

Identifying engineering (im)possibilities with geometry and topology¹

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Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

Feedback stabilizability

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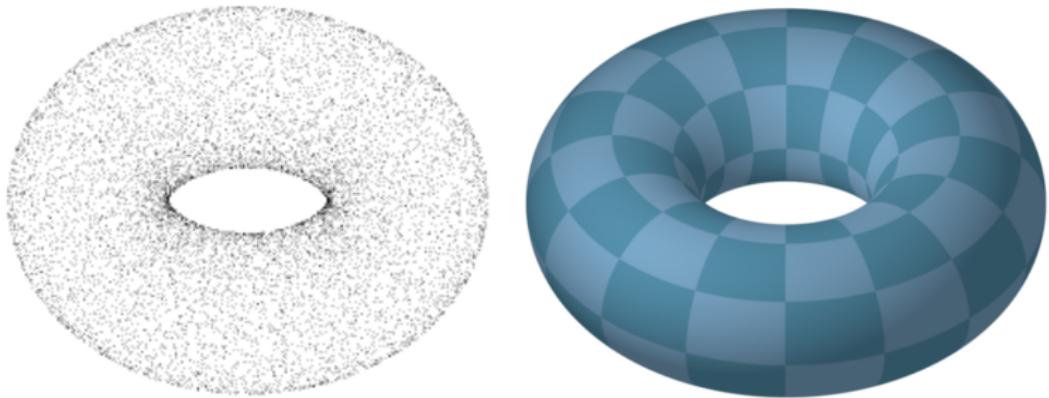
They should not work, and yet they do: resolving the paradox

Training implications: L^p but not L^∞ error can be made small

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Dimensionality reduction of data

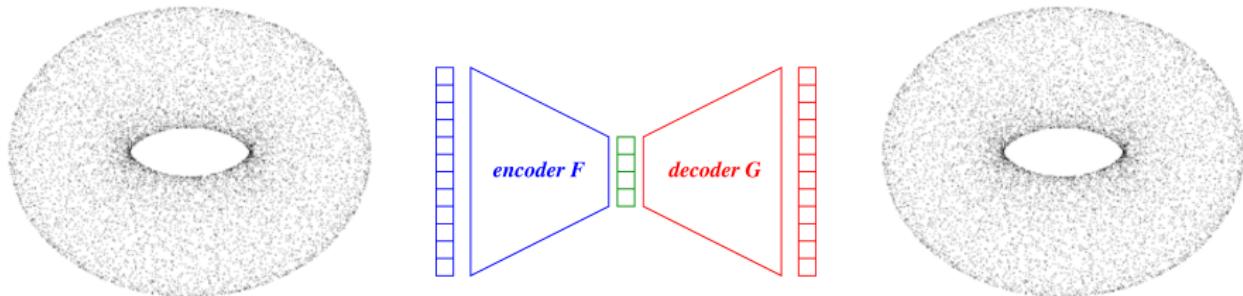


The “manifold hypothesis” postulates that a data set in \mathbb{R}^n lies on some k -dimensional submanifold $K \subset \mathbb{R}^n$.

⇒ data can be parametrized locally by $k < n$ real numbers.

Classical approaches like PCA to learn these parameters work well when K is linear, but not when K is nonlinear.

Autoencoding as a nonlinear dimensionality reduction approach (and why it should not work)



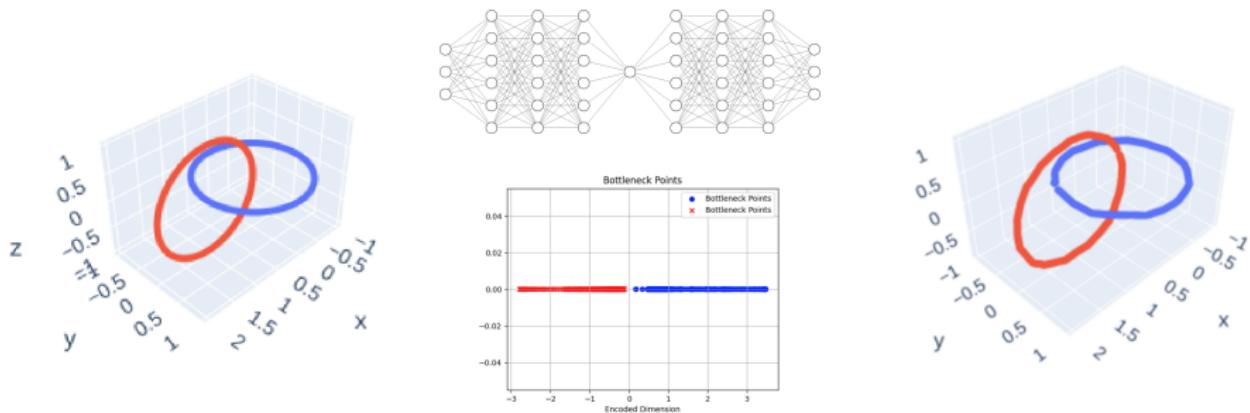
Popular nonlinear approach: seek **autoencoder** $G \circ F$, where the output of the **encoder** $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the desired k parameters, $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$ is the **decoder**, and F, G are continuous.

Often F, G are artificial neural network functions.

Ideal autoencoder: $G(F(x)) = x$ for all $x \in K$.

