

# When do Koopman embeddings exist?<sup>1</sup>

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<sup>1</sup>Funding from AFOSR award FA9550-24-1-0299 is gratefully acknowledged.

# Motivation

Given: a (possibly unknown) nonlinear system

$$\dot{x} = f(x).$$

- ▶ Extended Dynamic Mode Decomposition:<sup>2</sup> seeks  $y = h(x)$ , matrix  $A$  with *linear* dynamics

$$\dot{y} = Ay.$$

- ▶ To not lose information: want  $h$  one-to-one (1-1). Then

$$\begin{aligned}x(t) &= h^{-1}(y(t)) \\ &= h^{-1}(e^{At}h(x_0)).\end{aligned}$$

- ▶ Such **1-1 linearizing maps**  $h$  have also been called **Koopman embeddings, faithful linear representations** (Mezić 2021), **1-1 linear immersions** (Liu-Ozay-Sontag 2023, 2025).

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<sup>2</sup>Williams, Kevrekidis, and Rowley. J Nonlinear Science (2015)

# Main issue and question

To avoid practical issues: want  $h$  at least continuous.

**Main issue:** continuous 1-1 linearizing  $h$  **do not exist** in general.



**Main question:** when do they exist and when do they not?

# When do Koopman embeddings exist?

Positive results (sufficient conditions)

Negative results (necessary conditions)

A counterexample for multiple isolated equilibria

Necessary and sufficient conditions

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## Some positive results

Continuous 1-1 linearizing  $y = h(x)$  always exists:<sup>3</sup>

- ▶ near a hyperbolic equilibrium or limit cycle (Hartman-Grobman, Floquet);
- ▶ on the basin of any exponentially stable equilibrium or limit cycle (Lan-Mezić 2013, K-Revzen 2021);
- ▶ On the basin of ANY asymptotically stable equilibrium, not necessarily exponentially stable / hyperbolic (K-Sontag 2025)!

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<sup>3</sup>There are also  $C^k$  versions of all of these results.

# Global linearization for equilibria without hyperbolicity

Let  $x_*$  be asymptotically stable with basin  $B$  for

$$\dot{x} = f(x).$$

Assume  $f$  is continuous w/ unique trajectories defined for all time.

**Theorem (K-Sontag 2025).** There is a homeomorphism  $h: B \rightarrow \mathbb{R}^n$  such that  $y = h(x)$  satisfies

$$\dot{y} = Ay$$

and hence  $x(t) = h^{-1}(e^{At}h(x_0))$  for all  $t \in \mathbb{R}$ .

And if  $f \in C^{k \geq 1}$ ,  $n \neq 5$ :  $h$  is a  $C^k$  diffeomorphism on  $B \setminus \{x_*\}$ .

## Remarks

- ▶ Exponential stability / hyperbolicity of  $x_*$  is not needed.
- ▶ Proof relies on solutions to Poincaré conjecture (Smale, Perelman, Freedman). In fact:

**Proposition (K-Sontag 2025).** The  $C^k$  statement for  $n = 5$  in last theorem is true  $\iff$  the smooth 4-D Poincaré conjecture is true.

**Proposition (K 2025).** In the last theorem, if the vector field  $f_p$  depends continuously on parameter  $p \in P$ , there is a continuous family  $h_p : B \rightarrow \mathbb{R}^n$  of linearizing homeomorphisms if either (i)  $n > 5$  and  $\dim P = 1$  or (ii)  $n < 4$ .

- ▶ Proof of latter relies on corollary of:

**Theorem (K 2025).** The space of proper  $C^\infty$  Lyapunov-like functions on  $\mathbb{R}^n$  is path-connected and simply connected if  $n \neq 4, 5$  and weakly contractible if  $n < 4$ . (Same for  $C^{k \geq 1}$  & GAS vf)



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## Some negative results

Assume a connected state space to avoid trivialities. Then

$$\dot{x} = f(x)$$

does **not** have a continuous 1-1 linearizing  $y = h(x)$  if **either**:

- ▶ there is a non-global compact attractor (K-Arathoon 2024), **or**
- ▶ all forward trajectories are precompact, and there are  $\geq 2$  but at most countably many omega-limit sets, e.g., **multiple isolated equilibria** (Liu-Ozay-Sontag 2023, 2025).

