking limit $R \to \infty$ in (1) and using the above results we get

[(1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2}\, dx + 0 = \frac{-2\pi i^2}{4a^3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2}\, dx = \frac{\pi}{2a^3}. \quad \text{(Ans)}$$

$$\Rightarrow 2\int_0^{\infty} \frac{1}{(x^2 + a^2)^2}\, dx = \frac{\pi}{2a^3} \text{ ; by property of definite integral}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{(x^2 + a^2)^2}\, dx$$

$$= \frac{\pi}{4a^3}. \quad \text{(Ans)}$$

Solution – 30(i). Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \dfrac{1}{(1+z^2)^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above x-axis.

The poles of $f(z) = \dfrac{1}{(1+z^2)^2}$ are obtained from the equation

[$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি যেখানে $f(z) = \dfrac{1}{(1+z^2)^2}$ এবং C বদ্ধ কন্টুরটি গঠিতঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের ঊর্ধ্ব অর্ধ বৃত্ত, যাহা x অক্ষের ঊর্ধ্বে অবস্থিত।

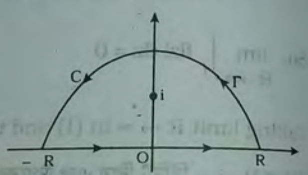
$f(z) = \dfrac{1}{(1+z^2)^2}$ এর পোল সমূহ নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$(1+z^2)^2 = 0$$

$$\Rightarrow 1 + z^2 = 0$$

$$\Rightarrow z^2 = -1 = i^2$$

$$\Rightarrow z = \pm i$$



Only the pole $z = i$ lies inside the contour which is a double pole [একমাত্র পোল $z = i$ কন্টুরের ভিতরে অবস্থিত যাহা একটি দ্বিপোল]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim\limits_{z \to i} \dfrac{1}{1!}\dfrac{d}{dz}\{(z-i)^2 f(z)\}$

$$= \lim_{z \to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{1}{(z^2+1)^2}\right\}$$

$$= \lim_{z \to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{1}{(z+i)^2(z-i)^2}\right\}$$

$$= \lim_{z \to i} \frac{d}{dz}\left\{\frac{1}{(z+i)^2}\right\}$$

$$= \lim_{z \to i} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(2i)^3}$$

$$= \frac{-1}{4i^3}$$

$$= \frac{-i}{4}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \text{ [Residue at } z = i]$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\left(\frac{-i}{4}\right) \ldots (1)$$

Where $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx$$

Also, $\lim\limits_{z \to \infty} zf(z) = \lim\limits_{z \to \infty} \dfrac{z}{(z^2+1)^2}; \dfrac{\infty}{\infty}$ form

$$= \lim_{z \to \infty} \frac{1}{2(z^2+1)\,.2z}; \text{ by Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

So, $\lim\limits_{R \to \infty} \int_\Gamma f(z)\,dz = 0$

Taking limit $R \to \infty$ in (1) and using the above results we get

[(1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx + 0$$

$$= \frac{-2\pi i^2}{4}$$

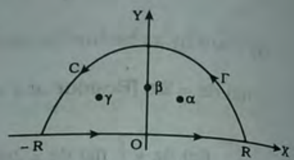
$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx$$

$$= \frac{\pi}{2} \textbf{ Ans.}$$

Solution–31. Consider the integral $\oint_C f(z)\,dz$, where $f(z)=\frac{1}{z^6+1}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z)=\frac{1}{z^6+1}$ are obtained from the equation

$$z^6+1=0$$

[$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি, যেখানে $f(z)=\frac{1}{z^6+1}$ এবং C একটি বদ্ধ কন্টুর যা গঠিত

(i) x অক্ষ, $-R$ হতে R, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের ঊর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উপরে অবস্থিত।

$f(z)=\frac{1}{z^6+1}$ এর পোল পাওয়া যাবে $z^6+1=0$ সমীকরণ হতে।]

$$\Rightarrow z^6=-1$$

$$\Rightarrow z=(-1)^{1/6}=(\cos\pi+i\sin\pi)^{1/6}$$

$$=\{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\}^{1/6}$$

$$=\cos\left(\frac{2n+1}{6}\right)\pi+i\sin\left(\frac{2n+1}{6}\right)\pi$$

$=e^{i(2n+1)\pi/6}$, where [যেখানে] $n = 0, 1, 2, 3, 4, 5$

$\therefore$ The poles are [পোলগুলি হল]

$$e^{i\pi/6},\ e^{i3\pi/6},\ e^{i5\pi/6},\ e^{i7\pi/6},\ e^{i9\pi/6} \text{ and } e^{i11\pi/6}$$

The amplitudes of the first three poles lie between 0 and π. These poles lie inside the contour and each is a simple pole. [প্রথম তিনটি পোলের কোণাঙ্ক 0 ও π এর মধ্যে। এই পোলগুলি কন্টুরের ভিতর এবং সরল পোল।]

Let [ধরি] $z=e^{i\pi/6}=\alpha$, $z=e^{i3\pi/6}=\beta$ and [এবং] $z=e^{i5\pi/6}=\gamma$.

Residue at $z=\alpha$ is [$z=\alpha$ অবশেষ] $\lim_{z\to\alpha}(z-\alpha)\,f(z)$

$$=\lim_{z\to\alpha}\frac{z-\alpha}{z^6+1};\ \frac{\infty}{\infty}\text{ form}$$

$$=\lim_{z\to\alpha}\frac{1}{6z^5};\ \text{by L. Hospital rule}$$

$$=\frac{1}{6\alpha^5}=\frac{\alpha}{6\alpha^6}=\frac{-\alpha}{6}$$

$$z^6+1=0$$
$$\Rightarrow \alpha^6+1=0$$
$$\beta^6+1=0$$
$$\gamma^6+1=0$$

Similarly, the residues at $z=\beta$ and $z=\gamma$ are respectively [অনুরূপে, $z=\beta$ ও $z=\gamma$ এ অবশেষ যথাক্রমে]

$$\frac{-\beta}{6}\text{ and }\frac{-\gamma}{6}$$

$\therefore$ Sum of the residues [অবশেষগুলির যোগফল] $=\frac{-1}{6}(\alpha+\beta+\gamma)$

$$=\frac{-1}{6}\left(e^{i\pi/6}+e^{i3\pi/6}+e^{i5\pi/6}\right)$$

$$=\frac{-1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}+\cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6}+\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

$$=\frac{-1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}+\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}-\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$=\frac{-1}{6}\left(2i\sin\frac{\pi}{6}+0+i\right)$$

$$=\frac{-i}{6}\left(2\cdot\frac{1}{2}+1\right)$$

$$=\frac{-i(1+1)}{6}=\frac{-2i}{6}=\frac{-i}{3}$$

Now by Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz=2\pi i\ \text{(Sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R}f(z)\,dz+\int_{\Gamma}f(z)\,dz=2\pi i\cdot\frac{-i}{3}\ \ldots\ldots (1)$$

When [যখন] $R\to\infty$ then [তখন] $\int_{-R}^{R}f(z)\,dz=\int_{-\infty}^{\infty}f(x)\,dx$

$$=\int_{-\infty}^{\infty}\frac{1}{x^6+1}\,dx$$

$$=2\int_0^{\infty}\frac{1}{x^6+1}\,dx,\ \text{since } f(x) \text{ is an even function.}$$

Also [আরো] $\lim\limits_{z\to\infty} z\, f(z) = \lim\limits_{z\to\infty} \frac{z}{z^6+1}$; $\frac{\infty}{\infty}$ form

$$= \lim_{z\to\infty} \frac{1}{6z^5} \text{ ; by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

Hence [অতএব] $\lim\limits_{R\to\infty} \int_\Gamma f(z)\, dz = 0$

Now taking limit $R \to \infty$ of both sides of (1) and using the above results we get [এখন (1) এর উভয় দিকে $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$2\int_0^\infty \frac{1}{x^6+1}\, dx + 0 = \frac{2\pi}{3}$$

$$\Rightarrow \int_0^\infty \frac{1}{x^6+1}\, dx = \frac{\pi}{3} \quad \text{(Ans)}$$

Solution–32. Consider $\oint_C f(z)\, dz$ where $f(z) = \frac{z^6}{(a^4+z^4)^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, lies above the x-axis.

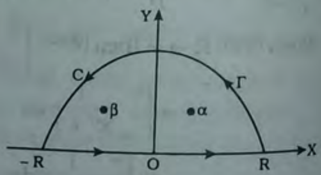
The poles of f(z) are given by the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{z^6}{(a^4+z^4)^2}$ এবং C বদ্ধ কন্টুরটি গঠিত :

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

f(z) এর পোলগুলি নিম্নের সমীকরণ হতে পাওয়া যায়।]



$$(a^4+z^4)^2 = 0$$

$$\Rightarrow z^4 + a^4 = 0$$

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = a(-1)^{1/4}$$

$$= a[\cos\pi + i\sin\pi]^{1/4}$$

$$= a[\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]^{1/4}$$

$$= a\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$$

$$= ae^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

The poles are [পোলগুলি হল] $ae^{i\pi/4}$, $ae^{i3\pi/4}$, $ae^{i5\pi/4}$ and $ae^{i7\pi/4}$

The amplitue of the frist two poles are greater than zero and less than π. So $z = ae^{i\pi/4}$ and $z = a^{i3\pi/4}$ lie inside the contour.

[প্রথম দুইটি পোলের কোনাঙ্ক শূন্য হতে বড় এবং π হতে ছোট, সুতরাং $z = ae^{i\pi/4}$ এবং $z = e^{i3\pi/4}$ কন্টুরের ভিতরে অবস্থিত।]

Let $z = ae^{i\pi/4} = \alpha$ and $z = a^{i3\pi/4} = \beta$

If [যদি] $z = \alpha + t$, then [তখন] $t \to 0$ and $f(z) = \frac{z^6}{(a^4+z^4)^2}$ becomes [দাঁড়ায়]

$$f(t+\alpha) = \frac{(\alpha+t)^6}{\{a^4+(\alpha+t)^4\}^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots + t^6}{(a^4+\alpha^4+4\alpha^3 t+6\alpha^2t^2+4\alpha t^3+t^4)^2}$$

$z^4 + a^4 = 0$
$\Rightarrow \alpha^4 + a^4 = 0$
since α is a root.

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots + t^6}{(4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots}{16\alpha^6 t^2\left(1+\frac{6t}{4\alpha}+\cdots\right)^2};$$ neglecting the terms of higher power of t.

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots}{16\alpha^6 t^2}\left(1+\frac{6t}{4\alpha}+\cdots\right)^{-2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots}{16\alpha^6 t^2}\left(1-2\cdot\frac{6t}{4\alpha}+\cdots\right)$$

At $z = \alpha$, Residue = Coefficient of $\frac{1}{t}$ in $f(t+\alpha)$ [$z = \alpha$ এ অবশেষ - $f(t+\alpha)$ এ $\frac{1}{t}$ এর সহগ]

$$= \frac{6\alpha^5}{16\alpha^6} - \frac{12\alpha^6}{16\cdot 4\alpha^7}$$

$$= \frac{2}{8\alpha} - \frac{3}{16\alpha} = \frac{6-3}{16\alpha} = \frac{3}{16\alpha}$$

Similarly, at $z = \beta$, Residue $= \frac{3}{16\beta}$ [একইভাবে $z = \beta$ এ অবশেষ $= \frac{3}{16\beta}$]

Sum of the residues [অবশেষগুলির যোগফল] $= \frac{3}{16}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$

$$= \frac{3}{16}\left(\frac{1}{ae^{i\pi/4}}+\frac{1}{ae^{i3\pi/4}}\right)$$

$$= \frac{3}{16a}\left(e^{-i\pi/4} + e^{-i3\pi/4}\right)$$

$$= \frac{3}{16a}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]$$

$$= \frac{3}{16a}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

$$= \frac{3}{16a}\left(-2i\sin\frac{\pi}{4}\right) = \frac{-3i}{8a}\cdot\frac{1}{\sqrt{2}} = \frac{-3i}{8\sqrt{2}a}$$

Now by Cauchy's residue theorem [এখন কচির অবশেষ উপপাদ্য হতে পাই]

$$\oint_C f(z)\,dz = 2\pi i \quad \text{[Sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\cdot\frac{-3i}{b\sqrt{2}a} \ \ldots\ldots (1)$$

When $R\to\infty$ then [যখন $R\to\infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{x^6}{(a^4+x^4)^2}\,dx$$

$$= 2\int_0^\infty \frac{x^6}{(a^4+x^4)^2}\,dx; \text{ since } f(x) \text{ is even.}$$

Also, $\lim\limits_{z\to\infty} z\,f(z) = \lim\limits_{z\to\infty}\frac{z^7}{(a^4+z^4)^2} = \lim\limits_{z\to\infty}\frac{1}{z\left(1+\frac{a^4}{z^4}\right)^2} = \frac{1}{\infty} = 0$

$$\therefore \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

Taking limit $R\to\infty$ in (1) and putting these values we get, [(1) এ লিমিট $R\to\infty$ নিয়ে এবং এই মানগুলি বসাইয়া পাই]

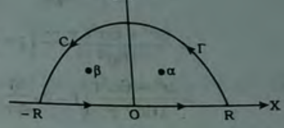
$$2\int_0^\infty \frac{x^6}{(a^4+x^4)^2}\,dx + 0 = \frac{3\pi}{4\sqrt{2}a}$$

$$\Rightarrow \int_0^\infty \frac{x^6}{(a^4+x^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16a}. \quad \text{(Ans)}$$

Solution–33. Let us consider the integral $\oint_C f(z)\,dz$, where $f(z) = \frac{z^6}{(z^4+1)^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis,



The poles of f(z) are given by the equation

[$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{z^6}{(z^4+1)^2}$ এবং C বদ্ধ কন্টুরটি হল :

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

f(z) এর পোলগুলি নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$(z^4+1)^2 = 0$$

$$\Rightarrow z^4 + 1 = 0$$

$$\Rightarrow z = (-1)^{1/4} = (\cos\pi + i\sin\pi)^{1/4}$$

$$= \{\cos(2n+1)\pi + i\sin(2n+1)\pi\}^{1/4}$$

$$= \cos\left(\frac{2n+1}{4}\right)\pi + i\sin\left(\frac{2n+1}{4}\right)\pi$$

$$= e^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

The poles are [পোলগুলি হল] $z = e^{i\pi/4}$, $z = e^{i3\pi/4}$, $z = e^{i5\pi/4}$ and $z = e^{i7\pi/4}$.

The amplitude of the first two poles are greater than 0 and less than π. So, $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$ lie inside the contour. If α is any one of them, let $z = \alpha + t$, then $t\to 0$ and f(z) becomes

[প্রথম দুইটি পোলের কোণাঙ্ক শূন্য হতে বড় এবং π হতে ছোট , সুতরাং $z = e^{i\pi/4}$ এবং $z = e^{i3\pi/4}$ কন্টুরের ভিতরে অবস্থিত। যদি তাদের একটি α হয়, ধরি $z = \alpha + t$ তখন $t\to 0$ এবং f(z) দাঁড়ায়]

$$f(t+\alpha) = \frac{(\alpha+t)^6}{\{(\alpha+t)^4+1\}^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots + t^6}{(\alpha^4 + 4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4 + 1)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots + t^6}{(4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \cdots}{16\alpha^6 t^2 \left(1 + \frac{6t}{4\alpha} + \cdots\right)^2}$$

Since α is a root of $z^4 + 1 = 0$
so, $\alpha^4 + 1 = 0$
neglecting the terms of higher power of t.

$$= \frac{(\alpha^6 + 6\alpha^5 t + \cdots)}{16\alpha^6 t^2}\left(1 + \frac{6t}{4\alpha} + \cdots\right)^{-2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t}{16\alpha^6 t^2}\left(1 - 2 \cdot \frac{6t}{4\alpha} + \cdots\right)$$

At $z = \alpha$, Residue = coefficient of $\frac{1}{t}$ in $f(t + \alpha)$ [$z = \alpha$ এ অবশেষ $= f(t + \alpha)$ এ $\frac{1}{t}$ এর সহগ]

$$= \frac{6\alpha^5}{16\alpha^6} - \frac{12\alpha^6}{16 \cdot 4\alpha^7} = \frac{3}{8\alpha} - \frac{3}{16\alpha} = \frac{6-3}{16\alpha} = \frac{3}{16\alpha}$$

When [যখন] $\alpha = e^{i\pi/4}$ then Residue [তখন অবশেষ] $= \frac{3}{16e^{i\pi/4}} = \frac{3}{16} e^{-i\pi/4}$

When [যখন] $\alpha = e^{i3\pi/4}$ then Residue [তখন অবশেষ]

$$= \frac{3}{16e^{i3\pi/4}} = \frac{3}{16} e^{-i3\pi/4}$$

$\therefore$ Sum of the residues [অবশেষগুলির যোগফল] $= \frac{3}{16}(e^{-i\pi/4} + e^{-i3\pi/4})$

$$= \frac{3}{16}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{3}{16}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \frac{3}{16}\left(-2i\sin\frac{\pi}{4}\right) = \frac{-3i}{8\sqrt{2}}$$

Now by Cauchy's residue theorem we have [এখন কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \times (\text{Sum of the residues})$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i \cdot \frac{-3i}{8\sqrt{2}} = \frac{3\pi}{4\sqrt{2}} \quad \ldots\ldots (1)$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{x^6}{(x^4+1)^2}\,dx$$

Also, $\lim\limits_{z\to\infty} z\,f(z) = \lim\limits_{z\to\infty} \frac{z^7}{(z^4+1)^2} = \lim\limits_{z\to\infty} \frac{1}{z\left(1 + \frac{1}{z^4}\right)^2} = \frac{1}{\infty} = 0$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$$

Taking limit $R \to \infty$ in (1) and using the above results we get [(1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{x^6}{(x^4+1)^2}\,dx + 0 = \frac{3\pi}{4\sqrt{2}} \Rightarrow 2\int_0^{\infty} \frac{x^6}{(x^4+1)^2}\,dx = \frac{3\sqrt{2}\pi}{8}$$

$$\therefore \int_0^{\infty} \frac{x^6}{(x^4+1)^2}\,dx = \frac{3\sqrt{2}\pi}{16}. \quad \text{(Ans)}$$

Solution–34. Consider $\oint_C f(z)\,dz$, where $f(z) = \frac{\log(z+i)}{z^2+1}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.

The poles of f(z) can be obtained by solving the equation

[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \frac{\log(z+i)}{z^2+1}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

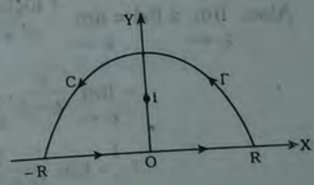
(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

f(z) এর পোলগুলি নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$z^2 + 1 = 0$$

$$\Rightarrow z^2 = -1 = i^2$$

$$\Rightarrow z = \pm i$$



Only the pole $z = i$ lies inside the contour which is a simple pole. [একমাত্র $z = i$ পোলটি কন্টুরের ভিতরে অবস্থিত যাহা একটি সরল পোল]

Residue at $z = i$ is [$z = i$ এ অবশেষ]

$$\lim_{z\to i}(z-i)\cdot f(z) = \lim_{z\to i}(z-i)\frac{\log(z+i)}{z^2+1}$$

$$= \lim_{z\to i}\left\{(z-i)\cdot\frac{\log(z+i)}{(z+i)(z-i)}\right\} = \lim_{z\to i}\frac{\log(z+i)}{z+i}$$

$$= \frac{\log(2i)}{2i} = \frac{\log 2 + \log i}{2i} = \frac{1}{2i}\left[\log 2 + \log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2i}[\log 2 + \log e^{i\pi/2}] = \frac{1}{2i}\left[\log 2 + \frac{i\pi}{2}\right]$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i\ \text{[Residue at } z = i]$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\cdot\frac{1}{2i}\left[\log 2 + \frac{i\pi}{2}\right] \cdots\cdots (1)$$

When $R\to\infty$ then [যখন $R\to\infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{\log(x+i)}{x^2+1}dx$$

$$= \int_{-\infty}^{\infty}\frac{\log\sqrt{x^2+1^2} + i\tan^{-1}\left(\frac{1}{x}\right)}{x^2+1}dx$$

$$\because \log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

$$= \int_{-\infty}^{\infty}\frac{\frac{1}{2}\log(x^2+1) + i\tan^{-1}\left(\frac{1}{x}\right)}{x^2+1}dx$$

Also, $\lim_{z\to\infty} z f(z) = \lim_{z\to\infty}\frac{z\log(z+i)}{z^2+1}$

$$= \lim_{z\to\infty}\frac{z}{z-i}\cdot\lim_{z\to\infty}\frac{\log(z+i)}{z+i};\ \frac{\infty}{\infty}\text{ form}$$

$$= \frac{1}{1}\cdot\lim_{z\to\infty}\frac{1}{(z+i)\cdot 1};\ \text{by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

Now taking limit $R\to\infty$ in(1) and putting the above values we get [এখন (1) এ $R\to\infty$ লিমিট নিয়ে এবং উপরের মানগুলি বসাইয়া পাই]

$$\int_{-\infty}^{\infty}\frac{\frac{1}{2}\log(x^2+1) + i\tan^{-1}\left(\frac{1}{x}\right)}{x^2+1}dx + 0 = \pi\cdot\log 2 + \frac{i\pi^2}{2}$$

Equating real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\log(x^2+1)}{x^2+1}dx = \pi\log 2$$

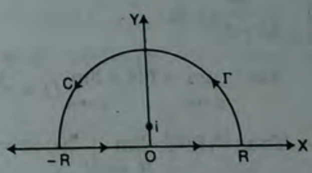
$$\frac{1}{2}\cdot 2\int_0^{\infty}\frac{\log(x^2+1)}{x^2+1}dx = \pi\log 2$$

$$\Rightarrow \int_0^{\infty}\frac{\log(x^2+1)}{x^2+1}dx = \pi\log 2.\ \text{(Ans)}$$

Solution–34(a). Consider $\oint_c f(z)\,dz$, where $f(z) = \frac{z^a}{(1+z^2)^2}$ and C is the closed contour consisting of

(i) The x-axis from $-R$ to R, where R is large.

(ii) The upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of f(z) are obtained by solving the equation [$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \frac{z^a}{(1+z^2)^2}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

f(z) এর পোলগুলি নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$1 + z^2 = 0$$

$$\Rightarrow z = \pm i$$

The only pole $z = i$ lies inside the contour which is a pole of order two. [একমাত্র পোল $z = i$ কন্টুরের ভিতর অবস্থিত যাহা দুই ক্রমের পোল]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim\limits_{z \to i} \frac{1}{\lfloor 1} \frac{d}{dz} \{(z-i)^2 f(z)\}$

$$= \lim_{z \to i} \frac{d}{dz} \left\{(z-i)^2 \cdot \frac{z^a}{(1+z^2)^2}\right\} = \lim_{z \to i} \frac{d}{dz} \left\{\frac{(z-i)^2 z^a}{(z-i)^2 (z+i)^2}\right\}$$

$$= \lim_{z \to i} \frac{d}{dz} \left\{\frac{z^a}{(z+i)^2}\right\} = \lim_{z \to i} \frac{(z+i)^2 a z^{a-1} - 2z^a (z+i)}{(z+i)^4}$$

$$= \lim_{z \to i} \frac{a(z+i) z^{a-1} - 2z^a}{(z+i)^3}$$

$$= \frac{a(2i)(i)^{a-1} - 2(i)^a}{(2i)^3} = \frac{2i^a (a-1)}{-8i}$$

$$= \frac{1-a}{4i} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^a \qquad \because i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$= \frac{1-a}{4i} e^{i\pi a/2}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_c f(z)\, dz = 2\pi i \text{ (sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz = 2\pi i \left[\frac{1-a}{4i} e^{i\pi a/2}\right] \ldots\ldots (1)$$

Now when $R \to \infty$ then [যখন $R \to \infty$ তখন] $\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{x^a}{(1+x^2)^2} dx$

Also, $\lim\limits_{z \to \infty} z f(z) = \lim\limits_{z \to \infty} \frac{z \cdot z^a}{(1+z^2)^2} = 0$

Thus [অতএব], $\lim\limits_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0$

Therefore, from (1) by taking limit $R \to \infty$ we get [অতএব, (1) হতে $R \to \infty$ লিমিট নিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{x^a}{(1+x^2)^2} dx + 0 = \frac{\pi}{2} (1-a) e^{i\pi a/2}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{x^a}{(1+x^2)^2} dx + \int_{0}^{\infty} \frac{x^a}{(1+x^2)^2} dx = \frac{\pi}{2} (1-a) e^{i\pi a/2} \ldots\ldots (2)$$

Let $x = -y$. Then $dx = -dy$ and when $x = -\infty, 0$ then $y = \infty, 0$.

The first integral of (2) becomes, [(2) এর প্রথম যোগজটি দাঁড়ায়]

$$\int_{-\infty}^{0} \frac{x^a}{(1+x^2)^2} dx = \int_{\infty}^{0} \frac{(-y)^a}{(1+y^2)^2} (-dy)$$

$$= \int_{0}^{\infty} \frac{y^a (-1)^a}{(1+y^2)^2} dy$$

$$= \int_{0}^{\infty} \frac{x^a (\cos \pi + i \sin \pi)^a}{(1+x^2)^2} dx; \quad \because \int_a^b f(x)\, dx = \int_a^b f(y)\, dy$$

$$= \int_{0}^{\infty} \frac{e^{i\pi a} x^a}{(1+x^2)^2} dx$$

Putting this value in (2) we get [(2) এ এইমান বসাইয়া পাই]

$$\int_{0}^{\infty} \frac{e^{i\pi a} x^a}{(1+x^2)^2} dx + \int_{0}^{\infty} \frac{x^a}{(1+x^2)^2} dx = \frac{\pi}{2} (1-a) e^{i\pi a/2}$$

$$\Rightarrow (e^{i\pi a} + 1) \int_{0}^{\infty} \frac{x^a}{(1+x^2)^2} dx = \frac{\pi}{2} (1-a) e^{i\pi a/2}$$

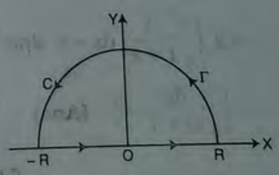
$$\Rightarrow \int_{0}^{\infty} \frac{x^a}{(1+x^2)^2} dx = \frac{\pi(1-a)}{2} \frac{e^{i\pi a/2}}{e^{i\pi a} + 1} = \frac{\pi(1-a)}{2} \frac{1}{e^{i\pi a/2} + e^{-i\pi a/2}}$$

$$= \frac{\pi(1-a)}{2} \frac{1}{2 \cos\left(\frac{\pi a}{2}\right)} = \frac{\pi(1-a)}{4} \sec\left(\frac{\pi a}{2}\right) \textbf{ (Ans)}$$

Solution–34(b). Consider $\oint_c f(z)\, dz$, where $f(z) = \frac{1}{1+z^2}$ and C is the contour consisting of

(i) The x-axis from $-R$ to R, where R is large

(ii) The upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of f(z) are obtained by solving the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{1}{1+z^2}$ এবং C কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

f(z) এর পোলগুলি নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$1 + z^2 = 0$$

$$\Rightarrow z = i, -i$$

The only pole $z = i$ lies in the contour which is a simple pole.

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim_{z \to i} (z - i) f(z)$ [একমাত্র পোল $z = i$ কন্টুরের ভিতরে অবস্থিত যাহা সরল পোল]

$$= \lim_{z \to i} (z - i) \cdot \frac{1}{1 + z^2} = \lim_{z \to i} \frac{(z - i)}{(z + i)(z - i)}$$

$$= \lim_{z \to i} \frac{1}{z + i} = \frac{1}{i + i} = \frac{1}{2i} = \frac{-i}{2}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_c f(z)\, dz = 2\pi i \quad \text{[sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz = 2\pi i \left[\frac{-i}{2}\right] \text{...... (1)}$$

when $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, du = \int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx$$

Also $\lim_{z \to \infty} z f(z) = \lim_{z \to \infty} \frac{z}{z^2 + 1} = 0$

Thus, (1) becomes, [অতএব, (1) দাঁড়ায়]

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx + 0 = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{1}{1 + x^2}\, dx = \pi, \text{ since } f(x) = \frac{1}{1 + x^2} \text{ is an even function.}$$

$$\therefore \int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2} \quad \textbf{(Ans)}$$

GROUP–C

Solution–35. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \dfrac{e^{iz}}{(z^2 + 1)(z^2 + 9)}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \dfrac{e^{iz}}{(z^2 + 1)(z^2 + 9)}$ can be found by solving the equation. [$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{iz}}{(z^2 + 1)(z^2 + 9)}$ এবং বদ্ধ কন্টুর C গঠিত :

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

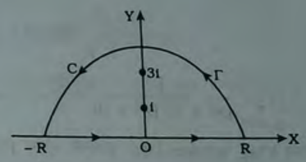
(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \dfrac{e^{iz}}{(z^2 + 1)(z^2 + 9)}$ এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$(z^2 + 1)(z^2 + 9) = 0$$

$$\therefore z^2 + 1 = 0 \Rightarrow z = \pm i$$

$$z^2 + 9 = 0 \Rightarrow z = \pm 3i$$



Only the simple pole $z = i$ and $z = 3i$ lie inside the contour. [একমাত্র $z = i$ এবং $z = 3i$ সরল পোল কন্টুরের ভিতর অবস্থিত]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim_{z \to i} (z - i) f(z)$

$$= \lim_{z \to i} \left\{(z - i) \cdot \frac{e^{iz}}{(z^2 + 1)(z^2 + 9)}\right\}$$

$$= \lim_{z \to i} \left\{(z - i) \cdot \frac{e^{iz}}{(z + i)(z - i)(z^2 + 9)}\right\}$$

$$= \lim_{z \to i} \frac{e^{iz}}{(z + i)(z^2 + 9)} = \frac{e^{-1}}{2i(i^2 + 9)} = \frac{e^{-1}}{16i}$$

Residue at $z = 3i$ is [$z = 3i$ এ অবশেষ] $\lim_{z \to 3i} (z - 3i) f(z)$

$$= \lim_{z \to 3i} \left\{(z - 3i) \cdot \frac{e^{iz}}{(z^2 + 1)(z^2 + 9)}\right\}$$

$$= \lim_{z \to 3i} \left\{ (z - 3i) \cdot \frac{e^{iz}}{(z^2 + 1)(z + 3i)(z - 3i)} \right\}$$

$$= \lim_{z \to 3i} \frac{e^{iz}}{(z^2 + 1)(z + 3i)} = \frac{e^{-3}}{(9i^2 + 1)\,6i} = \frac{e^{-3}}{-48i}$$

Sum of the residue [অবশেষগুলির যোগফল] $= \frac{e^{-1}}{16i} - \frac{e^{-3}}{48i} = \frac{3e^{-1} - e^{-3}}{48i}$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ [Sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz = 2\pi i \cdot \frac{3e^{-1} - e^{-3}}{48i} = \frac{\pi}{24}(3e^{-1} - e^{-3}) \text{ ...... (1)}$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)(x^2 + 9)}\, dx$$

$\lim_{z \to \infty} \frac{1}{(z^2 + 1)(z^2 + 9)} = \frac{1}{\infty} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where [$f(z)$ ফাংশন $e^{imz} F(z)$ আকারের যেখানে] $m = 1$ and $F(z) = \frac{1}{(z^2 + 1)(z^2 + 9)}$.

Therefore, by Jordan's lemma [অতএব, জর্ডানের লেমাদ্বারা]

$$\lim_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0$$

Now taking limit $R \to \infty$ in (1) and using the above results we get [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)(x^2 + 9)}\, dx + 0 = \frac{\pi}{24}(3e^{-1} - e^{-3})$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{(x^2 + 1)(x^2 + 9)}\, dx = \frac{\pi}{8}\left(\frac{e^{-1}}{1} - \frac{e^{-3}}{3}\right)$$

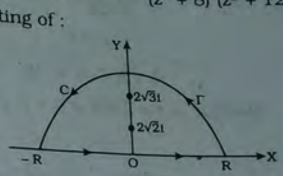
Equating real parts we get [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)(x^2 + 9)}\, dx = \frac{\pi}{8}\left(\frac{e^{-1}}{1} - \frac{e^{-3}}{3}\right). \text{ (Ans)}$$

Solution–36. Consider $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{iz}}{(z^2 + 8)(z^2 + 12)}$ and C is the closed contour consisting of :

(i) the axis from $-R$ to R, where R is large

(ii) the semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \frac{e^{iz}}{(z^2 + 8)(z^2 + 12)}$ are obtained from the equation.

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{e^{iz}}{(z^2 + 8)(z^2 + 12)}$ এবং বদ্ধ কন্টুর C গঠিত :

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{e^{iz}}{(z^2 + 8)(z^2 + 12)}$ এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$(z^2 + 8)(z^2 + 12) = 0$$

$$z^2 + 8 = 0 \Rightarrow z = \sqrt{-8} = \pm 2\sqrt{2}i$$

$$z^2 + 12 = 0 \Rightarrow z = \sqrt{-12} = \pm 2\sqrt{3}i$$

The poles are [পোলগুলি হল]

$z = 2\sqrt{2}i$, $z = -2\sqrt{2}i$, $z = 2\sqrt{3}i$, and $z = -2\sqrt{3}i$

Among these poles only the simple poles $z = 2\sqrt{2}i$ and $z = 2\sqrt{3}i$ lie inside the contour. [এই পোলগুলির মধ্যে মাত্র সরল পোল $z = 2\sqrt{2}i$ এবং $z = 2\sqrt{3}i$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = 2\sqrt{2}i$ is [$z = 2\sqrt{3}i$ এ অবশেষ] $\lim_{z \to 2\sqrt{2}i} (z - 2\sqrt{2}i) \cdot f(z)$

$$= \lim_{z \to 2\sqrt{2}i} \left\{ (z - 2\sqrt{2}i) \cdot \frac{e^{iz}}{(z^2 + 8)(z^2 + 12)} \right\}$$

$$= \lim_{z \to 2\sqrt{2}i} \left\{ (z - 2\sqrt{2}i) \cdot \frac{e^{iz}}{(z + 2\sqrt{2}i)(z - 2\sqrt{2}i)(z^2 + 12)} \right\}$$

$$= \lim_{z \to 2\sqrt{2}i} \frac{e^{iz}}{(z + 2\sqrt{2}i)(z^2 + 12)}$$

$$= \frac{e^{-2\sqrt{2}}}{4\sqrt{2}i(-8 + 12)} = \frac{e^{-2\sqrt{2}}}{16\sqrt{2}i}$$

Residue at $z = 2\sqrt{3}i$ is [$z = 2\sqrt{3}i$ এ অবশেষ] $\lim_{z \to 2\sqrt{3}i} (z - 2\sqrt{3}i) \cdot f(z)$

$$= \lim_{z \to 2\sqrt{3}i} \left\{(z - 2\sqrt{3}i) \cdot \frac{e^{iz}}{(z^2 + 8)(z + 2\sqrt{3}i)(z - 2\sqrt{3}i)}\right\}$$

$$= \lim_{z \to 2\sqrt{3}i} \frac{e^{iz}}{(z^2 + 8)(z + 2\sqrt{3}i)}$$

$$= \frac{e^{-2\sqrt{3}}}{(-12 + 8) \cdot 4\sqrt{3}i} = \frac{e^{-2\sqrt{3}}}{-16\sqrt{3}i}$$

Sum of the residues [অবশেষগুলির যোগফল]

$$= \frac{e^{-2\sqrt{2}}}{16\sqrt{2}i} - \frac{e^{-2\sqrt{3}}}{16\sqrt{3}i} = \frac{1}{16i}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}}\right)$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ [Sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz = 2\pi i \cdot \frac{1}{16i}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}}\right) \text{ ...... (1)}$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 8)(x^2 + 12)} dx$$

Here $\lim_{z \to \infty} \frac{1}{(z^2 + 8)(z^2 + 9)} = \frac{1}{\infty} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$ where [এবং $f(z)$ ফাংশন $e^{imz} F(z)$ আকারের] $m = 1$ and $F(z) = \frac{1}{(z^2 + 8)(z^2 + 12)}$.

Therefore, by Jordan's lemma we have [অতএব জডার্নের লেমাদ্বারা পাই] $\lim_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0$

Taking limit $R \to \infty$ in (1) and then using the above results we get, [(1) এ $R \to \infty$ লিমিট নিয়ে এবং তারপর উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 8)(x^2 + 12)} + 0 = \frac{\pi}{8}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{(x^2 + 8)(x^2 + 12)} = \frac{\pi}{8}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}}\right)$$

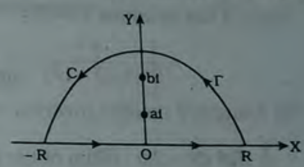
Equating real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 8)(x^2 + 12)} = \frac{\pi}{8}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}}\right) \text{ (Ans)}$$

Solution–37. Consider the integral $\oint_C f(z)$, where $f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R where R is large

(ii) the upper-semi circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$ can be obtained from the equation. [$\oint_C f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$ এর পোল নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$(z^2 + a^2)(z^2 + b^2) = 0$$

$$\Rightarrow (z + ai)(z - ai)(z - bi)(z + bi)$$

$$\Rightarrow z = ai, -ai, bi, -bi$$

$\therefore$ The poles are [পোলগুলি হল] $z = ai$, $z = -ai$, $z = bi$ and $z = -bi$.

Among these poles, the simple poles $z = ai$ and $z = bi$ lie inside the contour. [এই পোলগুলির মধ্যে সরল পোল $z = ai$ এবং $z = bi$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim\limits_{z\to ai} (z - ai) \cdot f(z)$

$$= \lim_{z\to ai} \left\{(z - ai) \cdot \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}\right\}$$

$$= \lim_{z\to ai} \left\{(z - ai) \cdot \frac{e^{iz}}{(z - ai)(z + ai)(z^2 + b^2)}\right\}$$

$$= \lim_{z\to ai} \frac{e^{iz}}{(z + ai)(z^2 + b^2)} = \frac{e^{-a}}{2ai(b^2 - a^2)}$$

Similarly, the residue at $z = bi$ is [একইভাবে, $z = bi$ এ অবশেষ]

$$\frac{e^{-b}}{2bi(a^2 - b^2)}$$

Sum of the residues [অবশেষগুলির যোগিফল]

$$= \frac{1}{2i}\left[\frac{e^{-b}}{b(a^2 - b^2)} - \frac{e^{-a}}{a(a^2 - b^2)}\right] = \frac{1}{2(a^2 - b^2)i}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বার পাই]

$$\oint_C f(z)\, dz = 2\pi i \quad [\text{Sum of the residues}]$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{1}{2(a^2 - b^2)i}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \quad \ldots\ldots (1)$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)}\, dx$$

$\lim\limits_{z\to\infty} \dfrac{1}{(z^2 + a^2)(z^2 + b^2)} = \dfrac{1}{\infty} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$ where [এবং $f(z)$ ফাংশন $e^{imz} F(z)$ আকারের] $m = 1$ and

$$F(z) = \frac{1}{(z^2 + a^2)(z^2 + b^2)}$$

Hence by Jordan's lemma [অতএব জর্ডানের লেমাদ্বারা পাই]

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0$$

Taking limit $R \to \infty$ in (1) and using the above results we get [(1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)}\, dx + 0 = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating the real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

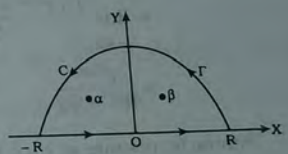
$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)}\, dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right). \quad \text{(Ans)}$$

Solution–38. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \dfrac{e^{i2\pi z}}{z^4 + z^2 + 1}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \dfrac{e^{i2\pi z}}{z^4 + z^2 + 1}$ can be found by solving the equation



[$\oint_C f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{i2\pi z}}{z^4 + z^2 + 1}$ এবং C বদ্ধ কন্টুর গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \dfrac{e^{i2\pi z}}{z^4 + z^2 + 1}$ এর পোল নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$z^4 + z^2 + 1 = 0$$

$$\Rightarrow (z^2 + 1)^2 - z^2 = 0$$

$$\Rightarrow (z^2 + z + 1)(z^2 - z + 1) = 0$$

$$z^2+z+1=0 \Rightarrow z=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$$

$$z^2-z+1=0 \Rightarrow z=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm i\sqrt{3}}{2}$$

The four poles are [চারটি পোল হল]

$$\frac{-1+i\sqrt{3}}{2},\ \frac{-1-i\sqrt{3}}{2},\ \frac{1+i\sqrt{3}}{2} \text{ and } \frac{1-i\sqrt{3}}{2}$$

Only the simple poles [একমাত্র সরল পোল] $z=\alpha=\dfrac{-1+i\sqrt{3}}{2}$ and $z=\beta$ $=\dfrac{1+i\sqrt{3}}{2}$

lie in the contour [কন্টুরের ভিতরে]

Residue at $z=\alpha$ is [$z=\alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha}(z-\alpha)\cdot f(z)$

$$=\lim_{z\to\alpha}\left\{(z-\alpha)\cdot\frac{e^{i2\pi z}}{z^4+z^2+1}\right\};\ \frac{0}{0} \text{ form}$$

$$=\lim_{z\to\alpha}\frac{1\cdot e^{i2\pi z}+i2\pi(z-\alpha)\cdot e^{i2\pi z}}{4z^3+2z};\ \text{by L. Hospital's rule}$$

$$=\frac{e^{i2\pi\alpha}}{4\alpha^3+2\alpha}$$

Similarly, residue at $z=\beta$ is [অনুরূপে, $z=\beta$ এ অবশেষ] $\dfrac{e^{i2\pi\beta}}{4\beta^3+2\beta}$

Now $i2\pi\alpha=2i\pi\cdot\dfrac{-1+i\sqrt{3}}{2}=-i\pi-\pi\sqrt{3}$

$$4\alpha^3+2\alpha=4\cdot\left(\frac{-1+i\sqrt{3}}{2}\right)^3+2\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$=4\cdot 1-1+i\sqrt{3}=3+i\sqrt{3}$$

$$4\beta^3+2\beta=4\left(\frac{1+i\sqrt{3}}{2}\right)^3+2\left(\frac{1+i\sqrt{3}}{2}\right)$$

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^3=\frac{-(1-3\sqrt{3}i+9i^2-3\sqrt{3}i^3)}{8}$$

$$=-\frac{1}{8}(1-3\sqrt{3}i-9+3i\sqrt{3})$$

$$=-\frac{1}{8}(-8)=1$$

$$=4(-1)+1+i\sqrt{3}=-3+i\sqrt{3}$$

$\therefore$ Sum of the residues [অবশেষগুলির যোগফল] $=\left(\dfrac{e^{i2\pi\alpha}}{4\alpha^3+2\alpha}+\dfrac{e^{i2\pi\beta}}{4\beta^3+2\beta}\right)$

$$=\left(\frac{e^{-i\pi-\pi\sqrt{3}}}{3+i\sqrt{3}}+\frac{e^{i\pi-\pi\sqrt{3}}}{-3+i\sqrt{3}}\right)$$

$$=\left(\frac{e^{-\pi\sqrt{3}}\cdot e^{-i\pi}}{3+i\sqrt{3}}-\frac{e^{-\pi\sqrt{3}}\cdot e^{i\pi}}{3-i\sqrt{3}}\right)$$

$$=e^{-\pi\sqrt{3}}\left(\frac{\cos\pi-i\sin\pi}{3+i\sqrt{3}}-\frac{\cos\pi+i\sin\pi}{3-i\sqrt{3}}\right)$$

$$=e^{-\pi\sqrt{3}}\left(\frac{-1-0}{3+i\sqrt{3}}-\frac{-1+0}{3-i\sqrt{3}}\right)$$

$$=e^{-\pi\sqrt{3}}\cdot\frac{-3+i\sqrt{3}+3+i\sqrt{3}}{3^2-3i^2}=\frac{e^{-\pi\sqrt{3}}}{12}\cdot 2i\sqrt{3}$$

$$=\frac{i\sqrt{3}}{6}e^{-\pi\sqrt{3}}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz=2\pi i.\ \text{[Sum of the residues]}$$

$$\Rightarrow\int_{-R}^{R}f(z)\,dz+\int_\Gamma f(z)\,dz=2\pi i\cdot\frac{i\sqrt{3}}{6}e^{-\pi\sqrt{3}}\ \ldots\ldots(1)$$

When $R\to\infty$ then [যখন $R\to\infty$ তখন]

$$\int_{-R}^{R}f(z)\,dz=\int_{-\infty}^{\infty}f(x)\,dx=\int_{-\infty}^{\infty}\frac{e^{i2\pi x}}{x^4+x^2+1}dx$$

Also, $\lim\limits_{z\to\infty}\dfrac{1}{z^4+z^2+1}=\dfrac{1}{\infty}=0$ and $f(z)$ is a function of the form $e^{imz}F(z)$, where [এবং $f(z)$ ফাংশন $e^{imz}F(z)$ আকারের] $m=2\pi$ and $F(z)=\dfrac{1}{z^4+z^2+1}$.

Hence by Jordan's lemma we have [অতএব জর্ডানের লেমাদ্বারা পাই]

$$\lim_{R\to\infty}\int_\Gamma f(z)\,dz=0$$

Taking limit $R \to \infty$ in (1) and then using the above results we get [(1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} f(x)\, dx + 0 = \frac{-\pi\sqrt{3}}{3} e^{-\pi\sqrt{3}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{i2\pi x}}{x^4 + x^2 + 1} dx = \frac{-\pi\sqrt{3}}{3} e^{-\pi\sqrt{3}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos 2\pi x + i \sin 2\pi x}{x^4 + x^2 + 1} dx = \frac{-\pi\sqrt{3}}{3} e^{-\pi\sqrt{3}}$$

Equating real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx = \frac{-\pi\sqrt{3}}{3} e^{-\pi\sqrt{3}}$$

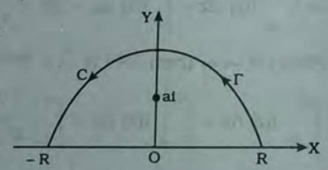
$$\Rightarrow 2\int_{-\infty}^{\infty} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx = \frac{-\pi}{\sqrt{3}} e^{-\pi\sqrt{3}}$$

$$\therefore \int_{0}^{\infty} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx = \frac{-\pi}{2\sqrt{3}} e^{-\pi\sqrt{3}} \text{ (Ans)}$$

Solution–39. Consider $\oint_C f(z)\, dz$, where $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of f(z) are obtained from the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$ এবং C বদ্ধ কন্টুরটি গঠিত

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

f(z) এর পোল সমূহ নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$z^2 + a^2 = 0$$

$$\Rightarrow z^2 = -a^2 = (ai)^2$$

$$\Rightarrow z = \pm ai$$

Only the simple pole $z = ai$ lies inside the contour. [একমাত্র সরল পোল $z = ai$ কন্টুরের ভিতর অবস্থিত]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim_{z \to ai} (z - ai) \cdot f(z)$

$$= \lim_{z \to ai} (z - ai) \frac{e^{imz}}{z^2 + a^2} = \lim_{z \to ai} (z - ai) \frac{e^{imz}}{(z + ai)(z - ai)}$$

$$= \lim_{z \to ai} \frac{e^{imz}}{z + ai} = \frac{e^{i^2 ma}}{2ai} = \frac{e^{-ma}}{2ai}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \quad \text{[Residue at } z = ai]$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{e^{-ma}}{2ai} \text{ ...... (1)}$$

Here $\lim_{z \to \infty} \dfrac{1}{z^2 + a^2} = \dfrac{1}{\infty} = 0$ and f(z) is a function of the form $e^{imz} F(z)$, where [এবং f(z) ফাংশনটি $e^{imz} F(z)$ আকারের যেখানে] $F(z) = \dfrac{1}{z^2 + a^2}$.

Hence by Jordan's lemma we have [অতএব জর্ডানের লেমাদ্বারা পাই]

$$\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2}$$

Now taking limit $R \to \infty$ in (1) and using the above results we get, [(1) এ $R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} dx + 0 = \frac{\pi}{a} e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma}$$

Equating real and imaginary parts we get, [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma} \text{ and } \int_{-\infty}^{\infty} \frac{\sin mx}{x^2 + a^2} dx = 0$$

$$\Rightarrow 2\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-ma} \Rightarrow \int_0^{\infty}\frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a}\,e^{-ma}$$

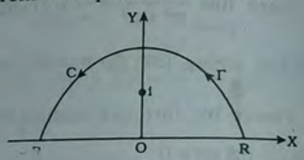
$$\left.\begin{aligned}\therefore \int_0^{\infty}\frac{\cos mx}{x^2+a^2}\,dx &= \frac{\pi}{2a}e^{-ma}\\ \text{and}\int_{-\infty}^{\infty}\frac{\sin mx}{x^2+a^2}\,dx &= 0\end{aligned}\right\}\text{(Ans)}$$

Solution–40. Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{e^{iaz}}{z^2+1}$ and C is the closed contour consisting of

(i) the x-axis from – R to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of f(z) are obtained from the equation $z^2 + 1 = 0$



[$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি যেখানে $f(z) = \dfrac{e^{iaz}}{z^2+1}$. বদ্ধ কন্টুর C গঠিত হবে

(i) x অক্ষ, – R হতে R, যেখানে R বৃহৎ।

(ii) $|z| = R$ বৃত্তের উর্ধ অর্ধবৃত্ত Γ, যাহা x অক্ষের উপরে অবস্থিত।

f(z) এর পোল $z^2 + 1 = 0$ সমীকরণ হতে পাওয়া যাবে।]

$$\Rightarrow z^2 = -1 = i^2$$

$$\Rightarrow z = \pm i$$

Only the pole z = i lies inside the contour which is a simple pole. [একমাত্র পোল z = i কন্টুরের ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at z = i is [z = i এ অবশেষ]

$$\lim_{z\to i}(z-i)\cdot f(z) = \lim_{z\to i}\left\{(z-i)\cdot\frac{e^{iaz}}{z^2+1}\right\}$$

$$= \lim_{z\to i}\left\{(z-i)\cdot\frac{e^{iaz}}{(z+i)(z-i)}\right\} = \lim_{z\to i}\frac{e^{iaz}}{z+i} = \frac{e^{i^2a}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i\ \text{[Residue at } z = i]$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz = 2\pi i\cdot\frac{e^{-a}}{2i}\ \ldots\ldots (1)$$

Here [এখানে] $\lim_{z\to\infty}\dfrac{1}{z^2+1} = \dfrac{1}{\infty} = 0$ and f(z) is a function of the form $e^{iaz}F(z)$, where $F(z) = \dfrac{1}{z^2+1}$. Hence by Jordan's lemma we have

$$\lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 0.$$

[এবং f(z) ফাংশন $e^{imz}F(z)$ আকারের, যেখানে] $F(z) = \dfrac{1}{z^2+1}$.

[অতএব জর্ডান লিমার দ্বারা পাই] $\lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 0$]

When [যখন] $R\to\infty$ then [তখন] $\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}\,dx$$

Now taking limit $R\to\infty$ in (1) and using the above results we get [এখানে (1) এ $R\to\infty$ লিমিট নিয়ে এবং উপরের ফলাফল ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}\,dx + 0 = \pi e^{-a} \Rightarrow \int_{-\infty}^{\infty}\frac{\cos ax + i\sin ax}{x^2+1}\,dx = \pi e^{-a}$$

Equating real and imaginary parts we get [বাস্তব এবং কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty}\frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}\ \text{and}\int_{-\infty}^{\infty}\frac{\sin ax}{x^2+1}\,dx = 0$$

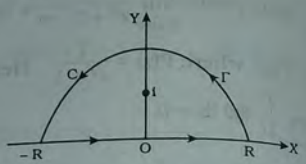
$$\Rightarrow 2\int_0^{\infty}\frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \Rightarrow \int_0^{\infty}\frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}$$

$$\left.\begin{aligned}\therefore \int_0^{\infty}\frac{\cos ax}{x^2+1}\,dx &= \frac{\pi}{2}e^{-a}\\ \text{and}\int_{-\infty}^{\infty}\frac{\sin ax}{x^2+1}\,dx &= 0\end{aligned}\right\}\text{(Ans)}$$

Solution–41. Consider $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{i2z}}{z^2+1}$ and C is the closed contour consisting of

(i) the x-axis from – R to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \frac{e^{i2z}}{z^2+1}$ are obtained from the equation

[$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \frac{e^{i2z}}{z^2+1}$ এবং C বদ্ধ কন্টুরটি গঠিতঃ

(i) x অক্ষ, – R হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{e^{i2z}}{z^2+1}$ এর পোল নিম্নের সমীকরণ হতে পাওয়া যায়।]

$z^2 + 1 = 0$

$\Rightarrow z^2 = -1 = i^2$

$\Rightarrow z = \pm i$

Only the pole $z = i$ lies inside the contour which is a simple pole. [একমাত্র পোল $z = i$ কন্টুরের ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z = i$ is[$z = i$ এ অবশেষ]

$$\lim_{z\to i} (z-i)\, f(z) = \lim_{z\to i} (z-i)\frac{e^{i2z}}{z^2+1}$$

$$= \lim_{z\to i} (z-i)\frac{e^{i2z}}{(z+i)(z-i)} = \lim_{z\to i}\frac{e^{i2z}}{z+i} = \frac{e^{-2}}{2i}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \cdot [\text{Residue at } z = i]$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{e^{-2}}{2i} = \pi e^{-2} \ \ldots\ldots (1)$$

Here $\lim_{z\to\infty} \frac{1}{z^2+1} = \frac{1}{\infty} = 0$ and f(z) is a function of the form $e^{i2z} F(z)$ where [এবং f(z) ফাংশন $e^{imz} F(z)$ আকারের, যেখানে] $F(z) = \frac{1}{z^2+1}$.

Hence by Jordan's lemma we have [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{e^{i2x}}{x^2+1}\, dx$$

Now taking limit $R \to \infty$ in (1) and then putting the above results we get [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^2+1}\, dx + 0 = \pi e^{-2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos 2x + i \sin 2x}{x^2+1} = \pi e^{-2}$$

Equating real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1} = \pi e^{-2}$$

$$\Rightarrow 2\int_0^{\infty} \frac{\cos 2x}{x^2+1}\, dx = \pi e^{-2};$$ By property of definite integral.

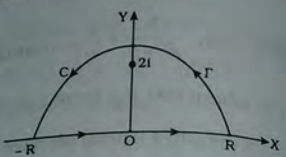
$$\therefore \int_0^{\infty} \frac{\cos 2x}{x^2+1}\, dx = \frac{\pi e^{-2}}{2} \quad \text{(Ans)}$$

Solution–42. Consider $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{i3z}}{z^2+4}$ and C is the closed contour consisting of

(i) the x-axis from – R to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \frac{e^{i3z}}{z^2+4}$ are obtained from the equation



[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \frac{e^{i3z}}{z^2+4}$ এবং C বদ্ধ কন্টুরটি গঠিতঃ

(i) x অক্ষ, – R হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{e^{i3z}}{z^2+4}$ এর পোল নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$z^2 + 4 = 0$$
$$\Rightarrow z^2 = -4 = (2i)^2$$
$$\Rightarrow z = \pm 2i$$

Only the simple pole $z = 2i$ lies inside the contour [একমাত্র সরল পোল $z = 2i$ কন্টুরের ভিতর অবস্থিত]

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z\to 2i} (z-2i)\, f(z)$

$$= \lim_{z\to 2i} (z-2i)\frac{e^{i3z}}{z^2+4}$$
$$= \lim_{z\to 2i} (z-2i)\cdot\frac{e^{i3z}}{(z+2i)(z-2i)}$$
$$= \lim_{z\to 2i} \frac{e^{i3z}}{z+2i} = \frac{e^{-6}}{4i}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \quad [\text{Residue at } z = 2i]$$
$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\cdot\frac{e^{-6}}{4i} = \frac{\pi e^{-6}}{2} \;\ldots\ldots (1)$$

When $R\to\infty$ then [যখন $R\to\infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{e^{i3x}}{x^2+4}\,dx$$

Here, $\lim\limits_{z\to\infty}\frac{1}{z^2+4} = \frac{1}{\infty} = 0$ and f(z) is a function of the form $e^{i3z}F(z)$, where [এবং f(z) ফাংশন $e^{imz}F(z)$ আকারের, যেখানে] $F(z) = \frac{1}{z^2+4}$.

Hence by Jordan's lemma we have [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

Now taking limit $R\to\infty$ in (1) and then putting the above results we get, [এখন (1) এ $R\to\infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty}\frac{e^{i3x}}{x^2+4}\,dx + 0 = \frac{\pi e^{-6}}{2}$$
$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos 3x + i\sin 3x}{x^2+4}\,dx = \frac{\pi e^{-6}}{2}$$

Equating real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

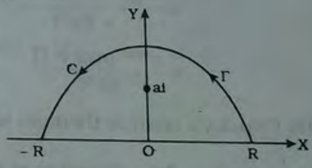
$$\int_{-\infty}^{\infty}\frac{\cos 3x}{x^2+4}\,dx = \frac{\pi e^{-6}}{2}.$$
$$\Rightarrow 2\int_0^{\infty}\frac{\cos 3x}{x^2+4}\,dx = \frac{\pi e^{-6}}{2}, \text{ By property of definite integral.}$$
$$\Rightarrow \int_0^{\infty}\frac{\cos 3x}{x^2+4}\,dx = \frac{\pi e^{-6}}{4}. \text{ (Ans)}$$

Solution–43. Consider $\oint_C f(z)\,dz$, where $f(z) = \frac{e^{imz}}{(z^2+a^2)^2}$ and C is the closed contour consisting of

(i) the x-axis from – R to R, R is very large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \frac{e^{imz}}{(z^2+a^2)^2}$ are obtained from the equation [$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \frac{e^{imz}}{(z^2+a^2)^2}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ



(i) x অক্ষ, – R হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{e^{imz}}{(z^2+a^2)^2}$ এর পোল সমূহ নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$(z^2+a^2)^2 = 0$$

$$(z+ia)^2 (z-ia)^2 = 0$$

$$\Rightarrow z = ai, -ai$$

Only the double pole $z = ai$ lies inside the contour. [একমাত্র দ্বিপোল $z = ai$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim\limits_{z\to ai} \frac{1}{1!}\frac{d}{dz}\{(z-ai)^2 \cdot f(z)\}$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{(z-ai)^2 \cdot \frac{e^{imz}}{(z^2+a^2)^2}\right\}$$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{(z-ai)^2 \cdot \frac{e^{imz}}{(z+ai)^2 (z-ai)^2}\right\}$$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{\frac{e^{imz}}{(z+ai)^2}\right\}$$

$$= \lim_{z\to ai} \frac{(z+ai)^2 \cdot im\, e^{imz} - e^{imz} \cdot 2(z+ai)}{(z+ai)^4}$$

$$= \lim_{z\to ai} \frac{e^{imz}(imz + ami^2 - 2)}{(z+ai)^3}$$

$$= \frac{e^{-ma}(-am - am - 2)}{-8a^3 i}$$

$$= \frac{-2e^{-ma}(am+1)}{-8a^3 i}$$

$$= \frac{e^{-ma}(ma+1)}{4a^3 i}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i. \quad [\text{Residue at } z = ai]$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i \cdot \frac{e^{-ma}(ma+1)}{4a^3 i} = \frac{\pi e^{-ma}}{2a^3}(ma+1) \quad \cdots (1)$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}\,dx$$

Here $\lim\limits_{z\to\infty} \frac{1}{(z^2+a^2)^2} = \frac{1}{\infty} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where [এবং $f(z)$ ফাংশন $e^{imz} F(z)$ আকারের, যেখানে]

$$F(z) = \frac{1}{(z^2+a^2)^2}.$$

Hence by Jordun's lemma [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0.$$

Now taking limit $R \to \infty$ in (1) and then setting the above results we get, [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}\,dx + 0 = \frac{\pi e^{-ma}}{2a^3}(ma+1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{(x^2+a^2)^2}\,dx = \frac{\pi e^{-ma}}{2a^3}(ma+1)$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{\pi e^{-ma}}{2a^3}(ma+1) \text{ and } \int_{-\infty}^{\infty} \frac{\sin mx}{(x^2+a^2)^2}\,dx = 0$$

$$\Rightarrow 2\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)}\,dx = \frac{\pi e^{-ma}}{2a^3}(ma+1)$$

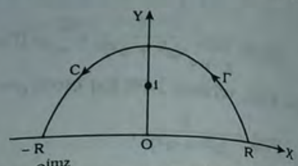
$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{\pi e^{-ma}}{4a^3}(ma+1)$$

$$\left.\begin{aligned} \therefore \int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx &= \frac{\pi e^{-ma}}{4a^3}(ma+1) \\ \text{and } \int_{-\infty}^{\infty} \frac{\sin mx}{(x^2+a^2)^2}\,dx &= 0 \end{aligned}\right\} \text{(Ans)}$$

Solution–44. Consider $\oint_C f(z)\,dz$, where $f(z) = \frac{e^{imz}}{(z^2+1)^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis



The poles of $f(z) = \dfrac{e^{imz}}{(z^2+1)^2}$ are obtained from the equation

[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{imz}}{(z^2+1)^2}$ এবং C বদ্ধ কন্টুরটি গঠিত :

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \dfrac{e^{imz}}{(z^2+1)^2}$ এর পোল সমূহ নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$(z^2+1)^2 = 0$$

$$\{(z+i)(z-i)\}^2 = 0$$

$$\Rightarrow z = i,\ z = -i$$

The double pole $z = i$ lies inside the contour. [দ্বিপোল $z = i$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim\limits_{z\to i} \dfrac{1}{1!}\dfrac{d}{dz}\{(z-i)^2 \cdot f(z)\}$

$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{e^{imz}}{(z^2+1)^2}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \frac{e^{imz}}{(z+i)^2 (z-i)^2}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{e^{imz}}{(z+i)^2}\right\}$$

$$= \lim_{z\to i} \frac{(z+i)^2 \cdot ime^{imz} - e^{imz} \cdot 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{e^{imz}\,(izm - m - 2)}{(z+i)^3}$$

$$= \frac{e^{-m}(-m-m-2)}{8i^3} = \frac{-2e^{-m}\,(m+1)}{-8i}$$

$$= \frac{e^{-m}}{4i}(m+1)$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i.\ \text{[Residue at } z = i]$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i \cdot \frac{e^{-m}(m+1)}{4i} = \frac{\pi(m+1)e^{-m}}{2} \quad \text{...... (1)}$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+1)^2}\,dx$$

Also, $\lim\limits_{z\to\infty} \dfrac{1}{(z^2+1)^2} = \dfrac{1}{\infty} = 0.$

Thus $f(z)$ is a function of the form $e^{imz} F(z)$, where [এবং $f(z)$ ফাংশন $e^{imz} F(z)$ আকারের, যেখানে]

$F(z) = \dfrac{1}{(z^2+1)^2}$. Hence by Jordan's lemma [অতএব জর্ডান লিমার দ্বারা পাই] $\lim\limits_{R\to\infty} \int_\Gamma f(z)\,dz = 0.$

Now taking limit $R \to \infty$ in (1) and then putting the above results we get, [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+1)^2}\,dx + 0 = \frac{\pi(m+1)e^{-m}}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{(x^2+1)^2}\,dx = \frac{\pi}{2}(m+1)\,e^{-m}$$

Equating real parts we get [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi}{2}(m+1)\,e^{-m}$$

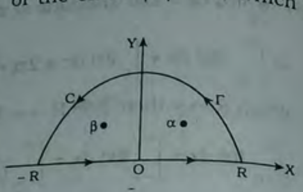
$$\Rightarrow 2\int_0^{\infty} \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi}{2}(m+1)\,e^{-m}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi}{4}(m+1)\,e^{-m} \quad \text{(Ans)}$$

Solution–45. Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{e^{imz}}{z^4+a^4}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \dfrac{e^{imz}}{z^4 + a^4}$ are obtained by solving the equation [$\oint_C f(z)\, dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{imz}}{z^4 + a^4}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \dfrac{e^{imz}}{z^4 + a^4}$ এর পোলগুলি নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$z^4 + a^4 = 0$$
$$\Rightarrow z^4 = -a^4$$
$$\Rightarrow z = a(-1)^{1/4}$$
$$= a[\cos\pi + i\sin\pi]^{1/4}$$
$$= a[\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{1/4}$$
$$= a\left[\cos\frac{(2n+1)\pi}{4} + i\sin\left(\frac{2n+1}{4}\right)\pi\right]$$
$$= ae^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3$$

The poles are [পোলগুলি হল] $z = ae^{i\pi/4}$, $z = ae^{i3\pi/4}$, $z = ae^{i5\pi/4}$ and $z = a^{i7\pi/4}$. The amplitude of the first two poles are greater than 0 and less

than π. So $z = ae^{i\pi/4}$ and $z = ae^{i3\pi/4}$ lie inside the contour which are simple pole. Let $z = ae^{i\pi/4} = \alpha$ and $z = ae^{i3\pi/4} = \beta$.

