

# Identifying engineering (im)possibilities with geometry and topology<sup>1</sup>

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# Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

Feedback stabilizability

# Identifying engineering (im)possibilities for:

## Deep neural network autoencoders

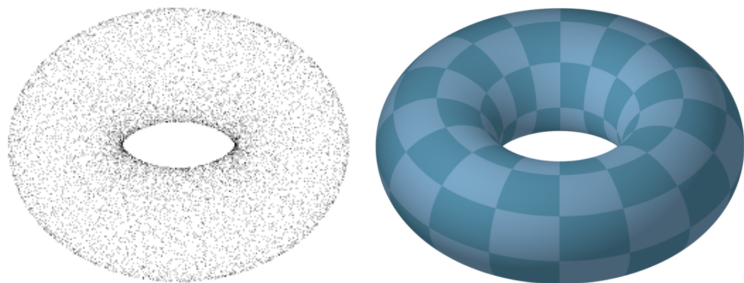
They should not work, and yet they do: resolving the paradox

Training implications:  $L^2$  but not  $L^\infty$  error can be made small

## Applied Koopman operator methods

## Feedback stabilizability

# Dimensionality reduction of data

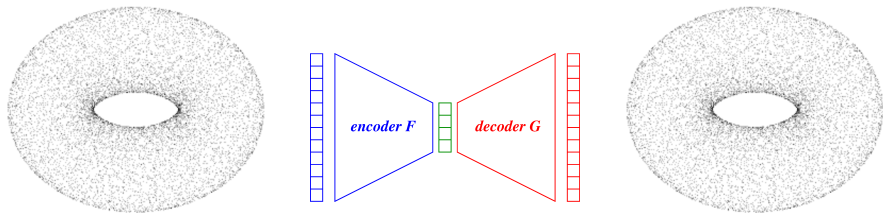


The “manifold hypothesis” postulates that a data set in  $\mathbb{R}^n$  lies on some  $k$ -dimensional submanifold  $K \subset \mathbb{R}^n$ .

$\implies$  data can be parametrized locally by  $k < n$  real numbers.

Classical approaches like PCA to learn these parameters work well when  $K$  is linear, but not when  $K$  is nonlinear.

## Autoencoding as a nonlinear dimensionality reduction approach (and why it should not work)



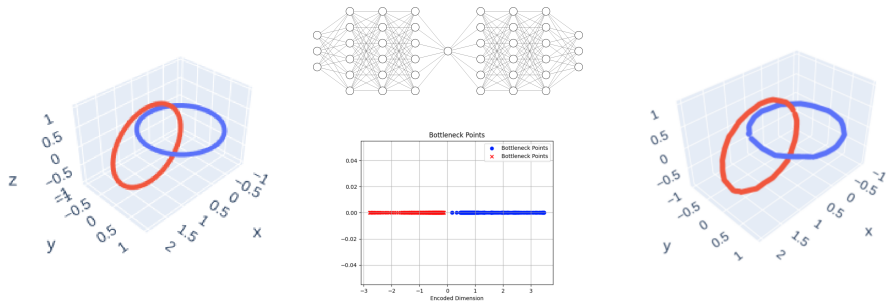
Popular nonlinear approach: seek **autoencoder**  $G \circ F$ , where the output of the **encoder**  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is the desired  $k$  parameters,  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$  is the **decoder**, and  $F, G$  are continuous.

Often  $F, G$  are artificial neural network functions.

**Ideal autoencoder:**  $G(F(x)) = x$  for all  $x \in K$ .

These **do not usually exist!** Existence  $\implies K$  is homeomorphic to a subset of  $\mathbb{R}^k$ , which is not true of most  $k$ -dimensional  $K$ .

If autoencoding should not work, how does it?<sup>2</sup> Example:



$K$  = a pair of circles in  $\mathbb{R}^3$ , after thickening then deleting small intervals, is diffeomorphic to a pair of disjoint intervals in  $\mathbb{R}$ .

**Encoder**  $F: \mathbb{R}^3 \rightarrow \mathbb{R}$  = any extension of this diffeomorphism.

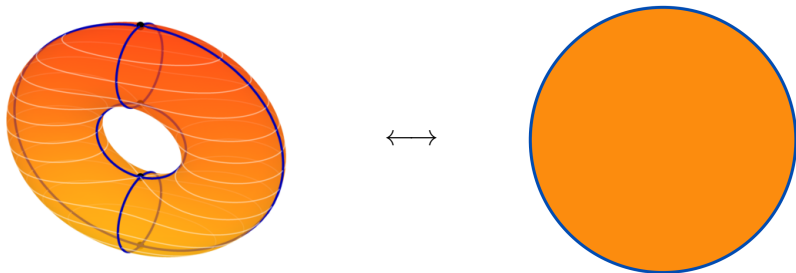
**Decoder**  $G: \mathbb{R} \rightarrow \mathbb{R}^3$  = any extension of inverse diffeomorphism.