These **do not usually exist!** Existence $\implies K$ is homeomorphic to a subset of \mathbb{R}^k , which is not true of most k -dimensional K .

If autoencoding should not work, how does it?² Example:



$K =$ a pair of circles in \mathbb{R}^3 , after thickening then deleting small intervals, is diffeomorphic to a pair of disjoint intervals in \mathbb{R} .

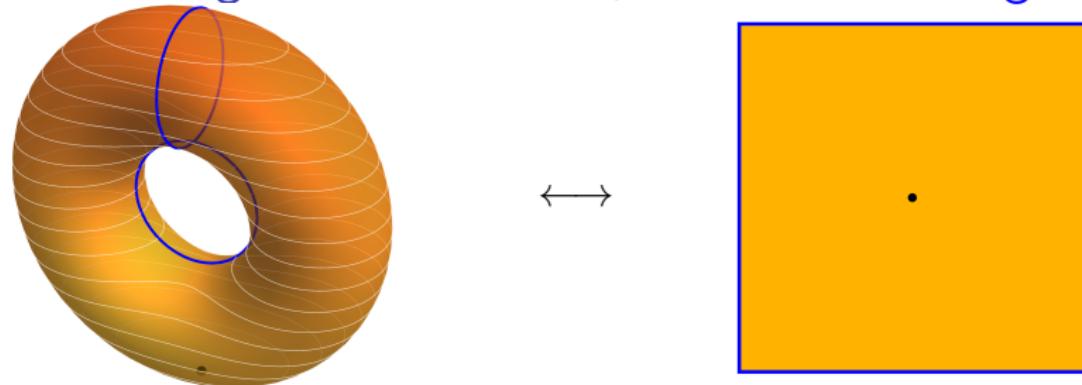
Encoder $F: \mathbb{R}^3 \rightarrow \mathbb{R}$ = any extension of this diffeomorphism.

Decoder $G: \mathbb{R} \rightarrow \mathbb{R}^3$ = any extension of inverse diffeomorphism.

Such small intervals disjoint from the data set always exist.

²MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

If autoencoding should not work, how does it? In general:



(figure courtesy of P. Arathoon)

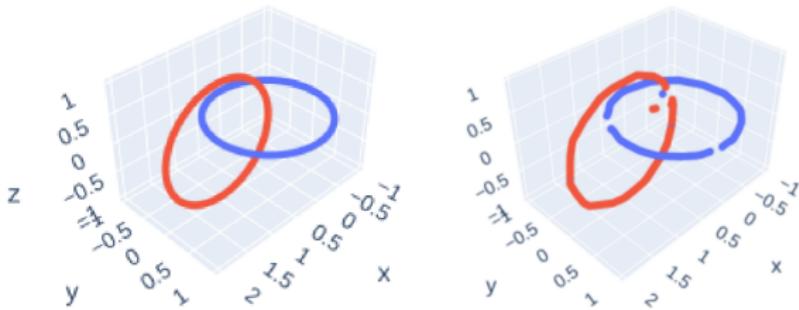
$K =$ a union of $\leq k$ -dimensional compact submanifolds of \mathbb{R}^n ,
after thickening then deleting the codimension > 0 “steepest
ascent disks” of a polar Morse function, is diffeomorphic to a
subset of \mathbb{R}^k .

Encoder $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$ = any extension of this diffeomorphism.

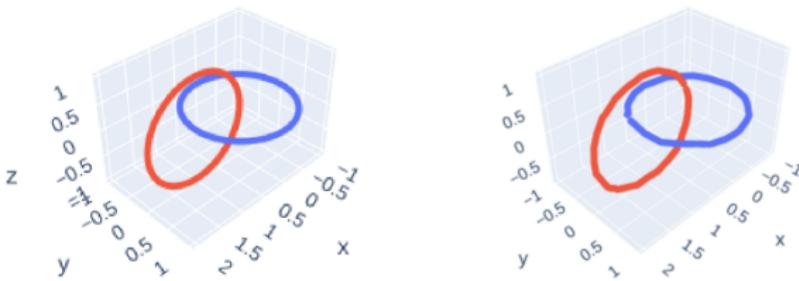
Decoder $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$ = any extension of inverse diffeomorphism.

Can always find such a “codim > 0 set” disjoint from the data.

Note: training sometimes yields disconnected “good” sets



In practice, random initialization/training leads to random outcomes, even those with disconnected “good sets”, despite the fact that arbitrarily large connected “good sets” (disks) exist.



Semi-global autoencoders always exist³

Let $\mathcal{F}^{\ell,m}$ be dense in the space of continuous functions $\mathbb{R}^\ell \rightarrow \mathbb{R}^m$, e.g., the collection of possible neural network outputs.

Theorem 1 (MDK and E D Sontag). Let $K \subset \mathbb{R}^n$ be finitely many disjoint compact $\leq k$ -dimensional submanifolds with(out) boundary, and let $\mu, \partial\mu$ be any smooth measures on $K, \partial K$. For each $\delta > 0$ and finite set $S \subset K$, there is a closed set $K_0 \subset K$ s.t.

