Flux in small noise dynamics: negative resistance and persistence[1]

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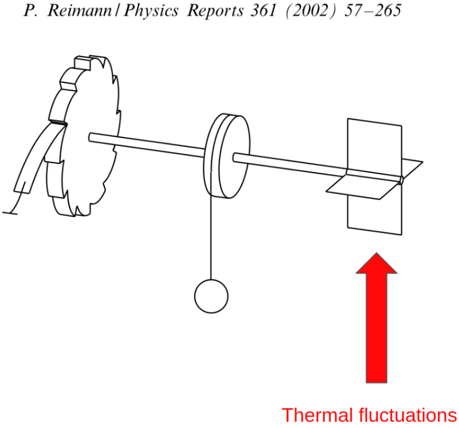
Outline

1. Introduction
   - **Q1**: What is flux, in general?
   - **Q2**: Is [flux $> 0 \iff$ tilt $+$ noise] true in general?
   - **Q3**: Can we quantify flux enough to rigorously explain negative resistance?
2. Flux in general
3. Flux in "tilted potential" systems
4. Main result
5. Some proof ideas

Ratchet and pawl[2]



Q0: Is it possible to gain useful work out of unbiased thermal fluctuations?

[2] Idea goes back at least to Smoluchowski (1912); popularized by Feynman (1963).

Analysis via a simplified model[3]

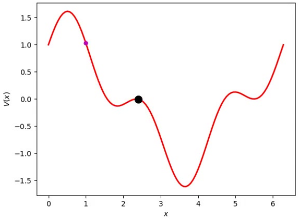
- Consider Brownian particle in 1D with coordinate $x(t)$ and mass m. Newton:

$$m\ddot{x}(t) = \underbrace{-V'(x)}_{\text{conservative force}} \quad \underbrace{-\eta\dot{x}}_{\text{damping}} + \underbrace{\xi(t)}_{\text{noise}}$$

- x represents angular position of ratchet, so $V(x+2\pi) = V(x)$. Heuristically, if $m \ll 1$:

$$\eta\dot{x} = -V'(x) + \xi(t)$$

$$\longrightarrow \qquad dX_t = \underbrace{-U'(X_t)}_{U:=V/\eta}\,dt + \underbrace{\sqrt{2\epsilon}}_{\textit{convenient}}\;dW_t \qquad \text{(Itô SDE)}$$



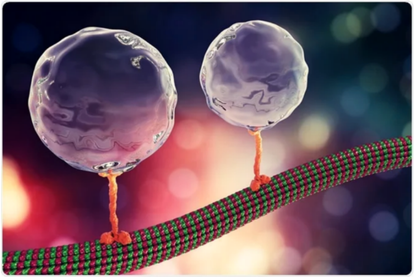
[3]Following Reimann. Physics Reports 361 (2002) 57–265.

Aside: example real-world applications of these kinds of models[4]

- Models like
$$dX_t = (-U'(X_t, f(t)) + y(t) + F)dt + \sqrt{2\epsilon}dW_t$$
have been used to describe:

Phase-locked loops, Josephson junctions, rotating dipoles in external fields, superionic conductors, charge density waves, synchronization phenomena, diffusion on crystal surfaces, particle separation by electrophoresis, biophysical processes such as intracellular transport,...

cf. Risken (1989), Reimann et al. (2001).



Intracellular transport, kinesin motor proteins transport molecules moving across microtubules, 3D illustration. Kateryna Kon / Shutterstock

[4] Image: www.news-medical.net/news/20190715/New-insights-into-molecular-motors-could-help-treat-neurological-disorders.aspx

Back to the ratchet & pawl **Q0**. Answer to **Q0**: no

1. Transition density $\rho(x_0, t, x)$ & **steady-state** $\rho_\epsilon(x)$:

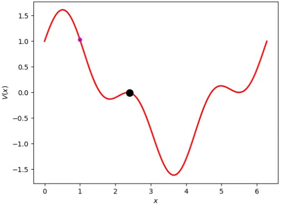
$$\partial_t \rho = \partial_x(\rho U' + \epsilon \partial_x \rho) \qquad \text{(Fokker-Planck)}$$

$$\implies \quad 0 = \partial_x(\rho_\epsilon U' + \epsilon \rho_\epsilon') \qquad \text{(s-s F-P)}$$

$$\implies \quad \rho_\epsilon = C e^{-\frac{1}{\epsilon}U} \qquad (C = \text{const.})$$

since $e^{-\frac{1}{\epsilon}U} U' + \epsilon \partial_x e^{-\frac{1}{\epsilon}U} = 0$.

