CS1810-S25

Assignment #1

Due: 11:59pm ET, February 14, 2025

Homework 1: Regression

Introduction

This homework is on different three different forms of regression: kernelized regression, nearest neighbors regression, and linear regression. We will discuss implementation and examine their tradeoffs by implementing them on the same dataset, which consists of temperature over the past 800,000 years taken from ice core samples.

The folder `data` contains the data you will use for this problem. There are two files:

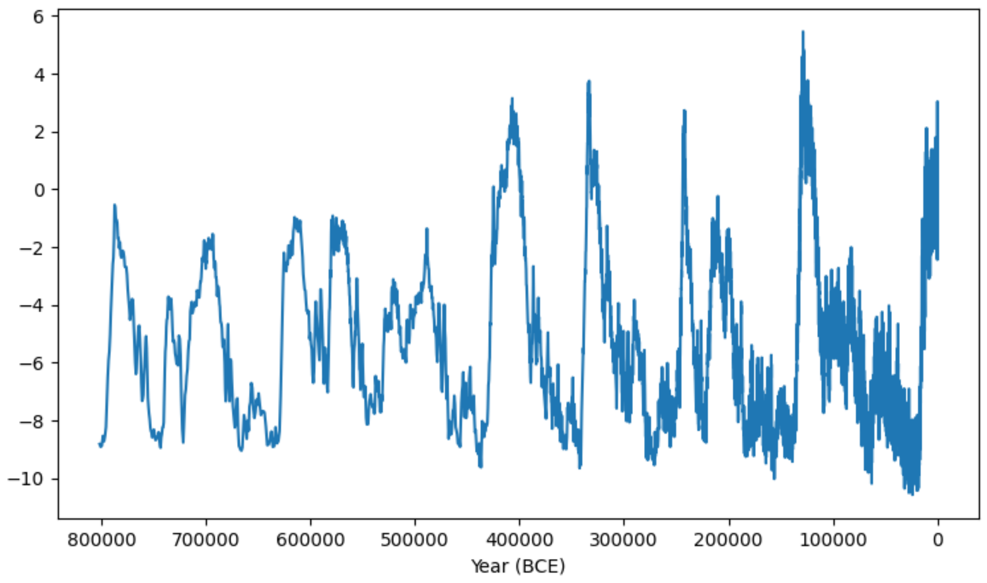
- `earth_temperature_sampled_train.csv`
- `earth_temperature_sampled_test.csv`

Each has two columns. The first column is the age of the ice core sample. The second column is the approximate difference in a year's temperature (K) from the average temperature of the 1,000 years preceding it. The temperatures were retrieved from ice cores in Antarctica (Jouzel et al. 2007)[1].

The following is a snippet of the data file:

```
# Age, Temperature
399946,0.51
409980,1.57
```

And this is a visualization of the full dataset:



Due to the large magnitude of the years, we will work in terms of thousands of years BCE in these problems. This is taken care of for you in the provided notebook.

[1]Retrieved from https://www.ncei.noaa.gov/pub/data/paleo/icecore/antarctica/epica_domec/edc3deuttemp2007.txt Jouzel, J., Masson-Delmotte, V., Cattani, O., Dreyfus, G., Falourd, S., Hoffmann, G., … Wolff, E. W. (2007). Orbital and Millennial Antarctic Climate Variability over the Past 800,000 Years. *Science, 317*(5839), 793–796. doi:10.1126/science.1141038

Resources and Submission Instructions

If you find that you are having trouble with the first couple problems, we recommend going over the fundamentals of linear algebra and matrix calculus (see links on website). The relevant parts of the cs181-textbook notes are Sections 2.1 - 2.7. We strongly recommend reading the textbook before beginning the homework.

We also encourage you to first read the Bishop textbook, particularly: Section 2.3 (Properties of Gaussian Distributions), Section 3.1 (Linear Basis Regression), and Section 3.3 (Bayesian Linear Regression). (Note that our notation is slightly different but the underlying mathematics remains the same!).

Please type your solutions after the corresponding problems using this LaTeX template, and start each problem on a new page. You may find the following introductory resources on LaTeX useful: LaTeX Basics and LaTeX tutorial with exercises in Overleaf

Homeworks will be submitted through Gradescope. You will be added to the course Gradescope once you join the course Canvas page. If you haven't received an invitation, contact the course staff through Ed.

Please submit the writeup PDF to the Gradescope assignment 'HW1'. Remember to assign pages for each question.

Please submit your LaTeXfile and code files to the Gradescope assignment 'HW1 - Supplemental'. Your files should be named in the same way as we provide them in the repository, e.g. `hw1.pdf`, etc.

Problem 1 (kNN and Kernels, 35pts)

You will now implement two non-parametric regressions to model temperatures over time.

Make sure to include all required plots in your PDF. Passing all test cases does not guarantee that your solution is correct, and we encourage you to write your own.