- ▶ $K_0 \cap S = \emptyset, \mu(K_0) < \delta, \partial\mu(K_0 \cap \partial K) < \delta;$
- ▶ $M \setminus K_0$ is connected for each component M of K ;
- ▶ For each $\varepsilon > 0$ there are functions $F \in \mathcal{F}^{n,k}, G \in \mathcal{F}^{k,n}$ s.t.

$$\sup_{x \in K \setminus K_0} \|G(F(x)) - x\| < \varepsilon.$$

⇒ data S can be reconstructed to order ε , and generalization error will also be uniformly smaller than ε with probability $1 - \delta$.

³MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

Almost-global autoencoders do not generally exist

Theorem 2 (MDK and EDS). Let $K \subset \mathbb{R}^n$ be a k -dimensional compact submanifold without boundary. For any continuous functions $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$,

$$\max_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\text{reach}} > 0.$$

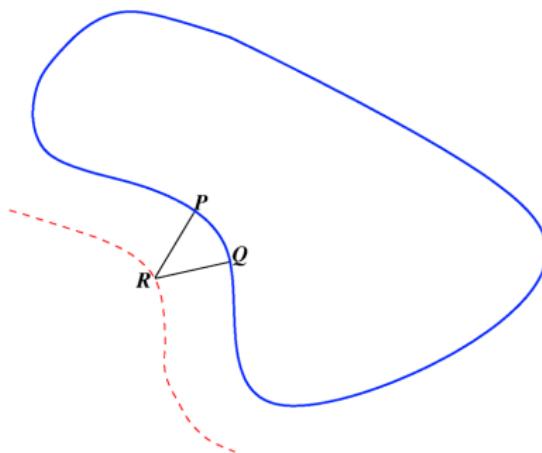


Figure: The **reach** $r_K > 0$ of K is the largest number such that any $x \in \mathbb{R}^n$ satisfying $\text{dist}(x, K) < r_K$ has a unique nearest point on K . Both line segments shown have length r_K .

Almost-global autoencoders do not generally exist⁴

Theorem 2 (MDK and EDS). Let $K \subset \mathbb{R}^n$ be a k -dimensional compact submanifold without boundary. For any continuous functions $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$,

$$\max_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\text{reach}} > 0. \quad (2)$$

Proof:

- ▶ $N := \{x \in \mathbb{R}^n : \text{dist}(x, K) < r_K\}$ contains line segment from $x \in N$ to nearest $\rho(x) \in K$; ρ is continuous.
- ▶ If (2) does not hold, $t \mapsto \rho \circ (tG \circ F|_K + (1-t)\text{id}_K)$ is a homotopy of id_K to $\rho \circ G \circ F|_K$, so $\deg_2(\rho \circ G \circ F|_K) = 1$.
- ▶ But this contradicts

$$0 = \deg_2(\rho \circ G \circ F|_K) = \deg_2(\rho \circ G|_{F(K)}) \underbrace{\deg_2(F|_K)}_0.$$

⁴MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

Example: $K = 2$ unit circles; max error > reach $r_K = 1$

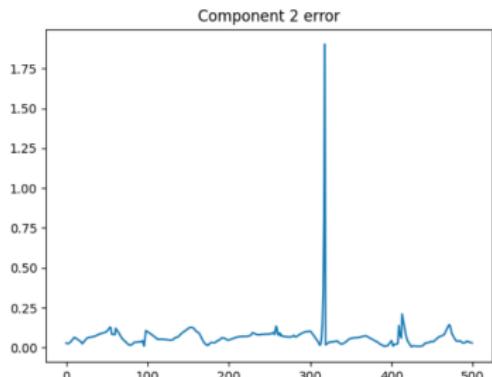
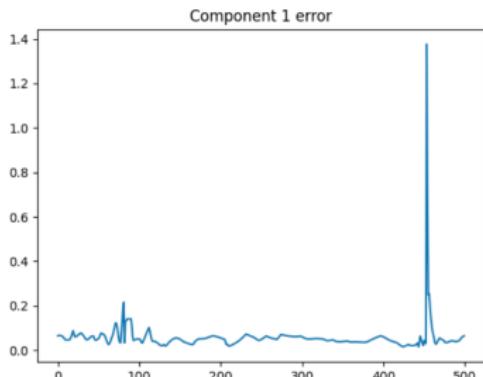
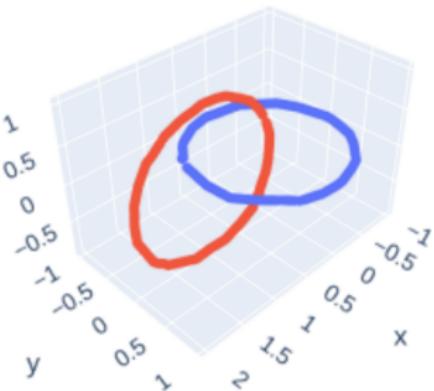
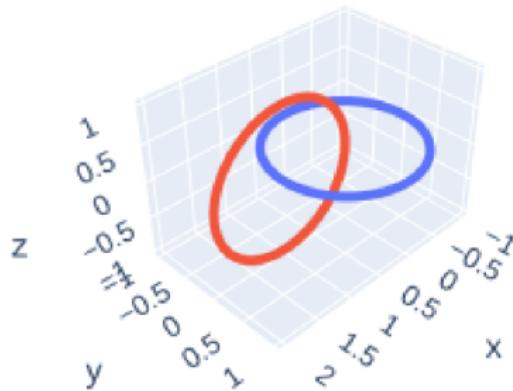
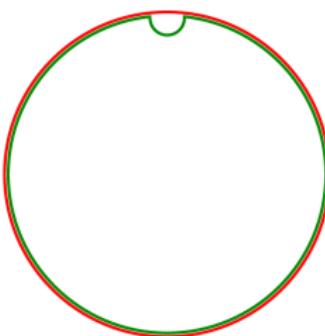


Figure: Errors $\|G(F(x)) - x\|$ on the two circles. The x-axis shows the index k representing the k -th evenly-spaced point on the respective circle.