Such small intervals disjoint from the data set always exist.

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<sup>2</sup>MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

If autoencoding should not work, how does it? In general:



$K$  = a union of  $\leq k$ -dimensional compact submanifolds of  $\mathbb{R}^n$ , after thickening then deleting the  $\text{codimension} > 0$  “steepest ascent disks” of a polar Morse function,<sup>3</sup> is diffeomorphic to a subset of  $\mathbb{R}^k$ .

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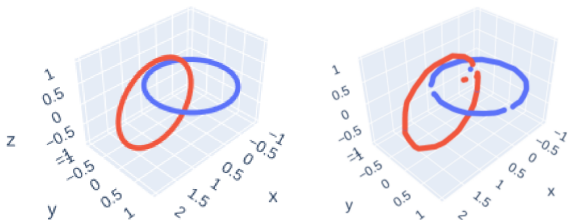
**Decoder**  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$  = any extension of inverse diffeomorphism.

Can always find such a “ $\text{codim} > 0$  set” disjoint from the data.

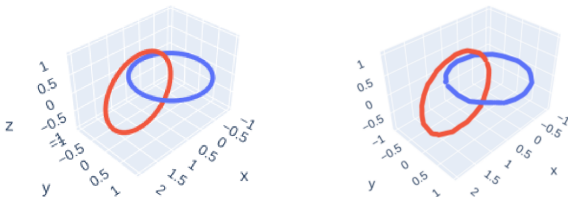
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<sup>3</sup>A *navigation function*, in the parlance of Rimon and Koditschek (1990).

Note: training sometimes yields disconnected “good” sets



In practice, random initialization/training leads to random outcomes, even those with disconnected “good sets”, despite the fact that arbitrarily large connected “good sets” (disks) exist.





## Semi-global autoencoders always exist<sup>4</sup>

Let  $\mathcal{F}^{\ell,m}$  be dense in the space of continuous functions  $\mathbb{R}^{\ell} \rightarrow \mathbb{R}^m$ , e.g., the collection of possible neural network outputs.

**Theorem 1 (MDK and E D Sontag).** Let  $K \subset \mathbb{R}^n$  be finitely many disjoint compact  $\leq k$ -dimensional submanifolds with(out) boundary, and let  $\mu, \partial\mu$  be any smooth measures on  $K, \partial K$ . For each  $\delta > 0$  and finite set  $S \subset K$ , there is a closed set  $K_0 \subset K$  s.t.

- ▶  $K_0 \cap S = \emptyset, \mu(K_0) < \delta, \partial\mu(K_0 \cap \partial K) < \delta$ ;
- ▶  $M \setminus K_0$  is connected for each component  $M$  of  $K$ ;
- ▶ For each  $\varepsilon > 0$  there are functions  $F \in \mathcal{F}^{n,k}, G \in \mathcal{F}^{k,n}$  such that

$$\sup_{x \in K \setminus K_0} \|G(F(x)) - x\| < \varepsilon.$$

$\implies$  data  $S$  can be reconstructed to order  $\varepsilon$ , and generalization error will also be uniformly smaller than  $\varepsilon$  with probability  $1 - \delta$ .

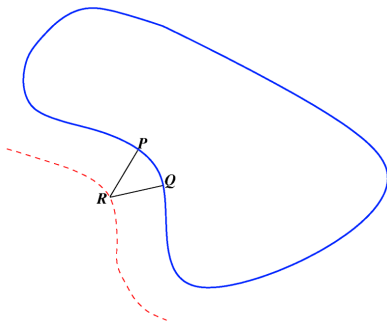
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<sup>4</sup>MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

## Almost-global autoencoders do not generally exist

**Theorem 2 (MDK and EDS).** Let  $K \subset \mathbb{R}^n$  be a  $k$ -dimensional compact submanifold without boundary. For any continuous functions  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  and  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\text{reach}} > 0.$$



**Figure:** The **reach**  $r_K > 0$  of  $K$  is the largest number such that any  $x \in \mathbb{R}^n$  satisfying  $\text{dist}(x, K) < r_K$  has a unique nearest point on  $K$ . Both line segments shown have length  $r_K$ .

