Discovering engineering (im)possibilities with geometry and topology[1]

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Discovering engineering (im)possibilities for:

Feedback stabilizability

Applied Koopman operator methods

Deep neural network autoencoders

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- Brockett's necessary condition and beyond
- A homotopy theorem beyond the Coron/Mansouri tests
- Periodic orbits can be easier to stabilize than equilibria

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Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \qquad (1)$$

where $M \ni x$ is a smooth manifold and f is smooth.

1. **Controllability problem**: Given $a, b \in M$, find $u(t)$ s.t. $x(T) = b$ if $x(0) = a$ for some $T > 0$.

$$a \rightsquigarrow b$$

2. **Stabilizability problem**: Given a compact subset $A \subset M$, find smooth $u(x)$ s.t. A is **asymptotically stable**[2] for the **closed-loop vector field** $F(x) = f(x, u(x))$. Link

[2] For every open $W \supset A$ there is an open $V \supset A$ s.t. all forward F-trajectories initialized in V are contained in W and converge to A.

The stabilization conjecture and Brockett's solution

Often $A = \{x_*\}$ is a point, $M = \mathbb{R}^n$ in the stabilization problem.

Stabilization conjecture (pre-1983): a reasonably strong form of controllability implies smooth stabilizability of a point.

Example: the "Heisenberg system" or "nonholonomic integrator"

$$\left.\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned}\right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?

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Theorem (Brockett). If a point is stabilizable, then image(f) is a neighborhood of 0. (In the example, $(0, 0, \varepsilon) \notin$ image(f).)

Other stabilizability work

- Exponential (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
- global (Byrnes 2008, Baryshnikov 2023),
- time-varying (Coron 1992), and
- discontinuous (Clarke, Ledyaev, Sontag, Subbotin 1997)

variants of the stabilization problem are not considered here.

Coron's and Mansouri's obstructions

Krasnosel'skiĭ and Zabreĭko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \colon f(x, u) \neq 0\}.$$

Theorem (Coron 1990). If $n > 1$ and a point is stabilizable,

$$f_*(H_{n-1}(\Sigma)) = H_{n-1}(\mathbb{R}^n \setminus \{0\}) \qquad (\cong \mathbb{Z}).$$

Theorem (Mansouri 2010). If a closed codimension > 1 submanifold $A \subset \mathbb{R}^n$ with Euler characteristic $\chi(A)$ is stabilizable,

$$f_*(H_{n-1}(\Sigma)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \qquad (\cong \chi(A) \cdot \mathbb{Z}).$$

Limitations of these results

The results of Brockett, Coron, Mansouri rely on parallelizability of $\mathbb{R}^n$ to view vector fields and control systems as $\mathbb{R}^n$-valued.

Furthermore, they apply only to the special case that A is a point or a closed submanifold of $\mathbb{R}^n$ with $\chi(A) \neq 0$.

But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

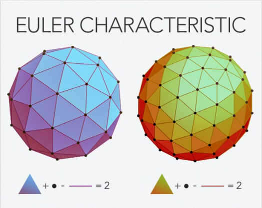
How to test for stabilizability in such general settings?[3]

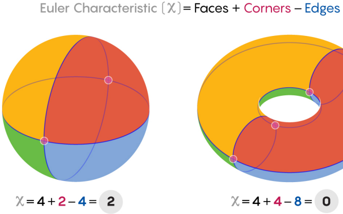
- **Generalization of Brockett's test** (MDK and Daniel E. Koditschek, J Geometric Mechanics, 2022).
- **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).

[3] An exposition of all stabilizability results here is in 2023 book *Topological Obstructions to Stability and Stabilization* by W. Jongeneel and E. Moulay.

A primer on the Euler characteristic[4]

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).





Notation: $\chi(Y) :=$ Euler characteristic of Y.

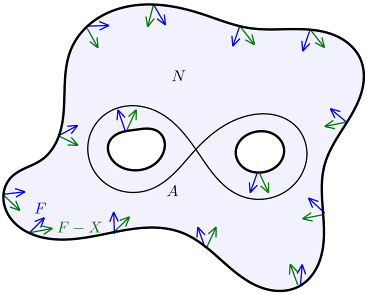
Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\text{figure } 8) = -1$

Theorem (Poincaré, Hopf): if N is a compact smooth manifold with boundary ∂N, then $\chi(N) = 0 \iff$ there exists a nowhere-zero smooth vector field on N pointing inward at ∂N.