1. Recall that kNN uses a predictor of the form

$$f(x^*) = \frac{1}{k}\sum_n y_n \mathbb{I}(x_n \texttt{ is one of k-closest to } x^*),$$

where $\mathbb{I}$ is an indicator variable.

   (a) The kNN implementation **has been provided for you** in the notebook. Run the cells to plot the results for $k = \{1, 3, N-1\}$, where N is the size of the dataset. Describe how the fits change with k. Please include your plot in your solution PDF.

   (b) Now, we will evaluate the quality of each model *quantitatively* by computing the error on the provided test set. Write Python code to compute test MSE for each value of k. Which solution has the lowest MSE?

2. *Kernel-based regression* techniques are another form of non-parametric regression. Consider a kernel-based regressor of the form

$$f_\tau(x^*) = \frac{\sum_n K_\tau(x_n, x^*)y_n}{\sum_n K_\tau(x_n, x^*)}$$

where $\mathcal{D}_{\texttt{train}} = \{(x_n, y_n)\}_{n=1}^N$ are the training data points, and x^* is the point for which you want to make the prediction. The kernel $K_\tau(x, x')$ is a function that defines the similarity between two inputs x and x'. A popular choice of kernel is a function that decays as the distance between the two points increases, such as

$$K_\tau(x, x') = \exp\left(-\frac{(x-x')^2}{\tau}\right)$$

where τ represents the square of the lengthscale (a scalar value that dictates how quickly the kernel decays).

   (a) First, implement the `kernel_regressor` function in the notebook, and plot your model for years in the range 800,000 BC to 400,000 BC at 1000 year intervals for the following three values of τ: $1, 50, 2500$. Since we're working in terms of thousands of years, this means you should plot $(x, f_\tau(x))$ for $x = 400, 401, \ldots, 800$. **In no more than 10 lines**, describe how the fits change with τ. Please include your plot in your solution PDF.

   (b) Denote the test set as $\mathcal{D}_{\texttt{test}} = \{(x'_m, y'_m)\}_{m=1}^M$. Write down the expression for MSE of f_τ over the test set as a function of the training set and test set. Your answer may include $\{(x'_m, y'_m)\}_{m=1}^M$, $\{(x_n, y_n)\}_{n=1}^N$, and K_τ, but not f_τ.

   (c) Compute the MSE on the provided test set for the three values of τ. Which model yields the lowest MSE? Conceptually, why is this the case? Why would choosing τ based on $\mathcal{D}_{\texttt{train}}$ rather than $\mathcal{D}_{\texttt{test}}$ be a bad idea?

   (d) Describe the time and space complexity of both kernelized regression and kNN with respect to the size of the training set N. How, if at all, does the size of the model—everything that needs to be stored to make predictions—change with the size of the training set N? How, if at all, do the number of computations required to make a prediction for some input x^* change with the size of the training set N?.

   (e) What is the exact form of $\lim_{\tau\to 0} f_\tau(x^*)$?

Solution

1. a)

   As τ increases, the model becomes smoother and less sensitive to noise. Even as τ gets extremely large, it still is able to capture some of the patterns in the data, which is not true for kNN; kNN converges to the mean of the dataset as $k \to N$. In this case, the kernelized model converges slower.

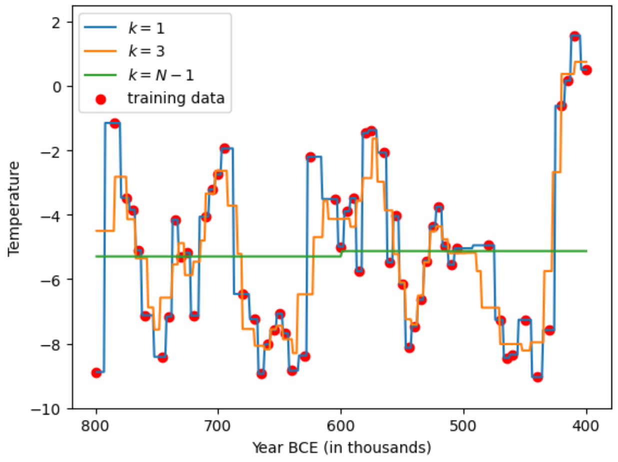


Figure 1: Plot of the kNN model fits for different values of k.

