Problem Set 3

Due Wednesday, February 19, 2025

Problem 1. (Exercise 2.2 in Rudin)
A complex number z is said to be *algebraic* if there exists an integer $n \geq 1$ and integers

$$a_0, a_1, \ldots, a_n \quad \text{with} \quad a_n \neq 0,$$

such that

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Prove that the set of all algebraic numbers is countable.
(You may use the fact that the above equation has at most n solutions.)

Solution 1. I begin by noting that any polynomial of degree n with integer coefficients has at most n complex roots. Thus, for each fixed degree n, each such polynomial yields only finitely many algebraic numbers.

Next, I observe that the set of all polynomials with integer coefficients is countable. Since the integers $\mathbb{Z}$ are countable and each polynomial is determined by a finite sequence of integers, I can list all such polynomials by partitioning them according to their degree (i.e., degree 1, degree 2, and so on). A countable union of finite sets is countable, so the set of all roots (i.e., the set of algebraic numbers) is a countable union of finite sets.

Thus, I conclude that the set of all algebraic numbers is countable.

$\square$

Problem 2. (Exercise 2.10 in Rudin)
Let X be an infinite set. For $p, q \in X$, define

$$d(p,q) := \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

Prove that d is a metric. Which subsets of the resulting metric space are open?

Solution 2. I first verify that d is a metric on X by checking the following properties for all $p, q, r \in X$:

1. **Non-negativity:** $d(p, q) \geq 0$ since $d(p, q)$ is either 0 or 1.

2. **Identity of indiscernibles:** $d(p, q) = 0$ if and only if $p = q$ by definition.

3. **Symmetry:** $d(p, q) = d(q, p)$ because the condition $p \neq q$ is symmetric.

4. **Triangle inequality:** I must show that

$$d(p, q) \leq d(p, r) + d(r, q).$$

If $p = q$, then $d(p, q) = 0$ and the inequality holds trivially. If $p \neq q$, then $d(p, q) = 1$. In this case, at least one of $d(p, r)$ or $d(r, q)$ must equal 1, so that $d(p, r) + d(r, q) \geq 1$. Thus, the triangle inequality is satisfied.

Next, I determine the open subsets in the metric space (X, d). In this discrete metric, for any $p \in X$ and any radius r satisfying $0 < r \leq 1$, the open ball centered at p is

$$B(p, r) = \{q \in X : d(p, q) < r\} = \{p\}.$$

Since every singleton $\{p\}$ is an open ball, every subset $A \subseteq X$ is open. Indeed, for each $p \in A$, choosing $r = 1/2$ gives $B(p, 1/2) = \{p\} \subseteq A$.

Thus, the topology induced by d is the discrete topology, in which every subset of X is open.

$\square$

Problem 3. (Exercise 2.8 in Rudin)
Prove or disprove the following statement:

"Every point of every open set $E \subseteq \mathbb{R}^k$ is a limit point of E."

Answer the same question for closed sets of $\mathbb{R}^k$.

Solution 3. Part 1: Open Sets

I claim that every point of every open set $E \subseteq \mathbb{R}^k$ is a limit point of E. To see this, let $x \in E$. Since E is open, there exists an $r > 0$ such that the open ball

$$B(x, r) = \{y \in \mathbb{R}^k : \|y - x\| < r\}$$

is contained in E. Because any open ball in $\mathbb{R}^k$ (with $r > 0$) contains infinitely many points other than x (in fact, uncountably many), every open ball $B(x, r)$ contains a point of E distinct from x. Therefore, x is a limit point of E.

Part 2: Closed Sets

I now consider whether every point of every closed set $E \subseteq \mathbb{R}^k$ is a limit point of E. This is not true. For a counterexample, consider the closed set

$$E = \{0\} \subset \mathbb{R}^k.$$

Since E consists of a single point, there exists an $r > 0$ (for instance, $r = 1$) such that

$$B(0, r) \cap E = \{0\}.$$

Thus, 0 is not a limit point of E because no point other than 0 lies in E.

In summary, while every point of an open set in $\mathbb{R}^k$ is a limit point of the set, the same is not true for closed sets.

$\square$

Problem 4. (Exercise 2.6 in Rudin)
Let $E \subseteq X$ be a subset of a metric space, and let E' be the set of all limit points of E. Prove that E' is closed.

Solution 4. I wish to show that the set E' of limit points of E is closed, meaning that it contains all its limit points. Let E'' denote the set of limit points of E'. It suffices to prove that $E'' \subseteq E'$.

So, let $p \in E''$. By definition, for every $r > 0$ there exists a point $q \in B(p, r) \setminus \{p\}$ with $q \in E'$. Since $q \in E'$, by the definition of a limit point of E, for every $s > 0$ there exists a point $x \in B(q, s) \setminus \{q\}$ such that $x \in E$.

Now, fix an arbitrary $r > 0$. Choose $q \in B(p, r/2) \setminus \{p\}$ (which exists because $p \in E''$), and then choose $s = r/2$. By the triangle inequality,

$$d(p, x) \leq d(p, q) + d(q, x) < \frac{r}{2} + \frac{r}{2} = r.$$

Thus, for every $r > 0$, the open ball $B(p, r)$ contains a point $x \in E$ with $x \neq p$. This shows that p is a limit point of E, i.e., $p \in E'$.

Since $p \in E''$ was arbitrary, I conclude that $E'' \subseteq E'$, and therefore,

$$\overline{E'} = E' \cup E'' = E'.$$

This means that E' is closed.

$\square$