On Professor Smale's legacy for asymptotic stability theory[1]

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Asymptotic stability

By default, finite-dim manifolds & maps between them are smooth.

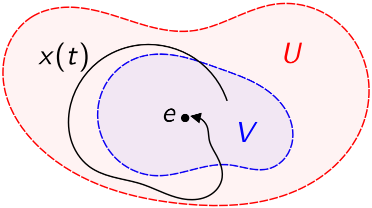
Consider vector field F on $\mathbb{R}^n$ and ODE

$$\dot{x}(t) = F(x(t)). \tag{1}$$

Let $e \in \mathbb{R}^n$ be an **equilibrium**: $F(e) = 0$.

We say that $e \in \mathbb{R}^n$ is (globally) **asymptotically stable** if

- every solution of (1) converges to e as $t \to \infty$, and
- for every open $U \ni e$ there is a smaller open $V \ni e$ such that all solutions of (1) starting in V stay in U for all $t \geq 0$.



Motivating question

Asymptotic stability models robust steady-state behavior; engineers try to achieve it through feedback control. ▸ Example

Goal: design stabilizing feedback depending on parameters that can later be adusted to create desired transient responses.

Question: if we have already done this for *some* parameters, when is it possible to extend our design to *all* parameters?

Answer to motivating question

The answer depends only on topological properties of the space

$$\mathcal{S}(\mathbb{R}^n) := \{\text{asymptotically stable vector fields on } \mathbb{R}^n\}$$

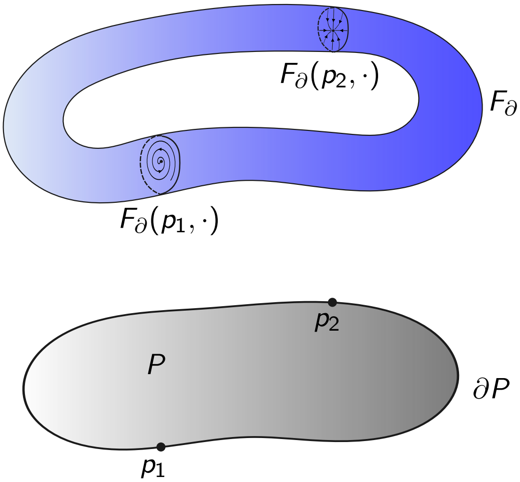
equipped with the compact-open C^∞ topology.

Main theorem (K 2025). $\mathcal{S}(\mathbb{R}^n)$ is both path-connected and simply connected if $n \neq 4, 5$, and is weakly contractible if $n < 4$.

$\Downarrow$

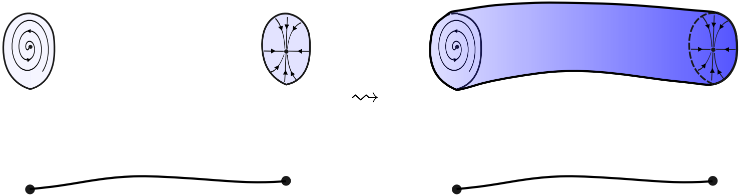
BVP existence theorem. Let P be a compact manifold with boundary ∂P and $F_\partial \colon \partial P \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy $F_\partial(p, \cdot) \in \mathcal{S}(\mathbb{R}^n)$. There is a map $F \colon P \times \mathbb{R}^n \to \mathcal{S}(\mathbb{R}^n)$ extending F_∂ and satisfying $F(p, \cdot) \in \mathcal{S}(\mathbb{R}^n)$ if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 3$.

Example boundary value problem ($n = 2 = \dim P$)



Previous theorem $\implies$ parametric family $F_\partial \colon \partial P \times \mathbb{R}^n \to \mathbb{R}^n$ of asymptotically stable vector fields has extension $F \colon P \times \mathbb{R}^n \to \mathbb{R}^n$.

Another example ($P = [0, 1]$) and Conley's question



Question (Conley 1978): When are isolated invariant sets having the same Conley index related by continuation?

Partial answer: "always" if $n \neq 4, 5$ and the sets are asymptotically stable equilibria; moreover, the continuation can be taken through equilibria of asymptotically stable vector fields.[2]

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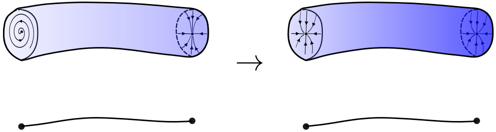
[2] See Reineck (1992) and Jongeneel (2024) for other partial answers.