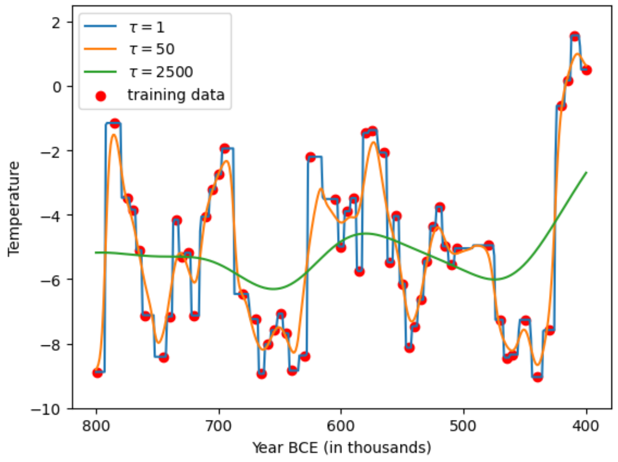


Figure 2: Plot of the kernel modle fits for different values of τ.

   b)

```
def model_mse(predictions, true):
"""
Calculate the MSE for the given model predictions, with respect to the true values

:param predictions: predictions given by the model
:param true: corresponding true values
```

```
        :return: the mean squared error
        """
        return np.mean((predictions - true) ** 2)

        k_values = [1, 3, len(year_train) - 1]
    for k in k_values:
        preds = predict_knn(year_test, k, year_train, temp_train)
        mse = model_mse(preds, temp_test)
        print(f"k = {k}, Test MSE = {mse}")
```

```
k = 1, Test MSE = 1.7406000000000004
k = 3, Test MSE = 3.8907662222222212
k = 56, Test MSE = 9.528571442602042
```

2.

$$\text{MSE} = \frac{1}{M}\sum_{m=1}^{M}\left(y'_m - \frac{\sum_{n=1}^{N} K_\tau(x_n, x'_m)\, y_n}{\sum_{n=1}^{N} K_\tau(x_n, x'_m)}\right)^2$$

3. After computing the test set MSEs for $\tau = 1$, $\tau = 50$, and $\tau = 2500$, we observed that the model with $\tau = 50$ yields the lowest MSE.

   Conceptually, this makes sense because:

   - A small τ leads to overfitting, as the model becomes too sensitive to noise in the training data.
   - A large τ leads to underfitting, as the model fails to capture the underlying patterns in the data; it just fits to the average values and does not weight the data points appropriately.

   Choosing τ based on $\mathcal{D}_{\texttt{train}}$ rather than $\mathcal{D}_{\texttt{test}}$ would be a bad idea because it would lead to overfitting on the training data. The test data is very helpful because it allows us to evaluate the model based on data it hasn't seen yet, but which is still representative of the real, unerlying data-generating process. If we choose τ based on the training data, the tau with the lowest MSE would be the smallest tau tested because it would be the closest fit to the training data, but would not generalize to unseen data.

4. d)

   Time complexity

   To determine the time complexity, we need to understand what is occuring for each data point. For each data point, we must calculate the kernel for every other data point. Thus, the time complexity is $O(N^2)$ because for every data point n in N, we have to do N calculations.$N \times N = N^2$.

   Space complexity is $O(N)$ because we need to store N data points.

5. e)

$$\lim_{\tau \to 0} f_\tau(x^*) = \frac{\sum_{n=1}^{N} \mathbb{I}\{x_n = x^*\}\, y_n}{\sum_{n=1}^{N} \mathbb{I}\{x_n = x^*\}},$$

   where $\mathbb{I}$ is the indicator function.

   If $x_n = x^*$, then $K_\tau(x_n, x^*) = 1$ and all other terms are 0. Thus, the limit is equal to the value of y_n for that data point.

Problem 2 (Deriving Linear Regression, 20pts)

We now seek to model the temperatures with a parametric method: linear regression. Before we implement anything, let's revisit the mathematical formulation of linear regression. Specifically, the solution for the least squares linear regression "looks" kind of like a ratio of covariance and variance terms. In this problem, we will make that connection more explicit.

Suppose we have some 2-D data where each observation has the form (x, y) and is independent and identically distributed according $x \sim p(x)$, $y \sim p(y|x)$. We will consider the process of fitting these data from this distribution with the best linear model possible, that is a linear model of the form $\hat{y} = wx$ that minimizes the expected squared loss $E_{x,y}[(y - \hat{y})^2]$.

Note: The notation $E_{x,y}$ indicates an expectation taken over the joint distribution $p(x, y)$. This essentially just means to treat x and y as random.

1. Derive an expression for the optimal w, that is, the w that minimizes the expected squared loss above. You should leave your answer in terms of moments of the distribution, e.g. terms like $E_x[x]$, $E_x[x^2]$, $E_y[y]$, $E_y[y^2]$, $E_{x,y}[xy]$ etc.
2. Note that while x, y are data that we have access to, $E_{x,y}[yx]$ is a theoretical constant. Keeping in mind the interpretation of expectations as average values, how could you use observed data $\{(x_n, y_n)\}_{n=1}^N$ to estimate $E_{x,y}[yx]$ and $E_x[x^2]$?
3. In general, moment terms like $E_{x,y}[yx]$, $E_{x,y}[x^2]$, $E_{x,y}[yx^3]$, $E_{x,y}[\frac{x}{y}]$, etc. can easily be estimated from the data (like you did above). If you substitute in these empirical moments, how does your expression for the optimal w^* in this problem compare with the optimal $\hat{\boldsymbol{w}}$ from Problem 4.3 of HW0?
4. Many common probabilistic linear regression models assume that variables x and y are jointly Gaussian. Did any of your above derivations rely on the assumption that x and y are jointly Gaussian? Why or why not?

