Problem Set 2

Due Wednesday, February 12, 2025

Problem 1. (Exercise 1.6(a) in Rudin) Fix a real number $b > 1$. If m, n, p, q are integers with $n > 0$, $q > 0$, and $\frac{m}{n} = \frac{p}{q}$, then prove that

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Proof. We wish to show that if

$$\frac{m}{n} = \frac{p}{q}$$

with $m, n, p, q \in \mathbb{Z}$, $n > 0$ and $q > 0$, then

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Step 1: Equate the rational exponents.
Since

$$\frac{m}{n} = \frac{p}{q},$$

cross-multiplication yields

$$mq = pn.$$

Step 2: Define the two expressions.
Let

$$x = (b^m)^{\frac{1}{n}} \quad \text{and} \quad y = (b^p)^{\frac{1}{q}}.$$

By definition, x is the unique positive real number satisfying

$$x^n = b^m,$$

and similarly, y is the unique positive real number satisfying

$$y^q = b^p.$$

Step 3: Raise both expressions to the power nq.
Raising x to the nq-th power, we obtain

$$x^{nq} = \left((b^m)^{\frac{1}{n}}\right)^{nq} = (b^m)^q = b^{mq}.$$

Similarly, raising y to the nq-th power yields

$$y^{nq} = \left((b^p)^{\frac{1}{q}}\right)^{nq} = (b^p)^n = b^{pn}.$$

Step 4: Use the equality from Step 1.
Since $mq = pn$, it follows that

$$x^{nq} = b^{mq} = b^{pn} = y^{nq}.$$

Step 5: Conclude the proof.
Because the function $f(t) = t^{nq}$ is strictly increasing on the positive real numbers, the equality $x^{nq} = y^{nq}$ implies $x = y$. Therefore,

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}},$$

which completes the proof. $\square$

Problem 2. (Exercise 1.6(b) in Rudin) It follows from the previous problem that it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$ for any rational number $r = \frac{m}{n}$. For any rational numbers r and s, prove that

$$b^{r+s} = b^r b^s.$$

Proof. Let $r = \frac{m}{n}$ and $s = \frac{p}{q}$, where $m, n, p, q \in \mathbb{Z}$ with $n, q > 0$. Then,

$$r + s = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}.$$

By the definition of rational exponents,

$$b^{r+s} = b^{\frac{mq+np}{nq}} = \left(b^{mq+np}\right)^{\frac{1}{nq}}.$$

Using the law of exponents for integers, we have

$$b^{mq+np} = b^{mq} \cdot b^{np}.$$

Thus,

$$b^{r+s} = (b^{mq} \cdot b^{np})^{\frac{1}{nq}}.$$

By the multiplicative property of nq-th roots,

$$b^{r+s} = (b^{mq})^{\frac{1}{nq}} \cdot (b^{np})^{\frac{1}{nq}}.$$

Notice that

$$(b^{mq})^{\frac{1}{nq}} = ((b^m)^q)^{\frac{1}{nq}} = (b^m)^{\frac{q}{nq}} = (b^m)^{\frac{1}{n}} = b^r,$$

and similarly,

$$(b^{np})^{\frac{1}{nq}} = ((b^p)^n)^{\frac{1}{nq}} = (b^p)^{\frac{n}{nq}} = (b^p)^{\frac{1}{q}} = b^s.$$

Therefore,

$$b^{r+s} = b^r \cdot b^s,$$

which completes the proof. □

Problem 3. (Exercise 1.6(c) in Rudin) For any real x, define

$$B(x) := \{b^t \mid t \in \mathbb{Q}, t \leq x\}.$$

Prove that

$$b^r = \sup B(r)$$

for any rational r.

Proof. Assume that $b > 1$. Notice that the function

$$t \mapsto b^t$$

is strictly increasing when $b > 1$. In other words, if $t_1, t_2 \in \mathbb{Q}$ with $t_1 \leq t_2$, then

$$b^{t_1} \leq b^{t_2},$$

and if $t_1 < t_2$, the inequality is strict:

$$b^{t_1} < b^{t_2}.$$

Now, fix a rational number r. By the definition of $B(r)$, every element in $B(r)$ is of the form b^t for some $t \in \mathbb{Q}$ with $t \leq r$. Since the function is strictly increasing, it follows that for every such t,

$$b^t \leq b^r.$$

Thus, b^r is an upper bound for the set $B(r)$.

Moreover, because r is a rational number and $r \leq r$, we have $b^r \in B(r)$. Hence, there is no upper bound of $B(r)$ smaller than b^r. Therefore, b^r is the least upper bound of $B(r)$; in other words,

$$\sup B(r) = b^r.$$

$\square$

Problem 4. (Exercise 1.6(d) in Rudin) It follows from the previous problem that it makes sense to define

$$b^x := \sup B(x)$$

for any real x. Prove that $b^{x+y} = b^x b^y$.

Proof. We begin by noting that the rational numbers are dense in $\mathbb{R}$. Hence, for any real numbers x and y, there exist sequences $\{r_n\}$ and $\{s_n\}$ of rational numbers such that

$$r_n \to x \quad \text{and} \quad s_n \to y \quad \text{as } n \to \infty.$$

In the previous problem it was shown that for any rational numbers r and s one has

$$b^{r+s} = b^r\, b^s.$$

Thus, for each n we have

$$b^{r_n+s_n} = b^{r_n}\, b^{s_n}.$$

By the definition of b^x as the supremum of the set $\{b^r : r \in \mathbb{Q},\, r < x\}$ and using the supremum approximation property (which states that for any $\varepsilon > 0$ there exists a rational number r with $x - \varepsilon < r < x$), we deduce that

$$\lim_{n\to\infty} b^{r_n} = b^x \quad \text{and} \quad \lim_{n\to\infty} b^{s_n} = b^y.$$

Similarly, since $r_n + s_n \to x + y$, it follows that

$$\lim_{n\to\infty} b^{r_n+s_n} = b^{x+y}.$$

Taking the limit in the identity $b^{r_n+s_n} = b^{r_n}\, b^{s_n}$, we obtain

$$b^{x+y} = b^x\, b^y.$$

This completes the proof. □