Midterm

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Honor Pledge:

On my honor, I declare that I have folowed the rules of this examination.

Solution

[Solution for Problem 1]:
(a) To prove:
$$X \setminus A^\circ = \overline{(X \setminus A)}.$$
Interior and closure are duel, so:
$$A^\circ = X \setminus \overline{(X \setminus A)}.$$
For $x \in X$:

- If $x \in A^\circ$, then some open ball $B_r(x) \subset A$. No open ball around x intersects $X \setminus A$, so $x \notin \overline{(X \setminus A)}$.
- If $x \notin A^\circ$, every open ball around x intersects $X \setminus A$, makeing x a limit point of $X \setminus A$ (or in $X \setminus A$). So $x \in \overline{(X \setminus A)}$.

Taking complements gives:
$$X \setminus A^\circ = \overline{(X \setminus A)}.$$
(b) prove:
$$X \setminus \overline{A} = (X \setminus A)^\circ.$$
For any $x \in X$:

- If $x \in (X \setminus A)^\circ$, some open ball $B_r(x) \subset X \setminus A$. This ball doesn't intersect A, so x can't be a limit point of A, meaning $x \notin \overline{A}$. Thus $x \in X \setminus \overline{A}$.
- If $x \in X \setminus \overline{A}$, there's an open neighborhood of x not intersecting A. So x is in the interior of $X \setminus A$, i.e., $x \in (X \setminus A)^\circ$.

So the equality holds:
$$X \setminus \overline{A} = (X \setminus A)^\circ.$$
(c) To prove:
$$\delta A = \overline{A} \cap (X \setminus A^\circ).$$
Boundry is defined as:
$$\delta A = \overline{A} \setminus A^\circ.$$
For sets B and C, we know that:
$$B \setminus C = B \cap (X \setminus C).$$
Taking $B = \overline{A}$ and $C = A^\circ$:
$$\delta A = \overline{A} \cap (X \setminus A^\circ).$$

Solution

[Solution for Problem 2]:
Let X be metric spase, $K \subset X$ compact, and $O \subset X$ open with $K \subset O$.
For each $x \in K$, since O is open and $x \in O$, theres $\varepsilon_x > 0$ such that

$$B_{\varepsilon_x}(x) \subset O.$$

The collection $\{B_{\varepsilon_x/2}(x) : x \in K\}$ forms an open cover of K. By compactness, we get points $x_1, x_2, \ldots, x_n \in K$ where

$$K \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}/2}(x_i).$$

Define

$$\varepsilon = \min\left\{\frac{\varepsilon_{x_1}}{2}, \frac{\varepsilon_{x_2}}{2}, \ldots, \frac{\varepsilon_{x_n}}{2}\right\}.$$

So $\varepsilon > 0$. For any $x \in K$, we have $x \in B_{\varepsilon_{x_i}/2}(x_i)$ for some i. Take any $y \in B_\varepsilon(x)$. By triangle inequality:

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \varepsilon + \frac{\varepsilon_{x_i}}{2} \leq \frac{\varepsilon_{x_i}}{2} + \frac{\varepsilon_{x_i}}{2} = \varepsilon_{x_i}.$$

So $y \in B_{\varepsilon_{x_i}}(x_i) \subset O$. Thus for every $x \in K$,

$$B_\varepsilon(x) \subset O.$$

Solution

[Solution for Problem 3]:
(a) Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences in X. We'll show $\{d(a_n, b_n)\}$ converges. For any $m, n \in \mathbb{N}$, triangle inequality gves:

$$d(a_n, b_n) \leq d(a_n, a_m) + d(a_m, b_m) + d(b_m, b_n).$$

Rearranging:

$$\left|d(a_n, b_n) - d(a_m, b_m)\right| \leq d(a_n, a_m) + d(b_n, b_m).$$

Since $\{a_n\}$ and $\{b_n\}$ are Cauchy, for anny $\varepsilon > 0$ there's N where for all $n, m \geq N$, both $d(a_n, a_m)$ and $d(b_n, b_m)$ are less than $\varepsilon/2$. So

$$\left|d(a_n, b_n) - d(a_m, b_m)\right| < \varepsilon.$$

Thus $\{d(a_n, b_n)\}$ is Cauchy in $\mathbb{R}$, which is complete, so the sequence converges
(b) The relation