↓ No tilt



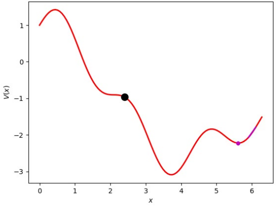
2. **Asymptotic winding rate** (ratchet speed):

$$\lim_{t\to\infty} \frac{X_t - X_0}{2\pi t} = \lim_{t\to\infty} \frac{1}{2\pi t} \int_0^t -U'(X_s)ds + \frac{\sqrt{2\epsilon}}{2\pi} \underbrace{\lim_{t\to\infty} \frac{W_t}{t}}_{0 \text{ a.s. by L.I.L.}}$$

$$\overset{\text{Birkhoff}}{\underset{\text{a.s}}{=}} \frac{1}{2\pi} \int_0^{2\pi} -U'(x) \underbrace{C e^{-\frac{1}{\epsilon}U(x)}}_{\rho_\epsilon(x)} dx \propto \int_0^{2\pi} \rho_\epsilon'(x) dx$$

$$= [\rho_\epsilon(2\pi) - \rho_\epsilon(0)] = 0 \implies \text{Answer: no.}$$

↓ Tilted potential $U(x) - Fx$



- *Answer* ⇝ *yes* with small constant "potential-tilting" force F : can **harvest work from the noise**.

$$dX_t = (-U'(X_t) + F)dt + \sqrt{2\epsilon} dW_t$$

Small applied tilting force $\to$ can harvest work from the noise

- *Answer* $\rightsquigarrow$ *yes* with small constant "potential-tilting" force F : can **harvest work from the noise**.

$$dX_t = (-U'(X_t) + F)dt + \sqrt{2\epsilon} dW_t$$

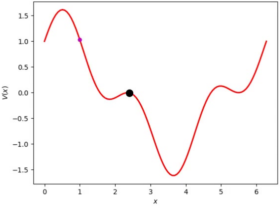
- In fact, one can solve the Fokker-Planck equation in closed form and deduce

$$\lim_{t\to\infty} \frac{X_t - X_0}{2\pi t} \overset{\text{a.s.}}{=} \frac{\overbrace{\epsilon(1 - e^{-\frac{F}{\epsilon}})}^{\text{foreshadowing: } \to 0 \text{ linearly as } \epsilon\to 0}}{\underbrace{\int_0^{2\pi}\int_x^{2\pi+x} e^{\frac{1}{\epsilon}(B(y)-B(x))}\, dy\, dx}_{\text{reciprocal } \to 0 \text{ exponentially as } \epsilon\to 0}},$$

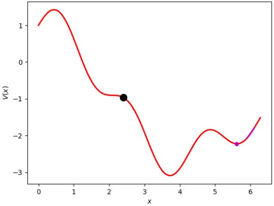
where $B(x) := U(x) - Fx$.

- Immediate: limit > 0 if $F > 0$.

$\downarrow$ No tilt



$\downarrow$ Tilted potential $U(x) - Fx$

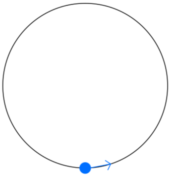


Asymptotic winding rate = flux

- Steady-state Fokker-Planck equation: $J_\epsilon = -U'\rho_\epsilon - \epsilon\rho_\epsilon' = \text{constant}$.
- Since $\int_0^{2\pi} \rho_\epsilon'(x)dx = \rho_\epsilon(2\pi) - \rho_\epsilon(0) = 0$,

$$\underbrace{\lim_{t\to\infty} \frac{X_t - X_0}{2\pi t} = \frac{1}{2\pi}\int_0^{2\pi} -U'(x)\rho_\epsilon(x)dx}_{\text{previous slide}} = \frac{1}{2\pi}\int_0^{2\pi} -U'(x)\rho_\epsilon(x) - \epsilon\rho_\epsilon'(x)dx = J_\epsilon.$$

- Hence asymptotic winding rate = **flux**.



Previous slide: **Flux $> \mathbf{0}$** $\iff$ **noise + tilt**.

Q2: does this generalize to higher-dimensional systems?

Wait...

Q1: What is flux, in general?

Positive resistance

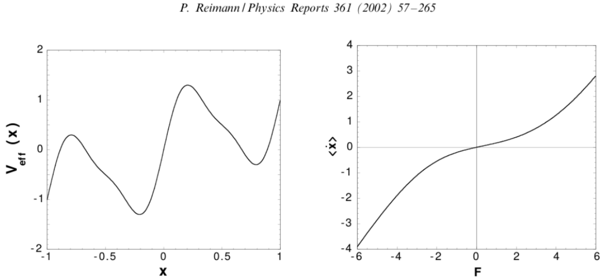
$$\lim_{t\to\infty} \frac{X_t - X_0}{2\pi t} \overset{\text{a.s.}}{=} \frac{\epsilon(1 - e^{-\frac{F}{\epsilon}})}{\int_0^{2\pi} \int_x^{2\pi + x} e^{\frac{1}{\epsilon}(B(y) - B(x))}\, dy\, dx},$$

where $B(x) := U(x) - Fx$.