[প্রথম দুইটি পোলের কোনাঙ্ক শূন্য হতে বড় এবং π হতে ছোট। সুতরাং $Z = ae^{i\pi/4}$ এবং $z = ae^{i3\pi/4}$ কন্টুরের ভিতরে অবস্থিত যাহা, সরল পোল। ধরি $Z = ae^{i\pi/4} = \alpha$ এবং $z = ae^{i3\pi/4} = \beta$]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha} (z - \alpha)\, f(z)$

$$= \lim_{z\to\alpha} (z-\alpha)\frac{e^{imz}}{z^4 + a^4}\,;\ \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{z\to\alpha} \frac{1 \cdot e^{imz} + (z-\alpha)\cdot ime^{imz}}{4z^3}\,; \text{ by L. Hospital's rule}$$
$$= \frac{e^{im\alpha} + 0}{4\alpha^3} = \frac{e^{im\alpha}}{4\alpha^3}$$

Similarly, the residue at $z = \beta$ is [একইভাবে, $z = \beta$ এ অবশেষ] $\dfrac{e^{im\beta}}{4\beta^3}$

Now $\alpha = ae^{i\pi/4} = a\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right) = \dfrac{a(1+i)}{\sqrt{2}}$

$$\beta = ae^{i3\pi/4} = a\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$= a\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{a(-1+i)}{\sqrt{2}}$$
$$\alpha^3 = (ae^{i\pi/4})^3 = a^3 e^{i3\pi/4} = -a^3 e^{-i\pi/4}$$
$$\beta^3 = (ae^{i3\pi/4})^3 = a^3 e^{i9\pi/4} = a^3 e^{i\pi/4}$$

$\therefore$ Sum of the residues [অবশেষগুলির যোগফল] $= \dfrac{e^{im\alpha}}{4\alpha^3} + \dfrac{e^{im\beta}}{4\beta^3}$

$$= \frac{e^{ima(1+i)/\sqrt{2}}}{-4a^3 e^{-i\pi/4}} + \frac{e^{ima(-1+i)/\sqrt{2}}}{4a^3 e^{i\pi/4}}$$
$$= \frac{-1}{4a^3}\left[e^{-ma/\sqrt{2}} \cdot e^{i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)} - e^{-ma/\sqrt{2}} \cdot e^{-i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)}\right]$$
$$= \frac{-e^{-ma/\sqrt{2}}}{4a^3}\left[e^{-i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)} - e^{-i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)}\right]$$
$$= \frac{-e^{-ma/\sqrt{2}}}{4a^3} \cdot 2i\sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

Now by Cauchy's residue theorem we have [এখন কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i.\ \text{(Sum of the residues)}$$
$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{-e^{-ma/\sqrt{2}}}{4a^3} \cdot 2i\sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$
$$= \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \cdot \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right) \text{...... (1)}$$

When $R \to \infty$ then [যখন $R \to \infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+a^4}$$

Also, $\lim\limits_{z\to\infty} \frac{1}{z^4+a^4} = \frac{1}{\infty} = 0$ and f(z) is a function of the form $e^{imz} \cdot F(z)$, where [এবং f(z) ফাংশন e^{imz} F(z) আকারের, যেখানে]

$F(z) = \frac{1}{z^4+a^4}$. Hence by Jordan's lemma we have [অতএব জর্ডন লিমার দ্বারা পাই]

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

Taking limit $R \to \infty$ in (1) and then using the above results we get, [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+a^4} + 0 = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \cdot \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x^4+a^4}\,dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \cdot \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

Equating real parts we get, [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4+a^4}\,dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \cdot \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\cos mx}{x^4+a^4}\,dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \cdot \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

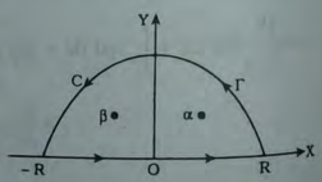
$$\Rightarrow \int_{0}^{\infty} \frac{\cos mx}{x^4+a^4}\,dx = \frac{\pi}{2a^3} e^{-ma/\sqrt{2}} \cdot \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right). \text{ (Ans)}$$

Solution-46. Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \frac{ze^{imz}}{z^4+a^4}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \frac{ze^{imz}}{z^4+a^4}$ can be obtained by solving the equation



[$\oint_C$ f(z) dz যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{ze^{imz}}{z^4+a^4}$ এবং বদ্ধ কন্টুর C গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের ঊর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের ঊর্ধ্বে অবস্থিত।

$f(z) = \frac{ze^{imz}}{z^4+a^4}$ এর পোলগুলি নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$z^4 + a^4 = 0$$

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = a(-1)^{1/4} = a(\cos\pi + i\sin\pi)^{1/4}$$

$$= a\left[\cos\left(\frac{2n+1}{4}\right)\pi + i\sin\left(\frac{2n+1}{4}\right)\pi\right]$$

$$= ae^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

$\therefore$ The poles are [পোলগুলি হল] $z = ae^{i\pi/4}$, $z = ae^{i3\pi/4}$, $z = ae^{i5\pi/4}$ and $z = ae^{i7\pi/4}$.

The amplitues of the first two poles are greater than zero and less than π. Therefore, the simple poles lie in the contour are $z = \alpha = ae^{i\pi/4}$ and $z = \beta = ae^{i3\pi/4}$. [প্রথম দুইটি পোলের কোনাঙ্ক শূন্য হতে বড় এবং π হতে ছোট। অতএব, কন্টুরের ভিতর অবস্থিত পোলগুলি $z = \alpha = ae^{i\pi/4}$ এবং $z = \beta = e^{i3\pi/4}$]

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha} (z-\alpha)\cdot f(z)$

$$= \lim_{z\to\alpha} \left\{(z-\alpha)\cdot\frac{ze^{imz}}{z^4+a^4}\right\}; \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{z\to\alpha} \frac{1\cdot ze^{imz} + (z-\alpha)\,(1\cdot e^{imz} + imze^{imz})}{4z^3};$$

by L. Hospital's rule

$$= \frac{\alpha e^{im\alpha} + 0}{4\alpha^3} = \frac{e^{im\alpha}}{4\alpha^2}$$

Similarly, the residue at $z = \beta$ is $\frac{e^{im\beta}}{4\beta^2}$

[একইভাবে, $z = \beta$ এ অবশেষ $\frac{e^{im\beta}}{4\beta^2}$]

Now $\alpha = ae^{i\pi/4} = a\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = a(1+i)/\sqrt{2}$

$$\beta = ae^{i3\pi/4} = a\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= a\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = a(-1+i)/\sqrt{2}$$

$$\alpha^2 = a^2\frac{(1+i)^2}{2} = \frac{a^2}{2}(1+2i+i^2) = \frac{a^2}{2}\cdot 2i = ia^2$$

$$\beta^2 = \frac{a^2}{2}(-1+i)^2 = \frac{a^2}{2}(1-2i+i^2) = \frac{a^2}{2}(-2i) = -ia^2$$

$\therefore$ Sum of the residues [অবশেষগুলির যোগফল] $= \dfrac{e^{im\alpha}}{4\alpha^2} + \dfrac{e^{im\beta}}{4\beta^2}$

$$= \frac{e^{ima(1+i)/\sqrt{2}}}{4ia^2} + \frac{e^{ima(-1+i)/\sqrt{2}}}{-4ia^2}$$

$$= \frac{1}{4ia^2}\left[e^{-ma/\sqrt{2}}\cdot e^{ima/\sqrt{2}} - e^{-ma/\sqrt{2}}\cdot e^{-ima/\sqrt{2}}\right]$$

$$= \frac{e^{-ma/\sqrt{2}}}{4ia^2}\left[e^{ima/\sqrt{2}} - e^{-ima/\sqrt{2}}\right]$$

$$= \frac{e^{-ma/\sqrt{2}}}{4ia^2}\cdot 2i\sin\frac{ma}{\sqrt{2}}$$

$$= \frac{e^{-ma/\sqrt{2}}}{2a^2}\sin\frac{ma}{\sqrt{2}}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i\ [\text{Sum of the residues}]$$

$$\int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z) = 2\pi i\cdot\frac{e^{-ma/\sqrt{2}}}{2a^2}\sin\frac{ma}{\sqrt{2}} \quad \ldots\ldots (1)$$

When $R\to\infty$ then [যখন $R\to\infty$ তখন]

$$\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{xe^{imx}}{x^4+a^4}\,dx$$

Also, $\lim\limits_{z\to\infty}\dfrac{z}{z^4+a^4} = \lim\limits_{z\to\infty}\dfrac{1}{z^3\left(1+\dfrac{a^4}{z^4}\right)} = \dfrac{1}{\infty} = 0$ and $f(z)$ is a function of the form $e^{imz}F(z)$, where [এবং $f(z)$ ফাংশন $e^{imz}F(z)$ আকারের, যেখানে] $F(z) = \dfrac{z}{z^4+a^4}$.

Hence by Jordan's lemma [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

Taking limit $R\to\infty$ in (1) and then using the above results we get [(1) এ $R\to\infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty}\frac{xe^{imx}}{x^4+a^4}\,dx + 0 = \frac{\pi i}{a^2}e^{-ma/\sqrt{2}}\cdot\sin\frac{ma}{\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{x(\cos mx + i\sin mx)}{x^4+a^4}\,dx = \frac{\pi i}{a^2}e^{-ma/\sqrt{2}}\cdot\sin\frac{ma}{\sqrt{2}}$$

Equating imaginary parts we get, [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^4+a^4}\,dx = \frac{\pi}{a^2}e^{-ma/\sqrt{2}}\cdot\sin\frac{ma}{\sqrt{2}}$$

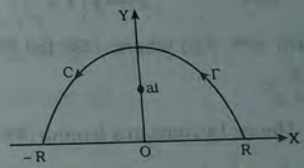
$$\Rightarrow 2\int_0^{\infty}\frac{x\sin mx}{x^4+a^4}\,dx = \frac{\pi}{a^2}e^{-ma/\sqrt{2}}\cdot\sin\frac{ma}{\sqrt{2}} \quad \text{[By property of definite integral]}$$

$$\Rightarrow \int_0^{\infty}\frac{x\sin mx}{x^4+a^4}\,dx = \frac{\pi}{2a^2}e^{-ma/\sqrt{2}}\cdot\sin\frac{ma}{\sqrt{2}}. \quad \text{(Ans)}$$

Solution–47. Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \dfrac{ze^{imz}}{z^2+a^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \dfrac{ze^{imz}}{z^2+a^2}$ will be obtained by solving the equation [$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \dfrac{ze^{imz}}{z^2+a^2}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, – R হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \dfrac{ze^{imz}}{z^2+a^2}$ এর পোলগুলি নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$z^2 + a^2 = 0 \Rightarrow z = \pm ai$$

Only the simple pole $z = ai$ lies inside the countour.
[একমাত্র সরল পোল $z = ai$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim_{z \to ai} (z - ai)\, f(z)$

$$= \lim_{z \to ai} \left\{(z - ai) \cdot \frac{ze^{imz}}{z^2 + a^2}\right\}$$

$$= \lim_{z \to ai} \left\{(z - ai) \cdot \frac{ze^{imz}}{(z + ai)\,(z - ai)}\right\}$$

$$= \lim_{z \to ai} \frac{ze^{imz}}{z + ai}$$

$$= \frac{aie^{-ma}}{2ai} = \frac{e^{-ma}}{2}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ [Sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{e^{-ma}}{2} \;\cdots\cdots (1)$$

When [যখন] $R \to \infty$ then [তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2}\, dx$$

$\lim_{z \to \infty} \frac{z}{z^2 + a^2} = \lim_{z \to \infty} \frac{1}{z\left(1 + \frac{a^2}{z^2}\right)} = \frac{1}{\infty} = 0$ and $f(z)$ is a function of the form e^{imz}. $F(z)$ where [এবং $f(z)$ ফাংশন e^{imz} $F(z)$ আকারের, যেখানে] $F(z) = \frac{z}{z^2 + a^2}$.

Hence by Jordan's lemma [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0$$

Taking limit $R \to \infty$ in (1) and then putting the above results we get, [(1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2}\, dx + 0 = i\pi e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos mx + i \sin mx)}{x^2 + a^2}\, dx = i\pi e^{-ma}$$

Equating real and imaginary parts [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে]

$$\int_{-\infty}^{\infty} \frac{x \cos mx}{x^2 + a^2}\, dx = 0 \text{ and } \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2}\, dx = \pi e^{-ma}$$

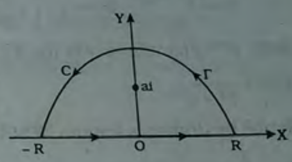
$\Rightarrow 2\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2}\, dx = \pi e^{-ma}$. By property of definite integral.

$\Rightarrow \int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2}\, dx = \frac{\pi}{2} e^{-ma}$. (Ans)

Solution–48. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{ze^{iz}}{a^2 + z^2}$ and C is the contour consisting of

(i) the x-axis from $-R$ to R, where R is lange

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \frac{ze^{iz}}{a^2 + z^2}$ can be obtained from the equation [$\oint_C f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{ze^{iz}}{a^2 + z^2}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{ze^{iz}}{a^2 + z^2}$ এর পোল নিম্নের সমীকরণ হতে পাওয়া যায়।]

$$a^2 + z^2 = 0$$

$$\Rightarrow z = \sqrt{-a^2} = \pm ai$$

Only the simple pole $z = ai$ lies inside the contour. [একমাত্র সরল পোল $z = ai$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ]

$\lim_{z \to ai} (z - ai) \cdot f(z)$

$$= \lim_{z \to ai} \left\{(z - ai) \cdot \frac{ze^{iz}}{a^2 + z^2}\right\}$$

$$= \lim_{z \to ai} \left\{ (z - ai) \cdot \frac{ze^{iz}}{(z + ai)(z - ai)} \right\}$$

$$= \lim_{z \to ai} \frac{ze^{iz}}{z + ai} = \frac{iae^{-a}}{2ai} = \frac{e^{-a}}{2}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \quad \text{[Residue at } z = ai]$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{e^{-a}}{2} = \pi i e^{-a} \ \ldots\ldots (1)$$

When [যখন] $R \to \infty$ then [তখন]

$$\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{xe^{ix}}{a^2 + x^2}\, dx$$

Here $\lim_{z \to \infty} \frac{z}{z^2 + a^2} = \lim_{z \to \infty} \frac{1}{z\left(1 + \frac{a^2}{z^2}\right)} = \frac{1}{\infty} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$ where [এবং $f(z)$ ফাংশন $e^{imz} F(z)$ আকারের, যেখানে] $m = 1$ and $F(z) = \frac{z}{z^2 + a^2}$.

Hence by Jordan's lemma [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0$$

Now taking limit $R \to \infty$ in (1) and then putting the above results we get [(1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{a^2 + x^2}\, dx + 0 = \pi i e^{-a}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)}{a^2 + x^2}\, dx = \pi i e^{-a}$$

Equating imaginary parts we get, [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{x \sin x}{a^2 + x^2}\, dx = \pi e^{-a}$$

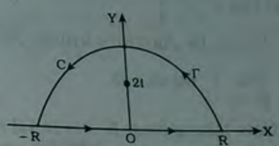
$$\Rightarrow 2\int_0^{\infty} \frac{x \sin x}{a^2 + x^2}\, dx = \pi e^{-a}, \text{ by property of definite integral.}$$

$$\Rightarrow \int_0^{\infty} \frac{x \sin x}{a^2 + x^2}\, dx = \frac{\pi}{2} e^{-a} \text{ (Ans).}$$

Solution–49. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{ze^{iz}}{z^2 + 4}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \frac{ze^{iz}}{z^2 + 4}$ are obtained by solving the equation [$\oint_C f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{ze^{iz}}{z^2 + 4}$ এবং C বদ্ধ কন্টুরটি গঠিত ঃ

(i) x অক্ষ, $-R$ হতে R পর্যন্ত, যেখানে R বৃহৎ

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধ বৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবস্থিত।

$f(z) = \frac{ze^{iz}}{z^2 + 4}$ এর পোল সমূহ নিম্নের সমীকরণ সমাধান করে পাওয়া যায়।]

$$z^2 + 4 = 0$$

$$\Rightarrow z = \sqrt{-4} = \pm 2i$$

Only the simple pole $z = 2i$ lies inside the contour [একমাত্র সরল পোল $z = 2i$ কন্টুরের ভিতরে অবস্থিত]

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim_{z \to 2i} (z - 2i) \cdot f(z)$

$$= \lim_{z \to 2i} \left\{ (z - 2i) \cdot \frac{ze^{iz}}{z^2 + 4} \right\}$$

$$= \lim_{z \to 2i} \left\{ (z - 2i) \cdot \frac{ze^{iz}}{(z + 2i)(z - 2i)} \right\}$$

$$= \lim_{z \to 2i} \frac{ze^{iz}}{z + 2i} = \frac{2ie^{-2}}{4i} = \frac{e^{-2}}{2}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i. \quad \text{[Residue at } z = 2i]$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz = 2\pi i \cdot \frac{e^{-2}}{2} = \pi i e^{-2} \ \ldots\ldots (1)$$

When $R \to \infty$ then $\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4}\,dx$

Here $\lim_{z\to\infty} \frac{z}{z^2+4} = \lim_{z\to\infty} \frac{1}{z\left(1+\frac{4}{z^2}\right)} = \frac{1}{\infty} = 0$ and f(z) is a function of the form $e^{imz} F(z)$, where [এবং f(z) ফাংশন $e^{imz} F(z)$ আকারের, যেখানে] $m = 1$ and $F(z) = \frac{z}{z^2+4}$.

Hence by Jordain's lemma [অতএব জর্ডান লিমার দ্বারা পাই]

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

Now taking limit $R \to \infty$ in (1) and then using the above results we get [এখন (1) এ $R \to \infty$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4}\,dx + 0 = \pi i e^{-2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+4}\,dx = \pi i e^{-2}$$

Equating the imaginary parts from both sides we get, [উভয় পক্ষ হেত কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4}\,dx = \pi e^{-2}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{x\sin x}{x^2+4}\,dx = \pi e^{-2}, \text{ by property of definite integral.}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x\sin x}{x^2+4}\,dx = \frac{\pi}{2} e^{-2} \quad \text{(Ans)}$$

GROUP-D

Solution-50. Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \frac{e^{iz}}{z}$.

$f(z) = \frac{e^{iz}}{z}$ has a singularity at $z = 0$ on the real axis where the integration is not possible and there are no singularity in the upper half plane. Thus, we consider the closed contour C consisting of

(i) the x-axis from $-R$ to $-r$

(ii) the small semi-circle γ of radius r, where $r \to 0$

(iii) the x-axis from r to R

(iv) the large semi-circle Γ of radius R of the circle $|z| = R$, where $R \to \infty$

As there is no pole inside the contour, so by Cauchy's residue theorem

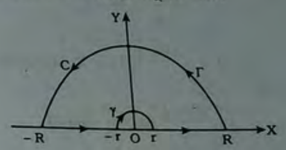
[$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{e^{iz}}{z}$. $z = 0$ তে $f(z) = \frac{e^{iz}}{z}$ এর ব্যতিচার বিন্দু আছে যাহা x অক্ষের উপর যেখানে যোগজীকরণ সম্ভব না এবং উপরের অর্ধতলে ইহার কোন ব্যতিচারিতা নাই। অতএব বিবেচনা করা যায় C কন্টুরটি গঠিত

(i) x অক্ষ, $-R$ হতে $-r$

(ii) r ব্যাসার্ধ বিশিষ্ট ক্ষুদ্র বৃত্ত γ যেখানে $r \to 0$

(iii) x অক্ষ, r হতে R

(iv) $|z| = R$ বৃত্তের বৃহৎ অর্ধবৃত্ত Γ যেখানে $R \to \infty$,



যেহেতু কন্টুনের ভিতর কোন পোল নাই, সুতরাং কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \times 0$$

$$\Rightarrow \int_{-R}^{-r} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{r}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 0 \quad \cdots\cdots (1)$$

Here [এখানে] $\lim_{z\to\infty} \frac{1}{z} = \frac{1}{\infty} = 0$ and $f(z) = e^{iz} \cdot \frac{1}{z}$, so by Jordan's lemma,

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

Again [আবার] $\lim_{z\to 0} z \cdot f(z) = \lim_{z\to 0} \frac{ze^{iz}}{z} = e^0 = 1$

$$\therefore \lim_{r\to 0} \int_{\gamma} f(z)\,dz = -i \cdot 1 \cdot (\pi - 0) = -i\pi$$

Thus, when $R \to \infty$ and $r \to 0$ then from (1), by using the above results we get [অতএব, যখন $R \to \infty$ এবং $r \to 0$ তখন উপরের ফলগুলি ব্যবহার করে (1) হতে পাই]

$$\int_{-\infty}^{0} f(x)\,dx - i\pi + \int_{0}^{\infty} f(x)\,dx + 0 = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx = i\pi$$

Equating the imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin x}{x} dx = \pi, \text{ by property of definite integral.}$$

$$\therefore \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{(Ans)}$$

Solution–51. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{imz}}{z}$.
Then f(z) has a singularity at $z = 0$ on the real axis where the integration is not possible and there are no singularity in the upper half plane. Thus we consider the closed contour C consisting of

(i) the x-axis from $-R$ to $-r$

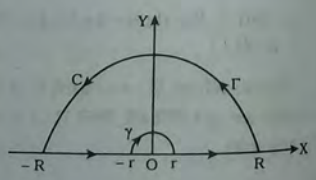
(ii) the small semi-circle γ of radius r, where $r \to 0$

(iii) the x-axis from r to R

(iv) the large semi-circle Γ of radius R of the circle $|z| = R$ where $R \to \infty$.

As there is no pole inside the contour, so by Cauchy's residue theorem we have

[$\oint_c f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \frac{e^{imz}}{z}$. তখন $f(z)$ এর ব্যতিচার বিন্দু $z = 0$ বাস্তব অক্ষের উপর হবে যেখানে যোজিতকরণ সম্ভব হবে না এবং উর্ধ অর্ধতলে আর কোন ব্যতিচার বিন্দু নাই। অতএব বদ্ধ কন্টুর C বিবেচনা করব যা গঠিত হবে



(i) x অক্ষ, $-R$ হতে $-r$

(ii) r ব্যাসার্ধ বিশিষ্ট ছোট অর্ধবৃত্ত γ যেখানে $r \to 0$

(iii) x অক্ষ, r হতে R.

(iv) R ব্যাসার্ধ বিশিষ্ট $|z| = R$ বৃত্তের বড় অর্ধবৃত্ত Γ যেখানে $R \to \infty$.

যেহেতু কন্টুরের ভিতরে কোন পোল নেই, সুতরাং কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \times 0$$

$$\Rightarrow \int_{-R}^{-r} f(x)\, dx + \int_{\gamma} f(z)\, dz + \int_{r}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 0 \quad \cdots\cdots (1)$$

Here [এখানে] $\lim_{z\to\infty} \frac{1}{z} = \frac{1}{\infty} = 0$ and [এবং] $f(z) = e^{imz} \cdot \frac{1}{z}$, so by Jordan's lemma

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0$$

Again [আবার] $\lim_{z\to 0} z \cdot f(z) = \lim_{z\to 0} z \cdot \frac{e^{imz}}{z} = \lim_{z\to 0} e^{imz} = e^0 = 1$

$$\therefore \lim_{r\to 0} \int_{\gamma} f(z)\, dz = -i \cdot 1\, (\pi - 0) = -i\pi$$

Thus, when $R \to \infty$ and $r \to 0$ then from (1), by using the above results we get [অতএব, যখন $R \to \infty$ এবং $r \to 0$ তখন উপরের ফলাফল ব্যবহার করে (1) হতে পাই]

$$\int_{-\infty}^{0} f(x)\, dx - i\pi + \int_{0}^{\infty} f(x)\, dx + 0 = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{x} = i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x} = i\pi$$

Equating the imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

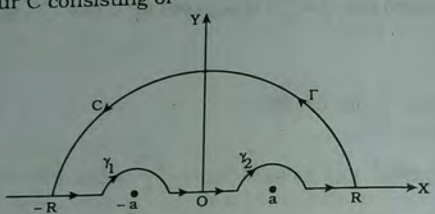
$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} = \pi$$

$$\Rightarrow 2\int_0^{\infty}\frac{\sin mx}{x} = \pi, \text{ by property of definite integral.}$$

$$\therefore \int_0^{\infty}\frac{\sin mx}{x} = \frac{\pi}{2} \quad \text{(Ans)}$$

Solution–52. Let us consider the integral $\oint_C f(z)\, dz$, where $f(z) = \dfrac{e^{iz}}{a^2 - z^2}$.

The singularities of f(z) are given by $a^2 - z^2 = 0 \Rightarrow z = \pm a$, which lie on the x-axis where integrations are not possible and there are no more singularity in the upper half plane. Thus we consider the closed contour C consisting of



(i) the x-axis from $-R$ to $-a + r_1$
(ii) the small semi-circle γ_1 of radius r_1 where $r_1 \to 0$
(iii) the x-axis from $-a - r_1$ to $a - r_2$
(iv) the small semi-circle γ_2 of radius r_2 where $r_2 \to 0$
(v) the x-axis from $a + r_2$ to R
(vi) the large semi-circle Γ of radius R of the circle $|z| = R$, where $R \to \infty$.

[$\oint_C f(z)\, dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{iz}}{a^2 - z^2}$

f(z) এর ব্যতিচার বিন্দুগুলি পাওয়া যায় $a^2 - z^2 = 0 \Rightarrow z = \pm a$, যাহা x অক্ষের উপর যেখানে যোজিতকরণ সম্ভব হবে না এবং উর্ধ্ব অর্ধতলে আর কোন ব্যতিচার বিন্দু নাই। অতএব বদ্ধ কন্টুর C বিবেচনা করব যা গঠিত হবে ঃ

(i) x অক্ষ, $-R$ হতে $-a + r_1$
(ii) r_1 ব্যাসার্ধ বিশিষ্ট ছোট অর্ধ বৃত্ত γ_1 যেখানে $r_1 \to 0$
(iii) x অক্ষ, $-a-r_1$ হতে $a - r_2$
(iv) r_2 ব্যাসার্ধ বিশিষ্ট ছোট অর্ধ বৃত্ত γ_2 যেখানে $r_2 \to 0$

(v) x অক্ষ, $a + r_2$ হতে R
(vi) R ব্যাসার্ধ বিশিষ্ট বড় বৃত্ত $|z| = R$ এর অর্ধ বৃত্ত Γ যেখানে $R \to \infty$]

By Cauchy's integral theorem we have [কচির যোজিত উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 0$$

$$\Rightarrow \int_{-R}^{-a-r_1} f(x)\, dx + \int_{\gamma_1} f(z)\, dz + \int_{-a+r_1}^{a-r_2} f(x)\, dx + \int_{\gamma_2} f(z)\, dz + \int_{a+r_2}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 0 \;\ldots\ldots (1)$$

When $R \to \infty$ and $r_1 \to 0$, $r_2 \to 0$ then

$$\lim_{z\to\infty} \frac{1}{a^2 - z^2} = \frac{1}{\infty} = 0 \text{ and } f(z) = e^{iz} \cdot \frac{1}{a^2 - z^2}, \text{ so by}$$

Jordan's lemma we have [জর্ডানের লেমাদ্বারা পাই] $\lim\limits_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0$

$$\text{Again, } \lim_{z\to -a} (z + a) \cdot f(z) = \lim_{z\to -a} (z + a) \cdot \frac{e^{iz}}{a^2 - z^2}$$

$$= \lim_{z\to -a} \frac{e^{iz}}{a - z} = \frac{e^{-ia}}{2a}$$

$$\therefore \lim_{r_1\to 0} \int_{\gamma_1} f(z)\, dz = -i(\pi - 0) \cdot \frac{e^{-ia}}{2a} = \frac{-i\pi e^{-ia}}{2a}$$

$$\text{Also, } \lim_{z\to a} (z - a)\, f(z) = \lim_{z\to a} (z - a) \cdot \frac{e^{iz}}{a^2 - z^2}$$

$$= \lim_{z\to a} \frac{e^{iz}}{-(a + z)} = \frac{e^{ia}}{-2a}$$

$$\therefore \lim_{r_1\to 0} \int_{\gamma_2} f(z)\, dz = -i(\pi - 0) \cdot \frac{e^{ia}}{-2a} = \frac{i\pi e^{ia}}{2a}$$

Taking limit $R \to \infty$, $r_1 \to 0$, $r_2 \to 0$ and then using the above results we get, [$R \to \infty$, $r_1 \to 0$, $r_2 \to 0$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{-a} f(x)\, dx + \frac{-i\pi e^{-ia}}{2a} + \int_{-a}^{a} f(x)\, dx + \frac{i\pi e^{ia}}{2a} + \int_a^{\infty} f(x)\, dx + 0 = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \frac{-i\pi}{2a}(e^{ia} - e^{-ia})$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2}\,dx = \frac{-i\pi}{2a} \cdot 2i \sin a$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{a^2 - x^2}\,dx = \frac{\pi}{a} \sin a$$

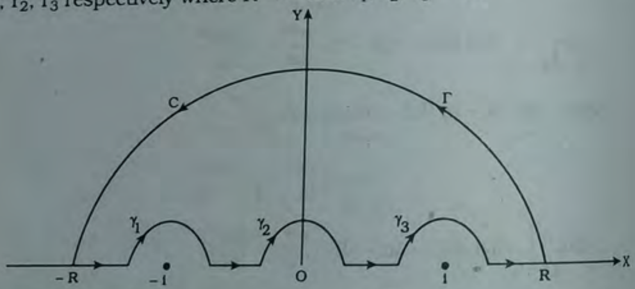
Equating the real parts we get. [বাস্তব অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi}{a} \sin a$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi}{a} \sin a; \text{ by property of definite integral}$$

$$\therefore \int_{0}^{\infty} \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi}{2a} \sin a. \qquad \text{(Ans)}$$

Solution–53. Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \dfrac{e^{i\pi z}}{z(1 - z^2)}$. The singularities of $f(z)$ are given by $z(1 - z^2) = 0$ $\Rightarrow z = -1, 0$ and 1, which are on the real axis. Thus, we consider the closed contour C consisting of a large upper semi-circle Γ of the circle $|z| = R$, indenting by three small circles $\gamma_1, \gamma_2, \gamma_3$ on the real axis, whose contres are at $z = -1, z = 0, z = 1$ with radius r_1, r_2, r_3 respectively where $R \to \infty$ and $r_1, r_2, r_3 \to 0$.



[$\oint_C f(z)\,dz$ যোগজটি বিবেচনা করি, যেখানে $f(z) = \dfrac{e^{i\pi z}}{z(1 - z^2)}$

$f(z)$ এর ব্যতিচার বিন্দুগুলি হল $z(1 - z^2) = 0 \Rightarrow z = -1, 0$ এবং 1, যাহা x অক্ষের উপর যেখানে যোজিতকরণ সম্ভব হবে না এবং উর্ধ্ব তলে আর কোন ব্যতিচার বিন্দু নাই। অতএব, বদ্ধ কন্টুর C হবে $|z| = R$ বৃত্তের উর্ধ্ব বড় অর্ধ বৃত্ত Γ যাহা ছোট তিনটি বৃত্ত $\gamma_1, \gamma_2, \gamma_3$ দ্বারা খাজকাঁটে (indenting) যাদের কেন্দ্র $z = -1, z = 0, z = 1$ এবং ব্যাসার্ধ যথাক্রমে r_1, r_2, r_3 যেখানে $R \to \infty$ এবং $r_1, r_2, r_3 \to 0$.]

As there is no pole inside the contour, so by Cauchy's integral theorem [যেহেতু কন্টুরের ভিতরে কোন পোল নাই, সুতরাং কচির ইনটিগরাল উপপাদ্য দ্বারা]

$$\oint_C f(z)\,dz = 0$$

$$\Rightarrow \int_{-R}^{-1-r_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-1+r_1}^{-r_2} f(x)\,dx + \int_{\gamma_2} f(z)\,dz$$

$$+ \int_{r_2}^{1-r_3} f(x)\,dx + \int_{\gamma_3} f(z)\,dz + \int_{1+r_2}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 0 \;\cdots\cdots (1)$$

When [যখন] $R \to \infty$ then [তখন] $\lim\limits_{z \to \infty} \dfrac{1}{z(1 - z^2)} = \dfrac{1}{\infty} = 0$

$\therefore$ By Jordan's lemma we have [জর্ডানের উপপাদ্য দ্বারা পাই]

$$\lim_{R \to \infty} \int_{\Gamma} f(z)\,dz = \lim_{R \to \infty} \int_{\Gamma} \frac{e^{i\pi z}\,dz}{z(1 - z^2)} = 0$$

When $r_1 \to 0$, $r_2 \to 0$ and $r_3 \to 0$ then

$$\lim_{z \to -1} (z + 1) \cdot f(z) = \lim_{z \to -1} (z + 1) \cdot \frac{e^{i\pi z}}{z(1 + z)(1 - z)} = \frac{e^{-i\pi}}{-1(2)} = \frac{-e^{-i\pi}}{2}$$

$$= -\frac{1}{2}(\cos \pi - i \sin \pi) = -\frac{1}{2}(-1 - 0) = \frac{1}{2}$$

$$\lim_{z \to 0} z \cdot f(z) = \lim_{z \to 0} z \cdot \frac{e^{i\pi z}}{z(1 - z^2)} = \frac{e^0}{1 - 0} = 1$$

$$\lim_{z \to 1} (z - 1) \cdot f(z) = \lim_{z \to 1} (z - 1) \cdot \frac{e^{i\pi z}}{z(1 - z^2)} = \lim_{z \to 1} \frac{e^{i\pi z}}{-z(1 + z)}$$

$$= \frac{e^{i\pi}}{-2} = -\frac{1}{2}(\cos \pi + i \sin \pi) = -\frac{1}{2}(-1 + 0) = \frac{1}{2}$$

$$\therefore \lim_{r_1 \to 0} \int_{\gamma_1} f(z)\,dz = -i(\pi - 0) \cdot \frac{1}{2} = \frac{-i\pi}{2}$$

$$\lim_{r_2 \to 0} \int_{\gamma_2} f(z)\,dz = -i(\pi - 0) \cdot 1 = -\pi i$$

$$\lim_{r_3 \to 0} \int_{\gamma_3} f(z)\,dz = -i(\pi - 0) \cdot \frac{1}{2} = \frac{-i\pi}{2}$$

Taking limit $R \to \infty$, $r_1 \to 0$, $r_2 \to 0$, $r_3 \to 0$ in (1) and then putting the above results we get, [(1) এ $R \to \infty$, $r_1 \to 0$, $r_2 \to 0$ লিমিট নিয়ে এবং অতপর উপরের ফলগুলি বসিয়ে পাই]

$$\int_{-\infty}^{-1} f(x)\,dx - \frac{i\pi}{2} + \int_{-1}^{0} f(x)\,dx - i\pi + \int_{0}^{1} f(x)\,dx - \frac{i\pi}{2} + \int_{1}^{\infty} f(x)\,dx + 0 = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x(1-x^2)}\,dx = 2i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos \pi x + i \sin \pi x}{x(1-x^2)}\,dx = 2i\pi$$

Equating the imaginary parts we get, [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = 2\pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = 2\pi, \text{ since } f(x) = \frac{\sin \pi x}{x(1-x^2)} \text{ is an even function}$$

$$\therefore \int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi \quad \text{(Ans)}$$

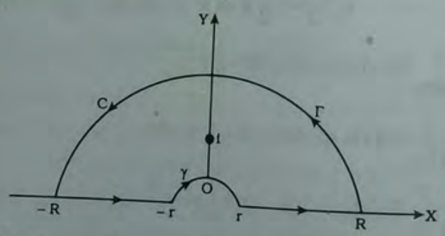
Solution–54. Consider the integral $\oint_C f(z)\,dz$, where $f(z) = \dfrac{(\log z)^2}{1+z^2}$. Then $z = 0$ is a branch point of $f(z)$ and poles of $f(z)$ are given by $1 + z^2 = 0 \Rightarrow z = \pm i$. We now consider the closed contour C as

(i) the x-axis from $-R$ to $-r$

(ii) the small semi-circle γ of radius r, where $r \to 0$

(iii) the x-axis from r to R

(iv) the upper semi-circle Γ of the circle $|z| = R$, where R is large



The simple pole $z = i$ lies inside the contour.

Residue at $z = i$ is $\lim_{z \to i} (z - i) \cdot f(z)$

$$= \lim_{z \to i} \left\{ (z-i) \cdot \frac{(\log z)^2}{1+z^2} \right\}$$

$$= \lim_{z \to i} \left\{ (z-i) \cdot \frac{(\log z)^2}{(z+i)(z-i)} \right\}$$

$$= \lim_{z \to i} \frac{(\log z)^2}{z+i} = \frac{(\log i)^2}{2i}$$

$$= \frac{1}{2i} \left\{ \log \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2} \right) \right\}^2 = \frac{1}{2i} \{\log(e^{i\pi/2})\}^2$$

$$= \frac{1}{2i} \left(\frac{i\pi}{2} \right)^2 = \frac{i^2 \pi^2}{8i} = \frac{-\pi^2}{8i}$$

Hence by Cauchy's residue theorem we have

$$\oint_C f(z)\,dz = 2\pi i \text{ [Residue at } z = i]$$

$$\Rightarrow \int_{-R}^{-r} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{r}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \cdot \frac{-\pi^2}{8i}$$

$$= \frac{-\pi^3}{4} \quad \ldots\ldots (1)$$

When $R \to \infty$ then $\lim_{z \to \infty} z \cdot f(z) = \lim_{z \to \infty} \dfrac{z(\log z)^2}{1+z^2}$

$$= \lim_{z \to \infty} \frac{(\log z)^2}{z} \cdot \lim_{z \to \infty} \frac{z^2}{1+z^2}$$

$$= 0 \cdot 1 = 0$$

$$\lim_{z \to \infty} \frac{(\log z)^2}{z} = \lim_{z \to \infty} \frac{2 \log z \cdot \frac{1}{z}}{1}$$

$$= 2 \cdot \lim_{z \to \infty} \frac{\log z}{z} = 2 \cdot 0 = 0$$

$$\therefore \lim_{R \to \infty} \int_{\Gamma} f(z)\,dz = \lim_{R \to \infty} \int_{\Gamma} \frac{(\log z)^2}{1+z^2}\,dz = 0$$

Again, $\lim_{z \to 0} z \cdot f(z) = \lim_{z \to 0} \dfrac{z(\log z)^2}{1+z^2}$

Putting $z = \dfrac{1}{t}$

$z = 0 \Rightarrow t = \infty$

$$= \lim_{t\to\infty} \frac{\frac{1}{t}\left(\log \frac{1}{t}\right)^2}{1+\frac{1}{t^2}}$$

$$= \lim_{t\to\infty} \frac{t(\log 1 - \log t)^2}{t^2+1}$$

$$= \lim_{t\to\infty} \frac{t(0 - \log t)^2}{1+t^2}$$

$$= \lim_{t\to\infty} \frac{t(\log t)^2}{1+t^2}$$

$= 0$, as above

$$\therefore \lim_{r\to 0} \int_\gamma f(z)\, dz = 0$$

Taking limit $R \to \infty$, $r \to 0$ in (1) and then putting the above results we get,

$$\int_{-\infty}^{0} f(x)\,dx + 0 + \int_0^\infty f(x)\,dx + 0 = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{(\log x)^2}{1+x^2}\,dx + \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{-\pi^3}{4} \quad \text{......} (2)$$

Now $\int_{-\infty}^{0} \frac{(\log x)^2}{1+x^2}\,dx$

Putting $x = -y$

$\Rightarrow dx = -dy$

and $x = -\infty, 0$

$\Rightarrow y = \infty, 0$

$$= \int_\infty^0 \frac{\{\log(-y)\}^2}{1+y^2} \cdot (-dy)$$

$$= \int_0^\infty \frac{\{\log(ye^{i\pi})\}^2}{1+y^2}\,dy \qquad \because -1 = \cos\pi + i\sin\pi = e^{i\pi}$$

$$= \int_0^\infty \frac{(\log y + i\pi)^2}{1+y^2}\,dy$$

$$= \int_0^\infty \frac{(\log x + i\pi)^2}{1+x^2}\,dx \qquad \because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

$$= \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + 2i\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx + i^2\pi^2 \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + i2\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx - \pi^2 \left[\tan^{-1} x\right]_0^\infty$$

$$= \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + i2\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx - \pi^2 \left(\frac{\pi}{2} - 0\right)$$

Putting this result in (2) we get

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + i2\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx - \frac{\pi^3}{2} + \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{-\pi^3}{4}$$

$$\Rightarrow 2\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + i2\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx = \frac{\pi^3}{4}$$

Equating real and imaginary parts we get,

$$2\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{4} \text{ and } 2\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$$

$$\therefore \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8} \text{ and } \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0 \text{ (Ans)}$$

Solution–55. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{\log z}{(1+z^2)^2}$. Then $z = 0$ is a branch point of $f(z)$ and poles of $f(z)$ are given by $(1+z^2)^2 = 0 \Rightarrow 1 + z^2 = 0 \Rightarrow z = \pm i$.

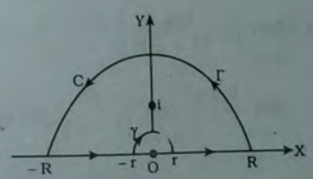
We now consider the closed contour C as

(i) the x-axis from $-R$ to $-r$

(ii) the small semi-circle γ of radius r where $r \to 0$

(iii) the x-axis from r to R

(iv) the upper semi-circle Γ of the circle $|z| = R$ where R is large.



Then the double pole $z = i$ lies inside the contour.

Residue at $z = i$ is $\lim_{z\to i} \frac{d}{dz}\{(z-i)^2 \cdot f(z)\}$

$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{\log z}{(1+z^2)^2}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{\log z}{(z+i)^2 (z-i)^2}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{\log z}{(z+i)^2}\right\}$$

$$= \lim_{z\to i} \frac{(z+i)^2 \cdot \frac{1}{z} - \log z \cdot 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{\frac{z+i}{z} - 2\log z}{(z+i)^3}$$

$$= \frac{\frac{2i}{i} - 2\log(i)}{(2i)^3} = \frac{2 - 2\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{-8i}$$

$$= \frac{1 - \log(e^{i\pi/2})}{-4i}$$

$$= \frac{1 - \frac{i\pi}{2}}{-4i} = \frac{1}{4}\left(\frac{\pi}{2} + i\right)$$

By Cauchy's residue theorem we have

$$\oint_C f(z)\,dz = 2\pi i.\ (\text{Residue at } z = i)$$

$$\Rightarrow \int_{-R}^{-r} f(x)\,dx + \int_\gamma f(z)\,dz + \int_r^R f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i \cdot \frac{1}{4}\left(\frac{\pi}{2} + i\right) \cdots (1)$$

Here $\lim_{z\to\infty} z \cdot f(z) = \lim_{z\to\infty} \left\{z \cdot \frac{\log z}{(1+z^2)^2}\right\}$

$$= \lim_{z\to\infty} \frac{\log z}{z} \cdot \lim_{z\to\infty} \frac{z^2}{(1+z^2)^2} = 0 \cdot 0 = 0$$

$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$

Again, $\lim_{z\to 0} z \cdot f(z) = \lim_{z\to 0} \frac{z\log z}{(1+z^2)^2}$

$$= \lim_{z\to 0} z\log z \cdot \lim_{z\to 0} \frac{1}{(1+z^2)^2} = 0 \cdot 1 = 0$$

$$\left[\because \lim_{z\to 0} z\log z = \lim_{t\to\infty} \frac{\log\frac{1}{t}}{t} = \lim_{t\to\infty} \frac{-\log t}{t} = 0; \text{ by putting } z = \frac{1}{t}\right]$$

$\therefore \lim_{r\to 0} \int_\gamma f(z)\,dz = 0$

Taking limit $R \to \infty$, $r \to 0$ in (1) and then putting the above results we get,

$$\int_{-\infty}^0 f(x)\,dx + 0 + \int_0^\infty f(x)\,dx + 0 = \frac{\pi i}{2}\left(\frac{\pi}{2} + i\right)$$

$$\Rightarrow \int_{-\infty}^0 \frac{\log x}{(1+x^2)^2}\,dx + \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{i\pi^2}{4} - \frac{\pi}{2} \cdots\cdots (2)$$

Now $\int_{-\infty}^0 \frac{\log x}{(1+x^2)^2}\,dx = \int_\infty^0 \frac{\log(-y)}{(1+y^2)^2} \cdot (-dy)$

Putting $x = -y$
$\Rightarrow dx = -dy$
$x = -\infty, 0$
$\Rightarrow y = \infty, 0$

$$= \int_0^\infty \frac{\log\{y(\cos\pi + i\sin\pi)\}}{(1+y^2)^2}\,dy$$

$$= \int_0^\infty \frac{\log(ye^{i\pi})}{(1+y^2)^2}\,dy$$

$$= \int_0^\infty \frac{\log y + i\pi}{(1+y^2)^2}\,dy$$

$$= \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + i\pi\int_0^\infty \frac{1}{(1+x^2)^2}\,dx$$

By property of definite integral

Putting this value in (2) we get,

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + i\pi\int_0^\infty \frac{1}{(1+x^2)^2}\,dx + \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{i\pi^2}{4} - \frac{\pi}{2}$$

Equating real parts we get,

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{-\pi}{2}$$

$$\Rightarrow 2\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{-\pi}{2}$$

$$\therefore \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{-\pi}{4} \quad \text{(Ans)}$$

GROUP–E

Solution–56. If $0 < p < 1$, then show that

(i) $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$.

(ii) $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$ **[RUH–1996, 2001]**

Ans (i) : Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{z^{p-1}}{1+z}$

Since $0 < p < 1$, so $z = 0$ is a branch point. Also, $f(z)$ has a simple pole $1 + z = 0 \Rightarrow z = -1$.

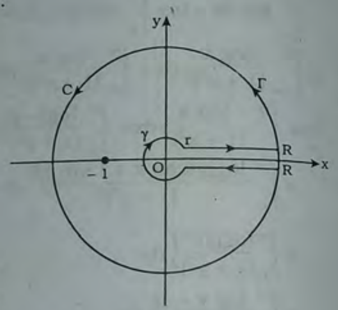
The contour consisting of :

(i) the x-axis from r to R

(ii) the large circle Γ : $|z| = R,\ R \to \infty$

(iii) the x-axis from R to r

(iv) the small circle γ : $|z| = r,\ r \to 0$



Thus C is a closed contour which excludes the origin.

On (i), amp $z = 0$ and on (iii), amp $z = 2\pi$

[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{z^{p-1}}{1+z}$. যেহেতু $0 < p < 1$, সুতরাং $z = 0$ একটি ব্রাঞ্চ বিন্দু। আরো, $f(z)$ এর $z = -1$ এ একটি সরল পোল আছে।

কন্টুরটি গঠিত হচ্ছে ঃ

(i) x অক্ষ, r হতে R

(ii) বৃহৎ বৃত্ত Γ : $|z| = R,\ R \to \infty$

(iii) x অক্ষ, R হতে r

(iv) ক্ষুদ্র বৃত্ত γ : $|z| = r,\ r \to 0$

অতএব, C কন্টুরটি আবদ্ধ যাহা মূলবিন্দুকে বাহিরে রাখে।

(i) এর উপর, amp $z = 0$ এবং (iii) এর উপর, amp $z = 2\pi$.]

Residue at $z = -1$ is [$z = -1$ এ অবশেষ]

$$\lim_{z \to -1} \left\{(z+1)\frac{z^{p-1}}{1+z}\right\} = (-1)^{p-1} = (e^{\pi i})^{p-1} = e^{i(p-1)\pi}$$

$\therefore$ By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\,dz = 2\pi i \cdot e^{i(p-1)\pi}$$

$$\Rightarrow \int_r^R f(x)\,dx + \int_\Gamma f(z)\,dz + \int_R^r f(x)\,dx + \int_\gamma f(z)\,dz = 2\pi i \cdot e^{i(p-1)\pi}$$

$$\Rightarrow \int_r^R \frac{x^{p-1}}{1+x}\,dx + \int_\Gamma \frac{z^{p-1}}{1+z}\,dz + \int_R^r \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx + \int_\gamma \frac{z^{p-1}}{1+z}\,dz$$

$$= 2\pi i \cdot e^{i(p-1)\pi} \text{ ...... (1)}$$

Now [এখন] $\displaystyle\int_r^R \frac{x^{p-1}}{1+x}\,dx + \int_R^r \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx$

$$= \int_r^R \frac{x^{p-1}}{1+x}\,dx - \int_r^R \frac{x^{p-1} \cdot e^{i2(p-1)\pi}}{1+x \cdot 1}\,dx$$

$$= \{1 - e^{i2(p-1)\pi}\} \int_r^R \frac{x^{p-1}}{1+x}\,dx \text{ ...... (2)}$$

We know [আমরা জানি] $|z_1 + z_2| \geq |z_1| - |z_2|$

Taking $z_1 = Re^{i\theta}$ and $z_2 = 1$ we get [$z_1 = Re^{i\theta}$ এবং $z_2 = 1$ নিয়ে পাই]

$$|Re^{i\theta} + 1| \geq |Re^{i\theta}| - |1|$$

$$\Rightarrow |Re^{i\theta} + 1| \geq |R|\,|e^{i\theta}| - 1$$

$$\Rightarrow |Re^{i\theta} + 1| \geq R \cdot 1 - 1; \quad \because |e^{i\theta}| = 1$$

$$\Rightarrow \frac{1}{|Re^{i\theta} + 1|} \leq \frac{1}{R-1} \text{ ...... (3)}$$

$$\therefore \left|\frac{z^{p-1}}{1+z}\right| = \frac{|z^{p-1}|}{|1+z|}$$

$$= \frac{|(Re^{i\theta})^{p-1}|}{|1+Re^{i\theta}|} = \frac{|R^{p-1}|\,|e^{i(p-1)\theta}|}{|1+Re^{i\theta}|}$$

$$= \frac{R^{p-1} \cdot 1}{|Re^{i\theta} + 1|}$$

$$\Rightarrow \left|\frac{z^{p-1}}{1+z}\right| \leq \frac{R^{p-1}}{R-1}; \text{ by (3)}$$

$$\therefore \left|\int_{\Gamma} \frac{z^{p-1}}{1+z}\, dz\right| = \left|\frac{z^{p-1}}{1+z}\right| \int_{\Gamma} |dz| \le \frac{R^{p-1}}{R-1} \cdot 2\pi R$$

$$\Rightarrow \left|\int_{\Gamma} \frac{z^{p-1}}{1+z}\, dz\right| \le \frac{2\pi R^p}{R-1} \;\ldots\ldots (4)$$

Since $0 < p < 1$, so when $R \to \infty$ then R^p is very small in comparison with $R - 1$, so $\frac{R^p}{R-1} \to 0$, as $R \to \infty$. [যেহেতু $0 < p < 1$, সুতরাং যখন $R \to \infty$ তখন $R - 1$ এর সাথে তুলনামূলকভাবে R^p খুবই ছোট। সুতরাং $\frac{R^p}{R-1} \to 0$ যখন $R \to \infty$.]

Thus, from (4) [অতএব (4) হতে]

$$\left|\int_{\Gamma} \frac{z^{p-1}}{1+z}\, dz\right| \to 0, \text{ when } R \to \infty$$

$$\Rightarrow \lim_{R\to\infty} \int_{\Gamma} \frac{z^{p-1}}{1+z}\, dz = 0 \;\ldots\ldots (5)$$

Similarly [অনুরূপে] $\left|\int_{\gamma} \frac{z^{p-1}}{1+z}\, dz\right| \le \frac{2\pi r^p}{1-r} \to 0$, when $r \to 0$

$$\Rightarrow \lim_{r\to 0} \int_{\gamma} \frac{z^{p-1}}{1+z}\, dz = 0 \;\ldots\ldots (6)$$

When $r \to 0$ and $R \to \infty$ then (2) becomes [যখন $r \to 0$ এবং $R \to \infty$ তখন (2) দাঁড়ায়]

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx + \int_{\infty}^{0} \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\, dx = \{1 - e^{i2(p-1)\pi}\} \int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx \;\ldots\ldots (7)$$

By (5), (6) and (7), (1) becomes [when $R \to \infty$, $r \to 0$]

[(5), (6) ও (7) দ্বারা (1) দাঁড়ায়]

$$\{1 - e^{2i(p-1)\pi}\} \int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx + 0 + 0 = 2\pi i \cdot e^{i(p-1)\pi}$$

$$\Rightarrow \int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{2\pi i \cdot e^{i(p-1)\pi}}{1 - e^{2i(p-1)\pi}} = \frac{2\pi i \cdot e^{ip\pi} \cdot e^{-i\pi}}{1 - e^{2ip\pi} \cdot e^{-2i\pi}} = \frac{2\pi i \cdot e^{ip\pi} \cdot (-1)}{1 - e^{2ip\pi} \cdot 1}$$

$$= \frac{-2\pi i}{\frac{1 - e^{2ip\pi}}{e^{ip\pi}}} = \frac{-2\pi i}{e^{-ip\pi} - e^{ip\pi}} = \frac{\pi}{\frac{e^{ip\pi} - e^{-ip\pi}}{2i}}$$

$$= \frac{\pi}{\sin p\pi} \;\ldots\ldots (8) \quad \text{(Showed)}$$

(ii) From Beta function we know that [বিটা ফাংশন হতে আমরা জানি]

$$\beta(p, q) = \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\, dx$$

$$\Rightarrow \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\, dx \;\ldots\ldots (9)$$

Let [ধরি] $p + q = 1$, then [তখন] $q = p - 1$

$$\therefore \text{(9) gives, } \frac{\Gamma(p)\,\Gamma(p-1)}{\Gamma(1)} = \int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx$$

$$\Rightarrow \Gamma(p)\,\Gamma(p-1) = \frac{\pi}{\sin p\pi}\text{; by (8) and since } \Gamma(1) = 1$$

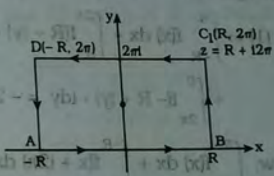
(2nd part. Proved)

Solution–57. Show that $\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\, dx = \frac{\pi}{\sin p\pi}$, where $0 < p < 1$

[RUH–1998, 2000, 2002, CUH–2001]

Ans : Consider $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{pz}}{1+e^z}$

The pole of $f(z)$ will be obtained by solving the equation



$$1 + e^z = 0$$

$$\Rightarrow e^z = -1$$

$$= \cos \pi + i \sin \pi$$

$$= \cos(2n\pi + \pi) + i \sin(2n\pi + \pi)$$

$$= \cos(2n+1)\pi + i \sin(2n+1)\pi$$

$$= e^{i(2n+1)\pi}$$

$\Rightarrow z = i(2n+1)\pi$, where $n = 0, \pm 1, \pm 2, \cdots$ etc.

The only pole enclosed by C is πi, which is a simple pole.

Thus C is a rectangle ABC_1D having sides consisting of the x-axis and the lines $x = \pm R$, $y = 2\pi$.