On the other hand, on the subject of multiple isolated equilibria...

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# Smooth 1-1 linearization despite multiple isolated equilibria

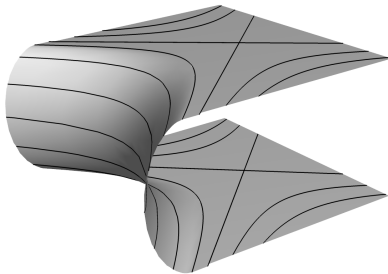
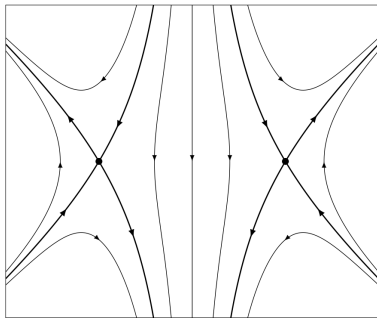
If we drop the assumption that forward trajectories are precompact, then (another positive result):

**Theorem (Arathoon-K 2023).** For any  $n > 1$  there is a smooth vector field on  $\mathbb{R}^n$  with any given finite number of isolated equilibria such that there exists a smooth 1-1 linearizing  $y = h(x)$ .

In fact,  $h$  is a smooth embedding, and can moreover be taken of the form  $h(x) = (x, g(x))!$

This theorem gives a family of strong counterexamples to an oft-repeated claim.

## Example



**Figure:** Smoothly embedding a nonlinear system on  $\mathbb{R}^2$  with two isolated equilibria as an invariant subset of a linear system on  $\mathbb{R}^3$ .

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# Where is the boundary between the positive and negative results?

We have now seen a variety of necessary conditions and sufficient conditions on

$$\dot{x} = f(x)$$

for a continuous 1-1 linearizing  $y = h(x)$  to exist.



**Fundamental question:** what are necessary **and** sufficient conditions on  $f$  for such an  $h$  to exist?

**Note:** without continuity, necessary and sufficient conditions for 1-1 linearizing  $h$  to exist were obtained by Mezić (2021).

# Preamble to answering the fundamental question

Assume a connected state space to avoid trivialities.

- ▶ **We can answer the fundamental question** for any continuous  $f$  with unique trajectories defined for all time **if there is at least one compact attractor**.
- ▶ Recall there does not exist such an  $h$  if there are  $\geq 2$  such attractors, or even a single non-global compact attractor (K-Arathoon 2024).



- ▶ Remains to consider case of a global compact attractor (can also restrict to basin of local attractor to apply next result).



## Torus preliminaries

The  **$m$ -torus**  $T = T^m$  is Lie group isomorphic to  $(\mathbb{R}/\mathbb{Z})^m$ , vectors w/  $m$  real entries but w/ addition defined elementwise modulo 1.

A **torus action** on a space  $S$  is a map  $\Theta: T \times S \rightarrow S$  satisfying  $\Theta^{\tau_1 + \tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$  for all  $s \in S$  and  $\tau_1, \tau_2 \in T$ .

A **1-parameter subgroup** of  $\Theta$  is a map  $\Phi: \mathbb{R} \times S \rightarrow S$  of the form  $\Phi^t(x) = \Theta^{\omega t}(x)$  for some  $\omega \in \mathbb{R}^m$ .

