When do Koopman embeddings exist?[1]

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Motivation

Given: a (possibly unknown) nonlinear system

$$\dot{x} = f(x).$$

- Extended Dynamic Mode Decomposition:[2] seeks $y = h(x)$, matrix A with *linear* dynamics

$$\dot{y} = Ay.$$

- To not lose information: want h one-to-one (1-1). Then

$$\begin{aligned} x(t) &= h^{-1}(y(t)) \\ &= h^{-1}(e^{At}h(x_0)). \end{aligned}$$

- Such **1-1 linearizing maps** h have also been called **Koopman embeddings**, **faithful linear representations** (Mezić 2021).

[2] Williams, Kevrekidis, and Rowley. J Nonlinear Science (2015)

Main issue and question

To avoid practical issues: want *h* at least continuous.

Main issue: continuous 1-1 linearizing *h* **do not exist** in general.

$$\Downarrow$$

Main question: when do they exist and when do they not?

When do Koopman embeddings exist?

Positive results (sufficient conditions)

Negative results (necessary conditions)

A counterexample for multiple isolated equilibria

Necessary and sufficient conditions

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Some positive results

Continuous 1-1 linearizing $y = h(x)$ always exists:[3]

- near a hyperbolic equilibrium or limit cycle (Hartman-Grobman, Floquet);
- on the basin of any exponentially stable equilibrium or limit cycle (Lan-Mezić 2013, K-Revzen 2021);
- On the basin of ANY asymptotically stable equilibrium, not necessarily exponentially stable / hyperbolic (K-Sontag 2025)!

[3] There are also C^k versions of all of these results.

Global linearization for equilibria without hyperbolicity

Let x_* be asymptotically stable with basin B for

$$\dot{x} = f(x).$$

Assume f is continuous w/ unique trajectories defined for all time.

Theorem (K-Sontag 2025). There is a homeomorphism $h\colon B \to \mathbb{R}^n$ such that $y = h(x)$ satisfies

$$\dot{y} = Ay$$

and hence $x(t) = h^{-1}(e^{At}h(x_0))$ for all $t \in \mathbb{R}$.
And if $f \in C^{k\geq 1}$, $n \neq 5$: h is a C^k diffeomorphism on $B \setminus \{x_*\}$.

Remarks

- Exponential stability / hyperbolicity of x_* is not needed.
- Proof relies on solutions to Poincaré conjecture (Smale, Perelman, Freedman). In fact:

Proposition (K-Sontag 2025). The C^k statement for $n = 5$ in last theorem is true $\iff$ the smooth 4-D Poincaré conjecture is true.

Proposition (K 2025). In the last theorem, if the vector field f_p depends continuously on parameter $p \in P$, there is a continuous family $h_p : B \to \mathbb{R}^n$ of linearizing homeomorphisms if either (i) $n > 5$ and $\dim P = 1$ or (ii) $n < 4$.

- Proof of latter relies on corollary of:

Theorem (K 2025). The space of proper C^∞ Lyapunov-like functions on $\mathbb{R}^n$ is path-connected and simply connected if $n \neq 4, 5$ and weakly contractible if $n < 4$. (Same for space of $C^{k \geq 1}$ GAS vf)

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Some negative results

Assume a connected state space to avoid trivialities. Then

$$\dot{x} = f(x)$$

does **not** have a continuous 1-1 linearizing $y = h(x)$ if **either**:

- there is a non-global compact attractor (K-Arathoon 2023), **or**
- all forward trajectories are precompact, and there are ≥ 2 but at most countably many omega-limit sets, e.g., multiple isolated equilibria (Liu-Ozay-Sontag 2023, 2025).

On the other hand, on the subject of multiple isolated equilibria...

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Smooth 1-1 linearization despite multiple isolated equilibria

If we drop the assumption that forward trajectories are precompact, then (another positive result):

Theorem (Arathoon-K 2023). For any $n > 1$ there is a smooth vector field on $\mathbb{R}^n$ with any given finite number of isolated equilibria such that there exists a smooth 1-1 linearizing $y = h(x)$.

In fact, h is a smooth embedding w/ *linear* (left) inverse h^{-1}!

This theorem gives a family of strong counterexamples to an oft-repeated claim (cf. SIAM DS 2023 debate).