- Check:

$$\frac{d}{dF}(\text{flux}) = \frac{d}{dF}\left(\lim_{t\to\infty} \frac{X_t - X_0}{2\pi t}\right) > 0.$$

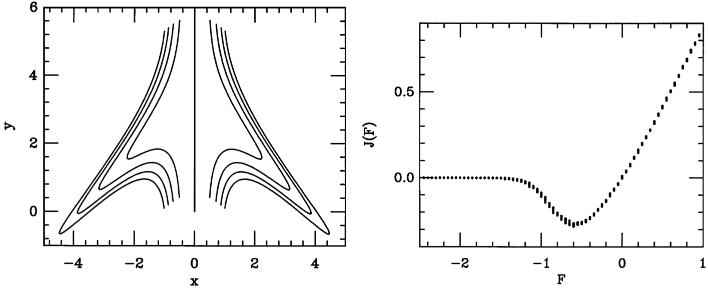
$\implies$ **positive "resistance"**/"conductance" (greater force $\to$ greater circulation).



- At least, in the one-dimensional case...

What about higher-dimensional systems? Negative resistance discovered

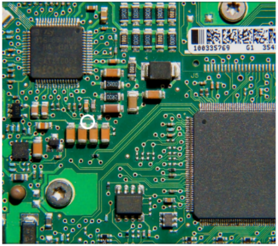
- Cecchi and Magnasco (1996) considered the following "herringbone" potential on $\mathbb{R}^2$; demonstrated negative resistance when tilted in the downward y-direction.

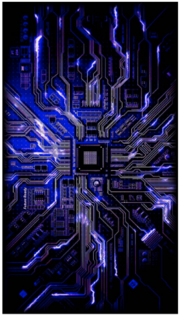


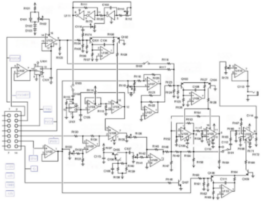
- C & M's analysis: heuristic and numerical considerations.
- **Q3:** can we quantify flux sufficiently well to understand negative resistance rigorously, in a general setting?

Why try to understand "Brownian resistance"?

- Electronic circuits → modern technology
- Circuits transporting non-electron particles might be fundamental for future (e.g., biological) technologies
- Before circuit design abstractions: need to characterize building blocks
- Electrical resistance: important property of electronic circuit components.
- "Brownian resistance": important property of "Brownian circuit" components
- A mathematical theory of "Brownian resistance" could reduce trial-and-error/ tuning in the design process





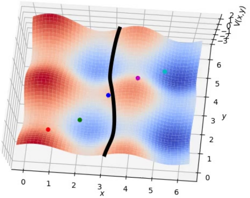


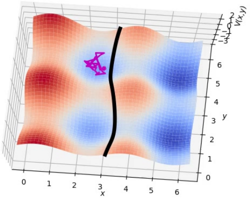
Why higher-dimensional “Brownian conductors”?

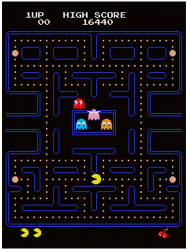
- E.g., coupled systems of n ratchets and pawls (angular coordinates) motivate systems on $T^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$,

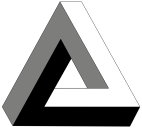
$$dX_t = (-\nabla U(X_t) + \underbrace{ce_1}_{F})dt + \sqrt{2\epsilon}dW_t.$$

- Instead of coupled angular variables, spatially periodic systems on $\mathbb{R}^n$ also lead to systems on T^n after symmetry reduction.
- State spaces M more general than T^n also arise naturally.









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General setup

- Everything is smooth; M is a closed (i.e., compact and boundaryless) connected manifold.
- For all $\epsilon > 0$, $(X_t^\epsilon, \mathbb{P}_x^\epsilon)$ is the diffusion process w/ generator $L_\epsilon \colon C^\infty(M) \to C^\infty(M)$,

$$L_\epsilon = \sum_i b_\epsilon^i(x) \frac{\partial}{\partial x^i} + \epsilon \sum_{i,j} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} \tag{1}$$

  in local coordinates, $a^{ij}(x)$ is symmetric positive definite (nondegenerate noise).
- Note: in local coordinates with $a(x) = \sigma(x)\sigma^T(x)$, can take

$$dX_t^\epsilon = b_\epsilon(X_t)dt + \sqrt{2\epsilon}\sigma(X_t)dW_t.$$

- A computation $\implies$ the $(a^{-1})_{ij}$ are the coordinate representations of a Riemannian metric g on M, w.r.t. which (for a suitable vector field $\mathbf{v}_\epsilon$)

$$L_\epsilon = \mathbf{v}_\epsilon + \epsilon\Delta.$$

- The **stationary probability density** $\rho_\epsilon \in C^\infty(M)$ w.r.t. the smooth measure induced by g satisfies

$$0 = \nabla \cdot \underbrace{(\rho_\epsilon \mathbf{v}_\epsilon - \epsilon \nabla \rho_\epsilon)}_{J_\epsilon} = \nabla \cdot J_\epsilon,$$

  where J_ϵ is the **steady-state probability current**.

