On Professor Smale's legacy for asymptotic stability theory[1]

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Slides are available at my website: mdkvalheim.github.io

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Motivation

- Asymptotic stability is a fundamental dynamical systems concept modeling robust steady-state behavior.
- Engineers often try to make systems asymptotically stable through feedback control. Example

Goal: allow stabilizing feedback control to depend on parameters to shape transient response (when perturbed from steady state).

Question: when can parametric stabilizing feedback be extended from subset of parameter space to all of parameter space?

Insight: given sufficient control authority, the answer depends only on the topology of the space of asymptotically stable vector fields.

Asymptotic stability

Consider an ordinary differential equation

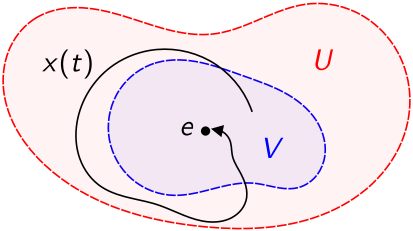
$$\dot{x}(t) = F(x(t)), \tag{1}$$

where F is a vector field on $\mathbb{R}^n$. Unless stated otherwise, F and everything else in this talk is smooth (C^∞).

Let $e \in \mathbb{R}^n$ be an **equilibrium**, meaning $F(e) = 0$.

We say that $e \in \mathbb{R}^n$ is (globally) **asymptotically stable** if

- every solution of (1) converges to e as $t \to \infty$.
- for every open $U \ni e$ there is a smaller open $V \ni e$ s.t. every solution of (1) starting in V at $t = 0$ stays in U for all $t \geq 0$.



Lyapunov functions

- A **Lyapunov function** for a vector field F with equilibrium e is a proper function $L\colon \mathbb{R}^n \to [0,\infty)$ such that $L^{-1}(0) = \{e\}$ and $dL(x)\cdot F(x) \leq 0$ for all x with equality iff $x = e$.
- **History:**
  - Lyapunov (1892) discovered:
    Lyapunov function exists $\implies$ e is asymptotically stable.
  - Massera (1956), Kurzweil (1956) proved converse:
    e is asymptotically stable $\implies$ (C^∞) Lyapunov function exists.
  - Wilson (1967) studied the topology of level sets of such Lyapunov functions.
- **In this talk**, we turn Wilson's idea on its head:

  We use level sets of Lyapunov functions to study the topology of the space of asymptotically stable vector fields.

Outline

Main results

Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Step 2: trivial homotopy groups of the nonlinear Grassmannian

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

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Results: topology and boundary value problems

$\mathcal{S}(\mathbb{R}^n) := \{$asymptotically stable vector fields on $\mathbb{R}^n\}$

$\mathcal{L}(\mathbb{R}^n) := \{$proper functions $\mathbb{R}^n \to [0,\infty)$ w/ unique critical value $= 0\}$

Equip both spaces with the compact-open C^∞ topology.

Theorem (K 2025). $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{L}(\mathbb{R}^n)$ are both path-connected and simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$.

$\Downarrow$

BVP existence theorem. For any compact manifold P and

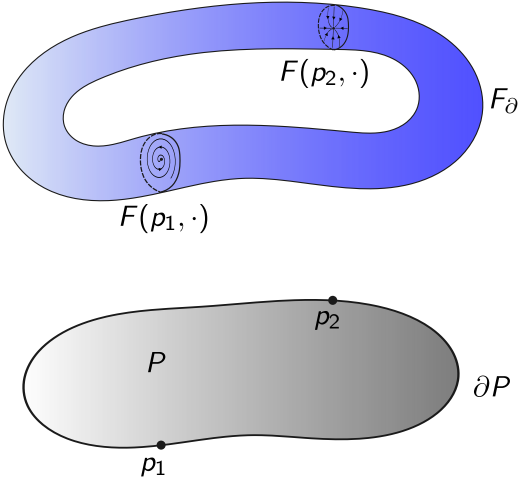
$$F_\partial \colon \partial P \times \mathbb{R}^n \to \mathbb{R}^n, \quad L_\partial \colon \partial P \times \mathbb{R}^n \to [0,\infty)$$

