Problem Set 6

Due Wednesday, April 9, 2025

Problem 1 (10 points)**.** Determine if the following series converges. If it converges, determine if it converges absolutely.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$$

Solution. We begin by writing the series in the form

$$\sum_{n=1}^{\infty} (-1)^n b_n, \quad \text{with} \quad b_n = \frac{1}{n+\sqrt{n}}.$$

Step 1. (Alternating Series Test)
The Alternating Series Test (see, e.g., Theorem 3.21 in *Principles of Mathematical Analysis* by Rudin) states that if $\{b_n\}$ is a decreasing sequence of nonnegative numbers with $\lim_{n\to\infty} b_n = 0$, then the series $\sum(-1)^n b_n$ converges. Here, note that:

$$n+\sqrt{n} \text{ is increasing } \implies b_n \text{ is decreasing,}$$

and clearly,

$$\lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 0.$$

Thus, the series converges.

Step 2. (Absolute Convergence)
Consider the absolute series:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n+\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}.$$

Since $n+\sqrt{n} \leq 2n$ for all $n \geq 1$, we have:

$$\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series and diverges. Hence, by the Comparison Test, the absolute series diverges.

Conclusion: The series converges *conditionally* but not absolutely.

Problem 2 (10 points)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies

$$\lim_{h\to 0} \Big(f(x+h) - f(x-h) \Big) = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous? Explain your answer.

Solution. The condition only concerns the symmetric difference $f(x+h)-f(x-h)$. It does not require that the one–sided limits of $f(x+h)$ and $f(x-h)$ individually approach $f(x)$.

Counterexample: Define the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

Then for every $x \neq 0$ and for all sufficiently small h (so that $x \pm h \neq 0$), we have

$$f(x+h) - f(x-h) = 1 - 1 = 0.$$

At $x = 0$, for any $h \neq 0$ we have

$$f(0+h) - f(0-h) = f(h) - f(-h) = 1 - 1 = 0.$$

Thus, the given condition holds for every $x \in \mathbb{R}$. However, f is discontinuous at $x = 0$ since

$$\lim_{x \to 0} f(x) = 1 \quad \text{but} \quad f(0) = 0.$$

Conclusion: The condition does not imply that f is continuous.

Problem 3 (10 points)**.** If $f : X \to Y$ is a continuous map between metric spaces X and Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every subset $E \subseteq X$.

Solution. Let $x \in \overline{E}$. By the definition of closure, for every $\delta > 0$ there exists a point $y \in E$ with $d_X(x, y) < \delta$.

Since f is continuous at x, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y\big(f(x), f(y)\big) < \varepsilon.$$

Thus, for every $\varepsilon > 0$, every neighborhood of $f(x)$ contains some point $f(y)$ with $y \in E$. This implies $f(x)$ is a limit point of $f(E)$, so $f(x) \in \overline{f(E)}$.

Therefore,

$$f(\overline{E}) \subseteq \overline{f(E)}.$$

Problem 4 (10 points)**.** Let f and g be continuous maps from a metric space X to a metric space Y, and let E be a dense subset of X (i.e. $\overline{E} = X$). Prove that if $f(p) = g(p)$ for all $p \in E$, then f and g are identical maps (i.e. $f(p) = g(p)$ for all $p \in X$).

Solution. Let $x \in X$. Because E is dense in X, there exists a sequence $\{p_n\} \subset E$ such that

$$\lim_{n \to \infty} p_n = x.$$

Since f and g are continuous, we have

$$\lim_{n \to \infty} f(p_n) = f(x) \quad \text{and} \quad \lim_{n \to \infty} g(p_n) = g(x).$$

But by the hypothesis, $f(p_n) = g(p_n)$ for all n. Therefore,

$$f(x) = \lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} g(p_n) = g(x).$$

Since x was arbitrary, we conclude that $f(x) = g(x)$ for all $x \in X$.

Problem 5 (Extra Credit; 10 points)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a complex sequence, and define its arithmetic means μ_n by

$$\mu_n := \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

(a) If $\lim_{n\to\infty} a_n = a$, prove that $\lim_{n\to\infty} \mu_n = a$.

(b) Put $d_n = a_{n+1} - a_n$ for $n \geq 1$. Assume that $\lim_{n\to\infty} n\, d_n = 0$ and that $\{\mu_n\}$ converges. Prove that $\{a_n\}$ converges. *Hint:* You can show that

$$a_n - \mu_n = \frac{1}{n}\sum_{k=1}^{n-1} k\, d_k.$$

Solution. **(a) Convergence of the Means.**
Since $\lim_{n\to\infty} a_n = a$, for every $\varepsilon > 0$ there exists N such that for all $n \geq N$,

$$|a_n - a| < \varepsilon.$$

Write the arithmetic mean as

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Split the sum into two parts:

$$\mu_n = \frac{1}{n}\sum_{k=1}^{N-1} a_k + \frac{1}{n}\sum_{k=N}^{n} a_k.$$

Then,

$$\mu_n - a = \frac{1}{n}\sum_{k=1}^{N-1}(a_k - a) + \frac{1}{n}\sum_{k=N}^{n}(a_k - a).$$

For the fixed finite sum, as $n \to \infty$ the term

$$\frac{1}{n}\sum_{k=1}^{N-1}(a_k - a)$$

tends to 0. For the other term, since $|a_k - a| < \varepsilon$ for all $k \geq N$,

$$\left|\frac{1}{n}\sum_{k=N}^{n}(a_k - a)\right| \leq \frac{1}{n}\sum_{k=N}^{n} \varepsilon \leq \varepsilon.$$

Hence, $|\mu_n - a|$ can be made arbitrarily small, proving that

$$\lim_{n\to\infty} \mu_n = a.$$

This result is sometimes known as the *Cesàro Mean Theorem.*

(b) Convergence of the Sequence $\{a_n\}$.

We are given the hint:

$$a_n - \mu_n = \frac{1}{n} \sum_{k=1}^{n-1} k\, d_k, \quad \text{with } d_k = a_{k+1} - a_k.$$

Assume that $\lim_{n\to\infty} n\, d_n = 0$. This means that for every $\varepsilon > 0$, there is an index N such that for all $k \geq N$,

$$|k\, d_k| < \varepsilon.$$

Now, split the sum in the formula for $a_n - \mu_n$ into two parts:

$$\frac{1}{n} \sum_{k=1}^{n-1} k\, d_k = \frac{1}{n} \sum_{k=1}^{N-1} k\, d_k + \frac{1}{n} \sum_{k=N}^{n-1} k\, d_k.$$

- The first term is a fixed finite sum divided by n, so it tends to 0 as $n \to \infty$. - For the second term, using the bound $|k\, d_k| < \varepsilon$ for all $k \geq N$, we have:

$$\left| \frac{1}{n} \sum_{k=N}^{n-1} k\, d_k \right| \leq \frac{1}{n} \sum_{k=N}^{n-1} |k\, d_k| < \frac{n-N}{n} \varepsilon \leq \varepsilon.$$

Thus, for sufficiently large n, $|a_n - \mu_n|$ can be made arbitrarily small.

Since by hypothesis $\{\mu_n\}$ converges (say to some limit L) and $|a_n - \mu_n| \to 0$, it follows that

$$a_n = \mu_n + (a_n - \mu_n) \to L.$$

Therefore, the sequence $\{a_n\}$ converges.

Conclusion: Under the stated conditions, the convergence of the arithmetic means and the control on the differences d_n ensure the convergence of the original sequence $\{a_n\}$.