Problem Set 1

Due Wednesday, February 5, 2025

Matt Krasnow

Problem 1. (Exercise 1.1 in Rudin)
If r is a non-zero rational number and x is an irrational number, prove that $r+x$ and rx are irrational.

Proof: We will prove both parts by contradiction.

Proof that $r+x$ is irrational

Step 1: Assume the contrary.
Suppose, for the sake of contradiction, that $r+x$ is rational. By definition, we can express:

$$r = \frac{m}{n}, \quad \text{with } m,n \in \mathbb{Z},\ n \neq 0,$$

and

$$r + x = \frac{a}{b}, \quad \text{with } a,b \in \mathbb{Z},\ b \neq 0.$$

Step 2: Solve for x.
Subtracting r from both sides gives:

$$x = \frac{a}{b} - \frac{m}{n}.$$

Step 3: Simplify the expression.
Combining the fractions, we have:

$$x = \frac{an - mb}{bn}.$$

Since $an - mb$ and bn are integers (with $bn \neq 0$), x is a rational number.

Step 4: Arrive at a contradiction.
This conclusion contradicts the assumption that x is irrational. Hence, the assumption that $r+x$ is rational is false, so:

$$r + x \text{ is irrational.}$$

Proof that rx is irrational

Step 1: Assume the contrary.
Suppose, for the sake of contradiction, that rx is rational. Then we can write:

$$rx = \frac{c}{d}, \quad \text{with } c,d \in \mathbb{Z},\ d \neq 0.$$

Step 2: Substitute the expression for r.
Since r is a nonzero rational number, write:

$$r = \frac{m}{n}, \quad \text{with } m, n \in \mathbb{Z},\ n \neq 0 \text{ and } m \neq 0.$$

Then, the equation becomes:

$$\frac{m}{n} \cdot x = \frac{c}{d}.$$

Step 3: Solve for x.
Multiplying both sides by $\frac{n}{m}$ (which is valid because $m \neq 0$) gives:

$$x = \frac{cn}{md}.$$

Step 4: Check rationality.
Since cn and md are integers (with $md \neq 0$), x is rational. That is, x can be written in the form:

$$x = \frac{f}{g}, \quad \text{with } f, g \in \mathbb{Z},\ g \neq 0.$$

Step 5: Arrive at a contradiction.
This contradicts the hypothesis that x is irrational. Therefore, the assumption that rx is rational must be false, so:

$$rx \text{ is irrational.}$$

$\square$

Problem 2. (Exercise 1.4 in Rudin) Let E be a non-empty subset of an ordered set. Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. I will prove this directly.
An ordered set is defined in Rudin 1.5 to have the following properties:

1. If $x, y \in S$, then exactly one of the following is true:
$$x \leq y, \quad x \geq y, \quad x = y$$
2. If $x, y, z \in S$, if
$$x < y \quad \text{and} \quad y < z$$
then
$$x < z.$$

By definition of a lower bound,
$$\forall x \in E, \quad \alpha \leq x.$$
By definition of an upper bound,
$$\forall x \in E, \quad \beta \geq x.$$
Let x_i be the ith value in E,
$$\alpha \leq x_i \quad \text{and} \quad x_i \leq \beta.$$
Thus, by the transitive property,
$$\alpha \leq \beta.$$
□

Problem 3. (Exercise 1.5 in Rudin) Let A be a non-empty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof: I will prove this directly.
By definition,

$$\inf(A) \leq X \quad \forall X \in A$$

$$\sup(-A) \geq X \quad \forall X \in -A$$

By negating a set, the order is reversed by definition.
Thus,
the least number becomes the largest, the largest becomes the least, etc.
Thus, it follows that

$$-\sup(-A) \leq X \quad \forall X \in -(-A)$$

because $\sup(-A)$ is the greatest value in the set A.
By properties of fields, $-(-X) = X$.
Thus, $-(-A) = A$.
It follows that

$$-\sup(-A) \leq X \quad \forall X \in A$$

This is the same form as the $\inf(A)$.
Thus,

$$\inf(A) = -\sup(-A)$$

□

Problem 4. Let $A \subset \mathbb{R}$ be a non-empty set of real numbers which is bounded above. Prove that, for any positive real number $\epsilon > 0$, there exists $x \in A$ such that $\sup A - x \leq \epsilon$.

Proof

I will prove this directly using case work.

Case 1: $\sup(A) \in A$

If $\sup(A) \in A$, this implies that

$$\sup(A) - \sup(A) \leq \epsilon$$

for any value $\epsilon \in \mathbb{R}_{>0}$ because

$$\sup(A) - \sup(A) = 0.$$

Case 2: $\sup(A) \notin A$

Since A is bounded from above and is a subset of the real numbers, the values in A must approach but never achieve $\sup(A)$.

This indicates that A is an infinitely long set which tends to $\sup(A)$, getting infinitely closer in value.

Since there exist infinitely close values to $\sup(A)$ which are in A,

$$\sup(A) - x \text{ must be less than any value of } \epsilon \in \mathbb{R}_{>0} \text{ for some value of } x.$$