In fact, true min-max error is usually bigger than the reach⁵



Red reach = 1 but green reach = $\varepsilon \ll 1$, so previous min max error “reach” is conservative. But green “**dewrinkled reach**” = $1 - \varepsilon$.

Corollary (MDK and EDS). Let $K \subset \mathbb{R}^n$ be a k -dimensional compact submanifold without boundary. For any continuous functions $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_{K,k}^*}_{\text{dewrinkled reach}} := \sup_{L \in \mathcal{M}_{n,k}, T \in C(L \rightarrow K)} \{r_L - \delta(T)\}.$$

⁵MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

Implications for autoencoder training error⁶

Theorem 1 $\implies F, G$ always exist making the $L^p(\mu)$ loss

$$\int_K \|G(F(x)) - x\|^p d\mu(x) < \varepsilon$$
$$\left(= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \|G(F(x_i)) - x_i\|^p\right)$$

arbitrarily small for any $p \in (0, \infty)$ (G can be modified off of $F(K \setminus K_0)$ to make the AE error smaller than some $C_K > 0$ on K_0 .)

However, Theorem 2 \implies for many K , the L^∞ loss

$$\max \|G(F(x)) - x\| \geq r_K > 0$$

is uniformly big, independent of F, G .

⁶We thank Dr. Joshua Batson for suggesting these observations.

Summary

Main representation result: data lying in a submanifold K of dimension k can be encoded through a bottleneck layer of the same dimension k , up to an arbitrarily small reconstruction error ε .

Moreover, the generalization error will also be uniformly smaller than ε with arbitrarily high probability $1 - \delta$.

Main necessity result: for many K (including all K without boundary), there is a geometric lower bound on the global reconstruction error.

Training implications: L^p error can always be made arbitrarily small for $p \in (0, \infty)$; L^∞ error cannot.

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Applied Koopman operator methods

Feedback stabilizability

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Applied Koopman operator methods

Many assume the dynamical system is globally linearizable

Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

Feedback stabilizability

“Applied Koopmanism”

“A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.”

— Brunton, Budišić, Kaiser, and Kutz. “Modern Koopman Theory for Dynamical Systems.” SIAM Review, 64.2 (2022)

They seek nonlinear measurements that separate points, to be able to invert / not to lose information. Also want measurements / inverse to be continuous for practical reasons.

More formally, they seek *embeddings* of *nonlinear* dynamical systems into *linear* ones as invariant subsets, so that existing theoretical and algorithmic linear tools can be utilized.

Linearizing embeddings

Let f be a locally Lipschitz vector field on a manifold M . Consider

$$\dot{x} = \frac{d}{dt}x = f(x),$$

assume this ODE's solutions $x(t) = \Phi^t(x_0)$ are defined for all time.

$F: M \rightarrow \mathbb{R}^n$ is a **topological embedding** if F is a one-to-one continuous map with a continuous inverse $F^{-1}: F(M) \rightarrow M$, and is a **smooth embedding** if additionally F, F^{-1} are smooth.

Such an embedding F is **linearizing** if $F \circ \Phi^t = e^{Bt} \circ F$ for some $n \times n$ matrix B . In the smooth case, $y = F(x)$ satisfies $\dot{y} = By$.

Fundamental question: when is (M, Φ) linearizable in this sense?

When is a dynamical system (M, Φ) globally linearizable?

- ▶ Not when M is connected, forward Φ -trajectories are precompact, and Φ has a countable number ≥ 2 of omega limit sets (Liu, Ozay, Sontag 2023).
- ▶ Not when M is connected and Φ has a non-global compact attractor $A \neq \emptyset$, since its open basin of attraction would also be closed (by the Jordan normal form theorem), hence empty.

Thus, we study linearizability of the restriction (S, Φ) of Φ to

1. compact invariant sets S , and
2. basins S of compact attractors A .

For these 2 cases we obtain **necessary and sufficient conditions for global linearizability** of (S, Φ) by an embedding, for the 2 cases of topological and smooth embeddings (4 cases total).⁷

⁷MDK and P. Arathoon, *Linearizability of flows by embeddings* (2023).

Torus preliminaries

The n -torus $T = T^n$ is (Lie group) isomorphic to $(\mathbb{R}/\mathbb{Z})^n$, vectors with n real entries but with addition defined elementwise modulo 1.

A **torus action** on S is a map $\Theta: T \times S \rightarrow S$ satisfying $\Theta^{\tau_1 + \tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

The flow (S, Φ) is a **1-parameter subgroup of a torus action** if $\Phi^t = \Theta^{\omega t \bmod 1}$ for some torus action Θ on S , $\omega \in \mathbb{R}^n$.

The linearizability theorem, case 1: compact, smooth

Observation: If (S, Φ) is linearizable with S compact, the Jordan normal form theorem implies (S, Φ) embeds into the flow on \mathbb{C}^n of a diagonal imaginary matrix, so (S, Φ) is a 1-parameter subgroup of restriction of standard torus action of T^n on \mathbb{C}^n to a subtorus.

