# Problem Set 5

## Due Wednesday, April 2, 2025

**Problem 1** (10 points)**.** Recall the definition of the upper limit $\limsup_{n\to\infty} a_n$ from Definition 3.16 (or its equivalent formulation in Theorem 3.17) of Rudin. This exercise explains why the upper limit is called "lim sup". Suppose that $\{a_n\}$ is a real sequence such that $\limsup_{n\to\infty} a_n \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \sup\{a_j \mid j \geq n\} = \limsup_{n\to\infty} a_n.$$

**Solution.** Let $\{a_n\}$ be a real sequence and let $a^* = \limsup_{n\to\infty} a_n$. We are given that $a^* \in \mathbb{R}$. Let $s_n = \sup\{a_j \mid j \geq n\}$. The sequence $\{s_n\}$ is non-increasing. Indeed, the set $\{a_j \mid j \geq n+1\}$ is a subset of $\{a_j \mid j \geq n\}$, so the supremum over the smaller set cannot exceed the supremum over the larger set, i.e., $s_{n+1} \leq s_n$. Since $\{s_n\}$ is a non-increasing sequence of real numbers, it converges to a limit $L \in \mathbb{R} \cup \{-\infty\}$. We want to prove that $L = a^*$.

We will use Theorem 3.17 from *Principles of Mathematical Analysis* by Walter Rudin (3rd Ed.), which states that $a^*$ is the unique real number such that:

(a) For every $\epsilon > 0$, there exists an integer $N$ such that $n \geq N$ implies $a_n < a^* + \epsilon$.

(b) For every $\epsilon > 0$, and for every integer $N$, there exists an integer $n \geq N$ such that $a_n > a^* - \epsilon$.

We will show that $L = \lim_{n\to\infty} s_n$ satisfies these two properties.

**Proof that $L$ satisfies property (a):** Since $s_k = \sup\{a_j \mid j \geq k\}$, we have $a_k \leq s_k$ for all $k$. Since $s_n \to L$ and $\{s_n\}$ is non-increasing, we have $s_n \geq L$ for all $n$. Also, because $s_n \to L$, for any $\epsilon > 0$, there exists an integer $N$ such that for all $n \geq N$, $s_n < L + \epsilon$. Combining these, for $n \geq N$, we have $a_n \leq s_n < L + \epsilon$. Thus, $L$ satisfies property (a) (with $L$ in place of $a^*$).

**Proof that $L$ satisfies property (b):** Let $\epsilon > 0$ and let $N$ be any integer. We need to show there exists $n \geq N$ such that $a_n > L - \epsilon$. Since $L = \lim_{k\to\infty} s_k = \inf_{k\geq 1} s_k$, we know $s_N = \sup\{a_j \mid j \geq N\} \geq L$. By the definition of the supremum, for the set $\{a_j \mid j \geq N\}$ and the number $s_N - \epsilon/2 < s_N$, there must exist an element $a_n$ in the set (so $n \geq N$) such that $a_n > s_N - \epsilon/2$. Since $s_N \geq L$, we have $a_n > s_N - \epsilon/2 \geq L - \epsilon/2$. Since $L - \epsilon/2 > L - \epsilon$, we have found an $n \geq N$ such that $a_n > L - \epsilon$. Thus, $L$ satisfies property (b).

**Conclusion:** Since $L = \lim_{n\to\infty} s_n$ satisfies both properties (a) and (b) from Theorem 3.17 of Rudin, and $a^*$ is the unique number with these properties, it must be that $L = a^*$. Therefore,

$$\lim_{n\to\infty} \sup\{a_j \mid j \geq n\} = \limsup_{n\to\infty} a_n.$$

This justifies calling the upper limit "lim sup".

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**Problem 2** (10 points)**.** For any two real sequences $\{a_n\}$ and $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

**Solution.** Let $a^* = \limsup_{n\to\infty} a_n$ and $b^* = \limsup_{n\to\infty} b_n$. Let $c^* = \limsup_{n\to\infty}(a_n+b_n)$. We want to show $c^* \le a^* + b^*$, assuming $a^* + b^*$ is well-defined (not $\infty - \infty$).

We use the characterization established in Problem 1 (which is also a standard definition of lim sup): $a^* = \lim_{n\to\infty} \sup\{a_k \mid k \ge n\}$ and $b^* = \lim_{n\to\infty} \sup\{b_k \mid k \ge n\}$. Let $s_n(x) = \sup\{x_k \mid k \ge n\}$. Then $a^* = \lim_{n\to\infty} s_n(a)$, $b^* = \lim_{n\to\infty} s_n(b)$, and $c^* = \lim_{n\to\infty} s_n(a+b)$.