Example

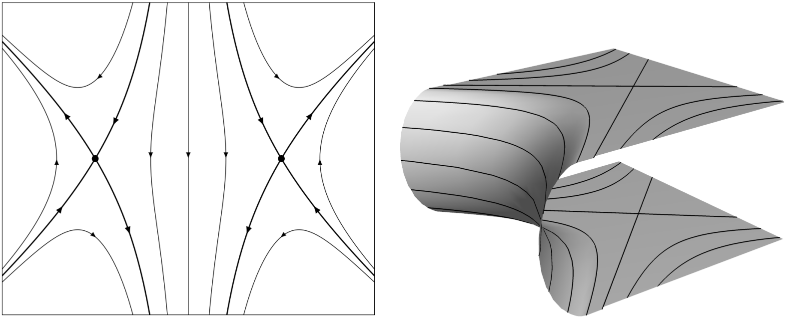

Figure: Smoothly embedding a nonlinear system on $\mathbb{R}^2$ with two isolated equilibria as an invariant subset of a linear system on $\mathbb{R}^3$.

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Where is the boundary between the positive and negative results?

We have now seen a variety of necessary conditions and sufficient conditions on

$$\dot{x} = f(x)$$

for a continuous 1-1 linearizing $y = h(x)$ to exist.

$$\Downarrow$$

Fundamental question: what are necessary **and** sufficient conditions on f for such an h to exist?

Preamble to answering the fundamental question

Assume a connected state space to avoid trivialities.

- **We can answer the fundamental question** for any continuous f with unique trajectories defined for all time **if there is at least one compact attractor**.
- Recall there does not exist such an h if there are ≥ 2 such attractors, or even a single non-global compact attractor (K-Arathoon 2025).
- $\implies$ remains to consider the case of a global compact attractor (can also restrict to basin of local attractor to apply next result).

Torus preliminaries

The m-**torus** $T = T^m$ is Lie group isomorphic to $(\mathbb{R}/\mathbb{Z})^m$, vectors w/ m real entries but w/ addition defined elementwise modulo 1.

A **torus action** on a space S is a map $\Theta\colon T \times S \to S$ satisfying $\Theta^{\tau_1+\tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

A **1-parameter subgroup** of Θ is a map $\Phi\colon \mathbb{R} \times S \to S$ of the form $\Phi^t(x) = \Theta^{\omega t}(x)$ for some $\omega \in \mathbb{R}^m$.

Θ has **finite orbit types** if there are only finitely many subgroups $H \subset T$ such that, for some $x \in S$,

$$H = \mathrm{Fix}(x) := \{\tau \in T \colon \Theta^\tau(x) = x\}.$$

Finishing the answer to the fundamental question

Assume f is continuous with unique trajectories defined for all time, so f generates a continuous flow $\Phi\colon \mathbb{R} \times X \to X$.[4]

Theorem (K-Arathoon 2023). Assume there is a global compact attractor A (or restrict to the basin of a local attractor). Then a continuous 1-1 linearizing $y = h(x)$ exists $\iff$

- $\Phi|_{\mathbb{R}\times A}$ is a 1-parameter subgroup of a continuous torus action with finite orbit types, and
- A has **continuous asymptotic phase** $P : X \to A$.

The latter condition means that for all $x \in X$, $t \in \mathbb{R}$,

$$P(\Phi^t(x)) = \Phi^t(P(x)).$$

$\implies$ $\operatorname{dist}(\Phi^t(x), \Phi^t(P(x))) \to 0$ as $t \to \infty$; x is "**asymptotically in phase with**" $P(x)$.

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[4] $t \mapsto \Phi^t(x_0)$ is the unique solution of $\dot{x} = f(x)$ satisfying $x(0) = x_0$.

Example: limit cycles

Previous theorem implies that dynamics on basin of limit cycle attractor admit a continuous 1-1 linearizing $y = h(x)$ if and only if there is continuous asymptotic phase (w/ level sets "isochrons").