This gives one implication below; the *Mostow-Palais equivariant embedding theorem* gives the other.

Theorem (MDK and P. Arathoon). If S is a compact embedded submanifold, (S, Φ) is linearizable by a **smooth** embedding \iff (S, Φ) is a 1-parameter subgroup of a **smooth** torus action.

We use this theorem to construct examples of smoothly linearizable (S, Φ) having isolated equilibria with e.g. $S =$ a sphere, torus, Klein bottle. On the other hand, regarding *nonlinearizability*...

Topological implications for case 1 (compact, smooth)

If (S, Φ) is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an $n \times n$ skew matrix B_e and a system of local coordinates on a neighborhood of each equilibrium $e \in S$ such that $\Phi^t \approx e^{B_e t}$. Hence if e is isolated then B_e is invertible, $n = \dim S$ is even, and the Hopf index of e is +1.

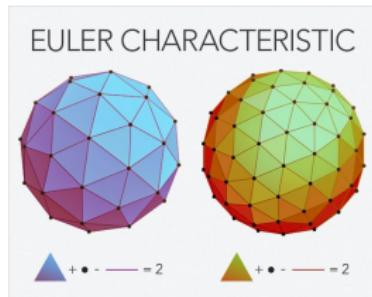
Corollary (MDK and PA). If S is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then (S, Φ) cannot be linearized by a smooth embedding.

Corollary (MDK and PA).⁸ If S is a compact submanifold containing at most finitely many equilibria such that (S, Φ) is linearizable by a smooth embedding, $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$.

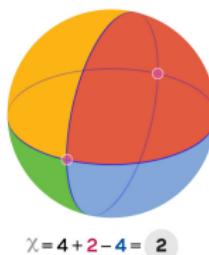
⁸ Apply the Poincaré-Hopf theorem.

A primer on the Euler characteristic⁹

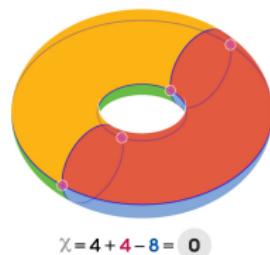
Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



$$\text{Euler Characteristic } (\chi) = \text{Faces} + \text{Corners} - \text{Edges}$$



$$\chi = 4 + 2 - 4 = 2$$



$$\chi = 4 + 4 - 8 = 0$$

Notation: $\chi(Y)$:= Euler characteristic of Y .

Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\Sigma_g) = 2 - 2g$



Σ_g for $g = 1, 2, 3$ (not linearizable for $g > 1$ if finite equilibria).

⁹Figures from Quanta Magazine and Wikipedia.

The linearizability theorem, case 2: compact, continuous

The theorem for case 2 is similar for case 1, but an additional assumption is needed to rule out a pathology not possible in case 1.

Theorem (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff (S, \Phi)$ is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

A torus action has **finitely many orbit types** if there are only finitely many subgroups $H \subset T$ such that

$H = \{\tau \in T : \Theta^\tau(s) = s\}$ is the fixed point set of some $s \in S$.

Another point of view: quasiperiodic pinched torus families

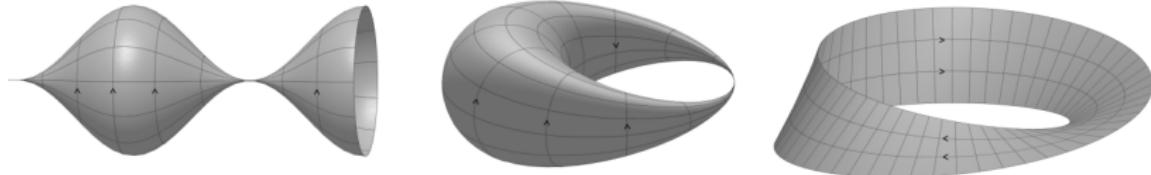


Figure: examples of quasiperiodic pinched torus families

Proposition (MDK and PA). If S is compact, (S, Φ) is linearizable by a [topological](#) embedding $\iff (S, \Phi)$ is a quasiperiodic pinched torus family.

Definition. P is a **pinched torus family** if there are $m, n \in \mathbb{N}$, closed subsets $C_1, \dots, C_n \subset B \subset T^m$, and a continuous group homomorphism $F: T^n \rightarrow T^m$ such that P is the quotient of $F^{-1}(B)$ by collapsing the j -th (\mathbb{R}/\mathbb{Z}) -factor of $F^{-1}(C_j) \subset T^n$ for all j . A pinched torus family P is **quasiperiodic** if it is equipped with the induced flow generated by any $\omega \in \mathbb{R}^n$ with $TF(\omega) = 0$.

The linearizability theorem, case 3: basin, continuous

If S is the basin of an asymptotically stable compact set $A \subset S$, A has continuous (smooth) **asymptotic phase**¹⁰ if there is a continuous (smooth) **asymptotic phase map** $P: S \rightarrow A$, i.e.,

$$P|_A = \text{id}_A, \quad P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

Theorem (MDK and PA). (S, Φ) is linearizable by a topological embedding $\iff A$ has continuous asymptotic phase and (A, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

Example. The basin of an asymptotically stable limit cycle is linearizable by a topological embedding \iff the cycle has continuous asymptotic phase. This is not always the case, but it is the case if $\Phi \in C^1$ and the cycle is hyperbolic.