s.t. $F_\partial(p,\cdot) \in \mathcal{S}(\mathbb{R}^n)$, $L_\partial(p,\cdot) \in \mathcal{L}(\mathbb{R}^n)$ $\forall p \in \partial P$, there exist

$$F \colon P \times \mathbb{R}^n \to \mathbb{R}^n, \quad L \colon P \times \mathbb{R}^n \to [0,\infty)$$

extending F_∂, L_∂ s.t. $F(p,\cdot) \in \mathcal{S}(\mathbb{R}^n)$, $L(p,\cdot) \in \mathcal{L}(\mathbb{R}^n)$ $\forall p \in P$ if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 3$.

Illustration of previous theorem (here $n = 2 = \dim P$)



Previous theorem: families of asymptotically stable vector fields on $\mathbb{R}^n$ can always be extended from the boundary ∂P to the entire parameter space P if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 3$.

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Main results

Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Relies on **Smale's h-cobordism theorem**

Step 2: trivial homotopy groups of the nonlinear Grassmannian

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

Topology of Lyapunov function sublevel sets for $n \neq 4, 5$

Key fact:
For any $L \in \mathcal{L}(\mathbb{R}^n)$, $L^{-1}([0,1])$ is diffeomorphic to D^n if $n \neq 4, 5$.

Proof:

- The flow of ∇L induces deformation retractions of $L^{-1}([0,1])$ to $L^{-1}(0)$ and of $\mathbb{R}^n \setminus \{L^{-1}(0)\}$ to $L^{-1}(1)$.
- Hence $L^{-1}([0,1])$ is a contractible manifold with boundary $L^{-1}(1)$ a homotopy sphere (Wilson 1967).
- Hence $L^{-1}([0,1])$ is diffeomorphic to D^n for $n \neq 4, 5$ by
  - classification of 1D and 2D manifolds for $n = 1, 2$,
  - solution to 3D Poincaré conjecture (Perelman 2003) for $n = 3$,
  - the **h-cobordism theorem (Smale 1962)** for $n > 5$. $\square$

The sublevel set map

Let $\mathcal{L}_0(\mathbb{R}^n) \subset \mathcal{L}(\mathbb{R}^n)$ be subspace of functions with min at $0 \in \mathbb{R}^n$.

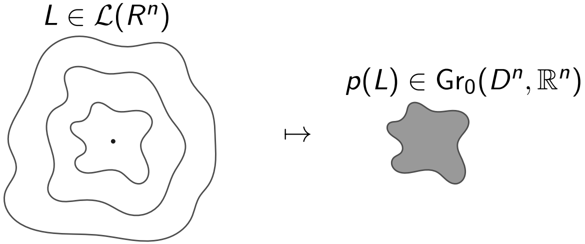
Consider the space

$$\mathrm{Gr}(D^n, \mathbb{R}^n) := \mathrm{Emb}(D^n, \mathbb{R}^n)/\mathrm{Diff}(D^n)$$

of submanifolds of $\mathbb{R}^n$ diffeomorphic to D^n, known as a **nonlinear Grassmannian**, and its open subspace $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ of submanifolds whose interiors contain $0 \in \mathbb{R}^n$.

By the previous slide, we have a well-defined **sublevel set map**

$$p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n), \qquad p(L) := L^{-1}([0,1]).$$



The sublevel set map is a weak homotopy equivalence

Theorem (K 2025). The sublevel set map

$$p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n), \qquad p(L) := L^{-1}([0,1])$$

is a fiber bundle w/ weakly contractible fibers (hence also a w.h.e.).

