Non-Adaptive Cryptanalytic Time-Space Lower Bounds via a Shearer-like Inequality for Permutations

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Abstract

The power of adaptivity in algorithms has been intensively studied in diverse areas of theoretical computer science. In this paper, we obtain a number of sharp lower bound results which show that adaptivity provides a significant extra power in cryptanalytic time-space tradeoffs with (possibly unlimited) preprocessing time.

Most notably, we consider the discrete logarithm (DLOG) problem in a generic group of N elements. The classical 'baby-step giant-step' algorithm for the problem has time complexity $T = O(\sqrt{N})$, uses $O(\sqrt{N})$ bits of space (up to logarithmic factors in N) and achieves constant success probability.

We examine a generalized setting where an algorithm obtains an advice string of S bits and is allowed to make T arbitrary non-adaptive queries that depend on the advice string (but not on the challenge group element for which the DLOG needs to be computed).

We show that in this setting, the $T = O(\sqrt{N})$ online time complexity of the baby-step giant-step algorithm cannot be improved, unless the advice string is more than $\Omega(\sqrt{N})$ bits long. This lies in stark contrast with the classical adaptive Pollard's rho algorithm for DLOG, which can exploit preprocessing to obtain the tradeoff curve $ST^2 = O(N)$. We obtain similar sharp lower bounds for the problem of breaking the Even-Mansour cryptosystem in symmetric-key cryptography and for several other problems.

To obtain our results, we present a new model that allows analyzing non-adaptive preprocessing algorithms for a wide array of search and decision problems in a unified way.

Since previous proof techniques inherently cannot distinguish between adaptive and non-adaptive algorithms for the problems in our model, they cannot be used to obtain our results. Consequently, we rely on information-theoretic tools for handling distributions and functions over the space S_N of permutations of N elements. Specifically, we use a variant of Shearer's lemma for this setting, due to Barthe, Cordero-Erausquin, Ledoux, and Maurey (2011), and a variant of the concentration inequality of Gavinsky, Lovett, Saks and Srinivasan (2015) for read-k families of functions, that we derive from it. This seems to be the first time a variant of Shearer's lemma for permutations is used in an algorithmic context, and it is expected to be useful in other lower bound arguments.

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1 Introduction

1.1 Background

Cryptanalytic time-space tradeoffs. We consider cryptanalytic time-space tradeoffs in the *preprocessing* (i.e., non-uniform) model. An algorithm in this model is divided into two phases. In the first phase, the preprocessing algorithm obtains access to an input defined by the setting of the problem (e.g., $f : [N] \to [N]$ in the function inversion problem), and produces an 'advice string' of S bits. In the second phase, the online algorithm receives a specific challenge it has to solve with respect to the given input (e.g., an output of the function to invert). The online algorithm is given the advice string, along with oracle access to the input. The complexity of the algorithm is measured in terms of the bit-length S of the advice and the time complexity T of the online phase, typically measured in terms of the number of oracle queries. Such an algorithm is called an (S, T)-algorithm.

The preprocessing model is very natural from a practical point of view, as in many scenarios, the adversary is willing to solve many specific instances of the problem and the cost of a one-time preprocessing is amortized over the multiple solved instances.

The first and most famous cryptanalytic algorithm in the preprocessing model is Hellman's algorithm [Hel80] for inverting a random function. For constant success probability, Hellman's algorithm obtains a time-space tradeoff of about $TS^2 = \tilde{O}(N^2)$ (where $\tilde{O}$ hides logarithmic factors in N) on a domain of size N. The algorithm was extended for inverting any function by Fiat and Naor [FN99].

In recent years, there has been a significant effort in the cryptographic community to understand the power of non-uniform algorithms by devising new algorithms and proving time-space lower bounds against them. The emphasis has been on proving lower bounds for various problems (see, for example, [DGK17, CDG18, CDGS18, CK18, CK19, BMZ19, GGH$^+$20, CHM20, CGLQ20, ACDW20, GLLZ21, FGK22, GGPS23, ABG$^+$24, AGL24, GGK24]).

Non-adaptive cryptanalytic time-space tradeoffs. An algorithm with oracle access to an input is called *adaptive* if its oracle queries may depend on the outcome of its previous queries. Otherwise it is called non-adaptive. In this paper, we will be interested in non-adaptive cryptanalytic algorithms receiving advice.

Almost all best-known cryptanalytic time-space tradeoff algorithms are adaptive. This includes (for example) Hellman's algorithm [Hel80] and the Fiat and Naor [FN99] algorithm for function inversion, and the algorithms by Mihalcik [Mih10], Bernstein and Lange [BL13] and Corrigan-Gibbs and Kogan [CK18] for discrete log (DLOG).

However, in most cases there is no proof that non-adaptive algorithms cannot perform as well as the best known adaptive ones. For example, for the function inversion problem, the best-known lower bound of $ST \geq \tilde{\Omega}(N)$ is far from the best-known tradeoff of $TS^2 = \tilde{O}(N^2)$. Yet, for non-adaptive algorithms, the gap is even larger, as the lower bound remains roughly $ST \geq \tilde{\Omega}(N)$ (see [GGK24]), but the best-known algorithm cannot do better than $\max(T, S) = \tilde{O}(N)$.

In 2019, Corrigan-Gibbs and Kogan [CK19] formally raised the question of the power of adaptivity in the context of the function inversion problem. The authors of [CK19] argued that the lack of progress in establishing such lower bounds can be explained by a classical barrier in complexity theory. Indeed, they showed that a significantly improved lower bound on strongly non-adaptive algorithms for function inversion would imply a circuit lower bound against Boolean circuits of linear size and logarithmic depth, thus resolving a 50-year old well-known open question of Valiant [Val77]. Later on, it was shown that such a lower bound would

also imply significant data structure lower bounds [DKKS21, GGH+20]. Following the work of Corrigan-Gibbs and Kogan [CK19], the power of adaptivity in the context of the function inversion problem was studied in a number of papers [CHM20, DKKS21, GGK24, GGPS23].

Our work. In this paper, we investigate the question of [CK19] in a more general context:

What is the power of adaptivity in non-uniform cryptanalytic algorithms?

We present sharp lower bound proofs which demonstrate the significant power of adaptivity in a variety of cryptographic problems, in the preprocessing model. More specifically, in the context of public-key cryptography, we present sharp non-adaptive time-space lower bounds for the generic DLOG problem and the square-DDH (decisional Diffie-Hellman) problem. We also prove a lower bound for DDH that is sharp in a part of the range. In the context of symmetric-key cryptography, we present sharp time-space lower bounds for key-recovery attacks against the classical Even-Mansour cipher.

From a technical viewpoint, we show in Section 1.3 and Appendix B that previous lower bound proof techniques for the problems we consider are inherently limited and cannot distinguish between adaptive and non-adaptive algorithms.

Definition of adaptivity in our model. In the problems we analyze, each query of the online adversary is processed by a translation function before it is passed to the oracle. We allow the pre-translated queries of the online algorithm to depend on the advice, but *not on the challenge*. In Appendix A, we argue that this is the natural definition of non-adaptivity for the problems we analyze. Intuitively, the reason is that the translation function adds an additional adaptivity round. In particular, after being processed by the translation function, some of the queries of the adversary to *the oracle* in our model *do depend on the challenge* (even though this dependency does not exist before translation). Nevertheless, we believe that our results are valuable regardless of the exact definition of adaptivity in our setting, which we do not consider to be particularly important.

Interestingly, our main lower bounds for DLOG and Even-Mansour are matched by classical and well-known algorithms, which are non-adaptive in our model.

Before presenting our results, we elaborate on the rich history of the problems we analyze.

The generic discrete logarithm problem. Given a group G of N elements with a fixed generator g, the DLOG problem in G asks for solving the equation $g^d = x$, for a given $x \in G$. It is widely believed that the problem is hard if the group is chosen carefully, and the security of various cryptosystems relies on the hardness of this problem and its variants.

In 1997, Shoup [Sho97] presented the *generic group model* (GGM), intended for proving lower bounds against a specific class of algorithms that are applicable to any group of a particular order. A major motivation for studying this model is that in some cryptographically relevant groups (such as some elliptic curve groups), generic algorithms are the best-known algorithms for DLOG.

The model uses an auxiliary set $\mathcal{W}$ of bit strings with $|\mathcal{W}| \geq N$. In the model, the elements of G are 'hidden' by mapping g^j to $\sigma(j)$, where $\sigma : \mathbb{Z}_N \to \mathcal{W}$ is a randomly chosen bijective function. (We show in the sequel that there is no loss of generality in stating our results with $\mathcal{W} = [N] = \{1, 2, \ldots, N\}$, and so we state them this way.) An algorithm for solving the problem receives as inputs the encodings $\sigma(1), \sigma(d)$ of g, x, and its goal is to find d. The oracle queries

allowed for the algorithm are of the form $\sigma(a \cdot d + b \mod N)$.[1] The complexity of the algorithm is the total number of queries it makes, and its success probability is the expectation of the probability that it outputs d, over all uniform choices of σ, d, and the internal randomness of the algorithm. In this model, Shoup proved that if N is prime, then the success probability of any algorithm for DLOG that makes T queries is at most T^2/N.

The $\Omega(\sqrt{N})$ lower bound asserted by Shoup on the complexity of any algorithm for DLOG in the GGM is matched by the classical *baby-step giant-step* algorithm proposed by Shanks [Sha71] in 1971. The algorithm is based on writing the equation $g^d = x$ in the form $g^{im+j} = x$, where $m = \lceil \sqrt{N} \rceil$ and $0 \le i, j < m$, which in turn can be written as $g^j = x(g^{-m})^i$. The algorithm computes g^j for $1 \le j \le m$ and stores the values in a table. Then, it computes $y = g^{-m}$ and tries to find matches between $x, xy, xy^2, \ldots$ and values in the table. Once a match of the form $g^j = x(g^{-m})^i$ is found, we know that $d = im + j$ is the solution of the equation $g^d = x$, that is, the discrete logarithm of x. The algorithm can be executed in the generic group model, by replacing each computation of g^j by the query $\sigma(j)$ and each computation of xy^i by the query $\sigma(im + d \mod N)$. The algorithm uses memory $O(\sqrt{N})$ words of $O(\log N)$ bits, and its query complexity is $O(\sqrt{N})$.

In 1978, Pollard [Pol78] introduced the *Pollard Rho* algorithm for DLOG, which obtains the same query complexity of $O(\sqrt{N})$ while using only a very small amount $\tilde{O}(1)$ of memory. The algorithm is based on finding collisions of the form $x^a g^b = x^{a'} g^{b'}$ and using them to find a solution of the equation $g^d = x$ via the extended Euclid's algorithm. The collisions are found using Floyd's cycle finding algorithm [Knu69], which allows finding a collision in a random function $f : [N] \to [N]$ in time $O(\sqrt{N})$, using only a few memory cells. This algorithm can also be executed in the GGM, as computations of $x^a g^b$ correspond to queries of the form $\sigma(a \cdot d + b \mod N)$, and the rest of the algorithm is based on comparisons.

The baby-step giant-step algorithm is clearly non-adaptive. On the contrary, Pollard's Rho algorithm heavily uses adaptivity, via Floyd's algorithm each of whose queries depends on the output of its previous query.

Time-space tradeoffs for the generic DLOG problem with preprocessing. The generic group model is naturally extended to algorithms with preprocessing. In this case, an algorithm A consists of a 'preprocessing' algorithm A_0 and an 'online' algorithm A_1. The algorithm A_0 has direct access to the encoding function σ (but not to the input $\sigma(d)$) and produces an advice string z. The algorithm A_1 is defined as in the standard generic group model described above, but also receives the advice string z as input.

The baby-step giant-step algorithm can be naturally viewed as an algorithm with preprocessing, as its first stage of computing $\{g^j : 0 \le j < m\}$ can be done before the value of x is known. However, it does not seem to fully exploit the power of preprocessing, since the time complexity of the preprocessing phase (which is allowed to be unbounded) is equal to its space complexity.

Pollard's Rho algorithm does not use preprocessing. Yet, as noted above, Mihalcik [Mih10], Bernstein and Lange [BL13] and Corrigan-Gibbs and Kogan [CK18] presented generic algorithms that solve DLOG with preprocessing with high probability whose time and space complexity satisfy $ST^2 \le \tilde{O}(N)$. In particular, if space of $\tilde{O}(N^{1/3})$ is allowed, then the online time complexity can be reduced to $O(N^{1/3})$. These algorithms are known to be best possible [CK18].

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[1] The original model of Shoup is slightly different but it is easy to show that this does not affect the analysis.

