Mathematical Formulation and Numerical Modeling of Transient Seepage Flow

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The following describes the background of seepage, the mathematical technique used to calculate components of the full hydraulic conductivity tensor and formulate the 2D transient seepage problem, and the numerical modeling of the problem implementing the finite difference scheme to enable evaluation of the impact of hydraulic conductivity on the seepage flow.

1 Seepage

In the presence of a hydraulic gradient, there will be flow. In the most general case, using conservation of mass, the governing equation of seepage can be found. In addition, assuming the density of water is constant, the conservation of mass turns into the conservation of volume. For an element of soil, this results in the following equation for transient seepage (Fredlund & Rahardjo, 1993):

$$\vec{\nabla} \cdot \vec{v} = -\frac{\partial \theta}{\partial t}, \tag{1}$$

where $\vec{v}$ is Darcy's velocity, otherwise known as discharge velocity, $\theta = nS_w$ is the volumetric water content, n is the soil porosity, and S_w is the degree of water saturation.

Variations in the volumetric water content can be found (Genetti Jr, 1999) as a function of the specific/elastic capacity (i.e., retention) of water, m_v, and temporal variations of the hydraulic head, h resulting in:

$$\vec{\nabla} \cdot \vec{v} = -m_v \frac{\partial h}{\partial t}, \tag{2}$$

where for unsaturated soils, the water-retention characteristics can be approximated to a two-segment polynomial with the two values for its slope, $m_v \approx 0.001\ m^{-1}$ for unsaturated soils and $m_v \approx 0.00001\ m^{-1}$ for saturated soils.

2 Mathematical formulation

Darcy's velocity $\vec{v}$ is the discharge velocity of groundwater flow, which can be calculated from Darcy's law:

$$\vec{v} = -k\vec{i} \tag{3}$$

where k is the hydraulic conductivity, $\vec{i} = \vec{\nabla} h$ is the hydraulic gradient and h is the total (hydraulic) head. The negative sign shows that the water moves from the higher total head towards a lower total head. Thus, Equation (3) can be rewritten as follows.

$$\vec{v} = -k\vec{\nabla} h. \tag{4}$$

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If the soil is isotropic (k is the same in all directions), then k could be treated as a scalar. If the soil is anisotropic, since hydraulic conductivity is different in different directions, k needs to be treated as a tensor, $\underline{\underline{k}}$.

Hence, if v_x, v_y, and v_z are the velocity component in X, Y, and Z directions respectively the Darcy's law can be written as follows:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = - \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} h, \tag{5}$$

where k_{xx}, k_{yy}, k_{zz} are diagonal elements and k_{xy}, k_{xz}, k_{yx}, k_{yz}, k_{zx}, and k_{zy} are non-diagonal elements of hydraulic conductivity tensor $\underline{\underline{k}}$. The non-diagonal elements of $\underline{\underline{k}}$ are symmetric.

If the system of coordinates is considered such that one plane (e.g., XY) is aligned with the stratigraphic plane on the mesoscale, the tensor $\underline{\underline{k}}$ can be assumed to be a diagonalized tensor. In other words, in these cases, seepage is analysed in an orthogonal system of coordinates aligned with the three axes of coordinates aligned with the three principal orientations (major: k_1, intermediate: k_2 and minor: k_3) of the tensor $\underline{\underline{k}}$ in which, k_3 is normal to the stratigraphic plane and k_1 and k_2 within stratigraphic planes. In most cases, soils are axisymmetric within the stratigraphic plane; hence, k_1 and k_2 are equal. Since the XY plane is almost always considered horizontal, and most soil stratigraphic planes are horizontal and axisymmetric, this represents most cases.

However, stratigraphic planes can be titled due to several reasons such as tectonic plate movements. In the case of tilted stratigraphic planes, nondiagonal elements of the hydraulic conductivity tensor $\underline{\underline{k}}$ must be considered. In 2D, if α be the angle of the degree of stratigraphic tilt then the hydraulic conductivity tensor $\underline{\underline{k}}$ will be nondiagonal and Darcy's velocity will be:

$$\begin{bmatrix} v_x \\ v_z \end{bmatrix} = - \begin{bmatrix} k_{xx} & k_{xz} \\ k_{zx} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} \end{bmatrix} h, \tag{6}$$

where

$$\underline{\underline{k}} = \begin{bmatrix} k_{xx} & k_{xz} \\ k_{zx} & k_{zz} \end{bmatrix},$$

considering Equation (5), k_{xx} and k_{xz} contribute to v_x through the impact of the X and Z components of the hydraulic gradients, respectively. On the other hand, k_{zx} and k_{zz} contribute to v_z through the impact of the X and Z components of the hydraulic gradients, respectively.