Residue at $z = \pi i$ is $\lim_{z\to\pi i} \{(z - \pi i) \cdot f(z)\}$

$$= \lim_{z\to\pi i} \left\{(z - \pi i) \cdot \frac{e^{pz}}{1+e^z}\right\}$$

$$= \lim_{z\to\pi i} e^{pz} \cdot \lim_{z\to\pi i} \frac{z-\pi i}{1+e^z}$$

$$= e^{ip\pi} \cdot \lim_{z\to\pi i} \frac{1}{e^z}\text{; by L. Hospital rule.}$$

$$= e^{ip\pi} \cdot \frac{1}{e^{\pi i}} = \frac{e^{ip\pi}}{\cos\pi + i\sin\pi} = \frac{e^{ip\pi}}{-1+0}$$

$$= -e^{ip\pi}$$

By Cauchy's residue theorem we have, $\oint_C f(z)\,dz = 2\pi i(-e^{ip\pi})$

$$\Rightarrow \int_{AB} f(z)\,dz + \int_{BC_1} f(z)\,dz + \int_{C_1D} f(z)\,dz + \int_{DA} f(z)\,dz = 2\pi i \cdot (-e^{ip\pi}) \;\ldots\ldots (1)$$

Along AB, $z = x \quad \Rightarrow dz = dx$

Along BC_1, $z = R + iy \quad \Rightarrow dz = idy$

Along C_1D, $z = x + i2\pi \quad \Rightarrow dz = dx$

Along DA, $z = -R + iy \quad \Rightarrow dz = idy$

$$\therefore (1) \Rightarrow \int_{-R}^{R} f(x)\,dx + \int_0^{2\pi} f(R+iy)\cdot idy + \int_R^{-R} f(x+i2\pi)\,dx + \int_{2\pi}^{0} f(-R+iy)\cdot idy = -2\pi i e^{ip\pi} \;\ldots\ldots (2)$$

Now, $\int_{-R}^{R} f(x)\,dx + \int_R^{-R} f(x+i2\pi)\,dx = \int_{-R}^{R} \frac{e^{px}}{1+e^x}\,dx - \int_{-R}^{R} \frac{e^{p(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx$

$$= \int_{-R}^{R} \frac{e^{px}}{1+e^x}\,dx - \int_{-R}^{R} \frac{e^{px}\cdot e^{2p\pi i}}{1+e^x}\,dx \qquad \left|\; \begin{aligned} &\because e^{2\pi i} \\ &= \cos 2\pi + i\sin 2\pi \\ &= 1 + 0 = 1 \end{aligned}\right.$$

$$= (1 - e^{2p\pi i}) \int_{-R}^{R} \frac{e^{px}}{1+e^x}\,dx \;\ldots\ldots (3)$$

$$|f(R+iy)| = \left|\frac{e^{p(R+iy)}}{1+e^{R+iy}}\right| = \frac{|e^{pR}|\,|e^{ipy}|}{|1+e^{R+iy}|} = \frac{e^{pR}}{|1+e^{R+iy}|} \;\ldots\ldots (4) \quad \because |e^{ipy}| = 1$$

We know, $|z_1 + z_2| \geq |z_1| - |z_2|$

Putting $z_1 = e^{R+iy}$, $z_2 = 1$ we get,

$$|e^{R+iy} + 1| \geq |e^{R+iy}| - |1|$$

$$\Rightarrow |e^{R+iy} + 1| \geq e^R - 1, \quad \because |e^{R+iy}| = |e^R|\,|e^{iy}| = e^R$$

$$\Rightarrow \frac{1}{|e^{R+iy}+1|} \leq \frac{1}{e^R - 1}$$

Putting this value in (4) we get,

$$|f(R+iy)| \leq \frac{e^{pR}}{e^R - 1}$$

$$\therefore \left|\int_0^{2\pi} f(R+iy)\,idy\right| \leq \frac{e^{pR}}{e^R - 1}[y]_0^{2\pi}; \quad \because |i| = 1$$

$$\Rightarrow \left|\int_0^{2\pi} f(R+iy)\,i\,dy\right| \leq \frac{e^{pR}\cdot 2\pi}{e^R - 1} \to 0, \text{ when } R\to\infty$$

$$\Rightarrow \lim_{R\to\infty} \int_0^{2\pi} f(R+iy)\cdot i\,dy = 0 \;\ldots\ldots (5)$$

Similarly, $\lim_{R\to\infty} \int_{2\pi}^{0} f(-R+iy)\,i\,dy = 0 \;\ldots\ldots (6)$

Thus, when $R\to\infty$ then by (2), (5) and (6); equation (3) becomes,

$$(1 - e^{i2p\pi}) \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx + 0 + 0 = -2\pi i e^{ip\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx = \frac{-2\pi i e^{ip\pi}}{1 - e^{2ip\pi}}$$

$$= \frac{2\pi i}{\dfrac{e^{2ip\pi} - 1}{e^{ip\pi}}}$$

$$= \frac{2\pi i}{e^{ip\pi} - e^{-ip\pi}}$$

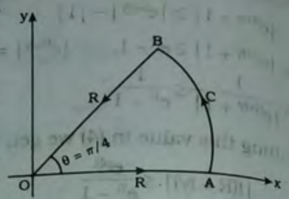
$$= \frac{\pi}{\dfrac{e^{ip\pi} - e^{-ip\pi}}{2i}}$$

$$= \frac{\pi}{\sin p\pi} \quad \text{(Showed)}$$

Solution–58. Show that $\int_0^{\infty} \sin x^2\,dx = \int_0^{\infty} \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

[RUH–1999]

Ans : Consider $\oint_C f(z)\,dz$ where $f(z) = e^{iz^2}$ ...... (1)



and C is the contour OABO as shown in the figure.

The contour consists of the following parts :

(i) the line OA on the x-axis from $x = 0$ to $x = R$, R is very large.

(ii) the arc AB of the circle $|z| = R$

$$\Rightarrow z = Re^{i\theta}, \text{ where } \theta = 0 \text{ to } \frac{\pi}{4}$$

(iii) the line BO, where $z = re^{i\theta}$, $r = R$ to $r = 0$

There is no singularity inside C.

$\therefore$ By Cauchy's integral formula we have

[$\oint_c f(z)\,dz$ বিবেচনা করি যেখানে $f(z) = e^{iz^2}$ ...... (1)

এবং C কন্টুরটি চিত্রে প্রদর্শিত OABO.

কন্টুরটি নিম্নের তিনটি অংশে নিয়ে গঠিত ঃ

(i) x অক্ষের উপর OA রেখা $x = 0$ হতে $x = R$, যেখানে R অনেক বড়।

(ii) $|z| = R$ বৃত্তের চাপ AB

$$\Rightarrow z = Re^{i\theta}, \text{ যেখানে } \theta = 0 \text{ হতে } \frac{\pi}{4}$$

(iii) BO রেখা, যেখানে $z = re^{i\theta}$, $r = R$ হতে $r = 0$.

C এর ভিতর কোন ব্যতিচারধর্মী (singularity) বিন্দু নাই।

$\therefore$ কচির যোজিত (integral) সূত্র দ্বারা পাই]

$$\oint_C f(z)\,dz = 0$$

$$\Rightarrow \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BO} f(z)\,dz = 0 \text{ ...... (2)}$$

Now on OA, we have $z = x$, and limit of $x = 0$ to $x = R$

[এখন OA রেখার উপর পাই $z = x$ এবং সীমা $x = 0$ হতে $x = R$]

$$\therefore \int_{OA} f(z)\,dz = \int_0^R f(x)\,dx = \int_0^R e^{ix^2}\,dx$$

$$= \int_0^\infty (\cos x^2 + i \sin x^2)\,dx \quad \text{when } R \to \infty$$

On AB, we have $z = Re^{i\theta}$ and limit of $\theta = 0$ to $\frac{\pi}{4}$

[AB রেখার উপর $z = Re^{i\theta}$ এবং সীমা $\theta = 0$ হতে $\frac{\pi}{4}$]

$$\therefore \int_{AB} f(z)\,dz = \int_{\theta=0}^{\pi/4} e^{iz^2}\,d(Re^{i\theta})$$

$$= \int_0^{\pi/4} e^{i(Re^{i\theta})^2} \cdot iRe^{i\theta}\,d\theta$$

$$= \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} \cdot iRe^{i\theta}\,d\theta$$

$$= \int_0^{\pi/4} e^{iR^2(\cos 2\theta + i \sin 2\theta)} \cdot i\,Re^{i\theta}\,d\theta$$

$$= \int_0^{\pi/4} e^{iR^2 \cos 2\theta} \cdot e^{-R^2 \sin 2\theta} \cdot iRe^{i\theta}\,d\theta$$

$$\Rightarrow \left|\int_{AB} f(z)\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2 \cos 2\theta} \cdot e^{-R^2 \sin 2\theta} \cdot iRe^{i\theta}\,d\theta\right|$$

$$= \left|\int_0^{\pi/4} e^{-R^2 \sin 2\theta} \cdot R\,d\theta\right| ; \quad \because |i| = 1,\ |e^{i\theta}| = 1,\ |e^{iR^2 \cos 2\theta}| = 1$$

$$= \left|\int_0^{\pi/2} e^{-R^2 \sin \phi} \cdot R \cdot \frac{1}{2}\,d\phi\right|$$

$$= \frac{R}{2}\left|\int_0^{\pi/2} e^{-R^2 \sin \phi}\,d\phi\right| \text{ ...... (3)}$$

Putting $2\theta = \phi$

$$\Rightarrow \theta = \frac{1}{2}\phi$$

$$\therefore d\theta = \frac{1}{2}\,d\phi$$

θ	0	$\pi/4$
ϕ	0	$\pi/2$

By Jordan's inequality we have [জর্ডানের অসমতা দ্বারা পাই]

$$\frac{2}{\pi} \le \frac{\sin \theta}{\theta} \le 1 \Rightarrow \frac{2\theta}{\pi} \le \sin \theta$$

$$\therefore \frac{2\phi}{\pi} \le \sin \phi; \quad \text{Replacng } \theta \text{ by } \phi$$

$$\Rightarrow -\frac{2\phi}{\pi} \geq -\sin\phi$$

$$\Rightarrow -R^2\frac{2\phi}{\pi} \geq -R^2 \sin\phi$$

$$\Rightarrow e^{-R^2\frac{2\phi}{\pi}} \geq e^{-R^2\sin\phi}$$

$$\Rightarrow e^{-R^2\sin\phi} \leq e^{-R^2\frac{2\phi}{\pi}}$$

Using this result in (3) we get [এইফল (3) এ ব্যবহার করে পাই]

$$\left|\int_{AB} f(z)\,dz\right| \leq \frac{R}{2}\left|\int_0^{\pi/2} e^{-R^2\frac{2\phi}{\pi}}\,d\phi\right|$$

$$\Rightarrow \left|\int_{AB} f(z)\,dz\right| \leq \frac{R}{2}\left|\left[\frac{e^{-R^2\frac{2\phi}{\pi}}}{-\frac{2R^2}{\pi}}\right]_0^{\pi/2}\right|$$

$$\Rightarrow \left|\int_{AB} f(z)\,dz\right| \leq \frac{\pi}{4R}\,|(1-e^{-R^2})|$$

when [যখন] $R \to \infty$ then [তখন] $\left|\int_{AB} f(z)\,dz\right| \leq \frac{\pi}{\infty}\,|(1-0)|$

$$\Rightarrow \left|\int_{AB} f(z)\,dz\right| \leq 0$$

$$\therefore \lim_{R\to\infty}\int_{AB} f(z)\,dz = 0 \;\cdots\cdots\; (4)$$

On BO : we have $z = re^{i\pi/4}$ and $r = R$ to 0

[BO রেখার উপরঃ আমরা পাই $z = re^{i\pi/4}$ এবং $r = R$ হতে 0]

$$\therefore \int_{BO} f(z)\,dz = \int_R^0 e^{i(re^{i\pi/4})^2}\cdot d(re^{i\pi/4})$$

$$= \int_R^0 e^{ir^2}\cdot e^{i\pi/2}\cdot e^{i\pi/4}\,dr$$

$$= \int_R^0 e^{ir^2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}\cdot e^{i\pi/4}\,dr$$

$$= e^{i\pi/4}\int_R^0 e^{ir^2(0+i)}\,dr$$

$$= -e^{i\pi/4}\int_0^R e^{-r^2}\,dr$$

When [যখন] $R \to \infty$ then [তখন] $\int_{BO} f(z)\,dz = -e^{i\pi/4}\int_0^\infty e^{-r^2}\,dr$

$$= -\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\int_0^\infty e^{-u}\cdot\frac{u^{-1/2}}{2}\,du$$

$$= -\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\cdot\frac{1}{2}\int_0^\infty e^{-u}\,u^{\frac{1}{2}-1}\,du$$

$$= \left(-\frac{1}{2\sqrt{2}}-\frac{i}{2\sqrt{2}}\right)\Gamma(1/2)$$

$$= \left(\frac{-1}{2\sqrt{2}}-\frac{i}{2\sqrt{2}}\right)\sqrt{\pi} \;\cdots\cdots\; (5)$$

Putting $r^2 = u$

$\Rightarrow r = \sqrt{u}$

$\Rightarrow dr = \frac{1}{2\sqrt{u}}\,du$

$= \frac{u^{-1/2}}{2}\,du$

r	0	∞
u	0	∞

Thus, when $R \to \infty$ then by (3), (4), (5) we have from (2)
[অতএব, যখন $R \to \infty$ তখন (3), (4), (5) দ্বারা (2) হতে পাই]

$$\int_0^\infty (\cos x^2 + i\sin x^2)\,dx + 0 + \left(\frac{-1}{2\sqrt{2}}-\frac{i}{2\sqrt{2}}\right)\sqrt{\pi} = 0$$

$$\Rightarrow \int_0^\infty \cos x^2\,dx + i\int_0^\infty \sin x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} + i\frac{\sqrt{\pi}}{2\sqrt{2}}$$

Equating real and imaginary parts we get [বাস্তব এবং কাল্পনিক াংশ সমীকৃত করে পাই]

$$\int_0^\infty \cos x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \text{ and } \int_0^\infty \sin x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Hence [অতএব] $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. (Showed)

Solution–59. Consider the integral $\oint_c f(z)\,dz$, where $f(z) = \frac{e^{az}}{\cos hz}$. The pole of f(z) will be obtained from the equation

$$\cos hz = 0$$

$$\Rightarrow e^z + e^{-z} = 0$$

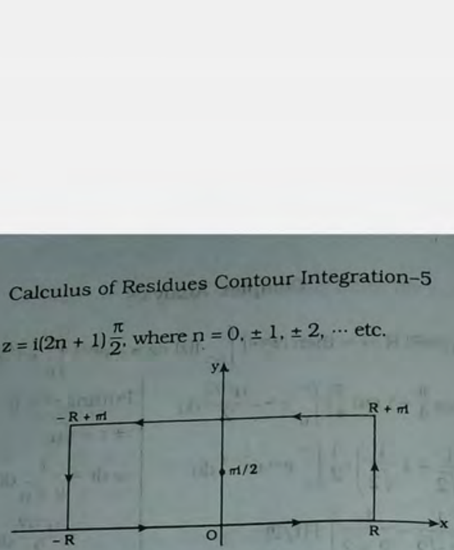
$$\Rightarrow e^z = -e^{-z} = -\frac{1}{e^z}$$

$$\Rightarrow e^{2z} = -1 = \cos\pi + i\sin\pi$$

$$\Rightarrow e^{2z} = \cos(2n+1)\pi + i\sin(2n+1)\pi = e^{i(2n+1)\pi}$$

$$\Rightarrow 2z = i(2n+1)\pi$$

$\Rightarrow z = i(2n+1)\frac{\pi}{2}$, where $n = 0, \pm 1, \pm 2, \cdots$ etc.



The only pole enclosed by C is $\frac{\pi i}{2}$ which is a simple pole and the contour C is a rectangle having vertices at $-R$, R, $R+\pi i$, $-R+\pi i$. Residue at $z = \frac{\pi i}{2}$ is

$$\lim_{z\to\pi i/2}\left(z-\frac{\pi i}{2}\right)f(z)$$

$$= \lim_{z\to\pi i/2}\left(z-\frac{\pi i}{2}\right)\frac{e^{az}}{\cos hz}$$

$$= \lim_{z\to\pi i/2} e^{az}\cdot\lim_{z\to\pi i/2}\frac{z-\frac{\pi i}{2}}{\cos hz}$$

$$= e^{a\pi i/2}\cdot\lim_{z\to\pi i/2}\frac{1}{\sin hz}, \quad \text{By L. Hospital rule}$$

$$= e^{a\pi i/2}\cdot\frac{1}{\sin h\frac{\pi i}{2}} = \frac{e^{a\pi i/2}}{i\sin\frac{\pi}{2}} = -ie^{a\pi i/2}$$

Now by Cauchy's residue theorem we have

$$\oint_c f(z)\,dz = 2\pi i\times\left[\text{Residue at } z = \frac{\pi i}{2}\right]$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_0^{\pi} f(R+iy)\,i\,dy + \int_R^{-R} f(x+\pi i)\,dx + \int_{\pi}^{0} f(-R+iy)\,i\,dy$$

$$= 2\pi i(-ie^{a\pi i/2}) \;\ldots\ldots (1)$$

$$\int_{-R}^{R} f(x)\,dx + \int_R^{-R} f(x+i\pi)\,dx$$

$$= \int_{-R}^{R}\frac{e^{ax}}{\cos hx}dx - \int_{-R}^{R}\frac{e^{a(x+i\pi)}}{\cos h(x+i\pi)}dx$$

$$= \int_{-R}^{R}\frac{e^{ax}}{\cos hx}dx - \int_{-R}^{R}\frac{2e^{ax}e^{ia\pi}}{e^{x+i\pi}+e^{-x-i\pi}}dx$$

$$= \int_{-R}^{R}\frac{e^{ax}}{\cos hx}dx - \int_{-R}^{R}\frac{2e^{ax}e^{ia\pi}}{e^{x}(-1)+e^{-x}(-1)}dx$$

$$= \int_{-R}^{R}\frac{e^{ax}}{\cos hx}dx + \int_{-R}^{R}\frac{e^{ax}e^{ia\pi}}{\cos hx}dx$$

$$= (1+e^{a\pi i})\int_{-R}^{R}\frac{e^{ax}}{\cos hx}dx \;\ldots\ldots (2)$$

$$f(R+iy) = \frac{e^{a(R+iy)}}{\cos h(R+iy)} = \frac{2e^{aR}e^{iay}}{e^{R+iy}+e^{-R-iy}}$$

$$\Rightarrow |f(R+iy)| \le \frac{2|e^{aR}|\,|e^{iay}|}{|e^{R}|\,|e^{iy}|+|e^{-R}|\,|e^{-iy}|}$$

$$\Rightarrow |f(R+iy)| \le \frac{2e^{aR}}{e^{R}+e^{-R}} \le \frac{2e^{aR}}{e^{R}-e^{-R}}$$

$$\Rightarrow |f(R+iy)| \le \frac{4e^{aR}}{e^{R}}, \text{ since } e^{R}-e^{-R} \ge \frac{1}{2}e^{R}$$

$$\therefore \left|\int_0^{\pi} f(R+iy)\,i\,dy\right| \le \int_0^{\pi}\frac{4e^{aR}}{e^{R}}dy = 4\pi e^{(a-1)R}$$

Since $|a| < 1$, so $e^{(a-1)R} \to 0$ when $R\to\infty$.

$$\therefore \left|\int_0^{\pi} f(R+iy)\,i\,dy\right| = 0 \Rightarrow \lim_{R\to\infty}\int_0^{\pi} f(R+iy)\,i\,dy = 0 \;\ldots\ldots (3)$$

Similarly, $\lim_{R\to\infty}\int_{\pi}^{0} f(-R+iy)\,i\,dy = 0 \;\ldots\ldots (4)$

Now taking limit $R\to\infty$ in (1) and (2), and then using the results of (3) and (4) we get

$$(1+e^{a\pi i})\int_{-\infty}^{\infty}\frac{e^{ax}}{\cos hx}dx + 0 + 0 = -2\pi i^2 e^{a\pi i/2}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ax}}{\cos hx}dx = \frac{2\pi e^{a\pi i/2}}{1+e^{a\pi i}} = \frac{2\pi}{e^{-a\pi i/2}+e^{a\pi i/2}}$$

$$= \frac{2\pi}{2\cos\frac{a\pi}{2}}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{e^{ax}}{\cos hx}\,dx + \int_{0}^{\infty} \frac{e^{ax}}{\cos hx}\,dx = \frac{\pi}{\cos\frac{a\pi}{2}} \quad \ldots\ldots (5)$$

Now, $\int_{-\infty}^{0} \frac{e^{ax}}{\cos hx}\,dx = \int_{\infty}^{0} \frac{e^{-ay}}{\cos h(-y)}(-dy)$

$= \int_{0}^{\infty} \frac{e^{-ay}}{\cos hy}\,dy$

Putting $x = -y$
$\Rightarrow dx = -dy$

x	0	$-\infty$
y	0	∞

$= \int_{0}^{\infty} \frac{e^{-ax}}{\cos hx}\,dx$ $\quad \because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

Putting this value in (5) we get,

$$\int_{0}^{\infty} \frac{e^{-ax}}{\cos hx}\,dx + \int_{0}^{\infty} \frac{e^{ax}}{\cos hx}\,dx = \frac{\pi}{\cos\frac{a\pi}{2}}$$

$$\Rightarrow \int_{0}^{\infty} \frac{e^{ax}+e^{-ax}}{\cos hx}\,dx = \frac{\pi}{\cos\frac{a\pi}{2}}$$

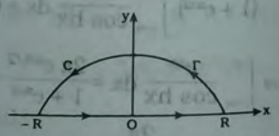
$$\Rightarrow \int_{0}^{\infty} \frac{2\cos h\,ax}{\cos hx}\,dx = \frac{\pi}{\cos\frac{a\pi}{2}}$$

$$\therefore \int_{0}^{\infty} \frac{\cos h\,ax}{\cos hx}\,dx = \frac{\pi}{2\cos\left(\frac{\pi a}{2}\right)}$$ Showed.

Solution-60. Consider the integral $\oint_c f(z)\,dz$ where $f(z) = \frac{e^{iz^2}-1}{z^2}$ and C is the closed contour consisting of

(i) The x-axis from $-R$ to R where R is very large,

(ii) The upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



We have $f(z) = \frac{e^{iz^2}-1}{z^2}$

$$= \frac{1}{z^2}\left[\left(1 + \frac{iz^2}{1!} + \frac{i^2z^4}{2!} + \frac{i^3z^6}{3!} + \cdots\right) - 1\right]$$

$$= \frac{1}{z^2}\left[iz^2 - \frac{z^4}{2!} - i\frac{z^6}{3!} + \cdots\right]$$

$$= i - \frac{z^2}{2!} - \frac{iz^4}{3!} + \cdots$$

This shows that f(z) has no poles with in the contour. As a matter f(z) is analytic for all finite values of z. Hence by Cauchy's residue theorem we have

$$\oint_c f(z)\,dz = 2\pi i \times 0$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 0 \cdots (1)$$

When $R \to \infty$ then $\int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{ix^2}-1}{x^2}\,dx$

$$= \int_{-\infty}^{\infty} \frac{\cos x^2 + i\sin x^2 - 1}{x^2}\,dx \ldots\ldots (2)$$

$$\left|\int_\Gamma f(z)\,dz\right| = \left|\int_\Gamma \frac{e^{iz^2}-1}{z^2}\,dz\right|$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \int_\Gamma \frac{|e^{iz^2}-1|}{|z^2|}\,|dz|$$

$|z| = R \Rightarrow z = Re^{i\theta}$
$\Rightarrow dz = iRe^{i\theta}\,d\theta$
$\Rightarrow |dz| = R\,d\theta$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \int_\Gamma \frac{|e^{iz^2}|+1}{R^2}\,R\,d\theta$$

$e^{iz^2} = e^{iR^2(\cos 2\theta + i\sin 2\theta)}$
$= e^{iR^2\cos 2\theta} \cdot e^{-R^2\sin 2\theta}$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \int_\Gamma \frac{e^{-R^2\sin 2\theta}+1}{R}\,d\theta$$

$\Rightarrow |e^{iz^2}| = e^{-R^2\sin 2\theta}$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \frac{2}{R}\int_0^{\pi/2} (e^{-4R^2\theta/\pi}+1)\,d\theta,$$ By Jordan's inequality.

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \frac{2}{R}\left[\frac{e^{-4R^2\theta/\pi}}{\frac{-4R^2}{\pi}}+\theta\right]_0^{\pi/2}$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \frac{2}{R}\left[\frac{-\pi}{4R^2}(e^{-R^2}-1)+\frac{\pi}{2}\right]$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \le \left[\frac{\pi}{R}+\frac{\pi}{2R^3}(1-e^{-2R^2})\right] \to 0 \text{ as } R\to\infty$$

$$\therefore \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0 \;\cdots\cdots (3)$$

Taking limit $R \to \infty$ in (1) and then using the results of (2) and (3) we get,

$$\int_{-\infty}^{\infty}\frac{\cos x^2 + i\sin x^2 - 1}{x^2}\,dx + 0 = 0$$

Equating real and imaginary parts we get

$$\int_{-\infty}^{\infty}\frac{\cos x^2 - 1}{x^2}\,dx = 0 \text{ and } \int_{-\infty}^{\infty}\frac{\sin x^2}{x^2}\,dx = 0$$

Adding these results we have

$$\int_{-\infty}^{\infty}\frac{\cos x^2 - 1}{x^2}\,dx + \int_{-\infty}^{\infty}\frac{\sin x^2}{x^2}\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos x^2 + \sin x^2 - 1}{x^2}\,dx = 0 \qquad \text{Proved.}$$

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. What is contour integration?

Ans : The process of integration along a contour is called contour integration.

2. What do you mean by a contour?

Ans : A contour is a curve consisting of finite number of smoth arcs joined end to end.

3. When a contour is called a closed contour?

Ans : If the starting point A of the first arc and the ending point B of the last arc coincide, then the contour is called a closed contour.

4. Write Jordan's inequality.

OR, Write down Jordan's inequality for contour integration. **[NUH–2013]**

Ans : The inequality $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$, where $0 \le \theta \le \frac{\pi}{2}$ is called the Jordan's inequality.

5. What is Jordan's lemma?

Ans : If f(z) is analytic except a finite number of singularities and $f(z) \to 0$ uniformly as $z \to \infty$, then the Jordan's lemma is

$$\lim_{R\to\infty}\int_\Gamma e^{imz} f(z)\,dz = 0,\ m > 0$$

where Γ is the semi-circle $|z| = R$, $\text{Im}(z) \ge 0$.

6. Under what condition $\lim_{R\to\infty}\int_\Gamma e^{imz} f(z)\,dz = 0,\ m > 0$. **[NUH–2013]**

Ans : $\lim_{R\to\infty}\int_\Gamma e^{imz} f(z)\,dz = 0,\ m > 0$, under the condition, if f(z) is analytic except a finite number of singularities and $f(z) \to 0$ uniformly as $z \to\infty$, where Γ is the semi-circle $|z| = R$, $\text{Im}(z) \ge 0$.

7. What do you know about indenting at a point?

Ans : When the poles of f(z) lie on the real axis and the integration is not possible at that poles, then we excluded those poles on the real axis by enclosing them with semi-circles of small radii. This procedure is known as indenting at a point.

8. On which the choice of contour depends?

Ans : The choice of contour depends on the nature of the function to be integrated.

9. IF C is are $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$ and if $\lim_{z \to \infty} zf(z) = A$ then what is the value of $\lim_{R \to \infty} \int_C f(z)\, dx$? **[NUH-2014]**

Ans. If C is arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$ and if $\lim_{z \to \infty} zf(z) = A$ then the value of $\lim_{R \to \infty} \int_C f(z)\, dz = i\, A$

$(\theta_2 - \theta_1)$.

EXERCISE-5

Part-A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. How can we found all the poles of a complex function f(z).
2. Write Cauchy's residue theorem.
3. What type of integrals we integrate round the unit circle?
4. In conttour integration when we evaluate $\int_{-\infty}^{\infty} f(x)\, dx$?
5. When the process of integration of the integrals of the form $\int_{-\infty}^{\infty} f(x)\, dx$ and $\int_{-\infty}^{\infty} f(x) \sin mx\, dx$ are same?

Part-B And C : Short & Broad Questions (সংক্ষিপ্ত ও বড় প্রশ্ন)

Prove the following by the method of contour integration

1. $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \pi$ **[RUH-2003]**
2. $\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$
3. $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a + \cos\theta}\, d\theta = 2\pi a\left(1 - \frac{a}{\sqrt{a^2 - 1}}\right)$, where $a > 1$
4. $\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{1 - 2p\cos 2\theta + p^2} = \pi \frac{1 - p + p^2}{1 - p}, 0 < p < 1$
5. $\int_0^{\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2}$ **[NUH-2005 (Phy)]**

   Ans : See solved problem-34(b)
6. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3} = \frac{3\pi}{8}$
7. $\int_0^{\infty} \frac{x^4\, dx}{x^6 - 1} = \frac{\pi\sqrt{3}}{6}$
8. $\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2 + a^2)^3} = \frac{\pi}{8a^3}$
9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx = \frac{5\pi}{12}$
10. $\int_0^{\infty} \frac{\cos mx}{x^2 + 1}\, dx = \frac{\pi}{2}$ **[RUMP-1984]**
11. $\int_0^{\infty} \frac{\cos x}{x^2 + a^2} = \frac{\pi e^{-a}}{2a}, a > 0$
12. $\int_0^{\infty} \frac{\cos x}{(1 + x^2)^2}\, dx = \frac{\pi}{2e}$ **[RUH-1972]**

13. $\int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b - a)$, where $a \geq b \geq 0$

14. $\int_0^{\infty} \frac{x \cos \pi x}{x^2 + 2x + 5} dx = \frac{\pi}{2} e^{-2\pi}$

15. $\int_0^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx = -\pi e^{-2\pi}$ [DUH-2005, RUMP-1984]

16. $\int_0^{\infty} \frac{\sin mx}{x(a^2 + x^2)^2} dx = \frac{\pi}{2a^4} - \frac{\pi}{4a^3} e^{-ma}\left(m + \frac{2}{a}\right)$, $m > 0, a > 0$

17. $\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4} dx = \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}$, where $m > 0, a > 0$

18. $\int_0^{\infty} \frac{\cos \pi x}{x(1 - x^2)} dx = 0$

19. $\int_0^{\infty} \frac{x^{a-1}}{1 - x} dx = \pi \cot a\pi$, $0 < a < 1$. [RUMP-1984]

20. $\int_0^{\infty} \frac{x^{m-1}}{1 + x} dx$, $0 < m < 1$ [NUH-1998]

Ans : Write $p = m$ in solved problem-56.

21. $\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} = \frac{2\pi}{3}$ [NUH-2006(Old)]

Ans : Solved problem-2.

22. $\int_0^{\infty} \frac{ln(1 + x)}{1 + x^2} dx = \frac{\pi}{2} \ln 2$. [NUH-2010]

23. Evaluate $\int_0^{2\pi} \frac{d\theta}{3 + 2 \cos \theta}$ [NUH-2010]

Ans : Solved problem-3(a).

CHAPTER-6

CONFORMAL MAPPING

6.1. Transformations or mappings :

Let us consider the complex valued function

$$w = f(z) = u + iv, \text{ where } z = x + iy.$$

Then $u = u(x, y)$, $v = (v(x, y)$ ...... (1)

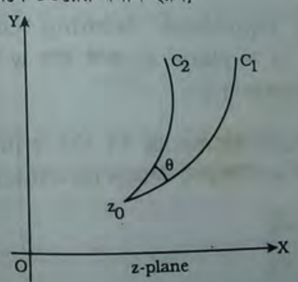
The set of equations in (1) defines a transformation or mapping which establishes a corresponce between points in the uv and xy planes. The equations (1) are called transformation equations.

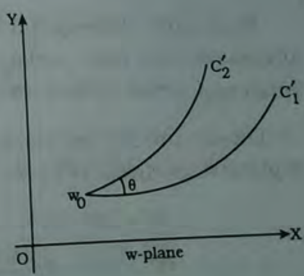
One-one correspondence : If to each point (x, y) of the z-plane there corresponds one and only one point (u, v) of the w-plane, and conversely, we say that the correspondence is one-one.

6.2. Conformal mapping : [NUH-1996, 1999, 2012, DUH-1996, 2001, 2005, KUH-2002, 2003]

Let the transformation $u = u(x, y)$, $v = v(x, y)$ map a point $z_0 = (x_0, y_0)$ of the z-plane to a point $w_0 = (u_0, v_0)$ of the w-plane. Also, let two curves C_1 and C_2 intersecting at z_0 are mapped C_1' and C_2' respectively. The mapping is called conformal if the angle at z_0 between C_1 and C_2 is equal to the angle at w_0 between C_1' and C_2' both in magnitude and sence.

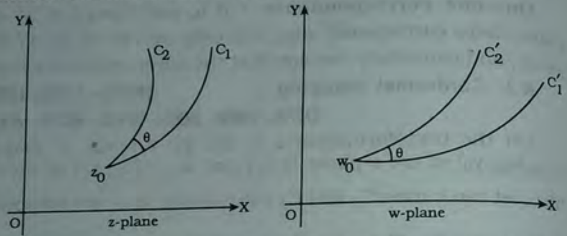
[মনে করি $u = u(x, y)$, $v = v(x, y)$ রূপান্তর z তলের একটি বিন্দু $z_0 = (x_0, y_0)$ কে w তলের একটি বিন্দু $w_0 = (u_0, v_0)$ এ চিত্রিত করে। আরো মনে করি z_0 বিন্দুতে ছেদকারী দুইটি বক্ররেখা C_1 ও C_2 যথাক্রমে C_1' ও C_2' চিত্রিত করে। চিত্রণটিকে conformal বলে যদি z_0 বিন্দুতে C_1 ও C_2 এর মধ্যবর্তী কোণ, w_0 বিন্দুতে C_1' ও C_2' এর মধ্যবর্তী কোণের মান ও অর্থে উভয়ের সমান হয়।]





Isogonal mapping : [DUH-2002]

Let the transformation $u = u(x, y)$, $v = v(x, y)$ map a point $z_0 = (x_0, y_0)$ of the z-plane to a point $w_0 = (u_0, v_0)$ of the w-Plane. Also, let two curves C_1 and C_2 intersecting at z_0 are mapped C_1' and C_2' respectively. The mapping is called isogonal if the angle at z_0 between C_1 and C_2 is equal to the angle at w_0 between C_1' and C_2' in magnitudes but not necessarily the sense.



6.3 Necessary condition for w = f(z) to be a conformal mapping. [NUH-2012, 2015, KUH-2003]

Theorem-1. If the mapping $w = f(z)$ is conformal, then $f(z)$ is an analytic function of z. [যদি $w = f(z)$ চিত্রণটি conformal হয় তখন $f(z)$ হল z এর বৈশ্লেষিক ফাংশন]

Proof : We have $w = f(z)$

$$\Rightarrow u + iv = f(z)$$

$$\Rightarrow u = u(x, y) \text{ and } v = v(x, y) \text{ ...... (1)}$$

Both are differentiable equations defining conformal transformation from z-plane to w-plane. [z-তলে হতে w-তলে বর্ণিত conformal রূপান্তর সহ উভয়ে অন্তরীকরণযোগ্য।]

Let ds and dσ be the length elements in the z-plane and w-plane respectively. [ধরি z-তল ও w-তলে দৈর্ঘ্য উপাদান ds এবং dσ]

$$\therefore ds^2 = dx^2 + dy^2 \text{ ...... (2)}$$

$$d\sigma^2 = du^2 + dv^2 \text{ ...... (3)}$$

From (1) we have [(1) হতে পাই] $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Putting these values in (3) we get, [এইমান (3) এ বসাইয়া পাই]

$$d\sigma^2 = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial u}{\partial y}\right)^2 dy^2 + 2\frac{\partial u}{\partial x}dx \cdot \frac{\partial u}{\partial y} \cdot dy$$

$$+ \left(\frac{\partial v}{\partial x}\right)^2 dx^2 + \left(\frac{\partial v}{\partial y}\right)^2 dy^2 + 2\frac{\partial v}{\partial x}dx \cdot \frac{\partial v}{\partial y}dy$$

$$\Rightarrow d\sigma^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]dx^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dy^2$$

$$+ 2\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]dx\,dy \text{ ...... (4)}$$

Given that $w = f(z)$ is conformal. So the ratio dσ : ds is independent of direction. Thus from (2) and (4) we have [দেওয়া আছে $w = f(z)$ conformal. সুতরাং dσ : ds অনুপাত দিক অনির্ভরশীল। অতএব (2) ও (4) হতে পাই]

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0} \text{ ...... (5)}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \text{ ...... (6)}$$

$$\text{and } \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \text{ ...... (7)}$$

Equation (6) is satisfied if [সমীকরণ (6) সিদ্ধ হয় যদি] $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\text{and } \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \text{ ...... (8)}$$

which is the Cauchy-Riemann equations. Hence the function $w = f(z)$ is an analytic function of z. [যাহা কচি-রীম্যান সমীকরণ। অতএব $w = f(z)$ ফাংশন হল z এর বৈশ্লেষিক ফাংশন]

The equation (7) is satisfied if [সমীকরণ (7) সিদ্ধ হয় যদি]

$$\frac{\partial u}{\partial x} = \frac{-\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ ...... (9)}$$

If we write $-v$ for v, then equation (9) reduces to equation (8).

In this case magnitude of the angle will be preserved but direction will be changed. Hence equation (9) correspond to an isogonal transformation but not comformal transformation Thus, we have established that if the transformation $w = f(z)$ is conformal then f(z) must be an analytic function of z.

[যদি আমরা v এর জন্য $-v$ লিখি, তখন সমীকরণ (9) হ্রাস পেয়ে সমীকরণ (8) হয়। এইক্ষেত্রে কোণের মান সংরক্ষিত থাকে কিন্তু দিক পরিবর্তন হবে। অতএব সমীকরণ (9) একটি এইসোগোনাল (isogonal) রূপান্তর সমন্ধ কিন্তু conformal রূপান্তর নয়। অতএব, আমরা প্রতিষ্ঠিত করেছি যে, যদি রূপান্তর $w = f(z)$ conformal হয় তখন f(z) অবশ্যই z এর বৈশ্লেষিক ফাংশন হবে।]

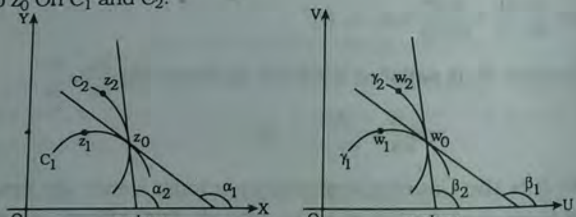
Sufficient condition for w = f(z) to be a conformal mapping. **[KUH–2002]**

Theorem–2. If f(z) is analytic and $f'(z) \neq 0$ in a region R. then the mapping $w = f(z)$ is conformal in R. **[NUH–1996, 1999]**

[একটি এলাকা R এ যদি f(z) বৈশ্লেষিক হয় এবং $f'(z) \neq 0$, তখন $w = f(z)$ চিত্রণটি R এ conformal.]

Proof : Let $w = f(z)$ be analytic and $f'(z) \neq 0$ in the region R. Also, let the continuous curves C_1 and C_2 intersects at z_0, and the tangents at z_0 of these curves make angles α_1, α_2 with the real axis.

Let z_1 and z_2 be two equal distance points from z_0 and very near to z_0 On C_1 and C_2.



z-plane   w-plane

If this distance is r where r is small, we can write

$$z_1 - z_0 = re^{i\theta_1} \text{ and } z_2 - z_0 = re^{i\theta_2}$$

When $r \to 0$ then the lines $z_1 z_0$ and $z_2 z_0$ will tend to be tangents at z_0 to the curve C_1 and C_2. Thus we have

$$\theta_1 \to \alpha_1 \text{ and } \theta_2 \to \alpha_2 \text{ when } r \to 0.$$

Let w_0, w_1, w_2 be the images in w-plane of the points z_0, z_1 and z_2 respectively in the z-plane. Since the point z_0 moves to z_1 along the curve C_1, the image point w_0 moves to ω, along the curve γ_1. Similarly, the point z_0 moves to z_2 along the curve C_2, the image point w_0 moves to w_2 along the curve γ_2.

Let us suppose that the tangents of the curves γ_1 and γ_2 at w_0 make angles β_1 and β_2 with the real axis. Then as above

[প্রমাণঃ মনেকরি R এলাকায় $w = f(z)$ বৈশ্লেষিক এবং $f(z) \neq 0$. আরো ধরি অবিচ্ছিন্ন বক্ররেখা C_1 ও C_2 পরস্পর z_0 বিন্দুতে ছেদ করে এবং z_0 বিন্দুতে এই বক্ররেখাগুলির স্পর্শক বাস্তব অক্ষের সাথে α_1, α_2 কোণ তৈরী করে।

মনেকরি C_1 ও C_2 এর উপর z_0 এর খুব নিকটবর্তী এবং z_0 হতে সমান দুরত্বে দুইটি বিন্দু z_1 ও z_2. যদি এই দূরত্ব r হয়, যেখানে r খুব ক্ষুদ্র তাহলে আমরা লিখতে পারি।

$z_1 - z_0 = re^{i\theta_1}$ এবং $z_2 - z_0 = re^{i\theta_2}$

যখন $r \to 0$ তখন $z_1 z_0$ ও $z_2 z_0$ রেখাদ্বয় C_1 ও C_2 বক্ররেখাতে z_0 এ স্পর্শক হবে। অতএব আমরা পাই $\theta_1 \to \alpha_1$ এবং $\theta_2 \to \alpha_2$ যখন $r \to 0$.

মনেকরি w তলে z_0, z_1, z_2 বিন্দুর পতিবিম্ব যথাক্রমে w_0, w_1, w_2. যেহেতু C_1 বক্ররেখা বরাবর z_1 বিন্দু z_0 এ যায়, γ_1 বক্ররেখা বরাবর প্রতিবিম্ব বিন্দু w_0 যায় w এ একইভাবে C_2 বক্ররেখা বরাবর z_0 যায় z_2 , γ_2 বক্ররেখা বরাবর প্রতিবিম্ব বিন্দু w_0 যায় w_2 এ।

ধরি w_0 বিন্দুতে γ_1 ও γ_2 বক্ররেখাদ্বয়েরে স্পর্শকদ্বয় বাস্তব অক্ষের সাথে β_1 ও β_2 কোণ তৈরী করে। তখন উপরে নিয়মে।]

$$w_1 - w_0 = r_1 e^{i\phi_1} \text{ and } w_2 - w_0 = r_1 e^{i\phi_2}$$

and $\phi_1 \to \beta_1$ and $\phi_2 \to \beta_2$ when $r_1 \to 0$

$$\therefore f'(z_0) = \lim_{z_1 \to z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0} = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} \quad [\because w = f(z)]$$

$$= \lim_{z_1 \to z_0} \frac{r_1 e^{i\phi_1}}{re^{i\theta_1}} = \lim_{z_1 \to z_0} \frac{r_1}{r} e^{i(\phi_1 - \theta_1)}$$

Since f(z) is analytic and $f'(z_0) \neq 0$, so we can write [যেহেতু f(z) বৈশ্লেষিক এবং $f'(z_0) \neq 0$ সুতরাং আমরা লিখতে পারি]

$$f'(z_0) = Re^{i\lambda}$$

$$\Rightarrow \lim_{z_1 \to z_0} \frac{r_1}{r} e^{i(\phi_1 - \theta_1)} = Re^{i\lambda}$$

Equating modulas and amplitude we get, [মানাঙ্ক ও কোনাঙ্ক সমীকৃত করে পাই]

$$\lim_{z_1 \to z_0} \frac{r_1}{r} = R \text{ and } \lim (\phi_1 - \theta_1) = \lambda$$

$$\Rightarrow \lim \phi_1 - \lim \theta_1 = \lambda$$

$$\Rightarrow \beta_1 - \alpha_1 = \lambda$$

$$\Rightarrow \beta_1 = \alpha_1 + \lambda$$

Similarly, we can show that [একইভাবে আমরা দেখাতে পারি যে] $\beta_2 = \alpha_2 + \lambda$

$$\therefore \beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 + \lambda) = \alpha_1 - \alpha_2$$

This shows that the angles between the curves γ_1 and γ_2 at w_0 is equal to the angle between the curves C_1 and C_2 at z_0. Also, from the figure and description it is clear that the above angles have the same sense. Hence the mapping $w = f(z)$ is conformal.

[ইহা দেখায় যে w_0 বিন্দুতে γ_1 ও γ_1 বক্ররেখাদ্বয়ের মধ্যবর্তী কোণ, z_0 বিন্দুতে C_1 ও C_2 বক্ররেখাদ্বয়ের মধ্যেবর্তী কোণের সমান । আরো, চিত্র এবং বর্ণনা হতে ইহা পরিষ্কার যে উপরের কোণদ্বয়ের একই অনুভূতি (sense). অতএব $w = f(z)$ চিত্রণটি conformal.]

The case $f'(z_0) = 0$, When $f'(z) = 0$ [$f'(z_0) = 0$ ক্ষেত্রটি যখন $f'(z_0) = 0$]

Suppose that $f'(z_0)$ has a zero of order $(n - 1)$ at the point z_0. Then $f'(z_0) = f''(z_0) = \cdots = f^{n-1}(z_0) = 0$ but $f^n(z_0) \neq 0$. Hence by Taylor's theorem in the neighborhood of z_0 we have [ধরি z_0 বিন্দুতে $(n - 1)$ ক্রমের $f'(z_0)$ এর একটি শূণ্য আছে। তখন $f'(z_0) = f''(z_0) = \cdots = f^{n-1}(z_0) = 0$ কিন্তু $f^n(z_0) \neq 0$ অতএব টেইলরের উপপাদ্য দ্বারা z_0 এর প্রতিবেশে পাই]

$$f(z) = f(z_0) + 0 + 0 + \cdots + 0 + a_n(z - z_0)^n + \cdots, \text{ where } a_n = \frac{f^n(z_0)}{n!} \neq 0.$$

$$\Rightarrow f(z) = f(z_0) + a_n(z - z_0)^n + \cdots$$

$$\therefore f(z_1) = f(z_0) + a_n(z_1 - z_0)^n + \cdots, \text{ Putting } z = z_1$$

$$\Rightarrow f(z_1) - f(z_0) = a_n(z_1 - z_0)^n + \cdots$$

$$\Rightarrow w_1 - w_0 = a_n(z - z_0)^n + \cdots \quad [\because w = f(z)]$$

$$\Rightarrow r_1 e^{i\phi_1} = |a_n| \cdot r^n \cdot e^{i(n\theta_1 + \lambda)} + \cdots, \quad \text{where } \lambda = \text{amp}(a_n)$$

$$\Rightarrow \text{Lim } \phi_1 = \lim(n\theta_1 + \lambda) = n\alpha_1 + \lambda$$

Similarly [একইভাবে], $\lim \phi_2 = n\alpha_2 + \lambda$

$$\therefore \lim(\phi_2 - \phi_1) = (n\alpha_2 + \lambda) - (n\alpha_1 + \lambda) = n(\alpha_2 - \alpha_1)$$

Thus, we see that the curves γ_1 and γ_2 still have tangents at w_0 but the magnitude of the angles between the tangents is not preserved.

Also the linear magnification $R = \lim \frac{r_1}{r} = 0$.

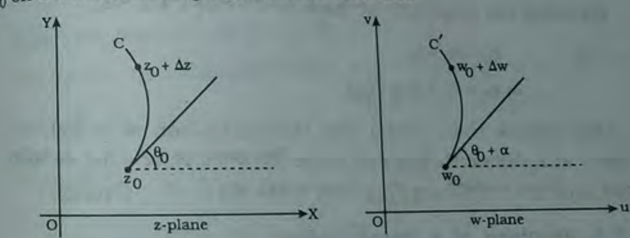
Hence the conformal property does not hold good at a point where $f'(z) = 0$. [অতএব, আমরা দেখি যে w_0 বিন্দুতে γ_1 ও γ_2 বক্ররেখার এখনও স্পর্শক আছে কিন্তু স্পর্শকদ্বয়ের মধ্যবর্তী কোণের মান সংরক্ষিত হয় নাই।

আরো, রৈখিক বিবর্ধন $R = \lim \frac{r_1}{r} = 0$.

অতএব একটি বিন্দু যেখানে $f'(z) = 0$ সেখানে conformal ধর্ম খাটে না।]

Theorem–3. Transformation $w = f(z)$ where $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$ the tangent at z_0 to any curve in the z-plane passing through z_0 is rotated through the angle arg $f'(z_0)$. **[RUH–1997, 2000]**

Proof : Let C be the curve in the z-plane and the tangent at z_0 on it makes an angle θ_0 with the real axis.



As a point moves from z_0 to $z_0 + \Delta z$ along the curve C, the image point moves along the curve C' in the w-plane from w_0 to $w_0 + \Delta w$. Here we use the parameter t to describe the curve. Then corresponding to the path $z = z(t)$ [or $x = x(t)$, $y = y(t)$] in the z-plane, we have the path $w = w(t)$ [or $u = u(t)$, $v = v(t)$] in the w-plane. The tangent vectors on C and C' are respectively the derivatives $\frac{dz}{dt}$ and $\frac{dw}{dt}$.

[প্রমাণ ঃ মনেকরি C বক্ররেখাটি z তলে এবং ইহার উপর z_0 বিন্দুতে স্পর্শক বাস্তব অক্ষের সাথে θ_0 কোণ তৈরী করে। যেহেতু C বক্ররেখা বরাবর বিন্দুটি z_0 হতে $z_0 + \Delta z$ এ যায়, প্রতিবিম্ব বিন্দু C' বক্ররেখা বরাবর w তলে w_0 হতে $w_0 + \Delta w$ এ যায়। এখানে বক্ররেখার বর্ণনার জন্য আমরা t পরামিতি ব্যবহার করেছি। তখন z তলে অনুসঙ্গী পথ $z = z(t)$ [অথবা $x = x(t), y = y(t)$]. আমরা পাই w তলে পথ $w = w(t)$ [অথবা $u = u(t), v = v(t)$]. C এবং C' এর উপর স্পর্শক ভেক্টর হল যথাক্রমে $\frac{dz}{dt}$ এবং $\frac{dw}{dt}$]

$$\text{Now } \frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} = \frac{d}{dz}f(z)\cdot\frac{dz}{dt} = f'(z)\frac{dz}{dt}$$

In particular, at z_0 and w_0 we have [নির্দিষ্ট ভাবে, z_0 ও w_0 এ পাই]

$$\left[\frac{dw}{dt}\right]_{w=w_0} = f'(z_0)\left[\frac{dz}{dt}\right]_{z=z_0} \quad \cdots\cdots (1)$$

provided $f(z)$ is analytic at $z = z_0$ [$z = z_0$ এ $f(z)$ বৈশ্লেষিক শর্তে]

Writting $\left[\frac{dw}{dt}\right]_{w=w_0} = \rho_0 e^{i\phi_0}$, $f'(z_0) = re^{i\alpha}$, $\left[\frac{dz}{dt}\right]_{z=z_0} = r_0 e^{i\theta_0}$

We have from (1) [(1) হতে আমরা পাই]

$$\rho_0 e^{i\phi_0} = re^{i\alpha}\, r_0 e^{i\theta_0} = r_0\, re^{i(\theta_0+\alpha)}$$

Equating the amplitude we have [কোণাঙ্ক সমীকৃত করে পাই]

$$\phi_0 = \theta_0 + \alpha$$

$$\Rightarrow \phi_0 = \theta_0 + \arg f'(z_0)$$

This shows that, under the transformation $w = f(z)$, the tangent at z_0 rotate through an angle [ইহা দেখায় যে, $w = f(z)$ এর অধীনে রূপান্তর, z_0 বিন্দুতে স্পর্শকটি $\arg f'(z_0)$ কোণে আবর্তন করে।] (Proved)

6.4. Jacobian of a transformation.

If $w = f(z) = u + iv$ is analytic in a region R, then $\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2$ [NUH-1999]

Proof : Given that $w = f(z) = u + iv$ is analytic. By Cauchy-Riemann equations we have [দেওয়া আছে $w = f(z) = u + iv$ বৈশ্লেষিক। কচি-রীম্যান সমীকরণ দ্বারা পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad \cdots\cdots (1)$$

Again, we have [আবার, আমাদের আছে] $w = f(z) = u + iv$

$$\therefore \frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \quad \text{by (1)}$$

$$\therefore |f'(z)| = \left|\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial u}{\partial y}\right)^2}$$

$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \quad \cdots\cdots (2)$$

$$\text{Now } \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{vmatrix}; \quad \text{by (1)}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = |f'(z)|^2, \text{ by (2)}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2 \qquad \text{(Proved)}$$

Prove that $\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1$.

Proof : We have $u = u(x, y)$, $v = v(x, y)$ ...... (1)

The inverse transformations are $x = x(u, v)$, $y = y(u, v)$ ...... (2)

From (1), $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$ ...... (3)

From (2), $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$, $dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$ ...... (4)

$$\therefore du = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$\Rightarrow du = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}\right)dv$$

$$\Rightarrow \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = 1, \quad \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = 0 \quad \cdots\cdots (5)$$

Similarly from the 2nd equation of (3) we get

[By interchanging u and v in (5)]

$$\frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = 1, \quad \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} = 0 \quad \cdots\cdots (6)$$

$$\text{Now } \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1 \qquad \text{(Proved)}$$

6.5. Bilinear transformation or Mobius transformation Linear transformation :

[NUH–1996, 2000, 2005, 2014, DUH–2005]

A transformation of the form $w = az + b$ is called a Linear transformation, where a, b are complex constants with $a \neq 0$.

Bilinear transformation : The transformation T defined by

$$w = T(z) = \frac{az + b}{cz + d}$$

is called a **bilinear transformation**, where a, b, c, d are complex constants and z, w are complex variables.

[**bilinear রূপান্তর** ঃ T দ্বারা বর্ণিত $w = T(z) = \frac{az + b}{cz + d}$ রূপান্তরকে bilinear রূপান্তর বলে, যেখানে a, b, c, d জটিল ধ্রুবক এবং z, w জটিল চলক।]

It is also called the **linear fractional transformation** or **Mobius transformation** after the name of A. F. Mobius (1790 – 1868) who first studied these transformations.

Determinent : We have $w = \frac{az + b}{cz + d} = \frac{a}{c} \cdot \frac{z + \frac{b}{a}}{z + \frac{d}{c}}$

If $\frac{b}{a} = \frac{d}{c} \Rightarrow ad - bc = 0$ then $w = \frac{a}{c}$ which is constant. Thus if $ad - bc = 0$ then for every value of z we have the same value of w. The expression $ad - bc$ is called the determinent of bilinear transformation.

Condition for one-one correspondence

Let $w_1 = \frac{az_1 + b}{cz_1 + d}$ and $w_2 = \frac{az_2 + b}{cz_2 + d}$

$$\therefore w_1 - w_2 = \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d}$$

$$= \frac{acz_1z_2 + adz_1 + bcz_2 + bd - acz_1z_2 - adz_2 - bcz_1 - bd}{(cz_1 + d)(cz_2 + d)}$$

$$= \frac{ad(z_1 - z_2) - bc(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

$$= \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

$\Rightarrow w_1 - w_2 = 0$ or $w_1 = w_2$ if $ad - bc = 0$ provided $z_1 \neq \frac{-d}{c}$ and $z_2 \neq \frac{-d}{c}$

Thus, w is constant if $ad - bc = 0$

and w becomes meaningless if $z_1 = \frac{-d}{c}$ or $z_2 = \frac{-d}{c}$.

Hence, the necessary condition for the bilinear transformation to set up a one-one correspondence between the points of the closed z-plane and closed w-plane is $ad - bc \neq 0$.

Critical points :

We have $w = \frac{az + b}{cz + d}$

$$\therefore \frac{dw}{dz} = \frac{d}{dz}\left(\frac{az + b}{cz + d}\right)$$

$$= \frac{(cz + d) \cdot a - (az + b) \cdot c}{(cz + d)^2}$$

$$= \frac{acz + da - acz - bc}{(cz + d)^2}$$

$$= \frac{ad - bc}{(cz + d)^2}$$

When $z = \frac{-d}{c}$ then $\frac{dw}{dz} = \infty$ and when $z = \infty$ then $\frac{dw}{dz} = 0$.

At the points $z = \frac{-d}{c}$ and $z = \infty$ conformal property does not hold good. These two points are called the critical points.

6.6. Some general transformations

(Geometrical interpretations of transformations)

1. Translation : By the transformation $w = z + \alpha$, the figures in the z-plane are displaced or translated in the direction of the vector α.

2. Rotation : By the transformation $w = e^{i\theta}z$, the figures in the z-plane are rotated through an angle θ. If $\theta > 0$, the rotation is counterclockwise and if $\theta < 0$, the rotation is clockwise.

3. Stretching (or Magnification) : By the transformation
$w = az$, the figures in the z-plane are stretched (or contracted) in the direction z if $a > 1$ (or $0 < a < 1$).

Contraction is a special case of stretching.

4. Inversion : $w = \frac{1}{z}$.

If $|z| = r$ and $\arg z = \theta$ then $|w| = \frac{1}{r}$ and $\arg w = -\theta$.

Prove that every bilinear transformation is the resultant of translation rotation, magnification and inversion. [RUH–1996, 1998, 2001]

Proof : Let us consider the bilinear transformation $w = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$, $a \neq 0$. [$w = \frac{az+b}{cz+d}$, যেখানে $ad - bc \neq 0$, $a \neq 0$ দ্বিরৈখিক রূপান্তরটি বিবেচনা করি]

By actual division we have [প্রকৃত ভাগের সাহায্যে পাই]

$$w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}$$

$$\Rightarrow w = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z + \frac{d}{c}}$$

$$cz + d \,\big)\, az + b \,\big(\, \frac{a}{c}$$
$$\underline{az + ad/c}$$
$$b - \frac{ad}{c}$$
$$= \frac{bc - ad}{c}$$

Consider the following transformation [নিম্নের রূপান্তরগুলি বিবেচনা করি]

(i) $z_1 = z + \frac{d}{c}$ Translation [স্থানান্তর]

(ii) $z_2 = \frac{1}{z_1}$ Inversion [বিপরীতকরণ]

(iii) $z_3 = \frac{bc-ad}{c^2} z_2 \Rightarrow z_3 = \beta z_2$, Rotation and Magnification

where [ঘূর্ণন এবং বিবর্ধনকরণ যেখানে] $\beta = \frac{bc-ad}{c^2}$

Hence [অতএব] $w = \frac{a}{c} + z_3$ which is again a translation. [যাহা পুনরায় একটি স্থানান্তর]

Thus every bilinear transformation is the resultant of translation, rotation, magnification and inversion. [অতএব প্রত্যেক দ্বিরৈখিক রূপান্তর হল স্থানান্তর, ঘূর্ণন, বিবর্ধন ও বিপরীতকরণের লব্ধি ।]

6.7. Cross ratio : [DUH–2002]

The cross ratio of four complex numbers z_1, z_2, z_3, z_4 taken in order is defined as $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$. It is denoted by (z_1, z_2, z_3, z_4).

$$\therefore (z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}.$$

Preservance of cross ratio under the bilinear transformation.

Theorem–4. If w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under the bilinear transformation then

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

that is, the cross ratio of w_1, w_2, w_3, w_4 = the cross ratio of z_1, z_2, z_3, z_4. [DUH–2002]

Proof : Let $w = \frac{az+b}{cz+d}$ be the bilinear transformation. Then

$$w_1 = \frac{az_1+b}{cz_1+d}, \quad w_2 = \frac{az_2+b}{cz_2+d}$$

$$w_3 = \frac{az_3+b}{cz_3+d}, \quad w_4 = \frac{az_4+b}{cz_4+d}$$

$$\therefore w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{acz_1z_2 + adz_1 + bcz_2 + bd - acz_1z_2 - bcz_1 - bd}{(cz_1 + d)(cz_2 + d)}$$

$$= \frac{ad(z_1 - z_2) - bc(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

$$\Rightarrow w_1 - w_2 = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \quad \ldots\ldots (1)$$

Similarly, $$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \quad \ldots\ldots (2)$$

$$w_3 - w_4 = \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)} \quad \ldots\ldots (3)$$

$$w_4 - w_1 = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)} \quad \ldots\ldots (4)$$

Now by (1), (2), (3) and (4) we have

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{\dfrac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \cdot \dfrac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)}}{\dfrac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \cdot \dfrac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}}$$

$$= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 \ z_1)}$$

$\therefore (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$.

Bilinear transformation for three given points.

Find the bilinear transformation which transforms the points z_1, z_2, z_3 of the z-plane into the points w_1, w_2, w_3, of w-plane.

Solution : Let $w = \dfrac{az + b}{cz + d}$ be the bilinear transformation.

Since w_1, w_2, w_3 are the images of z_1, z_2, z_3 so we have

$$w_1 = \frac{az_1 + b}{cz_1 + d},\ w_2 = \frac{az_2 + b}{cz_2 + d} \text{ and } w_3 = \frac{az_3 + b}{cz_3 + d}$$

$$\therefore w - w_1 = \frac{a + b}{cz + d} - \frac{az_1 + b}{cz_1 + d}$$

$$= \frac{acz_1z + adz + bc\,z_1 + bd - aczz_1 - adz_1 - bcz - bd}{(cz + d)(cz_1 + d)}$$

$$= \frac{ad(z - z_1) - bc(z - z_1)}{(cz + d)(cz_1 + d)}$$

$$\Rightarrow w - w_1 = \frac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)} \quad \ldots\ldots (1)$$

Similarly, $$w_1 - w_2 = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \quad \ldots\ldots (2)$$

$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \quad \ldots\ldots (3)$$

$$w_3 - w = \frac{(ad - bc)(z_3 - z)}{(cz_3 + d)(cz + d)} \quad \ldots\ldots (4)$$

Now by (1), (2), (3) and (4) we have

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{\dfrac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)} \cdot \dfrac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}}{\dfrac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \cdot \dfrac{(ad - bc)(z_3 - z)}{(cz_3 + d)(cz + d)}}$$

$$\Rightarrow \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$\Rightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.

which is our desired result.

6.8. Fixed or invariant points :

A fixed or invariant point of a transformation $w = f(z)$ is a point whose image is the same complex number. The fixed points are obtained from the equation $w = f(z) = z$.

