

When can hybrid systems operate safely?¹²

Matthew D. Kvalheim

University of Pennsylvania

kvalheim@seas.upenn.edu

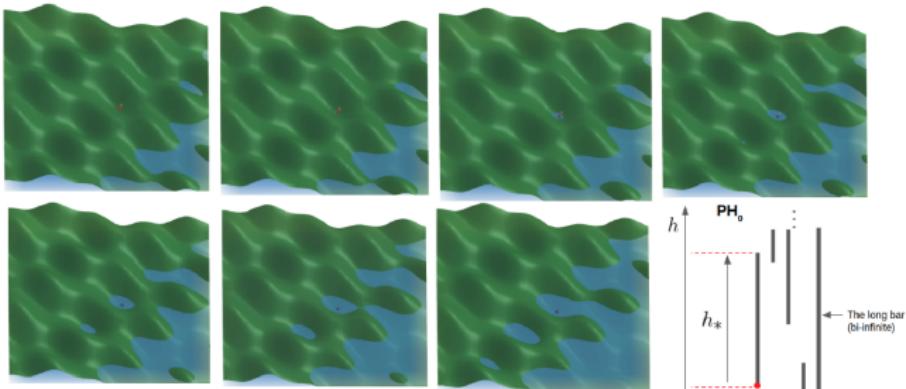
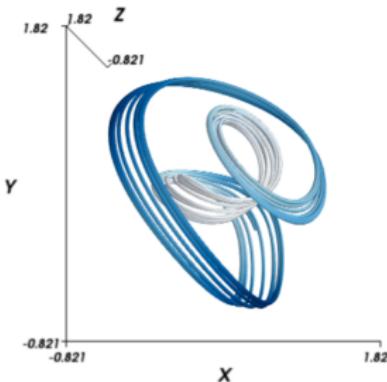
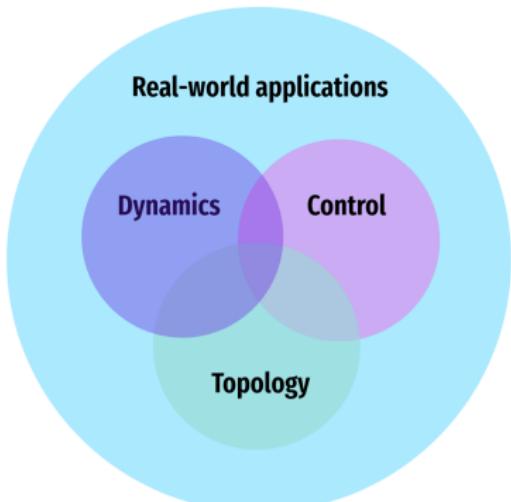
February 4, 2022

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¹Kvalheim, M. D. & Koditschek, D. E. "Necessary conditions for feedback stabilization and safety." J. Geometric Mechanics, in prep for invited resubmission (2022). arXiv:2106.00215.

²Kvalheim, M. D., Gustafson, P., & Koditschek, D. E. "Conley's fundamental theorem for a class of hybrid systems." SIAM J. Applied Dynamical systems, 20.2 (2021), pp. 784-825.

About my research



Acknowledgments

Necessary conditions for feedback stabilization and safety

with D E Koditschek.

Acknowledgments: Y Baryshnikov, W Clark, G Council, T Greco, R Gupta, E Lerman.

Dedication: to A M Bloch on the occasion of his 65th birthday. MDK would like to thank Bloch for his mentorship and, in particular, for introducing him to Brockett's necessary condition and to geometric mechanics during an inspiring course taught by Bloch at the University of Michigan in 2014. DEK would like to thank Bloch for his inspirational work and many decades of kind, unstinting tutorial wisdom.

Conley's fundamental theorem for a class of hybrid systems

with P Gustafson and D E Koditschek.

Acknowledgments: Y Baryshnikov, S A Burden, Z Cooperband, J Culbertson, D Guralnik, A M Johnson, E Lerman, P F Stiller.

Outline

1 Introduction

- **Q1:** Can we “test” for stabilizability and savability, in general, in nonlinear spaces?

2 Stabilizability and savability tests: the case of nonzero Euler characteristic

3 Stabilizability and savability tests: toward the case of zero Euler characteristic

4 Conley and the wild west of hybrid systems

Back to 1983: the stabilization conjecture and Roger W Brockett's solution



Stabilization conjecture³ (pre-1983): a reasonable form of local controllability implies the existence of a smooth time-independent control law asymptotically stabilizing a point.

Example: the “nonholonomic integrator” or “Heisenberg system”

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv\end{aligned}\right\} = f(\mathbf{x}, \mathbf{u}).$$

is accessible and controllable; e.g., any initial condition can be steered to the origin.⁴ →

Natural question: can the origin be stabilized by smooth time-independent feedback?

Brockett: **no.**

Brockett showed that **the stabilization conjecture is false** in general—and for this example—by formulating what is now known as **Brockett's necessary condition**.

³p. 2 of R W Brockett, “**Asymptotic stability and feedback stabilization.**” (1983).

⁴pp. 203–204 of A M Bloch, “**Nonholonomic mechanics and control**”, 2 ed. (2015).

Brockett's necessary condition⁵

Theorem (Brockett 1983): Assume $x_* \in \mathbb{R}^n$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via some C^1 feedback $u(x)$, where $f \in C^1$. Then for any $\xi \in \mathbb{R}^n$ with $\|\xi\|$ sufficiently small, there exists x_0, u_0 such that $\xi = f(x_0, u_0)$.

Proof: Assume such $u(x)$ exists.

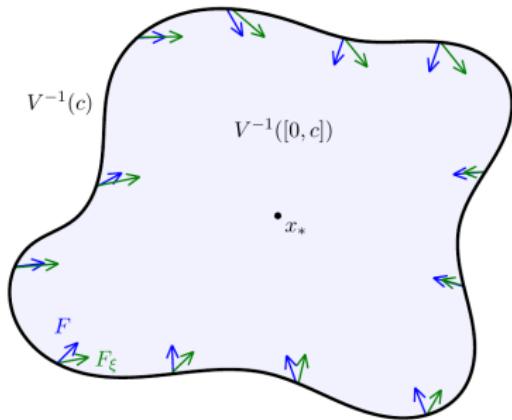
Define $F(x) := f(x, u(x))$.

Converse Lyap. theorem $\implies \exists C^\infty$ Lyapunov function $V: B(x_*) \rightarrow [0, \infty)$. Set $M_c := V^{-1}([0, c])$.

$\underbrace{}$
basin

Pick any $c \in (0, \infty)$. F points inward at $\partial M_c = V^{-1}(c)$. By continuity, so does $F_\xi := F - \xi$ if $\|\xi\|$ is small enough.

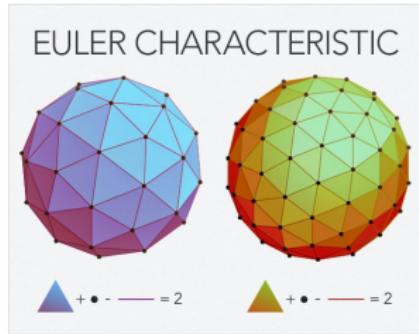
Since the Euler characteristic of M_c is nonzero, ...



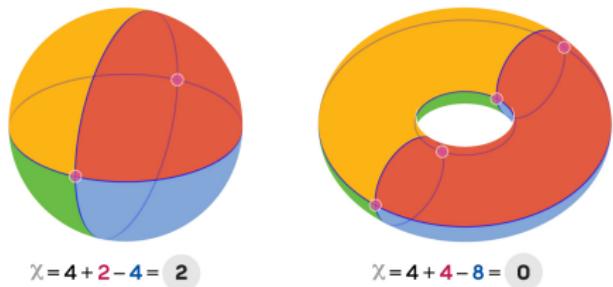
⁵Converse Lyap. theorem: F W Wilson, Jr. "Smoothing derivatives of functions and applications". (1969).

A brief primer on the Euler characteristic⁶

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



Euler Characteristic (χ) = Faces + Corners - Edges



Notation: $\chi(Y)$:= Euler characteristic of Y .

Examples: $\chi(\text{a point}) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\text{a figure eight}) = -1$, $\chi(\text{a genus-}g \text{ orientable closed surface}) = 2 - 2g$.

Theorem (Poincaré and Hopf): if N is a compact smooth manifold with boundary ∂N , then $\chi(N) = 0 \iff \exists \text{ a nowhere-zero } C^0 \text{ vector field on } N \text{ pointing inward at } \partial N$.

⁶Figures from Quanta Magazine.

Brockett's necessary condition for stabilizability⁷

Theorem (Brockett 1983): Assume $x_* \in \mathbb{R}^n$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via some C^1 feedback $u(x)$, where $f \in C^1$. Then for any $\xi \in \mathbb{R}^n$ with $\|\xi\|$ sufficiently small, there exists x_0, u_0 such that $\xi = f(x_0, u_0)$.

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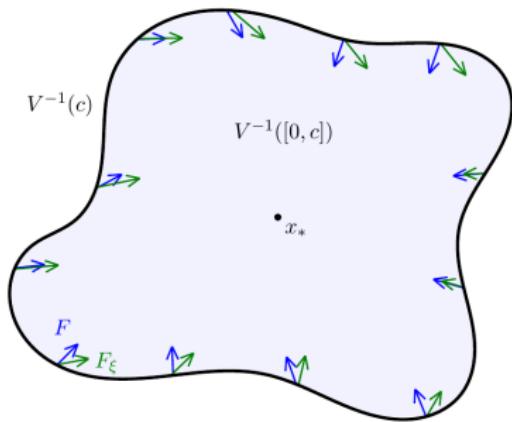
Define $F(x) := f(x, u(x))$.

