# Linearizability of dynamical systems by embeddings

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"Applied Koopmanism" assumes globally linearizable dynamics

Which dynamical systems are globally linearizable?

1-parameter subgroups of torus actions with asymptotic phase

# “Applied Koopmanism”

“*A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.*”

— Brunton, Budišić, Kaiser, and Kutz. “Modern Koopman Theory for Dynamical Systems.” SIAM Review, 64.2 (2022)

More formally, they seek *embeddings* of *nonlinear* dynamical systems into *linear* ones as invariant subsets, so that existing theoretical and algorithmic linear tools can be utilized.

# Linearizing embeddings

Let $f$ be a locally Lipschitz vector field on a manifold $M$. Consider

$$\dot{x} = \frac{d}{dt}x = f(x),$$

assume this ODE's solutions $x(t) = \Phi^t(x_0)$ are defined for all time.

$F\colon M \to \mathbb{R}^n$ is a **topological embedding** if $F$ is a one-to-one continuous map with a continuous inverse $F^{-1}\colon F(M) \to M$, and is a **smooth embedding** if additionally $F$, $F^{-1}$ are smooth.

Such an embedding $F$ is **linearizing** if $F \circ \Phi^t = e^{Bt} \circ F$ for some $n \times n$ matrix $B$. In the smooth case, $y = F(x)$ satisfies $\dot{y} = By$.

<u>**Fundamental question**</u>: when is $(M, \Phi)$ linearizable in this sense?

# When is a dynamical system $(M, \Phi)$ globally linearizable?

- Not when $M$ is connected and $\Phi$ has a countable number $\geq 2$ of omega limit sets (Liu, Ozay, Sontag 2023).
- Not when $M$ is connected and $\Phi$ has a non-global compact attractor $A$, since the open basin$(A)$ would also be closed (Jordan normal form theorem) and hence empty or all of $M$.

Thus, we study linearizability of the restriction $(S, \Phi)$ of $\Phi$ to

1. compact invariant sets $S$, and
2. basins $S$ of compact attractors $A$.

For these 2 cases we obtain **necessary and sufficient conditions for global linearizability** of $(S, \Phi)$ by an embedding, for the 2 cases of topological and smooth embeddings (4 cases total).[1]

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[1] MDK and P. Arathoon, *Linearizability of flows by embeddings (2023).*

# Torus preliminaries

The $n$-torus $T = T^n$ is Lie group isomorphic to $(\mathbb{R}/\mathbb{Z})^n$, vectors with $n$ real entries but with addition defined elementwise modulo 1.

A **torus action** on $S$ is a map $\Theta\colon T \times S \to S$ satisfying $\Theta^{\tau_1+\tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

The flow $(S, \Phi)$ is a **1-parameter subgroup of a torus action** if $\Phi^t = \Theta^{\omega t \bmod 1}$ for some torus action $\Theta$ on $S$, $\omega \in \mathbb{R}^n \cong \mathrm{Lie}(T^n)$.

# The linearizability theorem, case 1: compact, smooth

**Observation:** If $(S, \Phi)$ is linearizable with $S$ compact, the Jordan normal form theorem implies $(S, \Phi)$ embeds into the flow on $\mathbb{C}^n$ of a diagonal imaginary matrix, so $(S, \Phi)$ is a 1-parameter subgroup of restriction of standard torus action of $T^n$ on $\mathbb{C}^n$ to a subtorus.

This gives one implication below; the Mostow-Palais equivariant embedding theorem gives the other.

**Theorem (MDK and P. Arathoon).** If $S$ is a compact embedded submanifold, $(S, \Phi)$ is linearizable by a smooth embedding $\iff$ $(S, \Phi)$ is a 1-parameter subgroup of a smooth torus action.

We use this theorem to construct examples of smoothly linearizable $(S, \Phi)$ having isolated equilibria with e.g. $S =$ a sphere, torus, Klein bottle. On the other hand, regarding *non*linearizability...

# Topological implications for case 1 (compact, smooth)

If $(S, \Phi)$ is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an $n \times n$ skew matrix $B_e$ and a system of local coordinates on a neighborhood of each equilibrium $e \in S$ such that $\Phi^t \approx e^{B_e t}$. Hence if $e$ is isolated then $B_e$ is invertible, $n = \dim S$ is even, and the Hopf index of $e$ is $+1$.