Theorem on fixed (invariant) point : Prove that in general there are two values of z for which $w = z$ (invariant or fixed points) but there is only one if $(a - d)^2 + 4bc = 0$. Show that if there are distinct invariant points p and q the transformation may be put in the form

$$\frac{w - p}{w - q} = k\frac{z - p}{z - q}$$

and that if there is only one invariant point p, the transformation may be put in the form,

$$\frac{1}{w - p} = \frac{1}{z - p} + k.$$

Proof : Let $w = \dfrac{az + b}{cz + d}$ be the bilinear transformation.

Then for invariant points we have $w = z$

$$\Rightarrow \frac{az + b}{cz + d} = z$$

$$\Rightarrow cz^2 + dz = az + b$$

$$\Rightarrow cz^2 + (d-a)z - b = 0 \;\cdots\cdots (1)$$

$$\therefore z = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c} \;\cdots\cdots (2)$$

Since (1) is a quadratic equation, so we have two invariant (or fixed) points.

If there is a only one fixed point then (1) has equal roots and this happen when

$$(d-a)^2 + 4bc = 0$$

$$\Rightarrow (a-d)^2 + 4bc = 0$$

2nd Part : If p and q are two distinct invariant points then from (2), let

$$\left.\begin{aligned} P &= \frac{a-d+\sqrt{(a-d)^2+4bc}}{2c} \\ \text{and } q &= \frac{a-d-\sqrt{(a-d)^2+4bc}}{2c} \end{aligned}\right\} \cdots\cdots (3)$$

Since p and q are the roots of (1), so we have

$$cp^2 + (d-a)p - b = 0 \Rightarrow cp^2 - ap = b - pd \;\cdots\cdots (4)$$

$$\text{and } cq^2 + (d-a)q - b = 0 \Rightarrow cq^2 - aq = b - dq \;\cdots\cdots (5)$$

$$\therefore w - p = \frac{az+b}{cz+d} - p = \frac{(a-pc)z + b - pd}{cz+d}$$

$$= \frac{(a-pc)z + p(cp-a)}{cz+d}, \text{ by (4)}$$

$$\Rightarrow w - p = \frac{(a-pc)(z-p)}{cz+d} \;\cdots\cdots (6)$$

Similarly, $w - q = \dfrac{(a-qc)(z-q)}{cz+d}$ ..... (7) [Replacing p by q in (6)]

$$\therefore \frac{w-p}{w-q} = \frac{(a-pc)(z-p)}{cz+d} \cdot \frac{cz+d}{(a-qc)(z-q)}$$

$$= \frac{a-pc}{a-qc} \cdot \frac{z-p}{z-q}$$

$$\Rightarrow \frac{w-p}{w-q} = k\frac{z-p}{z-q}, \text{ where } k = \frac{a-pc}{a-qc}$$

3rd Part : If there is only one fixed point then from (3)

$$p = q = \frac{a-d}{2c} \Rightarrow d = a - 2cp \;\cdots\cdots (8)$$

$$\therefore \frac{1}{w-p} = \frac{1}{\dfrac{(a-pc)(z-p)}{cz+d}} = \frac{cz+d}{(a-pc)(z-p)}; \text{ by (6)}$$

$$= \frac{cz + a - 2cp}{(a-pc)(z-p)} \quad \text{by (8)}$$

$$= \frac{c(z-p) + a - cp}{(a-pc)(z-p)}$$

$$= \frac{c(z-p)}{(a-pc)(z-p)} + \frac{a-cp}{(a-pc)(z-p)}$$

$$= \frac{c}{a-cp} + \frac{1}{z-p}$$

$$= k + \frac{1}{z-p}, \text{ where } k = \frac{c}{a-cp}$$

$$\Rightarrow \frac{1}{w-p} = \frac{1}{z-p} + k \quad \text{(Showed)}$$

6.9. Group property of bilinear transformations :

Theorem–5. The set of all bilinear transformations form a non abelian group for the composition of product of transformation.

Proof : Let $w = T(z) = \dfrac{az+b}{cz+d}$ be a bilinear transformation, where a, b, c, d are complex constants and $ad - bc \neq 0$. Let S be the set of all bilinear transformations defined by the above formula.

(i) Closure law : If $T_1, T_2 \in S$, then $T_1(z) = \backslash F(a_1z + b_1, c_1z + d_1)$, $T_2(z) = \dfrac{a_2z + b_2}{c_2z + d_2}$

where $a_1d_1 - b_1c_1 \neq 0$ and $a_2d_2 - b_2c_2 \neq 0$.

$$\therefore T_1 T_2(z) = T_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)$$

$$= \frac{a_1\dfrac{a_2z+b_2}{c_2z+d_2} + b_1}{c_1\dfrac{a_2z+b_2}{c_2z+d_2} + d_1}$$

$$= \frac{a_1a_2z + a_1b_2 + b_1c_2z + b_1d_2}{a_2c_1z + b_2c_1 + c_2d_1z + d_1d_2}$$

$$= \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(a_2c_1 + c_2d_1)z + (b_2c_1 + d_1d_2)}$$

$$= \frac{AZ + B}{CZ + D}, \text{ say} \cdots\cdots (1)$$

where $A = a_1a_2 + b_1c_2$, $B = a_1b_2 + b_1d_2$

$C = a_2c_1 + c_2d_1$, $D = b_2c_1 + d_1d_2$

Now $AD - BC = (a_1a_2 + b_1c_2)(b_2c_1 + d_1d_2) - (a_1b_2 + b_1d_2)(a_2c_1 + c_2d_1)$

$= a_1a_2b_2c_1 + a_1a_2d_1d_2 + b_1b_2c_1c_2 + b_1c_2d_1d_2$

$\quad - a_1a_2b_2c_1 - a_1b_2c_2d_1 - a_2b_1c_1d_2 - b_1c_2d_1d_2$

$= a_1d_1(a_2d_2 - b_2c_2) - b_1c_1(a_2d_2 - b_2c_2)$

$= (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0.$

Hence (1) is a bilinear transformation.

Therefore $T_1, T_2 \in S \Rightarrow T_1 T_2 \in S$ and closure law is satisfied.

(ii) Associative law : We know that in the case of mapping

$$T_1(T_2T_3) = (T_1T_2)T_3$$

Hence associative law holds in S.

(iii) Identity law : The identity transformation $I(z) = z$ is the identity of the group.

(iv) Inverse law : Let $w = T(z) = \dfrac{az + b}{cz + d}$.

Then $az + b = czw + dw$

$\Rightarrow (a - cw)z = dw - b$

$\Rightarrow z = \dfrac{dw - b}{-cw + a}$

Also, $w = T(z) \Rightarrow z = T^{-1}(w) = \dfrac{dw - b}{-cw + a}$

Here $da - (-b)(-c) = ad - bc \neq 0$. Hence T^{-1} is a bilinear transformation.

$$\therefore T^{-1}T(z) = T^{-1}\left(\frac{az + b}{cz + d}\right)$$

$$= \frac{d \cdot \dfrac{az + b}{cz + d} - b}{-c \cdot \dfrac{az + b}{cz + d} + a}$$

$$= \frac{adz + bd - bcz - bd}{-acz - bc + acz + ad}$$

$$= \frac{(ad - bc)z}{ad - bc} = z = I(z)$$

$\Rightarrow T^{-1}T = I$

Similarly, $T\,T^{-1}(w) = T\left(\dfrac{dw - b}{-cw + a}\right)$

$$= \frac{a \cdot \dfrac{dw - b}{-cw + a} + b}{c \cdot \dfrac{dw - b}{-cw + a} + d}$$

$$= \frac{adw - ab - bcw + ab}{cdw - bc - cdw + ad}$$

$$= \frac{(ad - bc)w}{ad - bc} = w = I(w)$$

$\Rightarrow T\,T^{-1} = I$

$\therefore T^{-1}\,T = T\,T^{-1} = I$

Hence inverse law satisfied in S.

Thus, the set of all bilinear transformations form a group for the composition of product of transformation.

(v) We know that in general $T_1 T_2(z)$ is not equal to $T_2 T_1(z)$, that is, $T_1 T_2 \neq T_2 T_1$ always.

Hence the group is a non abelian group.

SOLVED PROBLEMS

Example–1. Why the transformation T :

$$T(z) = w = \frac{az + b}{cz + d}, \; ad - bc \neq 0 \cdots\cdots (1)$$

is called the bilinear transformation.

Solution : We have $w = \dfrac{az + b}{cz + d}$

$\Rightarrow czw + dw = az + b$

$\Rightarrow czw - az + dw - b = 0$

This equation is linear both in z and w.

For this reason the transformation $T(z) = w = \dfrac{az + b}{cz + d}$ is called bilinear transformation.

Example–2. Prove that the product of two bilinear transformations is again a bilinear transformation. **[DUH–1975]**

Solution : Let $T(z) = w = \frac{az+b}{cz+d}$, $(ad - bc \neq 0)$ ...... (1)

be a bilinear transformation.

If z in (1) under goes a transformation T' :

$$T'(z') = \frac{a'z'+b'}{c'z'+d'} \quad \text{......} (2)$$

where $a'd' - b'c' \neq 0$.

$$\therefore T(T'(z')) = T\left(\frac{a'z'+b'}{c'z'+d'}\right)$$

$$= \frac{a \cdot \frac{a'z'+b'}{c'z'+d'} + b}{c \cdot \frac{a'z'+b'}{c'z'+d'} + d}$$

$$= \frac{aa'z' + ab' + bc'z' + bd'}{ca'z' + b'c + c'dz' + dd'}$$

$$= \frac{(aa' + bc')z' + (ab' + bd')}{(ca' + c'd)z' + (b'c + dd')} \quad \text{......} (3)$$

Here $(aa' + bc')(b'c + dd') - (ab' + bd')(ca' + c'd)$

$= aa'b'c + aa'dd' + bb'cc' + bc'dd' - aa'b'c - ab'c'd - a'bcd' - bc'dd'$

$= aa'dd' + bb'cc' - ab'c'd - a'bcd'$

$= ad(a'd' - b'c') - bc(a'd' - b'c')$

$= (ad - bc)(a'd' - b'c') \neq 0$

Hence, (3) is a bilinear transformation. That is, the product of two bilinear transformations is again a bilinear transformation.

(Proved)

Example–3. The inverse of a bilinear transformation is unique. **[DUH–1975]**

Solution : Let T(z) be a bilinear transformation and I be the identity bilinear transformation.

Suppose T_1 and T_2 be two inverse of T. Then

$T T_1 = T_1 T = I$ ...... (1)

and $T T_2 = T_2 T = I$ ...... (2)

We shall show that $T_1 = T_2$.

Now, $T_1 = T_1 I$

$= T_1(T T_2)$; By (2)

$= (T_1 T) T_2$; By associative law of bilinear transformation.

$= IT_2$; By (1)

$= T_2$

Hence the inverse of a bilinear transformation is unique.

Example–4. Every bilinear transformation transforms a circle (or a straight line) of the z-plane into a circle (or a straight line) of the w-plane. **[NUH–97, DUH–77, 85, 90, 2000, 2003, 2006, RUH–1972, 1976, 1988, KUH–2002]**

Solution : We know that [আমরা জানি যে] $\left|\frac{z-p}{z-q}\right| = k$ ...... (1)

represents a circle in the z-plane with inverse points p, q. [z তলে p, q বিপরীত বিন্দুসহ একটি বৃত্ত নির্দেশ করে] If k = 1 the equation represent a line. [যদি k = 1 হয় তবে সমীকরণটি একটি সরলরেখা নির্দেশ করে ।]

Let [ধরি] $w = \frac{az+b}{cz+d}$, $(ad - bc \neq 0)$ ...... (2)

be a bilinear transformation. [একটি দ্বিরৈখিক রূপান্তর]

$(2) \Rightarrow az + b = wcz + wd$

$\Rightarrow az - cwz = dw - b$

$\Rightarrow z = \frac{dw-b}{-cw+a}$, $(ad - bc) \neq 0$ ...... (3)

Putting this value in (1) we get [এইমান (1) এ বসাইয়া পাই]

$$\left|\frac{\frac{dw-b}{-cw+a} - p}{\frac{dw-b}{-cw+a} - q}\right| = k$$

$$\Rightarrow \left|\frac{dw - b + cpw - ap}{dw - b + cqw - aq}\right| = k$$

$$\Rightarrow \left|\frac{(cp+d)w - (ap+b)}{(cq+d)w - (aq+b)}\right| = k$$

$$\Rightarrow \left|\frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}}\right| = k\left|\frac{cq+d}{cp+d}\right|$$

which represents the equation of a circle (or a straight line) in the w-plane with $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$ as inverse points. Hence proved. [যাহা z হলে $\frac{ap+b}{cp+d}$ এবং $\frac{aq+b}{cq+d}$ বিপরীত বিন্দুসহ একটি বৃত্ত (বা সরলরেখা) নির্দেশ করে। অতএব প্রমাণিত।]

Example–5. Every bilinear transformations with two finite fixed points α, β is of the form $\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}$.

[DUH–1976, RUH–1973, 1976]

Solution : Let us consider any bilinear transformation with α, β as two finite fixed points and suppose it maps any point $z = \gamma$ into the point $w = \delta$. Then the points $z = \alpha, \beta, \gamma$ are mapped into the points $w = \alpha, \beta, \delta$. [দুইটি সসীম স্থির বিন্দু সহ যেকোন দ্বিরৈখিক রূপান্তর বিবেচনা করি এবং ধরি ইহা যে কোন বিন্দু $z = \gamma$ কে $w = \delta$ বিন্দুতে চিত্রণ করে। তখন $z = \alpha, \beta, \gamma$ বিন্দুগুলি $w = \alpha, \beta, \delta$ বিন্দুতে চিত্রণ করে।]

Now by the preservation of cross ratio property we have [এখন দ্বৈত অনুপাতের সংরক্ষণ ধর্ম দ্বারা পাই]

$$(w, \alpha, \delta, \beta) = (z, \alpha, \gamma, \beta)$$

$$\Rightarrow \frac{(w-\alpha)(\delta-\beta)}{(\alpha-\delta)(\beta-w)} = \frac{(z-\alpha)(\gamma-\beta)}{(\alpha-\gamma)(\beta-z)}$$

$$\Rightarrow \frac{w-\alpha}{w-\beta} = \frac{(\alpha-\delta)(\gamma-\beta)}{(\delta-\beta)(\alpha-\gamma)} \cdot \frac{z-\alpha}{z-\beta}$$

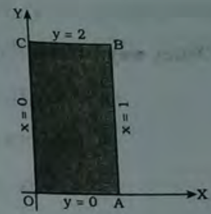
$$\Rightarrow \frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}, \text{ where } k = \frac{(\alpha-\delta)(\gamma-\beta)}{(\delta-\beta)(\alpha-\gamma)}$$

Example–6. Consider the linear transformation $w = (1+i)z + (2-i)$. Determine the region in the w-plane into which the rectangular region bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$ in the z-plane is mapped under this transformation : **[RUH–2004]**

Solution : The region in the z-plane bounded by the lines

$x = 0, y = 0, x = 1, y = 2$

is shown in the adjacent figure.



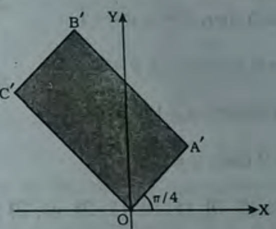
Given that $w = (1+i)z + (2-i)$

It is of the form $w = \alpha z + \beta$, where α is expansion or contraction and β is translation.

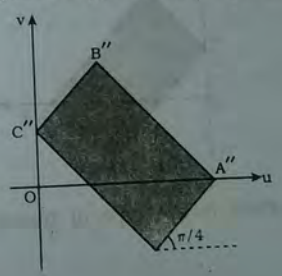
Here $\alpha = 1 + i \Rightarrow |\alpha| = |1+i| = \sqrt{1+1} = \sqrt{2}$

$$\therefore \alpha = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\, e^{i\pi/4}$$

So the given transformation is a expansion of z by $\sqrt{2}$ times a rotation by an angle $\frac{\pi}{4}$ which has shown in following figure.



Again the factor $2 - i$ represents a translation two units to the right and one unit downward. This is shown in the following figure which is our required region in the w-plane.



The transformation is the type of translation, rotation and magnification.

Other way : Given that $w = (1 + i)\, z + 2 - i$

$$\Rightarrow u + iv = (1 + i)\,(x + iy) + 2 - i$$

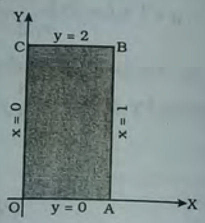
$$= x + iy + ix - y + 2 - i$$

$$= (x - y + 2) + i(x + y - 1)$$

Equating real and imaginary parts

$$u = x - y + 2,\ v = x + y - 1$$

In the z-plane the given lines intersect at O(0, 0), A(1, 0), B(1, 2) and C(0, 2).



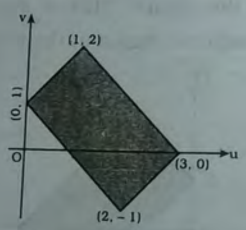
When $x = 0, y = 0$ then $u = 2, v = -1$

When $x = 1, y = 0$ then $u = 3, v = 0$

When $x = 1, y = 2$ then $u = 1, v = 2$

When $x = 0, y = 2$ then $u = 0, v = 1$

Thus the points (0, 0), (1, 0), (1, 2), (0, 2) respectively maps on the points (2, – 1), (3, 0), (1, 2) and (0, 1). The following is the transformed figure.



The transformation is the type of translation, rotation and magnification.

Example–7. If the transformation be $w = 3z$ then determine the region in the w-plane corresponding to triangular region bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ in the z-plane and name the type of transformation.

Solution : Given that $x = 0, y = 0, x + y = 1$.

$$w = 3z \Rightarrow u + iv = 3(x + iy)$$

Equating real and imaginary parts

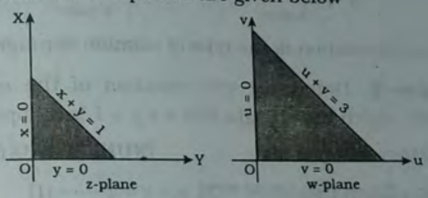
$$u = 3x,\ v = 3y$$

When $x = 0$ then $u = 0$

When $y = 0$ then $v = 0$

When $x + y = 1$ then $u + v = 3(x + y) = 3 \cdot 1 = 3$

The figures in the two planes are given below



The transformation is the type of magnification only.

Example–8. Let the transformation be $w = e^{i\pi/4}z$. Determine the region in the w-plane corresponding to triangular region bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ in the z-plane and name the type of transformation. **[RUH–2001]**

Solution : We have $w = u + iv = e^{i\pi/4}z = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)z$

$$= \frac{1}{\sqrt{2}}(1 + i)\,(x + iy)$$

$$= \frac{1}{\sqrt{2}}[(x - y) + i(x + y)]$$

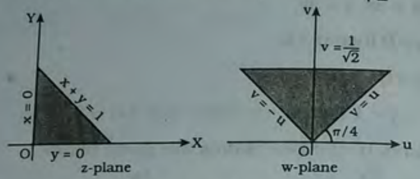
$$\therefore\ u = \frac{1}{\sqrt{2}}(x - y),\ v = \frac{1}{\sqrt{2}}(x + y)$$

When $x = 0$ then $u = -\frac{1}{\sqrt{2}}y$, $v = \frac{1}{\sqrt{2}}y$ $\quad \therefore v = -u$

When $y = 0$ then $u = \frac{1}{\sqrt{2}}x$, $v = \frac{1}{\sqrt{2}}x$ $\quad \therefore v = u$

When $x + y = 1$ then $v = \frac{1}{\sqrt{2}}$

$\therefore$ The given triangular region in z-plane is mapped on to a region in w-plane bounded by $v = u$, $v = -u$ and $v = \frac{1}{\sqrt{2}}$.



The transformation is the type of rotation through an angle $\frac{\pi}{4}$.

Example–9. Determine the equation of the curve in the w-plane into which the straight line $x + y = 1$ is mapped under the transformation $w = \frac{1}{z}$. **[NUH–2000(Old), 2004(Old)]**

Solution : Given that [দেওয়া আছে] $x + y = 1$ ...... (1)

and [এবং] $w = \frac{1}{z}$

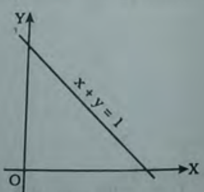
$$\Rightarrow z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{(u + iv)(u - iv)}$$

$$\Rightarrow x + iy = \frac{u - iv}{u^2 - i^2v^2}$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} + i\frac{-v}{u^2 + v^2}$$

Equating real and imaginary parts we get [বাস্তব এবং কাল্পনিক অংশ সমীকৃত করে পাই]

$$x = \frac{u}{u^2 + v^2} \text{ and [এবং] } y = \frac{-v}{u^2 + v^2}$$



Putting these values in (1) we get [এইমানগুলি (1) এ বসাইয়া পাই]

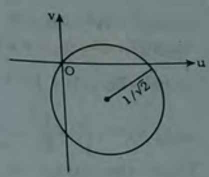
$$\frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2} = 1$$

$$\Rightarrow u^2 + v^2 = u - v$$

$$\Rightarrow u^2 + v^2 - u + v = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$



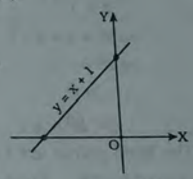
Thus the curve in the w-plane is a circle whose centre is [অতএব w তলে বক্রটি একটি বৃত্ত যার কেন্দ্র]

$\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius is [এবং ব্যাসার্ধ] $\frac{1}{\sqrt{2}}$.

Example–10. Find the image of the line $y = x + 1$ under the transformation $w = \frac{z + i}{z - 1}$

Solution : Given that $y = x + 1$ ...... (1)

and $w = \frac{z + i}{z - 1}$



$$\Rightarrow z + i = wz - w$$

$$\Rightarrow (1 - w)z = -w - i$$

$$\Rightarrow z = \frac{w + i}{w - 1} = \frac{u + iv + i}{(u - 1) + iv}$$

$$\Rightarrow x + iy = \frac{u + iv + i}{(u - 1) + iv} \times \frac{u - 1 - iv}{(u - 1) - iv}$$

$$= \frac{u^2 - u - iuv + iuv - iv + v^2 + iu - i + v}{(u - 1)^2 + v^2}$$

$$= \frac{u^2 - u + v^2 + v + i(-v + u - 1)}{(u - 1)^2 + v^2}$$

Equating real and imaginary parts we get,

$$x = \frac{u^2 + v^2 - u + v}{(u - 1)^2 + v^2} \text{ and } y = \frac{-v + u - 1}{(u - 1)^2 + v^2}$$

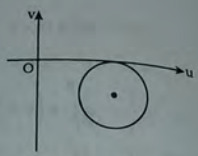
Putting the values of x and y in (1) we get,

$$\frac{-v + u - 1}{(u - 1)^2 + v^2} = \frac{u^2 + v^2 - u + v}{(u - 1)^2 + v^2} + 1$$

$$\Rightarrow -v + u - 1 = u^2 + v^2 - u + v + (u - 1)^2 + v^2$$

$$\Rightarrow u^2 + v^2 - u + v + u^2 - 2u + 1 + v^2 + v - u + 1 = 0$$

$$\Rightarrow 2u^2 + 2v^2 - 4u + 2v + 2 = 0$$
$$\Rightarrow 2(u^2 + v^2 - 2u + v + 1) = 0$$
$$\Rightarrow (u^2 - 2u + 1) + \left(v^2 + v + \frac{1}{4}\right) = \frac{1}{4}$$
$$\Rightarrow (u - 1)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$



Thus the image of the line $y = x + 1$ under the given transformation is the circle $u^2 + v^2 - 2u + v + 1 = 0$ whose centre is $\left(1, -\frac{1}{2}\right)$ and radius is $\frac{1}{2}$. Ans.

Example–11. Find the fixed points of the transformation $w = \frac{z-1}{z+1}$. **[DUH–1976]**

Solution : For a fixed point we have

$$w = f(z) = z$$
$$\Rightarrow \frac{z-1}{z+1} = z$$
$$\Rightarrow z^2 + z = z - 1$$
$$\Rightarrow z^2 = -1$$
$$= i^2$$
$$\therefore z = \sqrt{i^2} = \pm i$$

$\therefore$ The fixed points are $z = i$ and $z = -i$ Ans.

Example–12. Find a bilinear transformation which transforms points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

[NUH–2000, 2006 (Old), RUMP–1984, KUH–2004]

Solution : Here [এখানে] $z = 0, -i, -1$ and [এবং] $w = i, 1, 0$.

$\therefore$ By cross-ratio property, the bilinear transformation is [দ্বৈত অনুপাত ধর্ম দ্বারা দ্বিরৈখিক রূপান্তরটি]

$$\frac{w-i}{i-1}\,\frac{1-0}{0-w} = \frac{z-0}{0+i}\,\frac{-i+1}{-1-z}$$
$$\Rightarrow \frac{w-i}{i-1}\,\frac{-1}{w} = \frac{z}{i}\,\frac{-i+1}{-1-z}$$
$$\Rightarrow \frac{-w+i}{iw-w} = \frac{-iz+z}{-i-iz}$$
$$\Rightarrow (iw - w)(-iz + z) = (-w + i)(-i - iz)$$
$$\Rightarrow wz + iwz + iwz - wz = iw + iwz - i^2 - i^2z$$
$$\Rightarrow 2iwz - iw - iwz = 1 + z$$
$$\Rightarrow iwz - iw = 1 + z$$

$$\Rightarrow i(z-1)\,w = z + 1$$
$$\Rightarrow w = \frac{z+1}{i(z-1)}$$
$$\Rightarrow w = \frac{i(z+1)}{i^2(z-1)}$$
$$\Rightarrow w = -i\,\frac{z+1}{z-1}.$$ Ans.

Other way : Let [ধরি] $w = \frac{az+b}{cz+d}$ ...... (1)

be the bilinear transformation where [দ্বিরৈখিক রূপান্তরটি যেখানে] $ad - bc \neq 0$.

Putting $z = 0$, $w = i$ in (1) we get [(1) এ $z = 0$, $w = i$ বসাইয়া পাই]

$$i = \frac{0+b}{0+d}$$
$$\Rightarrow b = id \text{ ...... (2)}$$

Putting $z = -i$ and $w = 1$ in (1) we get [(1) এ $z = -i$ $w = 1$ বসাইয়া পাই]

$$1 = \frac{-ia+b}{-ic+d}$$
$$-ia + b = -ic + d \text{ ...... (3)}$$

Putting $z = -1$ and $w = 0$ in (1) we get [(1) এ $z = -1$ এবং $w = 0$ বসাইয়া পাই]

$$0 = \frac{-a+b}{-c+d}$$
$$\Rightarrow -a + b = 0$$
$$\Rightarrow a = b$$
$$\Rightarrow a = id \text{ ...... (4) ; by (2)}$$

Putting $a = id$, $b = id$ in (3) [(3) এ $a = id$, $b = id$ বসাইয়া]

$$-i(id) + id = -ic + d$$
$$\Rightarrow d + id = -ic + d$$
$$\Rightarrow id = -ic$$
$$\Rightarrow d = -c$$
$$\therefore c = -d$$

Putting $a = id$, $b = id$, $c = -d$ in (1) we get [(1) এ $a = id$, $b = id$, $c = -d$ বসাইয়া পাই]

$$w = \frac{idz + id}{-dz + d} = \frac{id(z+1)}{-d(z-1)} = -i\,\frac{z+1}{z-1}$$ Ans.

Example–12(i). Find a bilinear transformation that maps the points z = 0, 1, 2 of the z-plane into the points w = i, 0, 1, respectively of the w-plane. **[NUH–2013]**

Solution : Here [এখানে] z = 0, 1, 2 and [এবং] w = i, 0, 1.

∴ By cross ratio property, the bilinear transformation is [দ্বৈত অনুপাত দ্বারা দ্বিরৈখিক রূপান্তরটি]

$$\frac{(w-i)\,(0-1)}{(i-0)\,(1-w)}=\frac{(z-0)\,(1-2)}{(0-1)\,(2-z)}$$

$$\Rightarrow \frac{-(w-i)}{i(1-w)}=\frac{-z}{(z-2)}$$

$$\Rightarrow (w-i)\,(z-2)=iz(1-w)$$

$$\Rightarrow w(z-2)+izw=iz+i(z-2)$$

$$\Rightarrow w(z-2+iz)=i(2z-2)$$

$$\Rightarrow w=\frac{2i(z-1)}{(1+i)\,z-2}\quad \textbf{(Ans)}$$

Other way : Let [ধরি] $w=\dfrac{az+b}{cz+d}$ …… (1)

by the bilinear transformation where [দ্বিরৈখিক রূপান্তরটি যেখানে]

$$ad-bc\neq 0$$

Putting z = 0, w = i in (1) we get [(1) এ z = 0, w = i বসাইয়া পাই]

$$i=\frac{0+b}{0+d}\Rightarrow b=id \;\cdots\cdots (2)$$

Putting z = 1, w = 0 in (1) we get [(1) এ z = 1, w = 0 বসাইয়া পাই]

$$0=\frac{a+b}{c+d}\Rightarrow a=-b=-id \;\cdots\cdots (3),\ \text{by (2)}$$

Putting z = 2, w = 1 in (1) we get [(1) এ z = 2, w = 1 বসাইয়া পাই]

$$1=\frac{2a+b}{2c+d}$$

$$\Rightarrow 2c+d=2a+b$$

$$\Rightarrow 2c=2a+b-d$$

$$=-2id+id-d$$

$$=-id-d$$

$$\Rightarrow c=\frac{-1}{2}(1+i)\,d$$

Putting the value of a, b, c in (1) we get [a, b, c এর মান (1) এ বসাইয়া পাই]

$$w=\frac{-i\,dz+id}{-\frac{1}{2}(1+i)\,dz+d}$$

$$=\frac{-iz+i}{-\frac{1}{2}(1+i)\,z+1}=\frac{-2i(z-1)}{-(1+i)\,z+2}$$

$$=\frac{2i(z-1)}{(1+i)z-2}\cdot\quad \textbf{(Ans)}$$

Example–13. Find the bilinear transformation which transforms points 1, i, – 1 of the z-plane into the points 0, 1, ∞ of the w-plane respectively. **[NUH–1996, DUH–1977, 1985, 2002]**

Solution : By the cross-ratio property we have [দ্বৈত অনুপাত ধর্ম দ্বারা পাই]

$$(w,0,1,\infty)=(z,1,i,-1)$$

$$\Rightarrow \left(w,0,1,\frac{1}{w'}\right)=(z,1,i,-1),\ \text{where } \infty=\frac{1}{w'}\Rightarrow w'=\frac{1}{\infty}=0$$

$$\Rightarrow \frac{w-0}{0-1}\,\frac{1-\frac{1}{w'}}{\frac{1}{w'}-w}=\frac{z-1}{1-i}\,\frac{i+1}{-1-z}$$

$$\Rightarrow \frac{w}{-1}\,\frac{(w'-1)/w'}{(1-ww')/w'}=\frac{i+1}{-1+i}\,\frac{z-1}{z+1}$$

$$\Rightarrow \frac{w}{-1}\,\frac{w'-1}{1-ww'}=\frac{(1+i)\,(-1-i)}{(-1+i)\,(-1-i)}\,\frac{z-1}{z+1}$$

$$\Rightarrow \frac{w}{-1}\,\frac{0-1}{1-0}=\frac{-(1+i)^2}{(-1)^2-i^2}\,\frac{z-1}{z+1};\quad \therefore w'=0$$

$$\Rightarrow \frac{-w}{-1}=\frac{-(1+2i+i^2)}{1-i^2}\,\frac{z-1}{z+1}$$

$$\Rightarrow w=\frac{-(1+2i-1)}{1+1}\,\frac{z-1}{z+1}$$

$$\Rightarrow w=\frac{-2i}{2}\,\frac{z-1}{z+1}$$

$$\Rightarrow w=-i\,\frac{z-1}{z+1}\qquad \text{Ans.}$$

Other way [অন্যভাবে] **:** Let [ধরি] $w=\dfrac{az+b}{cz+d}$ …… (1)

be the bilinear transformation where [দ্বিরৈখিক রূপান্তরটি যেখানে]

$$ad-bc\neq 0.$$

Putting z = 1, w = 0 in (1) we get [(1) এ z = 1, w = 0 বসাইয়া পাই]

$$0 = \frac{a+b}{c+d}$$

$$\Rightarrow a + b = 0$$

$$\Rightarrow a = -b \cdots\cdots (2)$$

Putting z = i and w = 1 in (1) we get [(1) এ z = i এবং w = 1 বসাইয়া পাই]

$$1 = \frac{ai+b}{ci+d}$$

$$\Rightarrow ci + d = ai + b \cdots\cdots (3)$$

Putting z = − 1 and w = ∞ in (1) we get [(1) এ z = − 1 এবং w = ∞ বসাইয়া পাই]

$$\infty = \frac{-a+b}{-c+d}$$

$$\Rightarrow \frac{1}{0} = \frac{-a+b}{-c+d}$$

$$\Rightarrow -c + d = 0$$

$$\Rightarrow c = d \cdots\cdots (4)$$

By (2) and (4), (3) becomes [(2) এবং (4) দ্বারা (3) দাঁড়ায়]

$$di + d = -bi + b$$

$$\Rightarrow d(i+1) = b(-i+1)$$

$$\Rightarrow d = \frac{1-i}{1+i}b$$

$$\therefore c = d = \frac{1-i}{1+i}b$$

Putting the values of a, c, d in (1) we get [(1) এ a, c, d এর মানগুলি বসাইয়া পাই]

$$w = \frac{-bz+b}{\frac{1-i}{1+i}bz + \frac{1-i}{1+i}b}$$

$$= \frac{-b(z-1)}{\frac{1-i}{1+i}b(z+1)}$$

$$= \frac{-(1+i)}{1-i} \cdot \frac{z-1}{z+1}$$

$$= \frac{-(1+i)^2}{(1-i)(1+i)} \frac{z-1}{z+1}$$

$$= \frac{-(1+2i+i^2)}{1^2 - i^2} \frac{z-1}{z+1}$$

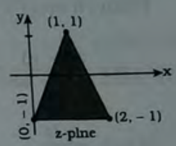
$$= \frac{-(1+2i-1)}{1+1} \frac{z-1}{z+1}$$

$$= \frac{-2i}{2} \frac{z-1}{z+1}$$

$$= -i\frac{z-1}{z+1} \qquad \text{Ans.}$$

Example–14. Find a bilinear transformation which maps the vertices 1 + i, − i, 2 − i of a triangle of the z-plane into the points 0, 1, i of the w-plane. **[DUH-2004, RUH-2000, KUH-2004]**

Solution : By cross ratio property we have [দ্বৈত অনুপাত ধর্ম দ্বারা পাই]



$$(w, 0, 1, i) = (z, 1+i, -i, 2-i)$$

$$\Rightarrow \frac{(w-0)(1-i)}{(0-1)(i-w)} = \frac{(z-1-i)(-i-2+i)}{(1+i+i)(2-i-z)}$$

$$\Rightarrow \frac{(1-i)w}{w-i} = \frac{-2(z-1-i)}{(1+2i)(2-i-z)}$$

$$\Rightarrow (1-i)(1+2i)(2-i-z)w = (-2z+2+2i)(w-i)$$

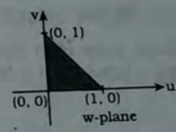
$$\Rightarrow (3+i)(2-i-z)w = (-2z+2+2i)(w-i)$$

$$\Rightarrow (6-3i-3z+2i+1-iz)w - (-2z+2+2i)w = -i(-2z+2+2i)$$

$$\Rightarrow (7-i-3z-iz+2z-2-2i)w = 2iz-2i+2$$

$$\Rightarrow (-z-iz+5-3i)w = i(2z-2-2i)$$

$$\Rightarrow i(iz-z-5i-3)w = i(2z-2-2i)$$



$$\Rightarrow w = \frac{2z-2-2i}{iz-z-3-5i}$$

$$\Rightarrow w = \frac{2z-2-2i}{(i-1)z-3-5i}$$

which is the required bilinear transformation. [যাহা প্রার্থিত দ্বিরৈখিক রূপান্তর] Ans.

Example–15. Find the bilinear transformation which maps 1, i, − 1 on i, 0, − i, respectively. **[NUH-2015]**

Solution : Let [ধরি] $w = \frac{az+b}{cz+d}$ be the bilinear transformation. [দ্বিরৈখিক রূপান্তরটি]

Putting $z = 1, i, -1$ and $w = i, 0, -i$ successively we get,
[$z = 1, i, -1$ এবং $w = i, 0, -i$ পর্যায়ক্রমে বসাইয়া পাই]

$$i = \frac{a+b}{c+d} \Rightarrow a + b = i(c + d) \;\cdots\cdots (1)$$

$$0 = \frac{ai+b}{ci+d} \Rightarrow ai + b = 0 \;\cdots\cdots (2)$$

and [এবং] $-i = \dfrac{-a+b}{-c+d} \Rightarrow -a + b = ic - id \;\cdots\cdots (3)$

From (2) [(2) হতে] $b = -ai \;\cdots\cdots (4)$

(1) + (3) gives [দেয়] $2b = 2ic \Rightarrow b = ic \;\cdots\cdots (5)$

(1) − (3) gives [দেয়] $2a = 2id. \Rightarrow d = -ia$

From (4) and (5) [(4) ও (5) হতে] $-ai + ic \Rightarrow c = -a$

$$\therefore w = \frac{az - ai}{-az - ai} = \frac{a(z-i)}{-a(z+i)} = -\frac{z-i}{z+i} = -\frac{-i\left(1 - \frac{z}{i}\right)}{i\left(1 + \frac{z}{i}\right)}$$

$$\Rightarrow w = \frac{1+iz}{1-iz} \qquad \text{Ans.}$$

Example-16. Find a bilinear transformation which map the points $i, -i, 1$ of the z-plane into the points $0, 1, \infty$ of w-plane respectively. **[NUH-2001, 2012, KUH-2005]**

Solution : Let [ধরি] $w = \dfrac{az+b}{cz+d}$ be the bilinear transformation where [দ্বিরৈখিক রূপান্তরটি যেখানে] $ad - bc \neq 0$. Since the points $i, -i, 1$ of z-plane map into the points $0, 1, \infty$ of the w-plane, so we have [যেহেতু z তলের $i, -i, 1$ বিন্দুগুলি w তলে $0, 1, \infty$ বিন্দুতে চিত্রণ করে, সুতরাং আমরা পাই]

$$0 = \frac{ai+b}{ci+d} \Rightarrow ai + b = 0 \Rightarrow b = -ai \;\cdots\cdots (1)$$

$$1 = \frac{-ai+b}{-ci+d} \Rightarrow -ai + b = -ci + d \;\cdots\cdots (2)$$

and [এবং] $\infty = \dfrac{a+b}{c+d} \Rightarrow c + d = 0 \Rightarrow c = -d \;\cdots\cdots (3)$

By (1) and (3), (2) gives [(1) ও (3) দ্বারা (2) দাঁড়ায়]

$$-ai - ai = di + d$$

$$\Rightarrow -2ai = d(i + 1)$$

$$\Rightarrow a = \left(\frac{1+i}{-2i}\right) d = \frac{(1+i)i}{2} d = \frac{-1+i}{2} d$$

$$\therefore b = -ai = \frac{i+1}{2} d$$

$$\therefore w = \frac{\left(\frac{-1+i}{2}\right)dz + \frac{(i+1)d}{2}}{-dz + d} = \frac{1}{2} \cdot \frac{(-1+i)z + (1+i)}{-z+1}$$

$$\Rightarrow \frac{(1-i)z - 1 - i}{2(z-1)} = \frac{z - iz - 1 - i}{2(z-1)} = \frac{1(z-i) - i(z-i)}{2(z-1)}$$

$$\Rightarrow w = \frac{(1-i)\,(z-i)}{2(z-1)} \qquad \text{Ans.}$$

Example-17. Show that the transformation $w = \dfrac{2z+3}{z-4}$ changes the circle $x^2 + y^2 - 4x = 0$ into the stright line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

[NUH-2005 (Old), 2014, NUP-2005, DUH-2005, RUP-1984, KUH-2002, 2004]

Solution : We have [আমাদের আছে] $w = f(z) = u + iv$

$$\Rightarrow \overline{w} = \overline{u + iv} = u - iv$$

$$\therefore w + \overline{w} = u + iv + u - iv = 2u \;\cdots\cdots (1)$$

Given [দেওয়া আছে] $w = \dfrac{2z+3}{z-4}$

$$\Rightarrow zw - 4w = 2z + 3$$

$$\Rightarrow zw - 2z = 4w + 3$$

$$\Rightarrow z(w - 2) = 4w + 3$$

$$\Rightarrow z = \frac{4w+3}{w-2} \;\cdots\cdots (2)$$

$$\therefore \overline{z} = \left(\overline{\frac{4w+3}{w-2}}\right) = \frac{4\overline{w}+3}{\overline{w}-2} \;\cdots\cdots (3)$$

Now [এখন] $x^2 + y^2 - 4x = 0$

$$\Rightarrow x^2 - iy^2 - 2(2x) = 0$$

$$\Rightarrow (x+iy)\,(x-iy) - 2(x + iy + x - iy) = 0$$

$$\Rightarrow z\overline{z} - 2(z + \overline{z}) = 0 \qquad [\because z = x + iy \Rightarrow \overline{z} = x - iy$$

$$\Rightarrow \frac{4w+3}{w-2} \cdot \frac{4\overline{w}+3}{\overline{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\overline{w}+3}{\overline{w}-2}\right) = 0$$

$$\Rightarrow (4w+3)\,(4\overline{w}+3) - 2\{(4w+3)\,(\overline{w}-2) + (4\overline{w}+3)\,(w-2)\} = 0$$

$$\Rightarrow 16w\overline{w} + 12w + 12\overline{w} + 9 - 2(4w\overline{w} - 8w + 3\overline{w} - 6$$
$$+ 4w\overline{w} - 8\overline{w} + 3w - 6) = 0$$
$$\Rightarrow 16w\overline{w} + 12w + 12\overline{w} + 9 - 16w\overline{w} + 10w + 10\overline{w} + 24 = 0$$
$$\Rightarrow 22w + 22\overline{w} + 33 = 0$$
$$\Rightarrow 11(2w + 2\overline{w} + 3) = 0$$
$$\Rightarrow 2w + 2\overline{w} + 3 = 0$$
$$\Rightarrow 2(w + \overline{w}) + 3 = 0$$
$$\Rightarrow 2(2u) + 3 = 0, \text{ by (1)}$$
$$\Rightarrow 4u + 3 = 0 \quad \text{Showed.}$$

2nd Part : The equation $4u + 3 = 0$ is of 1st degree, so it is the equation of straight line in the w-plane.

We know that $\left|\frac{z-p}{z-q}\right| = k$ represents a circle when $k > 0$ and $k \neq 1$, but when $k = 1$ then this equation represents a straight line. Thus, the straight line is a particular case of a circle (that is, it is possible that circle reduces to a straight line).

Example–18. Show that the relation $w = \frac{5-4z}{4z-2}$ transforms the circle $|z| = 1$ into a circle of radius 1 in the w-plane and find the centre of the circle. **[RUH–2000, RUMP–1985, KUH–2002]**

Solution : Given [দেওয়া আছে] $w = \frac{5-4z}{4z-2}$

$$\Rightarrow 4zw - 2w = 5 - 4z$$
$$\Rightarrow 4zw + 4z = 2w + 5$$
$$\Rightarrow (4w + 4)z = 2w + 5$$
$$\Rightarrow z = \frac{2w+5}{4w+4} \;\cdots\cdots (1)$$
$$\therefore \overline{z} = \overline{\left(\frac{2w+5}{4w+4}\right)} = \frac{2\overline{w}+5}{4\overline{w}+4} \;\cdots\cdots (2)$$

Also we have [আমাদের আরো আছে]

$$w = u + iv \Rightarrow \overline{w} = \overline{u+iv} = u - iv \;\cdots\cdots (3)$$

Now [এখন] $|z| = 1$

$$\Rightarrow |z|^2 = 1^2$$
$$\Rightarrow z\overline{z} = 1$$
$$\Rightarrow \frac{2w+5}{4w+4} \cdot \frac{2\overline{w}+5}{4\overline{w}+4} = 1$$

$$\Rightarrow (2w+5)(2\overline{w}+5) = (4w+4)(4\overline{w}+4)$$
$$\Rightarrow 4w\overline{w} + 10w + 10\overline{w} + 25 = 16w\overline{w} + 16w + 16\overline{w} + 16$$
$$\Rightarrow 9 = 12w\overline{w} + 6w + 6\overline{w}$$
$$= 12(u+iv)(u-iv) + 6(u + iv + u - iv)$$
$$= 12(u^2 - i^2v^2) + 6(2u)$$
$$= 12(u^2 + v^2) + 12u$$
$$= 12(u^2 + v^2 + u)$$
$$\Rightarrow u^2 + v^2 + u = \frac{9}{12}$$
$$\Rightarrow u^2 + v^2 + u = \frac{3}{4}$$
$$\Rightarrow \left(u + \frac{1}{2}\right)^2 + v^2 = \frac{3}{4} + \frac{1}{4}$$
$$\Rightarrow \left(u + \frac{1}{2}\right)^2 + v^2 = 1^2$$

This the equation of a circle in the w-plane, whose centre is $\left(-\frac{1}{2}, 0\right)$ and radius 1. [ইহা w তলে একটি বৃত্তের সমীকরণ যার কেন্দ্র $\left(-\frac{1}{2}, 0\right)$ এবং ব্যাসার্ধ 1] Showed.

Example–19. If $w = \frac{ze^{\alpha} - i}{z - ie^{\alpha}}$ and α is any real constant, then find the radius and the centre of the circle in the w-plane which corresponds to the real axis in the z-plane. **[RUMP–1988]**

Solution : Given $w = \frac{ze^{\alpha} - i}{z - ie^{\alpha}}$

$$\Rightarrow wz - iwe^{\alpha} = ze^{\alpha} - i$$
$$\Rightarrow wz - ze^{\alpha} = iwe^{\alpha} - i$$
$$\Rightarrow (w - e^{\alpha})z = i(we^{\alpha} - 1)$$
$$\Rightarrow z = \frac{i(we^{\alpha} - 1)}{w - e^{\alpha}}$$
$$\Rightarrow x + iy = \frac{i(we^{\alpha} - 1)}{w - e^{\alpha}} \cdot \frac{\overline{w} - e^{\alpha}}{\overline{w} - e^{\alpha}}$$
$$\Rightarrow x + iy = \frac{i(w\overline{w}e^{\alpha} - we^{2\alpha} - \overline{w} + e^{\alpha})}{w\overline{w} - we^{\alpha} - \overline{w}e^{\alpha} + e^{2\alpha}}$$

$$w = u + iv$$
$$\overline{w} = u - iv$$
$$w\overline{w} = (u+iv)(u-iv)$$
$$= u^2 - i^2v^2$$
$$= u^2 + v^2$$

$$= \frac{i[(u^2+v^2)e^{\alpha} - (u+iv)e^{2\alpha} - (u-iv) + e^{\alpha}]}{u^2 + v^2 - (u+iv)e^{\alpha} - (u-iv)e^{\alpha} + e^{2\alpha}}$$
$$= \frac{i[(u^2+v^2)e^{\alpha} - (u+iv)e^{2\alpha} - (u-iv) + e^{\alpha}]}{u^2 + v^2 - 2ue^{\alpha} + e^{2\alpha}}$$

Equating imaginary parts we get,

$$y = \frac{(y^2 + v^2)\, e^{\alpha} - ue^{2\alpha} - u + e^{\alpha}}{u^2 + v^2 - 2ue^{\alpha} + e^{2\alpha}} \;\ldots\ldots (1)$$

On the real axis (x-axis) we have y = 0, so (1) becomes

$$0 = \frac{(u^2 + v^2)e^{\alpha} - ue^{2\alpha} - u + e^{\alpha}}{u^2 + v^2 - 2ue^{\alpha} + e^{2\alpha}}$$

$$\Rightarrow (u^2 + v^2)e^{\alpha} - ue^{2\alpha} - u + e^{\alpha} = 0$$

$$\Rightarrow e^{\alpha}(u^2 + v^2 - ue^{\alpha} - ue^{-\alpha} + 1) = 0$$

$$\Rightarrow u^2 + v^2 - u(e^{\alpha} + e^{-\alpha}) + 1 = 0$$

$$\Rightarrow u^2 + v^2 - 2u \cos h\alpha + 1 = 0$$

which is the equation of a circle in the w-plane. it's centre = (cos hα, 0) and radius

$$= \sqrt{\cos h^2\alpha + 0 - 1} = \sqrt{\cos h^2\alpha - 1} = \sqrt{\sin h^2\alpha} = \sin h\alpha \quad \text{Ans.}$$

Example–20. Find the fixed points of the bilinear transformation $w = \frac{2z - 5}{z + 4}$. If a, b are these points, write the equation in the from $\frac{w - a}{w - b} = k\left(\frac{z - a}{z - b}\right)$, where k is a constant.

[NUH–1996, 2005]

Solution : For fixed points we have by definition [স্থির বিন্দুর জন্য সংজ্ঞা দ্বারা আমরা পাই]

$$w = z$$

$$\Rightarrow \frac{2z - 5}{z + 4} = z$$

$$\Rightarrow z^2 + 4z = 2z - 5$$

$$\Rightarrow z^2 + 2z + 5 = 0 \;\ldots\ldots (1)$$

$$\Rightarrow (z + 1)^2 = -4$$

$$\Rightarrow z + 1 = \sqrt{-4} = \pm 2i$$

$$\Rightarrow z = -1 \pm 2i$$

∴ The given transformation has two fixed points $-1 + 2i$ and $-1 - 2i$. [প্রদত্ত রূপান্তরের দুইটি স্থির বিন্দু $-1 + 2i$ এবং $-1 - 2i$ আছে।]

2nd Part : Let [ধরি] $a = -1 + 2i$ and [এবং] $b = -1 - 2i$.

Since a and b are roots of (1) so we have [যেহেতু a ও b, (1) এর মূল সুতরাং আমরা পাই]

$$a^2 + 2a + 5 = 0 \;\ldots\ldots (2)$$

$$b^2 + 2b + 5 = 0 \;\ldots\ldots (3)$$

Now [এখন] $w - a = \frac{2z - 5}{z + 4} - a$

$$= \frac{2z - 5 - az - 4a}{z + 4}$$

$$= \frac{2(z - a) - a(z - a) - 4a - 5 - a^2 + 2a}{z + 4}$$

$$= \frac{(z - a)\,(2 - a) - (a^2 + 2a + 5)}{z + 4}$$

$$= \frac{(2 - a)\,(z - a) - 0}{z + 4}\,;\ \text{by (2)}$$

$$\Rightarrow w - a = \frac{(2 - a)\,(z - a)}{z + 4}$$

Similarly [অনুরূপে] $w - b = \frac{(2 - b)\,(z - b)}{z + 4}$

$$\therefore \frac{w - a}{w - b} = \frac{(2 - a)\,(z - a)}{z + 4} \times \frac{z + 4}{(2 - b)\,(z - b)}$$

$$= \frac{2 - a}{2 - b}\left(\frac{z - a}{z - b}\right)$$

$\Rightarrow \frac{w - a}{w - b} = k\left(\frac{z - a}{z - b}\right)$, which is the required result [যাহা প্রার্থিত ফল যেখানে] where $k = \frac{2 - a}{2 - b}$

$$= \frac{2 - (-1 + 2i)}{2 - (-1 - 2i)} = \frac{3 - 2i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}$$

$= \frac{9 - 12i + 4i^2}{9 - 4i^2} = \frac{5 - 12i}{13}$, which is constant. [যাহা ধ্রুবক]

Example–21. Express the bilinear transformation $w = \frac{13iz + 75}{3z - 5i}$ in the form $\frac{w - a}{w - b} = k\,\frac{z - a}{z - b}$, where a, b and k are constants.

Solution : Given that $w = \frac{13iz + 75}{3z - 5i}$

$$\therefore \frac{w - a}{w - b} = \frac{\frac{13iz + 75}{3z - 5i} - a}{\frac{13iz + 75}{3z - 5i} - b}$$

$$= \frac{13iz + 75 - 3az + 5ia}{13iz + 75 - 3bz + 5ib}$$

$$= \frac{(13i - 3a)z + (75 + 5ia)}{(13i - 3b)z + (75 + 5ib)}$$

$$= \frac{13i - 3a}{13i - 3b}\left[\frac{z - \frac{75 + 5ia}{3a - 13i}}{z - \frac{75 + 5ib}{3b - 13i}}\right] \quad ...... (1)$$

Now let $\frac{75 + 5ia}{3a - 13i} = a$ and $\frac{75 + 5ib}{3b - 13i} = b$

$\Rightarrow 3a^2 - 13ia - 5ia + 75$ [from 1st equation]

$\Rightarrow 3a^2 - 18ia - 75 = 0$

$\Rightarrow a^2 - 6ia - 25 = 0$

$\Rightarrow a = \frac{6i \pm \sqrt{36i^2 + 100}}{2}$

$\Rightarrow a = \frac{6i \pm 8}{2}$

$\Rightarrow a = 3i \pm 4$

Similarly, $b = 3i \pm 4$

When $a = b$ then both sides are identical, so we choose $a \neq b$. Thus we may choose $a = 3i - 4$ and $b = 3i + 4$.

Also, $k = \frac{13i - 3a}{13i - 3b} = \frac{13i - 3(3i - 4)}{13i - 3(3i + 4)}$

$= \frac{13i - 9i + 12}{13i - 9i - 12} = \frac{4i + 12}{4i - 12}$

$= \frac{4(i + 3)}{4(i - 3)} = \frac{(i + 3)(i + 3)}{(i - 3)(i + 3)}$

$= \frac{i^2 + 6i + 9}{i^2 - 9} = \frac{6i + 8}{-10} = \frac{2(4 + 3i)}{-10}$

$= \frac{4 + 3i}{-5}$

Thus from (1) we have

$\frac{w - a}{w - b} = k\left(\frac{z - a}{z - b}\right)$, where $a = 3i - 4$, $b = 3i + 4$ and

$k = \frac{4 + 3i}{-5}$.

Example–22. Show that the transformation $w = \frac{1 + iz}{z + i}$ maps the real axis of the z-plane onto a circle in the w-plane. Find its centre and radius. **[NUH–2002, 2006, DUH–1993]**

Solution : On the real axis [বাস্তব অক্ষের উপর]

$y = 0$ and so [এবং সুতরাং] $z = x + iy = x + 0 = x$

$\therefore w = \frac{1 + iz}{z + i} = \frac{1 + i(x + iy)}{x + iy + i} = \frac{1 + i(x + 0)}{x + 0 + i} = \frac{1 + ix}{x + i}$

Taking modulas on both sides [উভয় দিকে মানাংক নিয়ে]

$$|w| = \left|\frac{1 + ix}{x + i}\right| = \frac{|1 + ix|}{|x + i|} = \frac{\sqrt{1 + x^2}}{\sqrt{x^2 + 1}} = 1$$

$\Rightarrow |w| = 1$ which is the equation of a circle. Thus the given transformation maps real axis of the z-plane into the circle $|w| = 1$ with centre (0, 0) and radius 1 in the w-plane. [যাহা একটি বৃত্তের সমীকরণ। অতএব প্রদত্ত রূপান্তরটি z তলের বাস্তব অক্ষকে w তলে $|w| = 1$ বৃত্তে চিত্রণ করে যার কেন্দ্র (0, 0) এবং ব্যাসার্ধ 1]

Example–23. In the transformation $w = i\,\frac{1 - z}{1 + z}$, show that

(i) The upper simi eirecle of the circle $|z| = 1$ is represented in the w-plane by the positive half of real axis.

(ii) The lower simi circle of the circle $|z| = 1$ is represented in the w-plane by the negative half of real axis.

(iii) The interior of the circle $|z| = 1$ is represented in the w-plane by the plane above the real axis.

Solution : Given that $w = i\,\frac{1 - z}{1 + z}$

$\Rightarrow u + iv = i\,\frac{1 - (x + iy)}{1 + x + iy}$

$= i\,\frac{1 - x - iy}{1 + x + iy} \times \frac{1 + x - iy}{1 + x - iy}$

$= \frac{i(1 + x - iy - x - x^2 + ixy - iy - ixy + i^2y^2)}{(1 + x)^2 - i^2y^2}$

$= \frac{i(1 - 2iy - x^2 - y^2)}{(1 + x)^2 + y^2}$

$= \frac{2y + i(1 - x^2 - y^2)}{(1 + x)^2 + y^2}$

Equating real and imaginary parts we get,

$$u = \frac{2y}{(1+x)^2 + y^2} \text{ ...... (1)}$$

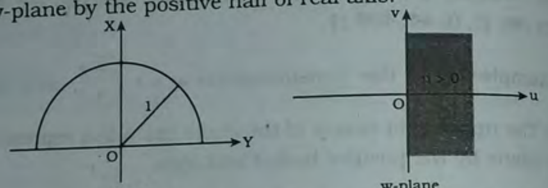
$$v = \frac{-(x^2 + y^2 - 1)}{(1+x)^2 + y^2} \text{ ...... (2)}$$

Now $v = 0$ gives $x^2 + y^2 - 1 = 0$

$$\Rightarrow x^2 + y^2 = 1 \Rightarrow |z| = 1.$$

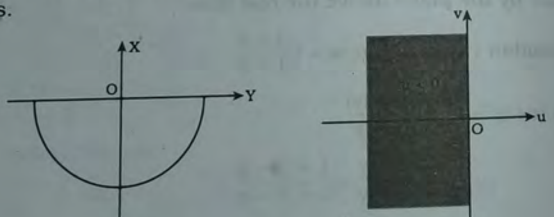
This shows that $|z| = 1$ corresponds to real axis $v = 0$ in the w-plane.

(i) For upper semi circle in z-plane $y > 0$ and so from (1) $u > 0$. Therefore, the upper semi circle of the circlr $|z| = 1$ is represented in the w-plane by the positive half of real axis.



z-plane w-plane

(ii) For the lower semi circle in z-plane $y < 0$ and so from (1) we have u is negative $\Rightarrow u < 0$. Therefore the lower semi circle of the circle $|z| = 1$ is represented in w-plane by the negative half of the real axis.



z-plane w-plane

(iii) For the interior of the circle $|z| = 1$ we have

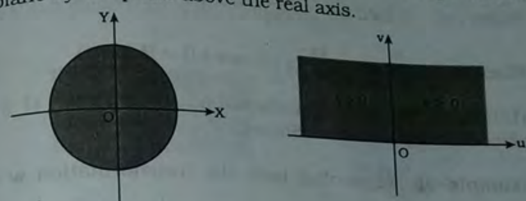
$$|z| < 1$$

$$\Rightarrow x^2 + y^2 < 1$$

$$\Rightarrow x^2 + y^2 - 1 < 0$$

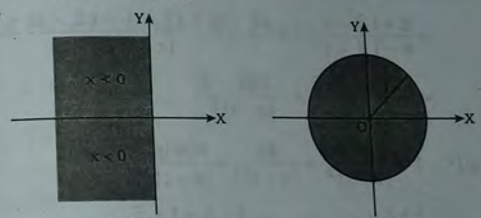
Thus, from (2), v = positive $\Rightarrow v > 0$.

Therefore, the interior of the circle $|z| = 1$ is represented in the w-plane by the plane above the real axis.



Example–24. Show that both the transformations $w = \frac{1+z}{1-z}$ and $w = \frac{z+1}{z-1}$ transform $|w| \le 1$ into half plane $Re(z) \le 0$.

Solution : When $w = \frac{1+z}{1-z}$ then



z-plane w-plane

$$w\overline{w} - 1 = \frac{1+z}{1-z}\overline{\left(\frac{1+z}{1-z}\right)} - 1$$

$$\Rightarrow |w|^2 - 1 = \frac{1+z}{1-z}\frac{\overline{1+z}}{\overline{1-z}} - 1$$

$$= \frac{(1+z)(1+\overline{z}) - (1-z)(1-\overline{z})}{|1-z|^2}$$

$$= \frac{1 + z + \overline{z} + z\overline{z} - 1 + \overline{z} + z - z\overline{z}}{|1-z|^2}$$

$$\Rightarrow |w|^2 - 1 = \frac{2(z+\overline{z})}{|1-z|^2} = \frac{4x}{|1-z|^2}$$

$[\because z = x + iy,\ \overline{z} = x - iy \Rightarrow z + \overline{z} = 2x]$

The same result is obtained when we choose the second transformation.

When $|w| = 1$ then $0 = \frac{4x}{|1-z|^2} \Rightarrow x = 0 \Rightarrow \mathrm{Re}(z) = 0$

When $|w| < 1$ then $\frac{4x}{|1-z|^2} < 0 \Rightarrow x < 0 \Rightarrow \mathrm{Re}(z) < 0$

Thus both the given transformation transforms $|z| \le 1$ into half plane $\mathrm{Re}(z) \le 0$. (Showed)

Example–25. Show that both the transformation $w = \frac{z+i}{z-i}$ and $w = \frac{i+z}{i-z}$ transform $|w| \le 1$ into the lower half plane $\mathrm{Im}(z) \le 0$.