What is flux, in general? Answer to **Q1**

Background:

- Recall: one-forms $\alpha\colon M \to \mathrm{T}^*M$ are dual versions of vector fields $M \to \mathrm{T}M$.
- A one-form α is *closed* if $d\alpha = 0$ and *exact* if $\alpha = df$ for some $f \in C^\infty(M)$.
- First de Rham cohomology $H^1_{\mathrm{dR}}(M) := \{\text{closed one-forms}\}/\{\text{exact one-forms}\}$.
- Example: $H^1_{\mathrm{dR}}(T^2) \approx \{a[dx] + b[dy] \colon a, b \in \mathbb{R}\} \approx \mathbb{R}^2$.

Proposition (motivation; follows from Poincaré duality)

Let $N \subset M$ be a closed hypersurface equipped with a unit normal vector field $\hat{n}$. There exists a unique $[\alpha] \in H^1_{\mathrm{dR}}(M)$ such that, for any $\alpha \in [\alpha]$ and any divergence-free vector field J on M,

$$\int_N \langle J, \hat{n}\rangle dy = \int_M \alpha(J) dx. \tag{2}$$

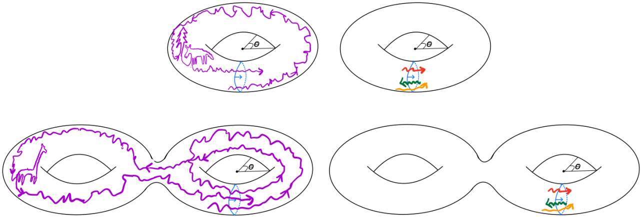
Flux (answer to **Q1**):

- Motivated by the proposition and Schwartzman (1957), we define the (steady-state) flux to be the linear map

$$\mathcal{F}_\epsilon \colon H^1_{\mathrm{dR}}(M) \to \mathbb{R}, \qquad \underbrace{\mathcal{F}_\epsilon([\alpha])}_{\text{“}[\alpha]\text{-flux”}} := \int_M \alpha(J_\epsilon) dx$$

$$\overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} \int_{X^\epsilon_{[0,t]}} \alpha \qquad \text{(Manabe 1982).}$$

Two flux examples: M a torus and M a genus-2 surface



- In these two examples, take $\alpha = \frac{1}{2\pi} d\theta$. The class $[\alpha] \in H^1_{\mathrm{dR}}(M)$ is Poincaré dual to the blue circle with indicated coorientation.
- Left: asymptotic winding rate
$$\lim_{t\to\infty} \frac{1}{t} \int_{X^{\epsilon}_{[0,t]}} \alpha$$
(= asymptotic oriented blue circle crossing rate).
- Right: flux, or
$$\#(\text{oriented crossings per unit time, in limit of large number of particles})$$
- These quantities are all equal (because of ergodicity; Manabe 1982)

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Answer to **Q2** is yes: [flux $> 0 \iff$ tilt $+$ noise], in general

Proposition (YB and MDK)

If $\underbrace{\mathbf{v}_\epsilon \equiv \alpha^\sharp}_{\langle \alpha^\sharp, \cdot \rangle = \alpha(\cdot)}$ is dual to a closed one-form, for the diffusion with generator $\mathbf{v}_\epsilon + \epsilon\Delta$:

$$\mathcal{F}_\epsilon([\alpha]) = \underbrace{\int_M \|\mathbf{v}_\epsilon - \nabla(\ln \rho_\epsilon)\|^2 \rho_\epsilon dx}_{\text{(aside: entropy production rate)}} = \int_M \frac{\|J_\epsilon\|^2}{\rho_\epsilon} dx \geq 0 \tag{3}$$

with equality iff α is exact.

$\implies$ Answer to **Q2** is **yes**.

- "tilted potential" gradients $\alpha^\sharp$ create flux in same "direction" as "tilt" in general, just like the 1D case.
- Let $\alpha = -dU + c\beta$. Earlier S^1 and T^2 special cases: $\alpha = dx$.
- *Proof of Proposition*: Since $J_\epsilon := \rho_\epsilon \mathbf{v}_\epsilon - \epsilon \nabla \rho_\epsilon$ satisfies $\nabla \cdot J_\epsilon = 0$,

  $$\|J_\epsilon\|^2/\rho_\epsilon = \langle \mathbf{v}_\epsilon - \nabla(\ln \rho_\epsilon), J_\epsilon \rangle = \alpha(J_\epsilon) - \nabla \cdot [(\ln \rho_\epsilon) J_\epsilon].$$