## Almost-global autoencoders do not generally exist<sup>5</sup>

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$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\text{reach}} > 0. \quad (2)$$

**Proof:**

- ▶  $N := \{x \in \mathbb{R}^n : \text{dist}(x, K) < r_K\}$  contains line segment from  $x \in N$  to nearest  $\rho(x) \in K$ ;  $\rho$  is continuous.

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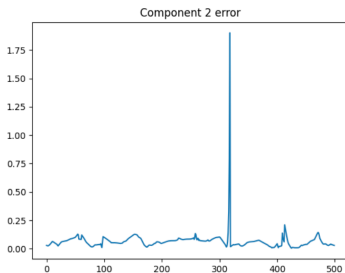
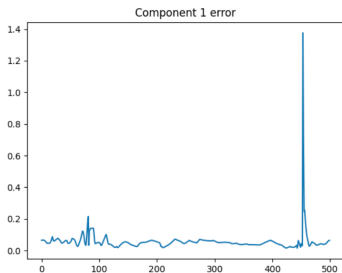
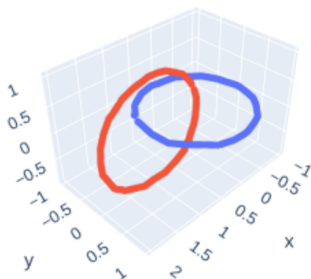
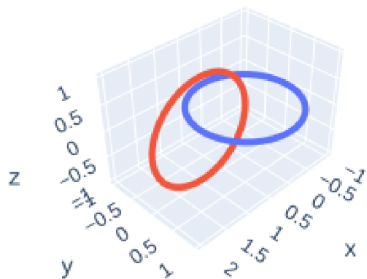
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- ▶ But this contradicts

$$0 = \deg_2(\rho \circ G \circ F|_K) = \deg_2(\rho \circ G|_{F(K)}) \underbrace{\deg_2(F|_K)}_0.$$

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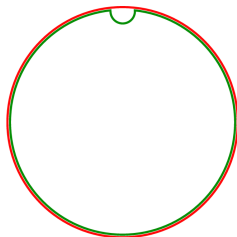
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Example:  $K = 2$  unit circles; max error  $>$  reach  $r_K = 1$



**Figure:** Errors  $\|G(F(x)) - x\|$  on the two circles. The  $x$ -axis shows the index  $k$  representing the  $k$ -th evenly-spaced point on the respective circle.

In fact, true min-max error is usually bigger than the reach<sup>6</sup>



Red reach = 1 but green reach =  $\varepsilon \ll 1$ , so previous min max error “reach” is conservative. But green “**dewrinkled reach**” =  $1 - \varepsilon$ .

**Corollary (MDK and EDS).** Let  $K \subset \mathbb{R}^n$  be a  $k$ -dimensional compact submanifold without boundary. For any continuous functions  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  and  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_{K,k}^*}_{\text{dewrinkled reach}} := \sup_{L \in \mathcal{M}_{n,k}, T \in C(L \rightarrow K)} \{r_L - \delta(T)\}.$$

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<sup>6</sup>MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

# Implications for autoencoder training error<sup>7</sup>

Theorem 1  $\implies$   **$F, G$  always exist making the  $L^2(\mu)$  loss**

$$\int_K \|G(F(x)) - x\|^2 d\mu(x) < \varepsilon$$
$$\left( = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \|G(F(x_i)) - x_i\|^2 \right)$$

**arbitrarily small** (any  $G$  can be modified off of  $F(K \setminus K_0)$  to make the autoencoder error smaller than a certain  $C_K > 0$  on  $K_0$ .)

**However,** Theorem 2  $\implies$  **for many  $K$ , the  $L^\infty$  loss**

$$\max \|G(F(x)) - x\| \geq r_K > 0$$

**is uniformly big, independent of  $F, G$ .**

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<sup>7</sup>We thank Dr. Joshua Batson for suggesting these observations.



## Summary

**Main representation result:** data lying in a submanifold  $K$  of dimension  $k$  can be encoded through a bottleneck layer of the same dimension  $k$ , up to an arbitrarily small reconstruction error  $\varepsilon$ .

**Moreover, the generalization error** will also be uniformly smaller than  $\varepsilon$  with arbitrarily high probability  $1 - \delta$ .

**Main necessity result:** for many  $K$  (including all  $K$  without boundary), there is a geometric lower bound on the global reconstruction error.