Solution : For $w = \frac{z+i}{z-i}$ we have

$$w\overline{w} - 1 = \frac{z+i}{z-i}\overline{\left(\frac{z+i}{z-i}\right)} - 1$$

$$= \frac{z+i}{z-i}\,\frac{\overline{z}-i}{\overline{z}+i} - 1 = \frac{z\overline{z} - iz + i\overline{z} + 1 - z\overline{z} - iz + i\overline{z} - 1}{(z-i)\,(\overline{z}+i)}$$

$$= \frac{-2iz + 2i\overline{z}}{(z-i)\,(\overline{z-i})} = \frac{-2i(z-\overline{z})}{|z-i|^2}$$

$$\Rightarrow |w|^2 - 1 = \frac{-2i(2iy)}{|z-i|^2} = \frac{4y}{|z-i|^2} = \frac{4\mathrm{Im}(z)}{|z-i|^2}$$

When $w = \frac{i+z}{i-z}$ then $w\overline{w} - 1 = \frac{i+z}{i-z}\,\frac{-i+\overline{z}}{-i-\overline{z}} - 1$

$$\Rightarrow |w|^2 - 1 = \frac{-i^2 + i\overline{z} - iz + z\overline{z} + i^2 + i\overline{z} - iz - z\overline{z}}{(i-z)\,(-i-\overline{z})}$$

$$= \frac{2i\overline{z} - 2iz}{(i-z)\,(\overline{i-z})} = \frac{-2i(z-\overline{z})}{|i-z|^2}$$

$$= \frac{-2i(2iy)}{|z-i|^2} = \frac{4y}{|z-i|^2} = \frac{4\mathrm{Im}(z)}{|z-i|^2}$$

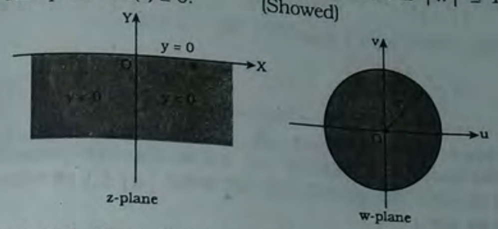
Thus in both cases

$$|w|^2 - 1 = \frac{4y}{|z-i|^2} = \frac{4\mathrm{Im}(z)}{|z-i|^2}$$

When $|w| = 1$ then $0 = \frac{4y}{|z-i|^2} \Rightarrow y = 0 \Rightarrow \mathrm{Im}(z) = 0$

When $|w| < 1$ then $\frac{4y}{|z-i|} < 0 \Rightarrow y < 0 \Rightarrow \mathrm{Im}(z) < 0$

Thus both the given transformation transform $|w| \le 1$ into the lower half plane $\mathrm{Im}(z) \le 0$. (Showed)



Example–26. Show that both the transformations $w = \frac{1-z}{1+z}$ and $w = \frac{z-1}{z+1}$ transform $|w| \le 1$ in the half plane $\mathrm{Re}(z) \ge 0$.

[CUH–2002, 2004]

Solution : When [যখন] $w = \frac{1-z}{1+z}$ then [তখন]

$$w\overline{w} - 1 = \frac{1-z}{1+z}\overline{\left(\frac{1-z}{1+z}\right)} - 1$$

$$\Rightarrow w\overline{w} - 1 = \frac{1-z}{1+z}\,\frac{1-\overline{z}}{1+\overline{z}} - 1$$

$$= \frac{1 - z - \overline{z} + z\overline{z} - 1 - z - \overline{z} - z\overline{z}}{(1+z)\,(\overline{1+z})} = \frac{-2(z+\overline{z})}{|1+z|^2}$$

$$\Rightarrow |w|^2 - 1 = \frac{-2(2x)}{|1+z|^2} = \frac{-4x}{|1+z|^2} \qquad [\because z + \overline{z} = 2x = 2\mathrm{Re}(z)$$

Again, when [আবার, যখন] $w = \frac{z-1}{z+1}$ then [তখন]

$$w\overline{w} - 1 = \frac{z-1}{z+1}\overline{\left(\frac{z-1}{z+1}\right)} - 1$$

$$\Rightarrow w\overline{w} - 1 = \frac{z-1}{z+1}\,\frac{\overline{z}-1}{\overline{z}+1} - 1$$

$$= \frac{z\overline{z} - z - \overline{z} + 1 - z\overline{z} - z - \overline{z} - 1}{(z+1)\,(\overline{z+1})}$$

$$\Rightarrow |w|^2 - 1 = \frac{-2(z+\overline{z})}{|z+1|^2} = \frac{-4x}{|1+z|^2}$$

Thus, in both coses we have [অতএব, উভয়ক্ষেত্রে পাই]

$$|w|^2 - 1 = \frac{-4x}{|1+z|^2} \;\ldots\ldots (1)$$

When [যখন] $|w| = 1$ then [তখন]

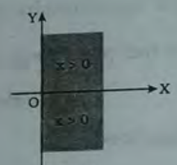
$$0 = \frac{-4x}{|1+z|^2} \Rightarrow x = 0 \Rightarrow Re(z) = 0 \;\cdots\cdots\; (2)$$

When [যখন] $|w| < 1$ then [তখন] $|w|^2 - 1 < 0$

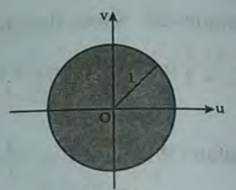
$$\Rightarrow \frac{-4x}{|1+z|^2} < 0$$

$$\Rightarrow -x < 0 \Rightarrow x > 0 \Rightarrow Re(z) > 0 \;\cdots\cdots\; (3)$$

Hence from (2) and (3) we see that both the given transformations transform $|w| \le 1$ in the half-plane $Re(z) \ge 0$. [অতএব (2) ও (3) হতে দেখা যায়, প্রদত্ত উভয় রূপান্তর $|w| \le 1$ কে অর্ধতল $Re(z) \ge 0$ তে রূপান্তর করে।]



z-plane



w-plane

Example–27. Show that both the transformations $w = \frac{z-i}{z+i}$ and $w = \frac{i-z}{i+z}$ transforms $|w| \le 1$ into upper half plane $Im(z) \ge 0$.

[DUH–1994, 2005, CUH–2003]

Solution : When [যখন] $w = \frac{z-i}{z+i}$ then [তখন] $w\overline{w} - 1 = \left(\frac{z-i}{z+i}\right)\overline{\left(\frac{z-i}{z+i}\right)}$

$$\Rightarrow w\overline{w} - 1 = \frac{z-i}{z+i}\,\frac{\overline{z}+i}{\overline{z}-i} - 1$$

$$= \frac{z\overline{z} + iz - i\overline{z} + 1 - z\overline{z} + iz - i\overline{z} - 1}{(z+i)\,(\overline{z}-i)}$$

$$\Rightarrow |w|^2 - 1 = \frac{2i(z-\overline{z})}{(z+i)\,\overline{(z+i)}}$$

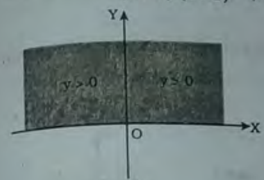
$$\Rightarrow |w|^2 - 1 = \frac{-4y}{|z+i|^2} \qquad [\because z - \overline{z} = 2iy]$$

When [যখন] $w = \frac{i-z}{i+z}$ then [তখন] $w\overline{w} - 1 = \frac{i-z}{i+z}\,\overline{\left(\frac{i-z}{i+z}\right)} - 1$

$$\Rightarrow w\overline{w} - 1 = \frac{i-z}{i+z}\,\frac{-i-\overline{z}}{-i+\overline{z}} - 1$$

$$= \frac{1 - i\overline{z} + iz + z\overline{z} - 1 - i\overline{z} + iz - z\overline{z}}{(i+z)\,(-i+\overline{z})}$$

$$\Rightarrow |w|^2 - 1 = \frac{2i(z-\overline{z})}{(i+z)\,\overline{(i+z)}} = \frac{-4y}{|z+i|^2}$$



z-plane

w-plane

Thus in both cases we have [অতএব উভয়ক্ষেত্রে পাই]

$$|w|^2 - 1 = \frac{-4y}{|z+i|^2} \;\cdots\cdots\; (1)$$

When [যখন] $|w| = 1$ then [তখন] $0 = \frac{-4y}{|z+i|^2} \Rightarrow y = 0$

$$\Rightarrow Im(z) = 0 \;\cdots\cdots\; (2)$$

When [যখন] $|w| < 1$ then [তখন] $|w|^2 - 1 < 0$

$$\Rightarrow \frac{-4y}{|z+i|^2} < 0$$

$$\Rightarrow -y < 0 \Rightarrow y > 0$$

$$\Rightarrow Im(z) > 0 \;\cdots\cdots\; (3)$$

Hence from (2) and (3) we see that both the given transformations transform $|w| \le 1$ into upper half plane $Im(z) \ge 0$ [অতএব (2) ও (3) হতে দেখা যায়, প্রদত্ত উভয় রূপান্তর $|w| \le 1$ কে অর্ধতল $Im(z) \ge 0$ তে রূপান্তর করে।]

Example–28. Prove that $w = \frac{1+iz}{1-iz}$ maps the real axis of the z-plane into a circle in the w-plane. Show by sketches the regions of the real axis of the z-plane and the circle of the w-plane that correspond.

[DUH–1993, CUH–2000]

Solution : We have [আমাদের আছে] $z = x + iy$. On the real axis of z-plane we have [বাস্তব অক্ষের উপর পাই] $y = 0 \therefore z = x + i0 = x$

$$\text{Now } w = \frac{1+iz}{1-iz} = \frac{1+i(x+iy)}{1-i(x+iy)} = \frac{1+ix}{1-ix}$$

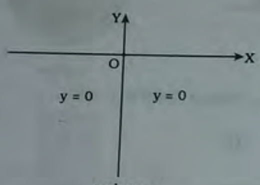
$$\therefore \overline{w} = \overline{\left(\frac{1+ix}{1-ix}\right)} = \frac{\overline{1+ix}}{1-ix} = \frac{1-ix}{1+ix}$$

$$\therefore w\bar{w} - 1 = \frac{1 + ix}{1 - ix}\frac{1 - ix}{1 + ix} - 1$$

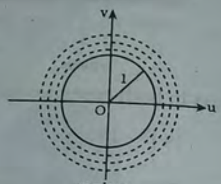
$$= \frac{1 + x^2}{1 + x^2} - 1 = 1 - 1 = 0$$

$$\Rightarrow |w|^2 = 1$$

$\Rightarrow |w| = 1$ which represents a circle in the w-plane. [যাহা w তলে একটি বৃত্ত নির্দেশ করে]



z-plane



w-plane

Example–29. Find all the mobius transformations which transform the unit circle (disc) $|z| \le 1$ into the unit circle (disc) $|w| \le 1$. Also verify your result.

[NUH–1998, DUH–2000, 2002, CUH–2000]

OR, Find a bilinear transformation which transform the unit circle (disc) $|z| \le 1$ into the unit circle $|w| \le 1$. **[NUH–2014]**

Solution : Let the transformation is [ধরি রূপান্তরটি]

$$w = \frac{az + b}{cz + d} = \frac{a}{c}\frac{z + \frac{b}{a}}{z + \frac{d}{c}} \quad \ldots\ldots (1)$$

This transform the circle $|z| = 1$ into the circle $|w| = 1$ [ইহা $|z| = 1$ বৃত্তকে $|w| = 1$ বৃত্তে রূপান্তর করে]

That is $|z|^2 = 1^2$ into $|w|^2 = 1^2$

$\Rightarrow z\bar{z} = 1$ into $w\bar{w} = 1$

$\Rightarrow z = \frac{1}{\bar{z}}$ and $w = \frac{1}{\bar{w}}$

Hence the points $z, \frac{1}{\bar{z}}$ inverse with respect to the circle $|z| = 1$ transform into the points $w, \frac{1}{\bar{w}}$ inverse with respect to the

circle $|w| = 1$. [অতএব $|z| = 1$ বৃত্তের সাপেক্ষে বিপরীত বিন্দুসমূহ $z, \frac{1}{\bar{z}}$ রূপান্তরিত হয় $|w| = 1$ বৃত্তের সাপেক্ষে বিপরীত বিন্দু সমূহ $w, \frac{1}{\bar{w}}$ এ]

When $w = 0$ then from (1) [যখন $w = 0$ তখন (1) হতে]

$$z + \frac{b}{a} = 0 \Rightarrow z = \frac{-b}{a} = \alpha, \text{ say}$$

When $w = \infty$ then from (1) [যখন $w = \infty$ তখন (1) হতে]

$$z + \frac{d}{c} = 0 \Rightarrow z = \frac{-d}{c}$$

Particularly, $w = 0, \infty$ corresponds to the points $\alpha, \frac{1^2}{\bar{\alpha}}$ respectively.

Hence from (1) it follows that [অতএব (1) হবে অনুসরণ করে]

$$\frac{-b}{a} = \alpha, \frac{-d}{c} = \frac{1^2}{\bar{\alpha}} = \frac{1}{\bar{\alpha}}$$

Thus (1) reduces to [ইহা (1) কে হ্রাস করে]

$$w = \frac{a}{c}\frac{z - \alpha}{z - \frac{1}{\bar{\alpha}}} = \frac{a\bar{\alpha}}{c}\frac{z - \alpha}{\bar{\alpha}z - 1} \quad \ldots\ldots (2)$$

$$\Rightarrow |w| = \left|\frac{a\bar{\alpha}}{c}\frac{z - \alpha}{\bar{\alpha}z - 1}\right|$$

$$\Rightarrow 1 = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{z - \alpha}{\bar{\alpha}z - 1}\right| \quad \because |w| = 1$$

$$\Rightarrow 1 = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1 - \alpha}{\bar{\alpha} - 1}\right| \quad [\because z = 1 \text{ corresponds } |w| = 1]$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\frac{|1 - \alpha|}{|1 - \bar{\alpha}|}$$

$$= \left|\frac{a\bar{\alpha}}{c}\right| \cdot 1 \quad [\because |1 - \alpha| = |1 - \bar{\alpha}|]$$

$$\Rightarrow \frac{a\bar{\alpha}}{c} = 1 \cdot e^{i\theta}, \text{ where } \theta \text{ is real}$$

$$\Rightarrow \frac{a}{c}\bar{\alpha} = e^{i\theta}$$

Putting this value in (2) we get, [এইমান (2) এ বসিয়ে পাই]

$$w = e^{i\theta}\frac{z - \alpha}{\bar{\alpha}z - 1} \quad \ldots\ldots (3)$$

This is the transformation which maps $|z| = 1$ into $|w| = 1$. Further, under this transformation the point $w = 0$ corresponds the point α which is the internal point of $|z| = 1$, therefore $|\alpha| < 1$.

[অতএব $|z| = 1$ রূপান্তরটি $|w| = 1$ এ চিত্রিত হয়। অধিকন্তু, রূপান্তরটির অধীনে $w = 0$ বিন্দুটি অনুসঙ্গী বিন্দু α দেয় যাহা $|z| = 1$ এর অন্তঃস্থ বিন্দু, অতএব $|\alpha| < 1$.]

Verification : Under this transformation [এই রূপান্তরের অধীন]

$$w\bar{w} - 1 = e^{i\theta}\frac{z-\alpha}{\bar{\alpha}z-1}\cdot e^{-i\theta}\frac{\bar{z}-\bar{\alpha}}{\alpha\bar{z}-1} - 1$$

$$\Rightarrow |w|^2 - 1 = \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(\bar{\alpha}z-1)(\alpha\bar{z}-1)} - 1$$

$$= \frac{z\bar{z} - \bar{\alpha}z - \alpha\bar{z} + \alpha\bar{\alpha} - \alpha\bar{\alpha}z\bar{z} + \alpha\bar{z} + \bar{\alpha}z - 1}{(\bar{\alpha}z-1)(\overline{\bar{\alpha}z-1})}$$

$$= \frac{|z|^2 + |\alpha|^2 - |\alpha|^2|z|^2 - 1}{|\bar{\alpha}z-1|^2}$$

$$= \frac{|z|^2(1-|\alpha|^2) - 1(1-|\alpha|^2)}{|\bar{\alpha}z-1|^2}$$

$$= \frac{-(1-|z|^2)(1-|\alpha|^2)}{|\bar{\alpha}z-1|^2}, \text{ where } |\alpha| < 1$$

When $|z| < 1$ and $|\alpha| < 1$ then $|w|^2 - 1 < 0 \Rightarrow |w| < 1$.

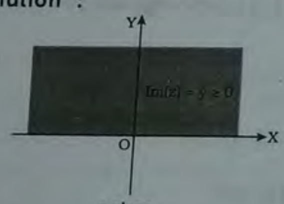
Hence $w = e^{i\theta}\dfrac{z-\alpha}{\bar{\alpha}z-1}$ is the required transformation, [যাহা পার্থিত রূপান্তর]

which transform $|z| \le 1$ into the circle (disc) $|w| \le 1$.

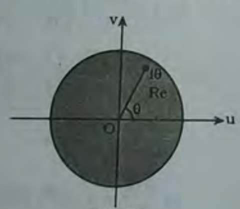
Example–30. Find all the bilinear (Mobius) transformation of the half-plane $\text{Im}(z) \ge 0$ into the unit circle.

[DUH–1993, 2003, 2005, RUH–2002, 2004, 2006]

Solution :



z-plane



w-plane

Let the bilinear (Mobius) transformation is

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\,\frac{z+\frac{b}{a}}{z+\frac{d}{c}}, \text{ where } ad - bc \ne 0 \;\cdots\cdots (1)$$

Also $c \ne 0$, otherwise points at infinity will correspond. This transforms $\text{Im}(z) = 0$ into $|w| = 1$.

That is, real axis in z-plane transform into unit circle in w-plane.

Therefore, points w, $\frac{1}{\bar{w}}$ inverse point with respect to the unit circle $|w| = 1$ transform respectively into the points z, $\bar{z}$ symmetrical with respect to the real axis in z-plane. In particular the points $w = 0, \infty$ correspond to the points α, $\bar{\alpha}$.

Hence from (1), we have $\frac{-b}{a} = \alpha, \frac{-d}{c} = \bar{\alpha}$ and then (1) reducesto

$$w = \frac{a}{c}\cdot\frac{z-\alpha}{z-\bar{\alpha}} \;\cdots\cdots (2)$$

The point $z = 0$ corresponds to the point $|w| = 1$; hence taking modulas of (2) and putting $z = 0$, $|w| = 1$ we have

$$|w| = \left|\frac{a}{c}\cdot\frac{z-\alpha}{z-\bar{\alpha}}\right|$$

$$\Rightarrow 1 = \left|\frac{a}{c}\right|\left|\frac{z-\alpha}{z-\bar{\alpha}}\right| = \left|\frac{a}{c}\right|\left|\frac{0-\alpha}{0-\bar{\alpha}}\right| = \left|\frac{a}{c}\right|\left|\frac{\alpha}{\bar{\alpha}}\right|$$

$$\Rightarrow 1 = \left|\frac{a}{c}\right|, \because |\alpha| = |\bar{\alpha}|$$

$$\Rightarrow \frac{a}{c} = e^{i\theta}, \text{ where } \theta \text{ is real}$$

$$\Rightarrow a = ce^{i\theta}$$

$$\therefore \text{From (2) we get, } w = \frac{z-\alpha}{z-\bar{\alpha}}e^{i\theta}$$

This is the transformation which transforms $\text{Im}(z) = 0$ into $|w| = 1$.

Verification : Further, under this transformation,

$$\text{we have } w\bar{w} - 1 = \frac{z-\alpha}{z-\bar{\alpha}}e^{i\theta}\cdot\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}e^{-i\theta} - 1$$

$$\Rightarrow |w|^2 - 1 = \frac{(z-\alpha)(\bar{z}-\bar{\alpha}) - (z-\bar{\alpha})(\bar{z}-\alpha)}{|z-\bar{\alpha}|^2}$$

$$\Rightarrow |w|^2 - 1 = \frac{z\bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha} - z\bar{z} + \alpha z + \bar{\alpha}\bar{z} - \alpha\bar{\alpha}}{|z-\bar{\alpha}|^2}$$

$$= \frac{\alpha(z-\bar{z}) - \bar{\alpha}(z-\bar{z})}{|z-\bar{\alpha}|^2}$$

$$= \frac{(z-\bar{z})(\alpha-\bar{\alpha})}{|z-\bar{\alpha}|^2}$$

$$= \frac{2i\,\text{Im}(z)\cdot 2i\,\text{Im}(\alpha)}{|z-\bar{\alpha}|^2}$$

$$= \frac{-4\,\mathrm{Im}(z)\cdot \mathrm{Im}(\alpha)}{|z-\bar{\alpha}|^2}$$

Since $w = 0$ corresponds to α, therefore $\mathrm{Im}(\alpha) > 0$. Hence $|w|^2 - 1 < 0$ for $\mathrm{Im}(z) > 0$

$\Rightarrow |w|^2 < 1$

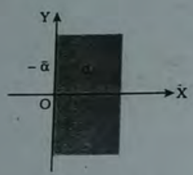
$\Rightarrow |w| < 1$ corresponds to $\mathrm{Im}(z) > 0$.

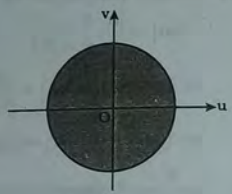
Hence the required transformations is $w = \frac{z-\alpha}{z-\bar{\alpha}} e^{i\theta}$

Example–31. Find all the bilinear transformation which transforms the half plane $\mathrm{Re}(z) \geq 0$ into the unit circle $|w| \leq 1$.

[DUH–2004, DUMP–1989, RUH–1995]

Solution :





Let $w = \frac{az+b}{cz+d} = \frac{a}{c}\cdot\frac{z+\frac{b}{a}}{z+\frac{d}{c}}$, $(ad - bc \neq 0)$ ...... (1)

be the required transformation.

This transforms $\mathrm{Re}(z) = 0$ into $|w| = 1$

That is, imaginary axis in z-plane transforms into the unit circle in w-plane.

Hence the points, $w, \frac{1}{\bar{w}}$ inverse with respect to the unit circle in w-plane transforms into the points $z, -\bar{z}$ symmetrical with respect to the imaginary axis in z-plane.

Particularly, the points $w = 0, \infty$ corresponds to the points $\alpha, -\bar{\alpha}$. Hence from (1) it follows that

$$\frac{-b}{a} = \alpha, \frac{-d}{c} = -\bar{\alpha}$$

$\therefore (1) \Rightarrow w = \frac{a}{c}\cdot\frac{z-\alpha}{z+\bar{\alpha}}$ ...... (2)

The point $z = 0$ corresponds to the point $|w| = 1$.

$\therefore$ Putting $z = 0$ and taking modulas of (2) we get,

$$|w| = \left|\frac{a}{c}\cdot\frac{-\alpha}{\bar{\alpha}}\right| = \left|\frac{a}{c}\right|\left|\frac{-\alpha}{\bar{\alpha}}\right|$$

$$\Rightarrow 1 = \left|\frac{a}{c}\right|\cdot\frac{|\alpha|}{|\bar{\alpha}|}$$

$$\Rightarrow 1 = \left|\frac{a}{c}\right|; \quad \because |\alpha| = |\bar{\alpha}|$$

$\Rightarrow 1 = \frac{a}{c} \Rightarrow \frac{a}{c} = e^{i\theta}$, where θ is real.

$$\therefore w = \frac{z-\alpha}{z+\bar{\alpha}} e^{i\theta}$$

This is the transformation which maps $\mathrm{Re}(z) = 0$ into $|w| = 1$. Further under this transformation the point $w = 0$ corresponds to α, therefore, $\mathrm{Re}(\alpha) > 0$.

Also, $w\bar{w} - 1 = \frac{z-\alpha}{z+\bar{\alpha}} e^{i\theta}\cdot\frac{\bar{z}-\bar{\alpha}}{\bar{z}+\alpha} e^{-i\theta} - 1$

$$\Rightarrow |w|^2 - 1 = \frac{z\bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha} - z\bar{z} - z\alpha - \bar{\alpha}\bar{z} - \alpha\bar{\alpha}}{|z+\bar{\alpha}|^2}$$

$$= -\frac{z\bar{\alpha} + \alpha\bar{z} + \alpha z + \bar{\alpha}\bar{z}}{|z+\bar{\alpha}|^2}$$

$$= -\frac{\alpha(z+\bar{z}) + \bar{\alpha}(z+\bar{z})}{|z+\bar{\alpha}|^2}$$

$$= -\frac{(z+\bar{z})(\alpha+\bar{\alpha})}{|z+\bar{\alpha}|^2}$$

$$= -\frac{2\mathrm{Re}(z)\cdot 2\mathrm{Re}(\alpha)}{|z+\bar{\alpha}|^2}, \text{ where } \mathrm{Re}(\alpha) > 0$$

Hence $|w|^2 - 1 < 0$. for $\mathrm{Re}(z) > 0$

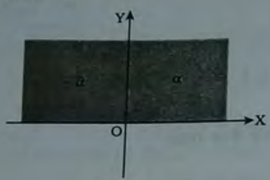
$\Rightarrow |w|^2 < 1$

$\Rightarrow |w| < 1$, Corresponds to $\mathrm{Re}(z) > 0$.

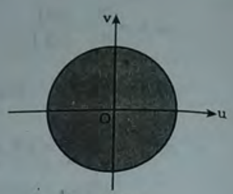
Hence $w = \frac{z-\alpha}{z+\bar{\alpha}} e^{i\theta}$ is the required transformation **Ans.**

Example–32. Find a bilinear transformation which maps the upper half of the z-plane in to the unit circle in the w-plane in such a way that $z = i$ is mapped into $w = 0$ while the point at infinity is mapped into $w = -1$. **[DUH–1975, DUM–1989, 1990]**

Solution :



z-plane



w-plane

If α is in the upper half of the z-plane, then $w = \dfrac{z - \alpha}{z - \bar{\alpha}} e^{i\theta}$ ... (1) is a bilinear transformation which maps the upper half of the z-plane into the unit circle $|w| = 1$ in the w-plane. Now $w = 0$ corresponds to $z = i$ and $w = -1$ corresponds to $z = \infty$.

Putting $w = 0$, $z = i$ in (1) we get,

$$0 = \frac{i - \alpha}{i - \bar{\alpha}} e^{i\theta}$$

$$\Rightarrow 0 = \frac{i - \alpha}{i - \bar{\alpha}}$$

$$\Rightarrow i - \alpha = 0 \Rightarrow \alpha = i \qquad \therefore \bar{\alpha} = \bar{i} = -i$$

Again, putting $z = \infty$ and $w = -1$ in (1) we get,

$$-1 = \lim_{z\to\infty} \frac{z - \alpha}{z - \bar{\alpha}} e^{i\theta}$$

$$= \lim_{z\to\infty} \frac{1 - \frac{\alpha}{z}}{1 - \frac{\bar{\alpha}}{z}} e^{i\theta}$$

$$= \frac{1 - 0}{1 - 0} e^{i\theta} = e^{i\theta}$$

$$\therefore e^{i\theta} = -1$$

Now putting $\alpha = i$, $\bar{\alpha} = -i$, $e^{i\theta} = -1$ in (1) we get,

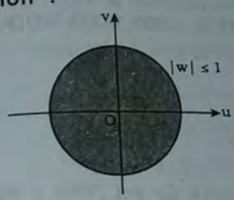
$$w = \frac{z - i}{z + i}(-1)$$

$$\Rightarrow w = \frac{i - z}{i + z}$$

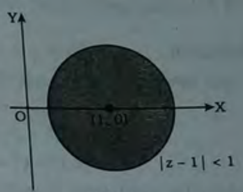
which is the required bilinear transformation . Ans.

Example–33. Find the bilinear transformation which maps the circle $|w| \le 1$ into the circle $|z - 1| < 1$ and maps $w = 0$, $w = 1$ respectively $z = \frac{1}{2}$, $z = 0$. **[RUH–1988]**

Solution :



w-plane



z-plane

Let the bilinear transformation is

$$w = \frac{az + b}{cz + d}, \ (ad - bc \ne 0) \ \cdots\cdots (1)$$

Let z be the one inverse point of the circle $|z - 1| = 1$, then the other inverse point will be $1 + \dfrac{1}{\bar{z} - 1}$.

The corresponding inverse point in the w-plane are w, $\dfrac{1}{\bar{w}}$ with respect to the circle $|w| = 1$.

Here $w = 0, \infty$ correspond to

$$z = \frac{1}{2}, \ 1 + \frac{1}{\frac{1}{2} - 1} \Rightarrow z = \frac{1}{2}, -1$$

Putting $z = \frac{1}{2}$, $w = 0$ in (1) we get,

$$0 = \frac{a \cdot \frac{1}{2} + b}{c \cdot \frac{1}{2} + d}$$

$$\Rightarrow \frac{a}{2} + b = 0 \Rightarrow a = -2b$$

Putting $z = -1$, $w = \infty$ in (1) we get,

$$\infty = \frac{-a + b}{-c + d}$$

$$\Rightarrow -c + d = 0 \Rightarrow c = d$$

Again, $w = 1$ corresponds $z = 0$.

$\therefore$ From (1) we get, $1 = \dfrac{0 + b}{0 + d} = \dfrac{b}{d} \Rightarrow b = d$

Now putting the value of $a = -2b$, $b = d$, $c = d$ in (1) we get $w = \dfrac{-2bz + b}{dz + d} = \dfrac{-2dz + d}{dz + d} = \dfrac{-2z + 1}{z + 1}$

which is our required transformation. Ans.

Example–34. A square S in the z-plane has vertices at (0, 0), (1, 0), (1, 1) and (0, 1). Determine the region in the w-plane into which S is mapped under the transformation $w = z^2$.

[NUH–1999, 2003, NU(Pre)–2011]

Solution : Let [ধরি] $z = x + iy$ and [এবং] $w = u + iv$.

Then [তখন] $w = z^2$ gives [দেয়]

$$
\begin{aligned}
u + iv &= (x + iy)^2 \\
&= x^2 + i^2y^2 \ 2ixy \\
&= x^2 - y^2 + i2xy \qquad [\because i^2 = -1]
\end{aligned}
$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$u = x^2 - y^2 \; \cdots\cdots \; (1)$$

$$v = 2xy \; \cdots\cdots \; (2)$$

At (0, 0) we have [(0, 0) এ পাই] $u = 0 - 0 = 0$, $v = 2 \cdot 0 \cdot 0 = 0$

At (1, 0) we have, $u = 1^2 - 0 = 1$, $v = 2 \cdot 1 \cdot 0 = 0$

At (1, 1) we have, $u = 1^2 - 1^2 = 0$, $v = 2 \cdot 1 \cdot 1 = 2$

At (0, 1) we have, $u = 0 - 1^2 = -1$, $v = 2 \cdot 0 \cdot 1 = 0$

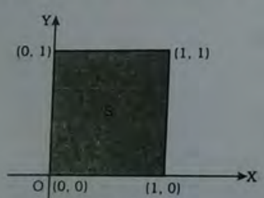
$\therefore$ The image of the point (0, 0) in the z-plane is the point (0, 0) in the w-plane. [z তলের (0, 0) বিন্দুর w তলে প্রতিবিম্ব (0, 0)]

$\therefore$ The image of the point (1, 0) in the z-plane is the point (1, 0) in the w-plane. [z তলের (1, 0) বিন্দুর w তলে প্রতিবিম্ব (1, 0)]

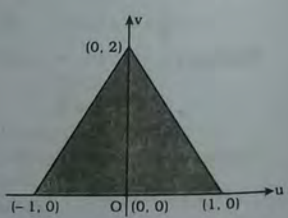
$\therefore$ The image of the point (1, 1) in the z-plane is the point (0, 2) in the w-plane. [z তলের (1, 1) বিন্দুর w তলে প্রতিবিম্ব (0, 2)]

$\therefore$ The image of the point (0, 1) in the z-plane is the point (– 1, 0) in the w-plane. [z তলের (0, 1) বিন্দুর w তলে প্রতিবিম্ব (– 1, 0)]

The transformation has shown in the following figure. [রূপান্তরটি নিম্নের চিত্রে দেখানো হয়েছে।]



z-plane



w-plane

Example–35. Show that $w = \dfrac{1 - z}{1 + z}$ maps $|z| < 1$ conformally into real part $w > 0$. Determine the curves in the w-plane which correspond to $|z| = 1$ and to $\arg z = \dfrac{\pi}{4}$. **[NUH–1998]**

Solution : Given [দেওয়া আছে] $w = \dfrac{1 - z}{1 + z}$

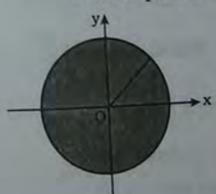
$$
\begin{aligned}
&\Rightarrow w + wz = 1 - z \\
&\Rightarrow wz + z = 1 - w \\
&\Rightarrow z(w + 1) = 1 - w \\
&\Rightarrow z = \frac{1 - w}{1 + w}
\end{aligned}
$$

Now [এখন] $|z| < 1 \Rightarrow \left|\dfrac{1 - w}{1 + w}\right| < 1$

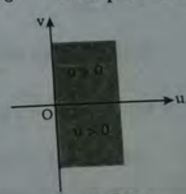
$$
\begin{aligned}
&\Rightarrow |1 - w| < |1 + w| \\
&\Rightarrow |1 - u - iv| < |1 + u + iv| \text{ where } w = u + iv \\
&\Rightarrow |(1 - u) - iv| < |(1 + u) + iv| \\
&\Rightarrow \sqrt{(1 - u)^2 + v^2} < \sqrt{(1 + u)^2 + v^2} \\
&\Rightarrow (1 - u)^2 + v^2 < (1 + u)^2 + v^2 \text{ ; by squaring} \\
&\Rightarrow (1 - u)^2 < (1 + u)^2 \\
&\Rightarrow 1 - 2u + u^2 < 1 + 2u + u^2 \\
&\Rightarrow -2u < 2u \\
&\Rightarrow 0 < 2u + 2u \\
&\Rightarrow 0 < 4u \\
&\Rightarrow 4u > 0 \\
&\Rightarrow u > 0 \\
&\Rightarrow \text{Real part of } w > 0
\end{aligned}
$$

$w = u + iv$

Real part of $w = u$



z-plane



w-plane

2nd part : $|z| = 1$

$$\Rightarrow \left|\frac{1-w}{1+w}\right| = 1$$

$$\Rightarrow |1 - w| = |1 + w|$$

$$\Rightarrow -2u = 2u \text{ ; as above}$$

$$\Rightarrow 4u = 0$$

$$\Rightarrow u = 0$$

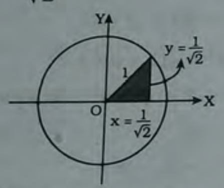
Also [আরো] $\arg z = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}$

$$\Rightarrow \frac{y}{x} = \tan\frac{\pi}{4} = 1 \Rightarrow y = x$$

Also [আরো] $|z| = 1 \Rightarrow |z|^2 = 1$

$$\Rightarrow x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2}}$$

$$\therefore x = y = \frac{1}{\sqrt{2}}$$



Example–36. Show that the transformation $w = \sin z$ transforms the infinite strip $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$, $y \ge 0$ in one-one manner onto the upper-half $v \ge 0$ of the w-plane. **[NUH–2013]**

Solution : We have $w = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$$\Rightarrow u + iv = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$z = x + iy$
$\Rightarrow iz = ix - y$
and $-iz = -ix + y$

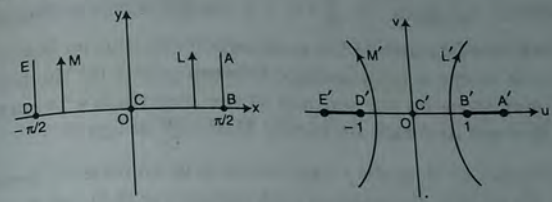
$$= (\cos x + i \sin x)\frac{e^{-y}}{2i} - (\cos x - i \sin x)\frac{e^{y}}{2i}$$

$$= \sin x\left(\frac{e^{-y} + e^{y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$\Rightarrow u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\Rightarrow u = \sin x \cosh y,\ v = \cos x \sinh y \ \ldots\ldots (1)$$

We first show that the boundary of the strip is mapped in a one to one manner onto the real axis in the w-plane, as shown in the following figure. The image of the line segment BA is found by writting $x = \frac{\pi}{2}$ in equation (1).



and restricting y to be negative. Since $u = \cosh y$ and $v = 0$ when $x = \frac{\pi}{2}$, a typical point $\left(\frac{\pi}{2}, y\right)$ on BA is mapped into the point $(\cosh y, 0)$ in the w-plane, and that image must move to the right from B′ along the u axis as $\left(\frac{\pi}{2}, y\right)$ moves upward from B. A point $(x, 0)$ on the horizontal segment DB has image $(\sin x, 0)$, which moves to the right from D′ to B′ as x increases from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, or as $(x, 0)$ goes from D to B. Finally, as a point $\left(-\frac{\pi}{2}, y\right)$ on the line segment DE moves upward from D, its image $(-\cosh y, 0)$ moves to the left from D′.

One way to see how the interior of the strip is mapped onto the upper half $v > 0$ of the w-plane is to examine the images of certain vertical half lines. If $0 < c_1 < \frac{\pi}{2}$, points on the line $x = c_1$ are transformed into points on the curve

$$u = \sin c_1 \cosh y,\ v = \cos c_1 \sinh y,\ (-\infty < y < \infty) \ \ldots\ldots (2)$$

which is the right-hand branch of the hyperbola

$$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1 \ \ldots\ldots (3)$$

with foci at the points $w = \pm\sqrt{\sin^2 c_1 + \cos^2 c_1} = \pm 1$.

As a point on the line moves upward, its image on the hyperbola also moves upward. In particular, there is a one to one mapping of the top half ($y > 0$) of the line onto the top half ($v > 0$) of the hyperbola's branch. Such a half line L and its image L′ are shown in the figure. If $-\frac{\pi}{2} < c_1 < 0$, the line $x = c_1$ is mapped onto the left-hand branch of the same hyperbola, and as before, there is a one to one correspondence between points on the top of the line and those on the top half of the hyperbola's branch. This half line and its image are labeled M and M′ in figure.

The line $x = 0$, or the y-axis, needs to be considered separately. From (1) we have the image of each point $(0, y)$ is $(0, \sin hy)$. Hence y axis mapped onto the v axis in a one to one manner, the positive y axis corresponding to the positive v axis.

Now each point in the interior $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y > 0$ of the strip lies on one of the above mentioned half lines, and it is important to notice that the images of those half lines are distinct and constitute the entire half plane $v > 0$. More precisely, if the upper half L of a line $x = c_1 (0 < c_1 < \pi/2)$ is thought of as moving to the left toward the positive y-axis, the right-hand branch of the hyperbola containing L′ is opening up wider and its vertex $(\sin c_1, 0)$ is tending toward the origin $w = 0$. Hence L′ tends to become the positive v-axis. One the other hand, as L approaches the segment BA of the boundary of the strip, the branch of the hyperbola closes down around the segment B′A′ of the u-axis and the vertex $(\sin c_1, 0)$ tends towards the point $w = 1$. Similar statements can be made regarding the half line M and its image M′ in the figure. We may conclude that the image of each point in the interior of the strip lies in the upper half plane $v > 0$ and that each point in the half plane is the image of exactly one point in the interior of the strip.

This completes our desnonstration that the transformation $w = \sin z$ is a one to one mapping of the strip $-\pi/2 \le x \le \frac{\pi}{2}$, $y \ge 0$ onto the half plane $v \ge 0$.

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. Write (or what is) necessary condition for $w = f(z)$ to be a conformal mapping. **[NUH-2015]**

   Ans : The necessary condition for $w = f(z)$ to be a conformal mapping is that $f(z)$ is an analytic function of z.

2. Write sufficient condition for $w = f(z)$ to be a conformal mapping.

   Ans : If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R, then the mapping $w = f(z)$ is conformal in R.

3. What is bilinear transformation? **[NUH-2012]**

   Ans : The transformation T defined by $w = T(z) = \frac{az + b}{cz + d}$ is called a bilinear transformation, where a, b, c, d are complex constants and z, w are complex variables.

4. Define Mobius transformation.

   Ans : The transformation T defined by $w = T(z) = \frac{az + b}{cz + d}$ is called Mobius transformation, where a, b, c, d are complex constants and z, w are complex variables.

5. What is the difference between bilinear transformation and Mobius transformation?

   Ans : The bilinear transformation and the Mobius transformation are same thing. There is no difference between them.

6. What are the critical points for conformal mapping.

   Ans : The conformal property of $w = \frac{az + b}{cz + d}$ does not hold good at the points $z = \frac{-d}{c}$ and $z = \infty$. These two points are called the critical points.

7. Define cross ratio of complex numbers. **[NUH-2012]**

Ans : The cross ratio of four complex numbers z_1, z_2, z_3, z_4 is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

8. If w_1, w_2, w_2, w_4 are the images of z_1, z_2, z_3, z_4 respectively, then what relation exists among all w's and all z's.

Ans : The cross ratio of w_1, w_2, w_3, w_4 = the cross ratio of z_1, z_2, z_3, z_4.

That is, $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$.

9. Define fixed point of a transformation.

Ans : A fixed point of a transformation $w = f(z)$ is a point whose image is the same complex number. The fixed points are obtained from the equation $w = f(z) = z$.

10. Why the transformation $w = \frac{az + b}{cz + d}$ is called the bilinear transformation?

Ans : $w = \frac{az + b}{cz + d} \Rightarrow czw - az + dw - b = 0$

This equation is linear both in z and w. For this reason $w = \frac{az + b}{ca + d}$ is called bilinear transformation.

11. How many inverse of a bilinear transformation?

Ans : The inverse of a bilinear transformation is unique.

12. Define transformations.

OR, What is transformation or mapping. **[NUH-2015]**

Ans : See art-6.1

13. Define conformal mapping.

Ans : See art-6.2

EXERCISE-6

Part-A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. Define a transformation for a complex valued func ion f(z).
2. When a correspondence is said to be one-one?
3. Define a isogonal mapping.
4. Let f(z) is analytic and $f'(z) = 0$. Does the conformal property hold in this case?
5. When the conformal property of an analytic function does not hold?

Part-B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. Define bilinear transformation. Why it is called bilinear transformation? **[NUH-2000, 2005, 2006 (Old), DUH-2000, RUH-1995, KUH-2002]**

Ans : See art-6.5 and solved expl-1.

2. Find the condition that the transformation $w = \frac{az + b}{cz + d}$ transforms the unit circle in the w-plane into a straight line in the z-plane. **[CUH-2004]**

Ans : See solved problem-4

3. Prove that a bilinear transformation $w = \frac{az + b}{cz + d}$, $ad - bc \neq 0$, transform the circles of the z-plane into the either circles or straight lines of the w-plane. **[NUH-1997, KUH-2002]**

Ans : See solved problem-4.

4. If $w = f(z) = u + iv$ is analytic in a region R, prove that $\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2$. **[NUH-1999]**

Ans : See art 6.4.

5. If a and b are two distinct fixed points of a bilinear transformation. Prove that it can be written in the form $\frac{w-a}{w-b} = k\left(\frac{z-a}{z-b}\right)$, where k is a constant.

Ans : See art-6.8.

6. Prove that every bilinear transformation is the resultant of translation, rotation, magnification and inversion.

[RUH-1996, 1998, 2001, KUH-2004]

Ans : See in art-6.6.

7. Prove that a bilinear transformation can be considered as a combination of the transformation of translation, rotation, stretching and inversion. **[CUH-2003, KUH-2002]**

Ans : see in art-6.6.

8. Find all bilinear transformations which have fixed points as -1 and 1.

Ans : $T(z) = \frac{az+b}{bz+a}$, $a^2 - b^2 \neq 0$.

9. Show that the transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5$, $v = 2y - 3x - 4$.

10. Find the fixed points of the transformation

(a) $w = \frac{(z-1)}{(z+1)}$, (b) $z = \frac{(6z-9)}{z}$.

Ans : (a) $z = \pm i$, (b) $z = 3$.

11. Find the bilinear transformation which

(i) maps the points – 1, 0, 1 onto the points 0, i, 3i.

(ii) maps the points 2, i, – 2 onto the points 1, i, – 1

(iii) maps the points ∞, i, 0 onto the points 0, i, ∞.

Ans : (i) $w = -\frac{3i(z+1)}{z-3}$, (ii) $w = \frac{3z+2i}{iz+6}$, (iii) $w = -\frac{1}{z}$.

12. What is cross ratio? Show that cross ratio is invarient under bilinear transformation. **[KUH-2002]**

Ans : See art 6.7 and theorem-4

13. Define cross ratio. Show that it is invariant under any Mobius transformation. **[DUH-2002]**

Ans : See art 6.7 and theorem-4.

Part–C (Broad Questions) (বড় প্রশ্ন)

1. Show that if f(z) is analytic and $f'(z) \neq 0$ in a region R, then it is conformal in R. **[RUH-2006, KUH-2004]**

Ans : See theorem-2 in art-6.3.

2. Prove that the transformation $w = f(z)$ where f(z) is analytic at z_0 and $f'(z_0) \neq 0$ the tangent at z_0 to any curve in the z-plane passing through z_0 is rotated through the angle arg $f'(z_0)$.

[RUH-2000]

Ans : See theorem-3 in art-6.3.

3. Find a bilinear transformation which maps points z_1, z_2, z_3 of the z-plane into the points w_1, w_2, w_3 of the w-plane respectively. **[RUH-1994, 1997]**

Ans : See bilinear transformation for three given points in art-6. 7.

4. Explain conformal mappings and state the sufficient conditions for a function $w = f(z)$ to represent a conformal mapping. **[NUH-1999]**

Ans : See art-6.2 and theorem 2 of art-6. 3.

5. If a = b are two fixed points of a bilinear transformation, show that it can be written as $\frac{1}{w-a} = \frac{1}{z-a} + k$ where k is a constant. **[RUH-1997]**

Ans : See art-6.8.

6. Prove that the cross ratio is invariant under any bilinear transformation. **[CUH-2004]**

Ans : Se theorem-4 in art-6.7

7. Prove that the set of all mobius transformations form a group under composition. **[RUH-2003]**

Ans : See theorem-5 in art-6.9.

8. Prove that the most general bilinear transformation which maps $|z| = 1$ on to $|w| = 1$ is $w = e^{i\theta}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$, where α is a constant.

Also show that the transformation maps $|z| < 1$ onto (a) $|w| < 1$ if $|\alpha| < 1$ and (b) $|w| > 1$ if $|\alpha| > 1$. What happen if $|\alpha| = 1$?

9. Show that the transformation $w = \frac{z-i}{z+i}$ transforms the upper half plane $Im(z) \geq 0$ into $|w| \leq 1$. **[DUH-1994, 2005]**

Ans : See solved example-27.

10. What is invariance of the cross ratio? Applying this or otherwise find the Mobius transformation that make the points 2, 1 + i, 0 of the z-plane corresponds to the points 0, 1, ∞ of the w-plane.

Ans : 1st part – See theorem-4.

2nd part – do as example-14 or 16.

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COMPLEX ANALYSIS

Shah Jalal University (Course-222)-2006

Set-A

1. (a) Define complex number. Is complex number algebra? Why or why not explain it.

(b) Show that the modulus of the product of two complex numbers are equal to the product of their.

(c) Show that $\arg\left(\frac{z-1+i}{z+i}\right) = \frac{\pi}{4}$ represents a circle. Find its center and radius.

2. (a) Define limit. Prove that $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

(b) If $\lim_{z\to 0} f(z)$ exist then prove that it must be unique.

(c) Prove that the function $f(z) = u + iv$, where

$$f(x) = \begin{cases} \frac{(\bar{z})^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$ is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there.

3. (a) Define analytic function. State and prove necessary and sufficient condition for f(z) to be analytic.

(b) Find analytic function $f(z) = u + iv$ in terns of z whose imaginary part is $\frac{x-y}{x^2+y^2}$.

4. (a) Define harmonic function with example. Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

(b) If $f(z) = u + iv$ analytic, find v, where $u = e^{-x}(x \sin y - y \cos y)$.

(c) If u and v are conjugate harmonic functions, prove that $dv = \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx$.

5. (a) State and prove Cauchy's integral formula.

(b) Evaluate (i) $\int \frac{\sin nz^2 + \cos \pi z^2}{(z-1)(z+2)} dz$

(ii) $\int \frac{e^3}{(z+1)^3} dz$, where c is the circle $|z| = 3$.

Set-B

6. (a) Discuss zeros and different type of singularities from principal part of Laurent's theorem.

(b) Evaluate (i) $\oint \frac{z-3}{z^2+2z+5} dz$ where c is the circle $|z+1-i| = 2$.

(ii) $\oint \frac{z^2 - 4}{z(z^2 + 9)} dz$ where c is the circle $|z| = 1$.

7. (a) State and prove Morera's theorem.

(b) Evaluate $\oint_{f(z)}^{f'(z)} dz$ if c is the circle $|z| = \pi$, where

(i) $f(z) = \sin \pi z$, (ii) $f(z) = \tan \pi z$.

8. (a) Expand $f(z) = \frac{1}{(z + 1)(z + 3)}$ in a Laurent series valid for (i) $|z| > 3$, (ii) $0 < |z + 1| < 2$.

(b) State and prove that argument theorem.

9. Prove that by contour integration

(i) $\int_0^{2\pi} \frac{d\theta}{(5 - 3 \sin \theta)^2} = \frac{5\pi}{32}$ and (iii) $\int_0^{\infty} \frac{ln(x^2 + 1)}{x^2 + 1} dx = \pi \, ln \, 2$.

10. (a) Define conformal napping. Show that the transform $w = f(z) = z^2$ at point $z = 1 + i$ is conformal mapping.

(b) Define Bilinear transformation find a bilinear transformation which transform the unit circle.

$|z| = 1$ into real axis such a way that the point of the z-plane are mapped of thes.

Khulna University (Course–3105, Complex–I)–2006

Section–A

1. (a) If z_1 and z_2 are complex numbers, then prove that

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(b) Prove that $\left|\frac{z - 1}{z + 1}\right|$ = constant represents a circle.

(c) Describe graphically the locus represented by

$$|z - 3| - |z + 3| = 4.$$

2. (a) Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ dows not exist.

(b) Test the continuity of $f(z) = \begin{cases} z^2, & z \neq z_0 \\ 0, & z = z_0 \end{cases}$, where $z_0 \neq 0$ at $z_0 = z_0$.

(c) If = f(z) is analytic at z_0, prove that it must be continuous at z_0.

3. (a) If $w = f(z) = z^2$, find the value of a which corresponds to $z = 1 - 3i$ and show that point graphically.

(b) Find the zeros of cos z.

(c) Evaluate $\lim_{z \to z+i} (z^2 - 5z + 10)$.

(d) What are Cauchy-Riemann equations? Does the function e^{z^2} satisfy the Cauchy-Riemann equations?

4. (a) Define mapping. If $w = f(z) = u + iv$ is analytic in a region R, then prove that $\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2$.

(b) The straight lines $y = 2x$, $x + y = 6$ in the xy plane are mapped on to the w-plane by means of the transformation $w = z^2$. Show graphically the images of the straight lines in the w-plane.

(c) Find a bilinear transformation which maps the points i, – i, 1 of the z-plane into 0, 1, ∞ of the w-plane respectively.

Section–B

5. (a) Evaluate $\int (x^2 - iy^2) \, dz$ along (i) the parabola $y = 2x^2$ from (1, 1) to (2, 8). (ii) the straight line from (1, 1) to (2, 8).

(b) Evaluate $\int_C \bar{z} \, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by–

(i) $z = t^2 + it$, (ii) the line form $z = 0$ to $z = 2i$ and then the line form $z = 2i$ to $z = 4 + 2i$.

6. (a) If f(z) is analytic inside and on the boundary C of a simply-connected region R, then prove taht

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

(b) Evaluate $\oint_C \frac{e^{2x}}{(z + 1)^4} dz$, where C is the circle $|z| = 3$.

7. (a) State and prove Liouville's theorem.

(b) $\oint_C \frac{\sin 3z}{z + \pi/2} dz$, if C is the cirele $|z| = 5$.

8. (a) State and prove Rouche's theorem.

(b) Show that the roots of $z^4 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Khulna University (Course–3105)–2007

Section–A

1. (a) Prove that $|Z_1 - Z_2| \geq |Z_1| - |Z_2|$ and give a graphical representation.

(b) Describe graphically the region represented by

$$|Z + 2 - 3i| + |Z - 2 + 3i| < 10.$$

(c) Express – 3 – 4i polar form.

2. (a) State and prove a set of sufficient conditions for a function f(z) to be analytic in a region.

(b) Let a function $f(z) = u + iv$ be defined as follows :

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \text{ ; when } z \neq 0$$
$$= 0 \qquad \text{ ; when } z = 0$$

Prove that the four partial derivatives u_x, u_y, v_x, v_y satisfies Cauchy-Riemann equations at (0, 0) but $f'(0)$ does not exist.

3. (a) Define harmonic function. Is $e^{-2xy} \sin(x^2 - y^2)$ harmonic? If so, find its conjugate harmonic function.

(b) What do you mean by analytic function? Does $f(z) = ze^{-z}$ satisfy Cauchy-Riemann equation.

(c) Locate and name the singularities of $\frac{ln(z + 3i)}{z^2}$.

4. (a) Define conformal mapping. If f(z) is analytic and $f'(z) \neq 0$ in aregion R, prove that the mapping $w = f(z)$ is conformal at all points of R.

(b) Find a bilinear transformation which maps the vertices $l+i$, $-i$, $2-i$ of a triangle of the z-plane into the points 0, 1, i of the w-plane.

Section–B

5. (a) Define the complex line integral of f(z) along a curve.

(b) If f(z) is analytic and $f'(z)$ is continuous in a region R bounded by a simple closed curve C and on its boundary, then prove that $\oint_c f(z)\,dz = 0$.

(c) Evaluate $\oint_c \frac{dz}{z-a}$; where C is any simple closed curve and $z = a$ is (i) outside C (ii) inside C.

6. (a) State and prove Cauchy's integral formula.

(b) Evaluate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$.

7. (a) State and prove Liouville's theorem.

(b) Find the maximum value of $|f(z)|$ in $|z| \leq 1$ for the function $f(z) = z^4 + z^2 + 1$.

8. (a) If C is closed curve around the origin, prove that $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i} \cdot \frac{1}{n!} \oint_c \frac{a^n e^{az}}{z^{n+1}} dz$. Hence show that

$$\sum \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{2a \cos\theta}\, d\theta.$$

(b) State and prove Morera's theorem.

Khulna University (Course–3205, Complex–II)–2005

Section–A

1. (a) State and prove Taylor's theorem for complex variable.

(b) Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a laurent series valid for

(i) $1 < |z| < 2$ (ii) $|z| > 2$ (iii) $0 < |z-2| < 1$.

2. (a) Define conformal and isogonal mappings. Prove that, if $W = f(z)$ is analytic in a region **R** and $|f'(z_0) \neq 0|$ for all interior points z_0 of **R**, then the mapping is conformal.

(b) Show that the inverse of the point "a" with respect to the circle $|z - c| = R$ is the point $c + \frac{R^2}{\bar{a} - \bar{c}}$.

3. (a) Define the linear transformation. Find a bilinear transformation which maps the points i, – i, 1 of the z-plane into 0, 1, ∞ of the W-plane respectively.

(b) Let the rectangular region R in the z-plane be bounded by $x = 0$, $x = 2$, $y = 0$, $y = 1$. Determine the region R′ of the W-plane into which R is mappel under the transformations

(i) $W = z + (1 - 2i)$ (ii) $W = \sqrt{2}e^{i\pi/4} z$ (iii) $W = \sqrt{2}e^{i\pi/4} z + (1 - 2i)$

4. (a) Show that the resultant of any number of successive bilinear transformations is also a bilinear transformation.

(b) Find the condition that the transformation $W = \frac{az + b}{cz + d}$ transform the unit circle in the W-plane into a straight line in the z-plane.

(c) Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$.

Section–A

5. (a) Define residue. State and prove residue theorem for complex function.

(b) Find residues of $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2 + 4)}$ at all its poles in the finite plane.

6. (a) Let f(z) be analytic inside and on a simple closed curve C except at a pole "a" of order m inside C. Prove that the residue of f(z) at "a" is given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z - a)^m f(z)\}.$$

(b) Define the Pole of a function. Show that if f(z) has a pole at $z = a$ of order m, then $\frac{1}{f(z)}$ has a zero of order m at $z = a$.

7. (a) Define singularity of a function. Find the Laurent series about the indicated singularity for each of the following functions. Also name the singularity in each case.

(i) $\frac{e^{2z}}{(z-1)^3}$; $z = 1$ (ii) $(z-3)\sin\frac{1}{(z+2)}$; $z = -2$

(iii) $\frac{z}{(z+1)(z+2)}$; $z = -2$

(b) Prove that a function f(z) which has no singularity in the finite part of the plane and has a pole of order n at infinity is a polynomial.

8. Find the contour integration of the followings (any two)

(i) $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$ (ii) $\int_0^{\infty} \frac{\sin x^2}{x} dx$

(iii) $\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)}$ if $a > |b|$.

Khulna University (Course–3205, Complex–II)–2006

Section–A

1. (a) State and prove Taylor's theorem on the expansion of a function f(z) in a series.

(b) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for the region (i) $1 < |z| < 3$ (ii) $0 < |z+1| < 2$.

2. (a) If f(z) be analytic within and on a closed contour C except at a finite number of poles within C and if $f(z) \neq 0$ on C then prove that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P$

where N is the number of zeros and P is the number of poles inside C (A pole of zero of order m must be counted m times)

(b) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

3. (a) State and prove Cauchy's residue theorem.

(b) Evaluate the residues of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and infinity and show that their sum is zero.

4. Evaluate any three of the following integrals by the method of contour integration.

(i) $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$

(ii) $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$; where $a > 0, b > 0, a > b$

(iii) $\int_0^{\infty} \frac{(\log x)^2 dx}{x^2+1}$

(iv) $\int_0^{\infty} \frac{x^{n-1} dx}{x^2+1}$ where $0 < a < 1$.

Section–B

5. (a) If f(z) is analytic and $f'(z) \neq 0$ in a region R, prove that the mapping $w = f(z)$ is conformal at all points of R.

(b) Show that the transformation $w = \frac{2z+3}{z-4}$ changes the circle $x^2 + y^2 - 4x = 0$, into the straight $4u + 3 = 0$ and explain why the curve obtained is not a straight line.

6. (a) Show that under a bilinear transformation the cross-ratio of any four points remains invariant and hence deduce a bilinear transformation which transforms three distinct points into three specified distinct points.

(b) Find a bilinear transformation which maps the vertices $1 + i, -i, 2 - i$ of a triangle A of the z-plane into the points 0, 1, i of the w-plane.

7. (a) Find a function harmonic in the upper half of the Z-plane, $\text{Im}(z) > 0$, which takes the prescribed values on the x-axis given by, $G(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$.

(b) (i) Find the complex potential due to a source at $z = a$ and a sink at $z = a$ of equal strength k.

(ii) Determine the equipotential lines and stream lines and represent graphically.

(iii) Find the speed of the fluid at any point.

8. (a) Define Beta and Gamma finction. Prove that

$$\Gamma(m) = 2\int_0^{\infty} x^{2m-1} e^{-x^2} dx, \ m > 0.$$

(b) Find the general solution of $Y'' - 3Y' + 2Y = 0$ by the method of contour integrals.

Chittagong University (Course–302)–2006

1. What is Argand diagram? Show that the equation of a circle in the Argand plane can be put in the form $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$, where c is real and b is complex constant.

2. Define limit. Prove that $\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2} = 1 - \frac{1}{2}i$.

3. Derive the sufficient condition for $f(z) = u(xy) + iv(x, y)$ to be analytic in a region R.

4. Show that the function $u = 2x(1 - y)$ is harmonic and find a function v such that $f(z) = u + iv$ is analytic.

5. If $f(z) = u + iv$ is a regular function of z in any domain R, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(x)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2.$$

6. Define analytic function. Determine the analytic function $w = u + iv$ if $v = \log(x^2 + y^2) + x - 2y$.

7. State and prove Cauchy's integral formulae.

8. Show that $\oint_C \frac{e^{tz}}{z^2 + 1} dz = 2\pi i \sin t$, where c is the circle $|z| = 3$ and $t > 0$.

9. State and prove Liouville's theorem.

10. If C is a closed countour around the region,

prove that $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int \frac{a^n}{n!}\frac{e^{az}}{z^{n+1}} dz$, Hence deduce that–

$$\sum_{n=0}^{\infty} \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta$$

11. Expand $f(z) = \frac{1}{(z-2)^2}$ in a Laurent series valid for $|z| < 2$.

12. State and prove Cauchy's residue theorem.

13. Find the residue of $\frac{1}{(z^2 + a^2)^2}$ at $z = ia$.

14. Define the residue at a pole. If f(z) is analytic within and on a cloned contour C except at a finite number of poles, and is not zero on C, then prove that

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} dz = N - P.$$

where N is the number of zeros and P is the number of poles inside C.

