

# Existence and uniqueness of Koopman eigenfunctions near stable equilibria and limit cycles<sup>1</sup>

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## Relevant papers and slides available

- **Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits.** MDK and Shai Revzen. Physica D (2021), arXiv:1911.11996.
- **Generic properties of Koopman eigenfunctions for stable fixed points and periodic orbits.** MDK, David Hong, and Shai Revzen. IFAC-PapersOnline (2021; MTNS conference cancelled), arXiv:2010.04008.
- Slides available on my website: [mdkvalheim.github.io/assets/NOLTA2022.pdf](https://mdkvalheim.github.io/assets/NOLTA2022.pdf)

## Motivation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0, \quad 0 \text{ hyperbolically stable with basin } B \subset \mathbb{R}^n. \quad (1)$$

- $\psi: B \rightarrow \mathbb{C}$  is a **Koopman eigenfunction** if  $\exists \lambda \in \mathbb{C}$  s.t.

$$\psi(x(t)) = e^{\lambda t} \psi(x_0), \quad \text{or } \dot{\psi} = \lambda \psi \text{ if } \psi \in C^1. \quad (2)$$

- Sufficiently many “independent” eigenfunctions determine an invertible change of coordinates through which (1) becomes a linear system, a drastic simplification.
- How to find eigenfunctions? If  $\mu \in \mathbb{C}$  and  $g: B \rightarrow \mathbb{C}$  is such that the limit<sup>2</sup>

$$g_\mu^*(x_0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(x(t)) e^{-\mu t} dt \quad (3)$$

exists and is not identically zero, then  $g_\mu^*$  is an eigenfunction with  $\lambda = \mu$  in (2).

- **Questions:** Which one? Can there be more than one possible limit (modulo scalar multiplication)? Can the limit depend sensitively on  $g$ ? Other numerical issues?
- If we knew that eigenfunctions were unique, we could resolve these questions. **We will discuss uniqueness and more**, including **new convergence results** for (3).

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<sup>2</sup>**Laplace average:** see §3 of Mauroy, Mezić, Moehlis “Isostables...” 2013. See also Mezić “Analysis...” 2012.

## Principal eigenfunctions

- $C^1$  eigenfunctions determining a linearizing diffeomorphism must be **principal**:

$$d\psi_i(0) \neq 0.$$

- (**Fact**: if  $\psi_i$  is principal and  $\dot{\psi}_i = \lambda\psi_i$ ,  $d\psi_i(0)$  is left eigenvector of  $D_0f$  w/ e.val  $\lambda$ .)
- Thus, we will concentrate on existence & uniqueness of  $C^k$  principal eigenfunctions.
- In particular we will see that, under some conditions, principal eigenfunctions are uniquely determined by their derivatives at 0.
- (Later we will classify **all**  $C^\infty$  eigenfunctions under generic conditions, not just the principal ones.)

## Counterexamples $\implies$ some conditions are needed

- **Ex. 1.** Let  $k \geq 2$  be an integer,  $(x, y) \in \mathbb{R}^2$ ,

$$\dot{x} = -x, \quad \dot{y} = -ky.$$

Both

$$\psi_1(x, y) = y \text{ and } \psi_2(x, y) = y + x^k$$

are analytic principal eigenfunctions s.t.  $d\psi_1(0) = d\psi_2(0)$ ,

$$\dot{\psi}_i = \lambda \psi_i \text{ with } \lambda = -k.$$

$\implies$  **nonresonance assumptions needed** (explained later).

- **Ex. 2.** Let  $a > 1$  not be an integer,  $(x, y) \in \mathbb{R}^2$ ,

$$\dot{x} = -x, \quad \dot{y} = -ay.$$

Both

$$\psi_1(x, y) = y \text{ and } \psi_2(x, y) = y + |x|^a$$

are  $C^{\lfloor a \rfloor}$  principal eigenfunctions ( $\lfloor a \rfloor$  is the integer part of  $a$ ) s.t.  $d\psi_1(0) = d\psi_2(0)$ ,

$$\dot{\psi}_i = \lambda \psi_i \text{ with } \lambda = -a.$$

$\implies$  resonance not an issue here, but **spectral spread assumptions needed** (later).

