# On Professor Smale's legacy for asymptotic stability theory[1]

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# Motivation

- Asymptotic stability is a fundamental dynamical systems concept modeling robust steady-state behavior.
- Engineers often try to make systems asymptotically stable through feedback control. ▸ Example

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- Asymptotic stability is a fundamental dynamical systems concept modeling robust steady-state behavior.
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**Goal:** allow stabilizing feedback control to depend on parameters to shape transient response (when perturbed from steady state).

**Question:** when can parametric stabilizing feedback be extended from subset of parameter space to all of parameter space?

**Insight:** given sufficient control authority, the answer depends only on the topology of the space of asymptotically stable vector fields.

# Asymptotic stability

Consider an ordinary differential equation

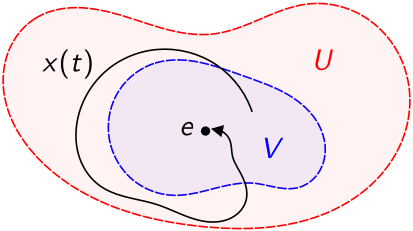
$$\dot{x}(t) = F(x(t)), \tag{1}$$

where $F$ is a vector field on $\mathbb{R}^n$. Unless stated otherwise, $F$ and everything else in this talk is smooth ($C^\infty$).

Let $e \in \mathbb{R}^n$ be an **equilibrium**, meaning $F(e) = 0$.

We say that $e \in \mathbb{R}^n$ is (globally) **asymptotically stable** if

- every solution of (1) converges to $e$ as $t \to \infty$.
- for every open $U \ni e$ there is a smaller open $V \ni e$ s.t. every solution of (1) starting in $V$ at $t = 0$ stays in $U$ for all $t \geq 0$.



# Lyapunov functions

- A **Lyapunov function** for a vector field $F$ with equilibrium $e$ is a proper function $L\colon \mathbb{R}^n \to [0,\infty)$ such that $L^{-1}(0) = \{e\}$ and $dL(x)\cdot F(x) \leq 0$ for all $x$ with equality iff $x = e$.
- **History:**
  - Lyapunov (1892) discovered:
    Lyapunov function exists $\implies$ $e$ is asymptotically stable.
  - Massera (1956), Kurzweil (1956) proved converse:
    $e$ is asymptotically stable $\implies$ ($C^\infty$) Lyapunov function exists.
  - Wilson (1967) studied the topology of level sets of such Lyapunov functions. In particular, they are $C^\infty$ spheres for $n > 5$ by the h-cobordism theorem of Smale (1962).
- **In this talk**, we turn Wilson's idea on its head:

  We use level sets of Lyapunov functions to study the topology of the space of asymptotically stable vector fields.

# Outline

Main results

Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Step 2: trivial homotopy groups of the nonlinear Grassmannian

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

# Outline

Main results

- Topology of $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$
- Boundary value problems

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# Results: topology and boundary value problems

$\mathcal{S}(\mathbb{R}^n) := \{$asymptotically stable vector fields on $\mathbb{R}^n\}$

$\mathcal{L}(\mathbb{R}^n) := \{$proper functions $\mathbb{R}^n \to [0, \infty)$ w/ unique critical value $= 0\}$

Equip both spaces with the compact-open $C^\infty$ topology.

**Theorem (K 2025).** $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{L}(\mathbb{R}^n)$ are both path-connected and simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$.

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$\Downarrow$

**BVP existence theorem.** For any compact manifold $P$ and $C^\infty$

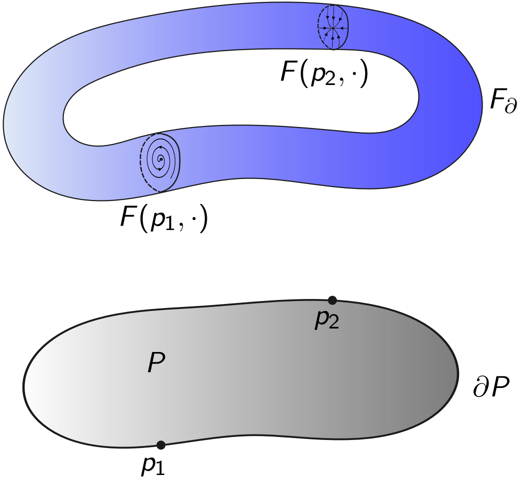
$$F_\partial \colon \partial P \times \mathbb{R}^n \to \mathbb{R}^n, \quad L_\partial \colon \partial P \times \mathbb{R}^n \to [0, \infty)$$

s.t. $F_\partial(p, \cdot) \in \mathcal{S}(\mathbb{R}^n)$, $L_\partial(p, \cdot) \in \mathcal{L}(\mathbb{R}^n)$ $\forall p \in \partial P$, there exist $C^\infty$

$$F \colon P \times \mathbb{R}^n \to \mathbb{R}^n, \quad L \colon P \times \mathbb{R}^n \to [0, \infty)$$

extending $F_\partial$, $L_\partial$ s.t. $F(p, \cdot) \in \mathcal{S}(\mathbb{R}^n)$, $L(p, \cdot) \in \mathcal{L}(\mathbb{R}^n)$ $\forall p \in P$ if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 3$.