Consider the term $s_n(a+b) = \sup\{a_k + b_k \mid k \ge n\}$. For any $k \ge n$, we have $a_k \le s_n(a)$ and $b_k \le s_n(b)$. Therefore, $a_k + b_k \le s_n(a) + s_n(b)$ for all $k \ge n$. This means that $s_n(a)+s_n(b)$ is an upper bound for the set $\{a_k + b_k \mid k \ge n\}$. By the definition of the supremum, $s_n(a+b)$ must be less than or equal to any upper bound. Thus,

$$s_n(a+b) \le s_n(a) + s_n(b).$$

This inequality holds for all $n$. Now we take the limit as $n \to \infty$. The sequences $s_n(a)$, $s_n(b)$, and $s_n(a+b)$ are non-increasing and hence their limits exist in the extended real number system $\mathbb{R} \cup \{+\infty, -\infty\}$. Let $L = \lim_{n\to\infty}(s_n(a) + s_n(b))$. By standard limit theorems (e.g., Theorem 3.20(b) in Rudin, adapted for extended real numbers), if the sum $\lim s_n(a) + \lim s_n(b) = a^* + b^*$ is defined (not $\infty - \infty$), then $L = a^* + b^*$. Using the inequality $s_n(a+b) \le s_n(a) + s_n(b)$ and taking the limit (Theorem 3.20(a) in Rudin, adapted), we get:

$$\lim_{n\to\infty} s_n(a+b) \le \lim_{n\to\infty}(s_n(a) + s_n(b)).$$

Substituting the definitions of $c^*$ and the limits of $s_n(a)$ and $s_n(b)$:

$$c^* \le a^* + b^*.$$

This holds provided $a^* + b^*$ is not of the form $\infty - \infty$.

Let's briefly check the infinite cases covered by this argument:

- If $a^* = \infty$, then $a^* + b^*$ is $\infty$ (since $b^* \ne -\infty$). The inequality $c^* \le \infty$ is always true.
- If $b^* = \infty$, then $a^* + b^*$ is $\infty$ (since $a^* \ne -\infty$). The inequality $c^* \le \infty$ is always true.
- If $a^* = -\infty$ and $b^* = -\infty$, then $a^* + b^* = -\infty$. The inequality becomes $c^* \le -\infty$, which implies $c^* = -\infty$. This is correct, as if $a_n \to -\infty$ and $b_n \to -\infty$, then $a_n + b_n \to -\infty$, so $c^* = -\infty$.
- If $a^* = -\infty$ and $b^*$ is finite, then $a^* + b^* = -\infty$. The inequality becomes $c^* \le -\infty$, implying $c^* = -\infty$. This is correct, as if $a_n \to -\infty$ and $b_n$ is bounded above for large $n$, then $a_n + b_n \to -\infty$, so $c^* = -\infty$. (Symmetrically if $a^*$ finite and $b^* = -\infty$).

The proof holds in all cases where the sum $a^* + b^*$ is defined.

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**Problem 3** (10 points)**.** Use the Root Test or the Ratio Test to determine which of the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$

**Solution.** We will use the Ratio Test (Theorem 3.34, *Principles of Mathematical Analysis* by Walter Rudin (3rd Ed.)) and the Root Test (Theorem 3.33, *Principles of Mathematical Analysis* by Walter Rudin (3rd Ed.)). Recall that for a series $\sum a_n$:

- Ratio Test: Examines $L = \lim_{n\to\infty} |a_{n+1}/a_n|$. Converges if $L < 1$, diverges if $L > 1$. Inconclusive if $L = 1$. More generally, converges if $\limsup |a_{n+1}/a_n| < 1$, diverges if $\liminf |a_{n+1}/a_n| > 1$.
- Root Test: Examines $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Converges if $\alpha < 1$, diverges if $\alpha > 1$. Inconclusive if $\alpha = 1$.

Absolute convergence implies convergence (Theorem 3.45, Rudin).

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ Let $a_n = 1/n!$. The terms are positive. We apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since the limit $L = 0 < 1$, the series converges by the Ratio Test.

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$ Let $a_n = (-1)^n/n^n$. We apply the Root Test to $|a_n| = 1/n^n$:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \limsup_{n\to\infty} \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \limsup_{n\to\infty} \frac{1}{n} = 0.$$

Since the limit $\alpha = 0 < 1$, the series $\sum a_n$ converges absolutely by the Root Test, and therefore converges.

