Problem Set 1

Due Wednesday, February 5, 2025

Matt Krasnow

Problem 1. (Exercise 1.1 in Rudin)
If r is a non-zero rational number and x is an irrational number, prove that $r+x$ and rx are irrational.

Proof. I will prove both parts by contradiction.

Proof that $r+x$ is irrational

Step 1: Assume the contrary.
Suppose, for the sake of contradiction, that $r+x$ is rational. By definition, I can express:

$$r = \frac{m}{n}, \quad \text{with } m, n \in \mathbb{Z},\ n \neq 0,$$

and

$$r + x = \frac{a}{b}, \quad \text{with } a, b \in \mathbb{Z},\ b \neq 0.$$

Step 2: Solve for x.
Subtracting r from both sides gives:

$$x = \frac{a}{b} - \frac{m}{n}.$$

Step 3: Simplify the expression.
Combining the fractions, I have:

$$x = \frac{an - mb}{bn}.$$

Since $an - mb$ and bn are integers (with $bn \neq 0$), x is a rational number.

Step 4: Arrive at a contradiction.
This conclusion contradicts the assumption that x is irrational. Hence, the assumption that $r+x$ is rational is false, so:

$$r + x \text{ is irrational.}$$

Proof that rx is irrational

Step 1: Assume the contrary.
Suppose, for the sake of contradiction, that rx is rational. Then I can write:

$$rx = \frac{c}{d}, \quad \text{with } c, d \in \mathbb{Z},\ d \neq 0.$$

Step 2: Substitute the expression for r.
Since r is a nonzero rational number, write:

$$r = \frac{m}{n}, \quad \text{with } m, n \in \mathbb{Z},\ n \neq 0 \text{ and } m \neq 0.$$

Then, the equation becomes:

$$\frac{m}{n} \cdot x = \frac{c}{d}.$$

Step 3: Solve for x.
Multiplying both sides by $\frac{n}{m}$ (which is valid because $m \neq 0$) gives:

$$x = \frac{cn}{md}.$$

Step 4: Check rationality.
Since cn and md are integers (with $md \neq 0$), x is rational. That is, x can be written in the form:

$$x = \frac{f}{g}, \quad \text{with } f, g \in \mathbb{Z},\ g \neq 0.$$

Step 5: Arrive at a contradiction.
This contradicts the hypothesis that x is irrational. Therefore, the assumption that rx is rational must be false, so:

$$rx \text{ is irrational.}$$

$\square$

Problem 2. (Exercise 1.4 in Rudin) Let E be a non-empty subset of an ordered set. Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. I prove the statement directly. Since E is non-empty, choose an arbitrary element $x \in E$. By the definition of a lower bound, I have

$$\forall x \in E, \quad \alpha \leq x.$$

Likewise, by the definition of an upper bound, it follows that

$$\forall x \in E, \quad x \leq \beta.$$

Thus, for my chosen $x \in E$, I obtain

$$\alpha \leq x \quad \text{and} \quad x \leq \beta.$$

By the transitive property of the order relation, I conclude that

$$\alpha \leq \beta.$$

□

Problem 3. (Exercise 1.5 in Rudin) Let A be a non-empty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof. I wish to show that $-\sup(-A)$ is the greatest lower bound of A.

Since A is bounded below, the set A has an infimum, say $m = \inf A$. Similarly, since A is non-empty and bounded below, the set $-A$ is non-empty and bounded above, so it has a supremum, say $s = \sup(-A)$.

Recall the following two properties:

1. For any $x \in A$, I have $m \leq x$.

2. For any $y \in -A$, I have $y \leq s$.

Note that for every $x \in A$, the corresponding element $-x$ belongs to $-A$. Thus, for every $x \in A$ I have

$$-x \leq s.$$

Multiplying the inequality by -1 (which reverses the inequality), I obtain

$$x \geq -s.$$

This shows that $-s$ is a lower bound for A; that is,

$$-\sup(-A) \leq x \quad \forall x \in A.$$

To establish that $-s$ is in fact the greatest lower bound (i.e., the infimum of A), suppose m' is any lower bound for A. Then for every $x \in A$, I have

$$m' \leq x.$$

Multiplying by -1 (and reversing the inequality) yields

$$-m' \geq -x.$$

Since this holds for all $x \in A$, I conclude that $-m'$ is an upper bound for $-A$. Hence, by the definition of supremum,

$$-m' \geq \sup(-A) = s.$$

Multiplying by -1 (again reversing the inequality) gives

$$m' \leq -s.$$

Thus, any lower bound m' of A satisfies $m' \leq -s$. Since I already showed that $-s$ is itself a lower bound for A, it follows that

$$\inf A = -s = -\sup(-A).$$

$\square$

Problem 4. Let $A \subset \mathbb{R}$ be a non-empty set of real numbers which is bounded above. Prove that, for any positive real number $\varepsilon > 0$, there exists $x \in A$ such that $\sup A - x \leq \varepsilon$.

Proof. I prove the statement by considering two cases.

Case 1: $\sup(A) \in A$
If $\sup(A) \in A$, I choose $x = \sup(A)$. Then,

$$\sup(A) - x = \sup(A) - \sup(A) = 0 \leq \varepsilon,$$

which holds for every $\varepsilon > 0$.

Case 2: $\sup(A) \notin A$
Since A is bounded above by $\sup(A)$ and $\sup(A) \notin A$, by the definition of the supremum, for every $\varepsilon > 0$ the number $\sup(A) - \varepsilon$ is not an upper bound for A. Hence, there exists some $x \in A$ such that

$$x > \sup(A) - \varepsilon.$$

This inequality can be rewritten as

$$\sup(A) - x < \varepsilon.$$

Thus, for any given $\varepsilon > 0$, I have found an $x \in A$ satisfying $\sup A - x < \varepsilon$, which completes the proof. □