Solution

Solution

1. a)

   We want to minimize

$$E_{x,y}\left[(y-\hat{y})^2\right]$$

$$E_{x,y}\left[(y-wx)^2\right]$$

$$E_{x,y}\left[y^2-2wxy+w^2x^2\right]=$$

$$E_{x,y}\left[y^2\right]-2wE_{x,y}\left[xy\right]+w^2E_{x,y}\left[x^2\right]$$

   We can find the minimum of this function by taking the derivative w.r.t. w

$$\frac{dMSE}{dw}=2wE_{x,y}\left[x^2\right]-2E_{x,y}\left[xy\right]=0$$

$$w=\frac{E_{x,y}\left[xy\right]}{E_{x,y}\left[x^2\right]}$$

   We can confirm this is the minimum by checking the second derivative.

$$\frac{d^2MSE}{dw^2}=2E_{x,y}\left[x^2\right] \text{ is always positive.}$$

   Thus, it is convex & this is a minimum.

2. b)

   We can use method of moments estimators to estimate the values for these values.

   MoM for $E_{x,y}\left[xy\right]$:

$$\approx\frac{1}{n}\sum_{i=1}^{n}x_iy_i$$

   MoM for $E_{x,y}\left[x^2\right]$:

$$\approx\frac{1}{n}\sum_{i=1}^{n}x_i^2$$

3. c)

   If we use method of moments estimators we get exactly the same as the closed-form OLS solution from HW0 for the one-dimensional (no-intercept) case. This is interesting because it shows that the theoretical derivation is equivalent to the practical implementation.

4. d)

No, none of my derivations relied on the fact or the assumption that variables x and y would be jointly Gaussian. We never made any claim that or any assumption that they were Gaussian. In this case, using method of moments estimators simply is nonparametric and you do not need to define or assume that the data comes from a specific data generating process to use them and to find the values. In problem 2.1, the derivation for the optimal w is just found from deriving it algebraically without any assumptions.

Problem 3 (Basis Regression, 30pts)

Having reviewed the theory, we now implement some linear regression models for the temperature. If we just directly use the data as given to us, we would only have a one dimensional input to our model, the year. To create a more expressive linear model, we will introduce basis functions.

Make sure to include all required plots in your PDF.

1. We will first implement the four basis regressions below. (The first basis has been implemented for you in the notebook as an example.) Note that we introduce an addition transform f (already into the provided notebook) to address concerns about numerical instabilities.

   (a) $\phi_j(x) = f(x)^j$ for $j = 1, \ldots, 9$. $f(x) = \frac{x}{1.81 \cdot 10^2}$.

   (b) $\phi_j(x) = \exp\left\{-\frac{(f(x) - \mu_j)^2}{5}\right\}$ for $\mu_j = \frac{j+7}{8}$ with $j = 1, \ldots, 9$. $f(x) = \frac{x}{4.00 \cdot 10^2}$.

   (c) $\phi_j(x) = \cos(f(x)/j)$ for $j = 1, \ldots, 9$. $f(x) = \frac{x}{1.81}$.

   (d) $\phi_j(x) = \cos(f(x)/j)$ for $j = 1, \ldots, 49$. $f(x) = \frac{x}{1.81 \cdot 10^{-1}}$. [a]

   * Note: Please make sure to add a bias term for all your basis functions above in your implementation of the `make_basis`.

   Let

$$\phi(\mathbf{X}) = \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_N) \end{bmatrix} \in \mathbb{R}^{N \times D}.$$

   You will complete the `make_basis` function which must return $\phi(\mathbf{X})$ for each part (a) - (d). You do NOT need to submit this code in your LaTeXwriteup.

   Then, create a plot of the fitted regression line for each basis against a scatter plot of the training data. Boilerplate plotting code is provided in the notebook—you will only need to finish up a part of it. **All you need to include in your writeup for this part are these four plots.**

2. Now we have trained each of our basis regressions. For each basis regression, compute the MSE on the test set. Discuss: do any of the bases seem to overfit? Underfit? Why?

3. Briefly describe what purpose the transforms f serve: why are they helpful?

4. As in Problem 1, describe the space and time complexity of linear regression. How does what is stored to compute predictions change with the size of the training set N and the number of features D? How does the computation needed to compute the prediction for a new input depend on the size of the training set N? How do these complexities compare to those of the kNN and kernelized regressor?