  Since $\int_M \nabla \cdot [(\ln \rho_\epsilon) J_\epsilon] dx = 0$ by the divergence theorem,

  $$\mathcal{F}_\epsilon([\alpha]) := \int_M \alpha(J_\epsilon) dx = \int_M \|J_\epsilon\|^2/\rho_\epsilon dx$$

  as desired. To finish, one verifies that $J_\epsilon \equiv 0 \iff \mathbf{v}_\epsilon = \nabla U$ for some U. $\square$

Flux > 0 $\iff$ tilt + noise

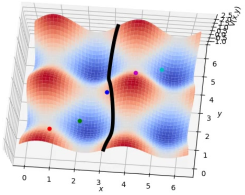
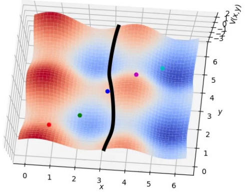


Figure: No noise, no tilt: no flux



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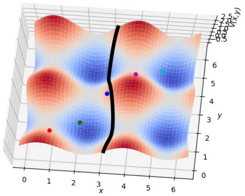

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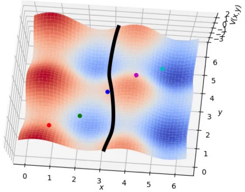


Figure: Tilt + noise: flux harvested from the noise

Asymptotic winding rate $> 0 \iff$ tilt $+$ noise

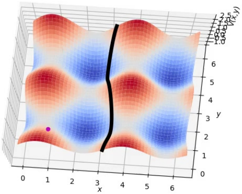
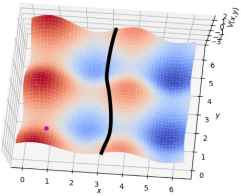


Figure: No noise, no tilt: no asymptotic winding



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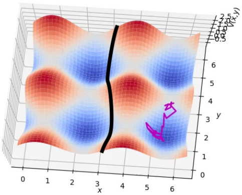

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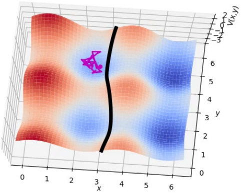

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Main result for tilted potential flux

Assumption (U is generic)

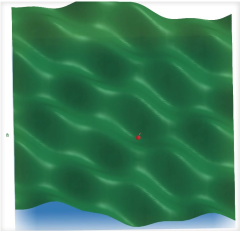
$U \in C^\infty(M)$ has a unique global minimizer, U takes distinct values on distinct index-1 critical points, and U is Morse-Smale.

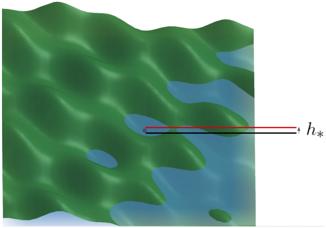
- If α is C^1-close to $-dU$, there is a unique zero v_* of α close to the global min. of U.
- Assume α is closed but not exact; assume $\mathbf{v}_\epsilon \to \alpha^\sharp$ uniformly as $\epsilon \to 0$.

Theorem (YB and MDK)

If α is sufficiently C^1-close to $-dU$, the steady-state $[\alpha]$-flux of the diffusion process with generator $\mathbf{v}_\epsilon + \epsilon\Delta$ satisfies

$$\lim_{\epsilon\to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_* \qquad \left(\text{hence } \mathcal{F}_\epsilon([\alpha]) = e^{-\frac{1}{\epsilon}(h_* + o(1))}; \quad \mathcal{F}_\epsilon([\alpha]) \asymp e^{-\frac{1}{\epsilon}h_*}\right).$$



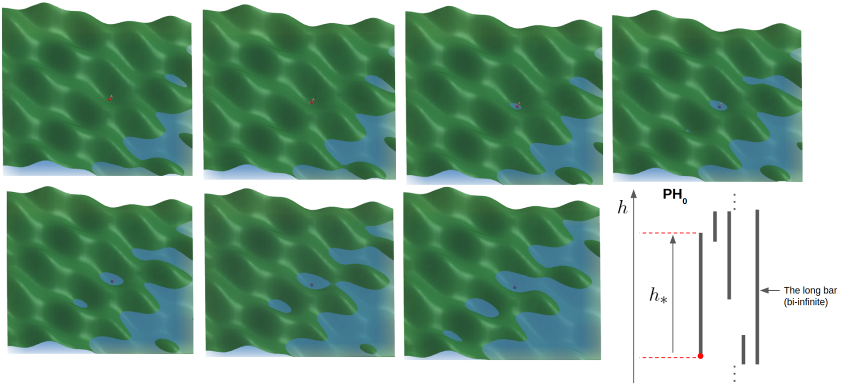


Tilted potential flux $\asymp \exp(-\frac{1}{\epsilon}(\text{critical PH}_0 \text{ bar length}))$

$$\lim_{\epsilon \to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_*$$

- "Unwrap" M: consider any cover $\pi : \tilde{M} \to M$ such that $\pi^* \alpha = -df$ is exact.
- Consider the 0th persistent homology (# components) of the filtration $\{f < h\}_{h \in \mathbb{R}}$.
- Choose a lift $\tilde{v}_* \in \pi^{-1}(v_*)$. In the zeroth persistent homology "barcode",[5]

$$h_* = \text{length(bar corresponding to } \tilde{v}_*).$$



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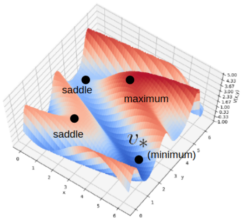