**(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}}{n!}$ Let $a_n = n^{100}/n!$. The terms are positive. We apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{(n+1)^{100}/(n+1)!}{n^{100}/n!} \\
&= \lim_{n\to\infty} \frac{(n+1)^{100}}{(n+1)!} \cdot \frac{n!}{n^{100}} \\
&= \lim_{n\to\infty} \frac{(n+1)^{100}}{(n+1)n!} \cdot \frac{n!}{n^{100}} \\
&= \lim_{n\to\infty} \frac{(n+1)^{100}}{(n+1)n^{100}} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \left( \frac{n+1}{n} \right)^{100} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \left( 1 + \frac{1}{n} \right)^{100}.
\end{aligned}$$

As $n \to \infty$, $\frac{1}{n+1} \to 0$ and $(1+1/n)^{100} \to (1+0)^{100} = 1$. So the limit is $L = 0 \times 1 = 0$. Since $L = 0 < 1$, the series converges by the Ratio Test.

**(d)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$ Let $a_n = (-1)^n n^n/n!$. We consider $|a_n| = n^n/n!$. Let's try the Ratio Test on $|a_n|$:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} \\
&= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\
&= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n!} \cdot \frac{n!}{n^n} \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left( \frac{n+1}{n} \right)^n \\
&= \lim_{n\to\infty} \left( 1 + \frac{1}{n} \right)^n = e.
\end{aligned}$$

Since the limit $L = e \approx 2.718 > 1$, the series $\sum |a_n|$ diverges by the Ratio Test. The Ratio Test result $L > 1$ implies that $|a_n|$ does not tend to 0. In fact, since $\lim |a_{n+1}|/|a_n| = e > 1$, $|a_n| \to \infty$. Since the terms $a_n = (-1)^n |a_n|$ do not converge to 0 (they oscillate between large positive and negative values), the series $\sum a_n$ diverges by the Term Test (Theorem 3.23, Rudin).

(a) Converges. (b) Converges. (c) Converges. (d) Diverges.

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**Problem 4** (10 points)**.** Let $\{a_n\}$ be a sequence of non-negative real numbers. Prove that the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}.$$

**Solution.** We are given that $a_n \geq 0$ for all $n$ and that the series $\sum_{n=1}^{\infty} a_n$ converges. We want to show that the series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

We use the Arithmetic Mean - Geometric Mean (AM-GM) inequality, which states that for non-negative real numbers $x$ and $y$, $\sqrt{xy} \leq \frac{x+y}{2}$. Let $x = a_n$ and $y = \frac{1}{n^2}$. Both are non-negative. Applying the AM-GM inequality:

$$\sqrt{a_n \cdot \frac{1}{n^2}} \leq \frac{a_n + \frac{1}{n^2}}{2}$$

$$\frac{\sqrt{a_n}}{\sqrt{n^2}} \leq \frac{a_n}{2} + \frac{1}{2n^2}$$

Since $n \geq 1$, $\sqrt{n^2} = n$. So,

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}a_n + \frac{1}{2n^2}.$$

We know that $\sum_{n=1}^{\infty} a_n$ converges by hypothesis. We also know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This is a p-series with $p = 2 > 1$ (see Rudin, Theorem 3.28). Since $\sum a_n$ and $\sum 1/n^2$ converge, their scalar multiples also converge, and their sum converges (Rudin, Theorem 3.47):

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}a_n + \frac{1}{2n^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus, the series $\sum(\frac{1}{2}a_n + \frac{1}{2n^2})$ converges.

Let $b_n = \frac{\sqrt{a_n}}{n}$ and $c_n = \frac{1}{2}a_n + \frac{1}{2n^2}$. We have shown that $0 \leq b_n \leq c_n$ for all $n \geq 1$, and that $\sum_{n=1}^{\infty} c_n$ converges. By the Comparison Test (Rudin, Theorem 3.25(a)), since the terms $b_n$ are non-negative and are bounded above by the terms of a convergent series $\sum c_n$, the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ must also converge.

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**Problem 5** (Extra Credit; 10 points)**.** Let $\{a_n\}$ be a sequence of positive numbers and let $\{b_n\}$ be a convergent sequence of positive numbers with nonzero limit. Prove that

$$\limsup_{n\to\infty} a_n b_n = \limsup_{n\to\infty} a_n \lim_{n\to\infty} b_n.$$

**Solution.** Let $a^* = \limsup_{n\to\infty} a_n$. Since $a_n > 0$, we have $a^* \geq 0$. Note that $a^*$ could be $+\infty$. Let $L = \lim_{n\to\infty} b_n$. We are given that $b_n > 0$ for all $n$, and $L \in \mathbb{R}$ with $L > 0$. Let $c^* = \limsup_{n\to\infty}(a_n b_n)$. We want to prove $c^* = a^* L$.

We will prove the equality by showing $c^* \leq a^* L$ and $c^* \geq a^* L$.