15. Apply Calculus of residues to prove that

$$\int_0^{\infty} \frac{x \sin ax}{x^2 + k^2} dx = \frac{\pi}{2} e^{-ak}, \text{ where } a > 0, k > 0.$$

16. Show that $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$

17. Show that $\int_0^{\infty} \frac{dx}{1 + x^2} = \frac{1}{2}\pi$

18. Find the singularities of the function $\frac{e^{c/(z-a)}}{e^{z/a-1}}$, indicating the character of each singularities.

19. Find the Mobius transformation which transforms the circle $|z| = 1$ onto $|w| = 1$ and makes the point $z = 1, -1$ correspond to $w = 1, -1$ respectively.

20. Show that the transformation $w = \frac{2z + 3}{z - 4}$ transforms the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$ and explain the curve obtained is not a circle.

Chittagong University (Course–302)–2008

1. If z_1 and z_2 are two non-zero complex numbers, prove that the modulus of can or difference is always less than or equal to the sum of this moduli.

2. Show that the origin and the point representing the roots of the equation $z^2 + pz + q = 0$ form an equilateral triangle if $p^2 = 3q$.

3. Show that the inverse of a point 'a' with respect to the circle $|z - c| = r$ is the point

$$c + \frac{r^2}{\bar{a} - \bar{c}}.$$

4. Define limit at infinity. If $f(z) = z^2$. Prove that

$$\lim_{z \to z_0} f(z) = z_0^2.$$

5. State and derive the necessary condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R.

6. Define regular function. Determine the regular function $w = u + iv$ if $v = \log(x^2 + y^2) + x - 2y$.

7. If $f(z) = u + iv$ is a regular function is a Region R, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2.$$

8. Show that $u = \frac{1}{2} \log (x^2 + y^2)$ satisfies the Laplace's equation and find u if $f(z) = u + iv$ is analytic.

9. Let f(z) be continuos in a simply-connected region R and suppose that $\oint_c f(z)\, dz = 0$ around every simple closed curve C in R. Then prove that f(z) is analytic in R.

10. Using the definition of the integral of f(z) on a given path, evaluate $\int_{-2+i}^{5+3i} z^3\, dz$.

11. State and prove the caucliy's integral formula.

12. State and rove Liouville's theorem.

13. Show that $\oint_c \frac{\sin^6 z}{z - \pi/6}\, dz = \frac{\pi i}{32}$, where C is the eirele $|z| = 1$.

14. Show that every polynomial of degree n has exactly n zeros.

15. If the function f(z) is analytic and single valued in $|z - a| < R$. Prove that when $0 < r < R$, $f'(a) = \frac{1}{\pi r} \int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$, where P(G) is a real part of $f[a + ra^{i\theta})$

16. State Tayler's and Laurent's theorem. Expand $f(z) = \frac{1}{z^2(z-1)(z+2)}$ in a Laurent series in the region $|z| > 2$.

17. Show that $\oint_c \frac{e^{tz}}{(z^2+1)^2}\, dz = \pi i(\sin t - t \cos t)$, where C is the circle $|z| = 3$ and $t > 0$.

18. By Cauchy's residue theorem, show that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \frac{\pi}{12}.$$

19. By the method of contour integration. Prove that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \pi.$$

20. Apply the calculus of residues to prove that $\int_0^{2\pi} e^{\cos\theta} \cos (\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$, where n is a positive integer.

21. Define singularities of f(z). Discuss the different types of singularities.

22. Show that both the transformations $w = \frac{1-z}{1+z}$ and $w = \frac{z-1}{z+1}$ transform $|W| \le 1$ in the half plane $Re(z) \ge 0$.

23. Find the Mobius transformations which transforms the unit circle $|z| \le 1$ into unit circle $|w| \le 1$.

24. Define bilinear transformation. Prove that every bilinear transformation maps circles or straight lines into circles or straight tines.

25. Find transformation which maps outside $|z| = 1$, on the half plane $R(w) \ge 0$. So that the points $z = -1, -i, 1$ correspond to $w = i, 0, -i$ respectively.

Pure Math, RUH (Course-302)-2006

1. (a) Define a complex number. Describe the region $|z - 1| \le |z + 1|$ geometrically.

(b) Show that the triangle inequality holds in 1.

(c) Define a complex function f(z). When a function f(z) is said to be continuous in a region. Prove that the differentiability of f(z) implies continuity but the converse is not true in general.

2. (a) If $w = f(z) = u + iv$ is an analytic function, show in polar form the Cauchy Reimann equations are–

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

(b) Prove that the real and iimaginary parts of on analytic function of a complex variabel when expressed in polar form satisfy the equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial r^2} = 0.$$

(c) Prove that an analytic function with constant modulus is constant.

3. (a) State and prove Cauchy's fundamental theorem for the case of a triangle.

(b) Define complex line integral. Prove that if f(z) in integrable along a curve C having finits length L and if there exists a positive number M such that $|f(z)| \le M$ on C, then

$$\left| \int_C f(z)\, dz \right| \le ML.$$

4. (a) If f(t) is analytic inside and on a simple closed curve C and a is any point inside C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a}\, dz.$$

(b) Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$, where C is $|z| = 3$ and $t > 0$.

(c) If F(z) is contiguous in a simple connected region R and if $\oint_C f(z)\,dz = 0$ around every simple closed curve C in R, then prove that f(z) is analytic in R.

5. (a) If f(z) is analytic inside and on a simple closed curve C except for a pole of order m at z = a inside C, prove that

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)}\cdot\frac{d^{m-1}}{dz^{m-1}}\{z(z-a)^m f(z)\}$$

(b) Evaluate $\oint_C \frac{1}{z^2 - 2z(2n^2+1)+1}dz$, where C is the unit circle and $a > 0$.

(c) If C is a closed contour around the origin, prove that

$$\left(\frac{a^n}{n i}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{n i z^{n+1}}dz.$$

6. (a) If f(z) is analytic incide a circle C with contre at a, then prove that for all z inside C,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{\lfloor 2}f''(a) + \cdots$$

(b) Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$.

(c) Expand $f(z) = \frac{1}{z^2-3z+2}$ in a Laurent series valid for $1 < |7| < 2$.

7. (a) Evaluate $\int_0^\infty \frac{x^{-1/2}}{x+1}dx$ by contour integration.

(b) When a mapping is said to be conformal at a point? Show that if f(z) is analytic and $f'(z) \neq 0$ in a region R, then it is conformal in R.

(c) Find all the bilinear transformation which maps the half plane $I(z) \geq 0$ into the unit disc $|w| \leq 1$.

8. (a) Define analytic continuation of on analytic function. Give an example of it.

(b) Show that if f(z) is analytic in a region R and f(z) = 0 at all points on an are PQ inside R, then f(z) = 0 throughout R.

(c) State the Remain Naing theorem.

Applied Math, RUH (Course-302)-2009

1. (a) Find the equation of the circle on the join of points z_1 and z_2 as diameter.

(b) The modulus of sum of difference of two complex numbers is always less than or equal to sum of their moduli.

(c) If $z = x + iy$, then prove that $|x| + |y| \leq \sqrt{2}\,|x + ry|$.

2. (a) Show that a function which is differentiable is necessarily continuous, but the converse is not true.

(b)If $f(z) = \frac{2z-1}{3z+2}$, prove that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{7}{(3z_0+2)^2},\ z_0 \neq -\frac{2}{3}.$$

(c) Show that $f(z) = \frac{z^2+1}{z^2-3z+2}$ is continuous for all z out side $|z| = 2$.

3. (a) Define Analytic function; Simple closed curve; Harmonic function and simply connected region.

(b) If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R and if u and v have continuous second order partial derivatives in R, then u and v are harmonic in R.

(c) Find an analytic function f(z) such that $\operatorname{Im}\{f'(z)\} = 6xy + 4x$, $f'(0) = 0$ and $f(1 + i) = 0$.

4. (a) State and prove the fundamental theorem of algebra.

(b) Prove that $f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-a)^2}$, where c is the boundary of a simply connected region R.

(c) If C is the circle $|z| = 3$ and $t > 0$, evaluate $\frac{1}{2\pi i}\oint_c \frac{e^{tz}}{(z^2+1)^2}dz$.

5. (a) State and prove the Laurentiz's theorem. What do you mean by analytic and principal part of Laurentz's series? What happens when principal part is zero?

(b) If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$ by using Rouche's theorem.

(c) If c is the circle $|z| = \pi$, then show that $\oint_c \frac{f'(z)}{f(z)}dz = 12\pi i$ if $f(z) = \cos \pi z$.

6. (a) Define singularity of f(z). Discuss various types of singularity.

(b) Locate and name all the singularitics of

$$f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}.$$

Also determine where f(z) is analytic.

(c) State argument theorem. If $\frac{1}{2\pi i}\oint_c \frac{dz}{z+1} = 1$ and if c is the circle $|z| = 3$, then show that $\oint_c \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} = 2\pi$.

7. Evaluate the following integrals (any three) by using contour integration

(i) $\int_0^\infty \frac{dx}{x^4 + x^2 + 1}$ (ii) $\int_0^\infty \frac{(\ln x)^2}{x^4 + 1}\,dx$ (iii) $\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx,\ m > 0$

(iv) $\int_0^\infty \frac{x^{P-1}}{1+x}\,dx,\ 0 < P < 1$ (v) $\int_0^\infty \frac{\sin x}{x}\,dx$.

8. (a) Prove that under the transformation $w = f(z)$ where f(z) is analytic at z_0 and $f'(z_0) \neq 0$ the tangent at z_0 to any curve c is the z plane passing through z_0 is rotated through the angle arg $f'(z_0)$.

(b) Show that the bilinear transformation $W = e^{iz}\frac{az + b}{\bar{a} + bz}$ maps inside of the circle $|z| = 1$ on the inside of the circle $|w| = 1$.

(c) Show that the transformation $w = \frac{2z+3}{z-4}$ transforms the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$.

Jahangirnagor University (Course–303, Complex–I)–2006

1. (a) Obtain a set of necessary and sufficient conditions of a complex function f(z) to be analytic in a region R. Interpret them geometrically.

(b) Define harmonic function. Show that $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic and find the corresponding analytic function.

2. Let f(z) be analytic and bounded for all z in the entire complex plane. Show that f(z) must be a constant.

Establish, giving all details, the fundamental theorem of algebra.

Show, stating necessary conditions, that

$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$. Also evaluate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{z^3 - 7z + 12}\,dz$, where $C : z = 6e^{i\theta},\ 0 \le \theta \le 2\pi$.

3. Establish Taylor's series for a complex function. Classify singularities. Give an example of each type refering, wherever possible, to the principal part of Laurent's expansion.

4. Prove, under conditions to be stated, the two theorems given by

$$\oint_c f(z) = dz = \begin{cases} 2\pi i\ 2RC \\ 0 \end{cases}$$

where the symbols have their usual meanings.

5. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for

(i) $|z| > 3$, (ii) $1 < |z| < 3$, (iii) $|z| < 1$, (iv) $0 < |z+1) < 2$.

6. Let f(z) be analytic inside and on a simple closed curve C except at a pole z_0 of order m inside C. Prove, in two different ways, that the residue of f(z) at z_0 is given by

$$b_1 = \lim_{z \to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z).$$

Evaluate $\frac{1}{2\pi i}\oint_c \frac{e^{zt}\,dz}{z^2(z^2 + 2z + 2)}$ around the circle C with equation $z = 3$.

7. (a) Prove that $w = \frac{iz+2}{4z+1}$ transforms the real axis in the z-plane into a circle in the w-plane. Find the centre and radius of the circle and the point in the z-plane which is mapped on the centre of the circle.

(b) Define a conformal mapping. Prove that at each point z of a domain D where f(z) is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

8. Evaluate any two of the following by contour integration :

(i) $\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx,\ 0 < \alpha < 1$

(ii) $\int_0^\infty \frac{\sinh(ax)}{\sinh x}\cos bx\,dx,\ |a| < 1,\ b > 0$

(iii) $\int_0^{2\pi} \frac{\cos^2 3x\,dx}{5 - 4\cos 2x}$.

DUH : DHAKA UNIVERSITY HONOURS
DUH (Course-302)–2002

1. (a) State Laurent's theorem. Find Laurent series for $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$ and $g(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ about $z = 2$. [লঁরার উপপাদ্য বর্ণনা কর। $f(z) = \frac{e^{2z}}{(z-1)^3}$ কে $z = 1$ এবং $g(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ কে $z = 2$ এর প্রেক্ষিতে লঁরা ধারায় বিস্তৃত কর।]

(b) Define with example : singularity, isolated singularity, essential singularity, pole and residue at a pole of a complex function $f(z)$. [জটিল ফাংশন $f(z)$ এর ব্যতিক্রম বিন্দু, নিঃসঙ্গ ব্যতিক্রম বিন্দু, গুরু ব্যতিক্রম বিন্দু, পোল এবং পোলে রেসিডিউ-এর উদাহরণসহ সংজ্ঞা দাও।]

The only singularities of a single valued function $f(z)$ are poles of order 2 and 1 at $z = 1$ and $z = 2$, with residues of these poles 1 and 3 respectively. If $f(0) = 3/2$, $f(-1) = 1$, determine the function. [একটি এককমান বিশিষ্ট ফাংশন $f(z)$ এর ব্যতিক্রম বিন্দু কেবল $z = 1$ এবং $z = 2$ বিন্দুতে যথাক্রমে 2 এবং 1 ক্রম বিশিষ্ট পোল যাহাদের রেসিডিউ যথাক্রমে 1 এবং 3, যদি $f(0) = 3/2$, $f(-1) = 1$ হয়, তবে ফাংশনটি নির্ণয় কর।]

2. (a) State and prove the maximum modulus principle. [ম্যাক্সিমাম মড্যুলাস নীতি' বর্ণনা কর ও প্রমাণ দাও।]

(b) State and prove the argument principle. ['আরগুমেন্ট নীতি' বর্ণনা কর ও প্রমাণ দাও।]

(c) State Rouche's theorem. Determine the number of roots, counting multiplicities, of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \le |z| \le 2$. ['রুসের উপপাদ্য' বর্ণনা কর। অতঃপর $1 \le |z| < 2$ এ্যানুলাসে $2z^5 - 6z^2 + z + 1 = 0$ সমীকরণের মূলের সংখ্যা, যথাযথ গুণিতাঙ্ক সমেত গণ্য, নির্ণয় কর।]

3. State Cauchy's residue theorem. Evaluate the following by contour integration (any two) : [কসি-র রেসিডিউ নীতি' বর্ণনা কর। কনটুর ইন্টিগ্রেশন পদ্ধতিতে নির্ণয় কর (যে কোন দুইটি)।]

(i) $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{4 + 3\cos\theta}$ (ii) $\int_0^{\infty} \frac{\sin mx}{x} dx \; (m > 0)$.

(iii) $\int_0^{\infty} \frac{x^{-a}}{1+x} dx \; (0 < a < 1)$.

4. (a) Distinguish between isogonal and conformal mapping. Discuss the effect of the transformation $f(z) = 1/z$ on a straight line and a circle of the z-plane. [আইসোগোনাল এবং কনফর্মাল ম্যাপিং এর মধ্যে পার্থক্য উল্লেখ কর। z তলে একটি সরলরেখা ও একটি বৃত্তের উপর $f(z) = 1/z$ রূপান্তরের ক্রিয়া আলোচনা কর।]

(b) Discuss the effect of the transformation $f(z) = z^2$ on the first quadrant and on the hyperbola $xy = c$ of the z-plane. [z তলার প্রথম চতুর্ভাগ এবং z তলে $xy =$ হাইপারবোলার উপর $f(z) = z^2$ রূপান্তরের ক্রিয়া আলোচনা কর।

(c) Given $w = \left(\frac{z+c}{z-c}\right)^2$, where c is a positive constant, find the region of the z-plane which is conformally mapped onto the upper half of the w-plane. [প্রদত্ত $w = \left(\frac{z+c}{z-c}\right)^2$, যেখানে c একটি ধনাত্মক ধ্রুবক, এর জন্য z তলের ক্ষেত্রটি নির্ণয় কর যাহা w তলের উপর অর্ধতলে কনফর্মালি চিত্রিত হয়।]

5. (a) Find all Möbius transformations which transform the unit disc $|z| \le 1$ onto the unit disc $|w| \le 1$. [একক চাকতি $|z| \le 1$ কে একক চাকতি $|w| \le 1$ এ রূপান্তরিত করে এমন সকল মোবিয়াস রূপান্তর নির্ণয় কর।]

(b) Define cross-ratio. Show that it is invariant under any Möbius transformation. ['ক্রস রেশিও'-এর সংজ্ঞা দাও। দেখাও যে, মোবিয়াস রূপান্তরের ফলে 'ক্রস রেশিও' অপরিবর্তিত থাকে।]

(c) Find Möbius transformation which maps the points $1, i, -1$ of the z-plane onto the points $0, 1, \infty$ of the w-plane, respectively. [z তলের $1, i, -1$ বিন্দুগুলোকে w তলে যথাক্রমে $0, 1, \infty$ বিন্দুত্রয়ে রূপান্তরিত করে এমন মোবিয়াস রূপান্তর নির্ণয় কর।]

6. (a) When is an infinite product of complex numbers said to be absolutely convergent? Show that $\prod_{z=1}^{\infty}\left(1 - \frac{z}{c+n}\right)e^{z/n}$ converges absolutely for all z, provided c is not negative. [জটিল সংখ্যায় অসীম গুণনকে কখন পরমভাবে অভিসারী বলা হয়? দেখাও যে, c ঋণাত্মক না হলে সকল z এর জন্য $\prod_{z-1}^{\infty}\left(1 - \frac{z}{c+n}\right)c^{z/n}$ পরমভাবে অভিসারী হবে।]

(b) If $f(z)$is an entire function which never vanishes, then show that there exists an entire function $g(z)$ such that $f(z) = e^{g(z)}$, for all z. [$f(z)$ এমন একটি এনটায়ার ফাংশন, যার মান কখনও শূন্য নয়, হলে দেখাও যে অন্য একটি এনটায়ার ফাংশন $g(z)$ বিদ্যমান যেন সকল z এর জন্য $f(z) = e^{g(z)}$ হয়।]

(c) Given a sequence of non-zero complex numbers (z_n) tending to infinity, show that it is possible to construct and entire function which vanishes at each of the points z_n and nowhere else. [দেওয়া আছে (z_n) অসীমে অভিসৃত শূন্য নয় এমন জটিল সংখ্যার পর্যায়ক্রম। প্রমাণ কর যে, এমন একটি যোগজ-ফাংশন গঠন করা হইলে যাহা z_n বিন্দুগুলোর প্রত্যেকটিতে বিলীন হয় এবং অন্য কোথাও নয়।]

DUH (Course-302)–2003

1. (a) For complex numbers z_1, z_2, z_3, z_4 prove that

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \le \frac{|z_1|+|z_2|}{|z_3|-|z_4|}, \text{ where } |z_3| \ne |z_4|.$$

(b) Define differentiability of a complex functions at a point z_0. Show that $f(z) = |z|^2$ is differentiable at the origin, but nowhere else.

(c) Let $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & ; \text{ where } z \ne 0 \\ 0 & ; \text{ where } z = 0 \end{cases}$

Show that f(z) Statistics Cauchy-Riemann equations at the origin, but $f'(0)$ does not exist.

2. (a) Represent graphically the set of values of z for which

(i) $\left|\frac{z-5}{z+5}\right| = 4$ (ii) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

(b) Using polar form of complex numbers show that

$$i(1 - i\sqrt{3})(\sqrt{3} + i) = 2 + 2i\sqrt{3}$$

(c) Define a neighbourhood of a point z_0 in the complex plane.

Are $A = \{z \in \mathbb{C} \mid |z - a < r, r > 0\}$ and

$B = \{z \in \mathbb{C} \mid |z - b| \le r, r > 0\}$

neighbourhood? If so what are the centres and radii of those? Display A and B in the complex plane.

3. (a) When is a complex function said to be analytic at a point. State and prove the necessary and sufficient conditions for a complex function to be analytic.

(b) Define a harmonic function. If $f(z) = u + iv$ is analytic, then show that both of u and v are harmonic. Prove that the function $u(x, y) = 2x - x^3 + 3xy^2$.

is harmonic. Find its harmonic conjugate and also the corresponding analytic function.

4. (a) What is inverse point? Prove that a bilinear transformation transforms a circle into a circle and points into inverse point.

(b) Consider the transformation $w = \frac{t-i}{t+i}$. Show that this transformation maps upper half of t-plane on the interior of the circle in w-plane.

5. (a) Derive in usual notations the formula

$$f^m(z_0) = \frac{\lfloor n}{z\pi i}\oint_C \frac{(fz)\,dz}{(z-z_0)^{m+1}}, m = 1, 2, 3, \cdots$$

under certain conditions to be static.

(b) Prove that the equation $z + \lambda - e^z = 0, \lambda > 1$ has only one real root in the left half plane $\text{Re}z < 0$.

(c) Identity the singularitics of the following functions.

(i) $\dfrac{1}{\cos\frac{1}{z}}$ (ii) $\dfrac{\cot \pi z}{(z-a)^2}$, at $z = 0$

6. (a) Prove that a bounded entire function is constant Liouville's.

(b) State and prove the fundamental theorem of algebra.

(c) State Laurent's theorem. Expand $f(z) = \frac{z-1}{z^2}$ in laurent's series in the domain $|z - 1| > 1$.

7. (a) State and prove Cauchy's residue theorem.

(b) State and prove Rouche's theorem. Determine the number of roots, counting multiplicities, of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \le |z| < 2$.

8. Evaluate any two of the following by contour integration :

(i) $\displaystyle\int_0^\pi \frac{2 + \cos x}{3 + 2\cos x}\,dx$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx$

(iii) $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\,dx, 0 < \alpha < 1$ (iv) $\displaystyle\int_0^\pi \frac{a\,d\theta}{a^2 + \sin^2\theta}\,(a > 0)$

9. (a) Find all Möbius transformations which transform the half-plane $\text{Im}(z) \ge 0$ into the unit disc $|w| \le 1$. Verify your result.

(b) Show that the line $3y = x$ is mapped into a circle under the transformation $w = \frac{iz + 2}{4z + i}$. Find also the center and radius of the image circle.

10. (a) Define infinite product of complex numbers. When does an infinite product converge? Test the convergence of the following infinite products :

(i) $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots$

(ii) $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots$

(b) When is an infinite product of complex numbers said to be absolutely convergent? Show that $\prod_{n=1}^{\infty} \left(1 - \frac{z}{c + n}\right) e^{z/n}$ converges absolutely for all z, provided C is not negative.

(c) If f(z) is an entire function which never vanishes, then show that there exists an entire function g(z) such that $f(z) = e^{g(z)}$ for all z.

DUH (Course-302)–2004

1. (a) Define differentiability of a complex function. If f(z) is differentiable at z_0, show that it must be continuous at z_0. Give an example to show that the converse is not true.

(b) Define analytic function. Show that the function $f(z) = |z|^2$ is differentiable but not analytic at $z = 0$.

(c) For any analytic function f(z), prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 - 4|f'(z)|^2$$

2. (a) If $w = f(z) = u + iv$ an analytic function, show that in polar form the Cauchy-Riemann equations are.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = \frac{1 \partial u}{r\, \partial \theta}.$$

(b) Define harmonic function. Prove that $\psi = ln[(x - 1)^2 + (y - 2)^2]$ is harmonic in every region which does not include the point (1, 2). Find f(z) and $\phi(x, y)$ so that $f(z) = \phi + i\psi$ is analytic.

3. (a) Evaluate $\int_C f(z)\, dz$, where $f(z) = y - 31x^2$ and C (i) is the line segment from $z = 0$ to $z = 1 + 1$ (ii) consists of two line segments one from $z = 0$ to $z = i$ and the other from $z = i$ to $z = 1 + i$.

(b) State and prove the Cauchy's Integral formula. Hence evaluate $\frac{1}{2\pi i}\int_C \frac{\cos \pi z}{z^2 - 1} dz$, where C is the circle (i) $|z - 1| = 1$. (ii) $|z| = 2$, (iii) $|z - i| = 1$.

4. (a) If f(z) is analytic inside and on the boundary C of a simply-connected region R, prove that

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - a)^2} dz$$

(b) Evaluate $\frac{1}{2\pi i}\int_C \frac{c^n}{(z^2 + 1)^2} dz$, if $t = 0$ and C is the circle $|z| = 3$.

(c) State and prove Liouville's theorem.

5. (a) If $f(z_0)$ is analytic inside and on a simple closed curve C, then show that the maximum value of $|f(z)|$ occurs on C, unless f(z) is a constant. Find the points in $R : |z| = 1$, where $|f(z)|$ has its maximum value if $f(z) = z^2 + 3z + 2$.

(b) State and prove the fundamental theorem of algebra. Using it show that a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ of degree n has exactly n zeroes.

6. (a) State Laurents theorem. Expand $f(z) = \frac{1}{(z + 1)(z + 3)}$ in a Laurent series valid for (i) $1 < |z| < 3$, (ii) $|z| > 3$.

(b) Define residue of a function at a point. Let f(z) be analytic inside and on a simple closed curve C except at a pole 'a' of order m inside C. Prove that the residue of f(z) at a is given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(m - 1)(m - 2)(m - 3) \cdots 1} \frac{d^{m-1}}{dz^{m-1}} \{(z - a)^m f(z)\}.$$

7. Evaluate any two of the following by contour integration

(i) $\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta}, 0 < a < 1$ (ii) $\int_0^{\infty} \frac{x^2\, dx}{(1 + x^2)^2}$

(iii) $\int_0^{\infty} \frac{\sin x}{x} dx$

8. (a) Define bilinear transformation. Find all the bilinear transformation which transforms the half plane $Re(z) \geq 0$ into unit circle $|w| \leq 1$.

(b) Find a bilinear transformation which maps the vertices $1 + i, -i, 2 - i$ of a triangle of the z-plane into the points 0, 1, i, of the w-plane. Sketch the region into which the interior of triangle T is mapped under the transformation obtained above.

9. (a) Define a meromorphic function. Show that a single-valued function f(z) is meromorphic if and only if f(z) can be written as a quotient $\frac{g(z)}{h(z)}$ of two entire functions.

(b) Determine the radius of convergence of each of the following power series.

(i) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2} z^n$

(ii) $z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots+(-1)^{n+1}\frac{z^n}{n}+\cdots$

10. (a) Define conformal mapping. Show that under the transformation $w=\frac{1}{z}$, the images of the lines $y = x - 1$ and $y = 0$ arc $u^2+v^2-u-v=0$ and $v = 0$ respectively. Sketch the four curves. Determine corresponding direction along them, verify conformality of the mapping at $z = 1$.

(b) When an infinite product of complex numbers is said to be convergent? Test the convergence of

(i) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{3/2}}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+(-1)^{n+1}\frac{1}{n}\right)$.

DUH (Course-302)–2005

1. (a) Prove that

(i) $|z_1+z_2| \le |z_1|+|z_2|$ (ii) $||z_1|-|z_2|| \le |z_1-z_2|$

for any two complex numbers z_1 and z_2.

(b) Given $z_1=\frac{1+i}{1-i}$, $z_2=\frac{\sqrt{2}}{1-i}$

find the modulus and argument of z_1, z_2 and z_1+z_2 and plot them on an Argand diagram.

(c) Sketch the locus of the set of points determined by

(i) $|2z+1-i|=2$ (ii) $|z|>4, -\pi<\arg z<\pi$.

2. (a) Define differentiability of a complex function. Ascertain whether the function f(z) defined by

$$f(z)=\begin{cases}\frac{(\bar{z})^2}{z}, & z\neq 0\\ 0, & z=0\end{cases}$$ is differentiable at $z = 0$.

(b) Define analytic function. State and prove necessary conditions for a complex function to be analytic in a region. Give an example to show that these conditions are not sufficient.

3. (a) Define harmonic function and harmonic conjugate function. Show that the function $u(x, y) = e^x(x\cos y - y\sin y)$ is harmonic. Find its conjugate function $v(x, y)$ and the corresponding analytic function $f(z) = u + iv$.

(b) Let f(z) be a piecewise continuous function defined on a contor C. Then show that $\left|\int_C f(z)\,dz\right| \le ML$, where L is the length of the contor C and $|f(z)| < M$ whenever z is on C. Use it to show that if C is the arc of the circle $|z| = 3$ from $z = 3$ to $z = 3i$ that lie in the first quadrant, then

$$\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{\pi}{8}.$$

4. (a) Evaluate $\int_C [(5x+6y-3)\,dx+(3x-4y+2)\,dy]$, where C is the triangle with vertices (0, 0), (4, 0) and (4, – 3).

(b) If f(z) is analytic within and on a closed contour C and z_0 is any point within C, then prove that $f'(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$, where C is taken in the positive direction. Hence $\int_C \frac{e^2}{(z^2+1)^2}\,dz$, where C is the circle $|z-i|=1$.

5. (a) Establish Taylor series for a complex function.

(b) Classify singularities. Give an example of each kind referring, wherever possible, to the principal part of the Laurent expansion.

6. (a) Define bilinear transformation. Find all bilinear transformations that map the half plane $\text{Im} z \ge 0$ onto the disk $|\omega| \le 1$.

(b) Find the image of the semiinfinite strip $x > 0$, $0 < y < 1$ under the transformation $\omega=\frac{i}{z}$. Sketch them.

7. (a) Calculate the residue of

(i) $f(z)=\frac{z^2-2z}{(z+1)^2(z^2+16)}$

and (ii) $f(z)=\text{cosec}^2 z\cdot\exp(z)$ at their poles.

(b) Expand $f(z) = \frac{1}{(z-1)(2-z)}$ in a Laurent series valid for (i) $|z| > 2$, (ii) $0 < |z-2| < 1$.

8. Evaluate any two of the following :

(i) $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}, a > |b|$

(ii) $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx$

(iii) $\int_0^{\infty} \frac{\ln(x^2+1)}{(x^2+1)} dx$

9. (a) Define conformal mapping and bilinear transformation. Show that the transformation $\omega = \frac{2z+3}{z-4}$ changes the circle $x^2 + y^2 - 4x = 0$ into the straight line $4x + 3 = 0$.

(b) Show that the transformations $\omega = \frac{z-i}{z+i}$ and $\omega = \frac{i-z}{i+z}$ transform the upper half plane $Im(z) \geq 0$ of the z-plane to the interior of the circle $|\omega| \leq 1$ of the ω-plane.

10. (a) Define a meromorphic function. Show that a single-valued function f(z) is meromorphic if and only if f(z) can be expressed as a quotient $\frac{g(z)}{h(z)}$ of two entire functions.

(b) Determine the radius of convergence of each of the following power series :

(i) $\sum_{n=1}^{m} \left(1 + \frac{1}{n}\right)^{n^3} z^n$

(ii) $z + \frac{z^2}{2} + \frac{z^3}{4} + \cdots + (-1)^{n+1}\frac{z^n}{n} + \cdots$

DUH (Course-302)–2006

1. (a) Prove that [প্রমাণ কর যে] ঃ

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{\left||z_3| - |z_4|\right|},$$

where z_1, z_2, z_3, z_4 are complex numbers and $|z_3| \neq |z_4|$. [যেখানে z_1, z_2, z_3, z_4 জটিল সংখ্যা এবং $|z_3| \neq |z_4|$]

(b) Sketch the locus of the set of points determined by [নির্ধারিত বিন্দুসেটের দ্বারা সঞ্চারপথ অংকন কর।]

(i) $\arg\left(\frac{z-1}{z+1}\right) = -\frac{\pi}{4}$ (ii) $Im\left(\frac{i}{z}\right) < \frac{1}{2}$

(c) Determine all the values of $(-1 - i\sqrt{3})^{1/4}$ and represent these graphically. [$(-1 - i\sqrt{3})^{1/4}$ এর সকল মান নির্ণয় কর এবং জটিল সমতলে উহাদের অবস্থান লেখচিত্রে দেখাও।]

2. (a) Obtain sufficient conditions for a complex function f(z) to be analytic in a region D of the complex plane. Interpret them geometrically. [জটিল সমতলের D এলাকায় একটি জটিল ফাংশন f(z) বৈশ্লেষিক হওয়ার পর্যাপ্তি শর্তসমূহ বের কর। এদের জ্যামিতিক ব্যাখ্যা দাও।]

(b) Prove that the function $f(z) = u(x, y) + iv(x, y)$ [প্রমাণ কর যে $f(z) = u(x, y) + iv(x, y)$ ফাংশন]

where [যেখানে] $f(z) = \begin{cases} \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

is continous and that the C–R equations are satisfied at the origin, yet $f'(z)$ does not exist there. [মূল-বিন্দুতে অবিচ্ছিন্ন এবং C–R এর সমীকরণসমূহ সিদ্ধ করে, তথাপি সেখানে $f'(z)$ বিদ্যমান নয়।]

3. (a) Prove that the real and imaginary parts of an analytic function of a complex variable, when expressed in polar form, satisfy the equation [প্রমাণ কর যে যখন জটিল চলকের বৈশ্লেষিক ফাংশনের বাস্তব ও কাল্পনিক অংশদ্বয়কে পোলার আকারে প্রকাশ করা হয়, তখন এরা সমীকরণকে সিদ্ধ করে।]

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2\psi}{\partial\theta^2} = 0$$

(b) Let $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$. Show that u(x, y) is harmonic. Find f(z) and v(x, y) so that $f(z) = u + iv$ is analytic. [ধরি $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ এবং দেখাও যে u(x, y) হারমনিক ফাংশন। f(z) এবং v(x, y) নির্ণয় কর যেন $f(z) = u + iv$ বৈশ্লেষিক হয়।]

(c) Evaluate $\oint_\Gamma \frac{dz}{(z - z_0)^m}$, where m is any integer positive, negative or zero and Γ denotes the circle $|z - z_0| = r$ described in the positive sense. [$\oint_\Gamma \frac{dz}{(z - z_0)^m}$ এর মান নির্ণয় কর, যেখানে m যোগাত্মক, ঘৃনাত্মক অথবা শূন্য যেকোন পূর্ণসংখ্যা এবং Γ যোগবোধকভাবে প্রকাশিত বৃত্ত $|z - z_0| = r$ প্রকাশ করে।]

4. (a) Stating necessary conditions prove that [প্রয়োজনীয় শর্তসমূহ উল্লেখপূর্বক প্রমাণ কর যে]

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

(b) Show that [দেখাও যে] $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i} \oint_C \frac{a^n e^{az}}{n!\, z^{n+1}} dz$,

where C is the closed contour around the origin. [যেখানে C মূলবিন্দুর চতুর্দিকে আবদ্ধ কন্টুর।]

(c) Let γ be a closed curve and $a \notin \gamma$, then define winding number $\eta(\gamma, a)$ of γ about a. For any closed curve γ and $a \notin \gamma$, prove that $\eta(\gamma, a)$ is an integer. [ধরি γ একটি আবদ্ধ বক্ররেখা এবং $a \notin \gamma$, এখন a এর প্রেক্ষিতে γ এর উইন্ডিং সংখ্যা $\eta(\gamma, a)$ এর সংজ্ঞা দাও। প্রমাণ কর যে, যেকোন আবদ্ধ বক্ররেখা γ এবং $a \notin \gamma$ এর জন্য $\eta(\gamma, a)$ একটি পূর্ণসংখ্যা।]

5. (a) State and prove Cauchy's inequality. [কসি-অসমতা বর্ণনা কর ও প্রমাণ দাও।]

(b) Expand $f(z) = \frac{z+3}{z(z^2 - z - 2)}$, in powers of z, when [z এর শক্তিতে সম্প্রসারণ কর] (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

6. (a) If f(z) is analytic within and on a closed contour C except at a finite numbers of poles and is not zero on C, then prove that [যদি বদ্ধ কন্টুর C এর নির্দিষ্ট সংখ্যক মেরু বিন্দু ব্যতীত C তে বৈশ্লেষিক হয় এবং C তে শূন্য না হয়, তাহলে প্রমাণ কর যে]

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

where N is the number of zeros and P is the number of poles inside C. [যেখানে N হচ্ছে শূন্যের সংখ্যা এবং P হচ্ছে মেরুর সংখ্যা।]

(b) Show that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. [দেখাও যে $z^7 - 5z^3 + 12 = 0$ এর মূলগুলি বৃত্ত $|z| = 1$ এবং $|z| = 2$ এর মাঝে অবস্থান করে।]

7. (a) Prove that a power series can be differentiated term by term as often as we please at any point within the circle of convergence. [প্রমাণ কর যে অভিসৃতির বৃত্তের অন্তর্গত যেকোন বিন্দুতে ইচ্ছামাফিক শক্তি ধারা পদসমূহকে অন্তরীকরণ করা যায়।]

(b) Investigate the behaviour of $\sum \frac{z^{4n}}{4n+1}$ on the circle of convergence. [অভিসৃতির বৃত্তের উপর $\sum \frac{z^{4n}}{4n+1}$ এর চারিত্রিক বৈশিষ্ট পর্যালোচনা কর।]

8. (a) What is bilinear transformation? Prove that bilinear transformation transfer two points which are inverse w. r. t. a circle into two points which are inverse w. r. t. transformed circle. [দ্বিযোগাশ্রয়ী রূপান্তর কি? প্রমাণ কর যে দ্বিযোগাশ্রয়ী রূপান্তর একটি বৃত্তের দুইটি বিপরীত বিন্দুকে রূপান্তরীত বৃত্তের দুইটি বিপরীত বিন্দুতে রূপান্তর করে।]

(b) Show that the bilinear transformation which carries the points $z = i, 0, -i$ respectively into $w = 0, -1$, and ∞, 0 maps [দেখাও যে, যে দ্বিযোগাশ্রয়ী রূপান্তর $z = i, 0, -i$ বিন্দুগুলিকে যথাক্রমে $w = 0, -1$ এবং ∞ তে নেয় তা]

(i) The real axis [বাস্তব অক্ষ] $Im(z) = 0$ on $|w| = 1$.

(ii) The upper half plane [উর্ধ্ব সমতল] $Im(z) > 0$ on $|w| < 1$

(iii) The lower half plane [অধঃ সমতল] $Im(z) < 0$ on $|w| > 1$ [এ রূপান্তর করে।]

9. (a) Using the theorem on expansion of meromorphic function prove that [আনুপাতিক ফাংশন প্রসারণের উপপাদ্য ব্যবহার করে প্রমাণ কর যে]

$$\cot z = \frac{1}{z} + 2z \sum_{1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

(b) Prove that the product [প্রমাণ কর যে গুন]

$$\pi \left\{ \left(1 - \frac{z}{c+n}\right) e^{z/n} \right\}$$

converges absolutely for any values of z, provided that C is not a negative integer. [যেখানে z মানের জন্য সম্পূর্ণরূপে অভিসারীত, যদি C ঘৃনাত্মক পূর্ণসংখ্যা না হয়।]

10. Evaluate any two of the following [নিম্নের যেকোন দুটির মান নির্ণয় কর] ঃ

(i) $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$

(ii) $\int_0^{\infty} \frac{x^{-a}}{1+x} dx,\ 0 < a < 1$

(iii) $\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} dx,\ m \geq 0,\ a > 0.$

NATIONAL UNIVERSITY QUESTIONS AND SOLUTION INDEX

NU(Phy) : NATIONAL UNIVERSITY PHYSICS

NU(Phy)–2003

১। (ক) কখন একটি ফাংশনকে এ্যানালাইটিক বলা যাবে? [When is a function analytic?] **[Art–2.5, Def^n]**

(খ) প্রমাণ কর যে, $u = e^{-x}(x \sin y - y \cos y)$ একটি হারমোনিক ফাংশন। [Prove that $u = e^{-x}(x \sin y - y \cos y)$ is a harmonic function] **[Ch–2, Expl–50]**

(গ) যদি $f(z) = u + iu$ একটি এ্যানালাইটিক ফাংশন হয় তবে সেক্ষেত্রে u এর মান নির্ণয় কর। [Find u such that $f(z) = u + iu$ is analytic]

২। (ক) কচী রেসিডিউ তত্ত্বটি লিখ। [State the Cauchy's Residue Theorem] **[Ch–4, Th^m–6]**

(খ) প্রমাণ কর যে [Prove that] $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$ **[Ch–5, Prob–1]**

(গ) $\frac{z^2}{z^2 + a^2}$ ফাংশনটির রেসিডিউ নির্ণয় কর (z এর শুধু ধনাত্মক মান গ্রহণ কর)। [Calculate the Residue of the functin $\frac{z^2}{z^2 + a^2}$ (take only the positive value of z)] **[Ch–4, Prob–10(a)]**

NU(Phy)–2004

১। (ক) কমপ্লেক্স নাম্বার কি? **[Ch–1, Page–2, in line 14 & 15]**

(খ) প্রমাণ কর যে, একটি রিজিয়ান R এ $f(z) = u(x, y) + iv(x, y)$ এনালাইটিক হওয়ার একটি প্রয়োজনীয় শর্ত হল Cauchy-Riemann সমীকরণ সিদ্ধ হওয়া। **[Ch–2, Th^m–4]**

(গ) যদি $f(z) = u + iv$ এনালাইটিক হয় এবং এর ডেরিভেটিভ $f'(z)$ একটি বদ্ধ বক্ররেখার উপরে ও অভ্যন্তরে সকল বিন্দুতে অবিচ্ছিন্ন হয় তাহলে কচির উপপাদ্য $\oint_C f(z)\, dz = 0$ প্রমাণ কর। **[Ch–3, Th^m–5]**

২। (ক) ফাংশনের রেসিডিউ কি? **[Ch–4, Art–4.5, Def^n]**

(খ) কচির রেসিডিউ উপপাদ্যটি প্রমাণ কর। **[Ch–4, Th^m–6]**

(গ) রেসিডিউ কৌশল প্রয়োগ করে দেখাও যে, $\oint_C \frac{e^z}{(z^2 + \pi^2)^2} dz = \frac{1}{\pi}$, যেখানে c হল একটি বৃত্ত $|z| = 4$. **[Ch–4, Prob–14]**

NU(Phy)–2005

১। (ক) কখন একটি ফাংশনকে এনালাইটিক বলা যাবে? **[Art–2.5, Def^n]**

(খ) ওয়লারের $a\frac{\partial^2 u}{\partial x^2} + 2h\frac{\partial^2 u}{\partial x\, \partial y} + b\frac{\partial^2 u}{\partial y^2} = 0$ সমীকরণটি সমাধান কর, যেখানে $ab = h^2 = 0$; এখানে a, b ও h ধ্রুবক।

(গ) প্রমাণ কর যে, অপেক্ষক $f(z) = z^2 + 5iz + 3 - i$ কচি-রিম্যান সমীকরণ সিদ্ধ করে। **[Ch–2, Expl–35(a)]**

২। (ক) এক-মানি অপেক্ষক single valued function বলতে কি বুঝ? **[Ch–2, Art–2.1, Def^n]**

(খ) রেসিডিউ কৌশল ব্যবহার করে সমাধান কর ঃ

(i) $\int_0^\infty \frac{x \sin mx}{x^2 + a^2} dx,\ a > 0$ **[Ch–5, Sol^n–47(ii)]**

(ii) $\int_0^\infty \frac{dz}{z^2 + 1}$ **[Ch–5, Sol^n–34(b)]**

NU(Phy)–2006

১। (ক) কমপ্লেক্স নাম্বার কি? [What is complex number?] **[Ch–1, Page–2, in line 14, 15]**

(খ) কচি-রিম্যান সমীকরণ প্রতিপাদন কর এবং দেখাও যে, সমীকরণগুলো ল্যাপ্লাসের সমীকরণ মেনে চলে। [Deduce Cauchy-Rieman equations and show that the equations obey the Laplace's equation.]

(গ) দেখাও যে, অপেক্ষক $z = |z|$ বিশ্লেষিক নয়। [Show that, the function $z = |z|$ is not analytic anywhere.]

২। (ক) রেসিডিউ বলতে কি বুঝ? [What is meant by residue?]

(খ) কচি'র রেসিডিউ তত্ত্ব বিবৃত ও প্রমাণ কর। [State and prove the Cauchy's residue theorem.]

(গ) রেসিডিউ কৌশল প্রয়োগ করে দেখাও যে [By using residue technique show that]

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin \alpha\pi}; 0 < a < 1.$$ **[Ch-5, Prob-56(i), Put a = p]**

NU(Phy)–2007

১। (ক) বিশ্লেষিক অপেক্ষকের সংজ্ঞা দাও।

(খ) একটি অপেক্ষক বিশ্লেষিক হওয়ার প্রয়োজনীয় ও পর্যাপ্ত শর্তাবলী নির্ণয় কর।

২। (ক) হারমোনিক অপেক্ষক কি? প্রমাণ কর যে, $u = e^{-x}(x \sin y - y \cos y)$ একটি হারমোনিক অপেক্ষক।

(খ) একটি জটিল চলরাশির অপেক্ষকের জন্য কচির সমাকলন উপপাদ্য বিবৃত ও প্রমাণ কর।

NU(Phy)–2008

১। (ক) জটিল সংখ্যা ও ফাংশনের সিংগুলারিটির সংজ্ঞা দাও। [Define complex number and singularity of a function.]

(খ) প্রমাণ কর যে, অপেক্ষ $f(z) = \cos 2z$ কচি-রিম্যান সমীকরণ সিদ্ধ করে। [Prove that, the function $f(z) = \cos 2z$ satisfies Cauchy-Riemann equation.]

(গ) যদি $f(z) = u + iv$ একটি এ্যানালাইটিক ফাংশন হয় তবে সেক্ষেত্রে v এর মান নির্ণয় কর। [If $f(z) = u + iv$ is an analytic function, find the value of v.]

২। (ক) ফাংশনের রেসিডিউ কি? [What is Residue of a function?]

(খ) কচির অবশেষ তত্ত্বটি বিবৃত ও প্রমাণ কর। [State and prove Cauchy's Residue theorem.]

(গ) রেসিডিউ কৌশল অবলম্বন করে প্রমাণ কর যে [Proof, using the technique of Residue]

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

NU(Pre) : NATIONAL UNIVERSITY PRELIMINARY

NU(Pre)–2005

৬। (ক) যদি একটি সীমাবদ্ধ ফাংশন f এর [a, b] ব্যবধিতে সসীম সংখ্যক বিচ্ছিন্নতা বিন্দু থাকে তবে প্রমাণ কর যে, f রীমান যোগজীকরণ যোগ্য।

(খ) একটি ফাংশন f(x), $\left[0, \frac{\pi}{4}\right]$ ব্যবধিতে নিম্নরূপে সংজ্ঞায়িত ঃ

$$f(x) = \begin{cases} \cos x \text{ যখন } x \text{ মুলদ} \\ \sin x \text{ যখন } x \text{ অমূলদ} \end{cases}$$

দেখাও যে, f(x) ঐ ব্যবধীতে রীমান যোগজীকরণ যোগ্য নয়।

৭। (ক) z_0 বিন্দুতে জটিল ফাংশন f(z) অন্তরীকরণযোগ্য তার এক সেট প্রয়োজনীয় শর্ত নিরুপন কর। তোমার দেয়া শর্তগুলি কি অন্তরীকরণ যোগ্যতার জন্য পর্যাপ্ত? তোমার উত্তরের সপক্ষে যুক্তি দাও।

(খ) বৈশ্লেষিক ফাংশন বলতে কি বুঝ? দেখাও যে কোন ডোমেনে ধ্রুব মানাংক বিশিষ্ট একটি বিশ্লেষিক ফাংশন ধ্রুবক।

৮। (ক) কোন ফাংশনের বিভিন্ন প্রকারের ব্যতিক্রম বিন্দুর প্রকৃতি নির্দেশ কর। এবং উদাহরণ দাও, z_0 যদি f(z) এর একটি পোল হয় তবে দেখাও যে $|f(z)| \to \infty$ যখন $z \to z_0$.

(খ) ফাংশন f(z) যদি $|z| < \pi$ ডোমেনে বৈশ্লেষিক হয় তবে দেখাও যে f(z) ডোমেনে $\sum_0^{\infty} a_n z^n$ রূপে বিস্তৃত করা যায়,

যেখানে $a_n = \frac{1}{2\pi l''} \frac{f(z)}{2^n + 1} dz$ এবং কন্টুরটি এর অভ্যন্তরে অবস্থিত ও মুল বিন্দুকে কেবলমাত্র একবার বেষ্টন করে।

৯। (ক) নিম্নোক্ত কন্টুর যোগজীকরণের যে কোন দুটির মান নির্ণয় কর।

(i) $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}$ (ii) $\int_0^{\infty} \frac{dx}{x^4 + 1}$ (iii) $\int_0^{\infty} \frac{\ln(1 + x^2)}{1 + x^2} dx$

১০। (ক) (i) $|z| < 1$ (ii) $1 < |z| < 2$

(iii) $|z| > z$ এলাকায় $f(z) = \frac{1}{(z^2 + 1)(z + 2)}$ ফাংশনের লরেন্ট ধারা নির্ণয় কর।

(খ) দেখাও যে $w = \frac{1 - z}{1 + z}$ রূপান্তর $|z| < 1$ এলাকাকে $\text{Re}w > 0$ এলাকায় চিত্রিত করে $|z| = 1$ এবং $\arg z = \frac{\pi}{4}$ রেখা দুটি w সমতলে যে যে রেখায় মিলিত হয় তাহা নির্ধারণ কর।

NU(Pre)–2006

৭। (ক) বিশ্লেষণযোগ্য ফাংশন বলতে কি বুঝায়? $w_1 = f_1(z) = |z|^2$ এবং $w_2 = f_2(z) = \frac{1}{2}$ এর বিশ্লেষণযোগ্যতা পরীক্ষা কর। [What is meant by an analytic function? **[Ch–2, Art–2.5, Def^n]** Test for analyticity of

$w_1 = f_1(z) = |z|^2$ and $w_2 = f_2(z) = \frac{1}{2}$.] **[Ch–2, Expl–35(b)]**

(খ) কশি-রীমান অন্তরক সমীকরণ কি? বিশ্লেষণযোগ্য ফাংশনের সঙ্গে এদের কি সম্পর্ক?। [Write down the Cauchy. Riemann partial differential equations. How are they related to analytic function.]

[Ch–2, Expl–35(c)]

(গ) হারমোনিক ফাংশনের সংজ্ঞা দাও। $u = u(x, y) = e^{-x}(x \sin y - y \cos y)$ ফাংশনটি হারমোনিক কি না নির্ণয় কর। [Define a harmonic function. **[Ch–2, Art–2.7, Def^n]** Determine whether or not $u = u(x, y) = e^{-x}(x \sin y - y \cos y)$ is harmonic.] **[Ch–2, Expl–50]**

৮। (ক) ত্রিভুজের জন্য কশি-গৌড়সাট উপপাদ্যটি প্রমাণ কর। [Prove the Chauchy-Goursat theorem for the sides of a triangle.]

[Ch–3, Th^m–2]

(খ) মনে কর f(z) ফাংশনটি দুইটি সরল বদ্ধ বক্ররেখা c_1 এবং c_2 এর মাধ্যমে আবৃত অঞ্চল R এর ভিতরে এবং অধিকন্ত বক্ররেখাদ্বয় c_1 ও c_2 এর উপরে বিশ্লেষিত। তাহলে প্রমাণ কর যে, $\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz$ হবে, যেখানে c_1 এবং c_2 তাদের দিক সাপেক্ষে ধনাত্মক দিক বরাবর অতিক্রান্ত হয়। [Let f(z) be analytic in a region R, bounded by two simple closed evrves c_1 and c_2 and also on c_1 and c_2. Prove $\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz$ where c_1 and c_2 are both traversed in the positive sense relative to their interiors.] **[Ch–2, Th^m–6]**

৯। (ক) যদি সকল z এর জন্য সম্পূর্ণ জটিল তলে f(z) বৈশ্লেষিক এবং সীমায়িত হয়, তবে প্রমাণ কর যে, f(z) অবশ্যই একটি ধ্রুবক। [If for all z in the entire complex plane f(z) is analytic and bounded then prove that f(z) must be a constant.] **[Ch–2, Expl–35(D) Or, Ch–3, Th^m–11]**

(খ) (i) $|z| < 1$ এবং $|z| > 1$ এলাকায় $f(z) = \frac{3z - 3}{(2z - 1)(z - 2)}$ কে লরেন্ট ধারায় বিস্তৃত কর। [Expand $f(z) = \frac{3z - 3}{(2z - 1)(z - 2)}$ in a Laurent series for the region $|z| < 1$ and $|z| > 1$.] **[Ch–4, Prob–40]**

(ii) $f(z) = \frac{z^2}{(z + 1)^2} \sin\left(\frac{1}{z - 1}\right)$ ফাংশনটির ব্যতিচার বিন্দুসমূহ নির্ণয় কর এবং এদের প্রকৃতি নিরূপণ কর। [Find the singular points for the function $f(z) = \frac{z^2}{(z + 1)^2} \sin\left(\frac{1}{z - 1}\right)$ and determine their nature.]

[Ch–4, Prob–8]

১০। (ক) একটি জটিল ফাংশনের রেসিডিউ-এর সংজ্ঞা দাও। m অর্ডারের পোলে কোন ফাংশনের রেসিডিউ নির্ণয়ের সূত্রটি নির্ণয় কর। [Define the residue of a complex function. **[Ch–4, Art–4.5, Def^n]** Derive the formula to find the residue of a function at a pole of order m.] **[Ch–4, Th^m–5]**

(খ) কন্টুর যোগজীকরণের মাধ্যমে যে-কোন দুইটির মান নির্ণয় কর। [Evaluate Contour integration any two of the following] :

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 3 \sin \theta}$ **[Ch–5, Expl–4]**

(ii) $\int_0^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)^2}$ **[Ch–5, Expl–24]**

(iii) $\int_0^{\infty} \frac{\cos x}{a^2 - x^2}\,dx$ **[Ch–5, Expl–52]**

NU(Pre)–2007

৭। (ক) (i) যদি $z = x + iy$ হয়, তবে প্রমাণ কর যে, $|x| + |y| \le \sqrt{2}\,|x + iy|$.

(ii) নিম্নলিখিত এলাকাগুলোর চিত্র অংকন কর ঃ

$1 < |z - zi| < 2$ এবং $|z - 4| > |z|$.

(খ) $(1 + i)^{1/4}$ এর সকল মান বাহির কর।

(গ) দেখাও যে, $\left|\frac{z_1}{z_2 + z_1}\right| \le \frac{|z_1|}{||z_2| - |z_3||}$, $|z_2| \ne |z_3|$, যেখানে z_1, z_2, z_3 হচ্ছে জটিল সংখ্যা।

৮। (ক) কোন এলাকায় বিশ্লেষণযোগ্য ফাংশনের পর্যাপ্ত শর্ত বর্ণনা ও প্রমাণ কর।

(খ) দেখাও যে, $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ একটি হারমনিক ফাংশন। অনুবন্ধী হারমনিক u(x, y) বাহির কর।

৯। (ক) লোরেন্টের উপপাদ্য বর্ণনা কর ও উহা প্রতিষ্ঠা কর।

(খ) $f(x) = \frac{x^2 + 1}{(z + 1)(z - 2)}$ ফাংশনটিকে (i) $1 < |z| < 2$; (ii) $0 < |z| < 1$ এলাকায় লরেন্ট'স বিস্তৃতি লিখ।

১০। (ক) একটি জটিল ফাংশনের পোল ও রেসিডিউ এর সংজ্ঞা দাও। রেসিডিউ উপপাদ্য বর্ণনা ও প্রমাণ কর।

(খ) কন্টুর যোগজীকরণের মাধ্যমে যে-কোন দুইটির মান বাহির কর ঃ

(i) $\int_0^\infty \frac{dx}{1+x^2}$, (ii) $\int_0^\infty \frac{x \sin x}{x^2+4} dx$, (iii) $\int_0^\infty \frac{x^{p-1}}{1+x} dx;\ 0<p<1$

NU(Pre)–2008

৭। (ক) z_1 ও z_2 দুইটি জটিল সংখ্যা হলে, দেখাও যে [For any two complex numbers z_1 and z_2, show that]

$|z_1+z_2| \le |z_1| + |z_2|$ **[Ch–1, Expl–22]**

(খ) $z = x + iy$ হলে, দেখাও যে, [If $z = x + iy$, then show that]

$\sqrt{2}|z| \ge |\text{Re}(z)| + |\text{Im}(z)|$] **[Ch–1, Expl–19]**

(গ) $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ দ্বারা নির্দেশিত এলাকার চিত্র অঙ্কন কর। [Draw the graph of the region represented by $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$.] **[Ch–1, Expl–33(xii)]**

৮। (ক) দেখাও যে, ধ্রুব মানাঙ্ক বিশিষ্ট একটি বিশ্লেষণযোগ্য ফাংশন ধ্রুব। [Show that an analytic funciton with constant modulus is constant.] **[Ch–2, Expl–26]**

(খ) হারমোনিক ফাংশনের সংজ্ঞা দাও। দেখাও যে, বৈশ্লেষিক ফাংশনের বাস্তব ও কাল্পনিক অংশ প্রত্যেকে হারমোনিক ফাংশন। [Define harmonic functon. **[Ch–2, Art–2.7]** Show that the real and imaginary parts of an annlytic function are harmonic functions.] **[Ch–2, Expl–28]**

(গ) প্রমাণ কর যে, $u = 2x(1-y)$ ফাংশনটি হারমোনিক। একটি ফাংশন u নির্ণয় কর, যেন $f(z) = u + iv$ বৈশ্লেষিক হয়। $f(z)$ কে z এর মাধ্যমে প্রকাশ কর। [Prove the function $u = 2x(1-y)$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Express $f(z)$ in terms of z.]**[Ch–2, Expl–48]**

৯। (ক) কশির ইন্টিগ্রাল সূত্র বর্ণনাসহ প্রমাণ কর। এই সূত্র প্রয়োগ করে $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$ এর মান নির্ণয় কর, যেখানে c দ্বারা ধনাত্মক দিকে বর্ণিত $|z| = 2$ বৃত্তকে বোঝায়। [State and prove Cauchy's integral formula. **[Ch–3, Th^m–7]** Using the formula, evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$, where c is the circle $|z| = 2$ described in the positive sense.]**[Ch–3, Expl–11]**

(খ) লিউভিলের উপপাদ্য বর্ণনাসহ প্রমাণ কর। [State and prove Liouville's theorem.] **[Ch–3, Th^m–11]**

১০। (ক) m ক্রমের পোলবিশিষ্ট একটি ফাংশনের রেসিডিউ নির্ণয়ের একটি সূত্র বাহির কর। [Derive a formula to find the residue of a function at a pole of order m.] **[Ch–4, Th^m–5]**

(খ) কন্টুর যোগজীকরণের মাধ্যমে যে-কোন দুইটির মান নির্ণয় কর। [Evaluate by contour integration (any two)]

(i) $\int_0^{2\pi} \frac{\text{cso } 3\theta}{5 - 4\cos\theta} d\theta$ **[Ch–5, Expl–10]**

(ii) $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx$ **[Ch–5, Expl–34]**

(iii) $\int_{-\infty}^\infty \frac{e^{px}}{1+e^x} dx$, where $0<p<1$ **[Ch–5, Expl–57]**

NU(Pre)–2011

৭। (ক) $f(z)$ ফাংশন বিশ্লেষণযোগ্য হওয়ার প্রয়োজনীয় শর্তসমূহ বর্ণনা কর এবং প্রমাণ কর।

(খ) দেখাও যে $f(z) = \sqrt{xy}$ ফাংশনটি মূলবিন্দুতে regular নয়; যদিও কসি-রীম্যান সমীকরণগুলি (0, 0) বিন্দুতে সিদ্ধ হয়।

৮। (ক) প্রমাণ কর যে, $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

উপরোক্ত ফলাফলটি জ্যামিতিকভাবে ব্যাখ্যা কর এবং প্রতিপন্ন কর যে,

$|\alpha + \sqrt{\alpha^2-\beta^2}| + |\alpha - \sqrt{\alpha^2-\beta^2}| = |\alpha+\beta| + |\alpha-\beta|$.

(খ) $|w| \le 1$ বৃত্তকে $|z-1| < 1$ বৃত্তে চিত্রণের bilinear রূপান্তর কর এবং $w = 0$, $w = 1$ এর চিত্রণ যথাক্রমে $z = \frac{1}{2}$ এবং $z = 0$ হবে প্রমাণ কর।

৯। (ক) প্রথম অন্তরক সহগের জন্য কসির সমাকলণ সূত্রটি বর্ণনা ও প্রমাণ কর।

(খ) দেখাও যে, $\oint_c \frac{e^{iz}}{z^2+1} dz = 2\pi i \sin t$, যেখানে c হচ্ছে বৃত্ত $|z| = 3$ এবং $t > 0$.