[4] Figures from Quanta Magazine.

Generalization of Brockett's test

Theorem (MDK & Koditschek 2022): Let $A \subset M$ be compact & stabilizable. Then $\chi(A)$ is well-defined. If $\chi(A) \neq 0$, then for any sufficiently small vector field X, $X(x_0) = f(x_0, u_0)$ for some x_0, u_0.



Proof: Assume $\exists$ stabilizing $u(x)$ and define $F(x) := f(x, u(x))$. Lyapunov function theory $\implies \exists$ compact smooth domain $N \supset A$ s.t. F points inward at ∂N and $\chi(A) = \chi(N) \neq 0$. Continuity $\implies F - X$ points inward at ∂N if X is small $\implies F - X$ has a zero by Poincaré-Hopf $\implies \exists x_0$ s.t. $X(x_0) = F(x_0) = f(x_0, u(x_0))$.

Examples

Heisenberg system

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \tag{2}$$

Kinematic differential drive robot

$$\begin{aligned} \dot{x} &= u\cos\theta \\ \dot{y} &= u\sin\theta \\ \dot{\theta} &= v \end{aligned} \tag{3}$$

The right side of (2) $\neq X_\varepsilon := (0, 0, \varepsilon)$ for any $\varepsilon > 0$.

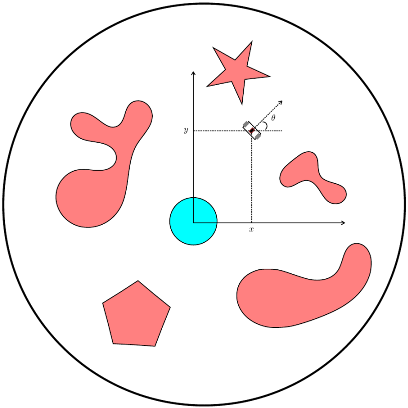
The right side of (3) $\neq X_\varepsilon := (\varepsilon\sin\theta, -\varepsilon\cos\theta, 0)$ for any $\varepsilon > 0$.

Thus, our result $\implies$ A is not stabilizable if $\chi(A) \neq 0$. E.g., if A is a stabilizable compact submanifold, A is a union of circles and tori.

Other applications: any stabilizable compact set has zero Euler characteristic for satellite orientation with ≤ 2 thrusters, for nonholonomic dynamics with ≥ 1 global constraint 1-form,...

Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some "bad" set immediately enter some "good" set.



E.g., impossible for this differential drive robot to aim within ± 179 degrees of the origin while "strictly" avoiding obstacles via $u(x)$.

Homotopy theorem & generalized Coron, Mansouri tests

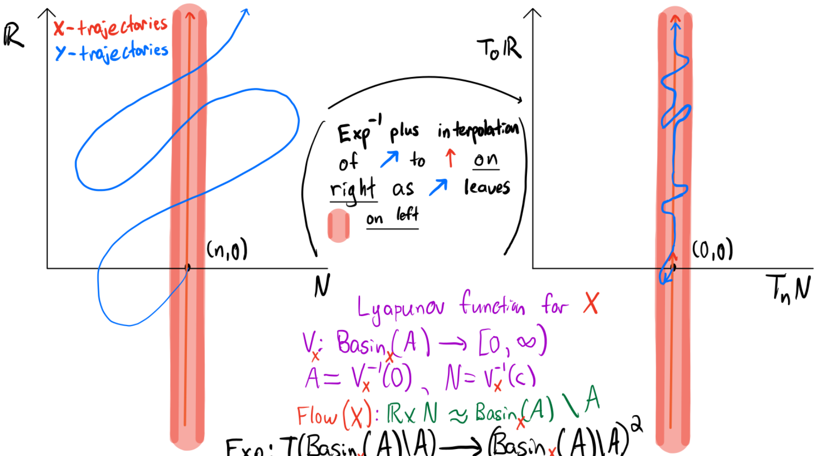
Homotopy theorem (MDK 2023). Let X, Y be smooth vector fields on a manifold M with a compact set $A \subset M$ asymptotically stable for both. There is an open set $U \supset A$ such that $X|_{U\setminus A}$, $Y|_{U\setminus A}$ are homotopic through nowhere-zero vector fields.