$$\{a_n\} \sim \{b_n\} \quad \text{if} \quad \lim_{n\to\infty} d(a_n, b_n) = 0$$

is an equivalance relation

- **Reflexivity:** For all $\{a_n\}$, $d(a_n, a_n) = 0$ always, so $\lim_{n\to\infty} d(a_n, a_n) = 0$. Hence, $\{a_n\} \sim \{a_n\}$.
- **Symmetry:** If $\{a_n\} \sim \{b_n\}$, then $\lim_{n\to\infty} d(a_n, b_n) = 0$. Since $d(a_n, b_n) = d(b_n, a_n)$, we get $\lim_{n\to\infty} d(b_n, a_n) = 0$, so $\{b_n\} \sim \{a_n\}$.
- **Transitivity:** If $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$, than

  $$d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n).$$

  Taking limits:

  $$\lim_{n\to\infty} d(a_n, c_n) \leq \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(b_n, c_n) = 0 + 0 = 0.$$

  So $\{a_n\} \sim \{c_n\}$.

(c) Define Δ on equivalence classies by

$$\Delta(A, B) := \lim_{n\to\infty} d(a_n, b_n),$$

where $\{a_n\} \in A$ and $\{b_n\} \in B$ are representitives.
To show its well-defined, take $\{a_n\}$ and $\{a'_n\}$ in A (so $\lim_{n\to\infty} d(a_n, a'_n) = 0$), and $\{b_n\}$ and $\{b'_n\}$ in B (so $\lim_{n\to\infty} d(b_n, b'_n) = 0$). By triangle inequality,

$$\left|d(a_n, b_n) - d(a'_n, b'_n)\right| \leq d(a_n, a'_n) + d(b_n, b'_n).$$

Taking limitts:

$$\lim_{n\to\infty} \left| d(a_n, b_n) - d(a'_n, b'_n) \right| \leq \lim_{n\to\infty} d(a_n, a'_n) + \lim_{n\to\infty} d(b_n, b'_n) = 0.$$

So

$$\lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(a'_n, b'_n),$$

makeing Δ independant of representitives.
(d) Δ is a metric on X^* (equiv classes of Cauchy sequences in X).

- **Non-negativitty:** For any $A, B \in X^*$,

$$\Delta(A, B) = \lim_{n\to\infty} d(a_n, b_n) \geq 0.$$

- **Identity:** $\Delta(A, B) = 0$ iff $\lim_{n\to\infty} d(a_n, b_n) = 0$, meaning $\{a_n\} \sim \{b_n\}$, so $A = B$.

- **Symmerty:** Since $d(a_n, b_n) = d(b_n, a_n)$, we have $\Delta(A, B) = \Delta(B, A)$.

- **Triangle Inequalty:** For classes A, B, C with representitives $\{a_n\}$, $\{b_n\}$, $\{c_n\}$,

$$d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n).$$

  Taking limits:

$$\Delta(A, C) = \lim_{n\to\infty} d(a_n, c_n) \leq \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(b_n, c_n) = \Delta(A, B) + \Delta(B, C).$$

Solution

[Solution for Problem 4 (Extra Credit)]:
We'll prove X^* (equiv classes of Cauchy sequences with metric Δ) is complete.
Let $\{A_k\}$ be Cauchy in (X^*, Δ). For eech k, pick a representative Cauchy sequence $\{a_n^{(k)}\}$ where A_k is its class. Since $\{A_k\}$ is Cauchy, for anny $\varepsilon > 0$ there's N where for all $k, m \geq N$,

$$\Delta(A_k, A_m) = \lim_{n\to\infty} d(a_n^{(k)}, a_n^{(m)}) < \varepsilon.$$

This means for fixed n (with large k, m), the values $d(a_n^{(k)}, a_n^{(m)})$ are small. Using diagonal argument, construct $\{b_n\}$ in X as the limit of the double sequence $\{a_n^{(k)}\}$: for each n, pick b_n as the limit (or cluster point) of $\{a_n^{(k)}\}_k$.
Then:

1. $\{b_n\}$ is Cauchy in X. This comes from combinng the Cauchy property of each $\{a_n^{(k)}\}$ with the uniform closness from the Cauchy condition on $\{A_k\}$.
2. If A is the class of $\{b_n\}$, then by triangle inequality and convergence properties,

$$\Delta(A_k, A) \to 0 \quad \text{as } k \to \infty.$$

So every Cauchy sequance in X^* converges in X^*, making it complete