Example. Using polar coordinates (r, θ) on $\mathbb{R}^2$, the system

$$\dot{r} = -(r-1)^3, \qquad \dot{\theta} = r$$

generates a smooth flow Φ on $\mathbb{R}^2 \setminus \{0\}$ with globally asymptotically stable limit cycle $A = \{r = 1\}$. Closed-form expression for $\Phi \implies$

$$\mathrm{dist}(\Phi^t(x), \Phi^t(y)) \not\to 0 \quad \text{as} \quad t \to \infty$$

for any $x \notin A$, $y \in A$, so A does not have continuous asymptotic phase, so a continuous 1-1 linearizing $y = -h(x)$ **does not exist**.

What about smooth linearizations?

- **Natural question**: when does there exist a smooth 1-1 linearizing $y = h(x)$ with smooth inverse $x = h^{-1}(y)$ $(y \in \mathrm{image}(h))$?
- Such an h is called a **smooth embedding**.
- So far, less satisfying answer in this case. But in particular, have the following necessary conditions:

Theorem (K-Arathoon 2023). Assume $\dot{x} = f(x)$ has a global compact attractor $A \subset X$ and is linearizable by a smooth embedding. Then:

- A is a smoothly embedded submanifold and normally hyperbolic,
- A has smooth asymptotic phase, and
- $\Phi|_{\mathbb{R}\times A}$ is a 1-parameter subgroup of a smooth torus action.

Answer to fundamental question for compact invariant sets

If state space X is compact, can view $A = X$ as a (trivial) compact attractor (with basin $B = A = X$). For this special case, we have:

Theorem (K-Arathoon 2023). Assume f generates a smooth (resp. continuous) flow Φ and X is compact. Then there is a smooth (resp. continuous) linearizing embedding $y = h(x) \iff$ Φ is a 1-parameter subgroup of a smooth (resp. continuous w/ finite orbit types) torus action.

For X noncompact, can still apply this theorem by restricting to a compact invariant set.

Surprising (?) examples with continuous 1-1 linearizations

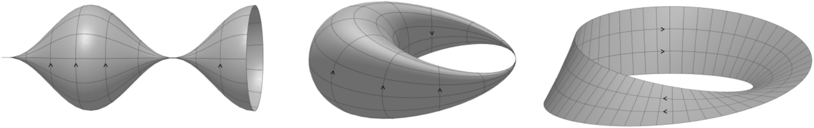


Figure: For all of these flows, a continuous 1-1 linearizing $y = h(x)$ exists (easy to see using preceding theorem).

Some corollaries of preceding theorem

Corollary (K-Arathoon 2023). If X is a compact smooth manifold and f has at most finitely many equilibria, and if there exists a smooth linearizing embedding $y = h(x)$, then

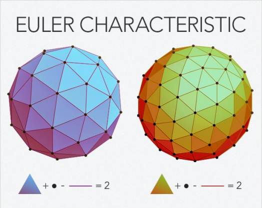
$$\underbrace{\chi(X)}_{\text{Euler characteristic}} = \#\{\text{equilibria}\} \geq 0.$$

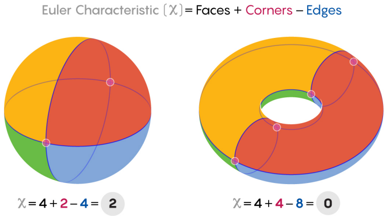
Corollary (K-Arathoon 2023). If X is an odd-dimensional compact smooth manifold and f has at least one isolated equilibrium, then there does not exist a smooth linearizing embedding $y = h(x)$.

Proof sketch. Using previous theorem, Bochner's linearization theorem for fixed points of torus actions $\implies$ the Hopf index of any equilibrium is $+1$. Apply the Poincaré-Hopf theorem to deduce the first corollary. Deduce the second corollary from the first using $\chi(X) = 0$ if X is an odd-dimensional compact manifold.

A primer on the Euler characteristic[5]

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).





Notation: $\chi(Y) :=$ Euler characteristic of Y.

Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\Sigma_g) = 2 - 2g$

Σ_g for $g = 1, 2, 3$ (not linearizable by smooth embedding for $g > 1$ if there is an isolated equilibrium).

[5] Figures from Quanta Magazine and Wikipedia.

Thank you for your time and attention.

References for (non)existence of 1-1 linearizing $y = h(x)$

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Linearizability of dynamical systems by embeddings

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Necessary and sufficient conditions