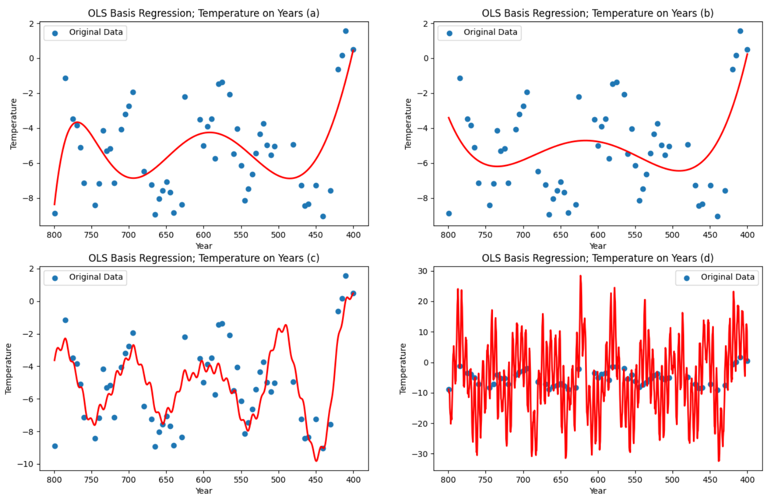
5. Briefly compare and constrast the different regressors: kNN, kernelized regression, and linear regression (with bases). Are some regressions clearly worse than others? Is there one best regression? How would you use the fact that you have these multiple regression functions?

Note: Recall that we are using a different set of inputs $\mathbf{X}$ for each basis (a)-(d). Although it may seem as though this prevents us from being able to directly compare the MSE since we are using different data, each transformation can be considered as being a part of our model. Contrast this with transformations (such as standardization) that cause the variance of the target $\mathbf{y}$ to be different; in these cases the MSE can no longer be directly compared.

[a]For the trigonometric bases (c) and (d), the periodic nature of cosine requires us to transform the data such that the lengthscale is within the periods of each element of our basis.

Solution

Solution



1.

2. Well, the MSE for the first three models is relatively similar to each other. Model number one seems to have a decent fit to the data. Model two is pretty underfit to the data. Model three is relatively well fit to the data.

   It's obvious that the fourth model underfits the data because the test MSE is so high.

3. The purpose of the F transforms is to map the data, which is not necessarily linear, into a space that is linear so that we can apply the properties of linear regression. this allows us to use the very powerful tool of linear regression in a case where we normally would not be able to. One of the key assumptions of linear regression is that the data must be linear. This is extremely restrictive; using F transforms allows us to use this power, simple, and interpretable model.

4. The space complexity of lienar regression is $O(ND)$ because we need to store N data points and D features.
   Since X is a matrix of size N x w, finding $(X'X)$ is $O(Nw^2)$ and finding the inverse is $O(w^3)$
   $(X'Y)$ takes$O(n * w^2)$ time and gives us a wxw matrix
   finally the matrix multiplication takes $O(w^3)$ time
   so the overall time is $O(w^2 * (n + w))$

   The storage does not change with the size of the training set N; we only need to store the calucated coefficeints for the features w. So we only need to store w numbers.
   The time to compute also does not change based onthe training size–the computation only relies on the final parameters w – we are calcualting the dot product of $y = w^T x$

   kNN
   Storage complexity is $O(N)$ because we need to store N data points.
   Time complexity is $O(Nw)$ or maybe $O(Nlog(w))$ because we need to find the k nearest neighbors – the distance formula is the complex part.

Kernelized regression
Storage complexity is $O(N)$ because we need to store N data points.
Prediction time we need to calculate the kernel for all of the points to each of hte other points. If the kernel function is $O(w)$, then the time complexity is $O(N * w)$

5. We have three different kinds of regressors.

   I don't think there is an objective "best" and "worst", but each is definetley situational and only applicable in certain cases.

   kNN is certainly limited in the sense that it is not able to extrapolate. It can only predict based on the data it has seen.

   Kernelized regression is a bit more flexible because it is able to use a kernel(a weighting function to determine different distances) but it is still limited in the sense that it is not able to extrapolate. It can only predict based on the data it has seen.

   Linear regression is able to extrapolate, but it is a parametric model and the data may need to be massaged (transformed) into being linear to fit the assumptions. There are quite a few assumptions for lienar regression which makes it a little less flexible.

Problem 4 (Probablistic Regression and Regularization, 30pts)

Finally, we will preview Bayesian regression and explore its connection to regularization for linear models. Then, we will fit a regularized model to the temperature data. Although the content is related, you do not need to know the material from the lectures on frequentist model selection and Bayesian model selection to solve this problem.