Converse Lyap. theorem $\implies \exists C^\infty$ Lyapunov function $V: \underbrace{B(x_*)}_{\text{basin}} \rightarrow [0, \infty)$. Set $M_c := V^{-1}([0, c])$.

Pick any $c \in (0, \infty)$. F points inward at $\partial M_c = V^{-1}(c)$. By continuity, so does $F_\xi := F - \xi$ if $\|\xi\| \ll 1$.

Since Euler characteristic $\chi(M_c) = \chi(\{x_*\}) = 1 \neq 0$,

Theorem (Poincaré-Hopf) $\implies F_\xi$ has at least one zero x_0 in $M_c \implies \xi = f(x_0, u(x_0))$. \square



⁷Converse Lyap. theorem: F W Wilson, Jr. "Smoothing derivatives of functions and applications". (1969).

Back to 1983: the stabilization conjecture and Roger W Brockett's solution



Stabilization conjecture⁸ (pre-1983): a reasonable form of local controllability implies the existence of a smooth time-independent control law asymptotically stabilizing a point.

Example: the “nonholonomic integrator” or “Heisenberg system”

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} = f(\mathbf{x}, \mathbf{u}).$$

is accessible and controllable.⁹ → **Natural question:** can the origin be stabilized by smooth time-independent feedback? Brockett: **no**, since $(0, 0, \epsilon) \notin \text{image}(f)$ for all $\epsilon \neq 0$.

Remark: finding $\xi = (0, 0, \epsilon)$ was easy!

Theorem (Brockett 1983): Assume $x_* \in \mathbb{R}^n$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via some C^1 feedback $u(x)$, where $f \in C^1$. Then for any $\xi \in \mathbb{R}^n$ with $\|\xi\|$ sufficiently small, there exists x_0, u_0 such that $\xi = f(x_0, u_0)$.

⁸p. 2 of R W Brockett, “Asymptotic stability and feedback stabilization.” (1983).

⁹pp. 203–204 of A M Bloch, “Nonholonomic mechanics and control”, 2 ed. (2015).

Q1: Can we test for **stabilizability** of subsets **more general than points**?

What about the “dual” question of **savability** (safety; avoiding a “bad” set)?

Why care?¹⁰

Hard to *directly* show control **Lyapunov/barrier** functions don't exist; this limits the utility of converse theorems based directly on these. **We need more flexible invariants.**

For a broad class of systems, existence of a **control Lyapunov** (resp. **barrier**) **function** implies existence of a continuous stabilizing (resp. safe) feedback law.

Hence “no-go” theorems can rule out existence of a control Lyapunov (resp. barrier) function. Ruling out the need to search for one might **save valuable time and computational resources.**

Other motivation from biology: might enable **ruling out candidate “templates”** from hypothesized neuromechanical control architectures.

¹⁰Sontag (1989); Ames, Xu, Grizzle, Tabuada (2017); Full and Koditschek (1999); Revzen, Kod., Full (2009)

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Studying Brockett's proof so that we can generalize (next slide)

Theorem (Brockett 1983): Assume $x_* \in \mathbb{R}^n$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via some C^1 feedback $u(x)$, where $f \in C^1$. Then for any $\xi \in \mathbb{R}^n$ with $\|\xi\|$ sufficiently small, there exists x_0, u_0 such that $\xi = f(x_0, u_0)$.

Proof: Assume such $u(x)$ exists.

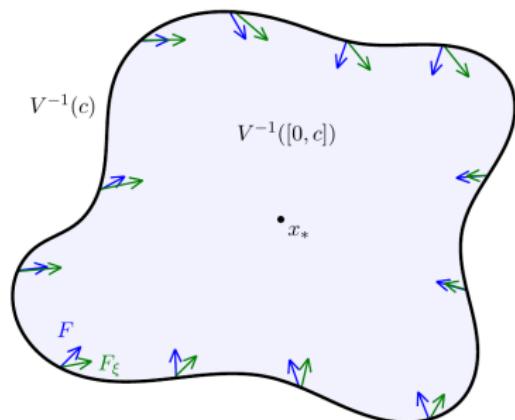
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Converse Lyap. theorem $\implies \exists C^\infty$ Lyapunov function $V: B(x_*) \rightarrow [0, \infty)$. Set $M_c := V^{-1}([0, c])$.
underbrace{B(x_*)}_{\text{basin}}

Pick any $c \in (0, \infty)$. F points inward at $\partial M_c = V^{-1}(c)$. By continuity, so does $F_\xi := F - \xi$ if $\|\xi\| \ll 1$.

Since Euler characteristic $\chi(M_c) = \chi(\{x_*\}) = 1 \neq 0$,

Theorem (Poincaré-Hopf) $\implies F_\xi$ has at least one zero x_0 in $M_c \implies \xi = f(x_0, u(x_0))$. \square



A necessary condition for stabilization of general $A^{\text{compact}} \subset M^{\text{smooth manifold}}$

Theorem (MDK, Koditschek 2022): Assume $A \subset M$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via C^1 feedback $u(x)$, where $f \in C^1$. Then $\chi(A)$ is well-defined.

Assume $\chi(A) \neq 0$. Then there exists a neighborhood $\mathcal{V} \subset TM$ of 0_{TM} such that, for any continuous vector field $X: M \rightarrow \mathcal{V} \subset TM$, there exists x_0, u_0 such that $X(x_0) = f(x_0, u_0)$.

Proof: Assume such $u(x)$ exists.

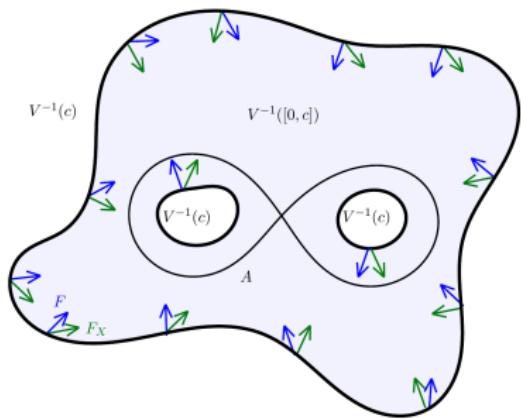
Define $F(x) := f(x, u(x))$.

Converse Lyap. theorem $\implies \exists C^\infty$ Lyapunov function $V: \underbrace{B(A)}_{\text{basin}} \rightarrow [0, \infty)$. Set $M_c := V^{-1}([0, c])$.

Pick any $c \in (0, \infty)$. F points inward at $\partial M_c = V^{-1}(c)$. By continuity, so does $F_X := F - X$ if all X values are in a small enough nbhd \mathcal{V} of 0_{TM} .

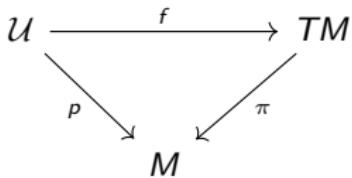
Since Euler characteristic $\chi(M_c) = \chi(A) \neq 0$,

Theorem (Poincaré-Hopf) $\implies F_X$ has at least one zero x_0 in $M_c \implies X(x_0) = f(x_0, u(x_0))$. \square



Technical remarks

- ① $u, f \in C^1$ is unnecessary; all that matters is that the closed-loop vector field F is **continuous and uniquely integrable** (e.g., locally Lipschitz).¹¹
- ② $f(x, u)$ can have the usual interpretation, or can represent a fiber-preserving map¹²



via arbitrarily general version of the “bundle picture” of control (only F matters).

- ③ If A is a compact polyhedron (or locally contractible, more generally), $\chi(A)$ is the usual Euler characteristic; otherwise, $\chi(A)$ must be defined using **CAS**¹³ cohomology.
- ④ More general theorem statement (next slide).

¹¹PH theorem works with merely continuous vector fields (cf. Pugh 1968).

¹²Brockett (1977, 1983), van der Schaft (1981), Grizzle & Marcus (1985), Lerman (2018).

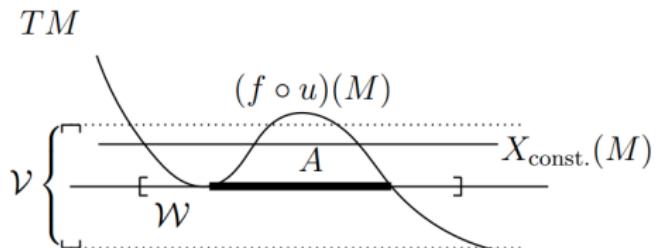
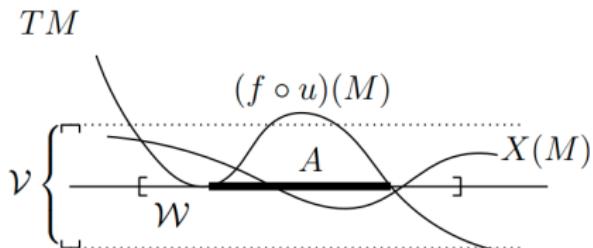
¹³Čech-Alexander-Spanier (cf. Spanier 1966, Warner 1971, Massey 1978/91, Gobbino 2001).

Intuition and a visual comparison with Brockett's proof

Theorem 1. Let (\mathcal{U}, M, p, f) be a control system and $A \subset M$ be a compact subset. Assume that A is stabilizable.

- Then the Euler characteristic $\chi(A)$ of A is well-defined according to Def. 1.
- Assume additionally that $\chi(A) \neq 0$. Then for any neighborhood $\mathcal{W} \subset M$ of A , there exists a neighborhood $\mathcal{V} \subset TM$ of the zero section $0_{TM} \subset TM$ such that, for any continuous vector field $X: \mathcal{W} \rightarrow \mathcal{V} \subset TM$ on \mathcal{W} taking values in \mathcal{V} ,

$$(8) \quad f(p^{-1}(\mathcal{W})) \cap X(\mathcal{W}) \neq \emptyset.$$



What about safety? (MDK and Koditschek 2022)

Definition. $S \subset M$ is **strictly positively invariant** for C^0 & uniquely integrable F if, for all $x_0 \in \text{cl}(S)$, the trajectory $x(t)$ with $x(0) = x_0$ satisfies $x(t) \in \text{int}(S)$ for all $t > 0$.