Proof sketch:

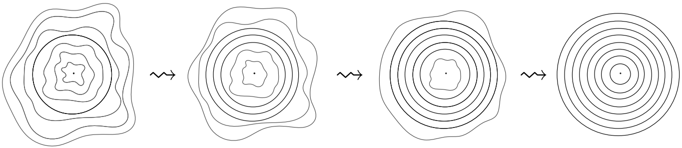
- p is continuous by implicit function theorem; surjective by disc theorem, which also implies $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is path-connected.
- Each $M \in \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ has neighborhood $U \subset \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ and map $\Psi\colon U \to \mathrm{Diff}(\mathbb{R}^n)$ s.t. $\Psi(N)(M) = N$ for all $N \in U$.
- Define $f\colon p^{-1}(U) \to \mathcal{F} := p^{-1}(M)$ by $f(L) := L \circ \Psi(p(L))$.
- Check: $(p, f)\colon p^{-1}(U) \to U \times \mathcal{F}$ is a homeomorphism.
- To show that $\mathcal{F}$ is weakly contractible...

Weak contractibility of $\mathcal{F}$

- Since $p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is a fiber bundle over a path-connected base, it suffices to check that $\mathcal{F} = p^{-1}(M)$ is weakly contractible for $M = D^n$. In this case,

$$\mathcal{F} = \{L \in \mathcal{L}_0(\mathbb{R}^n)\colon L^{-1}([0,1]) = D^n\}.$$

- Any parametric family $P \to \mathcal{F}$ is nullhomotopic to $P \to \{x \mapsto x^2\}$ by "parting the sea" of level sets away from $\partial D^n = S^{n-1}$, replacing the sea with level sets of $x \mapsto x^2$. $\square$



Outline

Main results

Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Step 2: trivial homotopy groups of the nonlinear Grassmannian

Relies on **Smale's theorem** that $\mathrm{Diff}_\partial(D^2)$ is contractible and Hatcher's proof of the **Smale conjecture** for $\mathrm{Diff}_\partial(D^3)$.

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

Toward homotopy groups of the nonlinear Grassmannian

- $\mathcal{L}_0(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}(\mathbb{R}^n)$ and $\mathrm{Gr}_0(D^n, \mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathrm{Gr}(D^n, \mathbb{R}^n)$, so to prove main theorem for $\mathcal{L}(\mathbb{R}^n)$ it suffices to show that the appropriate homotopy groups of $\mathrm{Gr}(D^n, \mathbb{R}^n)$ are trivial.
- The natural quotient map

$$\mathrm{Emb}^+(D^n, \mathbb{R}^n) \to \mathrm{Gr}(D^n, \mathbb{R}^n), \quad f \mapsto f(D^n)$$

is a principal $\mathrm{Diff}^+(D^n)$-bundle,[2] hence also a Serre fibration, so there is a long exact sequence of homotopy groups:

$$\cdots \pi_k \mathrm{Diff}^+(D^n) \longrightarrow \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \longrightarrow \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \cdots$$

[2] Gay-Balmaz and Vizman (2014) proved this result generalizing a theorem of Binz and Fischer (1981) based on an idea implicit in Weinstein (1971).

Analyzing the long exact sequence, part 1

That long exact sequence contains the segment

$$\begin{tikzcd} \pi_k \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[r] & \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \arrow[dll, rounded corners] \\ \pi_{k-1} \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n) & \end{tikzcd}$$

in which the indicated arrows are surjective because the diagram

$$\begin{tikzcd} \mathrm{Diff}^+(D^n) \arrow[rr, hook] \arrow[dr, "\text{surjective on } \pi_k"'] & & \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[dl, "\simeq"] \\ & GL^+(n) & \end{tikzcd}$$

commutes. The diagonal arrows are "evaluate derivative at point".

Analyzing the long exact sequence, part 2

Surjectivity, exactness $\implies$ other arrows are 0, injective:

$$\begin{tikzcd}
\pi_k \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[r, "0"] & \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \arrow[dll, hook, "\text{injective}"] \\
\pi_{k-1} \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n) &
\end{tikzcd}$$

- Hence $\pi_1 \mathrm{Gr}(D^n, \mathbb{R}^n) = \{*\}$ for $n > 5$, since then $\pi_0 \mathrm{Diff}^+(D^n) = \{*\}$ by the pseudoisotopy theorem (Cerf 1970).
- We already knew $\pi_0 \mathrm{Gr}(D^n, \mathbb{R}^n) = \{*\}$ by the disc theorem (Palais 1960, Cerf 1961).
- Remains to show $\pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) = \{*\}$ for all k when $n < 4$; suffices to show above surjections are bijections.