The Decisional Diffie Hellman (DDH) and the square-DDH (sqDDH) problems.
The DDH problem in the GGM asks to distinguish between the triples $(\sigma(x), \sigma(y), \sigma(xy))$ and $(\sigma(x), \sigma(y), \sigma(z))$, where $x, y, z \in G$ are random elements. This corresponds to the situation in the Diffie-Hellman key exchange scheme (and many other cryptosystems) where the adversary sees the elements g^x, g^y sent between the parties and wants to decide whether a candidate element is the shared secret key g^{xy} or a random element g^z. When introducing the GGM, Shoup [Sho97] showed that for a prime N, any algorithm for the DDH problem in the GGM with $|G| = N$ that makes T queries, has a success probability of at most $\frac{1}{2} + \frac{T^2}{N}$.

The sqDDH problem [MW99] is the special case of DDH in which $x = y$. Namely, it asks to distinguish between the pairs $(\sigma(x), \sigma(x^2))$ and $(\sigma(x), \sigma(z))$, where $x, z \in G$ are uniform elements. While there are groups in which sqDDH is significantly easier than DDH (see [JN03]), it can be shown that in the GGM, any algorithm for sqDDH that makes T queries, has a success probability of at most $\frac{1}{2} + \frac{T^2}{N}$, just like for DDH (see [CDG18]).

As in the case of the DLOG problem, the complexity of algorithms for DDH and sqDDH can be significantly reduced if preprocessing is allowed. Indeed, for the DDH problem with preprocessing, the generic algorithms of Mihalcik [Mih10], Bernstein and Lange [BL13], and Corrigan-Gibbs and Kogan [CK18] achieve success probability of $\frac{1}{2} + \tilde{\Omega}(ST^2/N)$. For the sqDDH problem, Corrigan-Gibbs and Kogan [CK18] presented a generic algorithm with preprocessing that achieves an even higher success probability of $\frac{1}{2} + \tilde{\Omega}(\sqrt{ST^2/N})$. In the other direction, Corrigan-Gibbs and Kogan [CK18] showed that any algorithm with preprocessing for the DDH problem or for the sqDDH problem in the GGM has success probability of $\frac{1}{2} + \tilde{O}(\sqrt{ST^2/N})$. Interestingly, the correct tradeoff formula for the DDH problem remained open for several years, until recently Akshima, Besselman, Guo, Xie and Ye [ABG$^+$24] proved that it is equal to $\frac{1}{2} + \tilde{\Theta}(ST^2/N)$ by presenting a tighter proof.

The Even-Mansour (EM) cryptosystem. The EM cryptosystem was proposed in 1991 by Even and Mansour [EM97] as the 'simplest possible' construction of a block cipher from a public permutation. It is defined as $EM(m) = k_2 \oplus \sigma(k_1 \oplus m)$, where $k_1, k_2 \in \{0, 1\}^n$ are n-bit keys and $\sigma : \{0, 1\}^n \to \{0, 1\}^n$ is a publicly known permutation. The EM cryptosystem has become a central element in block cipher constructions and the security level of EM and of its iterative variants was studied extensively (see, e.g., [CLL$^+$18, CLS15, DSST17]).

An adversary is allowed to make encryption/decryption queries to EM and public queries to σ or σ^{-1}. We consider key-recovery attacks which aim at recovering (k_1, k_2). Even and Mansour showed that if σ is chosen uniformly at random, then such an attack which makes T_2 encryption/decryption queries to EM and T_1 queries to σ or σ^{-1}, has a success probability of $O(T_1 T_2/2^n)$. Dunkelman, Keller and Shamir [DKS15] showed that the same result holds for the seemingly weaker *Single-key EM* cryptosystem, defined as $SEM(m) = k_1 \oplus \sigma(k_1 \oplus m)$ (i.e., EM with $k_2 = k_1$).

Both adaptive and non-adaptive attacks matching these bounds are known [BW00, Dae91, DKS15]. Like in the case of DLOG, it was shown that the online complexity can be reduced significantly in the preprocessing model. Specifically, a key recovery attack with complexity $S = T_1 = T_2 = \tilde{O}(2^{n/3})$ was presented in [FJM14] and an upper bound of $\frac{1}{2} + \tilde{O}(\sqrt{S(T_1 + T_2)T_2/2^n} + T_1 T_2/2^n)$ on the success probability of any distinguishing attack, which matches the attack in the constant success probability setting, was shown in [CDG18]. The previous works on EM are described in detail in Section 6.

1.2 Our results

Main motivation. The main motivation behind this work is seeking a better understanding of classical generic algorithms for DLOG and other problems. Although Pollard's algorithm for DLOG is over 45 years old, we are not aware of any formal statement that justifies its need for adaptivity. Indeed, proving such a statement seems to be related to the major open problem of proving strong time-space lower bounds against short-output oblivious branching programs,[2] for which the best-known result was obtained by Babai, Nisan and Szegedy [BNS92] more than 30 years ago. Yet, given the historical significance of Pollard's algorithm, we view the question of formally justifying its adaptivity in an alternative (but standard) computational model as fundamental. We make progress on this question by showing that adaptivity does provide a significant advantage for DLOG algorithms in the generic group model, in the *preprocessing* (auxiliary input) setting. We also obtain similar results for several other major problems.

In particular, we present a new model, which we call *permutation challenge* (PC), which allows proving lower bounds on non-adaptive time-space tradeoffs for a wide variety of problems, including DLOG, DDH, sqDDH, and breaking the EM cryptosystem. We describe the model as part of the proof overview in Section 1.4. We now describe the results we obtain for the specific problems.

Sharp lower bound for non-adaptive algorithms for DLOG with preprocessing. Our main result concerns non-adaptive generic algorithms with preprocessing for the DLOG problem. We show that unlike the adaptive setting, for non-adaptive algorithms preprocessing cannot be exploited better than in the baby-step giant-step algorithm – indeed, its $O(\sqrt{N})$ online time complexity cannot be reduced even if a preprocessing phase of unbounded complexity is allowed, as long as the size of the advice z is $\tilde{O}(\sqrt{N})$.

Theorem 1. *Let $A = (A_0, A_1)$ be a non-adaptive (S, T)-algorithm for the DLOG problem in the generic group model, over a group G with a prime number N of elements. Denote by $\boldsymbol{MaxS}_{\mathrm{DLOG}}(T)$ the optimal success probability of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success probability of A is at most*

$$2 \cdot \boldsymbol{MaxS}_{\mathrm{DLOG}}(T) + \frac{4\log(2)ST}{N} + \frac{T^2}{N} \leq \frac{3T^2}{N} + \frac{4\log(2)ST}{N}.$$

The bound on $\boldsymbol{MaxS}_{\mathrm{DLOG}}(T) \leq \frac{T^2}{N}$ is by Shoup's theorem [Sho97] for any non-preprocessing DLOG algorithm.

Lower bounds for non-adaptive algorithms for DDH and sqDDH with preprocessing. Our next results analyze decisional problems in the GGM.

Theorem 2. *Let $A = (A_0, A_1)$ be a non-adaptive (S, T) algorithm for the DDH (resp. sqDDH) problem in the generic group model, over a group G with a prime number N of elements. Denote by $\boldsymbol{MaxS}_{\mathrm{DDH}}(T)$ the optimal success probability of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success probability of A is at most*

$$\boldsymbol{MaxS}_{\mathrm{DDH}}(T) + \sqrt{\frac{2\log_e(2)ST}{N}} + \frac{T^2}{N} \leq \frac{1}{2} + \frac{2T^2}{N} + \sqrt{\frac{2\log_e(2)ST}{N}}.$$

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[2]Technically, the open problem is to prove slightly super-linear time-space tradeoff lower bounds for oblivious branching programs. In our case, since the best algorithm runs in time $O(\sqrt{N})$, the analogous problem is to prove a time-space tradeoff of the form $TS \geq N^{1/2+\epsilon}$ for some $\epsilon > 0$.

where the inequality is by Shoup's theorem [Sho97] which bounds $\boldsymbol{MaxS}_{\mathrm{DDH}}(T) \leq \frac{1}{2} + \frac{T^2}{N}$ for any non-preprocessing DDH algorithm (the bound for sqDDH follows by a similar proof).

For the sqDDH problem, the theorem is sharp, as a simple non-adaptive variant of the adaptive algorithm of [CK18] attains its success probability bound (see Appendix C for a sketch of this algorithm).

For the DDH problem, we conjecture that the bound on the success probability is not sharp, and the 'right' bound is $\frac{1}{2} + \tilde{O}(\frac{T^2}{N} + \frac{ST}{N})$. Possibly, the techniques used in the recent result of Akshima, Besselman, Guo, Xie and Ye [ABG+24] which determined the maximal success rate of adaptive algorithms can be combined with our techniques to show this improved bound in the non-adaptive setting.

Sharp lower bound for non-adaptive key-recovery attacks on Even-Mansour with preprocessing. Our last result shows that unbounded preprocessing does not allow speeding up non-adaptive key-recovery attacks on EM (unless $S \geq \tilde{\Omega}(\sqrt{N})$), in stark contrast with adaptive attacks. In the theorem, we denote by $T = T_1 + T_2$ the total number of queries the online phase of the algorithm makes.

Theorem 3. *Let $A = (A_0, A_1)$ be a key-recovery, non-adaptive (S, T)-adversary for the Even-Mansour cryptosystem, which can query only the public and encryption oracles and not the decryption oracle. Denote by $\boldsymbol{MaxS}_{\mathrm{EM}}(T)$ the optimal success probability of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success probability of A is at most*

$$2 \cdot \boldsymbol{MaxS}_{\mathrm{EM}}(T) + \frac{4\log(2)S(T+1)}{N} + \frac{T^2}{N} \leq \frac{3T^2}{N} + \frac{4\log(2)S(T+1)}{N}.$$

Moreover, the theorem also holds for the single-key variant where $k_1 = k_2$.

The inequality is by [DKS15, EM97]. We remark that by symmetry of the EM construction, we may allow querying the decryption (and not the encryption) oracle. The theorem is tight by the non-adaptive attacks of [BW00, Dae91, DKS15] that can be viewed as preprocessing attacks in the setting of the theorem.[3]

1.3 Our techniques

Our bounds are obtained using techniques from information theory, as summarized below.

Entropy and Shearer's lemma. For a finite set $\mathcal{X}$, the entropy of a random variable X assuming values in $\mathcal{X}$ is $H(X) = \sum_{x \in \mathcal{X}} \Pr[X = x] \log_e(1/\Pr[X = x])$. The entropy, which measures the amount of uncertainty associated with the possible outcomes of X, is one of the central notions in information theory.

A basic property of entropy is *subadditivity*: For any random variables $X_1, X_2, \ldots, X_N$, the entropy of the Cartesian product $(X_1, X_2, \ldots, X_N)$ (which assumes values in $\mathcal{X} \times \ldots \times \mathcal{X}$) satisfies the relation $H(X_1, \ldots, X_N) \leq \sum_{i=1}^{N} H(X_i)$. Shearer's lemma [CGFS86] is a classical inequality which generalizes this property to the relation between the entropy of a set of random variables to the entropies of collections of its subsets.

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[3]Similarly to baby-step giant-step for DLOG, these algorithms have a challenge-independent phase that can be performed during preprocessing, and whose outcome can be viewed as an advice string.

Theorem 4 (Shearer's lemma). *Let $X_1, \ldots, X_N$ be random variables, and let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be subsets of $[N] = \{1, 2, \ldots, N\}$, such that each $i \in [N]$ belongs to at least k of them. Then*

$$H(X_1, \ldots, X_N) \leq \frac{1}{k} \sum_{i=1}^{m} H(X_{\mathcal{U}_i}),$$

where for $\mathcal{U} = \{i_1, \ldots i_\ell\} \subset [N]$, $H(X_{\mathcal{U}}) := H(X_{i_1}, X_{i_2}, \ldots, X_{i_\ell})$.

Shearer's lemma was used to obtain numerous results in combinatorics, theoretical computer science, probability theory, and other areas (see the survey [Gal14]).

KL-divergence. In our applications, it will be convenient to use Shearer's lemma via its version for the *Kullback-Leibler (KL) divergence*, which measures the amount of dissimilarity between two distributions.