$\implies$ **Theorem (MDK 2023).** Let the compact set $A \subset M$ be asymptotically stable for *some* smooth vector field Y on M. If A is stabilizable for $\dot{x} = f(x, u)$, then for all small enough open $U \supset A$,

$$H_\bullet(T(U \setminus A) \setminus 0) \supset \underbrace{f_* H_\bullet(\Sigma) \supset Y_* H_\bullet(U \setminus A)}_{\text{cf. Coron, Mansouri}}.$$

These are stronger than all preceding results: there is an example (MDK 2023) for which non-stabilizability is detected by each of these theorems but not by any of the preceding theorems.

Proof of the homotopy theorem



Lyapunov function for X

$V_X\colon \mathrm{Basin}_X(A) \to [0,\infty)$

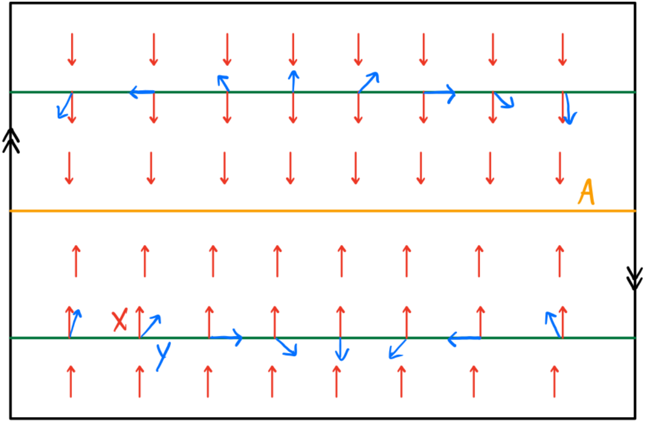
$A = V_X^{-1}(0)$, $N = V_X^{-1}(c)$

$\mathrm{Flow}(X)\colon \mathbb{R}\times N \approx \mathrm{Basin}_X(A)\setminus A$

$\mathrm{Exp}\colon T(\mathrm{Basin}_X(A)\setminus A) \longrightarrow (\mathrm{Basin}_X(A)\setminus A)^2$

w.r.t. product of any metric on N with Euclidean metric on $\mathbb{R}$

Möbius strip example



X $\not\simeq$ y since y ↻ twice around ◯ w.r.t. X while X ↻ zero times w.r.t. X $\Rightarrow$ A is not asymptotically stable for y by the homotopy theorem.

Can these results detect stabilizability of periodic orbits?

If A is the image of a periodic orbit with the same orientation for X and Y, the straight-line homotopy over a sufficiently small open $U \supset A$ satisfies the homotopy theorem's conclusion regardless of whether A is attracting, repelling, or neither for X or Y.

$\implies$ **homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, **preceding results also give no information.**

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...Could it be that periodic orbits might be "easy" to stabilize?

Periodic orbits are sometimes easier to stabilize

Indeed, at least sometimes:

Theorem (Anthony M. Bloch & MDK, in preparation).
For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!

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- Many assume the dynamical system is globally linearizable
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“Modern Koopman theory for dynamical systems”

$$\dot{x} = \frac{d}{dt}x = f(x), \qquad x \in M$$

- “A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.”[5]

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- I.e., find an embedding $F\colon M \hookrightarrow \mathbb{R}^n$ such that $y = F(x)$ satisfies $\dot{y} = By$ for some $n \times n$ matrix B, or equivalently

$$\forall t \in \mathbb{R}\colon F \circ \Phi^t = e^{Bt} \circ F, \quad \text{where } \Phi = \text{flow}(f).$$

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- **Fundamental question**: when is (M, Φ) globally linearizable?

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When is a dynamical system (M, Φ) globally linearizable?[7]

Not when M is connected and Φ has a compact non-global attractor A, since basin(A) would then be closed (by the Jordan normal form theorem) in addition to open, hence clopen, so basin$(A) = M$.[6]

Hence we study global linearizability of the restriction (S, Φ) of Φ to a basin S of a compact attractor A; we also study the important case that S is any compact invariant set for Φ.

In these cases we obtain **necessary and sufficient conditions for global linearizability** of (S, Φ) by an embedding, for the two cases of topological and smooth embeddings.

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[6]This observation is complementary to Cor. 3 of Liu-Ozay-Sontag (2023).

[7]MDK and P. Arathoon, *Linearizability of flows by embeddings (2023).*

Preliminaries

$F: S \to \mathbb{R}^n$ is a **topological embedding** if F is a one-to-one continuous map with a continuous inverse $F^{-1}: F(S) \to S$, and is a **smooth embedding** if additionally F and F^{-1} are smooth.