¹⁰This notion has roots in oscillator theory and more generally NHIM theory.

The linearizability theorem, case 4: basin, smooth

Theorem (MDK and PA). (S, Φ) is linearizable by a smooth embedding $\iff A$ is an embedded submanifold with smooth asymptotic phase, (A, Φ) is a 1-parameter subgroup of a smooth torus action, and for some open $U \supset A$, (U, Φ) embeds in a reducible linear flow covering Φ on some vector bundle over A .

When does the final condition hold? Classical linearization theorems and recent linearizing semiconjugacy theorems (MDK and Revzen, 2023) give answers in the special cases that A is an equilibrium or periodic orbit, and some things are known if A is quasiperiodic, but the general case seems to be an open problem.

A necessary condition for A to satisfy all conditions of the theorem is that A be a (eventually relatively ∞ -)normally hyperbolic invariant manifold. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

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Brockett's necessary condition and beyond

A homotopy theorem beyond the Coron/Mansouri tests

Periodic orbits can be easier to stabilize than equilibria

Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \quad (1)$$

where $M \ni x$ is a smooth manifold and f is smooth.

1. **Controllability problem:** Given $a, b \in M$, find $u(t)$ s.t.
 $x(T) = b$ if $x(0) = a$ for some $T > 0$.

$$a \rightsquigarrow b$$

2. **Stabilizability problem:** Given a compact subset $A \subset M$,
find smooth $u(x)$ s.t. A is **asymptotically stable**¹¹ for the
closed-loop vector field $F(x) = f(x, u(x))$. [▶ Link](#)

¹¹For every open $W \supset A$ there is an open $V \supset A$ s.t. all forward F -trajectories initialized in V are contained in W and converge to A .

The stabilization conjecture and Brockett's solution

Often $A = \{x_*\}$ is a point, $M = \mathbb{R}^n$ in the stabilization problem.

Stabilization conjecture (pre-1983): a reasonably strong form of controllability implies smooth stabilizability of a point.

Example: the “Heisenberg system” or “nonholonomic integrator”

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv\end{aligned}\right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?

Theorem (Brockett). If a point is stabilizable, then $\text{image}(f)$ is a neighborhood of 0. (In the example, $(0, 0, \varepsilon) \notin \text{image}(f)$.)

Other stabilizability work

- ▶ Exponential (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
 - ▶ global (Byrnes 2008, Baryshnikov 2023),
 - ▶ time-varying (Coron 1992), and
 - ▶ discontinuous (Clarke, Ledyayev, Sontag, Subbotin 1997)
- variants of the stabilization problem are not considered here.

Coron's and Mansouri's obstructions

Krasnosel'skiĭ and Zabreiko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : f(x, u) \neq 0\}.$$

Theorem (Coron 1990). If $n > 1$ and a point is stabilizable,

$$f_*(H_{n-1}(\Sigma)) = H_{n-1}(\mathbb{R}^n \setminus \{0\}) \quad (\cong \mathbb{Z}).$$

Theorem (Mansouri 2010). If a closed codimension > 1 submanifold $A \subset \mathbb{R}^n$ with Euler characteristic $\chi(A)$ is stabilizable,

$$f_*(H_{n-1}(\Sigma)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \quad (\cong \chi(A) \cdot \mathbb{Z}).$$

Limitations of these results

The results of Brockett, Coron, Mansouri rely on parallelizability of \mathbb{R}^n to view vector fields and control systems as \mathbb{R}^n -valued.

Furthermore, they apply only to the special case that A is a point or a closed submanifold of \mathbb{R}^n with $\chi(A) \neq 0$.

But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

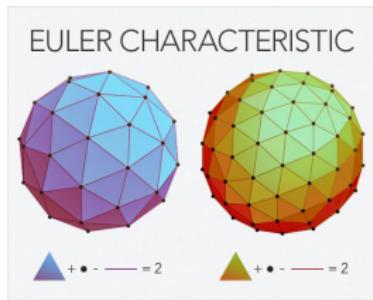
How to test for stabilizability in such general settings?¹²

- ▶ **Generalization of Brockett's test** (MDK and Daniel E. Koditschek, J Geometric Mechanics, 2022).
- ▶ **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).

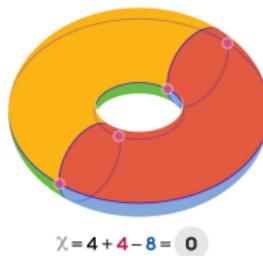
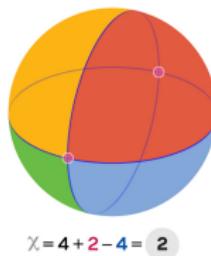
¹²An exposition of all stabilizability results here is in 2023 book *Topological Obstructions to Stability and Stabilization* by W. Jongeneel and E. Moulay.

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Notation: $\chi(Y)$:= Euler characteristic of Y .