# Illustration of previous theorem (here $n = 2 = \dim P$)




Previous theorem: families of asymptotically stable vector fields on $\mathbb{R}^n$ can always be extended from the boundary $\partial P$ to the entire parameter space $P$ if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 3$.

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Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Relies on **Smale's h-cobordism theorem**

Step 2: trivial homotopy groups of the nonlinear Grassmannian

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

# Topology of Lyapunov function sublevel sets for $n \neq 4, 5$

**Key fact:**
For any $L \in \mathcal{L}(\mathbb{R}^n)$, $L^{-1}([0,1])$ is diffeomorphic to $D^n$ if $n \neq 4, 5$.

**Proof:**

- The flow of $\nabla L$ induces deformation retractions of $L^{-1}([0,1])$ to $L^{-1}(0)$ and of $\mathbb{R}^n \setminus \{L^{-1}(0)\}$ to $L^{-1}(1)$.
- Hence $L^{-1}([0,1])$ is a contractible manifold with boundary $L^{-1}(1)$ a homotopy sphere (Wilson 1967).
- Hence $L^{-1}([0,1])$ is diffeomorphic to $D^n$ for $n \neq 4, 5$ by
  - classification of 1D and 2D manifolds for $n = 1, 2$,
  - solution to 3D Poincaré conjecture (Perelman 2003) for $n = 3$,
  - the **h-cobordism theorem (Smale 1962)** for $n > 5$. $\square$

# The sublevel set map

Let $\mathcal{L}_0(\mathbb{R}^n) \subset \mathcal{L}(\mathbb{R}^n)$ be subspace of functions with min at $0 \in \mathbb{R}^n$.

Consider the space

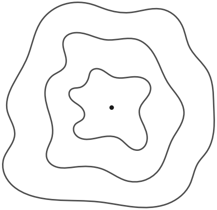
$$\mathrm{Gr}(D^n, \mathbb{R}^n) := \mathrm{Emb}(D^n, \mathbb{R}^n)/\mathrm{Diff}(D^n)$$

of submanifolds of $\mathbb{R}^n$ diffeomorphic to $D^n$, known as a **nonlinear Grassmannian**, and its open subspace $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ of submanifolds whose interiors contain $0 \in \mathbb{R}^n$.

By the previous slide, we have a well-defined **sublevel set map**

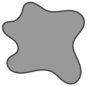
$$p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n), \qquad p(L) := L^{-1}([0,1]).$$

$L \in \mathcal{L}(R^n)$



$\mapsto$

$p(L) \in \mathrm{Gr}_0(D^n, \mathbb{R}^n)$



# The sublevel set map is a weak homotopy equivalence

**Theorem (K 2025).** The sublevel set map

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is a fiber bundle w/ weakly contractible fibers (hence also a w.h.e.).

**Proof sketch:**

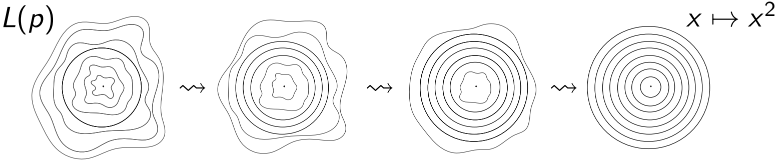
- $p$ is continuous by implicit function theorem (Hildebrandt and Graves 1927, Abraham 1967).
- Disc theorem (Palais 1960, Cerf 1961) implies $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is path-connected and $p$ is surjective.
- Each $M \in \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ has neighborhood $U \subset \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ and map $\Psi\colon U \to \mathrm{Diff}(\mathbb{R}^n)$ s.t. $\Psi(M)(N) = N$ for all $N \in U$.
- Define $f\colon p^{-1}(U) \to \mathcal{F} := p^{-1}(M)$ by $f(L) := L \circ \Psi(p(L))$.
- Check: $(p, f)\colon p^{-1}(U) \to U \times \mathcal{F}$ is a homeomorphism.
- To show that $\mathcal{F}$ is weakly contractible...