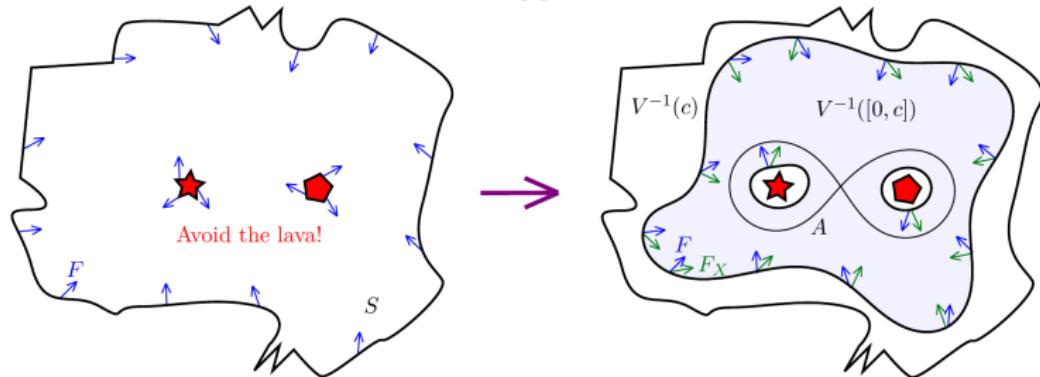
Definition. Given $\dot{x} = f(x, u)$ and precompact $S \subset M$, say S is **savable** (or can be **rendered safe**) if there exists a control law rendering S strictly positively invariant in the closed-loop.

Theorem 2. Let (\mathcal{U}, M, p, f) be a control system and $S \subset M$ be a precompact subset. Assume that S is savable.

- Then the Euler characteristic $\chi(S)$ is well-defined according to Def. 1.
- Assume additionally that $\chi(S) \neq 0$. Then there exists a neighborhood $\mathcal{V} \subset TM$ of 0_{TM} such that, if X is any continuous vector field taking values in \mathcal{V} , then

$$f(p^{-1}(S)) \cap X(S) \neq \emptyset.$$

Proof:



Examples¹⁴

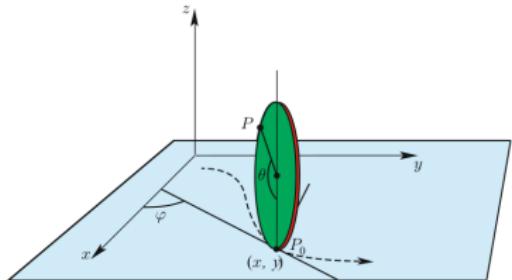


FIGURE 1.4.1. The geometry of the rolling disk.



Satellite orientation

Heisenberg system

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{z} = yu - xv$$

"Kinematic" diff.-drive robot

$$\dot{x} = \cos(\theta)u$$

$$\dot{y} = \sin(\theta)u$$

$$\dot{\theta} = v$$

$$\dot{R} = R\hat{\omega}$$

$$\dot{\omega} = (I\omega) \times \omega + g_1(R, \omega)u_1 + g_2(R, \omega)u_2.$$

Vertical rolling disk

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$

$$a^1 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) := \dot{x} - R(\cos \varphi)\dot{\theta} = 0$$

$$a^2 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) := \dot{y} - R(\sin \varphi)\dot{\theta} = 0,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = u_\varphi g^\varphi + u_\theta g^\theta + \lambda_1 a^1 + \lambda_2 a^2,$$

¹⁴Figs: Bloch "Nonholonomic mechanics and control" 2 ed. (2015); Satellite Industry Association (sia.org).

Examples

Satellite orientation (g_1, g_2 lin. indep. for simplicity)

Heisenberg system

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{z} = yu - xv$$

"Kinematic" diff.-drive robot

$$\dot{x} = \cos(\theta)u$$

$$\dot{y} = \sin(\theta)u$$

$$\dot{\theta} = v$$

$$\begin{aligned}\dot{R} &= R\hat{\omega} \\ \dot{\omega} &= (I\omega) \times \omega + g_1(R, \omega)u_1 + g_2(R, \omega)u_2.\end{aligned}\tag{3}$$

(1) **Vertical rolling disk**

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$

$$a^1 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) := \dot{x} - R(\cos \varphi)\dot{\theta} = 0\tag{4}$$

$$a^2 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) := \dot{y} - R(\sin \varphi)\dot{\theta} = 0,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = u_\varphi g^\varphi + u_\theta g^\theta + \lambda_1 a^1 + \lambda_2 a^2,$$

Consider the **adversaries**: $X_\epsilon = (0, 0, \epsilon)$ for (1), $X_\epsilon = \epsilon(\sin \theta, -\cos \theta, 0)$ for (2), $X_\epsilon = (0, g_1 \times g_2)$ for (3), $X_\epsilon = [\text{any lift of } \partial_x \text{ to } \mathcal{D}]$ for (4). Note $X_\epsilon \rightarrow 0$ uniformly on compact sets as $\epsilon \rightarrow 0$.

Hence MDK& Koditschek (2022) \implies in all examples above, A^{cpct} cannot be asymptotically stabilized via C^1 feedback if $\chi(A) \neq 0$; $S^{\text{pre-cpct}}$ cannot be rendered safe if $\chi(S) \neq 0$.

Remark: it was very easy to find the X_ϵ , hence to apply MDK & Koditschek (2022).

Examples: remarks

Satellite orientation (g_1, g_2 lin. indep. for simplicity)

Heisenberg system

$$\dot{x} = u$$

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$$\dot{x} = \cos(\theta)u$$

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$$\dot{\theta} = v$$

$$\begin{aligned}\dot{R} &= R\hat{\omega} \\ \dot{\omega} &= (I\omega) \times \omega + g_1(R, \omega)u_1 + g_2(R, \omega)u_2.\end{aligned}\tag{7}$$

(5) Vertical rolling disk

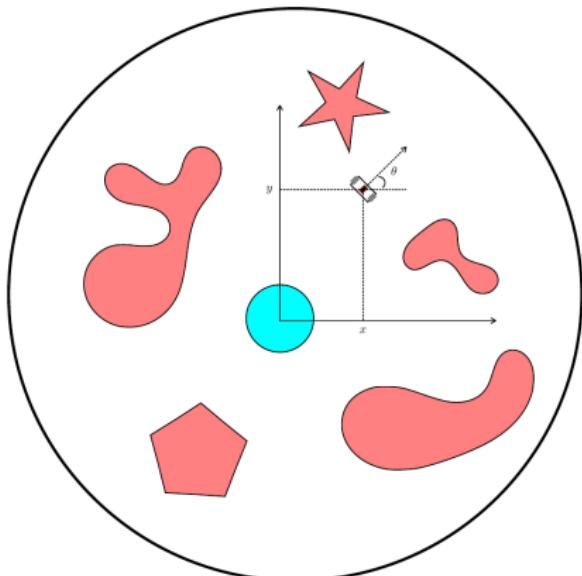
$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$

$$\begin{aligned}a^1 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) &:= \dot{x} - R(\cos \varphi)\dot{\theta} = 0 \\ a^2 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) &:= \dot{y} - R(\sin \varphi)\dot{\theta} = 0,\end{aligned}\tag{8}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = u_\varphi g^\varphi + u_\theta g^\theta + \lambda_1 a^1 + \lambda_2 a^2,$$

- ① If A^{cpct} is an asymptotically stable topological submanifold with (or without) boundary for (5), A is homeomorphic to either a circle, cylinder, Möbius band, or a 3-manifold with boundary. Ditto for (6).
- ② A very special case of our observation re: (7) is **Byrnes and Isodori (1991)**, Cor. 1.
- ③ **Bloch, Reyhanoglu, McClamroch (1992)** give sufficient conditions for A a boundaryless equilibrium manifold to be asymptotically stabilizable for (8). Our analysis \implies if A is compact, a necessary condition is A is a **torus or Klein bottle**.

Safety example



Differential drive robot + obstacles for simplicity assume we can directly control velocities:

$$\dot{x} = u_1 \cos \theta$$

$$\dot{y} = u_1 \sin \theta$$

$$\dot{\theta} = u_2$$

Goal: point camera within ± 179 degrees of the origin while “strictly” avoiding obstacles.

- We just saw that **MDK & Kod. (2022) safety theorem** \implies this is impossible.

Thus, if we insist on accomplishing the goal with pure state feedback, we must use discontinuous feedback. (This gives us additional motivation to study hybrid systems!)

Related work on stabilizability necessary conditions

Stabilizability of points

- Time-independent feedback: Brockett (1983); Zabczyk (1989); Coron (1990); Orsi, Praly, Mareels (2003); Ishikawa, Sampei (1998)
- Time-varying feedback: Coron (1992)
- exponential stabilizability: Gupta, Jafari, Kipka, Mordukhovich (2018); Christopherson, Mordukhovich, Jafari (2020)
- Discrete-time: Lin, Byrnes (1994); Kalabić, Gupta, Di Cairano, Bloch, Kolmanovsky (2017)
- Discontinuous systems: Ryan (1994); Clarke, Ledyaev, Stern (1998)

Stabilizability of more general subsets

- global asymptotic stability of compact $A \subset \mathbb{R}^n$: Byrnes (2008)
- local asymptotic stability of submanifolds $A \subset \mathbb{R}^n$: Mansouri (2007, '10, '13, '15)
- asymptotic stability of $A \subset M^{\text{smooth manifold}}$ with a fixed basin: Baryshnikov and Shapiro (2014), Baryshnikov (2021)

MDK & Kod (2022): arbitrary compact $A \subset M^{\text{smooth manifold}}$, basin not fixed; safety.