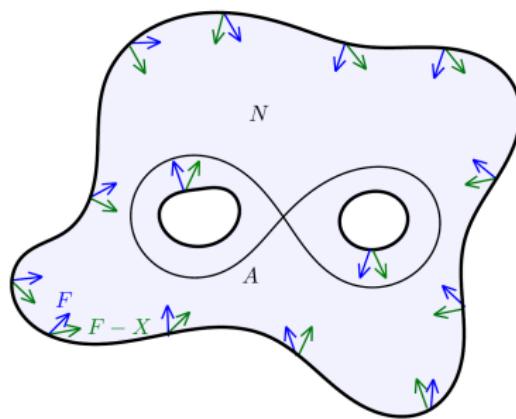
Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\text{figure 8}) = -1$

Theorem (Poincaré, Hopf): if N is a compact smooth manifold with boundary ∂N , then $\chi(N) = 0 \iff$ there exists a nowhere-zero smooth vector field on N pointing inward at ∂N .

¹³Figures from Quanta Magazine.

Generalization of Brockett's test

Theorem (MDK & Koditschek 2022): Let $A \subset M$ be compact & stabilizable. Then $\chi(A)$ is well-defined. If $\chi(A) \neq 0$, then for any sufficiently small vector field X , $X(x_0) = f(x_0, u_0)$ for some x_0, u_0 .



Proof: Assume \exists stabilizing $u(x)$ and define $F(x) := f(x, u(x))$.
Lyapunov function theory $\implies \exists$ compact smooth domain $N \supset A$ s.t. F points inward at ∂N and $\chi(A) = \chi(N) \neq 0$. Continuity $\implies F - X$ points inward at ∂N if X is small $\implies F - X$ has a zero by Poincaré-Hopf $\implies \exists x_0$ s.t. $X(x_0) = F(x_0) = f(x_0, u(x_0))$.

Examples

Heisenberg system

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv\end{aligned}\tag{2}$$

Kinematic differential drive robot

$$\begin{aligned}\dot{x} &= u \cos \theta \\ \dot{y} &= u \sin \theta \\ \dot{\theta} &= v\end{aligned}\tag{3}$$

The right side of (2) $\neq X_\varepsilon := (0, 0, \varepsilon)$ for any $\varepsilon > 0$.

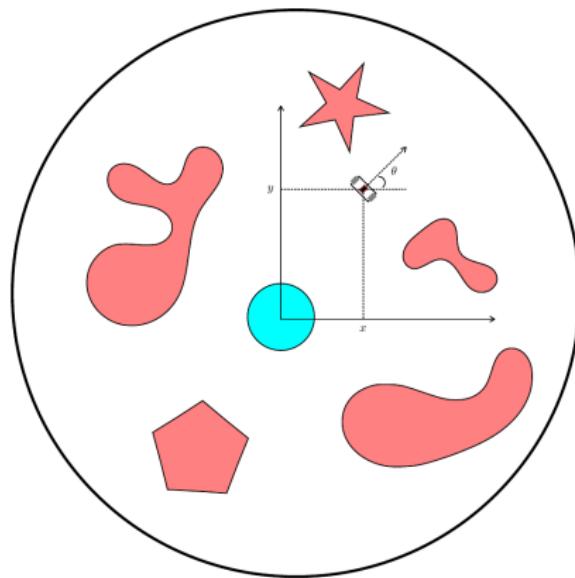
The right side of (3) $\neq X_\varepsilon := (\varepsilon \sin \theta, -\varepsilon \cos \theta, 0)$ for any $\varepsilon > 0$.

Thus, our result $\implies A$ is not stabilizable if $\chi(A) \neq 0$. E.g., if A is a stabilizable compact submanifold, A is a union of circles and tori.

Other applications: any stabilizable compact set has zero Euler characteristic for satellite orientation with ≤ 2 thrusters, for nonholonomic dynamics with ≥ 1 global constraint 1-form,...

Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some “bad” set immediately enter some “good” set.



E.g., impossible for this differential drive robot to aim within ± 179 degrees of the origin while “strictly” avoiding obstacles via $u(x)$.

Homotopy theorem & generalized Coron, Mansouri tests

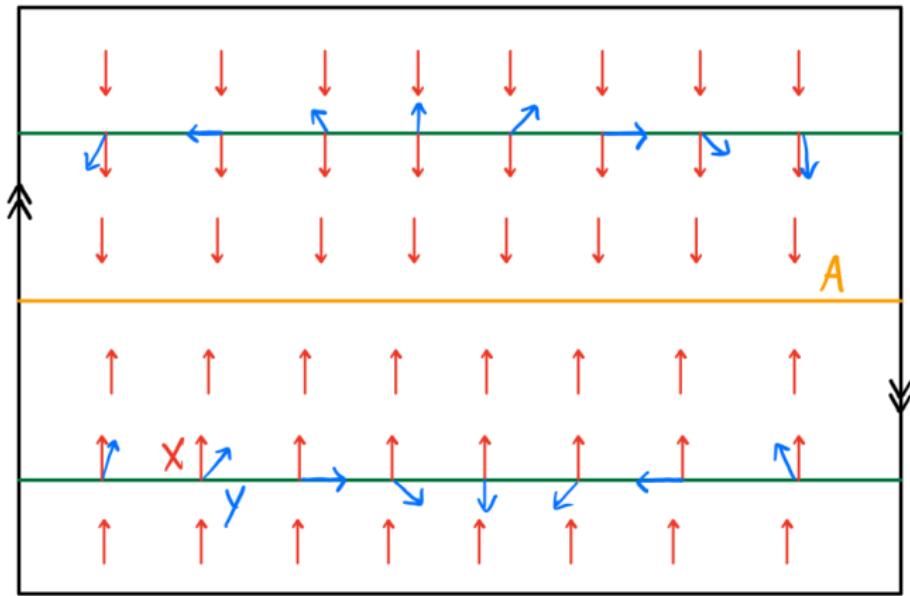
Homotopy theorem (MDK 2023). Let X, Y be smooth vector fields on a manifold M with a compact set $A \subset M$ asymptotically stable for both. There is an open set $U \supset A$ such that $X|_{U \setminus A}$, $Y|_{U \setminus A}$ are homotopic through nowhere-zero vector fields.