# Weak contractibility of $\mathcal{F}$

- Since $p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is a fiber bundle over a path-connected base, it suffices to check that $\mathcal{F} = p^{-1}(M)$ is weakly contractible for $M = D^n$. In this case,

$$\mathcal{F} = \{L \in \mathcal{L}_0(\mathbb{R}^n)\colon L^{-1}([0,1]) = D^n\}.$$

- Any parametric family $P \to \mathcal{F}$ is nullhomotopic to $P \to \{x \mapsto x^2\}$ by "parting the sea" of level sets of $L(p)$ away from $\partial D^n = S^{n-1}$, replacing with level sets of $x \mapsto x^2$. $\square$



# Outline

Main results

Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Step 2: trivial homotopy groups of the nonlinear Grassmannian

- Relies on **Smale's theorem** that $\mathrm{Diff}_\partial(D^2)$ is contractible and Hatcher's proof of the **Smale conjecture** for $\mathrm{Diff}_\partial(D^3)$.

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

# Toward homotopy groups of the nonlinear Grassmannian

- Without too much trouble,

$$\mathcal{L}_0(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}(\mathbb{R}^n) \text{ and } \mathrm{Gr}_0(D^n, \mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathrm{Gr}(D^n, \mathbb{R}^n),$$

  so to prove main theorem for $\mathcal{L}(\mathbb{R}^n)$, suffices to show that the appropriate homotopy groups of $\mathrm{Gr}(D^n, \mathbb{R}^n)$ are trivial.
- Generalizing a theorem of Binz and Fischer (1981) based on an idea implicit in work of Weinstein (1971), Gay-Balmaz and Vizman (2014) proved that the natural quotient map

$$\mathrm{Emb}^+(D^n, \mathbb{R}^n) \to \mathrm{Gr}(D^n, \mathbb{R}^n), \quad f \mapsto f(D^n)$$

  is a principal $\mathrm{Diff}^+(D^n)$-bundle, hence also a Serre fibration, so there is a long exact sequence of homotopy groups:

$$\cdots \pi_k \mathrm{Diff}^+(D^n) \longrightarrow \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \longrightarrow \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \cdots$$

# Analyzing the long exact sequence, part 1

That long exact sequence contains the segment

$$\begin{tikzcd}
\pi_k \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[r] & \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \arrow[dll, rounded corners] \\
\pi_{k-1} \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n) &
\end{tikzcd}$$

in which the indicated arrows are surjective because the diagram

$$\begin{tikzcd}
\mathrm{Diff}^+(D^n) \arrow[rr, hook] \arrow[dr, "\text{surjective on } \pi_k"'] & & \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[dl, "\simeq"] \\
& GL^+(n) &
\end{tikzcd}$$

commutes: the diagonal arrows are "evaluate derivative at 0", the right one is a homotopy equivalence by "zooming in", and the left one is $\pi_k$-surjective because the composition

$$\mathrm{Diff}^+(D^n) \longrightarrow GL^+(n) \xrightarrow{\text{Gram-Schmidt}} SO(n)$$

is $\pi_k$-surjective and Gram-Schmidt is a homotopy equivalence.

# Analyzing the long exact sequence, part 2: Cerf

$$\pi_k \mathrm{Diff}^+(D^n) \xrightarrow{\text{surjective}} \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \xrightarrow{0} \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \xrightarrow{\text{injective}} \pi_{k-1} \mathrm{Diff}^+(D^n) \xrightarrow{\text{surjective}} \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n)$$

- We have established surjectivity of the indicated arrows, so exactness $\implies$ other arrows are 0, injective.
- $\pi_0 \mathrm{Gr}(D^n, \mathbb{R}^n) = \{*\}$ by disc theorem (mentioned earlier).
- $\pi_0 \mathrm{Diff}^+(D^n) = \{*\}$ for $n > 5$ by Cerf's (1970) pseudoisotopy theorem, so taking $k = 1$ above yields $\pi_1 \mathrm{Gr}(D^n, \mathbb{R}^n) = \{*\}$.
- Remains only to show that $\pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) = \{*\}$ for all $k$ when $n < 4$. By exactness, suffices to show that the above surjections are bijections in this case.