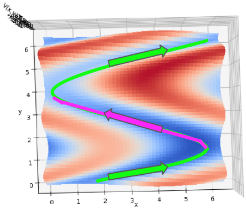

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Proof idea: enter Freidlin-Wentzell[6]

- Freidlin-Wentzell large deviations theory suggests we can discretize the problem and compute the small-noise flux asymptotics from an associated Markov chain (MC) on the set $V \subset M$ of attracting zeros of $\alpha^\sharp$.

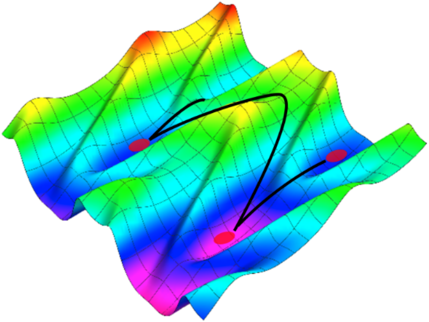


Figure: by Yuliy Baryshnikov.

- However: to detect flux, our MC needs to detect wrapping of trajectories around the manifold $\implies$ need a **finite set of vertices** V, but an **infinite set of directed edges** (we take one edge for each path homotopy class)...

[6] M I Freidlin and A D Wentzell, Random perturbations of dynamical systems, 3rd ed., 2012.

Path-homotopical refinement of Freidlin-Wentzell theory

- Given $T > 0$, the FW **action functional** $\mathcal{S}_T \colon C([0,T], M) \to [0, +\infty]$ is defined by

$$\mathcal{S}_T(\varphi) := \frac{1}{4} \int_0^T \|\dot{\varphi}(t) - \mathbf{v}(\varphi(t))\|^2 \, dt$$

  if φ is absolutely continuous and $\mathcal{S}_T(\varphi) := +\infty$ otherwise. Here $\mathbf{v} = \alpha^\sharp$.

- **Refined quasipotential** (YB, MDK): given **path homotopy class** $e \in \overbrace{\Pi(M)}^{\text{f. groupoid}}$,

$$Q_{\mathbf{v}}(e) := \inf\{\mathcal{S}_T(\varphi) \mid T > 0, [\varphi] = e\}.$$

$$\underbrace{\tilde{Q}_{\mathbf{v}}(e)}_{\text{restricted version}} := \inf\{\mathcal{S}_T(\varphi) \mid T > 0, [\varphi] = e \ \text{ and } \ (\varphi|_{\mathrm{int}(\mathrm{dom}(\varphi))})^{-1}(V) = \varnothing\}.$$

- Note: $Q_{\mathbf{v}} = \tilde{Q}_{\mathbf{v}}$ if $\dim(M) \geq 2$ (transversality).

Diffusion flux $\asymp$ Markov chain flux

- Consider the directed graph $\Gamma_\Pi = (V, E_\Pi, \mathfrak{s}, \mathfrak{t})$ where

$$V = \{\text{attracting zeros of } \alpha^\sharp\},$$
$$E_\Pi = \{[\varphi] \in \Pi(M) \colon \text{the ends of } \varphi \text{ are in } V\},$$

  and $\mathfrak{s}, \mathfrak{t} \colon E_\Pi \to V$ send path homotopy classes to their endpoints.
- Define MC on Γ_Π with transition probabilities

$$P(e) \coloneqq \exp(-\frac{1}{\epsilon}\tilde{Q}_{\mathbf{v}}(e))$$

  for $e \in E_\Pi$ and stationary distribution $\pi_\epsilon \colon V \to (0,1]$.
- Set $\alpha(e) \coloneqq \int_e \alpha$ for $e \in E_\Pi$ and $\alpha(E) \coloneqq \sum_{e \in E} \alpha(e)$ for finite $E \subset E_\Pi$; assume $\mathbf{v}_\epsilon \to \mathbf{v} = \alpha^\sharp$ uniformly as $\epsilon \to 0$.