For two distributions P, Q such that the support of P is contained in the support of Q, the KL divergence between P and Q is

$$D_{\mathrm{KL}}(P \| Q) = \sum_{x \in \mathrm{Support}(P)} P(x) \log \frac{P(x)}{Q(x)}.$$

The version of Shearer's lemma for KL-divergence reads as follows:

Theorem 5 (Shearer's lemma for KL-divergence [GLSS15]). *For a finite set $\mathcal{X}$, let $Q = (Q_1, Q_2, \ldots, Q_N)$ be the uniform distribution on $\mathcal{X}^N = \mathcal{X} \times \ldots \times \mathcal{X}$, and let $P = (P_1, \ldots, P_N)$ be another distribution on $\mathcal{X}^N$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be subsets of $[N]$, such that each $i \in [N]$ belongs to at most k of them. Then*

$$k \cdot D_{KL}(P \| Q) \geq \sum_{j \in [m]} D_{KL}(P_{\mathcal{U}_j} \| Q_{\mathcal{U}_j}),$$

where for $\mathcal{U} = \{i_1, \ldots i_\ell\} \subset [N]$, $P_{\mathcal{U}} = (P_{i_1}, \ldots, P_{i_\ell})$ is the marginal distribution of P on $\mathcal{X}^\ell$, and analogously for Q.

The concentration inequality of [GLSS15] for families of read-k functions. One of the directions in which Shearer's lemma was applied is showing that functions on a product space which depend on 'almost-disjoint' sets of variables behave, in some senses, similarly to independent functions. A formal manifestation of this phenomenon is the following.

Definition 1. *A family $\{f_1, f_2, \ldots, f_m\}$ of functions over a product space $\mathcal{X}^N$ is called a* read-k *family if for each $1 \leq i \leq N$, at most k of the functions depend on the i'th coordinate.*

An easy application of Shearer's lemma allows showing that if $f_1, \ldots, f_m : \{0, 1\}^n \to \{0, 1\}$ is a read-k family with $\Pr[f_i(x) = 1] = p$ for all i, then $\Pr[f_1(x) = f_2(x) = \ldots = f_m(x) = 1] \leq p^{m/k}$. That is, the probability that all functions are equal to 1 simultaneously is not much larger than p^m (which would be the probability if the functions were completely independent).

Gavinsky, Lovett, Saks and Srinivasan [GLSS15] used the KL-divergence variant of Shearer's lemma to prove a significantly stronger result in this direction – a variant of Chernoff's bound for read-once families of Boolean functions. Recall that for independent random variables $X_1, X_2, \ldots, X_m \in \{0, 1\}$ such that $\Pr[X_i = 1] = p$, Chernoff's inequality asserts $\Pr[\sum_{i=1}^{m} X_i \geq (p + \epsilon)m] \leq \exp(-2\epsilon^2 m)$. The authors of [GLSS15] proved the following.

Theorem 6 (The concentration bound of [GLSS15] for read-k families of Boolean functions)**.** *Let $f_1, \ldots, f_m : \{0,1\}^N \to \{0,1\}$ be a read-k family of functions. If $\Pr[f_i(x) = 1] = p$ for all i, then $\Pr[\sum_{i=1}^{m} X_i \geq (p + \epsilon)m] \leq \exp(-2\epsilon^2 m/k)$.*

In the 10 years since its introduction, this concentration inequality has become very useful and was applied to obtain various results in theoretical computer science and combinatorics (see, e.g., [HHTT21, HS24, Kah22]).

Variants of Shearer's lemma and the concentration inequality of [GLSS15] for distributions defined over permutations. The main technical tools we use in this paper are variants of the above results for distributions defined over the space S_N of permutations. It will be convenient for us to state the results for the space of bijections from $[N]$ to a set with N elements, which is obviously equivalent.

The following variant of Shearer's lemma for this non-product setting is a special case of a result that was proved (in equivalent forms) by Barthe, Cordero-Erausquin, Ledoux, and Maurey [BCELM11, Proposition 21] and by Caputo and Salez [CS24, Theorem 4].

Theorem 7 (Variant of Shearer's inequality for random bijections)**.** *Let $\mathcal{X}$ be a set of size N. Let $Q_X = Q_{X_1, \ldots, X_N}$ be the uniform distribution over bijections from $[N]$ to $\mathcal{X}$, and let $P_X = P_{X_1, \ldots, X_N}$ be another distribution over such bijections. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be subsets of $[N]$, such that each $i \in [N]$ belongs to at most k of them. Then*

$$2k \cdot \mathrm{D}_{KL}(P_X \| Q_X) \geq \sum_{j \in [m]} \mathrm{D}_{KL}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}),$$

where $P_{X_{\mathcal{U}}}$ is the distribution of the vector $X_{\mathcal{U}} := (X_i \mid i \in \mathcal{U})$ with respect to P (and analogously for Q).

Since the proofs of Theorem 7 in [BCELM11, CS24] are somewhat involved, we present an elementary proof of the theorem, albeit with a worse constant of 9, in Appendix D. We note that the constant 2 in the theorem cannot be improved to 1, as there are examples in which the ratio between the two sides of the inequality is $\frac{N}{N-1}$. It is not known whether the constant 2 is optimal in general.

We derive from Theorem 7 the following variant of the concentration inequality of Gavinsky, Lovett, Saks and Srinivasan for k-read families [GLSS15], for the setting of bijections.

Theorem 8 (Concentration for read-k families on bijections)**.** *Let $\mathcal{X}$ be a set of size N. Let $Q_X = Q_{X_1, \ldots, X_N}$ be the uniform distribution over bijections from $[N]$ to $\mathcal{X}$, and let $P_X = P_{X_1, \ldots, X_N}$ be another distribution over such bijections. Let $\{f_j\}_{j \in [m]}$ be a read-k family of functions, with $f_j : \mathcal{X}^N \mapsto [0,1]$ for all j. Denote $p_j = \mathbb{E}_{P_X}[f_j(X)]$ and let $p = \frac{1}{m} \cdot \sum_{j \in [m]} p_j$ be the average of the expectations. Similarly, denote $q_j = \mathbb{E}_{Q_X}[f_j(X)]$ and $q = \frac{1}{m} \cdot \sum_{j \in [m]} q_j$. Then*

$$2k \cdot \mathrm{D}_{KL}(P_X \| Q_X) \geq m \cdot \mathrm{D}_{KL}(p \| q),$$

where $\mathrm{D}_{KL}(p \| q) = p \log(\frac{p}{q}) + (1 - p) \log(\frac{1-p}{1-q})$ is the KL-divergence between two Bernoulli distributions with parameters p and q.

The proof of Theorem 8 is presented in Section 2. We remark that this theorem easily yields a concentration result in the form of Theorem 6 for read-k families on bijections, but we use the above variant as it is more useful to us.

8

We expect that, like Shearer's lemma and the concentration inequality of Gavinsky, Lovett, Saks and Srinivasan [GLSS15] for Boolean functions, the new variants will be useful in various other settings in theoretical computer science and combinatorics.

1.4 Proof overview

We begin by defining a model, referred to as *permutation challenge* (PC), for the problems analyzed in this paper. This model is inspired by the pre-sampling technique [BMZ19, CDG18, CDGS18, DGK17, Unr07] which allows bounding the success probability of preprocessing adversaries by analyzing non-preprocessing adversaries. Analogously, our model allows bounding the success probability of non-adaptive preprocessing adversaries (for a certain class of problems) by analyzing non-preprocessing adversaries.

While the pre-sampling technique is specialized to adaptive adversaries, our model allows optimizations for non-adaptive adversaries, and uses a completely different set of analytic techniques. The model abstracts away the internal details of the problem that are less relevant, and allows focusing on its most important aspects that allow proving time-space tradeoffs for non-adaptive adversaries. The fact that a single model can capture the wide array of search and decision problems we consider and allows analyzing them collectively is highly non-trivial.

The permutation challenge model. At a high-level, in a game in the PC model, there is an "inner permutation" $\sigma : [N] \mapsto [N]$ and a secret d selected from some space. An adversary A_1 is given direct oracle access to σ by issuing inner queries.

In addition, the secret d is used to implement an "outer function" which wraps the inner permutation. The adversary is allowed to query the outer function, while each query is translated using d to a query to σ using a *translation function*, and the output of σ is post-processed (again using d) and given back to the adversary as the oracle answer to the outer query.

The goal of A_1 is to "unwrap" the inner permutation (for example, by recovering the secret or a part of it) after interacting with the inner permutation via inner queries and with the outer function via outer queries.

In the preprocessing setting, A_1 is given an additional advice string $z = A_0(\sigma)$. The game is essentially the same, and we allow the queries of A_1 to depend on z. We say that an algorithm is non-adaptive if its inner and outer queries (before translation) depend only on z (but not on evaluations of σ on secret-dependent values).

Instantiations. The DLOG problem can be easily reduced to a setting where the encoding σ is a permutation on $[N]$, and the secret is the secret discrete-log d which the adversary has to compute. A query of the adversary is a pair of group elements (a, b), which is mapped to the group element $a \cdot d + b \mod N$ via a linear function applied to the discrete log. If $a \equiv 0 \pmod{N}$, then the query directly accesses the inner permutation, and otherwise, it is an outer query. Here, there is no post-processing of the answer of σ.

Our model further supports the DDH and sqDDH problems (although their definitions are more technical).

For the Even-Mansour construction, the inner permutation σ is the public permutation and the adversary is allowed to query it directly. The secret is the key (k_1, k_2), and the outer function corresponds to the encryption/decryption oracle. The translation function maps an outer query by XORing it with one of the keys, while the post-processing function XORs the other key to the output of σ.

9

The proof. The most important component of the model is the translation function that maps each outer query to an input of σ using the secret d.

A translation function is called *uniform* if it maps every outer query of A_1 to a (roughly) uniform input of σ for a uniformly chosen secret d. We show that if the translation function is uniform, then a non-adaptive preprocessing adversary obtains only a limited advantage (as a function of S and T and a uniformity parameter) over a non-preprocessing adversary.

Specifically, if the online non-adaptive adversary A_1 makes no inner queries, then fixing any preprocessing string z (which fixes the queries of the adversary), the uniformity of the translation function allows applying Theorem 8 to bound the advantage of A_1 over a non-preprocessing adversary. Moreover, we show that a similar bound holds even if we allow the adversary to issue inner queries that depend on the advice string. It is obvious that in this setting, Shearer-like inequalities are no longer directly applicable since some indices of the permutation may always be queried. Yet, we show that Theorem 8 can still be applied to the part of the permutation that has not been directly queried by an inner query. The overall proof is by a subtle hybrid argument. It works by defining an intermediate game that is positioned in between the preprocessing and non-preprocessing games, and allows dealing with inner queries.

We remark that one could potentially include the answers to the inner queries in the pre-processing string. However, this blows up the preprocessing string and gives a weaker result.

Comparison with other techniques. At a high level, there are currently three main generic techniques for proving cryptanalytic time-space tradeoffs: (1) compression arguments [CK18, FGK22, GT00, GGH+20, Wee05], (2) the pre-sampling technique [BMZ19, CDG18, CDGS18, DGK17, Unr07], and (3) concentration inequalities [ABG+24, ACDW20, AGL24, CHM20, CGLQ20, GLLZ21, IK10].

Most of these generic techniques were devised for adaptive algorithms, and it is not clear how to optimize them for non-adaptive algorithms for the problems we consider. In Appendix B, we discuss the limitations of previous techniques in detail and argue that (up to logarithmic factors) they cannot distinguish between adaptive and non-adaptive algorithms in our setting.

In addition to generic techniques, several papers developed specialized methods for proving time-space lower bounds for specific problems (some of which extended the generic techniques and combined additional methods). Examples of such papers include [CHM20, GGK24, GGPS23], which deal with non-adaptive algorithms for function inversion, as well as [CL23] which analyzes the 3SUM-Indexing problem. These specialized techniques seem inapplicable in our setting.

Future work. A natural goal to pursue, in view of our results, is to obtain time-space lower bounds for adaptive attacks with preprocessing, as a function of *the number of adaptivity rounds* for the problems considered in this paper. The most basic setting is to allow the queries of the adversary to depend on the challenge, corresponding to 2 adaptivity rounds (see Appendix A). For the DLOG problem, we conjecture that the success probability of an (S, T)-algorithm with r rounds of adaptivity is at most $\tilde{O}(T^2/N + rST/N)$. This matches our result for non-adaptive algorithms (i.e., 1 rounds of adaptivity), as well as the bounds of [CK18] for adaptive algorithms (i.e., T rounds of adaptivity). Furthermore, it would be sharp, as for any $1 \leq r \leq T$, it is matched by a variant of the adaptive algorithm of [BL13, CK18, Mih10], in which instead of constructing one chain of length T one constructs multiple chains of length r.