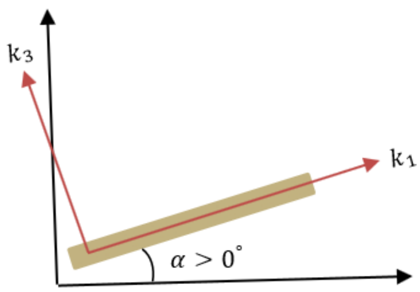

Figure 1: The schematic oblong rectangle ( ) shown on the figure represents the stratigraphic plane with a positive slope (used $\alpha > 0°$); k_1 aligns with the length of the schematic rectangle, representing the stratigraphic plane orientation.

In a 2D system of coordinates and the stratigraphic plane with a positive slope α shown in Figure 1, positive values of $\left(-\frac{\partial h}{\partial x}\right)$ *and* $\left(-\frac{\partial h}{\partial z}\right)$ can be projected on the k_1 and k_3 axes. Then, the flow velocities along the 1 and 3 directions can be simplified as follows.

$$v_1 = k_1 \left[\left(-\frac{\partial h}{\partial x} \right) \cos\alpha + \left(-\frac{\partial h}{\partial z} \right) \sin\alpha \right], \tag{7a}$$

$$v_3 = k_3 \left[\left(-\frac{\partial h}{\partial x} \right) \sin\alpha + \left(-\frac{\partial h}{\partial z} \right) \cos\alpha \right]. \tag{7b}$$

If the resultant of two vectors v_1 and v_3 are found and projected onto the X and Z directions, v_{xx} and v_{zz} can be computed and simplified as follows.

$$v_{xx} = v_{1x} + v_{3x} = -\left[\left(\frac{k_1 + k_3}{2}\right) + \left(\frac{k_1 - k_3}{2}\right)\cos 2\alpha\right]\frac{\partial h}{\partial x} - \left[\left(\frac{k_1 - k_3}{2}\right)\sin 2\alpha\right]\frac{\partial h}{\partial z}, \tag{8a}$$

$$v_{zz} = v_{1z} + v_{3z} = -\left[\left(\frac{k_1 - k_3}{2}\right)\sin 2\alpha\right]\frac{\partial h}{\partial x} - \left[\left(\frac{k_1 + k_3}{2}\right) + \left(\frac{k_1 - k_3}{2}\right)\cos 2\alpha\right]\frac{\partial h}{\partial z}. \tag{8b}$$

Comparing Equations (8) and (6), k_{xx}, k_{zz}, k_{xz}, and k_{zx} for a 2D case, can be written in terms of k_1 and k_3, which agrees with what was found by (Fanchi, 2008).

$$k_{xx} = \left(\frac{k_1 + k_3}{2}\right) + \left(\frac{k_1 - k_3}{2}\right)\cos 2\alpha, \tag{9a}$$

$$k_{xz} = k_{zx} = \left(\frac{k_1 - k_3}{2}\right)\sin 2\alpha, \tag{9b}$$

$$k_{zz} = \left(\frac{k_1 + k_3}{2}\right) - \left(\frac{k_1 - k_3}{2}\right)\cos 2\alpha. \tag{9c}$$

Substituting Darcy's law into Equation (2), the governing equation of seepage can be found.:

$$\vec{\nabla}\cdot\left(-\underline{\underline{k}}\vec{\nabla}h\right) = -m_v\frac{\partial h}{\partial t}. \tag{10}$$

Then, for a 2D case, Equation (10) takes the following form.

$$-\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}k_{xx} & k_{xz}\\ k_{zx} & k_{zz}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial z}\end{bmatrix}h = -m_v\frac{\partial h}{\partial t}, \tag{11}$$

where hydraulic conductivity tensor $\underline{\underline{k}}$ and its components value for any orientation angle α of soil stratigraphic planes given by equations (6) & (9). Equation (11) can be simplified as follows.