(গ) $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$ ফাংশনের বিশিষ্ট বিন্দু বের কর এবং তাদের বৈশিষ্ট্য নিরূপণ কর।

১০। (ক) জটিল ফাংশনের ক্ষেত্রে টেইলরের উপপাদ্যটি বর্ণনা ও প্রমাণ কর।

(খ) কনট্যুর সমাকলন পদ্ধতিতে যে কোনো দুটির মান নির্ণয় কর ঃ

(i) $\int_0^{\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}, a > b > 0$ (ii) $\int_{-\infty}^{\infty} \frac{e^{px}}{1 + e^x} dx, 0 < p < 1$

(iii) $\int_0^{\infty} \frac{\log_e (x^2 + 1)}{x^2 + 1} dx$ (iv) $\int_0^{\infty} \frac{dx}{x^6 + 1}$.

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7. (a) State and prove the necessary conditions for a function f(z) to be analytic. **[Ch–2, Th^m–4]**

(b) Show that the function $f(z) = \sqrt{xy}$ is not regular at the origin although Cauchy-Riemann equations are satisfied at the point (0, 0). **[Ch–2, Expl–33(a)]**

8. (a) Prove that, $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

Interprete the result geometrically and deduce that

$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$.

[Ch–1, Expl–28]

(b) Find the bilinear transformation which maps the circle $|w| \le 1$ into the circle $|z - 1| < 1$ and maps w = 0, w = 1 respectively $z = \frac{1}{2}, z = 0$. **[Ch–6, Expl–33]**

9. (a) State and prove Cauchy's sintegral formula for the first derivative. **[Ch–3, Th^m–8]**

(b) Show that, $\oint_c \frac{e^{tz}}{z^2 + 1} dz = 2\pi i \sin t$, where c in the circle $|z| = 3$ and $t > 0$. **[Ch–3, Expl–7]**

(c) Locate all the singularities of the function

$f(z) = \frac{z^8 + z^4 + 2}{(z - 1)^3 (3z + 2)^2}$ and ascertain their nature.

[Ch–4, Prob–10]

10. (a) State and prove Taylor's theorem for a complex function. **[Ch–4, Th^m–1]**

(b) Evaluate any two of the following contour integration :

(i) $\int_0^{\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}, a > b > 0$ **[Ch–5, Expl–19(i)]**

(ii) $\int_{-\infty}^{\infty} \frac{e^{px}}{1 + e^x} dx, 0 < p < 1$ **[Ch–5, Expl–57]**

(iii) $\int_0^{\infty} \frac{\log_e (x^2 + 1)}{x^2 + 1} dx$ **[Ch–5, Expl–34]**

(iv) $\int_0^{\infty} \frac{dx}{x^6 + 1}$. **[Ch–5, Expl–31]**

NU(Pre)–2012

৭। (ক) যদি $w = f(z) = u + iv$ একটি বৈশ্লেষিক ফাংশন হয়, তাহলে দেখাও যে, কসি-রীম্যান সমীকরণগুলি পোলার আকারে $\frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \theta}$ এবং $\frac{\delta u}{\delta \theta} = -r\frac{\delta v}{\delta r}$ হবে।

(খ) যদি f(z), z এর একটি বৈশ্লেষিক ফাংশন হয়, তাহলে প্রমাণ কর যে,

$\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right) |f(z)|^2 = 4|f'(z)|^2$.

৮। (ক) প্রমাণ কর যে, $|z_1 - z_2| \ge ||z_1| - |z_2|| \ge |z_1| - |z_2|$ যেখানে z_1, z_2 জটিল সংখ্যা।

(খ) দেখাও যে, $w = \frac{2z + 3}{z - 4}$ রূপান্তর $x^2 + y^2 - 4x = 0$ বৃত্তকে $4u + 3 = 0$ সরলরেখায় পরিবর্তন করে এবং কেন এই পরিবর্তিত বক্ররেখা একটি বৃত্ত নয় ব্যাখ্যা কর।

৯। লরেন্ট এর উপপাদ্যটি বর্ণনা ও প্রমাণ কর।

১০। (ক) কসির অবশেষ উপপাদ্যটি বর্ণনা ও প্রমাণ কর।

(খ) কন্ট্যুর সমাকলন পদ্ধতিতে যে কোনো দু'টির মান নির্ণয় কর ঃ

(i) $\int_0^{\infty} \frac{x \sin x}{x^2 + 4} dx$ (ii) $\int_0^{\infty} \frac{dx}{x^4 + 1}$

(iii) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ (iv) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)} dx$

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7. (a) If $w = f(z) = u + iv$ is an analytic function, then show that the Cauchy-Riemann equations in polar form are $\frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \theta}$ and $\frac{\delta u}{\delta \theta} = -r\frac{\delta v}{\delta r}$. **[Ch–2, Art–2.6]**

(b) If f(z) is analytic function of z, then prove that $\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right) |f(z)|^2 = 4|f'(z)|^2$. **[Ch–2, Expl–59]**

8. (a) Prove that $|z_1 - z_2| \geq ||z_1| - |z_2|| \geq |z_1| - |z_2|$, where z_1 and z_2 are complex numbers. **[Ch–1, Expl–24]**

(b) Show that the transformation $w = \frac{2z+3}{z-4}$ changes the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

[Ch–6, Expl–17]

9. State and prove Laurent's theorem. **[Ch–4, Th^m–2]**

10. (a) State and prove Cauchy's Residue Theorem.

[Ch–4, Th^m–6]

(b) Evaluate any two of the following contour integrations :

(i) $\int_0^\infty \frac{x \sin x}{x^2 + 4} dx$ **[Ch–5, Expl–49]**

(ii) $\int_0^\infty \frac{dx}{x^4 + 1}$ **[Ch–5, Expl–28]**

(iii) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ **[Ch–5, Expl–10]**

(iv) $\int_0^\infty \frac{\sin \pi x}{x(1 - x^2)} dx$ **[Ch–5, Expl–53]**

NU(Pre)–2013

৭। (ক) f(z) ফাংশন analytic হওয়ার প্রয়োজনীয় শর্তসমূহ বর্ণনা কর এবং প্রমাণ কর। **[Ch–2, Th^m–4]**

(খ) দেখাও যে $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ একটি harmonic ফাংশন। $u + iv$ ফাংশনটি analytic হওয়ার জন্য v এর মান নির্ণয় কর।

[Ch–2, Expl–42]

৮। (ক) কশির ইন্টিগ্রাল ফর্মুলা বর্ণনা ও প্রমাণ কর। **[Ch–3, Th^m–7]**

(খ) দেখাও যে, (1, 2) বিন্দু ব্যতীত প্রত্যেক এলাকায় $\psi = \ln[(x-1)^2 + (y-2)^2]$ হার্মোনিক, একটি ফাংশন ϕ নির্ণয় কর যেন $\phi + i\psi$ যেন বিশ্লেষণযোগ্য হয় এবং ইহাকে z এর মাধ্যমে প্রকাশ কর। **[Ch–2, Expl–62]**

৯। (ক) Liouville's উপপাদ্যটি বর্ণনা ও প্রমাণ কর। **[Ch–3, Th^m–11]**

(গ) ব্যতিচার বিন্দু ও পোলের সংজ্ঞা লিখ। **[Ch–4, art 4.1, and 4.2]**

$f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ ফাংশনটির সকল ব্যতিচার বিন্দু নির্ণয় করে প্রকৃতি নির্ণয় কর। **[Ch–4, Prob–10]**

১০। (ক) জটিল ফাংশনের রেসিডিউ সংজ্ঞায়িত কর **[Ch–4, art–4.5 def^n]** একটি ফাংশনের m মাত্রিক পোলে রেসিডিউ নির্ণয়ের সূত্র প্রতিপাদন কর।

[Ch–4, Th^m–5]

(খ) নিচের যে কোনো দুইটির মান 'কন্টুর ইন্টিগ্র্যাশন' পদ্ধতিতে নির্ণয় কর ঃ

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$ **[Ch–5, Expl–4]**

(ii) $\int_0^\infty \frac{\log(x^2 + 1)}{x^2 + 1} dx$ **[Ch–5, Expl–34]**

(iii) $\int_{-\infty}^\infty \frac{\cos ax}{1 + x^2} dx, a > 0$ **[Ch–5, Expl–40(i)]**

NU(Pre)–2014

৭। (ক) পোলার স্থানাংকে কশিরীম্যান সমীকরণগুলি বর্ণনা ও প্রমাণ কর।

(খ) দেখাও যে যদিও $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ যখন $z \neq 0$ এবং $f(0) = 0$ ফাংশনটি মূলবিন্দুতে কশিরীম্যান সমীকরণকে সিদ্ধ করে তথাপিও উহা মূলবিন্দুতে বৈশ্লেষক নয়।

৮। (ক) ত্রিভুজের জন্য কসি-গৌড়সাট উপপাদ্যটি প্রমাণ কর।

(খ) মনে করি f (z) ফাংশনটি দুইটি সরল বদ্ধ বক্ররেখা C_1 এবং C_2 এর মাধ্যমে আবৃত অঞ্চল R-এর ভিতরে এবং অধিকন্তু বক্ররেখাদ্বয় C_1 এবং C_2-এর উপরে বিশ্লেষিত। তা হলে প্রমাণ কর যে, $\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz$ হবে যেখানে C_1 এবং C_2 তাদের দিক সাপেক্ষে ধনাত্মক দিক বরাবর অতিক্রান্ত হয়।

৯। (ক) $|z| \leq 1$ এবং $|z| > 1$ এলাকায় $f(z) = \frac{3z-3}{(2z-1)(z-2)}$ কে লরন্টে ধারায় বিস্তৃত কর।

(খ) $f(z) = \frac{z^2}{(z+1)^2} \sin\left(\frac{1}{z-1}\right)$ ফাংশনটির ব্যতিচার বিন্দুসমূহ নির্ণয় কর এবং এদের প্রকৃতি নিরূপন কর।

১০। (ক) একটি জটিল ফাংশনের রেসিডিউ-এর সংজ্ঞা দাও। m অর্ডারের পোলে কোনো ফাংশনের রেসিডিউ নির্ণয়ের সূত্রটি নির্ণয় কর।

(খ) কন্টুর যোগজীকরণের মাধ্যমে নিম্নের যে কোনো দু'টির মান নির্ণয় কর ঃ

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$ (ii) $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$

(iii) $\int_0^{\infty} \frac{\cos x}{a^2 - x^2}\, dx.$

NU(Pre) [English Version]–2014

7. (a) State and prove Cauchy-Riemann equation in polar co-ordinate. **[Ch-2, Art-2.6]**

(b) Show that although the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ when $z \neq 0$ and $f(0) = 0$ satisfy the Cauchy Riemann equations at the origin yet it is not analytic at the origin. **[Ch-2, Expl-14]**

8. (a) Prove the Cauchy-Goursat theorem for the case of a triangle. **[Ch-3, Th^m-2]**

(b) Let f (z) be analytic in a region R, bounded by two simple closed curves C_1 and C_2 and also on C_1 and C_2. Prove that $\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz$, where C_1 and C_2 are both traversed in the positive sense relative to their interiors. **[Ch-3, Th^m-6]**

9. Expand. $f(z) = \frac{3z-3}{(2z-1)(z-2)}$ in a Laurent series for the region $|z| \leq 1$ and $|z| > 1$. **[Ch-4, Prob-40]**

(b) Find the singular points of the function $f(z) = \frac{z^2}{(z+1)^2} \sin\left(\frac{1}{z-1}\right)$ and determine their nature. **[Ch-4, Prob-8]**

10. (a) Define residue of a complex function **[Ch-4, Art-4.5]** Derive the formula to find the residue of a function at a pole of order m. **[Ch-4, Th^m-5]**

(b) Evaluate any two of the following by contour integrations :

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$ **[Ch-5, Expl-4]**

(ii) $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$ **[Ch-5, Expl-24]**

(iii) $\int_0^{\infty} \frac{\cos x}{a^2 - x^2}\, dx.$ **[Ch-5, Expl-52]**

NUH : NATIONAL UNIVERSITY HONOURS MATHEMATICS

NUH–1993

1. (a) Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ and deduce that $|\alpha\sqrt{\alpha^2-\beta^2}| + |\alpha^2 - \sqrt{\alpha^2-\beta^2}| = |\alpha+\beta| + |\alpha-\beta|$ all the numbers concerned being complex. **[Ch–1, Expl–28]**

(b) Prove that the function, $f(z) = \begin{cases} \dfrac{x^3(1+i) - y(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is continuous and the Cauchy–Riemann equations are satisfied at the origin yet $f'(0)$ does not exist. **[Ch–2, Expl–14]**

2. (a) Find all mobius transformations which transform the half-plane Im $z \geq 0$ into the unit dish $|w| \leq 1$. Verify your result.

(b) Prove that $w = \dfrac{1+iz}{i+z}$ maps the real axis of the z plane into a circle in the w-plane. Show that by sketches the resign of the real axis of the z-plane and the circle of the w-plane that correspond.

3. (a) State and prove a set of sufficient conditions for analyticity of a complex function. $w = f(z)$. **[Ch–2, Th^m–5]**

(b) Define harmonic **[Art–2.7]** and harmonic conjugate functions. **[Art–2.7]** Show that the function $u(x, y) = x^3 - 3xy^2$ is harmonic. Find its conjugate v such that $f(z) = u + iv$ is harmonic.

4. (a) Define zero **[Art–4.1]** singularity **[Art–4.2]** and residue of complex function **[Art–4.5]** classify the singularities with example. Drive the formula to find residue of a function at a pole of order m. **[Ch–4, Th^m–5]**

(b) State and prove the theorem on Laurent's series. **[Ch–4, Th^m–2]** Give two Laurent series expansions in powers of z, with regions of validity for the function $f(z) = \dfrac{1}{z^2(1-z)}$.

5. Evaluate any two of the following using contour integration

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ **[Ch–5, Solved Prob–1]**

(ii) $\displaystyle\int_0^{\infty} \frac{2x^2\,dx}{(x^2+9)(x^2+4)^2}$ **[Ch–5, Solved Prob–25]**

(iii) $\displaystyle\int_0^{\infty} \frac{\sin mx\,dx}{x}$, $(m > 0)$. **[Ch–5, Solved Prob–51]**

NUH–1994

1. (a) Prove that for any two complex numbers z_1 and z_2 .

(i) $|z_1| - |z_2| \geq ||z_1| - |z_2||$ **[Ch–1, Expl–24]**

(ii) $|z|\sqrt{2} \geq |\text{Re}z| + |\text{Im}z|$ **[Ch–1, Expl–19]**

(b) Draw each sketch of the following sets.

(i) $1 \leq |z+1+i| < 2$ **[Ch–1, Expl–33(xxxi)]**

(ii) $\text{Re}\left(\dfrac{1}{z}\right) > \dfrac{1}{2}$ **[Ch–1, Expl–33(xxxii)]**

(iii) $\text{Im}(z^2) > 2$ **[Ch–1, Expl–38(xxxiii)]**

(iv) $(|z+1+i| = |z-1+i|$ **[Ch–1, Expl–38(xxxiv)]**

2. (a) Define differentiability of a complex function **[Ch–2, Art–2.4]** show that the function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ for $z \neq 0$, $f(0) = 0$ is not differentiable at the origin although Cauchy's Riemann equation are satisfied at that point. **[Ch–2, Expl–14]**

(b) Define harmonic function **[Ch–2, Art–2.7]** and harmonic conjugate function **[Art–2.7]** show that the function $u(x, y) = e^x(\cos y - y\sin y)$ is harmonic find its conjugate v and corresponding analytic function $f(z) = u + iv$. **[Ch–2, Expl–49]**

3. (a) Show that the transformation $w = \dfrac{z-i}{z+i}$ transforms the upper half plane Im $z \geq 0$ into $|w| \leq 1$. **[Ch–6, Expl–27]**

(b) Define the complex line integral of f(z) along a curve. **[Ch–3, Art–3.3]**

Evaluate $\displaystyle\int_C \frac{e^{3z}}{z+\pi i}\,dz$ where is the circle $|z+1| = 4$ **[Ch–3, Expl–16]**

4. (a) Define singular points of a complex function. **[Ch–4, Art–4.1, Def^n–]** Find the singular points of the functon $f(z) = \dfrac{1}{(z^2-1)^2}\sin\dfrac{1}{z}$ and establish the nature.

(b) If f(z) is analytic inside and on a simple closed curve c except for a finite number of poles p and zeros N of f(z), then $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{f(z)}\,dz = N - p$.

5. (a) Evaluate any two of the following contour integration :

(i) $\int_0^{\infty} \frac{\sin x}{x} dx$ **[Ch–5, Prob–50]**

(ii) $\int_0^{\infty} \frac{x \sin x}{x^2+4} dx$ **[Ch–5, Prob–49]**

(iii) $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$ **[Ch–5, Prob–10]**

NUH–1995

1. (a) Describe the region determined by each of the following relation :

(i) $\frac{|z-1|}{|z+1|} = 2$ **[Ch–1, Expl–37]**

(ii) $|z+i| + |z-i| \leq 3$ **[Ch–1, Expl–33(xxix)]**

(b) Find all roods of the equation $\sinh z = i$ **[Ch–1, Expl–11]**

(c) Show that $z = 0$ is the sole accumulation point of the set.

$$\{x(1+i) : x = \frac{1}{n}, n = 1, 2, \cdots\}$$

2. (a) What is mean by saying that the function of a complex variable $f(z)$ is analytic at z_n? Show that validity of the Cauchy-Riemann equation is necessary condition for analyticity, but not a sufficient condition state a set of condition sufficient for analyticity. **[Exercise–2(25)]**

(b) State and prove Cauchy's integral formula. **[Ch–3, Th^m–7]**

3. (a) State Liuville's theorem. **[Ch–3, Th^m–11]** Apply it to prove the fundamental theorem of Algebra. **[Ch–3, Th^m–12]**

(b) Find the singular points of the function

$\frac{z^2}{(z+1)^2} \sin\left(\frac{1}{z-1}\right)$ and determine their nature. **[Ch–4, Prob–8]**

(c) Obtain Laurent expansion of $\frac{1}{z^2(z-3)^2}$. About the point $z = 3$. **[Ch–4, Prob–38]**

4. (a) State and prove Rouche's theorem. **[Ch–4, Th^m–10]**

(b) Show that the root of the equations $z^5 + z - 16i = 0$ all between the circles $|z| = 1$ and $|z| = 2$. Also show that the equation $z \tan z = i$ has infinitely many real roots but no complex root.

5. (a) Evaluate any two of the following integrals by the method of contour integration :

(i) $\int_0^{\infty} \frac{\cos 2x}{x^2+1} dx$ **[Ch–5, Prob–41]**

(ii) $\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta}$ **[Ch–5, Prob–3]**

(iii) $\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2+4)(x^2+9)}$ **[Ch–5, Prob–23]**

NUH–1996

1. (a) Discuss the locus represented by $\arg\left(\frac{z-z_1}{z_2-z_1}\right) = 0$ where z_1 and z_2 are two given points. **[Ch–1, Expl–41]** Does $\lim_{z\to 0} \frac{\bar{z}}{z}$ exist? **[Ch–2, Expl–8]**

(b) What is meant by the analyticity of a complex function at a point? Prove that the analyticity of a function at a point implies the continuity of the function at the point. Give an example to show that the converse is not necessarily true. **[Exercise–2(26)]**

(c) State the necessary and sufficient conditions for a function $f(z)$ to be $u = 2x(1-y)$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Express $f(z)$ in terms of z. **[Ch–2, Expl–48]**

2. (a) Explain conformal mapping. **[Ch–6, Art–6.2]** If $f(z)$ is analytic and $f(z) \neq 0$ in a region R. Prove that the mapping $w = f(z)$ is conformal in R. **[Ch–6, Th^m–2]**

(b) Define a bilinear transformation. **[Ch–6, Art–6.5]** Find a bilinear transformation which maps the points 1, –i, 1 of the z-plane into the points 0, 1 ∞ of w-plane respectively. **[Ch–6, Expl–13]**

(c) Find the fixed points of the bilinear transformation $w = \frac{2z-5}{z+4}$. If a b are these points, write the equation in the form $\frac{w-a}{w-b} = k\left(\frac{z-a}{z-b}\right)$, where K is a constant. **[Ch–6, Expl–20]**

3. (a) Let C denotes the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$ described in the positive sense. Determine :

(i) $\int_C \frac{1}{z^2+9} dz$ **[Ch-3, Expl-14(i)]**

(ii) $\int_C \frac{1}{z(z^2+9)} dz$ **[Ch-3, Expl-14(ii)]**

(iii) $\int_C \frac{1}{(z^2+1)(z^2+9)} dz$ **[Ch-3, Expl-14(iii)]**

4. (a) Classify the different kinds of singularities of a complex function and illustrate. **[Ch-4, Art-4.2]** Show that if z_0 is a pole of f(z), then $|f(z)| \to \infty$ as $z \to z_0$. **[Ch-4, Th^m-3]**

(b) Find the Laurent series expansions of $f(z) = \frac{3}{z(2-z-z^2)}$, in powers of z, valid in the region

(i) $0 < |z| < 1$ **[Ch-4, Prob-39(i)]**

(ii) $1 < |z| < 2$ **[Ch-4, Prob-39(ii)]**

(iii) $|z| > 2$. **[Ch-4, Prob-39(iii)]**

5. Using the method of Contour integration, evaluate any two of the following real integrals :

(a) $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx, a > 0$ **[Ch-5, Prob-30]**

(b) $\int_0^\infty \frac{\cos ax}{x^2+1} dx, a > 0$ **[Ch-5, Prob-40(i)]**

(c) $\int_0^{2\pi} \frac{1}{(1+a\sin x)} dx, 0 < a < 1$ **[Ch-5, Solved Prob-6]**

(d) $\int_0^\infty \frac{z^a}{(1+z^2)^2} dz, 0 < a < 1$ **[Ch-5, Prob-34(a)]**

NUH-1997

1. (a) Let **C** be the set of all complex number. Consider $z_1 = z_1 + iy_1$ and $z_2 = x_2 + iy_2 \in$ **C** with $x_1 < x_2$, $y_1 < y_2$. "Do you agree that $z_1 < z_2$? what about $|z_1| < |z_2|$? Prove that $\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j|$ and $\left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{\pi} |z_j|$ where $z_1, z_2 \cdots z_n$ are complex numbers.

[Ch-1, Expl-23]

(b) Prove that a bilinear transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. Transform the circles of the z-plane into the either circles or straight lines of the w-plane. **[Ch-6, Expl-4]**

2. (a) Show that Cauchy-Riemann equation in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. **[Ch-2, Art-2.6]**

(b) If f(z) is analytic function of z then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$. Also from it show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)^2| = 4 |f'(z)|^2$. **[Ch-2, Expl-57]**

3. (a) Suppose the complex valued function f(z) is analytic and $f'(z)$ is continuous within and on a closed contour C. Prove that $\int_C f(z)\, dz = 0$ can the restriction about continuity be removed?

[Ch-3, Th^m-5]

(b) What is Cauchy's integral formula? **[Ch-3, Th^m-7]** Using this formula evaluate $\int_C \frac{z\, dz}{(9-z^2)(z+1)}$ where c is the circle $|z| = 2$ described in the positive sense. **[Ch-3, Expl-11]** How significant is it in determining the value of a function?

4. Evaluate any two of the following by using contour integration :

(i) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$ **[Ch-5, Prob-9]**

(ii) $\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x} dx, 0 < p < 1$ **[Ch-5, Prob-57]**

(iii) $\int_0^\infty \frac{\log(x^2+1)}{x^2+1} dx$ **[Ch-5, Prob-34]**

(iv) $\int_0^\infty \frac{x\sin mx}{x^2+a^2} dx, a > 0, m \ge 0.$ **[Ch-5, Prob-47(ii)]**

NUH–1998

1. (a) (i) If $z = x + iy$ then prove that $|x| + |y| \le \sqrt{2}\,|x + iy|$. **[Ch–1, Expl–20]**

(ii) Draw the sketch of each of the following regions :
$1 < |z - 2i| < 2$, **[Ch–1, Expl–38]** and $|z - 4| > |z|$. **[Ch–1, Expl–33(i)]**

(b) Find a bilinear transformation which transforms the unit circle $|z| \le 1$ into the unit circle $|w| \le 1$. Verify your result. **[Ch–6, Expl–29]**

2. (a) Find the necessary and sufficient conditions for f(z) to be regular, where u and v both real. **[Ch–2, Th^m–4, 5]**

(b) Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace's equation and determine the corresponding regular function u + iv. **[Ch–2, Expl–42]**

3. (a) Show that $w = \frac{1 - z}{1 + z}$ maps $|z| < 1$ conformably into real part w > 0. Determine the curves in the w-plane which correspond to $|z| = 1$ and to $\arg z = \frac{\pi}{4}$. **[Ch–6, Expl–35]**

(b) Expand $\frac{1}{(z + 1)(z + 3)}$ in a Laurent series for $1 < |z| < 3$ and for $|z| > 3$. **[Ch–4, Expl–35(i), (ii)]**

4. (a) State and prove Cauchy's residue theorem. **[Ch–4, Th^m–6]**

(b) Evaluate any two of the following by contour integration :

(i) $\int_0^\infty \frac{\cos 2\pi x}{x^4 + x^2 + 1}$ **[Ch–5, Prob–38]**

(ii) $\int_0^\infty \frac{x^6}{(x^4 + 1)^2}\,dx$ **[Ch–5, Prob–33]**

(iii) $\int_0^\infty \frac{x^{m-1}}{1 + x}\,dx,\ 0 < m < 1$ **[Ch–5, Prob–56(i), Put p = m]**

(iv) $\int_0^\pi \frac{a\,d\theta}{a^2 + \cos^2\theta},\ a > 0$ **[Ch–5, Prob–17]**

NUH–1999

1. (a) Define a single valued and a multiple valued complex functions. **[Ch–2, Art–2.1]** Show that ln z has a branch point at z = 0. **[Ch–2, Prob–2]**

(b) Show that $\frac{1}{z}$ is not uniformly continuous in a region $|z| < 1$ **[Ch–2, Expl–22 (2nd part)]**

(c) A sauare S in the z plane has vertices at (0, 0), (1, 0), (1, 1), (0, 1). Determine the region in the w-plane into which S is mapped under the transformation $w = z^2$. **[Ch–6, Expl–34]**

2. (a) State and prove the converse theorem of Cauchy. **[Ch–3, Th^m–10]**

(b) Evaluated the value of $\oint_C \frac{e^{3z}}{z - \pi i}\,dz$, where C is a curve (i) $|z - 1| = 4$ and (ii) $|z - 2| + |z + 2| = 6$. **[Ch–3, Expl–6]**

3. (a) State and prove Cauchy's Integral formula and hence derive the expression for the derivative for an analytic function. **[Ch–3, Th^m–7, 8]**

(b) Evaluate by Contour integration :

(i) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$ **[Ch–5, Prob–10]** (ii) $\int_0^\infty \frac{\sin ax}{e^x + 1}\,dx$

4. (a) State and prove Morera's theorem. **[Ch–3, Th^m–10]**

(b) Explain conformal mappings **[Ch–6, Art–6.2]** and state the sufficient conditions for a function w = f(z) to represent a conformal mapping. **[Ch–6, Th^m–2]** If w = f(z) = u + iv is analytic in a region R. Prove that $\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2$. **[Ch–6, Art–6.4]**

NUH–2000

1. (a) (i) Find two complex numbers whose sum is 4 and whose product is 8. **[Ch–1, Expl–26]**

(ii) If z_1 and z_2 are two complex numbers, them prove that
$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ **[Ch–1, Expl–28]**

(iii) Draw the sketch of the region $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$ **[Ch-1, Expl-33(xiii)]**

(b) Define bilinear transformation. **[Ch-6, Art-6.5, Defn]** Why it is called bilinear transformation? **[Ch-6, Expl-1]** Find a bilinear transformation which transforms points $z = 0, -i, -1$, into $w = i, 1, 0$ respectively. **[Ch-6, Expl-12]**

2. (a) Define an analytic function. **[Art-2.5, Defn]** State and prove the necessary conditions for f(z) to be analytic. **[Ch-2, Thm-4]**

(b) Let $f(z) = u + iv = \frac{x^3 - 3xy + i(y^3 - 3x^2y)}{x^2 + y^2}$ when $z \neq 0$ and $f(0) = 0$ when $z = 0$. Show that f(z) continuous and the Cauchy-Riemann equations are satisfied but f(z) is not differentiable at $z = 0$. **[Ch-2, Expl-13]**

3. (a) If for all z in the entire complex plane (i) f(z) is analytic and (ii) bounded, then prove that f(z) must be a constant. **[Ch-3, Thm-11]**

(b) (i) Expand $f(z) = \frac{3z - 3}{(2z - 1)(z - 2)}$ in a Laurent series for the region $|z| < 1$ and $|z| > 1$. **[Ch-4, Prob-40]**

(ii) Find the singular points of the function $f(z) = \frac{z^2}{(z + 1)^2} \sin\left(\frac{1}{z - 1}\right)$ and determine their nature. **[Ch-4, Prob-8]**

4. (a) Define residue of a complex function **[Ch-4, Art-4.5, Defn]**. Derive the formula to find the residue of a function at a pole of order m. **[Art-4.5, Thm-5]**

(b) Evaluate any two of the following by contour integration :

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$ **[Ch-5, Prob-4]**

(ii) $\int_0^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)^2}$ **[Ch-5, Prob-24]**

(iii) $\int_0^{\infty} \frac{\cos x \, dx}{a^2 - x^2}$ **[Ch-5, Prob-52]**

NUH-2001

1. (a) For complex numbers z_1, z_2, z_3 and $|z_2| \neq |z_3|$. Show that $\left|\frac{z_1}{z_2 + z_3}\right| \leq \frac{|z_1|}{\left||z_2| - |z_3|\right|}$. **[Ch-1, Expl-25(i)]**

(b) Determine the set of complex numbers such that $\text{Re}\left(\frac{1}{z}\right) < 1$ represent the set in the complex plane. **[Ch-1, Expl-36]**

(c) Find all value of $(1 + i)^{1/4}$ **[Ch-1, Expl-9]**

2. (a) State and prove a set of sufficient condition for a function to be analytic in a region. **[Ch-2, Thm-5]**

(b) Define harmonic function. **[Art-2.7, Defn]** Show that the real and imaginary parts of an analytic function are harmonic functions. **[Ch-2, Thm-6]**

(c) Prove that the function $u = 2x(1 - y)$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Express f(z) in terms of z. **[Ch-2, Expl-48]**

3. (a) Find a bilinear transformation which maps the points, $i, -i, 1$ of the z-plane into the points $0, 1, \infty$ of w-plane respectively. **[Ch-6, Expl-16]**

(b) What is Cauchy's integral formula? **[Ch-3, Thm-6]** Using this formula evaluate $\int_C \frac{z\,dz}{(9 - z^2)(z + i)}$ where c is the circle $|z| = 2$ described in the positive sense. **[Ch-3, Expl-11]**

(c) State and prove Liouville's theorem. **[Ch-3, Thm-11]**

4. (a) Find the Laurent expansion of the function $f(z) = \frac{z^2 + 1}{(z + 1)(z - 2)}$ in each of the regions

(i) $1 < |z| < 2$ **[Ch-4, Prob-37(i)]**

(ii) $0 < |z| < 1$. **[Ch-4, Prob-37(ii)]**

(b) Evaluate any two of the following integrals using a suitable contour in each case :

(i) $\int_0^{2\pi} \frac{\sin 2\theta}{5 - 3\cos\theta} d\theta$ **[Ch-5, Prob-11]**

(ii) $\int_0^{\infty} \frac{dx}{x^4 + a^4}, a > 0$ **[Ch-5, Prob-29]**

(iii) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)} dx$ **[Ch-5, Prob-53]**

NUH–2002

1. (a) Determine the set of points in the complex plane which satisfy the inequality $|z + 1 - i| \leq |z - 1 + i|$, and sketch it. **[Ch–1, Expl–34]**

(b) Find all solutions of the equation $\cosh z = 2$ **[Ch–1, Expl–12]**

(c) Show that the transformation $w = \frac{1 + iz}{z + i}$ maps the real axes of the z-plane onto a circle in the w-plane. Find its center and radius. **[Ch–6, Expl–22]**

2. (a) What do you mean by saying that a complex function (i) is differentiable at z_0. **[Art–2.4, Def^n]** (ii) is analytic at z_0? State and prove necessary conditions for f(z) to be analytic at z_0. **[Ch–2, Th^m–4]** Show that the function $f(z) = |z|^2$ is differentiable at z = 0, but not analytic at that point. **[Ch–2, Expl–18]**

(b) Show that $u(x, y) = 3xy^2 + 2x^2 - y^3 - 2y^2$ is a harmonic function. Find a function v(x,y) such that u + iv is an analytic function. Hence express u + iv in terms of z = x + iy. **[Ch–2, Expl–43]**

3. (a) State and prove Cauchy's integral formula. **[Ch–3, Th^m–6]** Using the formula evaluate $\int_C \frac{dz}{z(z-2)^2}$, when C is the circle $|z| = 1$ describe counter clock wise. **[Ch–3, as Expl–15, Write 2 for 4]**

(b) Find the Laurent expansion of $f(z) = \frac{z - 1}{(z + 2)(z + 3)}$ in each of the following region–

(i) $|z| < 2$ **[Ch–4, Prob–36(i)]**
(ii) $2 < |z| < 3$ **[Ch–4, Prob–36(ii)]**
(iii) $|z| > 3$. **[Ch–4, Prob–36(iii)]**

4. (a) Define pole **[Ch–4, Art–4.2, Def^n]** and residue.**[Ch–4, Art–4.5, Def^n]** Derive the formula for finding the residue of a function at a pole of order m. **[Ch–4, Th^m–5]**

(b) State and prove Cauchy's residue theorem **[Ch–4, Th^m–6]**

(c) Evaluate any one by contour integration :

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ **[Ch–5, Prob–2]**

(ii) $\int_0^{\infty} \frac{dx}{x^4 + 1}$ **[Ch–5, Prob–28]**

NUH–2003

6.(a) For any complex numbers z_1 and z_2 prove that–
$|z_1 + z_2| \leq |z_1| + |z_2|$ **[Ch–1, Expl–22(i)]**

(b) Draw a sketch of $|z - i| = |z + i|$ in the complex plane. **[Ch–1, Expl–33(ii)]**

(c) A square s in the z-plane has vertices at (0, 0), (1, 0), (1, 1), (0, 1). Determine the region in the w-plane into which s is mapped under the transformation $w = z^2$. **[Ch–6, Expl–34]**

7. (a) Show that Cauchy-Riemann equation in polar form are
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ **[Ch–2, Art–2.6]**

(b) Let $f(z) = u + iv = \frac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$, when $z \neq 0$ and f(z) = 0, when z = 0. Show that f(z) is continuous and the Cauchy-Riemann equations are satisfied but f(z) is not differentiable at z = 0. **[Ch–2, Expl–13]**

8. (a) If f(z) is analytic inside and on the boundary C of a simply connected region R, then prove that–
$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - a)^2}dz$ **[Ch–3, Th^m–8]**

(b) Define singular point **[Ch–4, Art–4.1, Def^n]** and pole. **[Ch–4, Art–4.2m, Def^n]** Locate all the singularities of the function $f(z) = \frac{z^8 + z^4 + 2}{(z - 1)^3 (3z + 2)^2}$ and ascertain their nature. **[Ch–4, Prob–10]**

(c) Expand $f(z) = \frac{1}{(z + 1)(z + 3)}$ in a Laurent series for the region

(i) $1 < |z| < 3$ **[Ch–4, Prob–35(i)]**
(ii) $|z| > 3$ **[Ch–4, Prob–35(ii)]**
(iii) $0 < |z + 1| < 2$. **[Ch–4, Prob–35(iii)]**

9. (a) State and prove Morera's theorem. **[Ch–3, Th^m–10]**

(b) Evaluate any two of the following by using Contour integration : –

(i) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}d\theta$ **[Ch–5, Prob–10]**

(ii) $\int_0^{\infty} \frac{\log(x^2 + 1)}{x^2 + 1}dx$ **[Ch–5, Prob–34]**

(iii) $\int_0^{\infty} \frac{x \sin x}{a^2 + x^2}dx$. **[Ch–5, Prob–48]**

NUH–2004 (OLD)

6. (a) Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. Interpret the result geometrically and deduce that

$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$, all the numbers involved being complex. **[Ch–1, Expl–28]**

(b) Draw a sketch of $|z + 2i| + |z - 2i| = 6$. **[Ch–1, Expl–33(xxx)]**

(c) Determine the equation of the curve in the w-plane into which the straight line $x + y = 1$ is mapped under the transformation $w = \frac{1}{z}$. **[Ch–6, Expl–9]**

7. (a) Show that if the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = x + iy$ then the four partial derivatives u_x, v_x, u_y, v_y should exist and satisfy the equations $u_x = v_y$, $u_y = -v_x$. **[Exercise–2(34), Or Ch–2, Th^m–4]**

(b) If f(z) is an analytic function of z, then prove that

$$\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$ **[Ch–2, Expl–57]**

(c) Prove that the function $|z|^2$ is continuous everywhere, but nowhere differentiable except at the origin, where z is a complex number. **[Ch–2, Expl–15]**

8. (a) State and prove Cauchy's integral formula. **[Ch–3, Th^m–5]**

(b) Show that $\oint_C \frac{e^{tz}}{z^2 + 1} dx = 2\pi i \sin t$, where c is the circle $|z| = 3$ and $t > 0$. **[Ch–3, Expl–7]**

(c) Expand $f(z) = \frac{z^2}{(z-1)(z-2)}$ in a Laurent series for the region $1 < |z| < 2$. **[Ch–4, Prob–30]**

9. (a) State and prove Liouville's theorem. **[Ch–3, Th^m–11]**

(b) Evaluate any two of the following by using contour integration :

(i) $\int_0^\infty \frac{\cos 2x}{1 + x^2} dx$ **[Ch–5, Prob–41]**

(ii) $\int_0^\infty \frac{\sin mx}{x} dx,\ m > 0$ **[Ch–5, Prob–51]**

(iii) $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta$ **[Ch–5, Prob–20]**

NUH–2004 (NEW)

1. (a) If z_1 and z_2 are two complex numbers, then prove that $|z_1 + z_2| \le |z_1| + |z_2|$. Give the graphical representation of it. [z_1 এবং z_2 দুটি জটিল সংখ্যা হলে, প্রমাণ কর যে, $|z_1 + z_2| \le |z_1| + |z_2|$. ইহা গ্রাফ দ্বারা প্রকাশ কর।] **[Ch–1, Expl–22(i)]**

(b) Prove that the equation of any circle in the z-plane

$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, where α, γ are real constants, while β may be a complex constant. [প্রমাণ কর যে, জটিল তলে যে-কোন বৃত্তের সমীকরণ $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ যেখানে α, γ বাস্তব ধ্রুবক এবং β জটিল ধ্রুবক হতে পারে।]

[Ch–1, Expl–42]

(c) What is Branch Point? **[Art–4.2, Def^n]** Prove that $f(z) = \ln z$ has a branch point at $z = 0$. **[Ch–2, Expl–2]** [ব্রাঞ্চ-পয়েন্ট বরতে কি বুঝ? প্রমাণ কর যে, $f(z) = \ln z$ ফাংশনের $z = 0$ বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে।]

2. (a) Define conjugate harmonic function. **[Art–2.7, Def^n]** Show that $\psi(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ is a harmonic function in the region $C - \{0, 0\}$. Find the harmonic conjugate function $\phi(x, y)$ such that $f(z) = \phi + i\psi$ is analytic and also find f(z) in terms of z. **[Ch–2, Expl–44]** [অনুবন্ধী হারমোনিক ফাংশনের সংজ্ঞা দাও। দেখাও যে, $\psi(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ ফাংশনটি $C - (0, 0)$ এলাকায় হারমোনিক। ইহার অনুবন্ধী হারমোনিক ফাংশন $\phi(x, y)$ বাহির কর, যেন $f(z) = \phi + i\psi$ একটি এনালিটিক ফাংশন হয়। আবার f(z) কে z এর মাধ্যমে প্রকাশ কর।]

(b) State and prove the Cauchy-Riemann equations in polar form. [পোলার স্থানাংকে Cauchy-Riemann সমীকরণগুলোর বর্ণনা ও প্রমাণ কর।]

[Ch–2, Art–2.6]

3. (a) State and prove the Cauchy's theorem. **[Ch–3, Th^m–5]** Prove that if f(z) is integrable along a curve C having length L and $|f(z)| \le M$ on C, then [কচির উপপাদ্যের বর্ণনা এবং প্রমাণ দাও। প্রমাণ কর যে, f(z) একটি কার্ভ c এর উপর যোগজীকরণ হলে যার দৈর্ঘ্য L এবং $|f(z)| \le M$ তবে]

$$\left|\int_C f(z)\, dz\right| \le ML$$ **[Ch–3, Art–3.4, Th^m–1]**

(b) Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t > 0$ and C is the circle $|z| = 3$. [যদি $t > 0$ এবং $c : |z| = 3$ হলে দেখাও যে, $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$]

[Ch-3, Expl-7]

4. (a) State and prove the Rouche's theorem. **[Ch-4, Th^m-10]** If $a > e$, prove that the equation $az^n = e^z$ has n roots inside $|z| = 1$. **[Ch-4, Prob-18]** [কচি উপপাদ্যের বর্ণনা এবং প্রমাণ দাও। $a > e$ হলে প্রমাণ কর যে, $az^n = e^z$ সমীকরণের $|z| = 1$ এর মধ্যে n সংখ্যক মূল আছে।]

(b) What is Taylor's series? **[Ch-4, Th^m-1]** Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$ and determine the region of convergence. **[Ch-4, Prob-26(a)]** [টেলর ধারা বলতে কি বুঝ? $f(z) = \sin z$ ফাংশনকে $z = \frac{\pi}{4}$ বিন্দুতে টেলর ধারায় প্রকাশ কর।]

5. (a) State and prove that the Cauchy's integral formula. [Cauchy's integral formula বর্ণনা ও প্রমাণ কর।] **[Ch-3, Th^m-5]**

(b) Find the Laurent expansion of the function

$f(z) = \frac{z^2}{(z-1)(z-2)}$ is each regions [$f(z) = \frac{z^2}{(z-1)(z-2)}$ ফাংশনটির

(i) $1 < |z| < 2$ and (ii) $0 < |z| < 1$ **[Ch-4, Expl-30]**

এলাকার Laurent বিস্তার বাহির কর।]

(c) What do you mean by winding number? [Winding সংখ্যা বলতে কি বুঝ?] **[Ch-3, Art-3.11]** Discuss the properties of winding number. [ইহার ধর্মসমূহ আলোচনা কর।] **[Ch-3, Art-3.11]**

6. Evaluate any two of the following by Contour Integration method [নিম্নের যে-কোন দুইটির Contour Integration পদ্ধতিতে মান নির্ণয় কর।]

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ **[CH-5, Prob-2]**

(ii) $\int_0^{\infty} \frac{dx}{x^4 + a^4}$ **[CH-5, Prob-29]**

(iii) $\int_0^{\infty} \frac{\sin mx}{x}\,dx$. **[CH-5, Prob-51]**

NUH-2005 (OLD)

7. (a) Define an analytic function. **[Art-2.5, Def^n]** State and prove the necessary conditions for f(z) to be analytic. **[Ch-2, Th^m-4]**

(b) Show that the function

$$f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \text{ for } z \neq 0$$
$$f(0) = 0 \text{ for } z = 0$$

is not differentiable at the origin although Cauchy-Riemann equations are satisfied at that point. **[Ch-2, Expl-14]**

8. (a) State and prove Taylor's Theorem for expansion of an analytic function. **[Ch-4, Th^m-1]**

(b) Expand $f(z) = \frac{1}{(z+2)(z^2+1)}$ in Laurent series for the region $1 < |z| < 2$. **[Ch-4, Prob-34]**

9. (a) Define singular point **[Ch-4, Art-4.1, Def^n]** and pole **[Art-4.2, Def^n]**. Find the nature and location of the singularities of the function $f(z) = \frac{1}{z(e^z - 1)}$. **[Ch-4, Prob-9]**

(b) Show that the transformation $w = \frac{2z+3}{z-4}$ changes the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$ and explain why the curve obtained is not a straight line. **[Ch-6, Expl-17]**

10. (a) Define residue of a complex function. Derive the formula to find the residue of a function at a pole of order m. **Ch-4, Def^n and Th^m-5]**

(b) Evaluate any two of the following by contour integration :

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$ **[Ch-5, Prob-4]**

(ii) $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx,\ a > 0$ **[Ch-5, Prob-40(i)]**

(iii) $\int_0^{\infty} \frac{x \sin x}{x^2 + 4}\,dx$ **[Ch-5, Prob-49]**

(iv) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)}\,dx$. **[Ch-5, Prob-53]**

NUH–2005 (NEW)

1. (a) Describe the region determined by the relations [সম্পর্কদ্বয় দ্বারা প্রকাশিত অঞ্চলের বর্ণনা দাও ।]

(i) $\dfrac{|z-1|}{|z+1|}=2$ **[Ch–1, Expl–37]**

(ii) $|z+1|+|z-1|\le 3.$ **[Ch–1, Expl–33(xxiv)]**

(b) For any complex numbers z_1 and z_2 prove that [যে-কোন বাস্তব সংখ্যা z_1 এবং z_2 এর জন্য প্রমাণ কর যে]

$|z_1+z_2|\le|z_1|+|z_2|.$ **[Ch–1, Expl–22(i)]**

(c) Find all values of $(1+i)^{1/4}$. [$(1+i)^{1/4}$ এর সকল মান নির্ণয় কর ।]

[Ch–1, Expl–9]

2. (a) What do you mean by the analyticity of a complex function at a point? **[Art–2.5, Def^n]** State and prove the necessary conditions for f(z) to be analytic. **[Ch–2, Th^m–4]** [একটা জটিল ফাংশনের কোন বিন্দুতে analyticity বলতে কি বুঝ? f(z) ফাংশন analytic হওয়ার প্রয়োজনীয় শর্ত বর্ণনা কর এবং প্রমাণ কর ।]

(b) Let $f(z)=u+iv=\dfrac{x^3-3xy+i(y^3-3x^2y)}{x^2+y^2}$, when $z\ne 0$ and $f(0)=0$, when $z=0$. Show that f(z) is continuous and the Cauchy-Riemann equations are satisfied but f(z) is not differentiable at $z=0$. [ধরি $f(z)=u+iv=\dfrac{x^3-3xy+i(y^3-3x^2y)}{x^2+y^2}$, যখন $z\ne 0$ এবং $f(0)=0$, যখন $z=0$. প্রমাণ কর যে, f(z) অবিচ্ছিন্ন এবং কচি-রীমান সমীকরণসমূহ সিদ্ধ করে কিন্তু $z=0$ বিন্দুতে f(z) অন্তরীকরণযোগ্য নয় ।] **[Ch–2, Expl–13]**

3. (a) What is Cauchy's integral formula? **[Ch–3, Th^m–7]** Derive the formula to find residue of a function at a pole of order m. **[Ch–4, Th^m–5]** [কচির সমাকলন সূত্র কি? একটি ফাংশনের m মাত্রিক পোলে রেসিডিউ নির্ণয়ের ফর্মুলা প্রতিপাদন কর ।]

(b) Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for $1<|z|<3$ and $|z|>3$. [$f(x)=\dfrac{1}{(x+1)(z+3)}$ কে $1<|z|<3$ ও $|z|>3$ অঞ্চলে লরেন্ট ধারায় বিস্তার কর ।] **[Ch–4, Prob–35(i), (ii)]**

4. (a) Evaluate the integrals [এর মান নির্ণয় কর ।]

$$\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2}\,dz,\ c=\{z: z=1+2e^{i\theta},\ 0\le\theta\le 2\pi\}$$

using Cauchy's residue theorem. [কচির রেসিডিউ সূত্র ব্যবহার করে]

[Ch–4, Prob–23]

(b) What is bilinear transformation? **[Art–6.5, Def^n]** Why is it called so? **[Ch–6, Expl–1]** Find the fixed points of the bilinear transformation $W=\dfrac{2z-5}{z+4}$. If a and b are these points, write the equation in the form $\dfrac{w-a}{w-b}=k\left(\dfrac{z-a}{z-b}\right)$, where k is a constant. [বাইলিনিয়ার ট্রান্সফরমেশন কি? এটাকে তা বলা হয় কেন? $W=\dfrac{2z-5}{z+4}$ বাইলিনিয়ার ট্রান্সফরমেশনের স্থির বিন্দুসমূহ বের কর এবং $\dfrac{w-a}{w-b}=k\left(\dfrac{z-a}{z-b}\right)$ আকারে সমীকরণ লিখ, যেখানে k একটি ধ্রুবক যদি a এবং b এ বিন্দুসমূহ হয় ।]

[Ch–6, Expl–20]

5. Evaluate any two of the following by the contour integration method [নিম্নের যে-কোন দু'টির contour integration পদ্ধতিতে মান নির্ণয় কর ।]

(a) $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$ **[Ch–5, Prob–28]**

(b) $\displaystyle\int_0^\infty \sin x^2\,dx$ **[Ch–5, Prob–58]**

(c) $\displaystyle\int_0^\infty \frac{x^{p-1}\,dx}{1+x};\ 0<p<1.$ **[Ch–5, Prob–56]**

NUH–2006 (Old)

6. (a) (i) Find two complex numbers whose sum is 4 and product is 8. [দু'টি জটিল সংখ্যা নির্ণয় কর যাদের যোগফল 4 এবং গুণফল 8]

[Ch–1, Expl–26]

(ii) If z_1 and z_2 are two complex numbers, then prove that [যদি z_1 এবং z_2 দু'টি জটিল সংখ্যা হয় তবে প্রমাণ কর যে]

$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$ **[Ch–1, Expl–28]**

(iii) Draw the sketch of the region $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$. [$\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$ এলাকাটির চিত্র অংকন কর।] **[Ch–1, Expl–33(xiii)]**

(b) Define bilinear transformation **[Ch–6, Art–6.5, Def^n]** Why it is called bilinear transformation? Find a bilinear transformation which transforms points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively. **[Ch–6, Expl–12]** [দ্বি-রৈখিক রূপান্তরের সংজ্ঞা দাও। ইহাকে কেন দ্বি-রৈখিক রূপান্তর বলা হয়? একটি দ্বি-রৈখিক রূপান্তর বাহির কর যাহা $z = 0, -i, -1$ বিন্দুকে যথাক্রমে $w = i, 1, 0$ বিন্দুতে রূপান্তর করে।]

7. (a) Define an analytic function. **[Art–2.5, Def^n]** State and prove the necessary conditions for $f(z)$ to be analytic at z_0. **[Ch–2, Th^m–4]** [একটি বৈশ্লেষিক ফাংশনের সংজ্ঞা দাও। z_0 বিন্দুতে বৈশ্লেষিক হওয়ার জন্য প্রয়োজনীয় শর্তাবলী বর্ণনা ও প্রমাণ কর।]

(b) If $f(z)$ is an analytic function of z, then prove that [$f(z)$ যদি z এর বৈশ্লেষিক ফাংশন হয়, তবে প্রমাণ কর যে]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2.$$ **[Ch–2, Expl–59]**

(c) Prove that the function $|z|^2$ is continuous everywhere, but nowhere differentiable except at the origin, where z is a complex number. [প্রমাণ কর যে, ফাংশন $|z|^2$ সর্বত্র অবিচ্ছিন্ন, কিন্তু মূলবিন্দু ছাড়া কোথাও অন্তরীকরণযোগ্য নয়, যেখানে z একটি জটিল সংখ্যা।] **[Ch–2, Expl–15]**

8. (a) State and prove Cauchy's integral formula. [কশির যোগজী সূত্র বর্ণনা ও প্রমাণ কর।] **[Ch–3, Th^m–7]**

(b) Show that $\oint_c \frac{e^{tz}}{z^2+1} dz = 2\pi i \sin t$, where c is the circle $|z| = 3$ and $t > 0$. [দেখাও যে, $\oint_c \frac{e^{tz}}{z^2+1} dz = 2\pi i \sin t$, যেখানে c একটি বৃত্ত $|z| = 3$ এবং $t > 0$.] **[Ch–3, Expl–7]**

(c) Define singular point **[Ch–4, Art–4.1, Def^n]** and pole. **[Ch–4, Art–4.2, Def^n]** Locate all singularities of the function $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ and ascertain their nature. **[Ch–4, Prob–10]** [ব্যতিচার বিন্দু ও পোলের সংজ্ঞা দাও। $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ ফাংশনটির সকল ব্যতিচার বিন্দু নির্ণয় করে প্রকৃতি নিরূপণ কর।]

9. (a) Define residue of a complex function. **[Ch–4, Art–4.5, Def$_n$]** Derive the formula to find the residue of a function at a pole of order m. **[Ch–4, Th^m–5]** [একটি জটিল ফাংশনের রেসিডিউ-এর সংজ্ঞা দাও। m ক্রমের পোলে কোন ফাংশনের রেসিডিউ নির্ণয়ের সূত্রটি নির্ণয় কর।]

(b) Evaluate any two of the following integrals using suitable Contour in each case [উপযোগী কন্টুর ব্যবহার করে নিম্নের যে-কোন দু'টির মান নির্ণয় কর]

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ **[Ch–5, as Prob–4]**

(ii) $\int_0^{\infty} \frac{\cos ax}{x^2+1} dx, a > 0$ **[Ch–5, Prob–40(i)]**

(iii) $\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx$ **[Ch–5, Prob–53]**

(iv) $\int_0^{\infty} \frac{\log(x^2+1)}{x^2+1} dx$ **[Ch–5, Sol^n–34]**

NUH–2006 (New)

1. (a) For any complex number z_1, z_2 prove that [যে-কোন জটিল সংখ্যা z_1, z_2 এর জন্য প্রমাণ কর যে]

$$|z_1 - z_2| \le |z_1| + |z_2|$$ **[Ch–1, Expl–22(iii)]**

(b) Draw a sketch of $|z - i| = |z + i|$ in the complex plane. [জটিল সমতলে $|z - i| = |z + i|$ এর চিত্র অঙ্কন কর।] **[Ch–1, Expl–33(ii)]**

(c) Find all solutions of the equation $\cosh z = 2$. [$\cos hz = 2$ সমীকরণের সকল সমাধান নির্ণয় কর।] **[Ch–1, Expl–12]**

2. (a) What do you mean by the analytic function? **[Art–2.5, Def^n]** State and prove the sufficient conditions for $f(z)$ to be analytic. **[Ch–2, Th^m–5]** [Analytic ফাংশন বলতে কি বুঝ? $f(z)$ ফাংশন Analystic হওয়ায় যথেষ্ট শর্ত বর্ণনা কর এবং প্রমাণ কর।]

(b) Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic function. Find v such $u + iv$ in analytic. [দেখাও যে, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ একটি harmonic ফাংশন। $u + iv$ ফাংশনটি analytic হওয়ার জন্য v এর মান নির্ণয় কর।] **[Ch–2, Expl–42]**

3. (a) Derive Cauchy's integral formula for the derivative of an analytic function. [একটি Analytic ফাংশন-এর অন্তরকের জন্য ক'সির সমাকলন ফর্মুলাটি প্রতিষ্ঠা কর ।] **[Ch–3, Th^m–8]**

(b) Find the Laurent expansion of the function [নিম্নোক্ত প্রতিটি এলাকায় ফাংশনটির ল'রেন্ট ধারায় বিস্তৃতি নির্ণয় কর ।]

$$f(z) = \frac{z^2 - 1}{(z + 2)(z + 3)} \text{ in each of the region}$$

$$f(x) = \frac{z^2 - 1}{(z + 2)(z + 3)}$$

(i) $2 < |z| < 3.$ **[Ch–4, Prob–31(i)]**

(ii) $|z| < 2$ **[Ch–4, Prob–31(ii)]**

(iii) $|z| > 3.$ **[Ch–4, Prob–31(iii)]**

4. (a) Define pole [পোল] **[Ch–4, Art–4.2, Def^n]** and residue [রেসিডিউ] **[Ch–4, Art–4.5, Def^n]** State and prove Cauchy's residue theorem.**[Ch–4, Th^m–6]**. [এর সংজ্ঞা দাও। কশির রেসিডিউ উপপাদ্য বর্ণনা ও প্রমাণ কর ।]

(b) Define bi-linear, transformation. **[Ch–6, Art–6.5, Def^n]** Show that the transformation $w = \frac{1 + iz}{z + i}$ maps the real axes of the z-plane onto a circle in the w-plane. Find its centre and radius. **[Ch–6, Expl–22]** [বাই-লিনিয়ার রূপান্তরের সংজ্ঞা দাও। দেখাও যে, $w = \frac{1 + iz}{z + i}$ রূপান্তর z সমতলে বাস্তব অক্ষরসমূহ হতে z সমতলে একটি বৃত্তের ছবি তৈরি কর। এর কেন্দ্র এবং ব্যাসার্ধ বের কর ।]

5. Evaluate any two of the following by the Contour-integration method [নিচের যে-কোন দু'টির কন্টুর ইনটিগ্রেশন Contour integration পদ্ধতিতে মান নির্ণয় কর ।]

(i) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ **[Ch–5, Prob–10]**

(ii) $\int_0^{\infty} \frac{dx}{x^4 + a^4}$ **[Ch–5, Prob–29]**

(iii) $\int_0^{\infty} \frac{\sin mx}{x} dx.$ **[Ch–5, Prob–51]**

NUH–2007

1. (a) For any complex number z_1, z_2 prove that [যে-কোন জটিল সংখ্যা z_1, z_2 এর জন্য প্রমাণ কর যে]

$|z_1 + z_2| \leq |z_1| + |z_2|.$ **[Ch–1, Expl–22(i)]**

(b) Describe the region determined by the relations $1 < |z + i| \leq 2$. [$1 < |z + i| \leq 2$ সম্পর্ক দ্বারা প্রকাশিত অঞ্চলের বর্ণনা দও ।] **[Ch–1, Expl–33(xv)]**

2. (a) State and prove the necessary conditions for f(z) to be analytic. [f(z) ফাংশন analytic হওয়ার প্রয়োজনীয় শর্ত বর্ণনা কর এবং প্রমাণ কর ।] **[Ch–2, Th^m–4]**

(b) Prove that $f(z) = |z|^2$ is continuous everywhere but not differentiable except at the origin. [প্রমাণ কর যে, $f(z) = |z|^2$ সর্বত্র অবিচ্ছিন্ন কিন্তু মূল বিন্দু ব্যতীত অন্তরীকরণযোগ্য নয় ।] **[Ch–2, Expl–15]**

3. (a) State and prve the Cauchy's integral formula. [কচির সমাকলন সূত্রটি বর্ণনা কর এবং প্রমাণ কর ।] **[Ch–3, Th^m–7]**

(b) Whow that $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2 + 1} dz = \sin t$, if $t > 0$ and c is the circle $|z| = 3$. [$t > 0$ এবং $c : |z| = 3$ হলে দেখাও যে, $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2 + 1} dz = \sin t$] **[Ch–3, Expl–7]**

4. (a) State and prove the Rouche's theorem. [রচির উপপাদ্যের বর্ণনা কর এবং প্রমাণ কর ।] **[Ch–4, Th^m–10]**

(b) If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$. [যদি $a > e$ হয়, তবে দেখাও যে, $az^n = e^z$ সমীকরণের $|z| = 1$ এর মধ্যে n সংখ্যক মূল আছে।] **[Ch–4, Prob–17]**

5. (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$. [$f(z) = \sin z$ ফাংশনকে $z = \frac{\pi}{4}$ বিন্দুতে টেলর ধারায় প্রকাশ কর ।]**[Ch–4, Prob–26(a)]**

(b) What do you mean by Branch point? **[Ch–2, Art–4.2, Def^n]** Prove that $f(z) = \ln z$ has a branch point at $z = 0$. **[Ch–2, Expl–2]** [ব্রাঞ্চ পয়েন্ট বলতে কি বুঝ? প্রমাণ কর যে, $f(z) = \ln z$ ফাংশনের $z = 0$ বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে ।]

6. Evaluat any two of the following by the Contour integration method [নিচের যে-কোন দুটির কন্টর ইনটিগ্রেশন পদ্ধতিতে মান নির্ণয় কর ।]

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ **[Ch-5, Prob-2]**

(ii) $\int_0^{\infty} \sin x^2 \, dx$ **[Ch-5, Prob-58]**

(iii) $\int_0^{\infty} \frac{\sin mx}{x} dx$ **[Ch-5, Prob-51]**

NUH–2008

1. (a) Describe the region determinded by the relations

(i) $\frac{|z-1|}{|z+1|} = 2$ **[Ch-1, Expl-37]**

(ii) $|z+i| + |z-i| \le 3$ **[Ch-1, Expl-33(xxix)]**

(b) Find all values of $(1 + i)^{14}$ [$(1 + i)^{14}$ এর সকল মান নির্ণয় কর ।]

[Ch-1, Expl-9]

[উপরোক্ত সম্পর্কদ্বয় দ্বারা প্রকাশিত অঞ্চলের বর্ণনা দাও ।]

2. (a) If f(z) is analytic inside and on the boundary C of a simply connected region R, then prove that [যদি f(z) ফাংশনটি সরল সংযুক্ত অঞ্চল R এর অভ্যন্তরে এবং পরিসীমা C এর উপরে বিশ্লেষিত হয়, তবে প্রমাণ কর যে]

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2} dz.$$ **[Ch-3, Th^m–8]**

(b) Define singular point **[Art-4.1]** and pole. **[Art-4.2]** Locate all the singularities of the function $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ and ascertain their nature. [ব্যতিচার বিন্দু ও পোলের সংজ্ঞা দাও। $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ ফাংশনটির সকল ব্যতিচার বিন্দু নির্ণয় কর ও তাদের প্রকৃতি নিরূপণ কর ।] **[Ch-4, Prob-10]**

3. (a) Find the Laurent expansion of the function $f(z) = \frac{z^2}{(z-1)(z-2)}$ in each region (i) $1 < |z| < 2$ and (ii) $0 < |z| < 1$. [$f(z) = \frac{z^2}{(z-1)(z-2)}$ ফাংশনটির (i) $1 < |z| < 2$ ও (ii) $0 < |z| < 1$ এলাকার Laurent বিস্তার বের কর ।] **[Ch-4, Expl-30]**

(b) What do you mean by winding number? **[Art-3.11]** Discuss the properties of winding number. **[Art-3.11]** [Winding সংখ্যা বলতে কি বুঝ? ইহার ধর্মসমূহ আলোচনা কর ।]

4. (a) Establish Taylor's theorem for a complex function. [জটিল ফাংশনের জন্য টেলর-এর সিরিজটি প্রতিষ্ঠা কর ।] **[Ch-4, Th^m–1]**

(b) Classify singularitics. Give an example of each kind. [ব্যতিক্রমী বিন্দুগুলির শ্রেণীবিভাগ কর। প্রতিটি ধরণের একটি করে উদাহরণ দাও ।]

[Exercise-4 : 1]

5. (a) State and prove that Cauchy-Riemann equations in polar form. [পোলার স্থানাংক Cauchy-Riemann সমীকরণগুলোর বর্ণনা ও প্রমাণ কর ।] **[Art-2.6]**

(b) Evaluate the integrals $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} dz$, $C = \{z : z = 1 + 2e^{i0}, 0 \le 0 \le 2\pi\}$ using Cauchy's Residue theorem. [কচির রেসিডিউ সূত্র ব্যবহার করে $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} dz$, $C = \{z : z = 1 + 2e^{i0}, 0 \le 0 \le 2\pi\}$ এর মান নির্ণয় কর ।]

[Ch-4, Prob-23]

6. Evaluate any two of the following by Contour Integration method [নিচের যে-কোন দু'টির Contour Integration পদ্ধতিতে মান নির্ণয় কর ।]

(i) $\int_0^{\infty} \frac{dx}{x^4 + a^4}$ **[Ch-5, Prob-29]**

(ii) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ **[Ch-5, Prob-10]**

(iii) $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx; 0 < p < 1$ **[Ch-5, Prob-56]**

NUH–2009

1. (a) If $z = x + iy$, prove that $|x| + |y| \le \sqrt{2}\,|x + iy|$. [যদি $z = x + iy$ হয় তাহলে প্রমাণ কর $|x| + |y| \le \sqrt{2}\,|x + iy|$.] **[Ch-1, Expl-20]**

(b) Describe graphically the region represented by $1 < |z + i| \le 2$. [$1 < |z + i| \le 2$ দ্বারা নির্দেশিত রিজিয়নটিকে লৈখিকভাবে বর্ণনা কর ।]

[Ch-1, Expl-33(xv)]

(c) What is Branch Point? **[Def^n of Art–4.2]** Prove $f(z) = \ln z$ has a branch point at $z = 0$. **[Ch–2, Expl–2]** [ব্রাঞ্চ পয়েন্ট বলতে কি বুঝ? প্রমাণ কর যে, $f(z) = \ln z$ ফাংশনের $z = 0$ বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে।]

2. (a) Explain the meaning of the word analyticity at a point of domain of a complex function **[Def^n of Art–2.5]** and prove the necessary conditions for a complex function $f(z)$ to be analytic. **[Ch–2, Th^m–4]** [জটিল ফাংশনের ডোমেনের একটি বিন্দুতে analyticity শব্দটির ব্যাখ্যা কর এবং $f(z)$ জটিল ফাংশনটি analytic হবার necessary শর্তটি প্রমাণ কর।]

(b) Show that the function $\psi = \ln[(x - 1)^2 + (y - 2)^2]$ is a harmonic other than the point (1, 2) and find a function ϕ such that $\phi + i\psi$ is analytic. [দেখাও যে (1, 2) বিন্দু ব্যতীত $\psi = \ln[(x - 1)^2 + (y - 2)^2]$ ফাংশনটি harmonic এবং $\phi + i\psi$ analytic হলে ϕ ফাংশনটি নির্ণয় কর।] **[Ch–2, Expl–47]**

3. (a) State and prove Cauchy's theorem. **[Ch–3, Th^m–5]** If $f(z)$ is integrable along a curve having length L and $f(z) \le M$ on c, then prove [কচির উপপাদ্যের বর্ণনা এবং প্রমাণ দাও। প্রমাণ কর যে, $f(z)$ একটি কার্ভ c এর উপর যোগজীকরণ হলে যার দৈর্ঘ্য L এবং $f(z) \le M$ তবে]

$$\left|\int_C f(z)\, dz\right| \le ML. \qquad \textbf{[Ch–3, Th}^m\textbf{–1]}$$

(b) What do you mean by Winding number? **[Art–3.11]** Discuss the properties of winding number. **[Art–3.11]** [Winding সংখ্যা বলতে কি বুঝ? ইহার ধর্মসমূহ আলোচনা কর।]

4. (a) State and prove the Rouche's theorem. [রূচির উপপাদ্যের বর্ণনা দাও এবং প্রমাণ কর।] **[Ch–4, Th^m–10]**

(b) If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$. [যদি $a > e$ হয় তবে দেখাও যে, $az^n = e^z$ সমীকরণের $|z| = 1$ এর মধ্যে n সংখ্যক মূল আছে।] **[Ch–4, Expl–17]**

5. (a) State Taylor's theorem for the complex function $f(z)$. **[Ch–4, Th^m–1]** Expand $f(z) = \ln(1 + z)$ in Taylor's series about $z = 0$ and determine the region of convergence for the series. [$f(z)$ জটিল ফাংশনটির জন্য Taylor উপপাদ্যটি বর্ণনা কর। $z = 0$ বিন্দুর চারিদিকে $f(z) = \ln(1 + z)$ ফাংশনটিকে Taylor এর ধারায় বিস্তার কর এবং ধারাটির জন্য convergence রিজিয়ন নির্ণয় কর।]

(b) Define conformal mapping **[Art–6.2]** and the bilinear transformation. **[Art–6.5]** Prove that the bilinear transformation can be considered as a combination of the transformation of translation, rotation, stretching and inversion. **[Def^n of Art–6.6]** [Conformal mapping এবং bilinear রূপান্তরের সংজ্ঞা দাও। প্রমাণ কর যে, bilinear রূপান্তরকে translation, rotation, stretching এবং inversion এর combination হিসেবে বিবেচনা করা যেতে পারে।]

6. Evaluate any two of the following by the contour integration method [Contour integration method দ্বারা নিম্নের যে কোনো দু'টোর মান নির্ণয় কর।]

(i) $\displaystyle\int_0^{2\pi} \frac{1}{(5 - 3\sin\theta)^2}\, d\theta$ **[Ch–5, Group–A, Prob–8]**

(ii) $\displaystyle\int_0^{\infty} \frac{dx}{x^6 + 1}$ **[Ch–5, Group–B, Prob–31]**

(iii) $\displaystyle\int_0^{\pi/2} \frac{\sin z}{z}\, dz$ **[Ch–5, Expl–50 (Write z for x)]**

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১। (ক) z_1 এবং z_2 দুটি জটিল সংখ্যা হলে, প্রমাণ কর যে,
$|z_1 - z_2| \le |z_1| + |z_2|$.