# Completing the proof for $\mathcal{L}(\mathbb{R}^n)$, part 1

- Consider the earlier commutative diagram

$$\begin{tikzcd} \mathrm{Diff}^+(D^n) \arrow[rr, hook] \arrow[dr, "\text{surjective on } \pi_k"'] & & \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[dl, "\simeq"] \\ & GL^+(n) & \end{tikzcd} .$$

  The top arrow is $\pi_k$-surjective; need to show $\pi_k$-injective if $n < 4$. Suffices to show the same for the left diagonal arrow.
- "Evaluate derivative at 0" diagram is homotopic to "evaluate derivative at $e_1 \in \mathbb{R}^n$" diagram; left arrow is homotopic to

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{f} GL^+(n),$$

  composition of the restriction $\rho$ and map $f$ given by adjoining the value and derivative at $e_1$.

## Completing the proof for $\mathcal{L}(\mathbb{R}^n)$, pt 2: Smale and Hatcher

So we need to show $\pi_k$-injectivity of the composition

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{f} GL^+(n),$$

w/ $\rho$ restriction and $f$ adjoining the value and derivative at $e_1$.

- First fibration theorem of Cerf (1961) $\implies$ $\rho$ is fiber bundle; fiber over $\mathrm{id}_{S^{n-1}}$ is

  $$\mathrm{Diff}_\partial(D^n) := \{\text{diffeomorphisms of } D^n \text{ that are the identity on } \partial D^n\}.$$

- This fiber is contractible for:
  - $n = 1$ by a convexity argument,
  - $n = 2$ by a **theorem of Smale (1957)**, and
  - $n = 3$ by Hatcher's (1983) proof of the **Smale conjecture (1961)**.
- So it suffices to show that $f$ is $\pi_k$-injective.

# Completing the proof for $\mathcal{L}(\mathbb{R}^n)$, pt 3: Smale again

Need only show $f\colon \mathrm{Diff}^+(S^{n-1}) \to GL^+(n)$ is a w.h.e. for $n < 4$.
Trivial for $n = 1$, so assume $1 < n < 4$.
Identifying $GL^+(n)$ with $\underbrace{\mathrm{Fr}^+(TS^{n-1})}_{+\text{ frame bundle}}$, $f$ factors as the composition

$$\begin{tikzcd}
\mathrm{Diff}^+(S^{n-1}) \arrow[r] & \mathrm{Emb}^+(D_+^{n-1}, S^{n-1}) \arrow[r, "\simeq"] & \mathrm{Emb}^+(\mathrm{int}(D_+^{n-1}), S^{n-1}) \arrow[dll, "\simeq"] \\
\mathrm{Fr}^+(TS^{n-1}) & &
\end{tikzcd}$$

in which $D_+^{n-1}$ is upper hemisphere, first two arrows are restrictions, long arrow given by adjoining value & derivative at $e_1$.

- Indicated arrows are well known to be w.h.e.
- First arrow is fiber bundle again by Cerf (1961), with fiber

$$\mathrm{Diff}(S^{n-1} \text{ rel } D_+^{n-1}) \cong \mathrm{Diff}_\partial(D^{n-1})$$

  weakly contractible again for $n = 2$ by convexity and $n = 3$ by **theorem of Smale (1957)**. $\square$

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Step 2: trivial homotopy groups of the nonlinear Grassmannian

Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

Other applications

# Proving main theorem for $\mathcal{S}(\mathbb{R}^n)$ from $\mathcal{L}(\mathbb{R}^n)$ case

**Theorem (K 2025).** $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{L}(\mathbb{R}^n)$ are both path-connected and simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$.

We already sketched the proof for $\mathcal{L}(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}_0(\mathbb{R}^n)$.

To prove for $\mathcal{S}(\mathbb{R}^n)$, suffices to prove $\mathcal{S}(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}(\mathbb{R}^n)$:

- Consider the **negative gradient embedding**

$$-\nabla \colon \mathcal{L}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \qquad L \mapsto -\nabla L. \tag{2}$$

- Consider any compact family $P \to \mathcal{S}(\mathbb{R}^n)$ mapping $\partial P$ into $-\nabla(\mathcal{L}(\mathbb{R}^n))$.
- Wilson's (1969) converse Lyapunov theorem $\implies$ this is homotopic to a family $P \to -\nabla(\mathcal{L}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$.
- Use a cutoff function to make the homotopy stationary on $\partial P$.