## Towards $C^k$ existence and uniqueness, step 1: reduction to discrete-time

- Henceforth assume vector field  $f \in C^k$  is complete with  $C^k$  flow  $(t, x) \mapsto \Phi^t(x)$ .
- Can define eigenfunctions for a diffeomorphism  $F: B \rightarrow B$ :  $\psi(F(x)) = e^\lambda \psi(x)$ .
- If eigenfunctions for  $F = \Phi^1$  are unique, then they are unique for  $f$ .
- If a  $\lambda$ -eigenfunction  $\tilde{\psi}$  for  $F = \Phi^1$  exists, Sternberg's trick<sup>3</sup>  $\implies$

$$\psi = \int_0^1 e^{-\lambda t} \tilde{\psi} \circ \Phi^t dt$$

is a  $\lambda$ -eigenfunction for  $f$  and  $d\psi(0) = d\tilde{\psi}(0)$ .

- $\implies$  **suffices to consider discrete time**, i.e. prove existence & uniqueness for principal eigenfunctions of a diffeomorphism  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F(0) = 0$ , 0 hyperbolically stable with basin  $B$ .
- Existence & uniqueness for  $k < \infty$  plus bootstrapping yields existence & uniqueness for  $k = \infty$ , hence assume  $k < \infty$  for now.

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<sup>3</sup>cf. Lemma 4 of Sternberg, "Local contractions and a theorem of Poincaré" (1957).

## Step 2: nonresonance and solving polynomial equations

- If  $\mu \in \mathbb{C}$ ,  $k \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$  eigenvalues  $(D_0 F) = \lambda_1, \dots, \lambda_n$  repeated with multiplicity,  $(e^\lambda, D_0 F)$  is  **$k$ -nonresonant** if

$$e^\mu \neq \lambda_1^{m_1} \dots \lambda_n^{m_n}$$

whenever  $m_1, \dots, m_n \in \mathbb{N}_{\geq 0}$  satisfy  $2 \leq \sum_i m_i < k + 1$ .

- **Key fact:** If  $F \in C^k$  and  $\exists w \in \mathbb{R}^n$  s.t.  $w D_0 F = e^\lambda w$ ,  $k$ -nonresonance  $\implies$  invertibility of certain linear operators on polynomials  $\implies \exists!$  polynomial  $P: \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $P(0) = 0$  and

$$P \circ F = e^\lambda P + o(\|x\|^k), \quad \text{and } P \text{ is } \mathbb{R}\text{-valued if } e^\lambda \in \mathbb{R}.$$

- In other words,  $k$ -nonresonance  $\implies$  can Taylor expand and solve eigenfunction equation “order by order” to produce polynomial “eigenfunction up to order  $k$ ”  $P$ .
- Remains only to **find**  $o(\|x\|^k)$  **remainder**  $\varphi: \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $\psi = P + \varphi$  is an eigenfunction exactly.

### Step 3: spectral spread and contraction mapping to eliminate remainder

- **Spectral spread**  $\nu(e^\mu, D_0 F) := \min \left\{ r \in \mathbb{R} : |e^\mu| \geq \left( \max_{\lambda \in \text{evals}(D_0 F)} |\lambda| \right)^r \right\}$ .

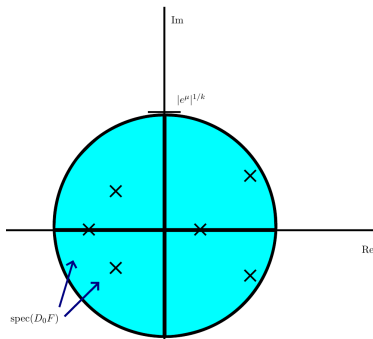


Figure: Illustration of  $\nu(e^\mu, D_0 F) < k$ .