(খ) জটিল সমতলে $|z - i| = |z + i|$ এর চিত্র অংকন কর।

(গ) $\cosh z = 2$ সমীকরণের সকল সমাধান কর।

২। (ক) এনালিটিক ফাংশনের সংজ্ঞা দাও। $f(z)$ ফাংশনটি এনালিটিক হওয়ার প্রয়োজনীয় শর্তগুলি বর্ণনা কর ও প্রমাণ কর।

(খ) প্রমাণ কর যে, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ ফাংশনটি harmonic। যদি $u + iv$ ফাংশনটি এনালিটিক হয় তবে y নির্ণয় কর।

(গ) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$ রিজিয়নটির বর্ণনা কর ও চিত্র আঁকা।

৩। (ক) যদি $f(z)$ ফাংশনটি simply connected region R এর ভিতরে C সীমারেখার উপরে ও ভিতরে এনালিটিক হয়, তবে প্রমাণ কর যে,

$$f'(a) = \frac{1}{2\pi} \cdot \oint_C \frac{f(z)}{(z - a)^2}\, dz.$$

(খ) $f(z) = \dfrac{z^2 - 1}{(z + 2)(z + 3)}$ ফাংশনটির (i) $2 < |z| < 3$, (ii) $|z| < 2$ ও (iii) $|z| > 3$ অঞ্চলের Laurent বিস্তার বাহির কর।

৪। (ক) $f(z) = \sin z$ ফাংশনকে $z = \dfrac{\pi}{4}$ বিন্দুতে টেলর ধারায় প্রকাশ কর।

(খ) বাই-লিনিয়ার রূপান্তরের সংজ্ঞা দাও। দেখাও যে, $w = \dfrac{1 + iz}{z + i}$ রূপান্তর z সমতলে বাস্তব অক্ষসমূহ হতে z সমতলে একটি বৃত্তের ছবি তৈরি করে। এর কেন্দ্র এবং ব্যাসার্ধ বের কর।

৫। (ক) পোলার স্থানাঙ্ক Cauchy-Riemann সমীকরণগুলোর বর্ণনা ও প্রমাণ কর।

(খ) কচির রেসিডিউ সূত্র ব্যবহার করে $\oint_C \dfrac{e^{-iz}}{(z + 3)(z - i)^2} dz$, $c = \{z : z = 1 + 2e^{i\theta}, 0 \le \theta \le 2\pi\}$ এর মান নির্ণয় কর।

৬। নিচের যে কোনো দুইটি Contour integral সমাধান কর ঃ

(ক) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta}$ (খ) $\displaystyle\int_0^{\infty} \frac{dx}{a^4 + x^4}, a > 0$

(গ) $\displaystyle\int_0^{\infty} \frac{\ln(1 + x)}{1 + x^2} dx$

[English Version]–2010

1. (a) For any complex number z_1, z_2, prve that, $|z_1 - z_2| \le |z_1| + |z_2|$. **[Ch–1, Expl–22]**

(b) Draw a sketch of $|z - i| = |z + i|$ in the complex plane. **[Ch–1, Ex–33(ii)]**

(c) Find all solutions of the equation $\cosh z = 2$. **[Ch–1, Expl–12]**

2. (a) Define analytic function. **[Ch–2, Art–2.5]** State and prove the necessary conditions for f(z) to be analytic **[Ch–2, Th^m–4]**

(b) Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Find v such that u + iv is analytic. **[Ch–2, Expl–42]**

(c) Describe the region $\text{Re}\left(\dfrac{1}{z}\right) < \dfrac{1}{2}$ and sketch it. **[Ch–1, Expl–33(xiii)]**

3. (a) If f(z) is analytic inside and on the boundary C of a simply connected region R, then prove that–

$$f'(a) = \frac{1}{2\pi}\cdot\oint_C \frac{f(z)}{(z - a)^2} dz.$$ **[Ch–3, Th^m–8]**

(b) Find the Laurent expansion of the function $f(z) = \dfrac{z^2 - 1}{(z + 2)(z + 3)}$ in each region (i) $2 < |z| < 3$, (ii) $|z| < 2$ and (iii) $|z| > 3$. **[Ch–3, Expl–31]**

4. (a) Expand $f(z) = \sin z$ in a Taylor series, about $z = \dfrac{\pi}{4}$. **[Ch–3, Expl–26(a)]**

(b) What is bilinear transformation? **[Ch–6, Art–6.5]** Show that the transformation $w = \dfrac{1 + iz}{z + i}$ maps the real axes of the z-plane onto a circle in the w-plane. Find its centre and radius. **[Ch–6, Expl–22]**

5. (a) State and prove the Cauchy-Riemann equation in polar form. **[Ch–2, Art–2.6]**

(b) Evaluate the integral $\oint_C \dfrac{e^{-iz}}{(z + 3)(z - i)^2} dz$, $c = \{z : z = 1 + 2e^{i\theta}, 0 \le \theta \le 2\pi\}$ using Cauchy's residue theorem. **[Ch–4, Expl–23]**

6. Evaluate any two of the following by the method of contour integration :–

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta}$ **[Ch–5, Expl–12 (Put a = 3, b = 2)]**

(b) $\displaystyle\int_0^{\infty} \frac{dx}{a^4 + x^4}, a > 0$ **[Ch–5, Expl–29]**

(c) $\displaystyle\int_0^{\infty} \frac{\ln(1 + x)}{1 + x^2} dx$ **[Ch–5, Exercise–22]**

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১। (ক) দেখাও যে, $\left|\dfrac{z_1}{z_2 + z_3}\right| \le \dfrac{|z_1|}{||z_2| - |z_3||}$, যেখানে z_1, z_2, z_3 হইল জটিল সংখ্যা এবং $|z_2| \ne |z_3|$.

(খ) z একটি জটিল সংখ্যা হলে $\left|\frac{z-1}{z+1}\right|$ = ধ্রুব এবং amp $\left(\frac{z-1}{z+1}\right)$ = ধ্রুব-এর সঞ্চার পথসমূহ নির্ণয় কর এবং দেখাও যে, উহারা পরস্পরকে লম্বভাবে ছেদ করে।

(গ) $\left(\frac{2+i}{3-i}\right)^2$ জটিল সংখ্যাটির মডুলাস ও আর্গুমেন্ট বাহির কর।

২। (ক) দেখাও যে, ধ্রুব মানাংকবিশিষ্ট একটি বিশ্লেষণযোগ্য ফাংশন ধ্রুব।

(খ) দেখাও যে, যদিও $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ যখন $z \neq 0$ এবং $f(0) = 0$ ফাংশনটি মূলবিন্দুতে কচিরাইম্যান সমীকরণকে সিদ্ধ করে তথাপিও উহা মূলবিন্দুতে অন্তরীকরণযোগ্য নয়।

(গ) হারমোনিক ফাংশনের সংজ্ঞা দাও। $u = e^{-x}(x \sin y - y \cos y)$ ফাংশনটি হারমোনিক কিনা নির্ণয় কর।

৩। (ক) কচির উপপাদ্যটি বর্ণনা ও প্রমাণ দাও। প্রমাণ কর যে, f(z) একটি বক্ররেখা C এর উপর যোগজীকরণ হলে যার দৈর্ঘ্য L এবং $|f(z)| \leq M$ তবে

$$\left|\int_c f(z)\, dz\right| \leq ML.$$

(খ) যদি $t > 0$ এবং $c : |z| = 3$ হলে দেখাও যে, $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2+1}\, dz = \sin t.$

৪। (ক) লিউভিলের উপপাদ্যটি বর্ণনা ও প্রমাণ কর।

(খ) বিভিন্ন প্রকার ব্যতিক্রমী বিন্দুগুলির বর্ণনা দাও।

৫। (ক) একটি জটিল ফাংশনের রেসিডিউ-এর সংজ্ঞা দাও। m ক্রমের পোলে কোনো ফাংশনের রেসিডিউ নির্ণয়ের সূত্রটি নির্ণয় কর।

(খ) $f(z) = \frac{1}{(z+2)(z^2+1)}$ ফাংশনটির (i) $1 < |z| < 2$; এবং (ii) $|z| > 2$ অঞ্চলের Laurent বিস্তার বাহির কর।

৬। (ক) কচির রেসিডিউ উপপাদ্যটি বর্ণনা ও প্রমাণ কর।

(খ) নিচের যে কোনো একটির কন্টুর ইন্টিগ্রাল সমাধান কর ঃ

(i) $\int_0^\infty \frac{\cos mx}{x^2+1}\, dx,\ m > 0,$ (ii) $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx,\ 0 < p < 1$

(iii) $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta$

NUH [English Version]–2011

1. (a) Show that $\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{||z_2| - |z_3||}$, where z_1, z_2, z_3 are complex number with $|z_2| \neq |z_3|$. **[Ch–1, Expl–25]**

(b) If z is a complex number find the locuses of $\left|\frac{z-1}{z+1}\right|$ = constant and amp $\left(\frac{z-1}{z+1}\right)$ = constant. Show that they cut orthogonally each other. **[Ch–1, Ex–43]**

(c) Find the modulus and argument of the complex number $\left(\frac{2+i}{3-i}\right)^2$. **[Ch–1, Expl–4]**

2. (a) Show that an analytic function with constant modulus is constant. **[Ch–2, Expl–26]**

(b) Show that the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ if $z \neq 0$ and $f(0) = 0$ is not analytic at the origin although it satisfies the Cauchy-Riemann equations at the origin. **[Ch–2, Expl–14]**

(c) Define harmonic function. Determine whether or not $u = e^{-x}(x \sin y - y \cos y)$ is harmonic. **[Ch–2, Expl–50]**

3. (a) State and prove Cauchy's theorem. **[Ch–3, Th^{m}–5]**

Prove that if f(z) is integrable along a curve C having length L and $|f(z)| \leq M$ on C then $\left|\int_c f(z)\, dz\right| \leq ML$. **[Ch–3, Th^{m}–1]**

(b) Show that $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2+1}\, dz = \sin t$, where c is the circle $|z| = 3$ and $t > 0$. **[Ch–3, Expl–7]**

4. (a) State and prove Liouville's theorem. **[Ch–3, Th^{m}–11]**

(b) Discuss the different kinds of singularities. **[Ch–4, Art–4.2]**

5. (a) Define residues of a complex number. **[Ch–4, Art–4.5]** Derive the formula to find the residue of a function at a pole of order m. **[Ch–4, Th^{m}–5]**

(b) Find the Laurent expansion of the function

$f(z) = \dfrac{1}{(z+2)(z^2+1)}$ at the region (i) $1 < |z| < 2$; and (ii) $|z| > 2$. **[Ch-4, Prob-34]**

6. (a) State and prove Cauchy's residue theorem. **[Ch-4, Th^m-6]**

(b) Solve the Cauchy's integral any one of the following : –

(i) $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx,\ a > 0$ **[Ch-5, Expl-40(i)]**

(ii) $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx,\ 0 < p < 1$ **[Ch-5, Expl-56]**

(iii) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,d\theta$ **[Ch-5, Expl-20]**

NUH–2012 (Old)

১। (ক) যদি $z = x + iy$ হয়, তাহলে প্রমাণ কর ঃ $|x| + |y| \le \sqrt{2}\,|x + iy|$.

(খ) $1 < |z + i| \le 2$ দ্বারা নির্দেশিত রিজিয়নটিকে লৈখিকভাবে বর্ণনা কর।

(গ) $(1 + i)^{1/4}$ এর সকল মান নির্ণয় কর।

২। (ক) এনালিটিক ফাংশনের সংজ্ঞা দাও। $f(z)$ ফাংশনটি এনালিটিক হওয়ার প্রয়োজনীয় শর্তগুলি বর্ণনা কর ও প্রমাণ কর।

(খ) প্রমাণ কর যে, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ ফাংশনটি harmonic. যদি $u + iv$ ফাংশনটি এনালিটিক হয়, তবে v নির্ণয় কর।

(গ) $R_e\left(\dfrac{1}{z}\right) \le \dfrac{1}{2}$ রিজিয়নটির বর্ণনা কর ও চিত্র আঁক।

৩। (ক) জটিল ফাংশনের জন্য টেলর-এর সিরিজটি প্রতিষ্ঠা কর।

(খ) $f(z) = \sin z$ ফাংশনকে $z = \dfrac{\pi}{4}$ বিন্দুতে টেলর ধারায় প্রকাশ কর।

৪। (ক) কচির ইনটিগ্র্যাল সূত্রটি বর্ণনা কর এবং প্রমাণ কর।

(খ) যদি $a > e$ হয়, তবে দেখাও যে, $az^n = e^z$ সমীকরণের $|z| = 1$ এর মধ্যে n সংখ্যক মূল আছে।

৫। (ক) পোলার স্থানাংক Cauchy-Riemann সমীকরণগুলোর বর্ণনা ও প্রমাণ কর।

(খ) Winding সংখ্যা বলতে কি বুঝ? ইহার ধর্মসমূহ আলোচনা কর।

৬। কন্টুর ইন্টিগ্রেশন পদ্ধতিতে নীচের যে কোনো দুটির মান নির্ণয় কর ঃ

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$, (ii) $\displaystyle\int_0^\infty \frac{dx}{x^4 + a^4}$, (iii) $\displaystyle\int_0^{\pi/2} \frac{\sin z}{z}\,dz$

NUH [English Version]–2012 (Old)

1. (a) If $z = x + iy$, Prove that $|x| + |y| \le \sqrt{2}\,|x + iy|$. **[Ch-1, Expl-20]**

(b) Describe graphically the region represented by $1 < |z + i| \le 2$. **[Ch-1, Expl-33(xv)]**

(c) Find all values of $(1 + i)^{1/4}$. **[Ch-1, Expl-9]**

2. (a) Define analytic function. **[Ch-2, Art-2.5]** State and prove the necessary conditions for $f(z)$ to be analytic. **[Ch-2, Th^m-4]**

(b) Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Find v such that $u + iv$ analytic. **[Ch-2, Expl-42]**

(c) Describe the region $R_e\left(\dfrac{1}{z}\right) < \dfrac{1}{2}$ and sketch it. **[Ch-1, Expl-33(xiii)]**

3. (a) Establish Taylor's theorem for a complex function. **[Ch-4, Th^m-1]**

(b) Expand $f(z) = \sin z$ in a Taylor series about $z = \dfrac{\pi}{4}$. **[Ch-4, Prob-26(a)]**

4. (a) State and prove Cauchy's Integral Formula. **[Ch-3, Th^m-7]**

(b) If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$. **[Ch-4, Prob-17]**

5. (a) State and prove the Cauchy-Riemann equation in polar form. **[Ch-2, Art-2.6]**

(b) What do you mean by winding number? Discuss the properties of winding number. **[Ch-3, Art-3.11]**

6. Evaluate any two of the following by the Contour integration method :

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ **[Ch-5, Expl-2]**

(ii) $\displaystyle\int_0^{\infty} \frac{dx}{x^4 + a^4}$ **[Ch-5, Expl-29]**

(iii) $\displaystyle\int_0^{\pi/2} \frac{\sin z}{z}\,dz$ **[Ch-5, Expl-50]**

NUH–2012 (NEW)

ক বিভাগ

[যে কোন দশটি প্রশ্নের উত্তর দাও।] [$10 \times 1 = 10$]

১। (ক) একটি জটিল সংখ্যার মডুলাসের সংজ্ঞা দাও।

(খ) $z = x + iy$ হলে $|z|$ এবং $|\bar{z}|$ সম্বন্ধে মন্তব্য কর।

(গ) কোনো বিন্দুতে বিশ্লেষিক অপেক্ষকের সংজ্ঞা দাও।

(ঘ) পোলার স্থানাংক কচি-রীম্যান সমীকরণগুলি লিখ।

(ঙ) অনুবন্ধী হারমোনিক ফাংশনের সংজ্ঞা দাও।

(চ) কি শর্তে $\oint_c f(z)\,dz = 0$.

(ছ) মরিবার উপপাদ্য বর্ণনা কর।

(জ) উইনডিং সংখ্যা শূন্য হয় কখন?

(ঝ) একটি জটিল ফাংশনের ব্যতিচার বিন্দুর সংজ্ঞা দাও।

(ঞ) $f(z) = \dfrac{z^2 - 3z}{z - 2}$ এর পোল কত?

(ট) দ্বিযোগাশ্রয়ী রূপান্তর কি?

(ঠ) জটিল সংখ্যার Cross ratio এর সংজ্ঞা দাও।

খ বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [$5 \times 4 = 20$]

২। যে কোনো দুটি জটিল সংখ্যা z_1 ও z_2 এর জন্য প্রমাণ কর যে,

$|z_1 + z_2| \geq |z_1| + |z_2|$.

৩। দুটি জটিল সংখ্যা নির্ণয় কর যাদের সমষ্টি 4 এবং গুণফল 8.

৪। $f(z) = u(x, y) + iv(x, y)$ ফাংশনের বৈশ্লেষিক হওয়ার জন্য প্রয়োজনীয় শর্তাবলি বর্ণনা ও প্রমাণ কর।

৫। যদি $f(z)$, z এর একটি রেগুলার ফাংশন হয়, তবে দেখাও যে

$\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

৬। $\displaystyle\int_c \frac{1}{z^2 + 9}\,dz$ এর মান নির্ণয় কর যেখানে c হল ধনাত্মক দিকে $x = \pm 2$, $y = \pm 2$ রেখা দ্বারা বর্ণিত বর্গক্ষেত্র।

৭। $f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2 - 1)^2}$ এর সিংগুলারিটি নির্ণয় কর।

৮। $f(z) = \dfrac{z^2}{z^2 + a^2}$ ফাংশনটির রেসিডিউ নির্ণয় কর।

৯। z তলের $i, -i, 1$ বিন্দুগুলোকে w তলে যথাক্রমে $0, 1, \infty$ বিন্দুত্রয়ে রূপান্তরিত করে এমন দ্বি-যোগাশ্রয়ী রূপান্তর নির্ণয় কর।

গ বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [$5 \times 10 = 50$]

১০। প্রমাণ কর যে, $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ এবং এর জ্যামিতিক ব্যাখ্যা দাও। অতপর দেখাও যে, $|\alpha + \sqrt{\alpha^2 + \beta^2}| + |\alpha - \sqrt{\alpha^2 + \beta^2}| = |\alpha + \beta|\ |\alpha - \beta|$ যেখানে z_1, z_2, α, β সংখ্যাসমূহ জটিল।

১১। (ক) জটিল সমতলে $|z - i| = |z + i|$ এর চিত্র অংকন কর।

(খ) $\cosh z = 2$ সমীকরণের সকল সমাধান নির্ণয় কর।

১২। দেখাও যে, $u(x, y)$ একটি হারমনিক ফাংশন এর হারমোনিক কনজুগেট $v(x, y)$ এবং এদের অ্যানালেটিক ফাংশন $f(z)$ নির্ণয় কর,

যেখানে $u(x, y) = x^2 - y^2 + 2e^{-x}\sin y$.

১৩। যদি $f(z)$ ফাংশনটি সরল সংযুক্ত বক্ররেখা c এর উপরে ও ভিতরে এনালিটিক হয়, তবে প্রমাণ কর যে, $f'(a) = \dfrac{1}{2\pi i}\displaystyle\oint_c \frac{f(z)}{(z - a)^2}\,dz$ যেখানে c এর অভ্যন্তরস্থ একটি বিন্দু a.

১৪। রচির উপপাদ্যের বর্ণনা দাও ও প্রমাণ কর।

১৫। $f(z) = \dfrac{z^2 + 1}{(z + 1)(z - 2)}$ ফাংশনটির (i) $1 < |z| < 2$, (ii) $0 < |z| < 1$ অঞ্চলে Laurent বিস্তার বাহির কর।

১৬। নিম্নের যে কোনো দুটির Contour Integration পদ্ধতিতে মান নির্ণয় কর ঃ

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$, (ii) $\int_0^{\infty} \frac{dx}{x^4 + 1}$, (iii) $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1} dx$

১৭। Conformal mapping ব্যাখ্যা কর এবং $w = f(z)$ ফাংশন Conformal mapping হওয়ার জন্য প্রয়োজনীয় শর্ত বর্ণনা কর এবং প্রমাণ কর।

NUH [English Version]–2012 (New)

A Part

1. (a) Define modulus of a complex number. **[Ch–1, Quiz–15]**
   (b) If $z = x + iy$ then comment about $|z|$ and $|\bar{z}|$. **[Ch–1, Quiz–16]**
   (c) Define analytic function at a point. **[Ch–2, Quiz–1]**
   (d) Write the polar form of Cauchy-Riemann equations. **[Ch–2, Quiz–4]**
   (e) Define harmonic conjugate. **[Ch–2, Quiz–14]**
   (f) Under what conditions $\oint_c f(z)\, dz = 0$? **[Ch–3, Quiz–10]**
   (g) State Morera's theorem. **[Ch–3, Quiz–15]**
   (h) When the winding number is zero? **[Ch–3, Quiz–21]**
   (i) Define singular point of a complex function. **[Ch–4, Quiz–4]**
   (j) What is the pole of $f(z) = \frac{z^2 - 3z}{z - 2}$? **[Ch–4, Quiz–15]**
   (k) What is bilinear transformation? **[Ch–6, Quiz–3]**
   (l) Define cross ratio of complex numbers. **[Ch–6, Quiz–7]**

B Part

2. For any two complex numbers z_1 and z_2 prove that $|z_1 + z_2| \le |z_1| + |z_2|$. **[Ch–1, Expl–22]**
3. Find two complex numbers whose sum is 4 and whose product is 8. **[Ch–1, Expl–26]**
4. State and prove the necessary condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic. **[Ch–2, Th^m–4]**
5. If f(z) is a regular function of z, then show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$. **[Ch–2, Expl–58]**

6. Evaluate $\int_c \frac{1}{z^2 + 9} dz$ where c denotes the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$ described in the positive sense. **[Ch–3, Expl–14(i)]**
7. Identify the singularities of $f(z) = \frac{\sin\left(\frac{1}{z}\right)}{(z^2 - 1)^2}$. **[Ch–4, Prob–8(i)]**
8. Calculate the residue of the function $f(z) = \frac{z^2}{z^2 + a^2}$. **[Ch–4, Expl–10(a)]**
9. Find a bilinear transformation which maps the points $i, -i, 1$ of the z-plane into the points $0, 1, \infty$ of w-plane respectively. **[Ch–6, Expl–16]**

C Part

10. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and interprete it geometrically. Hence show that $|\alpha + \sqrt{\alpha^2 + \beta^2}| + |\alpha - \sqrt{\alpha^2 + \beta^2}| = |\alpha + \beta|\ |\alpha - \beta|$ where z_1, z_2, α, β are complex numbers. **[Ch–1, Expl–28]**
11. (a) Draw a sketch of $|z - i| = |z + i|$ in the complex plane. **[Ch–1, Expl–33(ii)]**
    (b) Find all solutions of the equation $\cosh z = 2$. **[Ch–1, Expl–12]**
12. Show that u(x, y) is harmonic. Find its harmonic conjugate v(x, y) and the corresponding analytic function $f(z) = u + iv$ where $u(x, y) = x^2 - y^2 + 2e^{-x} \sin y$. **[Ch–2, Expl–50(i)]**
13. If f(z) is analytic inside and on a sumple closed curve c then prove that $f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z - a)^2} dz$ where a is a point inside c. **[Ch–3, Th^m–8]**
14. State and prove Rouche's theorem. **[Ch–4, Th^m–10]**
15. Find the Laurent expansion of the function $f(z) = \frac{z^2 + 1}{(z + 1)(z - 2)}$ in each of the regions : (i) $1 < |z| < 2$, (ii) $0 < |z| < 1$ **[Ch–4, Prob–37]**

16. Evaluate any two of the following by the method of Contour Integration :

(i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ **[Ch-5, Expl-2]**

(ii) $\int_0^{\infty} \frac{dx}{x^4 + 1}$ **[Ch-5, Expl-28]**

(iii) $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1}\, dx$ **[Ch-5, Expl-41]**

17. Explain conformal mapping **[Ch-6, Art-6.2]** and state and prove necessary condition for the function $w = f(z)$ to be a conformal mapping. **[Ch-6, Art-6.3 (Th^m-1)]**

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ক বিভাগ

[যে কোন দশটি প্রশ্নের উত্তর দাও।] [$1 \times 10 = 10$]

১। (ক) দেখাও যে, $|z|^2 = z\bar{z}$.

(খ) দুইটি জটিল সংখ্যার সমতা বলতে কি বুঝ?

(গ) $-2 - i$ এর আর্গুমেন্ট বের কর।

(ঘ) জটিল সংখ্যার ল্যাপলাস সমীকরণটি পোলার আকারে লিখ।

(ঙ) হারমোনিক ফাংশনের উদাহরণ দাও।

(চ) কখন একটি ফাংশনকে বিশ্লেষণ যোগ্য বলা যাবে?

(ছ) জটিল ফাংশনের ব্রাঞ্চ বিন্দু বলতে কি বুঝ?

(জ) Pole কী?

(ঝ) Widing সংখ্যার দুইটি ধর্ম লিখ।

(ঞ) জটিল ফাংশনের রেসিডিউ বলতে কি বুঝ?

(ট) কন্টুর যোগজীকরণ এর জন্য জর্ডান অসমতাটি লিখ।

(ঠ) কি শর্তে $\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0,\ m > 0.$

খ বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [$4 \times 5 = 20$]

২। যে কোনো জটিল সংখ্যা z এর জন্য, প্রমাণ কর যে, $|z|\sqrt{2} \ge |Re(z)| + |Im(z)|$

৩। একটি জটিল ফাংশন f(z) এর বৈশ্লিষিক হওয়ার যথার্থ শর্তটি বর্ণনাসহ প্রতিষ্ঠিত কর।

৪। দেখাও যে, $\oint_C \frac{e^{tz}}{z^2 + 1}\, dz = 2\pi i \sin t$, যেখানে c হচ্ছে বৃত্ত $|z| = 3$ এবং $t > 0$।

৫। কসির উপপাদ্যটি বর্ণনাসহ প্রমাণ কর।

৬। $f(z) = \frac{1}{z(e^z - 1)}$ ফাংশনটির সকল ব্যতিক্রম বিন্দুর শ্রেণিবিভক্তি করিয়া তাহা নিরূপণ কর।

৭। রেসিডিউ উপপাদ্যের বর্ণনা কর এবং প্রমাণ কর।

৮। একটি সসীম তলে $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z+4)}$ ফাংশনটি সব কয়টি পোলে উক্ত ফাংশনের রেসিডিউ নির্ণয় কর।

৯। z তলের $z = 0, 1, 2$ বিন্দুকে যথাক্রমে w তলের $w = i, 0, 1$ বিন্দুতে চিত্রিত করে এমন একটি দ্বিরৈখিক রূপান্তর নিণয় কর।

গ বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [$10 \times 5 = 50$]

১০। একটি জটিল সংখ্যা z, উহার বাস্তব অংশ এবং কাল্পনিক অংশ, উহার আর্গুমেন্ট ও পরম মান এবং সংজ্ঞা দাও। $z = \pm\sqrt{3} \pm i$ সংখ্যা চতুষ্টয় আরগেন্ড ডায়াগ্রাম চিত্রায়িত কর। ইহাদের আর্গুমেন্ট ও পরম মান নির্ণয় কর।

১১। দেখাও যে, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ একটি হারমোনিক ফাংশন। ইহার হারমোনিক অণুবন্ধী ফাংশন v এবং সংশ্লিষ্ট বৈশ্লেষিক ফাংশন $f(z) = u + iv$ নির্ণয় কর।

১২। মনে কর $f(z) = u + iv = \frac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$ যখন $z \neq 0$ এবং $f(0) = 0$, যখন $z = 0$, দেখাও যে, f(z) ফাংশনটি $z = 0$ বিন্দুতে অবিচ্ছিন্ন এবং কশি-রীমন সমীকরণগুলো সিদ্ধ হয় কিন্তু f(z) অন্তরীকরণযোগ্য নয়।

১৩। কশির যোগজ সূত্রটি কি? এ সূত্র প্রয়োগ করে $\int_C \frac{z\, dz}{(9 - z^2)(z + i)}$ এর মান নির্ণয় কর, এখানে c হচ্ছে $|z| = 2$ দ্বারা নির্দেশিত বৃত্ত যা ধনাত্মক দিকে বর্ণিত। ফাংশনের মান নির্ণয়ের ক্ষেত্রে এ সূত্রটা কতটুকু তাৎপর্যপূর্ণ?

১৪। জটিল ফাংশনের জন্য টেলর উপপাদ্যটি বর্ণনাসহ প্রমাণ কর।

১৫। z এর ঘাতে $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ এর লরেন্ট ধারা বিস্তৃতি নির্ণয় কর যেগুলো–

(i) $2 < |z| < 3$, (ii) $|z| > 3$, (iii) $|z| < 2$.

১৬। কন্টুর যোগজীকরণের সাহায্যে নিম্নের যে কোনো দুইটির মান নির্ণয় কর ঃ

(i) $\int_0^{\pi} \frac{dx}{1 + \sin^2 x}$ (ii) $\int_0^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 4)} dx$

(iii) $\int_0^{\infty} \frac{ln(1 + x)}{1 + x^2} dx$

১৭। দেখাও যে, $W = \sin z$ রূপান্তরটি অসীম স্ট্রিপ, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $y \ge 0$ কে এক-এক ভাবে w তলের উর্দ্ধ অধতল ≥ 0 তে রূপান্তরিত করে।

NUH [English Version]–2013

A Part

1. (a) Show that $|z|^2 = z\bar{z}$. **[Ch–1, Quiz–17]**
   (b) What do you mean by equality of two complex number? **[Ch–1, art–1.2 or Quiz–20]**
   (c) Find the argument of $-2 - i$. **[Ch–1, Quiz–19]**
   (d) Write down the Laplace equation in Polar form of complex number. **[Ch–2, Quiz–5]**
   (e) Give an example harmonic function. **[Ch–2, Quiz–16]**
   (f) When is a function analytic? **[Ch–2, Quiz–15]**
   (g) What is branch point of complex number? **[Ch–4, Quiz–16]**
   (h) What is pole? **[Ch–4, Quiz–7]**
   (i) Write down the two properties of Winding number. **[Ch–3, Quiz–23]**
   (j) What is meant by residue of complex number? **[Ch–4, Quiz–17]**
   (k) Write down the Jordan's inequality for contour integration. **[Ch–5, Quiz–4]**
   (l) Under what conditions $\lim_{R \to \infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$, $m > 0$.

B Part

2. For any complex number z, prove that, $|z|\sqrt{2} \ge |Re(z)| + |Im(z)|$. **[Ch–1, Expl–19]**
3. State and establish the sufficient condition for the analyticity of a complex function f(z). **[Ch–2, Th^m–5]**
4. Show that, $\oint_C \frac{e^{tz}}{z^2 + 1} dz = 2\pi i \sin t$, where c in the circle $|z| = 3$ and $t \ge 0$. **[Ch–3, Expl–7]**
5. State and prove Cauchy's theorem. **[Ch–3, Th^m–5]**
6. Determine and classify all the singularities of the function $f(z) = \frac{1}{z(e^z - 1)}$. **[Ch–4, Prob–9]**
7. State and prove the Residue theorem. **[Ch–4, Th^m–4]**
8. Find the residues of $f(z) = \frac{z^2 - 2z}{(z + 1)^2 (z + 4)}$ at all its poles in a finite plane. **[Ch–4, Expl–11]**
9. Find a bilinear transformation that maps the points $z = 0, 1, 2$ of the z plane into the points $w = i, 0, 1$, respectively, of the w-plane. **[Ch–5, Expl–12(i)]**

C Part

10. Define a complex number z, the real and imaginary part of z, **[Ch–1, Art–1.1.1 & 1.1.2]** the argument and modulus of z. **[Ch–1, Art–1.4.1]** Plot the four complex numbers $z = \pm\sqrt{3} \pm i$. Find the arguments and the modulus of these z. **[Ch–1, Expl–4(i)]**
11. Show that, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is a harmonic function. Find its conjugate harmonic function v and the corresponding analytic function $f(z) = u + iv$. **[Ch–2, Expl–42]**
12. Let $f(z) = u + iv = \frac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$ when $z \ne 0$ and f(0) when $z = 0$ show that, f(z) is continuous and Cauchy-Riemann equations are satisfied but f(z) is not differentiable at $z = 0$. **[Ch–2, Expl–13]**

13. What is Cauchy's integral formula? Using this formula evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$ where c is the circle $|z| = 2$ described in the positive sense. How significant is it in determining the value of a function. **[Ch-3, Expl-11]**

14. State and prove Taylor's theorem for a complex function. **[Ch-4, Th^m-1]**

15. Find the Laurent series expansions of $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ in power of z, valid in the region

(i) $2 < |z| < 3$, (ii) $|z| > 3$, (iii) $|z| < 2$. **[Ch-4, Prob-31]**

16. Evaluate any two of the following by contour integration–

(i) $\int_0^{\pi} \frac{dx}{1 + \sin^2 x}$ **[Ch-5, Expl-18(i)]**

(ii) $\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)}\,dx$ **[Ch-5, Expl-23(i)]**

(iii) $\int_0^{\infty} \frac{\ln(1+x)}{1+x^2}\,dx$ **[Ch-5, Exercise-22]**

17. Whow that the transformation $W = \sin z$ transforms the infinite strip $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $y \ge 0$ in one-one manner onto the upper-half $y \ge 0$ of the w-plane. **[Ch-6, Expl-36]**

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ক-বিভাগ

[যে কোন দশটি প্রশ্নের উত্তর দাও।] [$10 \times 1 = 10$]

১। (ক) $\frac{1-i}{1+i}$ এর মুখ্য নতি কত ?

(খ) উদাহরণসহ Entire ফাংশনের সংজ্ঞা দাও।

(গ) কখন একটি জটিল ফাংশন $f(z)$ কে কোনো বিন্দুতে অন্তরীকরণযোগ্য বলা হয়?

(ঘ) কসির যোগজ উপপাদ্যটি বিবৃত কর।

(ঙ) মরিরার উপপাদ্যটি বিবৃত কর।

(চ) Meromorphic ফাংশনের সংজ্ঞা দাও।

(ছ) $f(z)$ ফাংশনের $z = a$ বিন্দুতে অবস্থিত m ক্রমের পোলে রেসিডিউ নির্ণয়ের সূত্রটি লিখ।

(জ) বৈশ্লেষিক ফাংশনের শূণ্য (zero) বলতে কি বোঝায়?

(ঝ) সরল আবদ্ধ এলাকা কি?

(ঞ) এমন একটি ফাংশন লিখ যায় দুইটি স্বতন্ত্র (isolated) ব্যতিক্রম বিন্দু রয়েছে।

(ট) যদি c দ্বারা $|z| = R$ বৃত্তের $\theta_1 \le \theta \le \theta_2$ চাপ বুঝায় এবং $\lim_{z \to \infty} z\, f(z) = A$ হয়, তবে $\lim_{R \to \infty} \int_c f(z)\,dz$ এর মান কত?

(ঠ) মোবিয়াস রূপান্তর $w = \frac{az+b}{cz+d}$ কে কেন বাইলিনিয়ার রূপান্তর বলা হয় ?

খ-বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [$5 \times 4 = 20$]

২। যে কোনো দুটি জটিল সংখ্যা z_1 ও z_2 এর জন্য প্রমাণ কর যে, $|z_1 - z_2| \ge \big||z_1| - |z_2|\big| \ge |z_1| - |z_2|$.

৩। দেখাও যে, পোলার আকারে কসি-রীম্যান সমীকরণ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ এবং $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

৪। হারমোনিক কনজুগেট এর সংজ্ঞা দাও। $u = 2x(1-y)$ এর হারমোনিক কনজুগেট নির্ণয় কর।

৫। $\int_c f(z)\,dz$ ইন্টিগ্র্যালটির মান নির্ণয় কর, যেখানে $f(z) = z^2$ এবং c হলো $z = 0$ হতে $z = 1$ এবং $z = 1$ হতে $z = 1 + i$ এর সংযোগ সরলরেখাদ্বয় দ্বারা গঠিত।

৬। লরেন্ট এর উপপাদ্য বিবৃত কর। $f(z) = \frac{1}{z^2 - 3z + 2}$ ফাংশনকে $1 < |z| < 2$ অঞ্চলে লরেনট ধারায় বিস্তার কর।

৭। যদি L দৈর্ঘ্যবিশিষ্ট বক্ররেখা c বরাবর $f(z)$ ফাংশন যোগজীকরনযোগ্য এবং $|f(z)| \le M$ হয়, তবে প্রমাণ কর যে, $\left|\int_c f(z)\,dz\right| \le ML$.

৮। কসির রেসিডিউ উপপাদ্যের সাহায্যে $\oint_c \frac{e^{3z}}{z + \pi i}$ এর মান নির্ণয় কর, যেখানে c হলো $|z + 1| = 4$ বৃত্ত।

৯। দেখাও যে, $w = \frac{2z+3}{z-4}$ রূপান্তরটি $x^2 + y^2 - 4x = 0$ বৃত্তকে $4u + 3 = 0$ সরলরেখায় পরিণত করে।

গ-বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [$5 \times 10 = 50$]

১০। (ক) ধর $\mathbb{C}$ দ্বারা সকল জটিল সংখ্যার সেটকে বোঝায় $|z_1 = x_1 + iy_1$ ও $z_2 = x_2 + iy_2 \in \mathbb{C}$ এবং $x_1 < x_2$, $y_1 < y_2$ হলে, $z_1 < z_2$ বিষয়ে তুমি কি সম্মত ? $|z_1| < |z_2|$ সম্পর্কে তোমার মতামত দাও।

(খ) $|z - i| \le |z + i|$ অসমতাকে সিদ্ধ করে জটিল তলে এরূপ বিন্দুসমূহের সেট নির্ণয় কর এবং চিত্র আঁক।

১১। প্রমাণ কর যে, যদি f(z) ফাংশনে কোনো বিন্দুতে অন্তরীকরণযোগ্য হয় তবে উক্ত বিন্দুতে অবিচ্ছিন্ন হবে। কিন্তু এর বিপরীত সর্বদা সত্য নয়।

১২। $w = f(z)$ ফাংশন বৈশ্লেষিক হওয়ার প্রয়োজনীয় শর্তসমূহ বর্ণনা ও প্রমাণ কর।

১৩। কসির যোগজ সূত্রটি বর্ণনা ও প্রমাণ কর। এই সূত্রটি ব্যবহার করে দেখাও যে, $\oint_c \frac{\sin 3z}{z + \frac{\pi}{2}} = 2\pi i$ যেখানে c হলো $|z| = 5$ বৃত্ত।

১৪। লিউভিলের উপপাদ্য বর্ণনা ও প্রমাণ কর।

১৫। কচি'র উপপাদ্য বর্ণনা কর। উপপাদ্যটির সাহায্যে দেখাও যে, $z^7 - 5z^3 + 12 = 0$ সমীকরণের সকল মূল $|z| = 1$ এবং $|z| = 2$ বৃত্তদ্বয়ের মধ্যে অবস্থিত।

১৬। কন্টুর যোগজীকরণের সাহায্যে নিম্নের যে কোনো দুইটির মান নির্ণয় করঃ

(i) $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$; (ii) $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}$;

(iii) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)} dx$.

১৭। বাইলিনিয়ার রূপান্তরের সংজ্ঞা দাও। একটি বাইলিনিয়ার রূপান্তর নির্ণয় কর যা $|z| \le 1$ একক বৃত্তকে $|w| \le 1$ একক বৃত্তে রূপান্তর করে।

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A–Part

1. (a) What is the principal argument of $\frac{1 - i}{1 + i}$. **[Ch–1, Quiz–26]**

(b) Define entire function with an example. **[Ch–4, Quiz–12]**

(c) When a complex function f(z) is said to be differentiable at a point? **[Ch–2, Quiz–9]**

(d) State Cauchy's integral theorem. **[Ch–3, Quiz–10]**

(e) State Morera's theorem. **[Ch–3, Quiz–15]**

(f) Define Meromorphic function? **[Ch–4, Quiz–11]**

(g) Write the formula to find the residue of f(z) at the pole z = a of order m. **[Ch–4, Quiz–18]**

(h) What is meant by zero of an analytic function? **[Ch–4, Quiz–1]**

(i) What is simply connected region? **[Ch–3, Quiz–24]**

(j) Write a function which has two isolated singularity. **[Ch–4, Quiz–19]**

(k) If c is arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$ and if $\lim_{z \to \infty} z f(z) = A$ then what is the value of $\lim_{R \to \infty} \int_c f(z)\, dz$? **[Ch–5, Quiz–9]**

(l) Why the Mobius transformation $w = \frac{az + b}{cz + b}$ is called bilinear transformation? **[Ch–6, Quiz–10]**

B–Part

2. For any two complex numbers z_1 and z_2 prove that $|z_1 - z_2| \ge ||z_1| - |z_2|| \ge |z_1| - |z_2|$. **[Ch–1, Expl–24]**

3. Show that Cauchy-Riemann equations in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ **[Ch–2, Art–2.6]**

4. Define harmonic conjugate**[Ch–2, art–2.7]**. Find the harmonic conjugate of $u = 2x(1 - y)$. **[Ch–2, Expl–48]**

5. Evaluate the integral $\int_c f(z)\, dz$ where $f(z) = z^2$ and c is the line segment from z = 0 to z = 1 and then z = 1 to z = 1 + i **[Ch–3, Expl–2(i)]**

6. State Laurent's theorem **[Ch–4, Th^m–2]**. Expand the function f(z) $\frac{1}{z^2 - 3z + 2}$ in a Laurent's series for the region $1 < |z| < 2$ **[Ch–4, Expl–30(i)]**

7. If f(z) is integrable along a curve c having length L and $|f(z)| < M$ on c, then prove that $\left|\int_c f(z)\, dz\right| \le ML$.**[Ch–3, Th^m–1]**

8. Evaluate by Cauchy's residue theorem $\oint_c \frac{e^{3z}}{z + \pi i}$ where c is the circle $|z + 1| = 4$. **[Ch–3, Expl–16]**

9. Show that the transformation $w = \frac{2z + 3}{z - 4}$ changes the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$ **[Ch–6, Expl–17]**

C–Part

10. (a) Let $\mathbb{C}$ be the set of all complex-numbers. Consider $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbb{C}$ with $x_1 < x_2$, $y_1 < y_2$. Do you agree that $z_1 < z_2$? What about $|z_1| < |z_2|$? **[Ch–1, Expl–23]**
(b) Determine the set of all points in the complex plane which satisfy the inequality $|z - i| \le |z + i|$ and draw the graph. **[Ch–1, Expl–35]**

11. Prove that, if f(z) is differentiable at a point, then it is continuous there. But the converse is not necessarily true. **[Ch–2, Th^m–3 and Expl–21 (2nd part)]**

12. State and prove the necessary conditions for the analyticity of the complex function $w = f(z)$ **[Ch–2, Th^m–4]**

13. State and prove Cauchy's integral formula. **[Ch–3, Th^m–7]** Using this formula show that $\oint_c \frac{\sin 3z}{z + \frac{\pi}{2}} = 2\pi i$ where c is the circle $|z| = 5$. **[Ch–3, Expl–5]**

14. State and prove Liouville's theorem. **[Ch–3, Th^m–11]**

15. State Rouche's theorem. **[Ch–4, Th^m–10]** Using the theorem show that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. **[Ch–4, Prob–16]**

16. Evaluate any two of the following by Contour integration :

(i) $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$; **[Ch–5, Expl–9]**

(ii) $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}$; **[Ch–5, Expl–30(i)]**

(iii) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)} dx$. **[Ch–5, Expl–53]**

17. Define bilinear transformation**[Ch–6, Art–6.5]**.
Find a bilinear transformation which transform the unit circle (dise) $|z| \le 1$ into the unit circle $|w| \le 1$. **[Ch–6, Expl–29]**

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ক বিভাগ

[যে কোন দশটি প্রশ্নের উত্তর দাও।] [10 × 1 = 10]

১। (ক) $(1 + i)^{2/3}$ এর কতটি মান আছে?
(খ) কোনো বিন্দুতে বৈশ্লেষিক ফাংশনের সংজ্ঞা দাও।
(গ) $w = f(z)$ ফাংশন বৈশ্লেষিক হওয়ার প্রয়োজনীয় শর্ত কি?
(ঘ) বদ্ধ কন্টুর এর সংজ্ঞা দাও।
(ঙ) এমন একটি জটিল ফাংশন লিখ যার $z = 1$ বিন্দুতে একটি শূন্য (zero) আছে কিন্তু কোনো ব্যতিক্রম বিন্দু নাই।
(চ) জটিল ফাংশনের পোলের সংজ্ঞা দাও।
(ছ) জটিল ফাংশনের পোলে রেসিডিউ এর সংজ্ঞা দাও।
(জ) রূপান্তর বা চিত্রণ কাকে বলে?
(ঝ) রসির উপপাদ্য বর্ণনা কর।
(ঞ) বীজগণিতের মৌলিক উপপাদ্য বর্ণনা কর।
(ট) সর্বোচ্চ মডুলাস নীতি বর্ণনা কর।
(ঠ) $w = f(z)$ কনফরমাল conformal চিত্রন হওয়ার প্রয়োজনীয় শর্ত কি?

খ বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [5 × 4 = 20]

২। জটিল সমতলে $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ অসমতাকে সিদ্ধ করে এমন সকল বিন্দুর সেট নির্ণয় কর এবং চিত্র আঁক।

৩। প্রমান কর যে, ধ্রুবক মডুলাসবিশিষ্ট প্রত্যেক বৈশ্লেষিক ফাংশন ধ্রুবক।

৪। Winding সংখ্যার সংজ্ঞা দাও। চিত্রসহ বণনা কর কখন Winding সংখ্যা ধনাত্মক, ঋণাত্মক ও শূন্য হয়।

৫। জটিল ফাংশন f(z) এর জন্য টেলর উপপাদ্য বর্ণনা কর। $z = 0$ বিন্দুর প্রতিবেশে $f(z) = \ln(1 + z)$ ফাংশনটিকে টেলর ধারায় বিস্তাব কর।

৬। প্রথম অন্তরজের জন্য কসির সমাকলন সূত্রটি বর্ণনা ও প্রমাণ কর।

৭। $f(z) = \frac{z^2}{(z + 1)^2} \sin\left(\frac{1}{z - 1}\right)$ ফাংশনের সকল ব্যতিক্রম বিন্দু এবং তাদের প্রকৃতি নির্ণয় কর।

৮। কসির রেসিডিউ উপপাদ্য বর্ণনা ও প্রমাণ কর।

৯। দ্বিযোগাশ্রয়ী রূপান্তর নির্ণয় কর যাহা 1, i, – 1 বিন্দুকে যথাক্রমে i, 0, – i এ রূপান্তর করে।

গ বিভাগ

[যে কোন পাঁচটি প্রশ্নের উত্তর দাও।] [5 × 10 = 50]

১০। প্রমাণ কর যে, $f(z) = |z|^2$ ফাংশনটি সর্বত্র অবিচ্ছিন্ন কিন্তু মূলবিন্দু ব্যতীত অন্য কোথাও অন্তরীকরণযোগ্য নয়।

১১। হারমোনিক ফাংশনের সংজ্ঞা দাও। প্রমাণ কর যে, একটি বৈশ্লেষিক ফাংশনের বাস্তব ও কাল্পনিক অংশ হারমোনিক। যদি $w = f(z) = u + iv$ বৈশ্লেষিক এবং $u - v = e^x (\cos y - \sin y)$ হয় তবে $f(z)$ কে z-এর মাধ্যমে প্রকাশ কর।

১২। কসির যোগজ উপপাদ্য, কসির যোগজ সূত্র এর উচ্চতর অন্তরজের কসির যোগজ সূত্রটি বিবৃত কর। $\frac{1}{2\pi i}\oint_c \frac{e^{tz}}{(z+1)^3}\, dz$ যোগজটির মান নির্ণয় কর, যেখানে c হলো $z = -1$ কে আবর্তনকারী সরল বদ্ধ বক্ররেখা এবং $t > 0$.

১৩। মরিরার উপপাদ্য বর্ণনা ও প্রমাণ কর।

১৪। জটিল ফাংশনের বিভিন্ন প্রকারের ব্যতিক্রম বিন্দুর শ্রেণিবিন্যাস কর এবং উদাহরণসহ ব্যাখ্যা কর।

১৫। লরেন্ট এর উপপাদ্য বর্ণনা ও প্রমাণ কর।

১৬। কন্টুর যোগজীকরণের সাহায্যে নিম্নের যে কোনো দুইটির মান নির্ণয় কর ঃ

(i) $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ (ii) $\int_0^{\infty} \frac{\sin mx}{x}\, dx,\ m > 0$; (iii) $\int_{-\infty}^{\infty} \frac{\sin ax}{x^2 + 1}\, dx$

১৭। $w = f(z)$ রূপান্তরটি conformal mapping হাওয়ার প্রয়োজনীয় শর্ত বর্ণনা কর ও প্রমাণ কর।

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A Part

1. (a) How many values have the expression $(1 + i)^{2/3}$? **[Ch-1, Quiz-27]**

(b) Define analytic function at a point. **[Ch-2, Quiz-1]**

(c) What is the necessary condition for analyticity of at the function $w = f(z)$? **[Ch-2, Th^m-4]**

(d) Define a closed contour. **[Ch-3, Quiz-6]**

(e) Write a complex function which has a zero at $z = 1$ but has no singularity **[Ch-4, Quiz-20]**

(f) Define pole of a complex function? **[Ch-4, Quiz-7]**

(g) Define residue at a pole of a complex function **[Ch-4, Quiz-17]**

(h) What is Transformation or Mapping? **[Ch-6, Quiz-12]**

(i) State Rouches theorem. **[Ch-4, Th^m-10]**

(j) State fundamental theorem of Algebra. **[Ch-3, Quiz-18]**

(k) State maximum modulus principle. **[Ch-4, Th^m-7]**

(l) What is the necessary condition for $w = f(z)$ to be a conformal mapping? **[Ch-6, Quiz-1]**

B Part

2. Determine the set of points on the complex plane which satisfy the inequality Re $\left(\frac{1}{z}\right) \le \frac{1}{2}$ and sketch it. **[Ch-1, Expl-33(xiii)]**

3. Prove that an analytic function with constant modulus is constant. **[Ch-2, Expl-26]**

4. Define winding numbers **[Ch-3, Art-3.11]**. Explain with figure when winding number are positive, negative and zero. **[Ch-3, Art-3.11]**

5. State Taylar's theorem for the complex function $f(z)$. Expand $f(z) = \ln(1 + z)$ in Taylor's series about $z = 0$ **[Ch-4, Prob-26(b)]**

6. State and prove Cauchy's integral formula for the first derivative. **[Ch-3, Th^m-8]**

7. Find the singular points of the function $f(z) = \frac{z^2}{(z+1)^2} \sin\left(\frac{1}{z-1}\right)$ and determine their nature. **[Ch-4, Prob-8]**

8. State and prove Cauchy's residue theorem. **[Ch-4, Th^m-6]**

9. Find the bilinear transformation which maps $1, i, -1$ on $i, 0, -i$ respectively **[Ch-6, Expl-15]**

C Part

10. Prove that the function $f(z) = |z|^2$ is continuous everywhere but no where differentiable except at the origin.
[Ch–2, Expl–15]

11. Define Harmonic function **[Ch–2, Art–2.6]**. Prove that real and imaginary parts of an analytic function are harmonic **[Ch–2, Th^m–6]**. If $w = f(z) = u + iv$ is analytic and $u - v = e^x (\cos y - \sin y)$, find f(z) in terms of z.
[Ch–2, Expl–56]

12. State Cauchy's integral theorem **[Ch–3, Th^m–5]**. Cauchy's integral formula **[Ch–3, Th^m–7]** and Cauchy's integral formula for higher derivatives [Ch–3 Th^m–3. Find the integral $\frac{1}{2\pi i}\oint_c \frac{ze^{tz}}{(z+1)^3}\,dz$, where c is any simple closed curve enclosing $z = -1$ and $t > 0$. **[Ch–3, Expl–8]**

13. State and prove Morera's theorem. **[Ch–3, Th^m–10]**

14. Classify the different kinds of singularities of a complex function and illustrate. **[Ch–4, Prob–3]**

15. State and prove Laurent's theorem. **[Ch–4, Th^m–2]**

16. Evaluate any two of the following by Contour Integration :

    (i) $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ **[Ch–5, Prob–1]**

    (ii) $\int_0^{\infty} \frac{\sin mx}{x}\,dx$, $m > 0$; **[Ch–5, Prob–51]**

    (iii) $\int_{-\infty}^{\infty} \frac{\sin ax}{x^2 + 1}\,dx$ **[Ch–5, Prob–40(ii)]**

17. State and prove the necessary condition for the transformation $w = f(z)$ to be a conformal mapping.
[Ch–6, Art–6.3 (Th^m–1)]

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