$\Theta$  has **finite orbit types** if there are only finitely many subgroups  $H \subset T$  such that, for some  $x \in S$ ,

$$H = \text{Fix}(x) := \{\tau \in T : \Theta^\tau(x) = x\}.$$

# Finishing the answer to the fundamental question

Assume  $f$  is continuous with unique trajectories defined for all time, so  $f$  generates a continuous flow  $\Phi: \mathbb{R} \times X \rightarrow X$ .<sup>4</sup>

**Theorem (K-Arathoon 2024).** Assume there is a global compact attractor  $A$  (or restrict to the basin of a local attractor). Then a continuous 1-1 linearizing  $y = h(x)$  exists  $\iff$

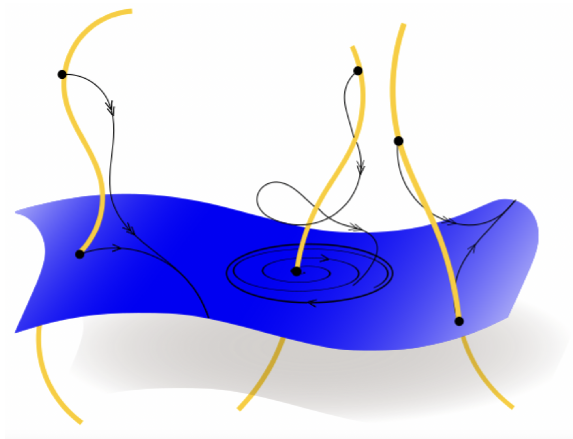
- ▶  $\Phi|_{\mathbb{R} \times A}$  is a 1-parameter subgroup of a continuous torus action with finite orbit types, and
- ▶  $A$  has continuous **asymptotic phase**  $P: X \rightarrow A$ .

Moreover, such  $h$  is automatically a proper topological embedding.

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<sup>4</sup> $t \mapsto \Phi^t(x_0)$  is the unique solution of  $\dot{x} = f(x)$  satisfying  $x(0) = x_0$ .

## Asymptotic phase



**Asymptotic phase** means: for all  $x \in X$ ,  $t \in \mathbb{R}$ ,

$$P(\Phi^t(x)) = \Phi^t(P(x)).$$

$\implies$  if  $P$  continuous, then  $\text{dist}(\Phi^t(x), \Phi^t(P(x))) \rightarrow 0$  as  $t \rightarrow \infty$ ;  
 $x$  is “**asymptotically in phase with**”  $P(x)$ .

## Example: limit cycles

Previous theorem implies that dynamics on basin of limit cycle attractor admit a continuous 1-1 linearizing  $y = h(x)$  if and only if there is continuous asymptotic phase (w/ level sets “isochrons”).

**Example.** Using polar coordinates  $(r, \theta)$  on  $\mathbb{R}^2$ , the system

$$\dot{r} = -(r - 1)^3, \quad \dot{\theta} = r$$

generates a smooth flow  $\Phi$  on  $\mathbb{R}^2 \setminus \{0\}$  with globally asymptotically stable limit cycle  $A = \{r = 1\}$ . Closed-form expression for  $\Phi \implies$

$$\text{dist}(\Phi^t(x), \Phi^t(y)) \not\rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

for any  $x \notin A$ ,  $y \in A$ , so  $A$  does not have continuous asymptotic phase, so a continuous 1-1 linearizing  $y = -h(x)$  **does not exist**.

## What about smooth linearizations?

- ▶ **Natural question:** when does there exist a smooth 1-1 linearizing  $y = h(x)$  with smooth inverse  $x = h^{-1}(y)$  ( $y \in \text{image}(h)$ )?
- ▶ Such an  $h$  is called a **smooth embedding**.
- ▶ So far, less satisfying answer in this case. But in particular, have the following necessary conditions:

**Theorem (K-Arathoon 2024).** Assume  $\dot{x} = f(x)$  has a global compact attractor  $A \subset X$  and is linearizable by a smooth embedding. Then:

- ▶  $A$  is a smoothly embedded submanifold and normally hyperbolic,
- ▶  $A$  has smooth asymptotic phase, and
- ▶  $\Phi|_{\mathbb{R} \times A}$  is a 1-parameter subgroup of a smooth torus action.