- **Key fact:** if  $\nu(e^\lambda, D_0 F) < k$ ,  $\exists$  adapted norm  $\|\cdot\|$  and  $\varepsilon > 0$  s.t. with  $N := B_\varepsilon(0)$

$$S: \{\varphi|_N \in C^k(N, \mathbb{C}) : \varphi|_N \in o(\|x\|^k)\} \circlearrowright$$

$$S(\varphi|_N) := -P|_N + e^{-\lambda}(P|_N + \varphi|_N) \circ F$$

is a contraction mapping  $\implies \exists! \varphi|_N$  s.t.  $S(\varphi|_N) = \varphi|_N = \lim_{m \rightarrow \infty} S^m(\tilde{\varphi}|_N)$ , i.e.

$$\underbrace{(P|_N + \varphi|_N)}_{\psi|_N} \circ F = e^\lambda \underbrace{(P|_N + \varphi|_N)}_{\psi|_N} \quad \text{and} \quad \psi|_N = \lim_{m \rightarrow \infty} e^{-\lambda m} P \circ F|_N.$$



## Step 4: globalization $\implies$ discrete-time existence and uniqueness result

- Can globalize  $\psi|_N: N \rightarrow \mathbb{C}$  to  $\psi: B \rightarrow \mathbb{C}$  as follows: set  $\psi(x) := e^{-m\lambda}\psi|_N \circ F^m(x)$  where  $m$  is large enough that  $F^m(x) \in N$ ; can show well-defined independent of  $m$ .<sup>4</sup>
- **Theorem:** let  $k \geq 1$ ,  $F \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ ,  $F(0) = 0$ ,  $0$  hyperbolically stable with basin  $B$ ,  $(e^\lambda, D_0F)$   $k$ -nonresonant,  $\nu(e^\lambda, D_0F) < k$ , and  $wD_0F = e^\lambda w$ . Then there **exists** a **unique**  $C^k$  principal eigenfunction  $\psi$  satisfying  $\psi \circ F = e^\lambda \psi$ , and moreover

$$\psi = \lim_{m \rightarrow \infty} e^{-\lambda m} P \circ F \quad C^k\text{-uniformly on compacts if } P \circ F = e^\lambda P + o(\|x\|^k). \quad (4)$$

- **Observation:** (4)  $\implies$  Theorem hypotheses  $\implies$  **convergence of Laplace average**

$$\psi = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M e^{-\lambda m} P \circ F.$$

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<sup>4</sup>Similar techniques are used in Lan and Mezić (2013); Kvalheim, Eldering, and Revzen (2018).

## Continuous-time existence and uniqueness with weaker nonresonance

- If  $\text{evals}(D_0 f) = \lambda_1, \dots, \lambda_n$  with multiplicity and  $F = \Phi^1$ , taking logarithm of  $e^\mu \neq e^{m_1 \lambda_1} \dots e^{m_n \lambda_n}$  implies that  $k$ -nonresonance of  $(e^\mu, D_0 F)$  is equivalent to

$$\mu \neq m_1 \lambda_1 + \dots + m_n \lambda_n + i 2\pi \ell \quad (5)$$

for any  $\ell \in \mathbb{Z}$  and any  $m_1, \dots, m_n \in \mathbb{N}_{\geq 0}$  satisfying  $2 \leq \sum_i m_i < k + 1$ .

- By replacing  $F = \Phi^1$  with  $F = \Phi^\tau$  for arbitrary  $\tau > 0$ , (5) becomes

$$\mu \neq m_1 \lambda_1 + \dots + m_n \lambda_n + i \frac{2\pi}{\tau} \ell \quad (6)$$

which can be violated for all  $\tau$  if and only if it is violated for  $\ell = 0$ .

- $\implies$  **Theorem**:<sup>5</sup> let  $k \geq 1$ , vector field  $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ ,  $f(0) = 0$ , 0 hyperbolically stable with basin  $B$ ,  $\nu(e^\lambda, e^{D_0 f}) < k$ ,  $\lambda$  not equal to any integer linear combination of eigenvalues of  $D_0 f$  with coefficient sum  $\geq 2$ , and  $w D_0 f = \lambda w$ . Then there **exists** a **unique**  $C^k$  principal eigenfunction  $\psi$  satisfying  $\psi \circ \Phi^t = e^{\lambda t} \psi$  for all  $t \in \mathbb{R}$ , and

$$\psi = \lim_{m \rightarrow \infty} e^{-\lambda t} P \circ \Phi^t \quad C^k\text{-uniformly on compacts if } P \circ \Phi^1 = e^\lambda P + o(\|x\|^k). \quad (7)$$

- Observation**: (4)  $\implies$  Theorem hypotheses  $\implies$  **convergence of Laplace average**

$$\psi = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-\lambda t} P \circ \Phi^t.$$

<sup>5</sup>see Remark 3 of Kvalheim and Revzen (2021) or Proposition 11 of Kvalheim, Hong, and Revzen (2021).