**Training implications:**  $L^2$  error can always be made arbitrarily small;  $L^\infty$  error cannot.

# Identifying engineering (im)possibilities for:

## Deep neural network autoencoders

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Training implications:  $L^2$  but not  $L^\infty$  error can be made small

## Applied Koopman operator methods

## Feedback stabilizability

# Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

Many assume the dynamical system is globally linearizable

Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

Feedback stabilizability

## “Applied Koopmanism”

*“A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.”*

— Brunton, Budišić, Kaiser, and Kutz. “Modern Koopman Theory for Dynamical Systems.” SIAM Review, 64.2 (2022)

They seek nonlinear measurements that separate points, to be able to invert / not to lose information. Also want measurements / inverse to be continuous for practical reasons.

More formally, they seek *embeddings of nonlinear* dynamical systems into *linear* ones as invariant subsets, so that existing theoretical and algorithmic linear tools can be utilized.

## Linearizing embeddings

Let  $f$  be a locally Lipschitz vector field on a manifold  $M$ . Consider

$$\dot{x} = \frac{d}{dt}x = f(x),$$

assume this ODE's solutions  $x(t) = \Phi^t(x_0)$  are defined for all time.

$F: M \rightarrow \mathbb{R}^n$  is a **topological embedding** if  $F$  is a one-to-one continuous map with a continuous inverse  $F^{-1}: F(M) \rightarrow M$ , and is a **smooth embedding** if additionally  $F, F^{-1}$  are smooth.

Such an embedding  $F$  is **linearizing** if  $F \circ \Phi^t = e^{Bt} \circ F$  for some  $n \times n$  matrix  $B$ . In the smooth case,  $y = F(x)$  satisfies  $\dot{y} = By$ .

**Fundamental question:** when is  $(M, \Phi)$  linearizable in this sense?

# When is a dynamical system $(M, \Phi)$ globally linearizable?

- ▶ Not when  $M$  is connected, forward  $\Phi$ -trajectories are precompact, and  $\Phi$  has a countable number  $\geq 2$  of omega limit sets (Liu, Ozay, Sontag 2023).
- ▶ Not when  $M$  is connected and  $\Phi$  has a non-global compact attractor  $A \neq \emptyset$ , since its open basin of attraction would also be closed (by the Jordan normal form theorem), hence empty.

Thus, we study linearizability of the restriction  $(S, \Phi)$  of  $\Phi$  to

1. compact invariant sets  $S$ , and
2. basins  $S$  of compact attractors  $A$ .

For these 2 cases we obtain **necessary and sufficient conditions for global linearizability** of  $(S, \Phi)$  by an embedding, for the 2 cases of topological and smooth embeddings (4 cases total).<sup>8</sup>

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<sup>8</sup>MDK and P. Arathoon, *Linearizability of flows by embeddings* (2023).

## Torus preliminaries

The  $n$ -torus  $T = T^n$  is (Lie group) isomorphic to  $(\mathbb{R}/\mathbb{Z})^n$ , vectors with  $n$  real entries but with addition defined elementwise modulo 1.

A **torus action** on  $S$  is a map  $\Theta: T \times S \rightarrow S$  satisfying  $\Theta^{\tau_1 + \tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$  for all  $s \in S$  and  $\tau_1, \tau_2 \in T$ .

The flow  $(S, \Phi)$  is a **1-parameter subgroup of a torus action** if  $\Phi^t = \Theta^{\omega t \bmod 1}$  for some torus action  $\Theta$  on  $S$ ,  $\omega \in \mathbb{R}^n$ .

## The linearizability theorem, case 1: compact, smooth

**Observation:** If  $(S, \Phi)$  is linearizable with  $S$  compact, the Jordan normal form theorem implies  $(S, \Phi)$  embeds into the flow on  $\mathbb{C}^n$  of a diagonal imaginary matrix, so  $(S, \Phi)$  is a 1-parameter subgroup of restriction of standard torus action of  $T^n$  on  $\mathbb{C}^n$  to a subtorus.

This gives one implication below; the *Mostow-Palais equivariant embedding theorem* gives the other.



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This gives one implication below; the *Mostow-Palais equivariant embedding theorem* gives the other.

**Theorem (MDK and P. Arathoon).** If  $S$  is a compact embedded submanifold,  $(S, \Phi)$  is linearizable by a smooth embedding  $\iff (S, \Phi)$  is a 1-parameter subgroup of a smooth torus action.

We use this theorem to construct examples of smoothly linearizable  $(S, \Phi)$  having isolated equilibria with e.g.  $S =$  a sphere, torus, Klein bottle. On the other hand, regarding *nonlinearizability*...

## Topological implications for case 1 (compact, smooth)

If  $(S, \Phi)$  is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an  $n \times n$  skew matrix  $B_e$  and a system of local coordinates on a neighborhood of each equilibrium  $e \in S$  such that  $\Phi^t \approx e^{B_e t}$ . Hence if  $e$  is isolated then  $B_e$  is invertible,  $n = \dim S$  is even, and the Hopf index of  $e$  is  $+1$ .