With the exception of B&S (2014), B (2021), all of these results require $\chi(A) \neq 0$.

Q2: What about stabilizability/safety tests for sets with **zero** Euler characteristic?

Why care?

Periodic orbits have zero Euler characteristic.

As do all **compact Lie groups** and **odd-dimensional closed manifolds!**

Periodic orbits are important in the theory of **locomotion**¹⁵¹⁶, among many other things.

¹⁵Westervelt, Grizzle, Chevallereau, Choi, Morris. "Feedback control of dynamic bipedal robot locomotion" (2007).

¹⁶Revzen and Kvalheim. "Data-driven models of legged locomotion." (2015).

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- **Q1:** Can we “test” for stabilizability and savability, in general, in nonlinear spaces?

2 Stabilizability and savability tests: the case of nonzero Euler characteristic

- **Answer to Q1:** yes!
- Examples
- **Q2:** What about stabilizability tests for periodic orbits (and more)?

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Toward the $\chi = 0$ case: inspiration from other literature on the $\chi \neq 0$ case

Theorem (Coron 1990, Mansouri 2007/10): Assume $A \subset \mathbb{R}^n$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via C^1 feedback $u(x)$, where $f \in C^1$. Then $\chi(A)$ is well-defined, and, defining $\Sigma_\epsilon := \{(x, u) : d(x, A) < \epsilon \text{ and } f(x, u) \neq 0\}$,

$$\forall \epsilon \in (0, +\infty] : f_*(H_{n-1}(\Sigma_\epsilon)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}.$$

Remark: like our results, this theorem gives zero information if $\chi(A) = 0$.

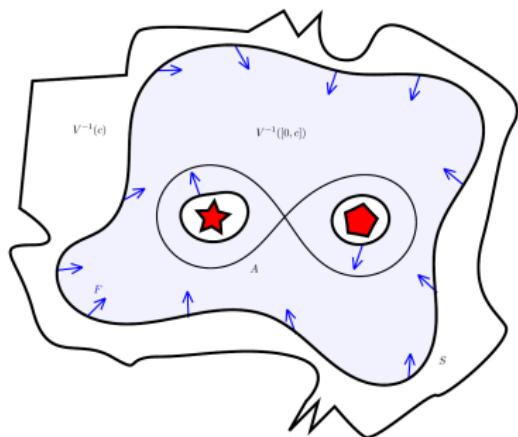
Proof: Assume such $u(x)$ exists and define

$$F(x) := f(x, u(x)).$$

Converse Lyap. theorem $\implies \exists C^\infty$ Lyapunov function $V: \underbrace{B(A)}_{\text{basin}} \rightarrow [0, \infty)$. Set $M_c := V^{-1}([0, c])$.

Pick any $c \in (0, \infty)$. F points inward at $\partial M_c = V^{-1}(c)$. By approximation we may assume F has isolated zeros.

Degree of a map $\partial W^k \rightarrow N^k$ is zero if map extends to W . Hence $\deg(\frac{F}{\|F\|}|_{\partial M_c} : \partial M_c \rightarrow \mathbb{S}^{n-1}) = \chi(M_c) = \chi(A)$ by this and the Poincaré-Hopf theorem. \square



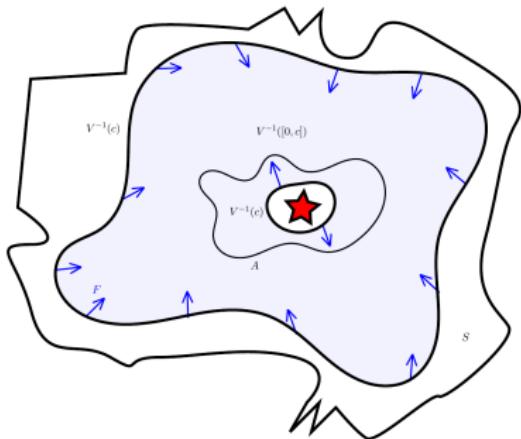
What to do when $\chi(A) = 0$? First of two results **hot off the press**

Observation: Coron's/Mansouri's formulations and proofs crucially depend on $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$; e.g., we needed this for the following map to be well-defined:

$$\frac{F}{\|F\|}|_{\partial M_c}: \partial M_c \longrightarrow \underbrace{\mathbb{S}(\mathbb{R}^n)}_{\text{unit sphere bundle}} \cong \mathbb{R}^n \times \mathbb{S}^{n-1} \xrightarrow{\text{pr}_2} \mathbb{S}^{n-1}.$$

Idea: the projection pr_2 is not defined in general and throws away information when it is. Let's not do that.

Define: $S(N) := TN_0 := TN \setminus 0_{TN}$ the **slit tangent bundle** of a smooth manifold N , and $\mathcal{W}_A := \mathcal{W} \setminus A$.

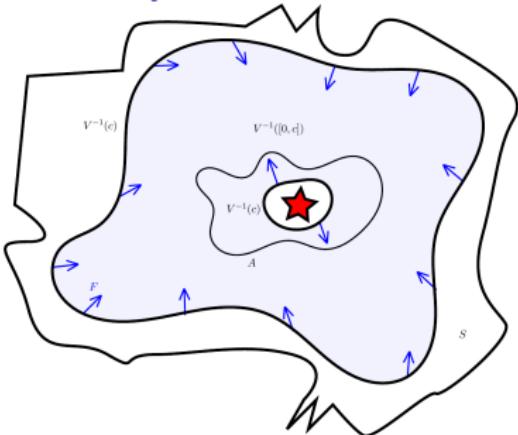


Theorem (MDK 2022, in prep). Assume $A \subset M^n$ can be rendered **asymptotically stable** for $\dot{x} = f(x, u)$ via C^1 feedback $u(x)$, where $f \in C^1$. Then the real (CAS) cohomology of A is finite-dimensional, and for each $\mathcal{W}^{\text{open}} \supset A$ there is an explicit $\sigma_{\mathcal{W}_A} \in H_{n-1}(S(\mathcal{W}_A))$

s.t. $\sigma_{\mathcal{W}_A} \in f_*(H_{n-1}(\Sigma_{\mathcal{W}_A}))$, where $\Sigma_{\mathcal{W}_A} := \{(x, u) : x \in \mathcal{W}_A, f(x, u) \neq 0\}$.

What to do when $\chi(A) = 0$? Two results **hot off the press**

Define: $S(N) := TN_0 := TN \setminus 0_{TN}$ the **slit tangent bundle** of a smooth manifold N , and $\mathcal{W}_A := \mathcal{W} \setminus A$.



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Moreover, $\mathcal{W} \mapsto \sigma_{\mathcal{W}_A}$ is natural w.r.t. inclusion (direct system of abelian groups).

Theorem (MDK 2022, in prep). Assume $S \subset M^n$ can be rendered **safe** for $\dot{x} = f(x, u)$ Then the real (CAS) cohomologies of $S, \partial S$ are finite-dim, & \exists explicit $\sigma_{\partial S} \in H_{n-1}(S(M)|_{\partial S})$ such that $\sigma_{\partial S} \in f_*(H_{n-1}(\Sigma_{\partial S}))$, where $\Sigma_{\partial S}, \Sigma_{\mathcal{W}_A}$ defined similarly.

Why am I excited about the homological results? Two reasons.

Theorem (stabilizability—Coron 1990, Mansouri 2007/10): ...homology...

Theorem (stabilizability—MDK 2022, in prep). ...homology...

Theorem (safety—MDK 2022, in prep). ...homology...

Tests of **Brockett, MDK-Koditschek** are simple to apply in many examples when we can “see what's going on”, but what to do beyond the limits of intuition, in high dimensions?

(1) **Automated numerical algorithms** can be designed to **algorithmically verify** the **homological conditions** via tools like simplicial approximation theorem, etc, or more refined methods.¹⁷ It's “just” **linear algebra**.

(2) *Finally...*recent two theorems yield tests for general A, S including A a **periodic orbit!**

¹⁷Kaczynski, Mischaikow, Mrozek. “**Computational homology**” (2004).

Example: periodic orbit non-stabilizability

Theorem (MDK 2022, in prep). Assume $A \subset M^n$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via C^1 feedback $u(x)$, where $f \in C^1$. Then the real (CAS) cohomology of A is finite-dimensional, and for each $\mathcal{W}^{\text{open}} \supset A$ there is an explicit $\sigma_{\mathcal{W}_A} \in H_{n-1}(S(\mathcal{W}_A))$

$$\text{s.t. } \sigma_{\mathcal{W}_A} \in f_*(H_{n-1}(\Sigma_{\mathcal{W}_A})), \quad \text{where} \quad \Sigma_{\mathcal{W}_A} := \{(x, u) : x \in \mathcal{W}_A, f(x, u) \neq 0\}.$$

Example (Artstein circles). Consider $z = x + iy \in \mathbb{C}$, $u \in \mathbb{R}$ and

$$\dot{x} = (x^2 - y^2)u, \quad \dot{y} = 2xyu, \quad \text{or} \quad \dot{z} = z^2u =: f(z, u). \quad (9)$$

Question: Can $A = \mathbb{S}^1$ (e.g., a periodic orbit) be asymptotically stabilized for (9) ?

Computations: Set $\mathcal{W} := \mathbb{C} \setminus \{0\}$, so $\mathcal{W}_A = \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\}) \simeq \mathbb{S}^1 \times \mathbb{S}^1$ and $S(\mathcal{W}_A) \simeq \mathbb{T}^2 \times \mathbb{T}^2$, with $\sigma_{\mathcal{W}_A} \cong (1, 1, 1, 1) \in \mathbb{Z}^2 \oplus \mathbb{Z}^2 \cong H_1(\mathbb{T}^2 \times \mathbb{T}^2) \cong H_1(S(\mathcal{W}_A))$.