Recall that the probabilistic version of linear regression states that

$$y_n = \mathbf{w}^\top \mathbf{x}_n + \epsilon_n, \quad \epsilon_n \sim \mathcal{N}(0, \sigma^2)$$

In Bayesian regression, we impose a prior $p(\mathbf{w})$ on the weights and fit the weights $\mathbf{w}$ through maximizing the posterior likelihood

$$p(\mathbf{w}|\mathbf{X}, \mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{w}, \mathbf{X})p(\mathbf{w})}{p(\mathbf{y}|\mathbf{X})}$$

Note: since we maximize with respect to $\mathbf{w}$, it suffices to just maximize the numerator.

1. Suppose $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \frac{\sigma^2}{\lambda}\mathbf{I})$. Show that maximizing the posterior likelihood is equivalent to minimizing

   $$\mathcal{L}_{ridge}(\mathbf{w}) = \frac{1}{2}||\mathbf{y} - \mathbf{X}\mathbf{w}||_2^2 + \frac{\lambda}{2}||\mathbf{w}||_2^2.$$

   Note that minimizing $\mathcal{L}_{ridge}(\mathbf{w})$ is exactly what ridge regression does.

   Hint: You don't need to solve for the maximizer/minimizer to show that the optimization problems are equivalent.

2. Solve for the value of $\mathbf{w}$ that minimizes $\mathcal{L}_{ridge}(\mathbf{w})$.

3. The Laplace distribution has the PDF

   $$L(a, b) = \frac{1}{2b} \exp\left(-\frac{|x - a|}{b}\right)$$

   Show that if all $w_d \sim L\left(0, \frac{2\sigma^2}{\lambda}\right)$, maximizing the posterior likelihood is equivalent to minimizing

   $$\mathcal{L}_{lasso}(\mathbf{w}) = \frac{1}{2}||\mathbf{y} - \mathbf{X}\mathbf{w}||_2^2 + \frac{\lambda}{2}||\mathbf{w}||_1.$$

   Note that minimizing $\mathcal{L}_{lasso}(\mathbf{w})$ is exactly what LASSO regression does.

4. Why is there no general closed form for the LASSO estimator, i.e. the value of $\mathbf{w}$ that minimizes $\mathcal{L}_{ridge}(\mathbf{w})$?

5. Since there is no general closed form for LASSO, we use numerical methods for estimating $\mathbf{w}$. One approach is to use *coordinate descent*, which works as follows:

   (a) Initialize $\mathbf{w} = \mathbf{w}_0$.

   (b) For each $d = 1, \ldots, D$ do the following 2 steps consecutively:

      i. Compute $\rho_d = \tilde{\mathbf{x}}_d^\top(\mathbf{y} - (\mathbf{X}\mathbf{w} - w_d\tilde{\mathbf{x}}_d))$. We define $\tilde{\mathbf{x}}_d$ as the d-th column of $\mathbf{X}$.

      ii. If $d = 1$, set $w_1 = \frac{\rho_1}{||\tilde{\mathbf{x}}_1||_2^2}$. Otherwise if $d \neq 1$, compute $w_d = \frac{\text{sign}(\rho_d)\max\left\{|\rho_d| - \frac{\lambda}{2}, 0\right\}}{||\tilde{\mathbf{x}}_d||_2^2}$.

   (c) Repeat step (b) until convergence or the maximum number of iterations is reached.

   Implement the `find_lasso_weights` function according to the above algorithm, letting $\mathbf{w}_0$ be a vector of ones and the max number of iterations be 5000. Then, fit models with $\lambda = 1, 10$ to basis (d) from Problem 3, plot the predictions, and compute the MSE's. You will need to do some preprocessing, but a completed helper function for this is already provided. How do the graphs and errors compare to those for the unregularized basis (d) model?

Solution

1. First, we need to write out the likelihood function:

$$y_n = \hat{\omega}^T x_n + \epsilon_n \quad \text{where} \quad \epsilon_n \sim \mathcal{N}(0, \sigma^2)$$

Thus,

$$\hat{y}_n | \hat{\omega} \sim \mathcal{N}(\hat{X}\hat{\omega}, \sigma^2 I)$$

$$P(\hat{y}_n|\hat{\omega}) = \left(\frac{1}{\sqrt{(2\pi\sigma^2)^N}}\right) \exp\left(-\frac{1}{2\sigma^2}\|\hat{y} - \hat{X}\hat{\omega}\|_2^2\right)$$

We have a prior for $\hat{\omega}$:

$$\hat{\omega} \sim \mathcal{N}(\hat{0}, \frac{\sigma^2}{\lambda} I)$$

$$P(\hat{\omega}) = \left(\frac{1}{\sqrt{(2\pi\frac{\sigma^2}{\lambda})^D}}\right) \exp\left(-\frac{\lambda}{2\sigma^2}\|\hat{\omega}\|_2^2\right)$$

Find the Posterior

$$P(\hat{\omega}|\hat{X}, \hat{y}) \propto \exp\left[-\frac{1}{2\sigma^2}\left(\|\hat{y} - \hat{X}\hat{\omega}\|_2^2 + \lambda\|\hat{\omega}\|_2^2\right)\right]$$

