Problem Set 9

Due Wednesday, April 30, 2025

Problem 1 (10 points)**.** Suppose $a \le x_0 \le b$ and

$$f(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

Prove that f is integrable on $[a, b]$ and that $\int_a^b f(x)\,dx = 0$.

Proof. By Theorem 6.13 of Rudin, a bounded function with only finitely many points of discontinuity on a closed interval is Riemann integrable. Here f is discontinuous only at the single point x_0, so f is integrable on $[a, b]$.

Next, given any $\varepsilon > 0$, choose a partition P of $[a, b]$ which includes x_0 as an endpoint of one subinterval of length $< \varepsilon$. On that small subinterval $\Delta x < \varepsilon$, the supremum $M_i = 1$ and on every other subinterval $M_j = 0$. Hence

$$U(f, P) = 1 \cdot \Delta x + 0 \le \varepsilon, \qquad L(f, P) = 0.$$

Since $\overline{\int_a^b} f \le U(f, P) < \varepsilon$ and $\underline{\int_a^b} f = 0$, letting $\varepsilon \to 0$ shows

$$\int_a^b f(x)\,dx = 0.$$

$\square$

Problem 2 (10 points)**.** Suppose f is continuous on $[a, b]$, $f \ge 0$, and $\displaystyle\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. By the Extreme Value Theorem (Rudin Thm 4.18), f attains a minimum and maximum on $[a, b]$. If there were some $c \in [a, b]$ with $f(c) = \delta > 0$, then by continuity there exists $\eta > 0$ such that $f(x) > \delta/2$ for all $x \in (c - \eta, c + \eta) \cap [a, b]$. Hence

$$\int_a^b f(x)\,dx \ge \int_{c-\eta}^{c+\eta} f(x)\,dx \ge \int_{c-\eta}^{c+\eta} \frac{\delta}{2}\,dx = \delta\eta > 0,$$

contradicting the hypothesis. Therefore no such c exists and $f \equiv 0$. $\square$

Problem 3 (10 points)**.** Let

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Prove that f is not integrable on $[a, b]$ for any $a < b$.

Proof. On every nonempty subinterval $[x_{i-1}, x_i] \subset [a, b]$, both rationals and irrationals are dense. Thus for each i, the supremum $M_i = 1$ and the infimum $m_i = 0$. For any partition P,

$$U(f, P) = \sum_i 1 \cdot \Delta x_i = b - a, \qquad L(f, P) = \sum_i 0 \cdot \Delta x_i = 0.$$

Hence the upper integral is $b - a$ and the lower integral is 0, so they do not agree. Therefore f is not Riemann integrable. □

Problem 4 (10 points)**.** Suppose f is bounded on $[a, b]$ and f^2 is integrable on $[a, b]$.

1. Does it follow that f is integrable on $[a, b]$?
2. Does the answer change if instead we assume f^3 is integrable?

Justify your answers.

Answer.

1. Not in general. Consider the function

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ -1, & x \notin \mathbb{Q}, \end{cases}$$

on $[a, b]$. Then $g^2(x) \equiv 1$, so g^2 is integrable, but g is discontinuous everywhere, hence not integrable.

2. Yes, if f^3 is integrable then f must be integrable. The mapping $\varphi(u) = u^3$ has derivative $\varphi'(u) = 3u^2$, which is bounded on the range of f since f is bounded. Thus φ is Lipschitz on that range, and by the Lipschitz composition theorem (Rudin Thm 6.15), $f^3 = \varphi \circ f$ integrable implies f integrable.

□

Problem 5 (Extra Credit; 10 points)**.** For any integrable f on $[a, b]$ define

$$\|f\|_2 := \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

For integrable f, g, h, prove the triangle inequality

$$\|f - h\|_2 \ \leq \ \|f - g\|_2 + \|g - h\|_2.$$

Proof. Let

$$u = f - g, \quad v = g - h,$$

so $f - h = u + v$. Then

$$\|u + v\|_2^2 = \int_a^b (u + v)^2 = \int_a^b u^2 + 2uv + v^2 = \|u\|_2^2 + 2\langle u, v\rangle + \|v\|_2^2.$$

By the Cauchy–Schwarz inequality (Rudin Thm 3.14),

$$|\langle u, v\rangle| \le \|u\|_2\,\|v\|_2,$$

so

$$\|u+v\|_2^2 \le \|u\|_2^2 + 2\|u\|_2\|v\|_2 + \|v\|_2^2 = \big(\|u\|_2 + \|v\|_2\big)^2.$$

Taking square roots yields

$$\|f-h\|_2 = \|u+v\|_2 \le \|u\|_2 + \|v\|_2 = \|f-g\|_2 + \|g-h\|_2.$$

□