**Case 1: $a^*$ is finite ($0 \le a^* < \infty$).**

*Proof of $c^* \le a^*L$:* Let $\epsilon > 0$. Since $L > 0$ and $a^* \ge 0$, we can choose $\delta > 0$ small enough for our purposes later. Since $a^* = \limsup a_n$, by Theorem 3.17(a) of Rudin, there exists $N_a$ such that for $n \ge N_a$, $a_n < a^* + \delta$. Since $b_n \to L$, there exists $N_b$ such that for $n \ge N_b$, $|b_n - L| < \delta$, which implies $L - \delta < b_n < L + \delta$. Since $L > 0$, we can choose $\delta$ small enough such that $L - \delta > 0$. So $0 < b_n < L + \delta$. Let $N = \max(N_a, N_b)$. For $n \ge N$, we have $a_n > 0$ and $b_n > 0$, so $a_n b_n > 0$. Also,

$$a_n b_n < (a^* + \delta)(L + \delta) = a^*L + \delta(a^* + L) + \delta^2.$$

Let $s_n = \sup\{a_k b_k \mid k \ge n\}$. For $n \ge N$, we have $s_n \le a^*L + \delta(a^* + L) + \delta^2$. Taking the limit as $n \to \infty$, using the result from Problem 1 ($c^* = \lim s_n$), we get

$$c^* = \lim_{n\to\infty} s_n \le a^*L + \delta(a^* + L) + \delta^2.$$

This inequality holds for any sufficiently small $\delta > 0$. We can make the term $\delta(a^* + L) + \delta^2$ arbitrarily small by choosing $\delta$ small. Specifically, for any $\epsilon > 0$, we can choose $\delta$ such that $\delta(a^* + L) + \delta^2 < \epsilon$. This implies $c^* \le a^*L + \epsilon$ for any $\epsilon > 0$. Therefore, we must have $c^* \le a^*L$.

*Proof of $c^* \ge a^*L$:* Since $a^* = \limsup a_n$, there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to a^*$ as $k \to \infty$ (Definition 3.16 and Theorem 3.7 of Rudin). Since $b_n \to L$, any subsequence of $\{b_n\}$ must also converge to $L$. In particular, $b_{n_k} \to L$ as $k \to \infty$. Consider the subsequence $\{a_{n_k} b_{n_k}\}$ of $\{a_n b_n\}$. By the limit properties for sequences (Theorem 3.20(d) of Rudin),

$$\lim_{k\to\infty}(a_{n_k} b_{n_k}) = (\lim_{k\to\infty} a_{n_k})(\lim_{k\to\infty} b_{n_k}) = a^*L.$$

We have found a subsequential limit of $\{a_n b_n\}$ that equals $a^*L$. The $\limsup$ of a sequence is the supremum of its subsequential limits (Definition 3.16, Rudin). Therefore,

$$c^* = \limsup_{n\to\infty}(a_n b_n) \ge a^*L.$$

Combining $c^* \le a^*L$ and $c^* \ge a^*L$, we conclude that $c^* = a^*L$ when $a^*$ is finite.

**Case 2: $a^* = +\infty$.** We need to show $c^* = \infty \cdot L = \infty$. Since $a^* = \limsup a_n = \infty$, there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to \infty$ as $k \to \infty$. As before, since $b_n \to L > 0$, the subsequence $b_{n_k} \to L$. Consider the subsequence $\{a_{n_k} b_{n_k}\}$. We want to show $a_{n_k} b_{n_k} \to \infty$. Since $b_{n_k} \to L > 0$, for $\epsilon = L/2 > 0$, there exists $K_1$ such that for $k \ge K_1$, $|b_{n_k} - L| < L/2$, which implies $b_{n_k} > L - L/2 = L/2$. Since $a_{n_k} \to \infty$, for any $M > 0$, there exists $K_2$ such that for $k \ge K_2$, $a_{n_k} > \frac{2M}{L}$. Let $K = \max(K_1, K_2)$. For $k \ge K$, we have

$$a_{n_k} b_{n_k} > \left(\frac{2M}{L}\right) \cdot \left(\frac{L}{2}\right) = M.$$

Since for any $M > 0$, we can find $K$ such that $a_{n_k} b_{n_k} > M$ for all $k \ge K$, this means $\lim_{k\to\infty} a_{n_k} b_{n_k} = +\infty$. We have found a subsequence of $\{a_n b_n\}$ that diverges to $+\infty$. This means $+\infty$ is a subsequential limit of $\{a_n b_n\}$. The $\limsup$ is the supremum of all subsequential limits. Therefore,

$$c^* = \limsup_{n\to\infty}(a_n b_n) \ge +\infty.$$

Since the $\limsup$ cannot exceed $+\infty$, we must have $c^* = +\infty$. This matches the expected result $a^* L = \infty \cdot L = \infty$ (since $L > 0$).

In both cases ($a^*$ finite and $a^* = \infty$), we have shown that $\limsup(a_n b_n) = (\limsup a_n)(\lim b_n)$.

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