## Classification of $C^\infty$ Koopman eigenfunctions

- **Key tool:**<sup>6</sup> under assumptions of preceding Theorem, if  $\varphi \in C^k(B, \mathbb{C})$  satisfies  $\varphi \circ \Phi^1 = e^\lambda \varphi$  and  $\varphi \in o(\|x\|^k)$ , then  $\varphi \equiv 0$ . In particular, if  $\varphi = \psi_1 - \psi_2$ ,  $\psi_1 = \psi_2$ .
- Key tool & preceding theorem can be used to prove the following.
- **Classification theorem:** let vector field  $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ ,  $f(0) = 0$ , 0 hyperbolically stable with basin  $B$ ,  $\lambda$  not equal to any integer linear combination of eigenvalues of  $D_0 f$  with coefficient sum  $\geq 2$ , and  $D_0 f$  diagonalizable over  $\mathbb{C}$ . Then
  - ▶ any Koopman  $\lambda$ -eigenfunction is a finite linear combination of products of  $n$  principal eigenfunctions and their complex conjugates.
  - ▶ In particular,  $\lambda$  is a linear combination of eigenvalues of  $D_0 f$ .

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<sup>6</sup>Proposition 1 from Kvalheim and Revzen (2021).

## Extension to periodic orbits

- Consider  $\dot{x} = f(x)$  having a hyperbolically stable  $\tau$ -periodic limit cycle with image  $\Gamma$ .
- Apply discrete-time versions of preceding theorems to a Poincaré map with section given by an isochron  $\implies$  existence and uniqueness theorems for  $C^k$  principal eigenfunctions (those with derivatives nonvanishing on  $\Gamma$ ).
- Corresponding classification theorem has a twist involving the unique  $C^\infty$  **asymptotic phase** eigenfunction  $\psi_\theta$  satisfying<sup>7</sup>  $\dot{\psi}_\theta = i\frac{2\pi}{\tau}\psi_\theta$ .
- **Classification theorem:** let  $f \in C^\infty$  and assume that no Floquet multiplier is an integer linear combination of the others with integer coefficient sum  $\geq 2$ . Then
  - ▶ **any** Koopman  $\lambda$ -eigenfunction is a finite linear combination of products of  $(n-1)$  principal eigenfunctions and  $\psi_\theta$  and their complex conjugates.
  - ▶ In particular,  $e^\lambda$  is a product of Floquet multipliers.

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<sup>7</sup>See Mauroy and Mezić “On the use of Fourier...” (2012), Kvalheim and Revzen (2021).

## Remarks on other results from Kvalheim and Revzen (2021)

- Results are given for both continuous-time and discrete-time.
- Main theorem is actually existence/uniqueness of general linearizing **semiconjugacies** (or **factors**): maps  $\psi: B \rightarrow \mathbb{C}^m$  s.t.  $\psi \circ \Phi^t = e^{At}\psi$  with  $A \in \mathbb{C}^{m \times m}$ .
- Application in paper: improvements of the Sternberg linearization and Floquet normal form theorems, with uniqueness statement, without assuming diagonalizable linearized dynamics.
- Paper considers  $\psi \in C^{k,\alpha}$ , i.e.  $\psi \in C^k$  such that  $D^k\psi$  is locally Hölder continuous with exponent  $\alpha$ . With this, results become fairly sharp (as examples in paper show).
- Stronger uniqueness-only statements in paper only require  $C^1$  (not  $C^k$ ) dynamics, but existence no longer guaranteed for merely  $C^1$  dynamics.
- Paper discusses in detail implications for **isostables** and **isostable coordinates** from literature.
- Also, see Schlosser and Korda “Sparsity structures for Koopman and Perron-Frobenius operators”, SIADS (2022) for an interesting application of the uniqueness results.

**Thank you for your time and attention, and thank you to the organizers Milan Korda, Alexandre Mauroy, Igor Mezić, and Yoshi Susuki for their kind invitation.**