Problem Set 1

Due Wednesday, February 5, 2025

Matt Krasnow

Problem 1. (Exercise 1.1 in Rudin) If r is a non-zero rational number and x is an irrational number, prove that $r + x$ and rx are irrational.

Proof: I will prove this by contradiction.

Proving that $r + x$ is irrational

Step 1: Assume the contrary. Assume, for contradiction, that $r + x$ is rational. By definition of rational numbers, we can write:

$$r = \frac{m}{n}, \quad m, n \in \mathbb{Z}, \quad n \neq 0$$

$$r + x = \frac{a}{b}, \quad a, b \in \mathbb{Z}, \quad b \neq 0$$

Step 2: Solve for x. Rearranging the equation:

$$x = \frac{a}{b} - \frac{m}{n}$$

Step 3: Simplify the expression. Since the rationals are closed under subtraction, we simplify:

$$x = \frac{an - mb}{bn}$$

Since integers are closed under multiplication and subtraction, the numerator and denominator are integers. Thus, x is of the form $\frac{c}{d}$ where $c, d \in \mathbb{Z}$ and $d \neq 0$, meaning x is rational.

Step 4: Contradiction. This contradicts our original assumption that x is irrational. Thus, our assumption that $r + x$ is rational must be false, and we conclude:

$$r + x \text{ is irrational.}$$

Proving that rx is irrational

Step 1: Assume the contrary. Assume, for contradiction, that rx is rational. By definition of rational numbers, we can write:

$$rx = \frac{c}{d}, \quad c, d \in \mathbb{Z}, \quad d \neq 0$$

Step 2: Express x explicitly. Since r is rational, we substitute $r = \frac{m}{n}$:

$$\frac{m}{n}x = \frac{c}{d}$$

Step 3: Solve for x. Rearranging,

$$x = \frac{cn}{md}$$

Step 4: Check rationality. Since integers are closed under multiplication, we see that x is of the form:

$$x = \frac{f}{g}, \quad f, g \in \mathbb{Z}, \quad g \neq 0$$

which implies x is rational.

Step 5: Contradiction. This contradicts our original assumption that x is irrational. Thus, our assumption that rx is rational must be false, and we conclude:

$$rx \text{ is irrational.}$$

$\square$

Problem 2. (Exercise 1.4 in Rudin) Let E be a non-empty subset of an ordered set. Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. I will prove this directly.
An ordered set is defined in Rudin 1.5 to have the following properties:

1. If $x, y \in S$, then exactly one of the following is true:
$$x \leq y, \quad x \geq y, \quad x = y$$

2. If $x, y, z \in S$, if
$$x < y \quad \text{and} \quad y < z$$
then
$$x < z.$$

By definition of a lower bound,
$$\forall x \in E, \quad \alpha \leq x.$$

By definition of an upper bound,
$$\forall x \in E, \quad \beta \geq x.$$

Let x_i be the ith value in E,
$$\alpha \leq x_i \quad \text{and} \quad x_i \leq \beta.$$

Thus, by the transitive property,
$$\alpha \leq \beta.$$

□

Problem 3. (Exercise 1.5 in Rudin) Let A be a non-empty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof: I will prove this directly.
By definition,

$$\inf(A) \leq X \quad \forall X \in A$$

$$\sup(-A) \geq X \quad \forall X \in -A$$

By negating a set, the order is reversed by definition.
Thus,
the least number becomes the largest, the largest becomes the least, etc.
Thus, it follows that

$$-\sup(-A) \leq X \quad \forall X \in -(-A)$$

because $\sup(-A)$ is the greatest value in the set A.
By properties of fields, $-(-X) = X$.
Thus, $-(-A) = A$.
It follows that

$$-\sup(-A) \leq X \quad \forall X \in A$$

This is the same form as the $\inf(A)$.
Thus,

$$\inf(A) = -\sup(-A)$$

□

Problem 4. Let $A \subset \mathbb{R}$ be a non-empty set of real numbers which is bounded above. Prove that, for any positive real number $\epsilon > 0$, there exists $x \in A$ such that $\sup A - x \leq \epsilon$.

Proof

I will prove this directly using case work.

Case 1: $\sup(A) \in A$

If $\sup(A) \in A$, this implies that

$$\sup(A) - \sup(A) \leq \epsilon$$

for any value $\epsilon \in \mathbb{R}_{>0}$ because

$$\sup(A) - \sup(A) = 0.$$

Case 2: $\sup(A) \notin A$

Since A is bounded from above and is a subset of the real numbers, the values in A must approach but never achieve $\sup(A)$.

This indicates that A is an infinitely long set which tends to $\sup(A)$, getting infinitely closer in value.

Since there exist infinitely close values to $\sup(A)$ which are in A,

$$\sup(A) - x \text{ must be less than any value of } \epsilon \in \mathbb{R}_{>0} \text{ for some value of } x.$$