Problem Set 4

Due Wednesday, March 5, 2025

Problem 1. (Exercise 2.12 in Rudin)
Let $K \subset \mathbb{R}$ consist of 0 and the numbers $\frac{1}{n}$ for $n = 1, 2, 3, \ldots$; that is,

$$K = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Prove directly from the definition (without using the Heine–Borel theorem) that K is compact.

Solution

Let $\mathcal{U}$ be an arbitrary open cover of K. Since $0 \in K$, there exists some open set $U_0 \in \mathcal{U}$ with $0 \in U_0$. Because U_0 is open, there exists $\varepsilon > 0$ such that

$$(-\varepsilon, \varepsilon) \subset U_0.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{1}{n} < \varepsilon.$$

Thus, for all $n \geq N$, we have $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subset U_0$.
The finitely many points $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{N-1}$ are each contained in some members $U_1, U_2, \ldots, U_{N-1}$ of $\mathcal{U}$. Therefore, the finite collection

$$\{U_0, U_1, U_2, \ldots, U_{N-1}\}$$

covers K. Since every open cover of K has a finite subcover, K is compact.

Problem 2. (Exercise 2.14 in Rudin)
Give an example of an open cover of the open interval $(0, 1) \subset \mathbb{R}$ which has no finite subcover (and prove that property).

Solution

Consider the collection

$$\mathcal{U} = \left\{ \left(\frac{1}{n}, 1 \right) : n \in \mathbb{N}, \, n \geq 2 \right\}.$$

For any $x \in (0,1)$, choose $n \in \mathbb{N}$ such that $n > \frac{1}{x}$. Then

$$\frac{1}{n} < x,$$

so $x \in \left(\frac{1}{n}, 1\right)$. Hence, $\mathcal{U}$ is an open cover of $(0,1)$.
Now, assume for contradiction that there exists a finite subcover

$$\left\{ \left(\frac{1}{n_1}, 1 \right), \left(\frac{1}{n_2}, 1 \right), \ldots, \left(\frac{1}{n_k}, 1 \right) \right\}.$$

Let $N = \max\{n_1, n_2, \ldots, n_k\}$. Then every interval in this finite collection is of the form $\left(\frac{1}{n_i}, 1\right)$ with $\frac{1}{n_i} \geq \frac{1}{N}$. Therefore, none of these intervals can cover any point less than or equal to $\frac{1}{N}$. In particular, the point

$$x = \frac{1}{N+1}$$

satisfies $x < \frac{1}{N}$ and hence is not contained in any $\left(\frac{1}{n_i}, 1\right)$, a contradiction. Thus, no finite subcover exists.

Problem 3. (Exercise 2.15 in Rudin)

1. Construct a collection of closed subsets $\{K_\alpha\}_{\alpha \in J}$ of $\mathbb{R}$ such that the intersection of every (nonempty) finite subcollection is nonempty, while

$$\bigcap_{\alpha \in J} K_\alpha = \emptyset.$$

2. Do the same with the word "closed" replaced by "bounded".

Solution

(a) Closed subsets:
For each $n \in \mathbb{N}$, define

$$K_n = [n, \infty).$$

Each K_n is closed in $\mathbb{R}$. For any finite subcollection $\{K_{n_1}, K_{n_2}, \ldots, K_{n_k}\}$, let

$$N = \max\{n_1, n_2, \ldots, n_k\}.$$

Then,

$$\bigcap_{i=1}^{k} K_{n_i} = [N, \infty),$$

which is nonempty. However,

$$\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} [n, \infty) = \emptyset,$$

since no real number is greater than or equal to every natural number.

(b) Bounded subsets:
For each $n \in \mathbb{N}$, define

$$B_n = \left(0, \frac{1}{n}\right).$$

Each B_n is bounded. For any finite subcollection $\{B_{n_1}, B_{n_2}, \ldots, B_{n_k}\}$, let

$$N = \max\{n_1, n_2, \ldots, n_k\}.$$

Then,

$$\bigcap_{i=1}^{k} B_{n_i} = \left(0, \frac{1}{N}\right),$$

which is nonempty. However, the intersection over all n is

$$\bigcap_{n=1}^{\infty} B_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n} \text{ for all } n \in \mathbb{N}\} = \emptyset,$$

since for any $x > 0$ there exists n such that $\frac{1}{n} < x$.

Problem 4. Let A and B be connected subsets of a metric space X. If $A \cap B \neq \emptyset$, prove that $A \cup B$ is connected.

Solution

Assume, to the contrary, that $A \cup B$ is disconnected. Then there exist nonempty disjoint open sets U and V in X such that

$$A \cup B = U \cup V.$$

Since A is connected, it must lie entirely in one of these open sets; without loss of generality, assume

$$A \subset U.$$

Similarly, since B is connected, we must have either $B \subset U$ or $B \subset V$. However, because $A \cap B \neq \emptyset$, there exists a point $x \in A \cap B$. Since $x \in A$ and $A \subset U$, it follows that $x \in U$. Thus, B cannot be entirely contained in V; hence, $B \subset U$ as well. This implies

$$A \cup B \subset U,$$

which contradicts the fact that V is nonempty. Therefore, $A \cup B$ must be connected.