Existence and uniqueness of Koopman eigenfunctions near stable equilibria and limit cycles[1]

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Relevant papers and slides available

- **Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits.** MDK and Shai Revzen. Physica D (2021), arXiv:1911.11996.

- **Generic properties of Koopman eigenfunctions for stable fixed points and periodic orbits.** MDK, David Hong, and Shai Revzen. IFAC-PapersOnline (2021; MTNS conference cancelled), arXiv:2010.04008.

- **Slides available on my website**: mdkvalheim.github.io/assets/NOLTA2022.pdf

Motivation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad f(0) = 0, \quad 0 \text{ hyperbolically stable with basin } B \subset \mathbb{R}^n. \quad (1)$$

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- How to find eigenfunctions? If $\mu \in \mathbb{C}$ and $g \colon B \to \mathbb{C}$ is such that the limit[2]

$$g^*_\mu(x_0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(x(t))e^{-\mu t}\,dt \tag{3}$$

exists and is not identically zero, then g^*_μ is an eigenfunction with $\lambda = \mu$ in (2).

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- **Questions:** Which one? Can there be more than one possible limit (modulo scalar multiplication)? Can the limit depend sensitively on g? Other numerical issues?

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- **Questions:** Which one? Can there be more than one possible limit (modulo scalar multiplication)? Can the limit depend sensitively on g? Other numerical issues?
- If we knew that eigenfunctions were unique, we could resolve these questions. **We will discuss uniqueness and more**, including **new convergence results** for (3).

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Principal eigenfunctions

- C^1 eigenfunctions determining a linearizing diffeomorphism must be **principal**:

$$d\psi_i(0) \neq 0.$$

- (**Fact**: if ψ_i is principal and $\dot{\psi}_i = \lambda \psi_i$, $d\psi_i(0)$ is left eigenvector of $D_0 f$ w/ e.val λ.)

- Thus, we will concentrate on existence & uniqueness of C^k principal eigenfunctions.

- In particular we will see that, under some conditions, principal eigenfunctions are uniquely determined by their derivatives at 0.

- (Later we will classify **all** C^∞ eigenfunctions under generic conditions, not just the principal ones.)

Counterexamples $\implies$ some conditions are needed

- **Ex. 1.** Let $k \geq 2$ be an integer, $(x, y) \in \mathbb{R}^2$,

$$\dot{x} = -x, \qquad \dot{y} = -ky.$$

Both

$$\psi_1(x, y) = y \text{ and } \psi_2(x, y) = y + x^k$$

are analytic principal eigenfunctions s.t. $d\psi_1(0) = d\psi_2(0)$,

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- **Ex. 2.** Let $a > 1$ not be an integer, $(x, y) \in \mathbb{R}^2$,

$$\dot{x} = -x, \qquad \dot{y} = -ay.$$

Both

$$\psi_1(x, y) = y \text{ and } \psi_2(x, y) = y + |x|^a$$

are $C^{\lfloor a \rfloor}$ principal eigenfunctions ($\lfloor a \rfloor$ is the integer part of a) s.t. $d\psi_1(0) = d\psi_2(0)$,

$$\dot{\psi}_i = \lambda\psi_i \text{ with } \lambda = -a.$$

$\implies$ resonance not an issue here, but **spectral spread assumptions needed** (later).

Towards C^k existence and uniqueness, step 1: reduction to discrete-time

- Henceforth assume vector field $f \in C^k$ is complete with C^k flow $(t, x) \mapsto \Phi^t(x)$.

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- If a λ-eigenfunction $\tilde{\psi}$ for $F = \Phi^1$ exists, Sternberg's trick[3] $\implies$

$$\psi = \int_0^1 e^{-\lambda t}\tilde{\psi} \circ \Phi^t \, dt$$

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- $\implies$ **suffices to consider discrete time**, i.e. prove existence & uniqueness for principal eigenfunctions of a diffeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$, $F(0) = 0$, 0 hyperbolically stable with basin B.

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- Existence & uniqueness for $k < \infty$ plus bootstrapping yields existence & uniqueness for $k = \infty$, hence assume $k < \infty$ for now.

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Step 2: nonresonance and solving polynomial equations

- If $\mu \in \mathbb{C}$, $k \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ eigenvalues$(D_0 F) = \lambda_1, \ldots, \lambda_n$ repeated with multiplicity, $(e^{\lambda}, D_0 F)$ is k-**nonresonant** if

$$e^{\mu} \neq \lambda_1^{m_1} \cdots \lambda_n^{m_n}$$

  whenever $m_1, \ldots, m_n \in \mathbb{N}_{\geq 0}$ satisfy $2 \leq \sum_i m_i < k + 1$.