**Corollary (MDK and PA).** If $S$ is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then $(S, \Phi)$ cannot be linearized by a smooth embedding.

**Corollary (MDK and PA).**[2] If $S$ is a compact submanifold containing at most finitely many equilibria such that $(S, \Phi)$ is linearizable by a smooth embedding, $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$.

[2] Apply the Poincaré-Hopf theorem.

# The linearizability theorem, case 2: compact, continuous

The theorem for case 2 is similar for case 1, but an additional assumption is needed to rule out a pathology not possible in case 1.

A torus action has **finitely many orbit types** if there are only finitely many subgroups $H \subset T$ such that $H = \{\tau \in T \colon \Theta^\tau(s) = s\}$ is the fixed point set of some $s \in S$.

**Theorem (MDK and PA).** If $S$ is compact, $(S, \Phi)$ is linearizable by a topological embedding $\iff$ $(S, \Phi)$ is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

# Another point of view: quasiperiodic pinched torus families

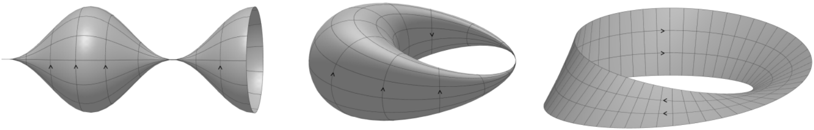

Figure: examples of quasiperiodic pinched torus families

**Proposition (MDK and PA).** If $S$ is compact, $(S, \Phi)$ is linearizable by a topological embedding $\iff$ $(S, \Phi)$ is a quasiperiodic pinched torus family.

**Definition.** $P$ is a **pinched torus family** if there are $m, n \in \mathbb{N}$, closed subsets $C_1, \ldots, C_n \subset B \subset T^m$, and a continuous group homomorphism $F\colon T^n \to T^m$ such that $P$ is the quotient of $F^{-1}(B)$ by collapsing the $j$-th $(\mathbb{R}/\mathbb{Z})$-factor of $F^{-1}(C_j) \subset T^n$ for all $j$. A pinched torus family $P$ is **quasiperiodic** if it is equipped with the induced flow generated by any $\omega \in \mathbb{R}^n$ with $\mathsf{T}F(\omega) = 0$.

# The linearizability theorem, case 3: basin, continuous

If $S$ is the basin of an asymptotically stable compact set $A \subset S$, $A$ has continuous (smooth) **asymptotic phase**[3] if there is a continuous (smooth) **asymptotic phase map** $P\colon S \to A$, i.e.,

$$P|_A = \mathrm{id}_A, \qquad P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

**Theorem (MDK and PA).** $(S, \Phi)$ is linearizable by a topological embedding $\iff$ $A$ has continuous asymptotic phase and $(A, \Phi)$ is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

**Example.** The basin of an asymptotically stable limit cycle is linearizable by a topological embedding $\iff$ the cycle has continuous asymptotic phase. This is not always the case, but it is the case if $\Phi \in C^1$ and the cycle is hyperbolic.

[3] This notion has roots in oscillator theory and more generally NHIM theory.

# The linearizability theorem, case 4: basin, smooth

**Theorem (MDK and PA).** $(S, \Phi)$ is linearizable by a smooth embedding $\iff$ $A$ is an embedded submanifold with smooth asymptotic phase, $(A, \Phi)$ is a 1-parameter subgroup of a smooth torus action, and for some open $U \supset A$, $(U, \Phi)$ embeds in a reducible linear flow covering $\Phi$ on some vector bundle over $A$.

**When does the final condition hold?** Classical linearization theorems and recent linearizing semiconjugacy theorems (MDK and Revzen, 2023) give answers in the special cases that $A$ is an equilibrium or periodic orbit, and some things are known if $A$ is quasiperiodic, but the general case seems to be an open problem.

**A necessary condition** for $A$ to satisfy all conditions of the theorem is that $A$ be a (eventually relatively $\infty$-)**normally hyperbolic invariant manifold**. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

Thank you for your time and attention.

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