Theorem (YB, MDK)

Under the hypotheses of the main theorem, the steady-state $[\alpha]$-flux of the diffusion with generator $\mathbf{v}_\epsilon + \epsilon\Delta$ satisfies

$$\lim_{\epsilon \to 0} (-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = \lim_{\epsilon \to 0} \left(-\epsilon \ln \underbrace{\sum_{e \in E_\Pi} \pi_\epsilon(\mathfrak{s}(e)) P(e) \alpha(e)}_{\text{s-s MC } \alpha\text{-flux}} \right)$$

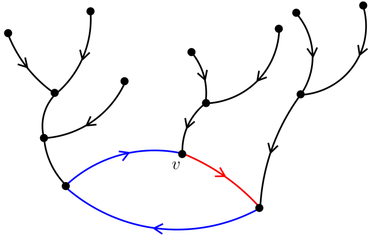
The Markov chain tree formula holds → cycle-rooted spanning tree formula

Theorem (YB, MDK)

Under the hypotheses of the main theorem, the steady-state $[\alpha]$-flux of the diffusion with generator $\mathbf{v}_\epsilon + \epsilon\Delta$ satisfies ($\mathbf{v} = \alpha^\sharp$)

$$\lim_{\epsilon\to 0}\left(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])\right) = \lim_{\epsilon\to 0}\left(-\epsilon \ln \underbrace{\sum_{e\in E_\Pi} \pi_\epsilon(\mathfrak{s}(e))P(e)\alpha(e)}_{\text{s-s MC } \alpha\text{-flux}}\right)$$

$$= \left(\min_{\substack{E\in \mathrm{CRST}(\Gamma_\Pi)\\ \alpha(\mathrm{cycle}(E))>0}} \sum_{e\in E} \tilde{Q}_{\mathbf{v}}(e)\right) - \left(\min_{E\in \mathrm{RST}(\Gamma_\Pi)} \sum_{e\in E} \tilde{Q}_{\mathbf{v}}(e)\right)$$



Tree formula, RSTs, CRSTs: Pitman & Tang (2018).

A glimpse of how $\tilde{Q}_{\mathbf{v}}$ might relate to h_* ($\tilde{Q}_{\mathbf{v}} = Q_{\mathbf{v}}$ if $\dim M \geq 2$)

Proposition (YB and MDK)

Under the hypotheses of the main theorem (with $\mathbf{v} = \alpha^\sharp$),

$$Q_{\mathbf{v}}(e) = 0 \quad \Longleftrightarrow \quad \text{e contains a piecewise } \mathbf{v}\text{-integral curve}$$

and

$$Q_{\mathbf{v}}(e) = \int_e (-\alpha) \quad \Longleftrightarrow \quad \text{e contains a piecewise } (-\mathbf{v})\text{-integral curve.}$$

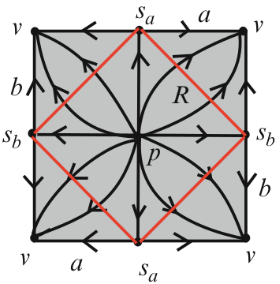
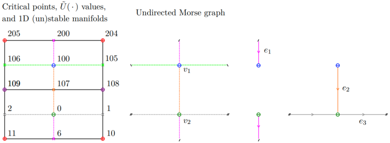


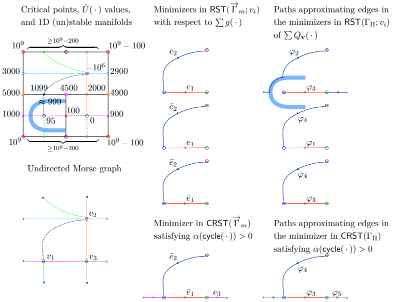
Figure: Points on red diamond index the piecewise integral curves $p \rightsquigarrow v$ in a Morse-Smale flow on T^2. From “An Invitation to Morse theory” by Nicolaescu, 2nd ed.

The "C^1-close" (small tilt) hypothesis is crucial

- Not enough for $\alpha^\sharp$ to be Morse-Smale without periodic orbits.
- When α is sufficiently C^1-close to $-dU$ for generic $U \in C^\infty(M)$, trajectories circulate around one "main highway" with high probability ($\rightarrow$ PH_0 h_* formula).
- When the tilt is large (α is not C^1-close to $-dU$), trajectories can "fall off" the main highway, and take exponentially long times to "climb back on".

- Explicit T^2 counterexamples (YB, MDK) show the hypotheses in the main theorem are fairly sharp, for multiple reasons.





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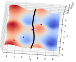
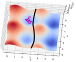
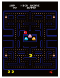
Flux in tilted potential systems: negative resistance and persistence[7]

Why higher-dimensional "Brownian conductors"?

- E.g., coupled systems of n ratchets and pawls (angular coordinates) motivate systems on $T^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$,

$$dX_t = (-\nabla U(X_t) + \underbrace{ce_1}_{F})dt + \sqrt{2\epsilon}dW_t.$$

- Instead of coupled angular variables, spatially periodic systems on $\mathbb{R}^n$ also lead to systems on T^n after symmetry reduction.
- State spaces M more general than T^n also arise naturally.