On the other hand, $f^{-1}(0) = \{z\text{-plane}\} \cup \{u\text{-axis}\}$, so $\Sigma_{\mathcal{W}_A} = (\mathcal{W}_A \times \mathbb{R}) \setminus f^{-1}(0) = \{u > 0, z \neq 0\} \cup \{u < 0, z \neq 0\} \simeq \mathbb{S}^1 \sqcup \mathbb{S}^1$, and

$$f_*(H_1(\Sigma_{\mathcal{W}_A})) = f_*(\mathbb{Z} \oplus \mathbb{Z}) = (1, 2, 1, 2)\mathbb{Z} \not\ni \sigma_{\mathcal{W}_A}.$$

Answer: from the theorem, no! **Answer to Q2:** yes, apparently we can detect periodic orbit non-stabilizability! **But wait a minute...** was it even possible to make \mathbb{S}^1 invariant? (Yes: $u \equiv 0$.) But what about make \mathbb{S}^1 a periodic orbit?

A no-go no-go theorem: no periodic orbit homotopical invariants beyond invariance

But wait a minute... was it even possible to make \mathbb{S}^1 invariant? (Yes: $u \equiv 0$.) But what about make \mathbb{S}^1 a periodic orbit? No. Perhaps we merely “detected” this?

Theorem: (MDK 2022, in prep). Assume $(\Gamma \approx \mathbb{S}^1) \subset M^n$ can be rendered a periodic orbit for $\dot{x} = f(x, u)$ via C^1 feedback $u(x)$, where $f \in C^1$, and define $F(x) := f(x, u(x))$. Then for all sufficiently small neighborhoods $\mathcal{W} \supset \Gamma$, $F|_{\mathcal{W}}: \mathcal{W} \rightarrow \mathcal{S}(\mathcal{W})$ is homotopic through such maps to one asymptotically stabilizing Γ . Similarly for $F|_{\mathcal{W}_{\Gamma}}$.

Proof: any such F is homotopic to $(\dot{\theta}, \dot{y}) = (1, 0)$ on sufficiently small tubular $\mathcal{W} \supset \Gamma$. \square

Intuitively: it follows that f will “pass” any “asymptotic stabilizability test” that depends only on the fiber-preserving homotopy class of $f|_{\Sigma_{\mathcal{W}}}: \Sigma_{\mathcal{W}} \rightarrow \mathcal{S}(\mathcal{W})$ (or $f|_{\Sigma_{\mathcal{W}_{\Gamma}}}$) if f can render Γ a periodic orbit (even an unstable one).

Future periodic orbit (and $\chi = 0$?) work: to test for asymptotic stabilizability of a PO we already know to be realizable, we need “harder” topological (or perhaps geometric) invariants! Homotopy classes of sections $\mathcal{W} \rightarrow \mathcal{S}(\mathcal{W})$ or $\mathcal{W}_A \rightarrow \mathcal{S}(\mathcal{W}_A)$ are too “soft” to give information beyond invariance for POs. Implies the same of all necessary conditions discussed in this talk.

What about **hybrid** systems?

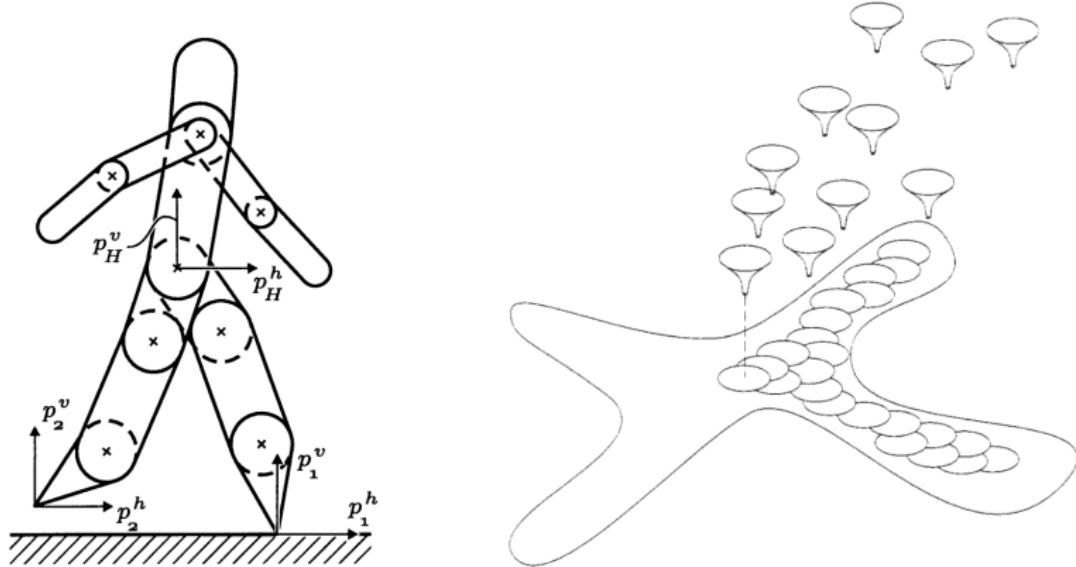


Figure: Left: Westervelt, Grizzle, Koditschek (2003). Right: Burridge, Rizzi, Koditschek (1999).

Q3: can we devise stabilizability/safety tests for hybrid systems?

Outline

1 Introduction

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2 Stabilizability and savability tests: the case of nonzero Euler characteristic

- **Answer to Q1:** yes!
- Examples
- **Q2:** What about stabilizability tests for periodic orbits (and more)?

3 Stabilizability and savability tests: toward the case of zero Euler characteristic

- **Answer to Q2:** yes! (in prep)
- **Q3:** How might such tests be devised for *hybrid* systems?

4 Conley and the wild west of hybrid systems

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4 Conley and the wild west of hybrid systems

Topological and metric hybrid systems (THS and MHS)

Definition

A **topological hybrid system (THS)** $H = (X, F, G, \varphi, r)$ consists of:

States a topological *state space* X whose points are the possible states of the system.

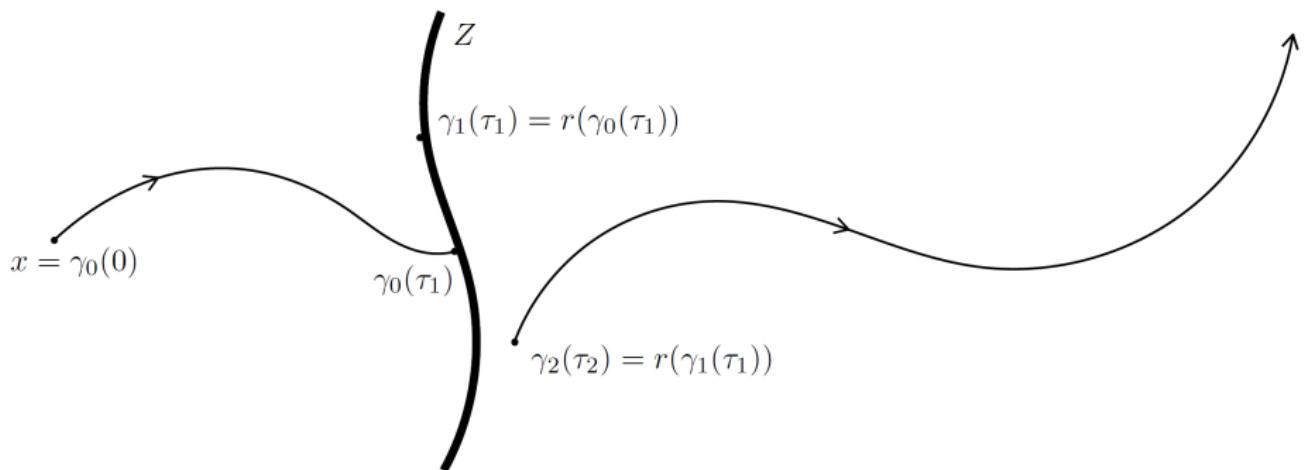
Continuous-time dynamics a continuous local semiflow φ on an open *flow set* $F \subseteq X$.

Discrete-time dynamics a closed *guard set* $G \subseteq X$ equipped with a continuous *reset map* $r : G \rightarrow X$.

If the topology of X arises from an extended metric $d(\cdot, X) : X \times X \rightarrow [0, \infty]$, we say that $(H, d(\cdot, \cdot))$ is a **metric hybrid system (MHS)**.

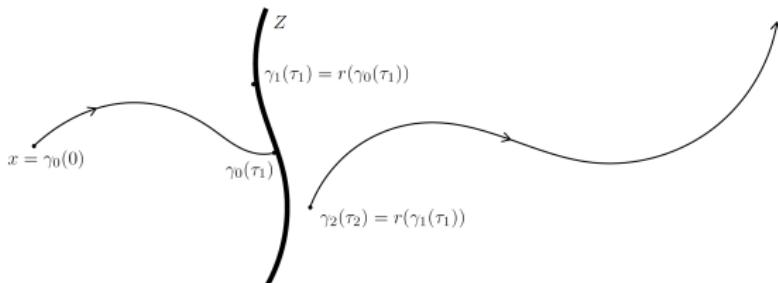
Motivated by Def. 2 of Johnson et al. ('16)—which is in turn motivated by, e.g., Simić et al. ('00,'05)—but we **discard all smoothness assumptions** unnecessary for topological dynamics.

Execution of a THS $H = (X, F, G, \varphi, r)$



Given a hybrid system from the literature defined in terms of **several “modes”**: taking the disjoint union of modes, guards, local semiflows, and resets **reduces to our definition** (“forgets” any underlying “directed graph structure”); see [KGK21, Rem. 1].