The n-torus $T = T^n$ is Lie group isomorphic to $(\mathbb{R}/\mathbb{Z})^n$, vectors with n real entries but with addition defined elementwise modulo 1.

A **torus action** on S is a map $\Theta: T \times S \to S$ satisfying $\Theta^{\tau_1+\tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

The flow (S, Φ) is a **1-parameter subgroup of a torus action** if $\Phi^t = \Theta^{\omega t \bmod 1}$ for some torus action Θ on S, $\omega \in \mathbb{R}^n \cong \mathrm{Lie}(T^n)$.

The linearizability theorem, part 1: compact invariant sets

Observation: If (S, Φ) is linearizable with S compact, Jordan normal form theorem implies (S, Φ) embeds into flow on $\mathbb{C}^n$ of a diagonal imaginary matrix, so (S, Φ) is 1-parameter subgroup of restriction of standard torus action of T^n on $\mathbb{C}^n$ to a subtorus.

This gives one implication below; Mostow-Palais gives the other.

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[8] I.e., there are only finitely many subgroups $H \subset T$ such that $H = \{\tau \in T \colon \Theta^\tau(s) = s\}$ is the set that fixes some $s \in S$.

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Theorem (MDK and P. Arathoon). If S is a compact submanifold, (S, Φ) is linearizable by a smooth embedding $\iff$ (S, Φ) is a 1-parameter subgroup of a smooth torus action.

Theorem (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff$ (S, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types[8].

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Implications concerning linearizability and topology

If (S, Φ) is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an $n \times n$ skew matrix B_e and a system of local coordinates on a neighborhood of each equilibrium $e \in S$ such that $\Phi^t \approx e^{B_e t}$. Hence if e is isolated then B_e is invertible, $n = \dim S$ is even, and the Hopf index of e is $+1$.

Corollary (MDK and PA). If S is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then (S, Φ) cannot be linearized by a smooth embedding.

Corollary (MDK and PA).[9] If S is a compact submanifold containing at most finitely many equilibria such that (S, Φ) is linearizable by a smooth embedding, $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$.

[9] Apply the Poincaré-Hopf theorem.

Another point of view: quasiperiodic pinched torus families

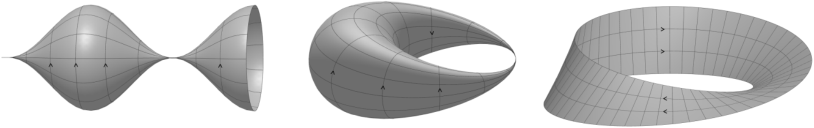

Figure: examples of quasiperiodic pinched torus families

Definition. P is a **pinched torus family** if there are $m, n \in \mathbb{N}$, closed subsets $C_1, \ldots, C_n \subset S \subset T^m$, and a continuous group homomorphism $F\colon T^n \to T^m$ such that P is the quotient of $F^{-1}(S)$ by collapsing the j-th $(\mathbb{R}/\mathbb{Z})$-factor of $F^{-1}(C_j) \subset T^n$ for all j. A pinched torus family P is **quasiperiodic** if it is equipped with the induced flow generated by any $\boldsymbol{\omega} \in \mathbb{R}^n$ with $\mathsf{T}F(\boldsymbol{\omega}) = 0$.

Proposition (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff$ (S, Φ) is a quasiperiodic pinched torus family.

The linearizability theorem, part 2: $S = \text{basin}(A)$

If S is the basin of an asymptotically stable compact set $A \subset S$, A has continuous (smooth) **asymptotic phase**[10] if there is a continuous (smooth) **asymptotic phase map** $P\colon S \to A$ satisfying

$$P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

Theorem (MDK and PA). (S, Φ) is linearizable by a topological embedding $\iff$ A has continuous asymptotic phase & (A, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

Theorem (MDK and PA). (S, Φ) is linearizable by a smooth embedding $\iff$ A is an embedded submanifold with smooth asymptotic phase, (A, Φ) is a 1-parameter subgroup of a smooth torus action, & locally $\Phi \hookrightarrow$ reducible lin. flow on vector bundle...

[10] This notion has roots in oscillator theory, and more generally NHIM theory.

Questions remain

Theorem (MDK and PA). (S, Φ) is linearizable by a smooth embedding $\iff$ ..., & for some open $U \supset A$, (U, Φ) embeds in a reducible linear flow covering Φ on some vector bundle over A.

