Problem Set 8

Due Wednesday, April 23, 2025

Problem 1 (10 points)**.** Suppose f is a real function on $\mathbb{R}$ such that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

Proof. We will prove that f is constant by showing that its derivative is zero at every point, and then applying the Mean Value Theorem.

Fix any point $a \in \mathbb{R}$. We need to show that $f'(a) = 0$. Consider the difference quotient:

$$\begin{aligned} \left| \frac{f(a+h) - f(a)}{h} \right| &\leq \frac{|f(a+h) - f(a)|}{|h|} && (1) \\ &\leq \frac{(a+h-a)^2}{|h|} \quad \text{(by the given condition)} && (2) \\ &= \frac{h^2}{|h|} && (3) \\ &= |h| && (4) \end{aligned}$$

Now, as $h \to 0$, we have $|h| \to 0$. Since the above inequality holds for all $h \neq 0$, we can take the limit as $h \to 0$ to get:

$$\lim_{h \to 0} \left| \frac{f(a+h) - f(a)}{h} \right| \leq \lim_{h \to 0} |h| = 0 \tag{5}$$

This means that the absolute value of the difference quotient approaches 0, which implies that the difference quotient itself must approach 0. Therefore:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0 \tag{6}$$

Since a was arbitrary, we have shown that $f'(x) = 0$ for all $x \in \mathbb{R}$.

Now we can use the Mean Value Theorem. For any two points $p, q \in \mathbb{R}$ with $p < q$, the Mean Value Theorem guarantees the existence of a point $c \in (p, q)$ such that:

$$f(q) - f(p) = f'(c)(q - p) \tag{7}$$

Since we've established that $f'(c) = 0$, we have:

$$f(q) - f(p) = 0 \cdot (q - p) = 0 \tag{8}$$

Therefore, $f(p) = f(q)$ for any two points $p, q \in \mathbb{R}$, which means that f is constant on $\mathbb{R}$. $\square$

Problem 2 (10 points)**.** Suppose g is a real, differentiable function on $\mathbb{R}$ with bounded derivative (i.e. g' is a bounded function). For $\epsilon > 0$, define

$$f_\epsilon(x) = x + \epsilon g(x).$$

Prove that f_ϵ is one-to-one (i.e. $f_\epsilon(x) = f_\epsilon(y)$ implies $x = y$) if ϵ is small enough.

Proof. We'll show that for sufficiently small $\epsilon > 0$, the function $f_\epsilon(x) = x + \epsilon g(x)$ is strictly monotonic, which implies that it's one-to-one.

Since g is differentiable on $\mathbb{R}$, f_ϵ is also differentiable on $\mathbb{R}$. Let's compute the derivative of f_ϵ:

$$f'_\epsilon(x) = \frac{d}{dx}[x + \epsilon g(x)] = 1 + \epsilon g'(x) \tag{9}$$

We're given that g' is bounded on $\mathbb{R}$. This means there exists a constant $M > 0$ such that $|g'(x)| \le M$ for all $x \in \mathbb{R}$. Therefore:

$$|f'_\epsilon(x) - 1| = |\epsilon g'(x)| = \epsilon |g'(x)| \le \epsilon M \tag{10}$$

This implies:

$$1 - \epsilon M \le f'_\epsilon(x) \le 1 + \epsilon M \tag{11}$$

Now, we want to ensure that $f'_\epsilon(x) > 0$ for all $x \in \mathbb{R}$, which would make f_ϵ strictly increasing and thus one-to-one. From the inequality above, we need:

$$1 - \epsilon M > 0 \tag{12}$$

$$\Rightarrow \epsilon M < 1 \tag{13}$$

$$\Rightarrow \epsilon < \frac{1}{M} \tag{14}$$

So if we choose $\epsilon < \frac{1}{M}$, then $f'_\epsilon(x) > 0$ for all $x \in \mathbb{R}$, making f_ϵ strictly increasing.

Now, to show that a strictly increasing function is one-to-one: suppose $f_\epsilon(x) = f_\epsilon(y)$ for some $x, y \in \mathbb{R}$. If $x < y$, then by the strict monotonicity of f_ϵ, we would have $f_\epsilon(x) < f_\epsilon(y)$, which contradicts our assumption. Similarly, if $x > y$, we'd have $f_\epsilon(x) > f_\epsilon(y)$, again contradicting our assumption. Thus, we must have $x = y$.