**This completes proof of theorem for** $\mathcal{S}(\mathbb{R}^n)$ $(\overset{\text{w.h.e.}}{\simeq} \mathcal{S}_0(\mathbb{R}^n))$. $\square$

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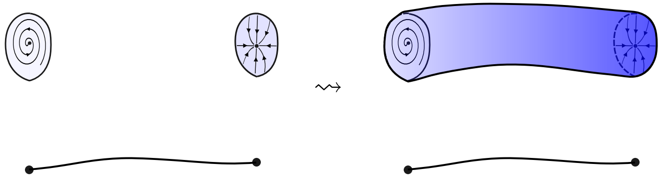
- Partial answer to question of Conley
- Parametric Hartman-Grobman theorem for non-hyperbolic asymptotically stable equilibria
- Parametric Morse lemma for degenerate minima of functions

# A question of Conley

- Conley (1978) defined the **Conley index** & proved that two compact isolated invariant sets $A$, $B$ for two flows $\Phi$, $\Psi$ have isomorphic Conley indices if they are related by *continuation*.
- In particular, this is the case if there is a continuous family $(\Theta_s)_{s\in[0,1]}$ of flows interpolating $\Phi$, $\Psi$ such that the $(\Theta_s)$-induced flow on $[0,1] \times$ (state space) has a compact isolated invariant set $C$ interpolating $A$, $B$.
- Conley asked to what extent the converse is true: Given a pair of compact isolated invariant sets with isomorphic Conley indices, when are they related by continuation?

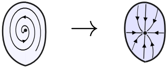
# Partial answers to Conley

- Reineck (1992): in many interesting cases (using **Smale's (1960) Morse fun./handle manipulation techniques!**).
- However, Reineck's results do not address a natural case of Conley's question: Given a pair of asymptotically stable vector fields, when can they be interpolated by asymptotically stable vector fields?
- Jongeneel (2024) proved that the associated (semi)flows can always be interpolated through asymptotically stable $C^0$ semiflows for state space dimension $n \neq 5$.
- This does not quite tell us about homotopy through $C^\infty$ such flows / vector fields, but the "path-connectedness" portion of our main theorem $\implies$ this is always possible for $n \neq 4, 5$.
- Cutoff functions, etc $\implies$ same answer for local version.



# Parametric Hartman-Grobman without hyperbolicity

- **Classical Hartman-Grobman (1960, 1959) theorem**: given a $C^1$ vector field $F$ with hyperbolic equilibrium $e$, there is a local homeomorphism identifying solutions of $\dot{x} = F(x)$ with those of $\dot{y} = Ay$ for some nonunique $A$ ($= DF(e)$ works).
- **Theorem (see K-Sontag 2025).** The hyperbolicity and "$C^1$" assumptions can be dropped if we assume that $e$ is asymptotically stable. Moreover, the homeomorphism is global (and a $C^k$ diffeo on complement of $e$ if $F \in C^k$).



- Proof uses **Smale (1962)**, Freedman (1982), Perelman (2003) to identify a Lyapunov function level set of one vector field with one of the other, then extendsidentification via flows.
- **Theorem (K-2025).** If in previous theorem $F_p$ depends continuously on parameter $p \in P$, there is a continuous family $h_p : B \to \mathbb{R}^n$ of linearizing homeomorphisms if either (i) $n > 5$ and $\dim P = 1$ or (ii) $n < 4$.

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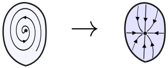

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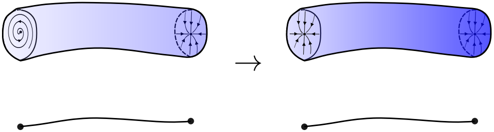

# Remarks on the 4-dimensional smooth Poincaré conjecture

By the way:

- The $C^{k\geq 1}$ statement from the "Hartman-Grobman without hyperbolicity" theorem holds for $n = 5 \iff$ the conjecture holds (K-Sontag 2025).
- If the path-connectedness portion of the main theorem holds for $n = 5$, then the conjecture is true (K-2025).

Of course, **Smale (1960) solved the topological Poincaré conjecture for smooth manifolds of all dimensions $> 4$, and was awarded the 1966 Fields medal.**

**Thank you for your attention.**

This talk is based on the preprint arXiv:2503.10828:

**"Differential topology of the spaces of asymptotically stable vector fields and Lyapunov functions"**, Kvalheim (2025).

link to preprint

# On Professor Smale's legacy for asymptotic stability theory

## Main results

Topology of $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$

Boundary value problems

## Proof step 1: equivalence of the space of Lyapunov functions and the nonlinear Grassmannian of unparametrized discs

Relies on **Smale's h-cobordism theorem**

## Step 2: trivial homotopy groups of the nonlinear Grassmannian

Relies on **Smale's theorem** that $\mathrm{Diff}_\partial(D^2)$ is contractible and Hatcher's proof of the **Smale conjecture** for $\mathrm{Diff}_\partial(D^3)$.

## Step 3: equivalence of the spaces of asymptotically stable vector fields and Lyapunov functions

## Other applications

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