When is this the case? MDK and Revzen, Physica D (2021) give fairly complete answers[11] in the special cases that A is an equilibrium or periodic orbit, and some is known when A is a quasiperiodic torus, but this remains an open question in general.

A necessary condition for A to satisfy all conclusions of the theorem is that A be an (eventually relatively ∞-)**normally hyperbolic invariant manifold**. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

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[11] involving "nonresonance" and "spectral spread" conditions on eigenvalues or Floquet multipliers of the infinitesimal linearization of the dynamics at A

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- They can't work and yet they do: resolving the paradox
- Just delete a thickened cut locus or top of a Morse complex

(deep NN) Autoencoding and topological obstructions to it

[data pts on low-dim submanifold] $\rightsquigarrow$

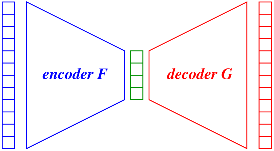

Figure: by E. D. Sontag

$\rightsquigarrow$ [compressed / decompressed data pts]

- "Manifold hypothesis" postulates data set $\subset \mathbb{R}^n$ lies on some k-dim submanifold K, describable locally by $k < n$ parameters
- For K linear, classical approaches like PCA / MDS work well
- K nonlinear, more challenging "manifold learning" problem

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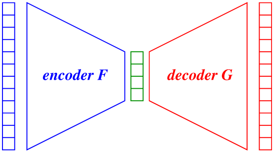

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- Popular approach: look for **autoencoder** $G \circ F$, where the **encoder** output $F\colon \mathbb{R}^n \to \mathbb{R}^k$ is the desired k-parameters, $G\colon \mathbb{R}^k \to \mathbb{R}^n$ is the **decoder**, and F, G are continuous
- **Ideal autoencoders**: $G(F(x)) = x$ for all $x \in K$

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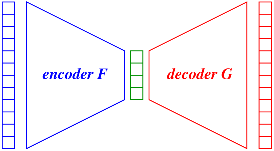

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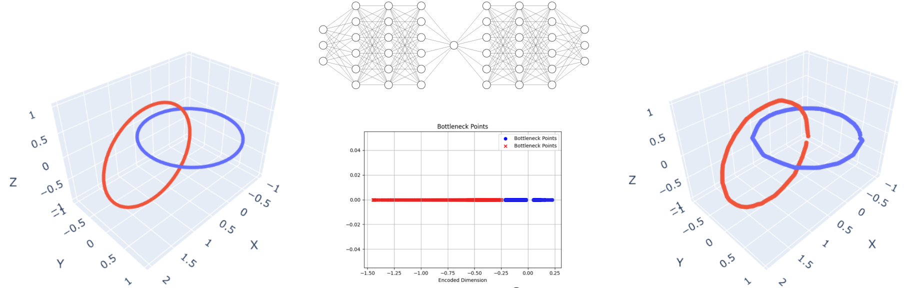
- **Ideal autoencoders**: $G(F(x)) = x$ for all $x \in K$
- These **do not usually exist!** Since existence $\implies$ k-dim K topologically embeds in $\mathbb{R}^k$, which is not true of most k-dim K

If autoencoding can't work, why does it?[12] Example:



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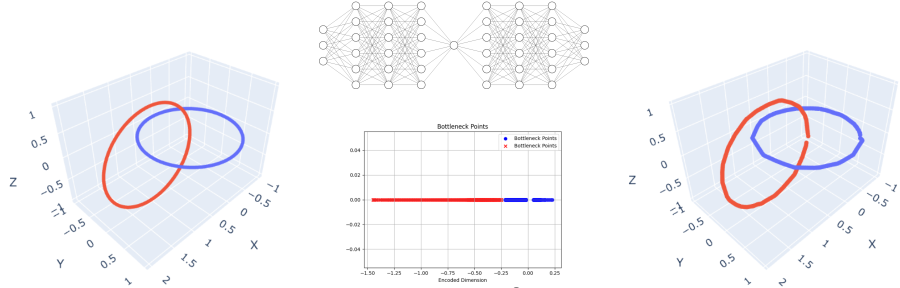

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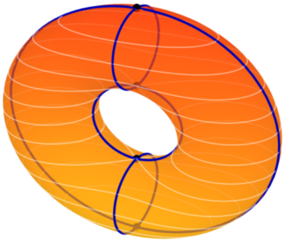
Explanation:

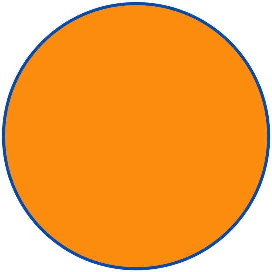
- A pair of circles $\subset \mathbb{R}^3$, after thickening then deleting small intervals, is diffeomorphic to a pair of intervals $\subset \mathbb{R}$
- **Encoder** $F\colon \mathbb{R}^3 \to \mathbb{R}$ can be any extension of this diffeomorphism. **Decoder** $G\colon \mathbb{R} \to \mathbb{R}^3$ can be any extension of the inverse diffeomorphism
- Can always find such small intervals disjoint from the data set

[12] MDK and E. D. Sontag, *Why do autoencoders work?* (2023).