Finding the Maximum A Posteriori is the same as minimizing the negative log-posterior:

$$-\log\left[P(\hat{\omega}|\hat{X}, \hat{y})\right] \propto \frac{1}{2\sigma^2}\left[\|\hat{y} - \hat{X}\hat{\omega}\|_2^2 + \lambda\|\hat{\omega}\|_2^2\right]$$

Dropping the multiplicative constant:

$$\propto \|\hat{y} - \hat{X}\hat{\omega}\|_2^2 + \lambda\|\hat{\omega}\|_2^2$$

Dividing both sides by 2:

$$\frac{1}{2}\|\hat{y} - \hat{X}\hat{\omega}\|_2^2 + \frac{\lambda}{2}\|\hat{\omega}\|_2^2$$

which is exactly the ridge regression loss function.

MAP is equal to min ridge.

2. Solving for $\hat{\mathbf{w}}$ which minimizes $L_{\text{ridge}}(\mathbf{w})$.

   Take derivative of result from 1:

$$\frac{dL_{\text{ridge}}(\mathbf{w})}{d\mathbf{w}}$$

$$L_{\text{ridge}}(\mathbf{w}) = \frac{1}{2}(\mathbf{y} - X\mathbf{w})^T(\mathbf{y} - X\mathbf{w}) + \frac{\lambda}{2}\mathbf{w}^T\mathbf{w}$$

   Derivative of the first term:

$$\nabla_{\mathbf{w}} f(\mathbf{w}) = -X^T(\mathbf{y} - X\mathbf{w})$$

   by the chain rule.

   Derivative of the second term:

$$\nabla_{\mathbf{w}}\left(\frac{\lambda}{2}\mathbf{w}^T\mathbf{w}\right) = \lambda\mathbf{w}$$

   Thus,

$$\nabla_{\mathbf{w}} L_{\text{ridge}}(\mathbf{w}) = -X^T(\mathbf{y} - X\mathbf{w}) + \lambda\mathbf{w}$$

   Rearrange:

$$\nabla_{\mathbf{w}} L_{\text{ridge}}(\mathbf{w}) = X^TX\mathbf{w} - X^T\mathbf{y} + \lambda\mathbf{w}$$

   Set equal to 0 and solve for $\mathbf{w}$:

$$X^T\mathbf{y} = (X^TX + \lambda I)\mathbf{w}$$

   Assume $X^TX + \lambda I$ is invertible, multiply both sides by its inverse to get:

$$\hat{\mathbf{w}} = (X^TX + \lambda I)^{-1}X^T\mathbf{y}$$

3. This is a very similar process to part 1 of this question. We will use the same setups before:

$$P(\hat{y}_n|\hat{w}) = \left(\frac{1}{\sqrt{(2\pi\sigma^2)^N}}\right)\exp\left(\frac{-1}{2\sigma^2}\|\hat{y} - X\hat{w}\|_2^2\right)$$

   However, now our prior is different:

$$P(\hat{w}) = \frac{\lambda}{\sigma^2}\exp\left(\frac{-|\hat{w}|}{2\sigma^2/\lambda}\right)$$

   Now,

$$P(\hat{w}|X, \hat{y}) \propto \left(\frac{1}{\sqrt{(2\pi\sigma^2)^N}}\right)\exp\left(\frac{-1}{2\sigma^2}\|\hat{y} - X\hat{w}\|_2^2\right)$$

$$\frac{\lambda}{\sigma^2}\exp\left(\frac{-|\hat{w}|}{2\sigma^2/\lambda}\right)$$

Taking the negative log of this and dropping multiplicative constants:

$$-\log P(\hat{w}|X,\hat{y}) \propto$$

$$\left(\frac{1}{2\sigma^2}\|\hat{y}-X\hat{w}\|_2^2\right)+\frac{\lambda|\hat{w}|}{2\sigma^2/\lambda}$$

Rearranging and dropping the σ^2 constant,

$$\propto \frac{1}{2}\|\hat{y}-X\hat{w}\|_2^2+\frac{\lambda}{2}\|\hat{w}\|_1$$

Thus, they are equivalent.

4. There is no general closed form for LASSO because it is a non differentiable function. It contains an absolute value term which is not differentiable at 0. Thus, we must use other methods to solve for it.