Completing the proof for $\mathcal{L}(\mathbb{R}^n)$, part 1

- Consider earlier commutative diagram

$$\begin{tikzcd} \mathrm{Diff}^+(D^n) \arrow[rr, hook] \arrow[dr, "\text{surjective on } \pi_k"'] & & \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[dl, "\simeq"] \\ & GL^+(n) & \end{tikzcd}.$$

  Need to show top arrow is w.h.e. if $n < 4$.
  Suffices to show same for left diagonal arrow.

- Left diagonal arrow is homotopic to composition

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{f} GL^+(n)$$

  of restriction ρ and map f given by adjoining the value and derivative at the north pole of S^{n-1}.

Completing the proof for $\mathcal{L}(\mathbb{R}^n)$, pt 2: Smale and Hatcher

So for $n < 4$, need to prove that following composition is a w.h.e.

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{f} GL^+(n)$$

- ρ is fiber bundle (Cerf 1961); fiber over $\mathrm{id}_{S^{n-1}}$ is

  $\mathrm{Diff}_\partial(D^n) :=$ {diffeomorphisms of D^n that are the identity on ∂D^n}.

- This fiber is contractible for:
  - $n = 1$ by convexity,
  - $n = 2$ by a **theorem of Smale (1957)**, and
  - $n = 3$ by Hatcher's (1983) proof of the **Smale conjecture (1961)**.
- Hence ρ is a w.h.e., so it suffices to show that f is a w.h.e. for $n < 4$ (trivial for $n = 1$).

Completing the proof for $\mathcal{L}(\mathbb{R}^n)$, pt 3: Smale again

Need to show $f\colon \mathrm{Diff}^+(S^{n-1}) \to GL^+(n)$ is a w.h.e. for $1 < n < 4$.

Identifying $GL^+(n)$ with $\underbrace{\mathrm{Fr}^+(TS^{n-1})}_{+\text{ frame bundle}}$, f factors as the composition

$$\mathrm{Diff}^+(S^{n-1}) \longrightarrow \mathrm{Emb}^+(D^{n-1}_+, S^{n-1}) \xrightarrow{\simeq} \mathrm{Emb}^+(\mathrm{int}(D^{n-1}_+), S^{n-1}) \xrightarrow{\simeq} \mathrm{Fr}^+(TS^{n-1})$$

in which D^{n-1}_+ is upper hemisphere, first two arrows are restrictions, long arrow adjoins value and derivative at north pole.

Similar to last slide, first arrow is a fiber bundle (Cerf 1961) with contractible fiber $\simeq \mathrm{Diff}_\partial(D^{n-1})$ **(Smale 1957)**, so it is a w.h.e. $\square$

Outline

Finishing proof of main theorem for $\mathcal{S}(\mathbb{R}^n)$

Theorem (K 2025). $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{L}(\mathbb{R}^n)$ are both path-connected and simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$.

- We already sketched the proof for $\mathcal{L}(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}_0(\mathbb{R}^n)$.
- To prove for $\mathcal{S}(\mathbb{R}^n)$, suffices to prove $\mathcal{S}(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}(\mathbb{R}^n)$.
- In fact, a w.h.e. is given by the **negative gradient embedding**

$$-\nabla \colon \mathcal{L}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \qquad L \mapsto -\nabla L.$$

- Proof uses converse Lyapunov theorem of Wilson (1969). $\square$

Remark. if path-connectedness statement of above theorem is true for $n = 5$, then the 4D smooth Poincaré conjecture is true.