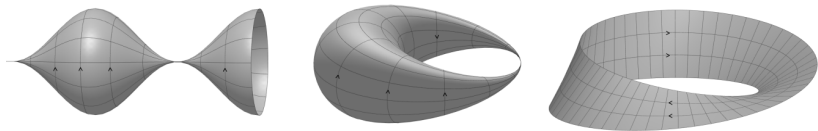
## Answer to fundamental question for compact invariant sets

If state space  $X$  is compact, can view  $A = X$  as a (trivial) compact attractor (with basin  $B = A = X$ ). For this special case, we have:

**Theorem (K-Arathoon 2024).** Assume  $f$  generates a smooth (resp. continuous) flow  $\Phi$  and  $X$  is compact. Then there is a smooth (resp. continuous) linearizing embedding  $y = h(x) \iff \Phi$  is a 1-parameter subgroup of a smooth (resp. continuous w/ finite orbit types) torus action.

For  $X$  noncompact, can still apply this theorem by restricting to a compact invariant set.

## Surprising (?) examples with continuous 1-1 linearizations



**Figure:** For all of these flows, a continuous 1-1 linearizing  $y = h(x)$  exists (easy to see using preceding theorem).

## Some corollaries of preceding theorem

**Corollary** (K-Arathoon 2024). If  $X$  is a compact smooth manifold and  $f$  has at most finitely many equilibria, and if there exists a smooth linearizing embedding  $y = h(x)$ , then

$$\underbrace{\chi(X)}_{\text{Euler characteristic}} = \#\{\text{equilibria}\} \geq 0.$$

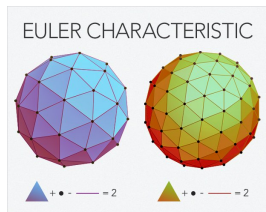
**Corollary** (K-Arathoon 2024). If  $X$  is an odd-dimensional compact smooth manifold and  $f$  has at least one isolated equilibrium, then there does not exist a smooth linearizing embedding  $y = h(x)$ .

**Proof sketch.** Using previous theorem, Bochner's linearization theorem for fixed points of torus actions  $\implies$  the Hopf index of any equilibrium is  $+1$ . Apply the Poincaré-Hopf theorem to deduce the first corollary. Deduce the second corollary from the first using  $\chi(X) = 0$  if  $X$  is an odd-dimensional compact manifold.

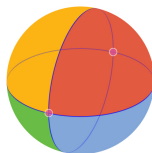


# A primer on the Euler characteristic<sup>5</sup>

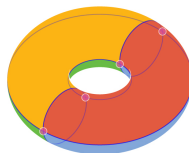
Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



Euler Characteristic ( $\chi$ ) = Faces + Corners - Edges



$$\chi = 4 + 2 - 4 = 2$$



$$\chi = 4 + 4 - 8 = 0$$

**Notation:**  $\chi(Y) :=$  Euler characteristic of  $Y$ .

**Examples:**  $\chi(\bullet) = 1$ ,  $\chi(S^1) = 0$ ,  $\chi(S^2) = 2$ ,  $\chi(\Sigma_g) = 2 - 2g$



$\Sigma_g$  for  $g = 1, 2, 3$  (not linearizable by smooth embedding for  $g > 1$  if there is an isolated equilibrium).

<sup>5</sup>Figures from Quanta Magazine and Wikipedia.

Thank you for your time and attention.

## References for (non)existence of 1-1 linearizing $y = h(x)$

- ▶ **Linearization in the large of nonlinear systems and Koopman operator spectrum.** Lan & Mezić. Phys D (2013)
- ▶ **Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits.** Kvalheim & Revzen. Phys D (2021)
- ▶ **Linearizability of flows by embeddings.** Kvalheim and Arathoon. arXiv:2305.18288 (2024)
- ▶ **Koopman embedding and super-linearization counterexamples with isolated equilibria.** Arathoon and Kvalheim. arXiv:2306.15126 (2023)
- ▶ **Properties of immersions for systems with multiple limit sets with implications to learning Koopman embeddings.** Liu, Ozay, & Sontag. Automatica (2025)
- ▶ **Global linearization without hyperbolicity.** Kvalheim and Sontag. arXiv:2502.07708 (2025)
- ▶ **Differential topology of the spaces of asymptotically stable vector fields and Lyapunov functions.** Kvalheim. arXiv:2503.10828 (2025)

# Linearizability of dynamical systems by embeddings

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Necessary and sufficient conditions