$$-\left[\frac{\partial k_{xx}}{\partial x}\frac{\partial h}{\partial x} + k_{xx}\frac{\partial^2 h}{\partial x^2} + \frac{\partial k_{xz}}{\partial x}\frac{\partial h}{\partial z} + k_{xz}\frac{\partial^2 h}{\partial x\partial z} + \frac{\partial k_{zx}}{\partial z}\frac{\partial h}{\partial x} + k_{zx}\frac{\partial^2 h}{\partial z\partial x} + \frac{\partial k_{zz}}{\partial z}\frac{\partial h}{\partial z} + k_{zz}\frac{\partial^2 h}{\partial z^2}\right] = -m_v\frac{\partial h}{\partial t}. \tag{12}$$

3 Numerical modeling

Using the MATLAB interface, a 2D numerical model was developed to solve the governing equation of transient seepage, Equation (10) to simulate the impacts of the nondiagonal $\underline{\underline{k}}$ tensor on the seepage flow for soils with tilted stratigraphic planes (i.e., there is a tilt between the X and k_1 axes and, in turn, Z and k_3 axes), types, and different orientations of soil stratigraphy. For our seepage model, two types of boundary conditions were considered: Neumann boundary conditions on most of the domain boundary, which stimulates impermeable boundaries, and Dirichlet boundary conditions at inlets and outlets, where hydraulic heads are constants: H_1 and H_2, respectively, to allow flow.

The finite difference (FD) with the forward difference in time derivatives and first-order space derivatives and central difference in higher-order space derivatives to discretize the time and space domains in order to linearize Equation (12) to the following form. In Equation (13), superscripts (t_k, t_{k+1}) represent time steps and subscripts $(i,\ j)$ represent space discretization. The implicit scheme was applied to h on the left side of Equation (12) and all h terms are considered at time t_{k+1}. Thus, $h_{i,j}^{t_k}$ represents the hydraulic head at Row i, Column j, i.e., Node (i, j) in the space, and at time instance t_k; $h_{i,j}^{t_{k+1}}$ is the hydraulic head at Node (i, j) at time t_{k+1} and so on.

$$\begin{aligned}&\frac{k_{xx(i,j)}}{(\Delta x)^2}h_{i,j-1}^{t_{k+1}} - \left(\frac{k_{xx(i,j+1)} + k_{xx(i,j)}}{(\Delta x)^2} + \frac{k_{xz(i,j+1)} + k_{xz(i,j)}}{\Delta x\Delta z} + \frac{k_{zx(i+1,j)} + k_{zx(i,j)}}{\Delta z\Delta x} + \frac{k_{zz(i+1,j)} + k_{zz(i,j)}}{(\Delta z)^2} + \frac{m_{v(i,j)}}{\Delta t}\right)h_{i,j}^{t_{k+1}}\\ &+ \left(\frac{k_{xx(i,j+1)}}{(\Delta x)^2} + \frac{k_{zx(i+1,j)} + k_{zx(i,j)}}{\Delta z\Delta x}\right)h_{i,j+1}^{t_{k+1}} + \left(\frac{k_{xz(i,j)}}{\Delta x\Delta z} + \frac{k_{zx(i,j)}}{\Delta z\Delta x}\right)h_{i-1,j-1}^{t_{k+1}} + \frac{k_{zz(i,j)}}{(\Delta z)^2}h_{i-1,j}^{t_{k+1}}\\ &+ \left(\frac{k_{xz(i,j+1)} - k_{xz(i,j)}}{\Delta x\Delta z} + \frac{k_{zz(i+1,j)}}{(\Delta z)^2}\right)h_{i+1,j}^{t_{k+1}} + \left(\frac{k_{xz(i,j)}}{\Delta x\Delta z} + \frac{k_{zx(i,j)}}{\Delta z\Delta x}\right)h_{i+1,j+1}^{t_{k+1}} = \frac{m_{v(i,j)}}{\Delta t}h_{i,j}^{t_k},\end{aligned} \tag{13}$$