⇒ **Theorem (MDK 2023).** Let the compact set $A \subset M$ be asymptotically stable for *some* smooth vector field Y on M . If A is stabilizable for $\dot{x} = f(x, u)$, then for all small enough open $U \supset A$,

$$H_\bullet(T(U \setminus A) \setminus 0) \supset \underbrace{f_* H_\bullet(\Sigma)}_{\text{cf. Coron, Mansouri}} \supset Y_* H_\bullet(U \setminus A).$$

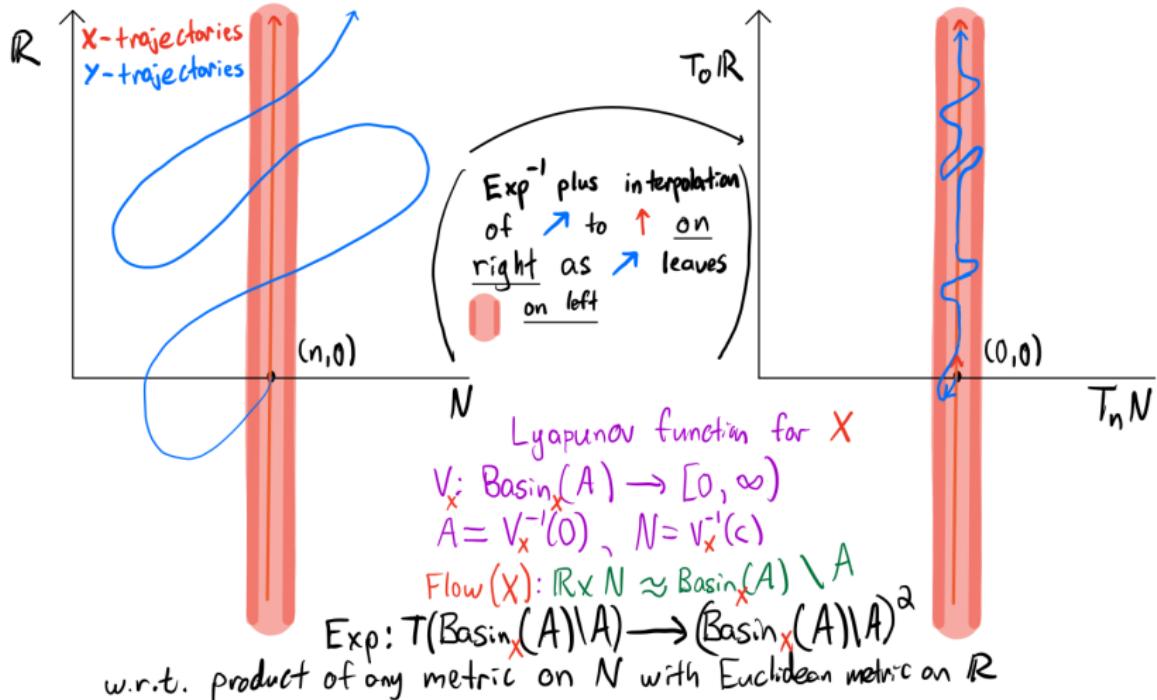
These are stronger than all preceding results: there is an example (MDK 2023) for which non-stabilizability is detected by each of these theorems but not by any of the preceding theorems.

Möbius strip example



$X \neq Y$ since Y \curvearrowright twice
around \bigcirc w.r.t. X while X \curvearrowright
zero times w.r.t. $X \Rightarrow A$ is not
asymptotically stable for Y by the
homotopy theorem.

Proof of the homotopy theorem



Can these results detect stabilizability of periodic orbits?

If A is the image of a periodic orbit with the same orientation for X and Y , the straight-line homotopy over a sufficiently small open $U \supset A$ satisfies the homotopy theorem's conclusion regardless of whether A is attracting, repelling, or neither for X or Y .

\implies **homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, preceding results also give no information.

...Could it be that periodic orbits might be “easy” to stabilize?

Periodic orbits are sometimes easier to stabilize

Indeed, at least sometimes:

Theorem (Anthony M. Bloch & MDK, in preparation).

For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!

Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

Feedback stabilizability

Brockett's necessary condition and beyond

A homotopy theorem beyond the Coron/Mansouri tests

Periodic orbits can be easier to stabilize than equilibria

Thank you for your attention.

Identifying engineering (im)possibilities for:

Deep neural network autoencoders

They should not work, and yet they do: resolving the paradox
Training implications: L^p but not L^∞ error can be made small

Applied Koopman operator methods

Many assume the dynamical system is globally linearizable

Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

Feedback stabilizability

Brockett's necessary condition and beyond

A homotopy theorem beyond the Coron/Mansouri tests

Periodic orbits can be easier to stabilize than equilibria