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Step 2: trivial homotopy groups of the nonlinear Grassmannian

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

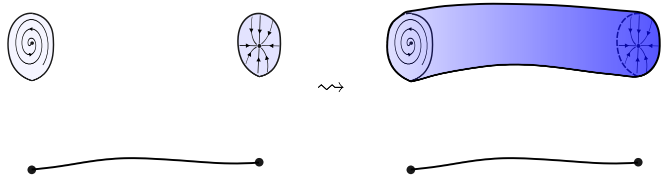
- Partial answer to question of Conley
- Parametric Hartman-Grobman theorem for non-hyperbolic asymptotically stable equilibria
- Parametric Morse lemma for degenerate minima of functions

A question of Conley

- Conley (1978) defined the **Conley index** & proved that two compact isolated invariant sets A, B for two flows Φ, Ψ have isomorphic Conley indices if they are related by *continuation*.
- In particular, this is the case if there is a continuous family $(\Theta_s)_{s\in[0,1]}$ of flows interpolating Φ, Ψ such that the (Θ_s)-induced flow on $[0,1]\times$ (state space) has a compact isolated invariant set C interpolating A, B.
- Conley asked to what extent the converse is true: Given a pair of compact isolated invariant sets with isomorphic Conley indices, when are they related by continuation?

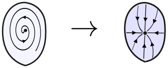
Partial answers to Conley

- Reineck (1992): in many interesting cases (using **Smale's (1960) Morse fun./handle manipulation techniques!**).
- However, Reineck's results do not address a natural case of Conley's question: Given a pair of asymptotically stable vector fields, when can they be interpolated by asymptotically stable vector fields?
- Jongeneel (2024) proved that the associated (semi)flows can always be interpolated through asymptotically stable C^0 semiflows for state space dimension $n \neq 5$.
- This does not quite tell us about homotopy through C^∞ such flows / vector fields, but the "path-connectedness" portion of our main theorem $\implies$ this is always possible for $n \neq 4, 5$.
- Cutoff functions, etc $\implies$ same answer for local version.



Hartman-Grobman without hyperbolicity

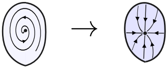
- **Classical Hartman-Grobman (1960, 1959) theorem**: given a C^1 vector field F with hyperbolic equilibrium e, there is a local homeomorphism identifying solutions of $\dot{x} = F(x)$ with those of $\dot{y} = Ay$ for some nonunique A ($= DF(e)$ works).
- **Theorem (see K-Sontag 2025).** The hyperbolicity and "C^1" assumptions can be dropped if we assume that e is asymptotically stable. Moreover, the homeomorphism is global (and a C^k diffeo on complement of e if $F \in C^k$).



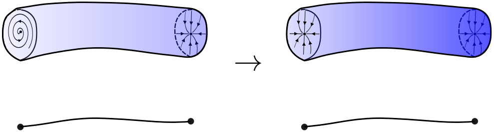
- **Proof.** use **Smale (1962)**, Freedman (1982), Perelman (2003) to identify a Lyapunov function level set of one vector field with one of the other, then extend identification via flows.
- **Remark.** the above $C^{k\geq 1}$ statement is true for $n = 5 \iff$ the 4D smooth Poincaré conjecture is true.

Parametric Hartman-Grobman without hyperbolicity

- **Theorem (see K-Sontag 2025).** The hyperbolicity and "C^1" assumptions can be dropped if we assume that e is asymptotically stable. Moreover, the homeomorphism is global (and a C^k diffeo on complement of e if $F \in C^k$).



- **Theorem (K-2025).** If in above theorem F_p depends continuously on parameter $p \in P$, there is a continuous family $h_p : B \to \mathbb{R}^n$ of linearizing homeomorphisms if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P = 1$.



- Proof: same, but instead of using Smale, Freedman, Perelman to identify Lyapunov function level sets for a pair of vector fields, use main theorem to parametrically identify Lyapunov function level sets for a pair of families of vector fields.

Thank you for your attention.

This talk is based on the preprint arXiv:2503.10828:

"Differential topology of the spaces of asymptotically stable vector fields and Lyapunov functions", Kvalheim (2025).

▸ Link to preprint

Slides are available at my website: mdkvalheim.github.io

On Professor Smale's legacy for asymptotic stability theory

Main results

Topology of $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$

Boundary value problems

Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Relies on **Smale's h-cobordism theorem**

Step 2: trivial homotopy groups of the nonlinear Grassmannian

Relies on **Smale's theorem** that $\mathrm{Diff}_\partial(D^2)$ is contractible and Hatcher's proof of the **Smale conjecture** for $\mathrm{Diff}_\partial(D^3)$.

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