Organization of the paper. In Section 2 we present in detail the definitions and results related to entropy and KL-divergence that we will use in our proofs, including the variants of Shearer's inequality and of the inequality of Gavinsky, Lovett, Saks and Srinivasan for functions over permutations. In Section 3 we present the permutation challenge (PC) game model and show how DLOG, DDH, sqDDH, and breaking EM fit into it. In Section 4 we prove our main theorem for the PC model. In Section 5 we present the bounds for the DLOG, DDH, and sqDDH problems, and in Section 6 we present the bound for attacks on the EM cryptosystem.

2   Information Theory

In this section we present definitions and results from information theory that will be used throughout the paper. For more background on information theory, see [CT06].

2.1   Definitions and notations

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be finite sets. Let $P_{X,Y,Z}$ and $Q_{X,Y,Z}$ be probability distributions over $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. Denote the projections of $P_{X,Y,Z}$ on $\mathcal{X} \times \mathcal{Y}, \mathcal{X}$, and $\mathcal{Z}$, by $P_{X,Y}, P_X$, and P_Z, respectively, and use similar notations for $Q_{X,Y,Z}$. A random variable X assuming values in $\mathcal{X}$ is said to be drawn from the distribution P_X if $\Pr[X = x] = P_X(x)$ for all $x \in \mathcal{X}$.

(a). The *entropy* of a random variable X drawn from P_X is

$$\mathrm{H}(X) = \underset{P_X(x)}{\mathbb{E}}[\log(1/P_X(x))] = \sum_{x \in \mathcal{X}} P_X(x) \log(1/P_X(x)).$$

Here and throughout the paper, all logarithms are in base e, unless explicitly stated otherwise. In case of ambiguity about the distribution, we may also write $\mathrm{H}(P_X)$.

(b). The *conditional entropy* of X given Y (drawn from P_Y) is

$$\mathrm{H}(X \mid Y) = \underset{P_Y(y)}{\mathbb{E}}[\mathrm{H}(X \mid Y = y)] = \sum_{y \in \mathcal{Y}} P_Y(y) \, \mathrm{H}(X \mid Y = y) = \mathrm{H}(X, Y) - \mathrm{H}(Y).$$

(c). The *Kullback-Leibler divergence* (KL-divergence) between two distributions P_X, Q_X is

$$\mathrm{D_{KL}}(P_X \| Q_X) = \underset{P_X(x)}{\mathbb{E}}[\log(P_X(x)/Q_X(x))] = \underset{P_X(x)}{\mathbb{E}}[\log(1/Q_X(x))] - \mathrm{H}(P_X),$$

where we assume that the support of P_X is contained in the support of Q_X (otherwise, the KL-divergence is infinite).

(d). The KL-divergence between P_X, Q_X conditioned on P_Z is

$$\mathrm{D_{KL}}(P_{X|Z} \| Q_{X|Z}) = \underset{P_Z(z)}{\mathbb{E}}[\mathrm{D_{KL}}(P_{X|Z=z} \| Q_{X|Z=z})].$$

(e). The KL-divergence between two Bernoulli distributions P_X, Q_X with parameters p, q, respectively (i.e., $\mathcal{X} = \{0, 1\}$, $P_X(1) = p, P_X(0) = 1 - p$, and similarly for Q_X) is denoted by

$$\mathrm{D_{KL}}(p \| q) = \mathrm{D_{KL}}(P_X \| Q_X).$$

(f). The *mutual information* between X and Y (drawn from P_X, P_Y) is

$$\mathrm{I}(X; Y) = \mathrm{D_{KL}}(P_{X,Y} \| P_X P_Y).$$

2.2 Basic properties

We shall use the following basic properties of entropy, KL-divergence and mutual information.

(1) Conditioning does not increase entropy, namely

$$H(X) \geq H(X \mid Y),$$

with equality if and only if X, Y are independent, i.e., $P_{X,Y} = P_X \times P_Y$.

(2) The *chain rule for entropy* asserts that

$$H(X, Y) = H(X) + H(Y \mid X).$$

Thus, $H(X, Y) = H(X) + H(Y)$ if and only if X, Y are independent.

(3) The entropy of X is upper-bounded by the logarithm of the size of its support, namely

$$H(X) \leq \log |\operatorname{Supp}(\mathcal{X})|.$$

(4) KL-divergence is non-negative, namely

$$D_{KL}(P_X \| Q_X) \geq 0.$$

(5) KL-divergence is convex, namely for $0 \leq \lambda \leq 1$,

$$D_{KL}(\lambda P_X + (1 - \lambda)P'_X \| \lambda Q_X + (1 - \lambda)Q'_X) \leq \lambda D_{KL}(P_X \| Q_X) + (1 - \lambda) D_{KL}(P'_X \| Q'_X).$$

(6) If Q_X is uniform over its support (which includes the support of P_X), then

$$D_{KL}(P_X \| Q_X) = \underset{P_X(x)}{\mathbb{E}}[\log(1/Q_X(x))] - H(P_X) = \underset{Q_X(x)}{\mathbb{E}}[\log(1/Q_X(x))] - H(P_X)$$
$$= H(Q_X) - H(P_X).$$

(7) The *chain rule for KL-divergence* asserts that

$$D_{KL}(P_{X,Y} \| Q_{X,Y}) = D_{KL}(P_X \| Q_X) + D_{KL}(P_{Y|X} \| Q_{Y|X}).$$

(8) The *data processing inequality* asserts that for a function $f : \mathcal{X} \mapsto \mathcal{Y}$,

$$D_{KL}(P_X \| Q_X) \geq D_{KL}(P_{f(X)} \| Q_{f(X)}).$$

(9) A special case of *Pinsker's inequality* states that

$$2(p - q)^2 \leq D_{KL}(p \| q).$$

(10) The mutual information satisfies

$$I(X; Y) = D_{KL}(P_{X,Y} \| P_X P_Y) = D_{KL}(P_{X|Y} \| P_X) = H(X) - H(X \mid Y) \leq H(X).$$

2.3 Lemmas

We shall use the following lemmas.

Proposition 1. *Let $p, \epsilon > 0$ such that $p + \epsilon \leq 1$. Then,*

$$\mathrm{D}_{KL}(p + \epsilon \| p) \geq \frac{\epsilon^2}{2(p + \epsilon)}.$$

Proof. The proof follows a standard analytic approach for proving inequalities on KL-divergence.

By Property (9), $\mathrm{D}_{\mathrm{KL}}(p + \epsilon \| p) \geq 2\epsilon^2$, implying the statement whenever $p + \epsilon \geq \frac{1}{4}$. Therefore, we may assume $p + \epsilon \leq \frac{1}{4}$. Let us define a function $f : [0, \epsilon] \to \mathbb{R}$ by

$$f(x) = \mathrm{D}_{\mathrm{KL}}(p + x \| p) - \frac{1}{2(p + \epsilon)} x^2.$$

Since $f(0) = 0$ and the statement is equivalent to $f(\epsilon) \geq 0$, it suffices to show that f is monotonically increasing. We have

$$f'(x) = \log\left(\frac{p + x}{p}\right) - \log\left(\frac{1 - (p + x)}{1 - p}\right) - \frac{1}{p + \epsilon} x.$$

Again, we may notice that $f'(0) = 0$. Therefore, in order to show that $f'(x) \geq 0$ for all $x \in [0, \epsilon]$ and finish the proof, it suffices to show that $f''(x) \geq 0$ for all $x \in [0, \epsilon]$. Since $p + \epsilon \leq \frac{1}{4}$, we obtain that for all $x \in [0, \epsilon]$ we have

$$f''(x) = \frac{1}{(1 - (p + x))(p + x)} - \frac{1}{p + \epsilon} \geq \frac{1}{(1 - (p + \epsilon))(p + \epsilon)} - \frac{1}{p + \epsilon} \geq 0,$$

completing the proof. ∎

Corollary 1. *Let $0 < p, q \leq 1$. Then,*

$$p \leq 2(q + \mathrm{D}_{KL}(p \| q)).$$

Proof. If $p - q < q$, then $p < 2q \leq 2(q + \mathrm{D}_{\mathrm{KL}}(p \| q))$, as the KL-divergence is non-negative. Otherwise, $p - q \geq q$. Applying Proposition 1, we obtain

$$\mathrm{D}_{\mathrm{KL}}(p \| q) \geq \frac{(p - q)^2}{2q} \geq \frac{(p - q)}{2}.$$

Therefore, $p \leq q + 2\,\mathrm{D}_{\mathrm{KL}}(p \| q) \leq 2(q + \mathrm{D}_{\mathrm{KL}}(p \| q))$.

∎

Proposition 2 ([GLSS15], Claim 2.6)**.** *Let Q_X be the uniform distribution over a set $\mathcal{X}$, let P_X be a distribution over $\mathcal{X}$, and let $f : \mathcal{X} \mapsto [0, 1]$. Denote $q = \mathbb{E}_Q[f(X)]$ and $p = \mathbb{E}_P[f(X)]$. Then*

$$\mathrm{D}_{KL}(P_X \| Q_X) \geq \mathrm{D}_{KL}(p \| q).$$

2.4 A variant of Shearer's inequality for functions over bijections

In this subsection we show how our formulation of the variant of Shearer's inequality for bijections (namely, Theorem 7) follows from [BCELM11, Proposition 21] of Barthe, Cordero-Erausquin, Ledoux and Maurey and [CS24, Theorem 4] of Caputo and Salez, which are stated in a very different form. Let us recall Theorem 7.

Theorem 7. Let $\mathcal{X}$ be a set of size N. Let $Q_X = Q_{X_1,\ldots,X_N}$ be the uniform distribution over bijections from $[N]$ to $\mathcal{X}$, and let $P_X = P_{X_1,\ldots,X_N}$ be another distribution over such bijections. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be subsets of $[N]$, such that each $i \in [N]$ belongs to at most k of them. Then

$$2k \cdot \mathrm{D_{KL}}(P_X \| Q_X) \geq \sum_{j \in [m]} \mathrm{D_{KL}}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}),$$

where $P_{X_{\mathcal{U}}}$ is the distribution of the vector $X_{\mathcal{U}} := (X_i \mid i \in \mathcal{U})$ with respect to P (and analogously for Q).

Since the equivalence between [BCELM11, Proposition 21] and [CS24, Theorem 4] is discussed in a remark after [CS24, Theorem 4], we show how Theorem 7 follows from [CS24, Theorem 4]. We first cite this result and then explain the notations involved.

Theorem 9 ([CS24, Theorem 4, Equation (34)]). *For any choice of a probability vector $(\theta_A, A \subseteq [N])$ on subsets of $[N]$ and every function $f : S_N \to \mathbb{R}_{\geq 0}$,*

$$\sum_{A \subseteq [N]} \theta_A \, Ent(E_A[f]) \leq (1 - \kappa) \, Ent(f),$$

where $\kappa = \min_{i \neq j} \sum_{A \supseteq \{i,j\}} \theta_A$.

The reader is referred to [CS24] for a full description of the functional Ent and its properties. Here we only state two basic observations, that follow immediately from its definitions:

Observation 1. *Let $\mathcal{X}$ be a set of size N. Let $Q_X = Q_{X_1,\ldots,X_N}$ be the uniform distribution over bijections from $[N]$ to $\mathcal{X}$, let $P_X = P_{X_1,\ldots,X_N}$ be another distribution over such bijections, and let $f : S_N \to \mathbb{R}$ be defined as $f(\sigma) = \frac{P(\sigma)}{Q(\sigma)}$. Let $A \subseteq [N]$ be a set of indices. Then:*

- $Ent(f) = \mathrm{D}_{KL}(P \| Q)$.

- $Ent(E_A[f]) = \mathrm{D}_{KL}(P_{X_{A^c}} \| Q_{X_{A^c}})$ *(note that the distributions are projected to the complement set $A^c = [N] \setminus A$ and not to A).*

Using these observations, we show that Theorem 7 follows directly from Theorem 9.