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- **Key fact**: If $F \in C^k$ and $\exists w \in \mathbb{R}^n$ s.t. $wD_0 F = e^\lambda w$, k-nonresonance $\implies$ invertibility of certain linear operators on polynomials $\implies$ $\exists!$ polynomial $P\colon \mathbb{R}^n \to \mathbb{C}$ such that $P(0) = 0$ and

$$P \circ F = e^\lambda P + o(\|x\|^k), \qquad \text{and } P \text{ is } \mathbb{R}\text{-valued if } e^\lambda \in \mathbb{R}.$$

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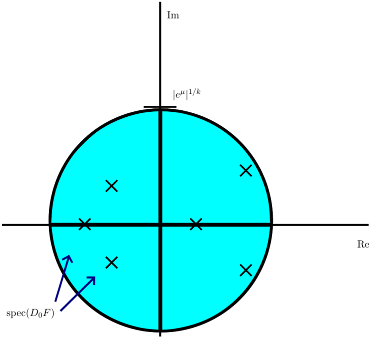
- Remains only to **find** $o(\|x\|^k)$ **remainder** $\varphi \colon \mathbb{R}^n \to \mathbb{C}$ such that $\psi = P + \varphi$ is an eigenfunction exactly.

Step 3: spectral spread and contraction mapping to eliminate remainder

- **Spectral spread** $\nu(e^{\mu}, D_0F) := \min\left\{r \in \mathbb{R} \colon |e^{\mu}| \geq \left(\max_{\lambda \in \mathrm{evals}(D_0F)} |\lambda|\right)^r\right\}$.



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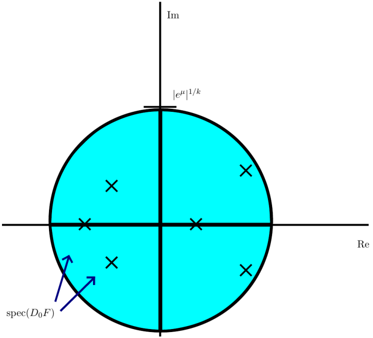

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- **Key fact**: if $\nu(e^{\lambda}, D_0F) < k$, $\exists$ adapted norm $\|\cdot\|$ and $\varepsilon > 0$ s.t. with $N := B_\varepsilon(0)$

$$S \colon \{\varphi|_N \in C^k(N, \mathbb{C}) \colon \varphi|_N \in o(\|x\|^k)\} \circlearrowleft$$

$$S(\varphi|_N) := -P|_N + e^{-\lambda}(P|_N + \varphi|_N) \circ F$$

is a contraction mapping $\implies$ $\exists!$ $\varphi|_N$ s.t. $S(\varphi|_N) = \varphi|_N = \lim_{m\to\infty} S^m(\tilde{\varphi}|_N)$, i.e.

$$\underbrace{(P|_N + \varphi|_N)}_{\psi|_N} \circ F = e^{\lambda}\underbrace{(P|_N + \varphi|_N)}_{\psi|_N} \quad \text{and} \quad \psi|_N = \lim_{m\to\infty} e^{-\lambda m} P \circ F|_N.$$

Step 4: globalization $\implies$ discrete-time existence and uniqueness result

- Can globalize $\psi|_N \colon N \to \mathbb{C}$ to $\psi \colon B \to \mathbb{C}$ as follows: set $\psi(x) := e^{-m\lambda}\psi|_N \circ F^m(x)$ where m is large enough that $F^m(x) \in N$; can show well-defined independent of m.[4]

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- **Observation**: (4) $\implies$ Theorem hypotheses $\implies$ **convergence of Laplace average**

$$\psi = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} e^{-\lambda m} P \circ F.$$

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Continuous-time existence and uniqueness with weaker nonresonance

- If $\text{evals}(D_0 f) = \lambda_1, \ldots, \lambda_n$ with multiplicity and $F = \Phi^1$, taking logarithm of $e^{\mu} \neq e^{m_1 \lambda_1} \cdots e^{m_n \lambda_n}$ implies that k-nonresonance of $(e^{\mu}, D_0 F)$ is equivalent to

$$\mu \neq m_1 \lambda_1 + \cdots m_n \lambda_n + i 2\pi \ell \tag{5}$$

  for any $\ell \in \mathbb{Z}$ and any $m_1, \ldots, m_n \in \mathbb{N}_{\geq 0}$ satisfying $2 \leq \sum_i m_i < k + 1$.

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- $\implies$ **Theorem**:[5] let $k \geq 1$, vector field $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, 0 hyperbolically stable with basin B, $\nu(e^{\lambda}, e^{D_0 f}) < k$, λ not equal to any integer linear combination of eigenvalues of $D_0 f$ with coefficient sum ≥ 2, and $wD_0 f = \lambda w$. Then there **exists** a **unique** C^k principal eigenfunction ψ satisfying $\psi \circ \Phi^t = e^{\lambda t}\Phi^t$ for all $t \in \mathbb{R}$, and

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which can be violated for all τ if and only if it is violated for $\ell = 0$.