**Corollary (MDK and PA).** If  $S$  is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then  $(S, \Phi)$  cannot be linearized by a smooth embedding.

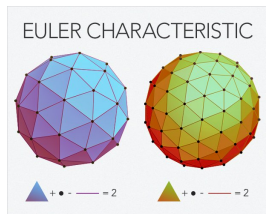
**Corollary (MDK and PA).**<sup>9</sup> If  $S$  is a compact submanifold containing at most finitely many equilibria such that  $(S, \Phi)$  is linearizable by a smooth embedding,  $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$ .

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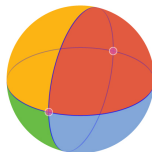
<sup>9</sup>Apply the Poincaré-Hopf theorem.

# A primer on the Euler characteristic<sup>10</sup>

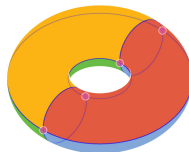
Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



Euler Characteristic  $\chi = \text{Faces} + \text{Corners} - \text{Edges}$



$$\chi = 4 + 2 - 4 = 2$$



$$\chi = 4 + 4 - 8 = 0$$

**Notation:**  $\chi(Y) :=$  Euler characteristic of  $Y$ .

**Examples:**  $\chi(\bullet) = 1$ ,  $\chi(\mathbb{S}^1) = 0$ ,  $\chi(\mathbb{S}^2) = 2$ ,  $\chi(\Sigma_g) = 2 - 2g$



$\Sigma_g$  for  $g = 1, 2, 3$  (not linearizable for  $g > 1$  if finite equilibria).

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<sup>10</sup>Figures from Quanta Magazine and Wikipedia.

## The linearizability theorem, case 2: compact, continuous

The theorem for case 2 is similar for case 1, but an additional assumption is needed to rule out a pathology not possible in case 1.

**Theorem (MDK and PA).** If  $S$  is compact,  $(S, \Phi)$  is linearizable by a **topological** embedding  $\iff (S, \Phi)$  is a 1-parameter subgroup of a **continuous** torus action with **finitely many orbit types**.

A torus action has **finitely many orbit types** if there are only finitely many subgroups  $H \subset T$  such that  $H = \{\tau \in T : \Theta^\tau(s) = s\}$  is the fixed point set of some  $s \in S$ .

## Another point of view: quasiperiodic pinched torus families

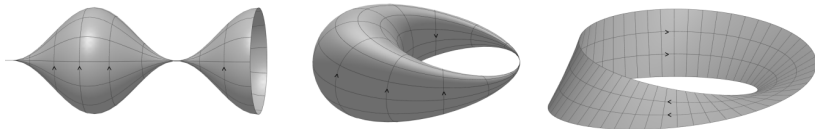


Figure: examples of quasiperiodic pinched torus families

**Proposition (MDK and PA).** If  $S$  is compact,  $(S, \Phi)$  is linearizable by a **topological** embedding  $\iff (S, \Phi)$  is a **quasiperiodic pinched torus family**.

**Definition.**  $P$  is a **pinched torus family** if there are  $m, n \in \mathbb{N}$ , closed subsets  $C_1, \dots, C_n \subset B \subset T^m$ , and a continuous group homomorphism  $F: T^n \rightarrow T^m$  such that  $P$  is the quotient of  $F^{-1}(B)$  by collapsing the  $j$ -th  $(\mathbb{R}/\mathbb{Z})$ -factor of  $F^{-1}(C_j) \subset T^n$  for all  $j$ . A pinched torus family  $P$  is **quasiperiodic** if it is equipped with the induced flow generated by any  $\omega \in \mathbb{R}^n$  with  $TF(\omega) = 0$ .

## The linearizability theorem, case 3: basin, continuous

If  $S$  is the basin of an asymptotically stable compact set  $A \subset S$ ,  $A$  has continuous (smooth) **asymptotic phase**<sup>11</sup> if there is a continuous (smooth) **asymptotic phase map**  $P: S \rightarrow A$ , i.e.,

$$P|_A = \text{id}_A, \quad P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

**Theorem (MDK and PA).**  $(S, \Phi)$  is linearizable by a **topological** embedding  $\iff A$  has **continuous** asymptotic phase and  $(A, \Phi)$  is a 1-parameter subgroup of a **continuous** torus action with **finitely many orbit types**.

**Example.** The basin of an asymptotically stable limit cycle is linearizable by a topological embedding  $\iff$  the cycle has continuous asymptotic phase. This is not always the case, but it is the case if  $\Phi \in C^1$  and the cycle is hyperbolic.