Executions of a THS $H = (X, F, G, \varphi, r)$

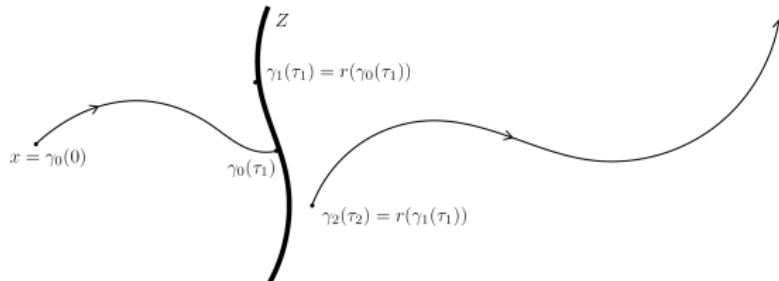


Definition

An **execution** in H is a tuple $\chi = (N, \tau, \gamma)$ of

- 1 **Jump times:** a nondecreasing sequence $\tau = (\tau_j)_{j=0}^{N+1} \subseteq \mathbb{R} \cup \{+\infty\}$ where $N \in \mathbb{N}_{\geq 0} \cup \{+\infty\}$, $\tau_0 = 0$, and $(\tau_j)_{j=0}^N \subseteq \mathbb{R}$.
- 2 **Flow arcs:** a sequence of continuous maps $\gamma = (\gamma_j: T_j \rightarrow I)_{j=0}^N$ with $[T_j, T_{j+1}] \subseteq T_j \subseteq [\tau_j, \tau_{j+1}] \cap \mathbb{R}$, $\gamma_j([\tau_j, \tau_{j+1}]) \subseteq F$, and such that the restriction $\gamma_j|_{[\tau_j, \tau_{j+1}]}: [\tau_j, \tau_{j+1}] \rightarrow F$ is a trajectory segment for the local semiflow φ . For all $0 \leq j < N$, we additionally require that $T_j = [\tau_j, \tau_{j+1}]$, $\gamma_j(\tau_{j+1}) \in G$, and $\gamma_{j+1}(\tau_{j+1}) = r(\gamma_j(\tau_{j+1}))$.

Definitions regarding a THS $H = (X, F, G, \varphi, r)$



The **stop time** of χ is

$$\tau^{\text{stop}} := \begin{cases} \tau_{N+1} & N < \infty \\ \lim_{j \rightarrow \infty} \tau_j & N = \infty \end{cases}.$$

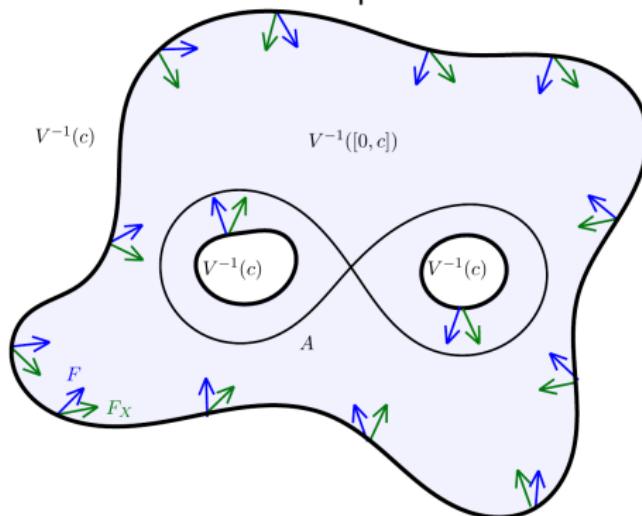
If $\tau^{\text{stop}} = \infty$, we say that χ is an **infinite execution**.

If $\tau^{\text{stop}} < \infty$ but χ has infinitely many jumps ($N = \infty$), then χ is a **Zeno execution**.

$\chi = (N, \tau, \gamma)$ is a **maximal execution** if, for any execution $\chi' = (N', \tau', \gamma')$ with $\gamma'_0(0) = \gamma_0(0)$ and each γ_j equal to $\gamma'_{j'}|_{T_j}$ for some $j' \in \{0, \dots, N'\}$, $\chi = \chi'$.

Motivation: recall the role of Lyapunov functions

Motivation: recall that every proof of every necessary condition for stability/safety in this talk involved a picture like this.



And every picture like this involves a Lyapunov function V . Thus, it seems like **converse Lyapunov theory is fundamentally important for deriving stab./safety necessary conditions.**

How about this, then: let's generalize the **ultimate converse Lyapunov theorem** to hybrid systems:

Conley's fundamental theorem of dynamical systems.

Conley's Fundamental Theorem of Dynamical Systems¹⁸

For this talk, just think of **chain recurrent set** as “generalized steady-state”; it includes equilibria, (quasi-)periodic orbits, chaotic attractors, etc.

Theorem (Conley '78, Franks '88, Hurley '95)

Let Φ be a continuous semi-dynamical system on a compact metric space X . Then the **chain recurrent set** $R(\Phi)$ admits a **Conley decomposition**:

$$X \setminus R(\Phi) = \bigcup \{B(A) \setminus A \text{ is asymptotically stable for } \Phi\}.$$

Furthermore, $x, y \in X$ are chain equivalent if and only if either $x, y \in A$ or $x, y \notin B(A)$ for every asymptotically stable set A .

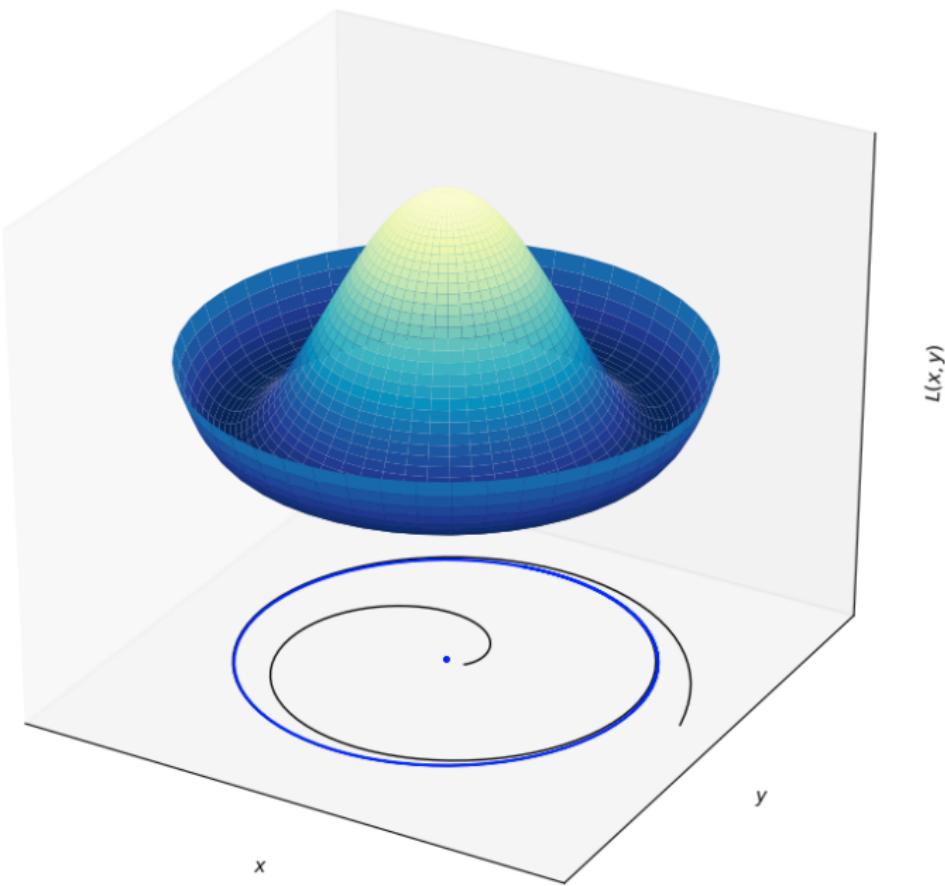
Theorem (Conley '78, Franks '88, Hurley '98, Patrão '11)

Every continuous semi-dynamical system on a compact metric space has a **complete Lyapunov function** which strictly decreases on trajectories outside $R(\Phi)$ and is constant on trajectories within $R(\Phi)$.

For **smooth** complete Lyapunov functions: see Fathi and Pageault (2019).

¹⁸Norton (1995) made a case for this title.

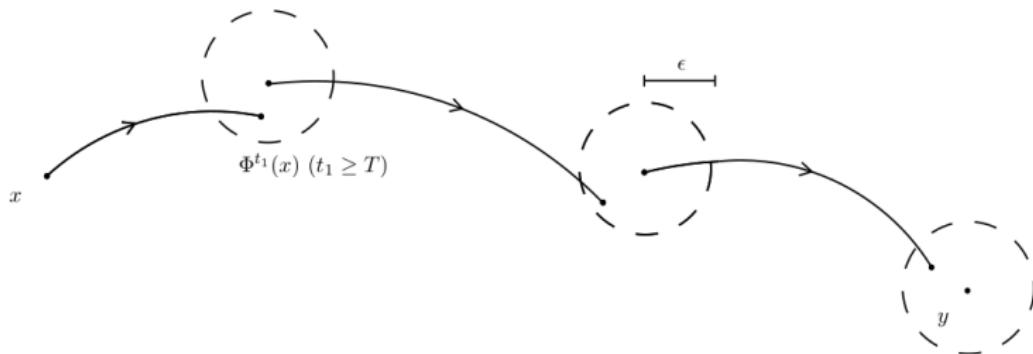
Example: $\dot{\rho} = \rho(1 - \rho)$, $\dot{\theta} = 1$; $R(H) = \{\mathbf{0}\} \cup S^1$ (blue)