Proof (of Theorem 7). Let P, Q, f be as in Observation 1. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be subsets of $[N]$, such that each $i \in [N]$ belongs to at most k of them. For every $A \subseteq [N]$, we denote $\theta_A = |\{j \mid \mathcal{U}_j = A^c\}|/m$. Applying Theorem 9 we obtain

$$\sum_{A \subseteq [N]} \theta_A \, \mathrm{Ent}(E_A[f]) \leq (1 - \kappa) \, \mathrm{Ent}(f),$$

where $\kappa = \min_{i \neq i'} \sum_{A \supseteq \{i,i'\}} \theta_A$. By Observation 1, an equivalent formula is

$$\sum_{A \subseteq [N]} \theta_A \, \mathrm{D_{KL}}(P_{X_{A^c}} \| Q_{X_{A^c}}) \leq (1 - \kappa) \, \mathrm{D_{KL}}(P \| Q).$$

Substituting the formula for θ_A, we obtain

$$\frac{1}{m}\sum_{j=1}^{m} \mathrm{D_{KL}}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}) \leq (1-\kappa)\,\mathrm{D_{KL}}(P\|Q). \tag{1}$$

It remains to bound κ. Notice that $\kappa = \min_{i\neq i'} \sum_{A \supseteq \{i,i'\}} \theta_A = \frac{1}{m} \min_{i\neq i'} |\{j \mid \{i,i'\} \subseteq \mathcal{U}_j^c\}|$. From the assumption on $\mathcal{U}_j$ we obtain that $|\{j \mid i \in \mathcal{U}_j^c\}| \geq m-k$ for all i, implying that $\kappa \geq \frac{m-2k}{m}$. The statement now follows from (1). $\blacksquare$

An elementary direct proof of a weaker version of Theorem 7, with the constant 9 instead of 2, is presented in Appendix D.

2.5 A variant of the inequality of [GLSS15] for functions over bijections

In this subsection we derive Theorem 8, namely, the variant of the concentration bound of Gavinsky, Lovett, Saks and Srinivasan [GLSS15] for read-k families of functions over bijections, from the variant of Shearer's lemma for the same setting (Theorem 7 above). Let us restate the theorem we prove:

Theorem 8. Let $\mathcal{X}$ be a set of size N. Let $Q_X = Q_{X_1,\ldots,X_N}$ be the uniform distribution over bijections from $[N]$ to $\mathcal{X}$, and let $P_X = P_{X_1,\ldots,X_N}$ be another distribution over such bijections. Let $\{f_j\}_{j\in[m]}$ be a read-k family of functions, with $f_j : \mathcal{X}^N \mapsto [0,1]$ for all j. Denote $p_j = \mathbb{E}_{P_X}[f_j(X)]$ and let $p = \frac{1}{m}\cdot\sum_{j\in[m]} p_j$ be the average of the expectations. Similarly, denote $q_j = \mathbb{E}_{Q_X}[f_j(X)]$ and $q = \frac{1}{m}\cdot\sum_{j\in[m]} q_j$. Then

$$2k \cdot \mathrm{D_{KL}}(P_X\|Q_X) \geq m\cdot\mathrm{D_{KL}}(p\|q),$$

where $\mathrm{D_{KL}}(p\|q) = p\log(\frac{p}{q}) + (1-p)\log(\frac{1-p}{1-q})$ is the KL-divergence between two Bernoulli distributions with parameters p and q.

Proof. Let $\{\mathcal{U}_j\}_{j\in[m]}$ be a family of index sets such that for each $j \in [m]$, f_j depends only on the coordinates in $\mathcal{U}_j$. Since for each $i \in [N]$, $|\{j \in [m] \mid i \in \mathcal{U}_j\}| \leq k$, Theorem 7 implies

$$2k \cdot \mathrm{D_{KL}}(P_X\|Q_X) \geq \sum_{j\in[m]} \mathrm{D_{KL}}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}).$$

For every $j \in [m]$, $Q_{X_{\mathcal{U}_j}}$ is uniform over its support (which includes the support of $P_{X_{\mathcal{U}_j}}$). Thus, we apply Proposition 2 with f_j and deduce

$$\mathrm{D_{KL}}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}) \geq \mathrm{D_{KL}}(p_j\|q_j).$$

Hence,

$$2k \cdot \mathrm{D_{KL}}(P_X\|Q_X) \geq \sum_{j\in[m]} \mathrm{D_{KL}}(p_j\|q_j) = m\sum_{j\in[m]} (1/m)\,\mathrm{D_{KL}}(p_j\|q_j) \geq m\cdot\mathrm{D_{KL}}(p\|q),$$

where the final inequality holds by convexity of the KL-divergence (Property (5)). $\blacksquare$

Remark 1. *Note that while the definition of a read-k family requires the functions $\{f_j\}$ to be defined over the entire domain $\mathcal{X}^N$, only their values on bijections are relevant in Theorem 8. Consequently, in the applications of this theorem we define each function only over bijections, while implicitly fixing it to 0 on inputs that are not bijections.*

3 The Permutation Challenge Model

3.1 Non-preprocessing setup

A permutation challenge (PC) game

$$PC := PC(N, \mathcal{D}, \mathcal{M}, tr, post, suc)$$

is defined as follows.

For an integer $N > 0$, let $\sigma : [N] \mapsto [N]$ be a permutation and let $\mathcal{D}$ be a space of secrets.

Let $tr : \mathcal{D} \times \mathcal{M} \to [N]$ be a translation function that obtains as inputs a secret $d \in \mathcal{D}$ (e.g., the discrete log in the DLOG problem) and an outer query $m \in \mathcal{M}$ (from the outer query space $\mathcal{M}$) and returns a translated query to σ, denoted $tr(d, m) \in [N]$. Let $post : \mathcal{D} \times [N] \to [N]$ be a post-processing function that receives the secret and an inner permutation output and returns a value in $[N]$ (the output space of the outer function).

Definition 2. *A translation function tr is called u-uniform if for every $m \in \mathcal{M}$ and $j \in [N]$,*

$$|\{d \in \mathcal{D} \mid tr(d, m) = j\}| \leq \frac{|\mathcal{D}|}{u}.$$

In several of our applications, the translation function tr will be N-uniform, which is the largest value possible.

Let A_1 be an adversary for PC. We assume for simplicity that A_1 is deterministic. As noted in Remark 2 below, this assumption is without loss of generality.

A_1 has oracle access to two related oracles. The inner oracle allows A_1 to issue an *inner query* $i \in [N]$ to σ and obtain $\sigma(i)$. In some applications, A_1 can also issue inner queries $i \in [N]$ to the inverse inner permutation and obtain $\sigma^{-1}(i)$.

The outer oracle allows A_1 to issue an *outer query* $m \in \mathcal{M}$ and obtain $post(d, \sigma(tr(d, m)))$.

After the interaction with the oracles which we denote in short by $\mathcal{O}(\sigma, d)$, $A_1^{\mathcal{O}(\sigma,d)}$ outputs a value v from some domain $\mathcal{V}$.

Let $suc := suc_{A_1^{\mathcal{O}(\sigma,d)}}(d)$ be a 0/1 success predicate that obtains d and has the same oracle accesses as $A_1^{\mathcal{O}(\sigma,d)}$. This predicate outputs 1 if A_1 succeeds and 0 otherwise.

Formally, suc has two phases. In the first phase, it simulates $A_1^{\mathcal{O}(\sigma,d)}$ and obtains its output. In the second phase, it outputs a value by applying a 0/1 predicate that receives as input the secret d, the output of A, and all the query answers obtained by A.[4] We may also allow suc to make additional queries to verify the success of A, which will be accounted for in the total time complexity (but we do not use this possibility in our applications).

We remark that when we write $PC := PC(N, \mathcal{D}, \mathcal{M}, tr, post, suc)$, the success predicate is viewed as an interface that is instantiated given an adversary A_1.

Let $P_{\Sigma,D} = P_\Sigma P_D$ be a distribution such that P_Σ is uniform over permutations and P_D is uniform over the secret space $\mathcal{D}$.

The success probability of A_1 is defined as

$$\mathbb{E}_{P_{\Sigma,D}} [suc_{A_1^{\mathcal{O}(\Sigma,D)}}(D)].$$

We measure the complexity of A_1 in terms of its number of queries to the inner permutation, T_1, and its number of queries to the outer function, T_2, and define $T = T_1 + T_2$.

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[4]In our applications, the predicate only needs the output of A, but we allow it to obtain the query answers obtained by A for generality.

Non-adaptivity. We say that A_1 is non-adaptive if its queries are fixed in advance and do not depend on σ.

3.2 Preprocessing setup

Let $A = (A_0, A_1)$ be a pair of deterministic preprocessing and online algorithms for $\boldsymbol{PC}$.

Remark 2. *In our context, we may assume that A is deterministic without loss of generality, as our lower bound proofs hold for every randomness string shared by A_0 and A_1.*

The preprocessing algorithm A_0 gets direct access to σ as input and outputs an advice string $z = A_0(\sigma)$. The online algorithm A_1 takes as an additional input the advice string z. We denote the algorithm A_1, when it is executed with a preprocessing string z, by $(A_1)_z$.

We call (A_0, A_1) an (S, T)-algorithm if the length of $z = A_0(\sigma)$ is bounded by S bits (for all σ) and A_1 makes at most T oracle queries to both the inner permutation and the outer function.

We extend the non-preprocessing success predicate $\boldsymbol{suc}$ to the preprocessing model as follows: it additionally obtains an advice string z. As previously, $\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma,d)}}(d)$ has two phases. In the first phase, it simulates $(A_1)_z^{\mathcal{O}(\sigma,d)}$ and obtains its output. In the second phase, it outputs a value by applying the same non-preprocessing $0/1$ predicate that receives as input the secret d, the output of A_1 and all the query answers obtained by A_1.[5]

We extend the distribution $P_{\Sigma,D}$ as $P_{\Sigma,Z,D} = P_{\Sigma,Z} \times P_D$, where $P_{\Sigma,Z} = P_\Sigma P_{Z|\Sigma}$ such that $P_{Z|\Sigma=\sigma}(z) = 1$ if $z = A_0(\sigma)$.

The success probability (A_0, A_1) in solving the problem is defined as

$$p := p(A_0, A_1) := \underset{P_{\Sigma,Z,D}}{\mathbb{E}} [\boldsymbol{suc}_{(A_1)_Z^{\mathcal{O}(\Sigma,D)}}(D)].$$

Remark 3. *Crucially, the predicate applied by $\boldsymbol{suc}$ after simulating $(A_1)_z^{\mathcal{O}(\sigma,d)}$ does not receive z as input. Namely, $\boldsymbol{suc}$ must be able to verify whether the output of $(A_1)_z$ is correct on (σ, d) independently of z. Technically, our analysis will sometimes fix z that is not equal to $A_0(\sigma)$, but we require that the success of $(A_1)_z$ will still be defined with respect to (σ, d).*

Another way to state this remark is that it is sufficient to define the predicate $\boldsymbol{suc}$ for non-preprocessing algorithms. The extension to the preprocessing model is well-defined and it simply simulates $(A_1)_z$ as a black-box and checks its success with respect to (σ, d) as in the non-preprocessing model.

Non-adaptivity. We say that A_1 is non-adaptive if given any z, its queries are fixed and do not (further) depend on σ. Note that if A_1 is non-adaptive then $\boldsymbol{suc} := \boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma,d)}}(d)$ is a non-adaptive predicate.

We denote by $\mathcal{S} \subseteq [N]$ the set of inner queries of A_1 (which may include inverse queries) and by $\mathcal{U} \subseteq \mathcal{M}$ the set of its outer queries. To simplify notation, we assume that $\mathcal{S}$ refers to queries to σ and not σ^{-1} using the conversion that if $\sigma^{-1}(i) = j$, then $\sigma(j) = i$ is a query to σ. Both $\mathcal{S}$ and $\mathcal{U}$ may depend on z, and we sometimes emphasize this by writing (for example) $\mathcal{S}(z)$.

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[5]In our applications, the predicate only compares the output of $(A_1)_z$ to some function of the secret.

3.3 Instantiation in the generic group model

We first define the general setting of GGM which is common to all problems in the model.

Let A_1 be a generic group algorithm in $\mathbb{Z}_N$ such that N is prime. We assume that A_1 knows the image of $\sigma : \mathbb{Z}_N \to \mathcal{W}$, as this knowledge can only increase its success probability. Thus, we restrict σ to a subset of $\mathcal{W}$ of size N. Furthermore, by renaming the symbols of the image of σ, we assume without loss of generality that $\sigma : \mathbb{Z}_N \to \mathbb{Z}_N$ is a uniform bijection from $\mathbb{Z}_N$ to itself. We represent the elements of $\mathbb{Z}_N$ using $[N]$, and we can thus write $\sigma : [N] \to [N]$. In particular, note that $N \bmod N = 0$.

We remark that in GGM, we do not allow A_1 to query σ^{-1}.

We now define the DLOG, DDH, and sqDDH problems as permutation challenge games.

Discrete-log. Let A_1 be a discrete-log algorithm for $\mathbb{Z}_N$. We define a permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{DL}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ for the discrete-log problem as follows.

Let $\mathcal{D} = [N]$ (identified with $\mathbb{Z}_N$) be the space of secrets of discrete logarithms. Let $\boldsymbol{post}(d, j) = j$ be the trivial post-processing function (as A_1 always sees direct outputs of σ).