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<sup>11</sup>This notion has roots in oscillator theory and more generally NHIM theory.

## The linearizability theorem, case 4: basin, smooth

**Theorem (MDK and PA).**  $(S, \Phi)$  is linearizable by a smooth embedding  $\iff A$  is an embedded submanifold with smooth asymptotic phase,  $(A, \Phi)$  is a 1-parameter subgroup of a smooth torus action, and for some open  $U \supset A$ ,  $(U, \Phi)$  embeds in a reducible linear flow covering  $\Phi$  on some vector bundle over  $A$ .

**When does the final condition hold?** Classical linearization theorems and recent linearizing semiconjugacy theorems (MDK and Revzen, 2023) give answers in the special cases that  $A$  is an equilibrium or periodic orbit, and some things are known if  $A$  is quasiperiodic, but the general case seems to be an open problem.

**A necessary condition** for  $A$  to satisfy all conditions of the theorem is that  $A$  be a (eventually relatively  $\infty$ -) **normally hyperbolic invariant manifold**. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

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- Brockett's necessary condition and beyond

- A homotopy theorem beyond the Coron/Mansouri tests

- Periodic orbits can be easier to stabilize than equilibria



# Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \quad (1)$$

where  $M \ni x$  is a smooth manifold and  $f$  is smooth.

1. **Controllability problem:** Given  $a, b \in M$ , find  $u(t)$  s.t.  $x(T) = b$  if  $x(0) = a$  for some  $T > 0$ .

$$a \rightsquigarrow b$$

2. **Stabilizability problem:** Given a compact subset  $A \subset M$ , find smooth  $u(x)$  s.t.  $A$  is **asymptotically stable**<sup>12</sup> for the **closed-loop vector field**  $F(x) = f(x, u(x))$ . [▶ Link](#)

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<sup>12</sup>For every open  $W \supset A$  there is an open  $V \supset A$  s.t. all forward  $F$ -trajectories initialized in  $V$  are contained in  $W$  and converge to  $A$ .

# The stabilization conjecture and Brockett's solution

Often  $A = \{x_*\}$  is a point,  $M = \mathbb{R}^n$  in the stabilization problem.

**Stabilization conjecture (pre-1983):** a reasonably strong form of controllability implies smooth stabilizability of a point.

**Example:** the “Heisenberg system” or “nonholonomic integrator”

$$\left. \begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?

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**Theorem (Brockett).** If a point is stabilizable, then  $\text{image}(f)$  is a neighborhood of 0. (In the example,  $(0, 0, \varepsilon) \notin \text{image}(f)$ .)

## Other stabilizability work

- ▶ Exponential (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
- ▶ global (Byrnes 2008, Baryshnikov 2023),
- ▶ time-varying (Coron 1992), and
- ▶ discontinuous (Clarke, Ledyaev, Sontag, Subbotin 1997)

variants of the stabilization problem are not considered here.

## Coron's and Mansouri's obstructions

Krasnosel'skiĭ and Zabreĭko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : f(x, u) \neq 0\}.$$

**Theorem (Coron 1990).** If  $n > 1$  and a point is stabilizable,

$$f_*(H_{n-1}(\Sigma)) = H_{n-1}(\mathbb{R}^n \setminus \{0\}) \quad (\cong \mathbb{Z}).$$

**Theorem (Mansouri 2010).** If a closed codimension  $> 1$  submanifold  $A \subset \mathbb{R}^n$  with Euler characteristic  $\chi(A)$  is stabilizable,

$$f_*(H_{n-1}(\Sigma)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \quad (\cong \chi(A) \cdot \mathbb{Z}).$$

## Limitations of these results

The results of Brockett, Coron, Mansouri rely on parallelizability of  $\mathbb{R}^n$  to view vector fields and control systems as  $\mathbb{R}^n$ -valued.

Furthermore, they apply only to the special case that  $A$  is a point or a closed submanifold of  $\mathbb{R}^n$  with  $\chi(A) \neq 0$ .

But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

How to test for stabilizability in such general settings?<sup>13</sup>

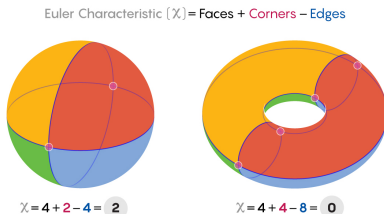
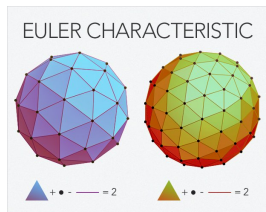
- ▶ **Generalization of Brockett's test** (MDK and Daniel E. Koditschek, J Geometric Mechanics, 2022).
- ▶ **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).