Other applications

- **Parametric Hartman-Grobman theorem without hyperbolicity.** Given continuous families $F_p, G_p \in \mathcal{S}(\mathbb{R}^n)$, $p \in P$, there is a continuous family of homeomorphisms $h_p \colon \mathbb{R}^n \to \mathbb{R}^n$ identifying trajectories of F_p with those of G_p if either[3] (i) $n < 4$ or (ii) $n > 5$ and $\dim P \leq 1$.



- **Relative homotopy groups.** Let $\mathcal{AN}(\mathbb{R}^n)$ be the space of "almost nonsingular" vector fields having exactly one equilibrium. Then

$$\pi_k(\mathcal{AN}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) \cong \pi_k \mathcal{AN}(\mathbb{R}^n)$$

if either (i) $n < 4$ or (ii) $n > 5$ and $k = 1, 2$.

[3] No restrictions on n needed for nonparametric case (see K-Sontag 2025).

Outline

Motivating question and answer

Other applications

Proof step 1: reduction to the space of Lyapunov functions

Proof step 2: reduction to the nonlinear Grassmannian of discs

Proof step 3: homotopy groups of the nonlinear Grassmannian

Outline

Lyapunov functions

Define the space of (global) **Lyapunov functions**

$$\mathcal{L}(\mathbb{R}^n) := \{\text{proper functions } \mathbb{R}^n \to [0,\infty) \text{ w/ unique critical value} = 0\}$$

equipped with the compact-open C^∞ topology, and its subspace $\mathcal{L}_0(\mathbb{R}^n)$ of functions equal to 0 at 0.

Proposition (K-2025). There are weak homotopy equivalences

$$\mathcal{L}_0(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{L}(\mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathcal{S}(\mathbb{R}^n)$$

(latter given by $-\nabla$; proof via Wilson's 1969 Lyapunov theorem).

$\implies$ suffices to prove main theorem for $\mathcal{L}_0(\mathbb{R}^n)$ instead of $\mathcal{S}(\mathbb{R}^n)$:

Theorem (K 2025). $\mathcal{L}_0(\mathbb{R}^n)$ is both path-connected and simply connected if $n \neq 4, 5$, and is weakly contractible if $n < 4$.

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Proof step 1: reduction to the space of Lyapunov functions

Proof step 2: reduction to the nonlinear Grassmannian of discs

Relies on **Smale's h-cobordism theorem**

Proof step 3: homotopy groups of the nonlinear Grassmannian

Plan

Wilson (1967) studied the topology of level sets of Lyapunov functions using dynamics of asymptotically stable vector fields.

To prove the main theorem, we turn that idea on its head:

We will use level sets of Lyapunov functions to study the space of asymptotically stable vector fields.

Topology of Lyapunov function sublevel sets

Proposition. For any $L \in \mathcal{L}(\mathbb{R}^n)$, the sublevel set $L^{-1}([0,1])$ is diffeomorphic to $D^n := \{x \in \mathbb{R}^n \colon \|x\| \leq 1\}$ if $n \neq 4, 5$.

Proof:

- The flow of ∇L induces deformation retractions of $L^{-1}([0,1])$ to $L^{-1}(0)$ and of $\mathbb{R}^n \setminus \{L^{-1}(0)\}$ to $L^{-1}(1)$.
- $\implies L^{-1}([0,1])$ is a contractible manifold with boundary $L^{-1}(1)$ a homotopy sphere (Wilson 1967).
- $\implies L^{-1}([0,1])$ is diffeomorphic to D^n for $n \neq 4, 5$ by
  - classification of 1D and 2D manifolds for $n = 1, 2$,
  - solution to 3D Poincaré conjecture (Perelman 2003) for $n = 3$,
  - the **h-cobordism theorem (Smale 1962)** for $n > 5$. $\square$

The sublevel set map

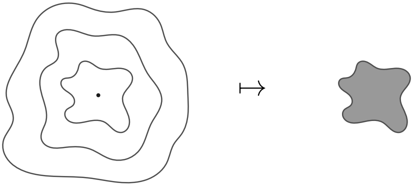
Define the space

$$\mathrm{Gr}(D^n, \mathbb{R}^n) := \mathrm{Emb}(D^n, \mathbb{R}^n)/\mathrm{Diff}(D^n)$$

of submanifolds of $\mathbb{R}^n$ diffeomorphic to D^n, known as a **nonlinear Grassmannian**, and its open subspace $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ consisting of neighborhoods of $0 \in \mathbb{R}^n$.