5.
```
def find_lasso_weights(lam, X, y):
    """
    Fit the weights of a LASSO linear regression through the coordinate descent a

    :param lam: the lambda parameter
    :param X: the design matrix with training set features (with first column as
    :param y: the training set labels
    :return: the fitted weights as a numpy array
    """
    n, d = X.shape
    w = np.ones(d)
    max_iter = 5000
    tol = 1e-6  # convergence tolerance

    col_norm_sq = np.sum(X**2, axis=0)

    for iteration in range(max_iter):
        w_old = w.copy()
        for j in range(d):
            # r_j = y - (Xw - w[j]*X[:, j])
            r_j = y - (X.dot(w) - w[j] * X[:, j])
            # rho_j = x_j^T r_j
            rho_j = np.dot(X[:, j], r_j)

            if j == 0:
                w[j] = rho_j / col_norm_sq[j]
            else:
                w[j] = np.sign(rho_j) * max(np.abs(rho_j) - lam/2, 0) / col_norm_s
        # convergence check
        if np.max(np.abs(w - w_old)) < tol:
            break

    return w

# Helper function for standardizing inputs to LASSO
def preprocess_lasso(X):
    X = make_basis(X, part='d')
    X[:, 1:] = (X[:, 1:] - X[:, 1:].mean(axis=0)) / X[:, 1:].std(axis=0)
    return X

# Fit the weights for both models
phi_x_train = preprocess_lasso(year_train)

lam1, lam2 = 1, 10

w1 = find_lasso_weights(lam1, phi_x_train, temp_train)
w2 = find_lasso_weights(lam2, phi_x_train, temp_train)

x_array = np.arange(400, 801)
phi_x_array = preprocess_lasso(x_array)

y_pred1 = phi_x_array.dot(w1)
y_pred2 = phi_x_array.dot(w2)

plt.figure(figsize=(10, 6))
plt.plot(x_array, y_pred1, label=f'LASSO (lambda = {lam1})', color='blue')
plt.plot(x_array, y_pred2, label=f'LASSO (lambda = {lam2})', color='green')
plt.scatter(year_train, temp_train, label="Training Data", color="red")
plt.xlabel("Year BCE (in thousands)")
plt.ylabel("Temperature")
plt.title("LASSO Regression using Coordinate Descent")
plt.xticks(np.arange(400, 901, 100))
plt.gca().invert_xaxis()
plt.legend()
plt.savefig("img_output/p4.5.png", bbox_inches="tight")
plt.show()

y_train_pred1 = phi_x_train.dot(w1)
y_train_pred2 = phi_x_train.dot(w2)
mse1 = np.mean((temp_train - y_train_pred1)**2)
mse2 = np.mean((temp_train - y_train_pred2)**2)

print("MSE for LASSO with lambda = 1:", mse1)
print("MSE for LASSO with lambda = 10:", mse2)
```

FINAL ANSWER FOR PART 5:

```
MSE for LASSO with lambda = 1: 1.9508996956712439
MSE for LASSO with lambda = 10: 3.136512716950103
```

Clearly, at a point, using LASSO with too high a lambda will degrade the quality of the model. We are essentially "overcorrecting" the fit of the data. As lambda is a hyperparameter, we would want to find an idealized lambda value which minimizes the validation / test MSE

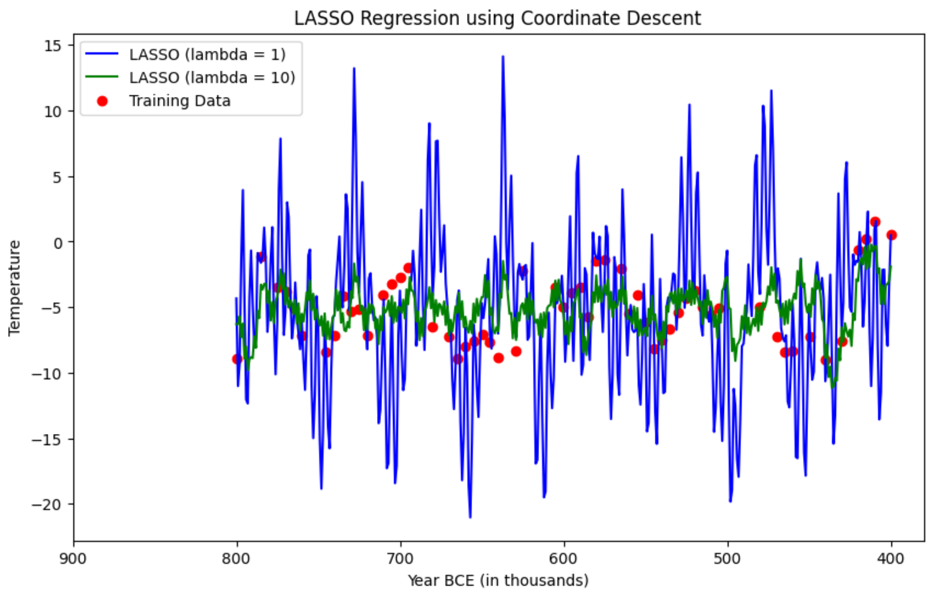


Figure 3: LASSO regression predictions using coordinate descent with $\lambda = 1$ and $\lambda = 10$.

Matt Krasnow

Collaborators and Resources

Whom did you work with, and did you use any resources beyond cs181-textbook and your notes?

I worked with me myself and I because im cool