Therefore, for any ϵ satisfying $0 < \epsilon < \frac{1}{M}$, the function f_ϵ is one-to-one. □

Problem 3 (10 points)**.** If

$$c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0,$$

where $c_0, \cdots, c_n$ are real constants, prove that the equation

$$c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Let's define the polynomial function:

$$P(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n$$

We need to prove that there exists some $r \in [0, 1]$ such that $P(r) = 0$.

Consider the definite integral of $P(x)$ from 0 to 1:

$$I = \int_0^1 P(x)\,dx \tag{15}$$

$$= \int_0^1 (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n)\,dx \tag{16}$$

$$\tag{17}$$

We can integrate each term separately:

$$I = \int_0^1 c_0\,dx + \int_0^1 c_1 x\,dx + \cdots + \int_0^1 c_{n-1}x^{n-1}\,dx + \int_0^1 c_n x^n\,dx \tag{18}$$

$$= c_0 \int_0^1 dx + c_1 \int_0^1 x\,dx + \cdots + c_{n-1} \int_0^1 x^{n-1}\,dx + c_n \int_0^1 x^n\,dx \tag{19}$$

$$\tag{20}$$

For each term, we compute:

$$\int_0^1 x^k\,dx = \left[\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{1}{k+1} \tag{21}$$

Therefore:

$$I = c_0 \cdot 1 + c_1 \cdot \frac{1}{2} + \cdots + c_{n-1} \cdot \frac{1}{n} + c_n \cdot \frac{1}{n+1} \tag{22}$$

$$= c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} \tag{23}$$

By the given condition, this sum equals zero. So we have $I = 0$.

Now, we apply the Mean Value Theorem for Integrals, which states that if a function f is continuous on a closed interval $[a, b]$, then there exists a point $c \in (a, b)$ such that:

$$\int_a^b f(x)\,dx = f(c) \cdot (b - a)$$

In our case, $P(x)$ is a polynomial, which is continuous on $[0, 1]$. Therefore, there exists a point $r \in (0, 1)$ such that:

$$\int_0^1 P(x)\,dx = P(r) \cdot (1 - 0) \tag{24}$$

$$\Rightarrow 0 = P(r) \cdot 1 \tag{25}$$

$$\Rightarrow P(r) = 0 \tag{26}$$

This proves that there exists at least one real value $r \in (0, 1)$ such that $P(r) = 0$, which means the equation $c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n = 0$ has at least one real root between 0 and 1. □

Problem 4 (10 points)**.** Suppose f is a real function defined and differentiable at every $x > 0$. Suppose that[1]

$$\lim_{x\to+\infty} f'(x) = 0.$$

Put $g(x) := f(x+1) - f(x)$. Prove that

$$\lim_{x\to+\infty} g(x) = 0.$$

Proof. We are given that f is differentiable for all $x > 0$ and that $\lim_{x\to+\infty} f'(x) = 0$. We need to prove that $\lim_{x\to+\infty} g(x) = 0$, where $g(x) = f(x+1) - f(x)$.

We'll apply the Mean Value Theorem to the function f on the interval $[x, x+1]$ for $x > 0$. The Mean Value Theorem states that if a function is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Since f is differentiable for all $x > 0$, it is continuous for all $x > 0$ as well. Therefore, for each $x > 0$, there exists a point $c_x \in (x, x+1)$ such that:

$$\begin{aligned} f'(c_x) &= \frac{f(x+1) - f(x)}{(x+1) - x} && (27) \\ &= \frac{f(x+1) - f(x)}{1} && (28) \\ &= f(x+1) - f(x) && (29) \\ &= g(x) && (30) \end{aligned}$$

So we have established that $g(x) = f'(c_x)$ for some $c_x \in (x, x+1)$.

Now, as $x \to +\infty$, we also have $c_x \to +\infty$ (since $c_x > x$). By the given limit condition, we know that $\lim_{x\to+\infty} f'(x) = 0$. Therefore:

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} f'(c_x) = \lim_{c_x\to+\infty} f'(c_x) = 0 \qquad (31)$$

The last equality follows from the fact that as $x \to +\infty$, $c_x \to +\infty$ as well, and we're given that $\lim_{x\to+\infty} f'(x) = 0$.

Therefore, we have proven that $\lim_{x\to+\infty} g(x) = 0$, as required. □

[1] In other words, for any $\epsilon > 0$, there exists M such that $|f'(x)| < \epsilon$ whenever $x > M$.