Previous slide $\implies$ we have a well-defined **sublevel set map**

$$p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n), \qquad p(L) := L^{-1}([0,1]).$$



The sublevel set map is a weak homotopy equivalence

Theorem (K 2025). The sublevel set map

$$p\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n), \qquad p(L) := L^{-1}([0,1])$$

is a fiber bundle w/ weakly contractible fibers (hence also a w.h.e.).

Proof sketch:

- p is continuous by implicit function theorem; surjective by disc theorem,[4] which also implies $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is path-connected.
- Each $M \in \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ has neighborhood $U \subset \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ and map $\Psi\colon U \to \mathrm{Diff}(\mathbb{R}^n)$ s.t. $\Psi(N)(M) = N$ for all $N \in U$.
- Define $f\colon p^{-1}(U) \to \mathcal{F} := p^{-1}(M)$ by $f(L) := L \circ \Psi(p(L))$.
- $(p, f)\colon p^{-1}(U) \to U \times \mathcal{F}$ is a homeomorphism, so p is bundle.
- To show that $\mathcal{F}$ is weakly contractible...

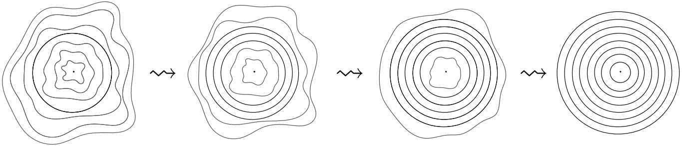
[4] Hildebrandt & Graves (1927), Abraham (1967); Palais (1960), Cerf (1961).

Weak contractibility of $\mathcal{F}$

- Since $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is path-connected, it suffices to check that $\mathcal{F} = p^{-1}(M)$ is weakly contractible for $M = D^n$, in which case

$$\mathcal{F} = \{L \in \mathcal{L}_0(\mathbb{R}^n) \colon L^{-1}([0,1]) = D^n\}.$$

- Any map $P \to \mathcal{F}$ is nullhomotopic to $P \to \{x \mapsto x^2\}$ by "parting the sea" of level sets away from $\partial D^n = S^{n-1}$, replacing the sea with level sets of $x \mapsto x^2$. $\square$



Outline

Motivating question and answer

Other applications

Proof step 1: reduction to the space of Lyapunov functions

Proof step 2: reduction to the nonlinear Grassmannian of discs

Proof step 3: homotopy groups of the nonlinear Grassmannian

Relies on **Smale's theorem** that $\mathrm{Diff}_\partial(D^2)$ is contractible and Hatcher's proof of the **Smale conjecture** for $\mathrm{Diff}_\partial(D^3)$.

Toward homotopy groups of the nonlinear Grassmannian

- Can easily show $\mathrm{Gr}_0(D^n, \mathbb{R}^n) \overset{\text{w.h.e.}}{\simeq} \mathrm{Gr}(D^n, \mathbb{R}^n)$, so to prove main theorem for $\mathcal{L}_0(\mathbb{R}^n)$ it suffices to show that the appropriate homotopy groups of $\mathrm{Gr}(D^n, \mathbb{R}^n)$ are trivial.
- The natural quotient map

$$\mathrm{Emb}^+(D^n, \mathbb{R}^n) \to \mathrm{Gr}(D^n, \mathbb{R}^n), \quad f \mapsto f(D^n)$$

  is a principal $\mathrm{Diff}^+(D^n)$-bundle,[5] so there is a long exact sequence of homotopy groups:

$$\cdots \pi_k \mathrm{Diff}^+(D^n) \longrightarrow \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \longrightarrow \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \cdots$$

[5] Gay-Balmaz and Vizman (2014) proved this result generalizing a theorem of Binz and Fischer (1981) based on an idea implicit in Weinstein (1971).

Analyzing the long exact sequence, part 1

This long exact sequence contains the segment

$$\pi_k \mathrm{Diff}^+(D^n) \xrightarrow{\text{surjective}} \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \xrightarrow{0} \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \xrightarrow{\text{injective}} \pi_{k-1} \mathrm{Diff}^+(D^n) \xrightarrow{\text{surjective}} \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n)$$

Claim: Gram-Schmidt $\implies$ the indicated arrows are surjective.