Diagonal elements of $\underline{\underline{k}}$: $k_{xx(i,j)}$, $k_{zz(i,j)}$, $k_{xx(i,j+1)}$, and $k_{zz(i+1,j)}$ are at Nodes (i, j), $(i, j + 1)$and $(i + 1, j)$ ccorrespondingly, to account for the flow velocity in one direction based on the hydraulic gradient in the same direction whereas nondiagonal elements $k_{xz(i,j)}$, $k_{zx(i,j)}$, $k_{xz(i,j+1)}$, $k_{zx(i+1,j)}$ are at Nodes (i, j), $(i, j + 1)$ and $(i + 1, j)$ accordingly to account for the flow velocity in one direction based on the hydraulic gradient in the orthogonal direction. Here, $i = 1, 2, \cdots\cdots M$ and $j = 1, 2, \cdots\cdots N$ are the number of nodes in the vertical and horizontal directions respectively. Values of M and N are forced to be odd to allow a midpoint row to accommodate symmetry. For chosen values, of M, N, the horizontal space grid size, Δx, and vertical space grid size, Δz calculated for a given time step Δt, horizontal length, L, and vertical length (thickness of soil), T.

The model simulates seepage flow for a domain consisting of two soil types deposited in two layers stacked up together with the same length, which can also be used to simulate the flow for a single layer of soil considering the fact that hydraulic conductivity for both of the types is the same. In addition, the code is flexible to simulate other scenarios, e.g., soil layers stacked either horizontally or vertically, and there could be any degree of tilt, α, of the soil stratigraphic planes with respect to the X (in this case horizontal) axis.

However, for the selected values of hydraulic conductivity along the major principal orientations, k_1 and minor principal orientations, k_3 of $\underline{\underline{k}}$ for both soils, the hydraulic conductivity tensor elements were calculated Equation (9). Furthermore, in this model, in the case of unsaturated soil, with each time step, both diagonal and nondiagonal components of the hydraulic-conductivity tensor are updated using the following soil water-retention formula.

$$k_{p,q} = \frac{k_{0p,q}}{1 + a_1 \left| h_{p,q} - z_{p,q} \right|^{a_2}}, \tag{14}$$

where $k_{0p,q}$ is the saturated hydraulic conductivity for the soil; two constant coefficients $a_1 = 1$ and $a_3 = 3$, are used in this formula; $k_{p,q}$ represent k_{xx}, k_{zz}, k_{xz}, and k_{zx} at Node $p = i-1$, i, $i+1$, $q = j-1$, j, $j+1$, and $z^t_{p,q}$ is the elevation head at any Node (p, q).

The term $h_{p,q}$ in the denominator of Equation (15) can be considered at either of Times t_k, t_{k+1}, or an average of the two. In here, a scheme similar to the Crank-Nicholson scheme (Chávez-Negrete, Domínguez-Mota, & Santana-Quinteros, 2018) is used, and the average of $h^{t_k}_{p,q}$, and $h^{t_{k+1}}_{p,q}$, was used in Equation (15).

$$k^{t_k}_{p,q} = \frac{k_{0p,q}}{1 + a_1 \left| \frac{h^{t_k}_{p,q} + h^{t_{k+1}}_{p,q}}{2} - z_{p,q} \right|^{a_2}}. \tag{15}$$

Due to the presence of the unknown $h^{t_{k+1}}_{p,q}$ in the denominator of Equation (16), if Equation (16) is substituted in Equation (13), the resulting equation will be nonlinear, which will defeat the purpose of linearization to enable solving the system of linear equations. Hence, instead of solving this resulting equation, all coefficients $k_{p,q}$ need to be found separately and substituted as known values in Equation (13) to maintain the linearity of Equation (13). Since neither $h^{t_{k+1}}_{p,q}$ nor $k_{p,q}$ are known, $k_{p,q}$ (k based on average values of h for the time increment between t_k and t_{k+1}) are found using a successive iteration scheme. Basically, for each time step, initially, $k_{p,q}$ is found using only $h^{t_k}_{p,q}$, the solver is used to find $h^{t_{k+1}}_{p,q}$, then, $k_{p,q}$ is updated using this newly calculated $h^{t_{k+1}}_{p,q}$, which is then used to solve and find updated $k_{p,q}$.This successive iteration is continued until the code converges to the best answer for $h^{t_{k+1}}_{p,q}$ and in turn, the best $k_{p,q}$. Only then, the code is ready to advance (i.e., "march to the next step in time") using $h^{t_{k+1}}_{p,q}$ through another successive iteration that converges to the best $h^{t_{k+2}}_{p,q}$ and corresponding $k_{p,q}$ (average for the increment between t_{k+1} and t_{k+2}). This applies to all elements of the hydraulic conductivity tensor. The code then advances to all following time steps. The total head at each time and space node is used to calculate Darcy's velocity and pore-water pressure values.

References

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