Answer to **Q2** is yes: [flux $> 0 \iff$ tilt + noise], in general

Proposition (YB and MDK)

If $\underbrace{\mathbf{v}_\epsilon \equiv \alpha^\sharp}_{\langle\alpha^\sharp,\,\cdot\,\rangle=\alpha(\,\cdot\,)}$ is dual to a closed one-form, for the diffusion with generator $\mathbf{v}_\epsilon + \epsilon\Delta$:

$$\mathcal{F}_\epsilon([\alpha]) = \underbrace{\int_M \|\mathbf{v}_\epsilon - \nabla(\ln\rho_\epsilon)\|^2\rho_\epsilon dx}_{\text{(aside: entropy production rate)}} = \int_M \frac{\|J_\epsilon\|^2}{\rho_\epsilon}dx \geq 0 \tag{3}$$

with equality iff α is exact.

$\implies$ Answer to **Q2** is **yes**.

- "tilted potential" gradients $\alpha^\sharp$ create flux in same "direction" as "tilt" in general, just like the 1D case.
- Let $\alpha = -dU + c\beta$. Earlier S^1 and T^2 special cases: $\alpha = dx$.
- *Proof of Proposition*: Since $J_\epsilon := \rho_\epsilon\mathbf{v}_\epsilon - \epsilon\nabla\rho_\epsilon$ satisfies $\nabla\cdot J_\epsilon = 0$,

$$\|J_\epsilon\|^2/\rho_\epsilon = \langle\mathbf{v}_\epsilon - \nabla(\ln\rho_\epsilon), J_\epsilon\rangle = \alpha(J_\epsilon) - \nabla\cdot[(\ln\rho_\epsilon)J_\epsilon].$$

Since $\int_M \nabla\cdot[(\ln\rho_\epsilon)J_\epsilon]dx = 0$ by the divergence theorem,

$$\mathcal{F}_\epsilon([\alpha]) := \int_M \alpha(J_\epsilon)dx = \int_M \|J_\epsilon\|^2/\rho_\epsilon dx$$

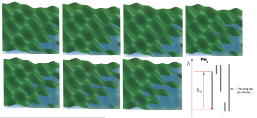
as desired. To finish, one verifies that $J_\epsilon \equiv 0 \iff \mathbf{v}_\epsilon = \nabla U$ for some U. $\square$

Tilted potential flux $\asymp \exp(-\frac{1}{\epsilon}(\text{critical } PH_0 \text{ bar length}))$

$$\lim_{\epsilon\to 0}(-\epsilon\ln\mathcal{F}_\epsilon([\alpha])) = h_*$$

- "Unwrap" M: consider any cover $\pi\colon \tilde{M} \to M$ such that $\pi^*\alpha = -df$ is exact.
- Consider the 0th persistent homology (# components) of the filtration $\{f < h\}_{h\in\mathbb{R}}$.
- Choose a lift $\tilde{v}_* \in \pi^{-1}(v_*)$. In the zeroth persistent homology "barcode",[5]

$$h_* = \text{length}(\text{bar corresponding to } \tilde{v}_*).$$



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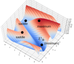

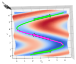

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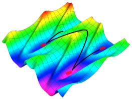

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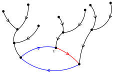

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THANK YOU

[7] Matthew Kvalheim (kvalheim@seas.upenn.edu) and Yuliy Baryshnikov. arXiv:2108.06431

Dual (probability-free) point of view: reformulation as PDE result

- Consider the singularly perturbed elliptic PDE on the closed Riemannian manifold M,

$$0 = \nabla \cdot (u\mathbf{v}_\epsilon) - \epsilon \Delta u = \nabla \cdot \underbrace{(u\mathbf{v}_\epsilon - \epsilon \nabla u)}_{J(u)}. \tag{4}$$

- Let $U \in C^\infty(M)$ be generic, α be a closed but not exact one-form C^1-close to $-dU$, $\mathbf{v}_\epsilon \to \alpha^\sharp$ uniformly as $\epsilon \to 0$.

Theorem (YB and MDK)

Given a family of positive solutions $(u_\epsilon)_{\epsilon>0}$ to (4) such that $\lim_{\epsilon\to 0} \epsilon \ln \left(\int_M u_\epsilon dx\right) = 0$,

$$\lim_{\epsilon \to 0} \left(-\epsilon \ln \int_M \alpha(J(u_\epsilon))dx\right) = h_*.$$

- Recall: if $[\alpha]$ is Poincaré dual to a cooriented hypersurface $N \subset M$,

$$\int_M \alpha(J(u_\epsilon))dx = \int_N \langle J(u_\epsilon), \hat{n}\rangle dy,$$

so the theorem gives asymptotics of the flux through N of some macroscopic "stuff" described by the steady-state advection-diffusion PDE (4).