Problem Set 7

Due Wednesday, April 16, 2025

Problem 1 (10 points)**.** Suppose $f : X \to Y$ is a uniformly continuous map between metric spaces X and Y. Prove that, if $\{x_n\}$ is a Cauchy sequence in X, then $f(x_n)$ is a Cauchy sequence in Y.

Solution. Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, x' \in X$,

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

Since $\{x_n\}$ is Cauchy in X, there exists N such that for all $m, n \geq N$, we have $d_X(x_m, x_n) < \delta$. By uniform continuity, $d_Y(f(x_m), f(x_n)) < \varepsilon$. Hence, $\{f(x_n)\}$ is Cauchy in Y.

Problem 2 (10 points)**.** Let f be a real-valued, uniformly continuous function on a bounded set $E \subset \mathbb{R}$. Prove that f is a bounded function.

Solution. Since E is bounded, it is totally bounded in $\mathbb{R}$ (Rudin Thm 2.36). Hence, for any $\delta > 0$, there exists a finite δ-net $\{x_1, \ldots, x_n\} \subset E$ such that

$$E \subset \bigcup_{i=1}^{n} B(x_i, \delta).$$

Let $\delta > 0$ correspond to uniform continuity of f, so that for all $x, y \in E$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$. Then for any $x \in E$, there exists x_i with $|x - x_i| < \delta$, so

$$|f(x)| \leq |f(x_i)| + 1.$$

Set $M = \max_i |f(x_i)| + 1$. Then $|f(x)| \leq M$ for all $x \in E$. So f is bounded.

Problem 3 (10 points)**.** Let E be a compact subset of $\mathbb{R}$ and let f be a real-valued function on E. The *graph* of f is defined as:

$$\Gamma_f := \{(x, f(x)) \in \mathbb{R}^2 \mid x \in E\}.$$

Prove that f is continuous if and only if its graph Γ_f is compact.

Solution. ($\Rightarrow$) If f is continuous, then $x \mapsto (x, f(x))$ is continuous from E to $\mathbb{R}^2$. Since E is compact and continuous images of compact sets are compact (Rudin Thm 2.35), Γ_f is compact.

($\Leftarrow$) Suppose Γ_f is compact. Let $x_n \to x$ in E, and consider $(x_n, f(x_n)) \in \Gamma_f$. Since Γ_f is compact, some subsequence $(x_{n_k}, f(x_{n_k})) \to (x', y') \in \Gamma_f$. But $x_{n_k} \to x$, so $x' = x$, and thus $y' = f(x)$. Hence, $f(x_n) \to f(x)$, so f is continuous.

Problem 4 (10 points)**.** Let $I = [0, 1]$ be the closed unit interval. Suppose $f : I \to I$ is continuous. Prove that $f(x) = x$ for some $x \in I$.

Solution. Define $g(x) = f(x) - x$. Then g is continuous on $[0, 1]$. Note:

$$g(0) = f(0) - 0 \geq 0, \quad g(1) = f(1) - 1 \leq 0.$$

By the Intermediate Value Theorem (Rudin Thm 4.23), there exists $x \in [0, 1]$ such that $g(x) = 0$, i.e., $f(x) = x$.

Problem 5 (Extra Credit; 10 points)**.** Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function on E. Prove that f has a continuous extension from E to X.

Solution. For each $x \in X$, I'll define:

$$F(x) = \lim_{n\to\infty} f(x_n), \quad \text{where } x_n \in E,\, x_n \to x.$$

To show well-definedness, let $x_n, y_n \to x$ in E. Since f is uniformly continuous, for $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x_n, y_n) < \delta \Rightarrow |f(x_n) - f(y_n)| < \varepsilon$. Since $x_n, y_n \to x$, I get $|f(x_n) - f(y_n)| \to 0$. Thus, the limit exists and is independent of sequence. I define $F(x)$ to be this limit.

Then for $x_n \to x$ in X, $F(x_n) \to F(x)$ by similar argument. So F is continuous, and extends f.