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<sup>13</sup>An exposition of all stabilizability results here is in 2023 book *Topological Obstructions to Stability and Stabilization* by W. Jongeneel and E. Moulay.

# A primer on the Euler characteristic<sup>14</sup>

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



**Notation:**  $\chi(Y) :=$  Euler characteristic of  $Y$ .

**Examples:**  $\chi(\bullet) = 1$ ,  $\chi(S^1) = 0$ ,  $\chi(S^2) = 2$ ,  $\chi(\text{figure 8}) = -1$

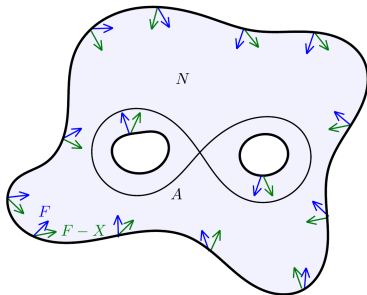
**Theorem (Poincaré, Hopf):** if  $N$  is a compact smooth manifold with boundary  $\partial N$ , then  $\chi(N) = 0 \iff$  there exists a nowhere-zero smooth vector field on  $N$  pointing inward at  $\partial N$ .

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<sup>14</sup>Figures from Quanta Magazine.

## Generalization of Brockett's test

**Theorem (MDK & Koditschek 2022):** Let  $A \subset M$  be compact & stabilizable. Then  $\chi(A)$  is well-defined. If  $\chi(A) \neq 0$ , then for any sufficiently small vector field  $X$ ,  $X(x_0) = f(x_0, u_0)$  for some  $x_0, u_0$ .



**Proof:** Assume  $\exists$  stabilizing  $u(x)$  and define  $F(x) := f(x, u(x))$ .  
Lyapunov function theory  $\implies \exists$  compact smooth domain  $N \supset A$   
s.t.  $F$  points inward at  $\partial N$  and  $\chi(A) = \chi(N) \neq 0$ . Continuity  
 $\implies F - X$  points inward at  $\partial N$  if  $X$  is small  $\implies F - X$  has a  
zero by Poincaré-Hopf  $\implies \exists x_0$  s.t.  $X(x_0) = F(x_0) = f(x_0, u(x_0))$ .



## Examples

### Heisenberg system

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv\end{aligned}\tag{2}$$

### Kinematic differential drive robot

$$\begin{aligned}\dot{x} &= u \cos \theta \\ \dot{y} &= u \sin \theta \\ \dot{\theta} &= v\end{aligned}\tag{3}$$

The right side of (2)  $\neq X_\varepsilon := (0, 0, \varepsilon)$  for any  $\varepsilon > 0$ .

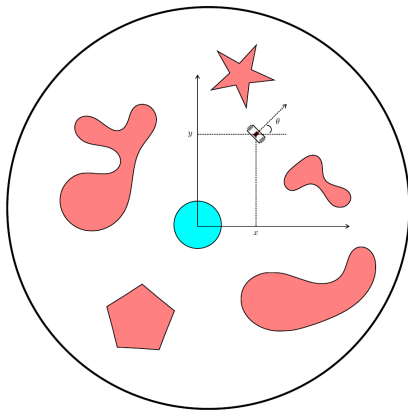
The right side of (3)  $\neq X_\varepsilon := (\varepsilon \sin \theta, -\varepsilon \cos \theta, 0)$  for any  $\varepsilon > 0$ .

Thus, our result  $\implies A$  is not stabilizable if  $\chi(A) \neq 0$ . E.g., if  $A$  is a stabilizable compact submanifold,  $A$  is a union of circles and tori.

**Other applications:** any stabilizable compact set has zero Euler characteristic for satellite orientation with  $\leq 2$  thrusters, for nonholonomic dynamics with  $\geq 1$  global constraint 1-form,...

## Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some “bad” set immediately enter some “good” set.



E.g., impossible for this differential drive robot to aim within  $\pm 179$  degrees of the origin while “strictly” avoiding obstacles via  $u(x)$ .