(ϵ, T) -chains, chain equivalence, and chain recurrence



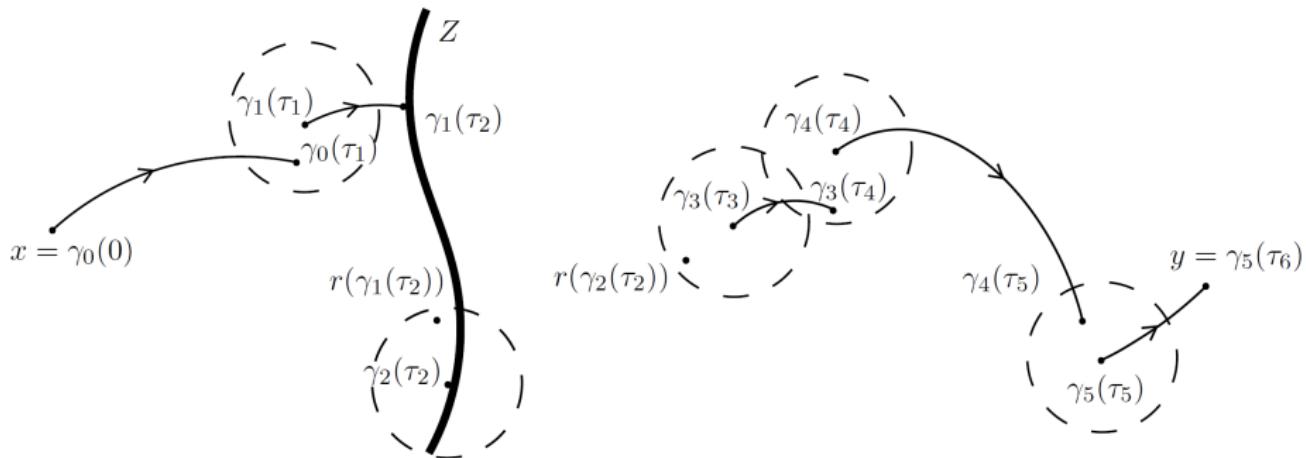
"If...**rough equations** are to be of use it is **necessary to study them in rough terms.**"—Charles C Conley.¹⁹



x is **chain equivalent** to y if $\forall \epsilon, T > 0, \exists (\epsilon, T)$ -chain from x to y and vice versa. x is **chain recurrent** if x is chain equivalent to itself.

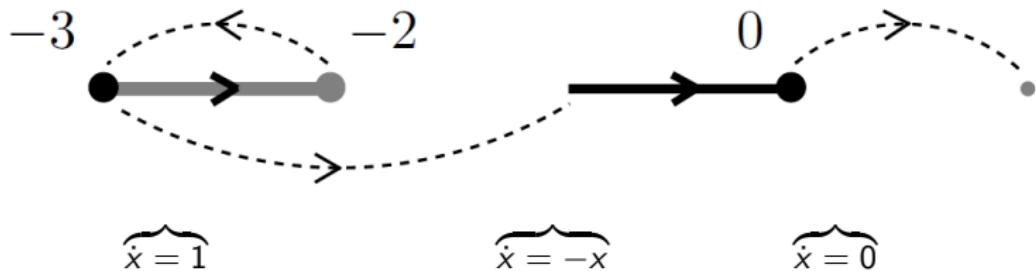
¹⁹"Isolated invariant sets and the Morse index." (1978).

Hybrid (ϵ, T) -chains and chain recurrence



Once hybrid chains are defined, the rest matches the classical definitions: x is **chain equivalent** to y if $\forall \epsilon, T > 0, \exists (\epsilon, T)$ -chain from x to y and vice versa. x is **chain recurrent** if x is chain equivalent to itself.

What can go wrong? Example of four bad things



- ① $\omega(-1) = \{0\}$ is not forward invariant.
- ② $\omega(-1) = \{0\}$ is not contained in $R(H)$.
- ③ $R(H)$ is not forward invariant since $r(r(-2)) = -1 \notin R(H)$.
- ④ $R(H) = (-3, -2] \cup \{1\}$ is not closed in X .

Eliminating the bad: the **trapping guard condition** on $H = (X, F, G, \varphi, r)$

- Define the **maximum flow time** $\mu: X \rightarrow [0, +\infty]$ by

$$\mu(x) := \begin{cases} \sup\{t \in [0, \infty) : (t, x) \in \text{dom}(\varphi)\}, & x \in F \\ 0, & x \notin F \end{cases}. \quad (10)$$

- H satisfies the **trapping guard condition** if G has a neighborhood $U \subseteq X$ with continuous retraction $\rho: U \rightarrow G$ ($\rho|_G = \text{id}_G$) satisfying:²⁰
 - ▶ $\mu|_U$ is continuous
 - ▶ For all $x \in U \cap F$:

$$\rho(x) := \lim_{t \nearrow \mu(x)} \varphi^t(x) \in G.$$

- Propositions 1, 2 of [KGK21] show that two broad, physically relevant classes of hybrid systems appearing in the literature satisfy this condition.

²⁰This def. is equivalent to the more general one in [KGK21, Def. 11] if G is Hausdorff.

Main results from Kvalheim, Gustafson, Koditschek (2021)

Let $H = (X, F, G, \varphi, r)$ be a deterministic MHS. Assume that X is compact and that G is a trapping guard (TG). Further suppose that, for every $x \in X$, there is an infinite or Zeno execution starting at x .

Theorem (Conley's decomposition theorem for MHS)

The hybrid chain recurrent set $R(H)$ admits a **Conley decomposition**:

$$X \setminus R(H) = \bigcup \{B(A) \setminus A \mid A \text{ is asymptotically stable for } H\}. \quad (11)$$

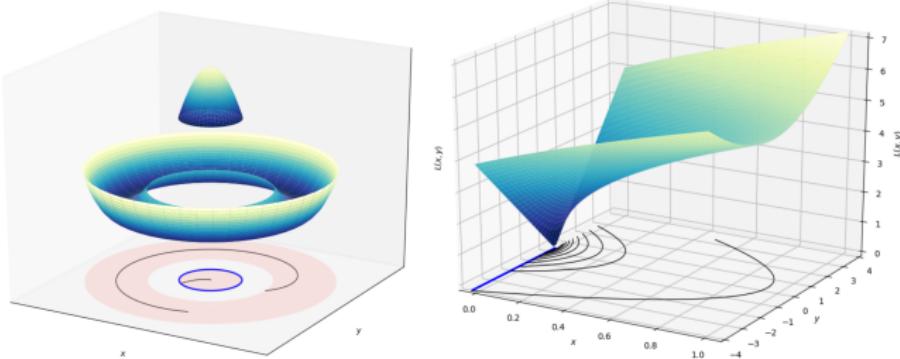
Furthermore, $x, y \in X$ are chain equivalent if and only if either $x, y \in A$ or $x, y \notin B(A)$ for every asymptotically stable set A .

Theorem (Conley's fundamental theorem for MHS)

There exists a **complete Lyapunov function** for H (def. on next slide).

Remarks. \exists examples where TG fails yet conclusions hold, but also examples where TG fails and conclusions fail \implies sufficient condition only. \rightarrow **future work**.

Complete Lyapunov function for MHS $H = (X, F, G, \varphi, r)$

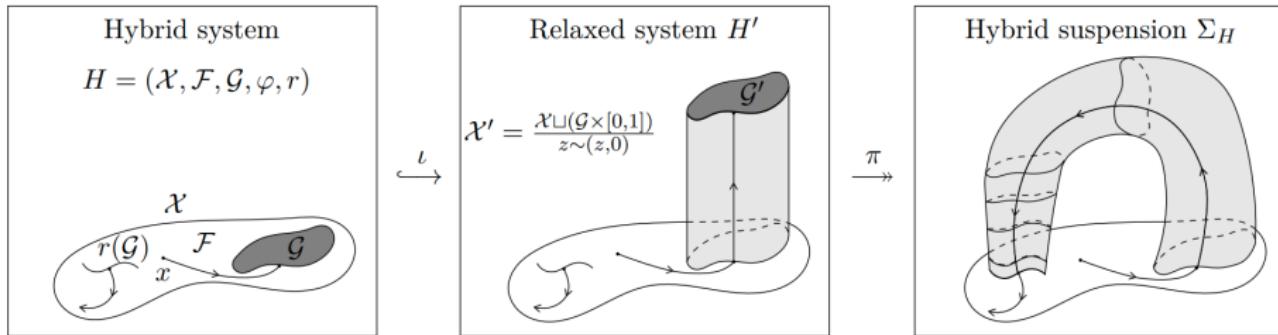


Definition (Hybrid complete Lyapunov function $L: X \rightarrow \mathbb{R}$)

L is continuous and satisfies the following.

- For every $x \in F \setminus R(H)$, $\chi \in \mathcal{E}_H(x)$, $t > 0$, and $y \in \chi(t)$, $L(y) < L(x)$.
- If $x \in G \setminus R(H)$, then $L(r(x)) < L(x)$.
- for all $x, y \in R(H)$: x and y are chain equivalent $\iff L(x) = L(y)$.
- $L(R(H))$ is nowhere dense in \mathbb{R} .