If autoencoding can't work, how does it? More generally:



$\longleftrightarrow$



Explanation:

- A k-dim manifold $\subset \mathbb{R}^n$, after thickening then deleting either (i) a cut locus or (ii) the "top" cells from the complex of a polar Morse function[13], is diffeomorphic to a k-dim disk $\subset \mathbb{R}^k$
- **Encoder** $F: \mathbb{R}^n \to \mathbb{R}^k$ can be any extension of this diffeomorphism. **Decoder** $G: \mathbb{R}^k \to \mathbb{R}^n$ can be any extension of the inverse diffeomorphism
- Can always find such a "disk boundary" disjoint from the data

[13] A *navigation function*, in the parlance of Rimon and Koditschek; these exist by Thm 3 in *Robot navigation functions on manifolds with boundary* (1990).

Almost-ideal autoencoders always exist

$\mathcal{F}^{\ell,m} \subset \{\text{continuous funcs } \mathbb{R}^\ell \to \mathbb{R}^m\}$ are possible neural outputs.

Theorem 1 (MDK and E. D. Sontag). Let $K \subset \mathbb{R}^n$ be a finite union of compact submanifolds with(out) boundary, each of dimension $\leq k$. For each $\varepsilon, \delta > 0$ there is a closed set $K_0 \subset K$ with $\underbrace{\text{intrinsic measure}}_{\text{length, surface area,...}}$ $\mu(K_0) < \delta$ and $F \in \mathcal{F}^{n,k}$, $G \in \mathcal{F}^{k,n}$ such that

$$\sup_{x \in K \setminus K_0} \|G(F(x)) - x\| < \varepsilon. \tag{4}$$

Moreover, K_0 can be chosen disjoint from any finite set $S \subset K$ and such that $M \setminus K_0$ is connected for each component M of K.

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Q. Can K_0 be taken "smaller", e.g., measure zero? **A.** No, by...

Optimality of the almost-ideal autoencoding theorem

Theorem 2 (MDK and EDS). Let $K \subset \mathbb{R}^n$ be a k-dimensional compact submanifold without boundary. For any continuous functions $F\colon \mathbb{R}^n \to \mathbb{R}^k$ and $G\colon \mathbb{R}^k \to \mathbb{R}^n$,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\textbf{reach}} > 0. \tag{2}$$

Proof:

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Proof:

- $N_{r_K}(K) := \{x \in \mathbb{R}^n \colon \mathrm{dist}(x, K) < r_K\}$ contains line segment from $x \in N_{r_K}(K)$ to nearest $\rho(x) \in N_{r_K}(K)$; ρ is continuous.

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- If (2) does not hold, $t \mapsto \rho \circ (tG \circ F|_K + (1-t)\,\mathrm{id}_K)$ is a homotopy of id_K to $\rho \circ G \circ F|_K$, so $\deg_{(2)}(\rho \circ G \circ F|_K) = 1$.

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- $\implies$ contradiction, since

$$0 = \deg_{(2)}(\rho \circ G \circ F|_K) \sim \underbrace{(\rho \circ G \circ F|_K)^*}_{\substack{\| \\ (F|_K)^* \circ \underbrace{(G|_{F(K)})^*}_{0} \circ \rho^*}} \colon \check{H}^k(K) \to \check{H}^k(K)$$

as $\mathrm{domain}[(G|_{F(K)})^*] = \check{H}^k(F(K)) = 0$ by duality theory.

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Deep neural network autoencoders

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Thank you for your time and attention.

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Brockett's necessary condition and beyond
A homotopy theorem beyond the Coron/Mansouri tests
Periodic orbits can be easier to stabilize than equilibria

Applied Koopman operator methods

Many assume the dynamical system is globally linearizable
Which ones are?
1-parameter subgroups of torus actions with asymptotic phase

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