# Homotopy theorem & generalized Coron, Mansouri tests

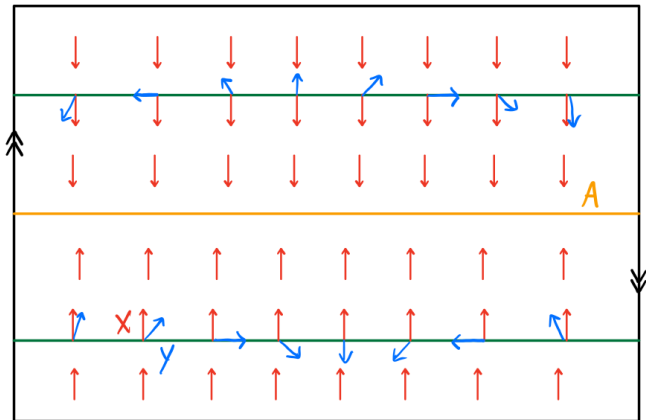
**Homotopy theorem (MDK 2023).** Let  $X, Y$  be smooth vector fields on a manifold  $M$  with a compact set  $A \subset M$  asymptotically stable for both. There is an open set  $U \supset A$  such that  $X|_{U \setminus A}, Y|_{U \setminus A}$  are homotopic through nowhere-zero vector fields.

$\implies$  **Theorem (MDK 2023).** Let the compact set  $A \subset M$  be asymptotically stable for *some* smooth vector field  $Y$  on  $M$ . If  $A$  is stabilizable for  $\dot{x} = f(x, u)$ , then for all small enough open  $U \supset A$ ,

$$H_{\bullet}(T(U \setminus A) \setminus 0) \supset \underbrace{f_{*}H_{\bullet}(\Sigma) \supset Y_{*}H_{\bullet}(U \setminus A)}_{\text{cf. Coron, Mansouri}}.$$

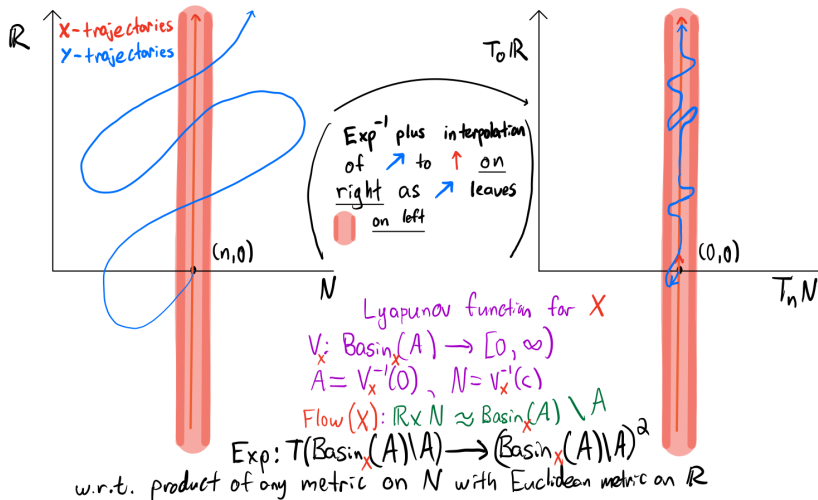
These are stronger than all preceding results: there is an example (MDK 2023) for which non-stabilizability is detected by each of these theorems but not by any of the preceding theorems.

# Möbius strip example



$X \neq y$  since  $y \curvearrowright$  twice  
 around  $\bigcirc$  w.r.t.  $X$  while  $X \curvearrowright$   
 zero times w.r.t.  $X \Rightarrow \underline{A \text{ is not}}  
asymptotically stable for  $y$  by the  
 homotopy theorem.$

# Proof of the homotopy theorem



## Can these results detect stabilizability of periodic orbits?

If  $A$  is the image of a periodic orbit with the same orientation for  $X$  and  $Y$ , the straight-line homotopy over a sufficiently small open  $U \supset A$  satisfies the homotopy theorem's conclusion regardless of whether  $A$  is attracting, repelling, or neither for  $X$  or  $Y$ .

**$\implies$  homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, preceding results also give no information.

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...Could it be that periodic orbits might be “easy” to stabilize?

# Periodic orbits are sometimes easier to stabilize

Indeed, at least sometimes:

**Theorem (Anthony M. Bloch & MDK, in preparation).**

For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!



# Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

## Feedback stabilizability

- Brockett's necessary condition and beyond

- A homotopy theorem beyond the Coron/Mansouri tests

- Periodic orbits can be easier to stabilize than equilibria

Thank you for your attention.

# Identifying engineering (im)possibilities for:

## Deep neural network autoencoders

They should not work, and yet they do: resolving the paradox  
Training implications:  $L^2$  but not  $L^\infty$  error can be made small

## Applied Koopman operator methods

Many assume the dynamical system is globally linearizable  
Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

## Feedback stabilizability

Brockett's necessary condition and beyond

A homotopy theorem beyond the Coron/Mansouri tests

Periodic orbits can be easier to stabilize than equilibria