Let $\mathcal{M} = [N-1] \times [N]$ be the space of outer queries and define

$$\boldsymbol{tr}(d, (a, b)) = a \cdot d + b \mod N$$

as the translation function that maps a query (a, b) (where $a \neq N$) to a group element according to the linear function $a \cdot d + b \mod N$. Note that in the representation, an inner query issued to σ corresponds to the pair (N, b) (but we view it as directly accessing $\sigma(b)$).

The adversary receives as input the values $\sigma(1)$ and $\sigma(d)$. However, for simplicity we assume that these are given as outputs of the queries $(N, 1)$ and $(1, N)$, and the adversary receives no input (we make similar assumptions about DDH and sqDDH, defined below).

Finally, $\boldsymbol{suc}_{A_1^{\mathcal{O}(\sigma, d)}}(d)$ simply simulates A_1 and returns 1 if it outputs the discrete-log secret d.

DDH. Let A_1 be a DDH generic algorithm for $\mathbb{Z}_N$. We define a permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{DDH}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ for the DDH problem as follows.

Let $\mathcal{D} = [N]^3 \times \{0, 1\}$ be the secret space, consisting of 3 group elements d_1, d_2, d_3 and a bit k that A_1 needs to output. Let $\boldsymbol{post}(d, j) = j$ be the trivial post-processing function.

Let $\mathcal{M} = [N]^4 \setminus (\{(N, N, N)\} \times [N])$ be the space of outer queries, which consists of 4 group elements that specify a multi-linear function, denoted by (a_1, a_2, a_3, b), where we do not allow $a_1 = a_2 = a_3 = N$ (as this corresponds to an inner query).

Define the translation function $\boldsymbol{tr}((d_1, d_2, d_3, k), (a_1, a_2, a_3, b))$ as follows:

$$\boldsymbol{tr}((d_1, d_2, d_3, k), (a_1, a_2, a_3, b)) = \begin{cases} a_1 \cdot d_1 + a_2 \cdot d_2 + a_3 \cdot d_3 + b \mod N & \text{if } k = 0, \\ a_1 \cdot d_1 + a_2 \cdot d_2 + a_3 \cdot (d_1 d_2) + b \mod N & \text{if } k = 1. \end{cases}$$

Note that if $k = 0$, the algorithm effectively receives $\sigma(d_1), \sigma(d_2), \sigma(d_3)$, while if $k = 1$, the algorithm effectively receives $\sigma(d_1), \sigma(d_2), \sigma(d_1 d_2 \bmod N)$, as in DDH.

As in the discrete-log game, an inner query can be viewed as directly accessing σ.

Finally, $\boldsymbol{suc}_{A_1^{\mathcal{O}(\sigma, d_1, d_2, d_3, k)}}(d_1, d_2, d_3, k)$ simply simulates A_1 and returns 1 if it outputs the bit k.

sqDDH. Let A_1 be a sqDDH generic algorithm for $\mathbb{Z}_N$. We define a permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{sqDDH}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ for the sqDDH problem as follows.

Let $\mathcal{D} = [N]^2 \times \{0, 1\}$ be the space of secrets, consisting of 2 group elements d_1, d_2 and a bit k that A_1 needs to output. Let $\boldsymbol{post}(d, j) = j$ be the trivial post-processing function.

Let $\mathcal{M} = [N]^3 \setminus (\{(N, N)\} \times [N])$ be the space of outer queries, which consist of 3 group elements that specify a multi-linear function, denoted by (a_1, a_2, b), where we do not allow $a_1 = a_2 = N$ (as this corresponds to an inner query).

Define the translation function $\boldsymbol{tr}((d_1, d_2, k), (a_1, a_2, b))$ as follows:

$$\boldsymbol{tr}((d_1, d_2, k), (a_1, a_2, b)) = \begin{cases} a_1 \cdot d_1 + a_2 \cdot d_2 + b \mod N & \text{if } k = 0, \\ a_1 \cdot d_1 + a_2 \cdot (d_1)^2 + b \mod N & \text{if } k = 1. \end{cases}$$

Note that if $k = 0$, the algorithm effectively receives $\sigma(d_1), \sigma(d_2)$, while if $k = 1$, the algorithm effectively receives $\sigma(d_1), \sigma((d_1)^2 \mod N)$, as in sqDDH.

Once again, an inner query is viewed as directly accessing σ.

Finally, $\boldsymbol{suc}_{A_1^{\mathcal{O}(\sigma, d_1, d_2, k)}}(d_1, d_2, k)$ simply simulates A_1 and returns 1 if it outputs the bit k.

3.3.1 Uniformity of the translation function in GGM.

In all the $\boldsymbol{PC}$ games in GGM defined above, the $\boldsymbol{tr}$ function is a multi-variate polynomial of degree 1 or 2, in up to 3 group elements, chosen uniformly at random from $[N]$. The coefficients of this polynomial are determined by the query. For such queries, the Schwartz–Zippel lemma [Sch80] immediately gives the following general result.

Lemma 1 (*u*-uniformity of $\boldsymbol{tr}$ in the GGM). *Let $\boldsymbol{PC}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ be a permutation challenge game in the GGM. Assume that $\boldsymbol{tr}$ is a multi-variate polynomial (with coefficients determined by the query) of degree $m > 0$ in v variables chosen uniformly at random from $[N]$ with N prime. Then, $\boldsymbol{tr}$ is $\frac{N}{m}$-uniform.*

3.4 Instantiation for the Even-Mansour cryptosystem

We represent plaintexts, ciphertexts and keys using $[N]$, and we can thus write $\sigma : [N] \to [N]$. Using this encoding, the bit representation of any $a \in [N]$ (used when XORing values in the domain) is the standard bit representation of $a - 1$.

We define a permutation challenge game for the key recovery. Let $\mathcal{D} = [N]^2$ be the space of secrets, consisting of pairs (k_1, k_2) (in case of the single-key scheme, $k_1 = k_2$ and $\mathcal{D} = [N]$). Let $\mathcal{M} = [N]$ be the space of outer queries, which are chosen encryption messages. For a secret $(k_1, k_2) \in [N]^2$ and a message (outer query) $m \in [N]$, define $\boldsymbol{tr}((k_1, k_2), m) = m \oplus k_1$, and let $\boldsymbol{post}((k_1, k_2), j) = j \oplus k_2$.

Note that an inner query that is issued to σ corresponds to a call to the public encryption oracle. Here, we allow the adversary to query σ^{-1} as well.

Let A_1 be a key-recovery algorithm for EM. For the permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{EM-KR}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$, the success predicate $\boldsymbol{suc}_{A_1^{\mathcal{O}(\sigma, (k_1, k_2))}}(k_1, k_2)$ simply simulates A_1 and returns 1 if it outputs the key (k_1, k_2).

4 Main General Result

In this section we prove our main result in the permutation challenge game model.

Theorem 10. *Let $PC := PC(N, \mathcal{D}, \mathcal{M}, tr, post, suc)$ be a permutation challenge game, such that tr is u-uniform, and suc compares the output of $(A_1)_z$ to some function of the secret. Let (A_0, A_1) be an (S, T) non-adaptive algorithm with preprocessing for PC. Denote by $MaxS(T)$ the optimal success probability (with respect to suc) of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success probability of (A_0, A_1) is at most*

$$\min\left(2 \cdot MaxS(T) + \frac{4\log(2)ST}{u} + \frac{T^2}{u}, MaxS(T) + \sqrt{\frac{\log(2)ST}{u}} + \frac{T^2}{2u}\right).$$

We note that a slightly better bound can be achieved for problems with a trivial post-processing function $post(d, j) = j$. For further details, see Appendix E.

4.1 Discussion

Before proving the theorem, several important remarks are due.

No need to analyze the preprocessing setting. The power of the theorem comes from the fact that there is no need to analyze the preprocessing setting. Indeed, it suffices to instantiate the permutation challenge game $PC := PC(N, \mathcal{D}, \mathcal{M}, tr, post, suc)$, to prove that tr is a u-uniform translation function (ideally, for $u \geq \Omega(N)$), and to bound $MaxS(T)$ (in the non-preprocessing setting). Plugging these values into Theorem 10 immediately bounds the success probability of an adversary in the preprocessing setting.

Generality of suc. We prove the theorem only for a limited class of success predicates that compare the output of $(A_1)_z$ to some function of the secret. Yet, almost all the steps of the proof apply to arbitrary non-adaptive success predicates (under the restrictions defined in Section 3), where the only exception that uses this restriction is Lemma 2. This lemma deals with a specific game in the non-preprocessing setting, and it is not possible to prove it in general for all predicates, as its statement is false for some artificial predicates.

However, for all "natural" predicates we are aware of, it is easy to extend the theorem tightly by extending the proof of Lemma 2 accordingly. This may require extending suc to make additional non-adaptive queries, which are accounted for in the parameter T. For example, it is possible to support selective forgery attacks, where the goal of the adversary is to predict the value of a predefined outer query (that the adversary is not allowed to make). This is done by extending suc to make this additional (non-adaptive) outer query, thus adjusting the total query complexity to $T + 1$. Now, suc verifies success by comparing the outcome of the query to the output of $(A_1)_z$. The proof is then adjusted by proving a corresponding variant of Lemma 2.

tr vs. $post$, and treatment of inner queries to σ^{-1}. The u-uniformity of tr is crucially used in the proof. On the other hand, there are no additional requirements on the function $post$, and indeed it does not play any direct role in the proof. Its relevance will be in applications of this theorem (in particular, it is important for bounding $MaxS(T)$). Similarly, the proof holds regardless of whether A_1 can issue inner queries to σ^{-1}, yet this fact may be important for bounding $MaxS(T)$.

4.2 Warm-up: no inner queries

Inner and outer queries. The proof of Theorem 10 combines two separate ideas. The first idea allows proving a restricted variant of the theorem under the assumption that A_1 makes no inner queries. The second idea shows how to lift this assumption and prove the general theorem. In order to isolate each of these ideas, we first prove the restricted variant of Theorem 10 and then extend it.

As we show below, the reason that it is easier to prove the restricted version is that if the algorithm makes no inner queries, then a variant of Shearer's lemma applies more directly, whereas for algorithms with inner queries, additional work is required.

Theorem 11 (Restricted variant of Theorem 10). *In the setting of Theorem 10, suppose that A_1 makes no inner queries. In addition, let $\boldsymbol{suc}$ be an arbitrary success predicate, as defined in Section 3. Then, the success probability of (A_0, A_1) is at most*

$$\min\left(2 \cdot \boldsymbol{MaxS}(T) + \frac{4\log(2)ST}{u}, \boldsymbol{MaxS}(T) + \sqrt{\frac{\log(2)ST}{u}}\right).$$

4.2.1 Proof overview.

- Recall that in the distribution $P_{Z,\Sigma}$, Σ is a uniform permutation and Z is determined by Σ as $Z = A_0(\Sigma)$. The proof essentially shows that a preprocessing adversary whose oracle is distributed as $P_{\Sigma|Z}$ cannot do much better that a non-preprocessing adversary whose oracle is distributed as P_Σ. Formally, this is done by defining a distribution Q, in which Σ and Z are independent, and the success probability of a non-preprocessing algorithm is bounded by $\boldsymbol{MaxS}(T)$ (Claim 1).

- We define κ_z to measure the amount of information that A_1 obtains if the preprocessing algorithm A_0 outputs the value z, compared to a non-preprocessing adversary under Q that obtains no information. We show that since z is of length S bits, on average over z, $\kappa_z \leq \log(2)S$ (Claim 2), confirming the intuitive insight that a string of S bits can provide at most S bits of information on average.

- Claim 3 is the heart of the proof. We fix $Z = z$, and show that the success probabilities of adversaries with and without z are very close (as a function of κ_z and T/u). This is done by observing that the translated outer queries (that are input to Σ) are u-uniform for a uniformly sampled secret d. It follows that the probability that A_1 queries any fixed $i \in [N]$ of Σ is at most $\frac{T}{u}$, which allows us to apply variants of Shearer's lemma.

- The theorem follows by averaging over Claim 3 and applying Claim 2 to bound this average.