$\implies$ other arrows are 0, injective by exactness.

Analyzing the long exact sequence, part 2: Cerf

$$
\begin{tikzcd}
\pi_k \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[r, "0"] & \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \arrow[dll, rounded corners, "\text{injective}"] \\
\pi_{k-1} \mathrm{Diff}^+(D^n) \arrow[r, "\text{surjective}"] & \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n) &
\end{tikzcd}
$$

$\implies$ $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is simply connected for $n > 5$, since then $\pi_0 \mathrm{Diff}^+(D^n) = \{*\}$ by the pseudoisotopy theorem of Cerf (1970).

Remains to show $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is contractible when $n < 4$; suffices to show above surjections become bijections.

Completing the proof of main theorem, part 1

- Previous surjections are induced by top arrow in diagram

$$\begin{tikzcd}
\mathrm{Diff}^+(D^n) \arrow[rr, hook] \arrow[dr] & & \mathrm{Emb}^+(D^n, \mathbb{R}^n) \arrow[dl, "\simeq"] \\
& GL^+(n) &
\end{tikzcd}$$

  in which diagonal arrows are "evaluate derivative at point". **Suffices to show left one is w.h.e. if $n < 4$.**

- Left diagonal arrow is homotopic to composition

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{f} GL^+(n)$$

  of restriction ρ and map f given by adjoining the value and derivative at the north pole of S^{n-1}.

Completing the proof, part 2: Smale and Hatcher

So for $n < 4$, need to prove that following composition is a w.h.e.

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{f} GL^+(n)$$

- ρ is fiber bundle (Cerf 1961); fiber over $\mathrm{id}_{S^{n-1}}$ is

  $\mathrm{Diff}_\partial(D^n) :=$ {diffeomorphisms of D^n that are the identity on ∂D^n}.
- This fiber is contractible for:
  - $n = 1$ by convexity,
  - $n = 2$ by a **theorem of Smale (1957)**, and
  - $n = 3$ by Hatcher's (1983) proof of **Smale conjecture (1961)**.
- Hence ρ is a w.h.e., so it suffices to show that f is a w.h.e. for $n < 4$ (trivial for $n = 1$).

Completing the proof, part 3: Smale again

Need to show $f\colon \mathrm{Diff}^+(S^{n-1}) \to GL^+(n)$ is a w.h.e. for $1 < n < 4$.

Identifying $GL^+(n)$ with $\underbrace{\mathrm{Fr}^+(TS^{n-1})}_{+\text{ frame bundle}}$, f factors as the composition

$$\mathrm{Diff}^+(S^{n-1}) \longrightarrow \mathrm{Emb}^+(D^{n-1}_+, S^{n-1}) \xrightarrow{\simeq} \mathrm{Emb}^+(\mathrm{int}(D^{n-1}_+), S^{n-1}) \xrightarrow{\simeq} \mathrm{Fr}^+(TS^{n-1})$$

in which D^{n-1}_+ is upper hemisphere, first two arrows are restrictions, long arrow adjoins value and derivative at north pole.

Similar to last slide, first arrow is a fiber bundle (Cerf 1961) with contractible fiber $\simeq \mathrm{Diff}_\partial(D^{n-1})$ **(Smale 1957)**, so it is a w.h.e. $\square$

Thank you for your attention

This talk is based on the preprint:

"Differential topology of the spaces of asymptotically stable vector fields and Lyapunov functions", Kvalheim (2025).

Link to preprint

Slides are available at mdkvalheim.github.io

On Professor Smale's legacy for asymptotic stability theory

Motivating question and answer

$\mathcal{S}(\mathbb{R}^n)$ is 1-connected for $n \neq 4, 5$, contractible for $n < 4$.

Boundary value problems

Other applications

Partial answer to question of Conley

Parametric Hartman-Grobman theorem without hyperbolicity

Relative homotopy groups

Proof step 1: reduction to the space of Lyapunov functions

Proof step 2: reduction to the nonlinear Grassmannian of discs

Relies on **Smale's h-cobordism theorem**

Proof step 3: homotopy groups of the nonlinear Grassmannian

Relies on **Smale's theorem** that $\mathrm{Diff}_\partial(D^2)$ is contractible and Hatcher's proof of the **Smale conjecture** for $\mathrm{Diff}_\partial(D^3)$.