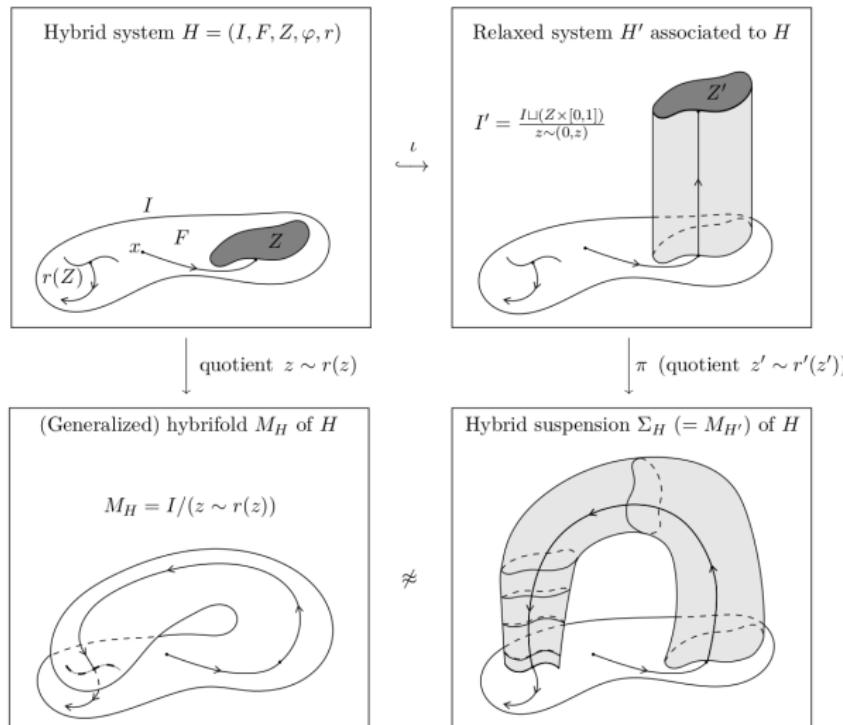
Proof idea: we introduce and use the hybrid suspension



- Generalizes the classical suspension of a discrete-time system (Smale '67). (Related work: Ames et al. '05, Burden et al. '15).
- Suspension semiflow Φ_H is well-defined & continuous \iff trapping guard condition holds (our motivation).
- Work required at the point-set level to show that
 - H chain equivalence classes correspond to Φ_H chain equivalence classes,
 - H attracting-repelling pairs correspond to Φ_H attracting-repelling pairs, etc.

In contrast: the “hybridfold” would **not** work

for several reasons; see [KGK21, Rem. 17, 20, App. A.4].



Hybridfold: introduced by Simić et al. ('00, '05).

Example: the bouncing ball $H = (X, F, G, \varphi, r)$

- $X := \{(x, \dot{x}) \mid x \geq 0\}$, $G := \{(0, \dot{x}) \mid \dot{x} \leq 0\}$, $F := X \setminus G$, $r(0, \dot{x}) := -d\dot{x}$ with $0 < d < 1$, φ determined by $\ddot{x} = -g$.
- Can directly show that the maximum flow time $\mu: X \rightarrow [0, +\infty]$ is

$$\mu(x, \dot{x}) = \frac{\dot{x} + \sqrt{\dot{x}^2 + 2xg}}{g},$$

is continuous, & also other parts of the trapping guard condition hold.

- Energy level sets $X_{E_0} := \{\frac{1}{2}\dot{x}^2 + gx \leq E_0\}$ are compact, forward invariant. Hence the hybrid system restricted to X_{E_0} is well-defined and satisfies the hypotheses of our main theorems.
- Alternatively, do less work using Prop. 2: trivially $\ddot{x} < 0$ when $x = \dot{x} = 0$, so Prop. 2
⇒ same conclusion.
- The only nontrivial attracting-repelling pair is $(\mathbf{0}, \emptyset)$, $R(H) = \{\mathbf{0}\}$, and the graph of a complete Lyapunov function is shown below.

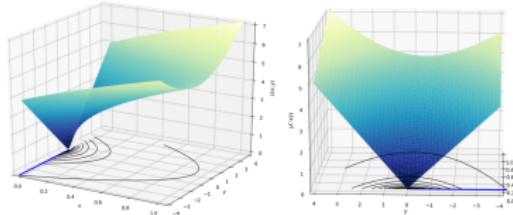


Figure: bouncing ball $H = (X, F, G, \varphi, r)$

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- **Answer to Q1:** yes!
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4 Conley and the wild west of hybrid systems

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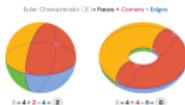
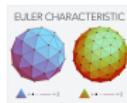
4 Conley and the wild west of hybrid systems

- **Answer to Q3:** I believe topological/Conley-theoretic methods are part of the puzzle.
- Another potential puzzle piece: Kvalheim. “**Poincaré-Hopf theorem for hybrid systems.**” arXiv:2108.07434 (2021).

When can hybrid (and non-hybrid) systems operate safely?

A brief primer on the Euler characteristic⁵

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



Notation: $\chi(Y)$:= Euler characteristic of Y .

Examples: $\chi(\text{a point}) = 1$, $\chi(S^1) = 0$, $\chi(S^2) = 2$, $\chi(\text{a figure eight}) = -1$, $\chi(\text{a genus-}g \text{ orientable closed surface}) = 2 - 2g$.

Theorem (Poincaré and Hopf): If N is a compact smooth manifold with boundary ∂N , then $\chi(N) = 0 \iff \exists \text{ a nowhere-zero } C^1 \text{ vector field on } N \text{ pointing inward at } \partial N$.

⁵Figures from Quanta Magazine.

What about safety? (MDK and Koditschek 2022)

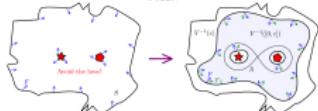
Definition: $S \subset M$ is strictly positively invariant for C^k & uniquely integrable F if, for all $x_0 \in c(S)$, the trajectory $x(t)$ with $x(0) = x_0$ satisfies $x(t) \in \text{int}(S)$ for all $t > 0$.

Definition: Given $\dot{x} = f(x, u)$ and precompact $S \subset M$, say S is rendered safe if there exists a control law rendering S strictly positively invariant in the closed-loop.

Theorem 2. Let (U, M, p, f) be a control system and $S \subset M$ be a precompact subset. Assume that S is stable.

- Then the Euler characteristic $\chi(S)$ is well-defined according to Def 1.
- Assume additionally that $\chi(S) \neq 0$. Then there exists a neighborhood $V \subset TM$ of 0_{TM} such that, if X is any continuous vector field taking values in V , then $f(p^{-1}(S)) \cap X(S) \neq \emptyset$.

Proof:



A necessary condition for stabilization of general $A^{\text{compact}} \subset M^{\text{smooth}}$ manifold

Theorem (MDK, Koditschek 2022): Assume $A \subset M$ can be rendered asymptotically stable for $\dot{x} = f(x, u)$ via C^1 feedback $u(x)$, where $f \in C^1$. Then $\chi(A) \neq 0$.

Assume $\chi(A) \neq 0$. Then there exists a neighborhood $V \subset TM$ of 0_{TM} such that, for any continuous vector field $X: M \rightarrow V \subset TM$, there exists x_0, u_0 such that $X(x_0) = f(x_0, u_0)$.

Proof: Assume such $u(x)$ exists.

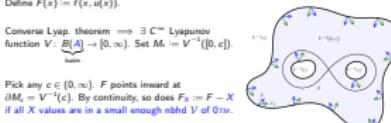
Define $F(x) := f(x, u(x))$.

Converse Lyapunov theorem $\implies \exists C^{\infty}$ Lyapunov function $V: \underbrace{B(A)}_{\text{neighborhood}} \rightarrow [0, \infty)$. Set $M := V^{-1}([0, c])$.

Pick any $c \in (0, \infty)$. F points inward at $\partial M = V^{-1}(c)$. By continuity, so does $F_x := F - X$ if all X values are in a small enough nbhd V of 0_{TM} .

Since Euler characteristic $\chi(M_c) = \chi(A) \neq 0$,

Theorem (Poincaré-Hopf) $\implies F_x$ has at least one zero x_0 in $M_c \implies X(x_0) = f(x_0, u(x_0))$. \square



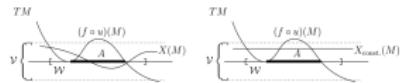
Intuition and a visual comparison with Brockett's proof

Theorem 1. Let (U, M, p, f) be a control system and $A \subset M$ be a compact subset. Assume that A is stabilizable.

- Then the Euler characteristic $\chi(A)$ is well-defined according to Def 1.
- Assume additionally that $\chi(A) \neq 0$. Then for any neighborhood $V \subset M$ of A , there exists a neighborhood $V \subset TM$ of the zero section 0_{TM} such that, for any continuous vector field $X: M \rightarrow V \subset TM$ on W taking values in V ,

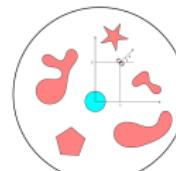
(8)

$$f(p^{-1}(W)) \cap X(W) \neq \emptyset.$$



Safety example

Safety example



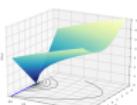
Goal: point camera within vicinity of the origin while "strictly" avoiding obstacles.
Safety theorem \implies this is impossible.

Thus, if we insist on accomplishing the goal with pure state feedback, we must use discontinuous feedback. (This gives us additional motivation to study hybrid systems!)

Differential drive robot + obstacles for simplicity assume we can directly control velocities:

$$\begin{aligned}\dot{x} &= \mu_1 \cos \theta \\ \dot{y} &= \mu_1 \sin \theta \\ \dot{\theta} &= \omega\end{aligned}$$

Complete Lyapunov function for MHS $H = (X, F, G, \varphi, r)$



Definition (Hybrid complete Lyapunov function $L: X \rightarrow \mathbb{R}$)

L is continuous and satisfies the following:

- For every $x \in F \setminus R(H)$, $x \in E_0(x)$, $t > 0$, and $y \in \chi(t)$, $L(y) < L(x)$.
- If $x \in G \setminus R(H)$, then $L(x) < L(y)$.
- For all $x, y \in R(H)$: x and y are chain equivalent $\iff L(x) = L(y)$.
- $L(R(H))$ is nowhere dense in \mathbb{R} .

THANK YOU

Kvalheim & Koditschek “Necessary conditions for feedback stabilization and safety.” arXiv:2106.00215 (2022).

Kvalheim, Gustafson, & Koditschek “Conley’s fundamental theorem for a class of hybrid systems.” SIADS (2021).