4.2.2 Proof of Theorem 11.

We define a distribution $Q_{\Sigma,Z,D}$, in which Σ and Z are distributed as in P, but are now sampled independently. Specifically, $Q_{\Sigma,Z} = P_\Sigma P_Z$. The conditioned distribution of D remains unchanged, i.e., $Q_{D|\Sigma,Z} = P_{D|\Sigma,Z}$. Intuitively, the distribution Q represents running the algorithms (A_0, A_1), where each algorithm queries a different bijection, effectively making A_1 a non-preprocessing algorithm. Denoting the success probability of (A_0, A_1) with respect to Q by $q(A) := \mathbb{E}_{Q_{\Sigma,Z,D}}[\boldsymbol{suc}_{(A_1)_Z^{\mathcal{O}(\Sigma,D)}}(D)]$, we obtain the following claim as a direct corollary:

21

Claim 1.
$$q(A) \leq \boldsymbol{MaxS}(T).$$

Let
$$\kappa_z = \mathrm{D_{KL}}(P_{\Sigma|Z=z} \| Q_{\Sigma|Z=z})$$

measure the amount of information A_1 obtains if the preprocessing algorithm A_0 outputs the value z. We have:

Claim 2.
$$\mathop{\mathbb{E}}_{P_Z(z)}[\kappa_z] \leq \log(2)S. \tag{2}$$

Proof. Recall that $Q_{Z,\Sigma} = P_Z P_\Sigma$. Therefore, by Property (10) above, we obtain:

$$\mathop{\mathbb{E}}_{P_Z(z)}[\kappa_z] =: \mathrm{D_{KL}}(P_{\Sigma|Z} \| Q_{\Sigma|Z}) = \mathrm{D_{KL}}(P_{\Sigma|Z} \| P_\Sigma) = \mathrm{D_{KL}}(P_{\Sigma,Z} \| P_\Sigma P_Z) \leq \mathrm{H}(P_Z).$$

From Property (3), we get $\mathrm{H}(P_Z) \leq \log |\mathrm{Supp}(\mathcal{Z})| \leq \log(2)S$, which completes the proof. ∎

We will consider the success probability of (A_0, A_1) when some of the random variables are fixed and denote the fixed values in subscript. In particular, let $p_z := \mathbb{E}_{P_{\Sigma,D|Z=z}}[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\Sigma,D)}}(D)]$ and $q_z := \mathbb{E}_{Q_{\Sigma,D|Z=z}}[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\Sigma,D)}}(D)]$ denote the success probabilities of (A_0, A_1) conditioned on $Z = z$ with respect to P and Q, respectively. Clearly, $p = \mathbb{E}_{P_Z(z)}[p_z]$.

The heart of the proof of Theorem 11 is the following claim.

Claim 3. *For any z in the support of P_Z,*

$$p_z \leq \min\left(2q_z + \frac{4\kappa_z T}{u}, q_z + \sqrt{\frac{\kappa_z T}{u}}\right).$$

The proof of Theorem 11 below simply averages both sides of this claim over $P_Z(z)$.

Proof (of Claim 3). We fix z, which fixes the queries of A_1.

Recall that $\mathcal{U} = \mathcal{U}(z)$ denotes the set of outer queries made by A_1, which includes all queries in this case. Define $|\mathcal{D}|$ sets $\{\mathcal{U}_d\}_{d \in \mathcal{D}}$, where each $\mathcal{U}_d = \{\boldsymbol{tr}(d, m) \mid m \in \mathcal{U}\}$ represents the set of indices of σ queried by A_1 after translation, given that the secret is d and the preprocessing string is z. Fix any $i \in [N]$.

Since $\boldsymbol{tr}$ is a u-uniform translation function, we have

$$|\{d \mid i \in \mathcal{U}_d\}| = |\cup_{m \in \mathcal{U}} \{d \mid \boldsymbol{tr}(d, m) = i\}| \leq \sum_{m \in \mathcal{U}} |\{d \mid \boldsymbol{tr}(d, m) = i\}| \leq \frac{|\mathcal{U}| \cdot |\mathcal{D}|}{u} = \frac{T \cdot |\mathcal{D}|}{u}. \tag{3}$$

For every $d \in \mathcal{D}$, define the indicator function $f_d : [N]^N \to \{0, 1\}$ as $f_d(\sigma) = \boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma,d)}}(d)$, which outputs 1 if (the deterministic algorithm) A_1 succeeds on input σ and secret d when hard-wired with z.

Recall that $p_{z,d} := \mathbb{E}_{P_{\Sigma|Z=z,D=d}}[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\Sigma,d)}}(d)]$, $q_{z,d} := \mathbb{E}_{Q_{\Sigma|Z=z,D=d}}[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\Sigma,d)}}(d)]$ denote the success probabilities of (A_0, A_1) conditioned on $Z = z$ and $D = d$, with respect to P and Q, respectively. Note that under Q, generally $z \neq A_0(\sigma)$, but the success predicate $\boldsymbol{suc}$ does not depend on z (see Remark 3) and remains accurate.

Thus,

$$\underset{P_{\Sigma|Z=z}}{\mathbb{E}}[f_d(\Sigma)] = p_{z,d}, \quad \underset{Q_{\Sigma|Z=z}}{\mathbb{E}}[f_d(\Sigma)] = q_{z,d},$$

and

$$\frac{1}{|\mathcal{D}|}\sum_{d \in \mathcal{D}} p_{z,d} = p_z, \quad \frac{1}{|\mathcal{D}|}\sum_{d \in \mathcal{D}} q_{z,d} = q_z.$$

Recall that for every $d \in \mathcal{D}$, $f_d(\sigma) = \boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma,d)}}(d)$ depends only on the secret and the part of σ queried by $(A_1)_z^{\mathcal{O}(\sigma,d)}$. Hence, it follows from (3) that $\{f_d\}_{d \in \mathcal{D}}$ form a read-$(T \cdot |\mathcal{D}|/u)$ family on the N variables of Σ.

Observe that Q_Σ is the uniform distribution function over bijections on $[N]$, and P_Σ is another distribution function over such bijections.

We apply Theorem 8 and deduce

$$(2 \cdot T \cdot |\mathcal{D}|/u) \cdot \kappa_z = (2 \cdot T \cdot |\mathcal{D}|/u) \cdot \mathrm{D_{KL}}(P_{\Sigma|Z=z} \| Q_{\Sigma|Z=z}) \ge |\mathcal{D}| \cdot \mathrm{D_{KL}}(p_z \| q_z).$$

Therefore, $\mathrm{D_{KL}}(p_z \| q_z) \le \frac{2 \cdot \kappa_z T}{u}$.

Applying Corollary 1 and Property (9), respectively, we conclude

$$p_z \le 2q_z + \frac{4\kappa_z T}{u}, \text{ and } p_z \le q_z + \sqrt{\frac{\kappa_z T}{u}},$$

as claimed. ∎

Finally, we prove Theorem 11.

Proof (of Theorem 11). We average both sides of the inequality in Claim 3 over $P_Z(z)$.

On the left-hand-side we obtain $\mathbb{E}_{P_Z(z)}[p_z] = p$, which is the success probability of (A_0, A_1) with respect to P.

On the right-hand-side, we consider q_z and κ_z separately. First, by Claim 1, we obtain

$$\underset{P_Z(z)}{\mathbb{E}}[q_z] = \underset{Q_Z(z)}{\mathbb{E}}[q_z] = q \le \boldsymbol{MaxS}(T).$$

Second, recall that by Claim 2, we have $\mathbb{E}_{P_Z(z)}[\kappa_z] \le \log(2)S$. Therefore, for the first term we obtain

$$\underset{P_Z(z)}{\mathbb{E}}\left[\frac{4\kappa_z T}{u}\right] \le \frac{\log(2)4ST}{u}.$$

For the second term, we use the concavity of the square root function to conclude

$$\underset{P_Z(z)}{\mathbb{E}}\left[\sqrt{\frac{\kappa_z T}{u}}\right] \le \sqrt{\frac{\mathbb{E}_{P_Z(z)}[\kappa_z]T}{u}} \le \sqrt{\frac{\log(2)ST}{u}},$$

completing the proof. ∎

4.3   Proof of Theorem 10

We now assume that A_1 makes $T_1 \ge 0$ inner queries and prove Theorem 10. The structure of the proof is similar to that of Theorem 11, albeit somewhat more involved.

We prove Theorem 10 by a hybrid argument. We refer to the hybrid as a middle game:

Definition 3. *A middle game **MID** has the same setting as **PC**, and it differs only in the allowed actions for the adversary. An adversary $\widehat{A}$ is a pair of algorithms $(\widehat{A}_0, \widehat{A}_1)$ that work as follows: $\widehat{A}_0$ receives only N as input, and outputs* constraints *which include a pair of sequences $I = (I_1, \ldots, I_{T_1})$ and $O = (O_1, \ldots, O_{T_1})$, each containing non-repeating elements from $[N]$. Then a permutation $\sigma : [N] \to [N]$ is sampled uniformly from the set of permutations satisfying $\sigma(I_j) = O_j$ for all j. $\widehat{A}_1$ is defined similarly to the algorithm A_1 from **PC**, but with respect to that permutation σ, and it is allowed to make only outer queries and no inner queries. The outer queries of $\widehat{A}_1$ may depend on I and O. The rest of the game is defined similarly. The running time T_2 of $\widehat{A}_1$ is defined as the number of queries it makes, and the total running time of $\widehat{A}$ is defined as $T := T_1 + T_2$.*

4.3.1 Proof overview.

MID can be thought of as a non-preprocessing game, where the adversary is allowed to choose a limited number of T_1 values of the permutation (while the other values are sampled uniformly). The proof is divided into two steps.

1. The first step shows (in Theorem 12 below) that the preprocessing model **PC** is not much stronger than **MID**. First, allowing to fix T_1 values of the permutation gives the adversary in **MID** enough power to simulate the inner queries of the online algorithm in **PC**, that depend on the preprocessing string. Once we have dealt with the inner queries, it remains to deal with the outer queries. We exploit the fact that in **MID** the non-fixed elements of the permutation are sampled uniformly from all the possible (remaining) values, hence we are left with a uniform permutation on a smaller space. Thus, the proof proceeds in similar way to the proof of Theorem 11 (where the online adversary only makes outer queries), considering the permutation on the smaller space. As Theorem 11, Theorem 12 is applicable to an arbitrary success predicate ***suc***, as defined in Section 3.

2. The second step shows (in Lemma 2 below) that fixing T_1 values of the permutation (independently of the secret) does not give the **MID** adversary much advantage over a non-preprocessing adversary. This lemma assumes that ***suc*** compares the output of $(A_1)_z$ to some function of the secret.

Combining the two above steps allows showing that in our setting, a preprocessing adversary is not much stronger than a non-preprocessing adversary, proving Theorem 10.

4.3.2 First step.

We first prove that the game **MID** is not much weaker than **PC**.

Theorem 12. *Let $\boldsymbol{PC} := \boldsymbol{PC}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ be a permutation challenge game, such that $\boldsymbol{tr}$ is u-uniform. Let (A_0, A_1) be an (S, T) non-adaptive algorithm with preprocessing for $\boldsymbol{PC}$. Denote by $\widehat{\boldsymbol{MaxS}}(T)$ the optimal success probability (with respect to $\boldsymbol{suc}$) of an adversary to $\boldsymbol{MID}$ with running time T. Then, the success probability of (A_0, A_1) is at most*

$$\min\left(2 \cdot \widehat{\boldsymbol{MaxS}}(T) + \frac{4\log(2)ST}{u}, \widehat{\boldsymbol{MaxS}}(T) + \sqrt{\frac{\log(2)ST}{u}}\right).$$

In order to prove the theorem, some preparation is needed.

First, let us recall a few notations we shall use in the proof. The set of inner queries is denoted by $\mathcal{S}$, and we denote $\overline{\mathcal{S}} = [N] \setminus \mathcal{S}$. Recall that $\mathcal{S}$ depends on z. The distribution P was defined in Section 3.2, as $P_{\Sigma,Z,D} = P_\Sigma P_{Z|\Sigma} P_D$, where P_Σ is the uniform distribution over permutations, P_D is the distribution over secrets D, and $P_{Z|\Sigma=\sigma}(z) = 1$ if $z = A_0(\sigma)$.

We define a distribution $Q'_{\Sigma,Z,D}$ which is identical to $P_{\Sigma,Z,D}$, with the exception that

$$Q'_{\Sigma|Z} = Q'_{\mathcal{S},\Sigma_{\mathcal{S}},\Sigma_{\overline{\mathcal{S}}}|Z} = Q'_{\mathcal{S},\Sigma_{\mathcal{S}}|Z} Q'_{\Sigma_{\overline{\mathcal{S}}}|Z,\mathcal{S},\Sigma_{\mathcal{S}}}$$

is defined by setting $Q'_{\mathcal{S},\Sigma_{\mathcal{S}}|Z} = P_{\mathcal{S},\Sigma_{\mathcal{S}}|Z}$, while $Q'_{\Sigma_{\overline{\mathcal{S}}}|Z,\mathcal{S},\Sigma_{\mathcal{S}}}$ is the uniform distribution over bijections mapping the elements of $\overline{\mathcal{S}}$ to those of $[N] \setminus \Sigma_{\mathcal{S}}$. On the other hand, $Q'_\Sigma = P_\Sigma$, $Q'_Z = P_Z$, $Q'_D = P_D$, and $Q'_{\Sigma,Z,D} = Q'_{\Sigma,Z} \times Q'_D$.