- $\Longrightarrow$ **Theorem**:[5] let $k \geq 1$, vector field $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, 0 hyperbolically stable with basin B, $\nu(e^\lambda, e^{D_0 f}) < k$, λ not equal to any integer linear combination of eigenvalues of $D_0 f$ with coefficient sum ≥ 2, and $wD_0 f = \lambda w$. Then there **exists** a **unique** C^k principal eigenfunction ψ satisfying $\psi \circ \Phi^t = e^{\lambda t}\Phi^t$ for all $t \in \mathbb{R}$, and

$$\psi = \lim_{m\to\infty} e^{-\lambda t} P \circ \Phi^t \quad C^k\text{-uniformly on compacts if} \quad P \circ \Phi^1 = e^\lambda P + o(\|x\|^k). \tag{7}$$

- **Observation**: (4) $\Longrightarrow$ Theorem hypotheses $\Longrightarrow$ **convergence of Laplace average**

$$\psi = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-\lambda t} P \circ \Phi^t.$$

[5] see Remark 3 of Kvalheim and Revzen (2021) or Proposition 11 of Kvalheim, Hong, and Revzen (2021).

Classification of C^∞ Koopman eigenfunctions

- **Key tool**:[6] under assumptions of preceding Theorem, if $\varphi \in C^k(B, \mathbb{C})$ satisfies $\varphi \circ \Phi^1 = e^\lambda \varphi$ and $\varphi \in o(\|x\|^k)$, then $\varphi \equiv 0$. In particular, if $\varphi = \psi_1 - \psi_2$, $\psi_1 = \psi_2$.

- Key tool & preceding theorem can be used to prove the following.

- **Classification theorem:** let vector field $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, 0 hyperbolically stable with basin B, λ not equal to any integer linear combination of eigenvalues of $D_0 f$ with coefficient sum ≥ 2, and $D_0 f$ diagonalizable over $\mathbb{C}$. Then
    - **any** Koopman λ-eigenfunction is a finite linear combination of products of n principal eigenfunctions and their complex conjugates.
    - In particular, λ is a linear combination of eigenvalues of $D_0 f$.

[6] Proposition 1 from Kvalheim and Revzen (2021).

Extension to periodic orbits

- Consider $\dot{x} = f(x)$ having a hyperbolically stable τ-periodic limit cycle with image Γ.

- Apply discrete-time versions of preceding theorems to a Poincaré map with section given by an isochron $\implies$ existence and uniqueness theorems for C^k principal eigenfunctions (those with derivatives nonvanishing on Γ).

[7] See Mauroy and Mezić "On the use of Fourier..." (2012), Kvalheim and Revzen (2021).

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- Corresponding classification theorem has a twist involving the unique C^∞ **asymptotic phase** eigenfunction ψ_θ satisfying[7] $\dot{\psi}_\theta = i\frac{2\pi}{\tau}\psi_\theta$.

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- **Classification theorem:** let $f \in C^\infty$ and assume that no Floquet multiplier is an integer linear combination of the others with integer coefficient sum ≥ 2. Then
  - **any** Koopman λ-eigenfunction is a finite linear combination of products of $(n-1)$ principal eigenfunctions and ψ_θ and their complex conjugates.
  - In particular, e^λ is a product of Floquet multipliers.

[7] See Mauroy and Mezić "On the use of Fourier..." (2012), Kvalheim and Revzen (2021).

Remarks on other results from Kvalheim and Revzen (2021)

- Results are given for both continuous-time and discrete-time.
- Main theorem is actually existence/uniqueness of general linearizing **semiconjugacies** (or **factors**): maps $\psi\colon B \to \mathbb{C}^m$ s.t. $\psi \circ \Phi^t = e^{At}\psi$ with $A \in \mathbb{C}^{m\times m}$.
- Application in paper: improvements of the Sternberg linearization and Floquet normal form theorems, with uniqueness statement, without assuming diagonalizable linearized dynamics.
- Paper considers $\psi \in C^{k,\alpha}$, i.e. $\psi \in C^k$ such that $D^k\psi$ is locally Hölder continuous with exponent α. With this, results become fairly sharp (as examples in paper show).
- Stronger uniqueness-only statements in paper only require C^1 (not C^k) dynamics, but existence no longer guaranteed for merely C^1 dynamics.
- Paper discusses in detail implications for **isostables** and **isostable coordinates** from literature.
- Also, see Schlosser and Korda "Sparsity structures for Koopman and Perron-Frobenius operators", SIADS (2022) for an interesting application of the uniqueness results.

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