To simplify notation, denote the random variables that A_1 obtains from inner queries to σ by $\Sigma^{in} := (\mathcal{S}, \Sigma_{\mathcal{S}})$.

Given Σ^{in}, the distribution function $Q'_{\Sigma_{\overline{\mathcal{S}}}|Z,\Sigma^{in}}$ does not depend on the value of Z. Therefore,

$$Q'_{\Sigma_{\overline{\mathcal{S}}}|Z,\Sigma^{in}} = Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}} = P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}}.$$

Note that if $T_1 = 0$ (i.e., if $\mathcal{S} = \emptyset$), then Q' is identical to Q defined above. However, in general we have $Q'_{\Sigma|Z} = Q'_{\Sigma^{in}|Z} Q'_{\Sigma_{\overline{\mathcal{S}}}|Z,\Sigma^{in}} = P_{\Sigma^{in}|Z} P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}}$, while $Q_{\Sigma|Z} = Q_\Sigma = P_\Sigma$.

Denote the success probability of (A_0, A_1) with respect to Q' by

$$q' := \mathop{\mathbb{E}}_{Q'_{\Sigma,Z,D}} [\boldsymbol{suc}_{(A_1)_Z^{\mathcal{O}(\Sigma,D)}}(D)].$$

Claim 4. *We have*

$$q' \leq \widehat{\boldsymbol{MaxS}}(T).$$

Proof. Recall that $Q'_{\Sigma_{\overline{\mathcal{S}}}|Z,\Sigma^{in}} = Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}} = P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}}$ is the uniform bijection mapping the elements of $\overline{\mathcal{S}}$ to those of $[N] \setminus \Sigma_{\mathcal{S}}$.

Since $(A_1)_z$ queries $\mathcal{S}$, the preprocessing string z does not give A_1 any additional information about Σ. Thus, A under Q' cannot do better than in the following game: First A_0 receives a uniform permutation σ', and chooses a set $\mathcal{S}$ of elements to fix to pre-chosen values. Then, these elements are fixed, and the other elements of σ' are re-shuffled to obtain σ. The rest of the game occurs with respect to σ. One can see that this game is equivalent to $\boldsymbol{MID}$.

Formally, given a preprocessing algorithm (A_0, A_1) we define a (randomized) adversary $(\widehat{A}_0, \widehat{A}_1)$ for $\boldsymbol{MID}$ that makes T queries and has success probability identical to (A_0, A_1) (i.e., q'). By definition, the success probability of $(\widehat{A}_0, \widehat{A}_1)$ is bounded by $\widehat{\boldsymbol{MaxS}}(T)$, hence this will complete the proof. It remains to define $(\widehat{A}_0, \widehat{A}_1)$.

$\widehat{A}_0$ receives N as input, samples a permutation σ', and calls A_0 to obtain z. Then, it calls A_1 with z as input and obtains the set $\mathcal{S}(z)$ of inner queries. Finally, $\widehat{A}_0$ outputs $I = \mathcal{S}(z)$ and $O = \sigma'_{\mathcal{S}(z)}$. $\widehat{A}_1$ receives oracle access to a permutation σ with $\sigma_{\mathcal{S}} = \sigma'_{\mathcal{S}}$ and independent values elsewhere. Notably, $\mathcal{S}, \sigma_{\mathcal{S}}$ and z are known to $\widehat{A}_1$, since $\widehat{A}_0$ and $\widehat{A}_1$ share the randomness tape and it is the only randomness source of $\widehat{A}_0$. Therefore, the distribution of σ conditioned on z is exactly $Q'_{\Sigma|Z}$. Therefore, $\widehat{A}_1$ can perfectly simulate A_1 by making the same (non-adaptive) outer queries as A_1 and returning its output. Thus, the success probability of $\widehat{A}$ is q', as required. ∎

Fixing z, σ^{in}, we can think of $\sigma_{\overline{\mathcal{S}}}$ drawn from either $P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in},Z=z}$ or $Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in}}$ as an extended bijection from $[N]$ to itself that maps the elements of $\mathcal{S}$ to those in $\sigma_{\mathcal{S}}$, and maps the elements of $\overline{\mathcal{S}}$ to those of $[N] \setminus \sigma_{\overline{\mathcal{S}}}$.

We can therefore think of σ^{in} as being hardwired into $(A_1)_z$ such that it only makes queries to the random variables $\Sigma_{\mathcal{S}}$ (after translation). Of course, some of the (translated) queries of $(A_1)_z$ may still fall in $\mathcal{S}$, but $\sigma^{in} = (\mathcal{S}, \sigma_{\mathcal{S}})$ are formally no longer random variables.

Denote
$$p_{z,\sigma^{in}} := \underset{P_{\Sigma_{\overline{\mathcal{S}}}, D | \Sigma^{in} = \sigma^{in}, Z = z}}{\mathbb{E}} \left[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma^{in}, \Sigma_{\overline{\mathcal{S}}}, D)}}(D) \right]$$

and
$$q'_{z,\sigma^{in}} := \underset{Q'_{\Sigma_{\overline{\mathcal{S}}}, D | \Sigma^{in} = \sigma^{in}, Z = z}}{\mathbb{E}} \left[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma^{in}, \Sigma_{\overline{\mathcal{S}}}, D)}}(D) \right].$$

Also, denote
$$\kappa'_{z,\sigma^{in}} = D_{\mathrm{KL}}(P_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}, Z = z} \| Q'_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}}) = D_{\mathrm{KL}}(P_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}, Z = z} \| P_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}}).$$

The heart of the proof of Theorem 12 is the following additional claim.

Claim 5. *For any z, σ^{in} in the support of $P_{Z, \Sigma^{in}}$,*

$$p_{z,\sigma^{in}} \leq \min \left(2q'_{z,\sigma^{in}} + \frac{4\kappa'_{z,\sigma^{in}} T_2}{u}, q'_{z,\sigma^{in}} + \sqrt{\frac{\kappa'_{z,\sigma^{in}} T_2}{u}} \right).$$

Proof. We fix z, σ^{in} (hence fixing the queries of A_1).

As in the proof of Claim 3, define $|\mathcal{D}|$ sets $\{\mathcal{U}_d\}_{d \in \mathcal{D}}$, where $\mathcal{U}_d = \{\boldsymbol{tr}(d, m) \mid m \in \mathcal{U}\}$. Recall that Σ^{in} and D are independent random variables (under both P and Q'), so for any value $\Sigma^{in} = \sigma^{in}$, D is still uniform.

For every $d \in \mathcal{D}$, define the indicator function $f_c : [N]^{N-T_1} \mapsto \{0,1\}$ by setting $f_d(\sigma_{\overline{\mathcal{S}}}) = \boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma^{in}, \sigma_{\overline{\mathcal{S}}}, d)}}(d)$, which outputs 1 if (the deterministic algorithm) A_1 succeeds with the secret d on input $\sigma_{\overline{\mathcal{S}}}$ when hard-wired with z, σ^{in}.

Recall that
$$p_{z,\sigma^{in},d} := \underset{P_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}, Z = z, D = d}}{\mathbb{E}} \left[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma^{in}, \Sigma_{\overline{\mathcal{S}}}, d)}}(d) \right]$$

and
$$q'_{z,\sigma^{in},d} := \underset{Q'_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}, Z = z, D = d}}{\mathbb{E}} \left[\boldsymbol{suc}_{(A_1)_z^{\mathcal{O}(\sigma^{in}, \Sigma_{\overline{\mathcal{S}}}, d)}}(d) \right]$$

denote the success probability of (A_0, A_1) conditioned on $Z = z, \Sigma^{in} = \sigma^{in}, D = d$ with respect to P and Q', respectively.

Thus,
$$\underset{P_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}, Z = z}}{\mathbb{E}} [f_d(\Sigma_{\overline{\mathcal{S}}})] = p_{z,\sigma^{in},d} \qquad \text{and} \qquad \underset{Q'_{\Sigma_{\overline{\mathcal{S}}} | \Sigma^{in} = \sigma^{in}, Z = z}}{\mathbb{E}} [f_d(\Sigma_{\overline{\mathcal{S}}})] = q'_{z,\sigma^{in},d}.$$

Also,
$$\frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} p_{z,\sigma^{in},d} = p_{z,\sigma^{in}} \qquad \text{and} \qquad \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} q'_{z,\sigma^{in},d} = q'_{z,\sigma^{in}}.$$

Recall that when σ^{in} is fixed, A_1 only makes queries in $\overline{\mathcal{S}}$ (after translation), and by similar calculation to (3), $\{f_d\}_{d \in \mathcal{D}}$ form a read-$(T_2 \cdot |\mathcal{D}|/u)$ family on the $N - T_1$ variables of $\Sigma_{\overline{\mathcal{S}}}$.

Note that $\sigma_{\overline{\mathcal{S}}}$ drawn from either $P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma_{\mathcal{S}}=\sigma_{\mathcal{S}},Z=z}$ or $Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in},Z=z} = Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in}}$ is a bijection from $\overline{\mathcal{S}}$ to $[N] \setminus \sigma_{\mathcal{S}}$, and $Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in}}$ is uniformly chosen. We apply Theorem 8 and deduce

$$(2 \cdot T_2 \cdot |\mathcal{D}|/u)\kappa'_{z,\sigma^{in}} = (2 \cdot T_2 \cdot |\mathcal{D}|/u)\,\mathrm{D}_{\mathrm{KL}}(P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in},Z=z}\|Q'_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in}})$$
$$\geq |\mathcal{D}|\,\mathrm{D}_{\mathrm{KL}}(p_{z,\sigma^{in}}\|q'_{z,\sigma^{in}}).$$

Therefore,

$$\mathrm{D}_{\mathrm{KL}}(p_{z,\sigma^{in}}\|q'_{z,\sigma^{in}}) \leq \frac{2 \cdot \kappa'_{z,\sigma^{in}}T_2}{u}.$$

Applying Corollary 1 and Property (9), respectively, we conclude

$$p_{z,\sigma^{in}} \leq 2q'_{z,\sigma^{in}} + \frac{4\kappa'_{z,\sigma^{in}}T_2}{u} \qquad \text{and} \qquad p_{z,\sigma^{in}} \leq q'_{z,\sigma^{in}} + \sqrt{\frac{\kappa'_{z,\sigma^{in}}T_2}{u}},$$

as claimed. ∎

Now, we are ready to prove Theorem 12.

Proof (of Theorem 12). We average both sides of the inequality of Claim 5 over $P_{Z,\Sigma^{in}}(z,\sigma^{in})$.
On the left-hand-side, we obtain $\mathbb{E}_{P_{Z,\Sigma^{in}}(z,\sigma^{in})}[p_{z,\sigma^{in}}] = p$.
On the right-hand-side, we consider $q'_{z,\sigma^{in}}$ and $\kappa'_{z,\sigma^{in}}$ separately.
First, since $P_{Z,\Sigma^{in}}(z,\sigma^{in}) = Q'_{Z,\Sigma^{in}}(z,\sigma^{in})$, we have

$$\mathbb{E}_{Q'_{Z,\Sigma^{in}}(z,\sigma^{in})}[q'_{z,\sigma^{in}}] = q' \leq \widehat{\boldsymbol{MaxS}}(T),$$

where the inequality is by Claim 4.
Second, we have

$$\mathbb{E}_{P_{Z,\Sigma^{in}}(z,\sigma^{in})}[\kappa'_{z,\sigma^{in}}] = \mathbb{E}_{P_{Z,\Sigma^{in}}(z,\sigma^{in})}[\mathrm{D}_{\mathrm{KL}}(P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in},Z=z}\|P_{\Sigma_{\overline{\mathcal{S}}}|\Sigma^{in}=\sigma^{in}})] = \mathrm{D}_{\mathrm{KL}}(P_{\Sigma|Z}\|P_{\Sigma}),$$

while by Property (3) and recalling that (A_0, A_1) is an (S,T) algorithm, we have

$$\mathrm{D}_{\mathrm{KL}}(P_{\Sigma|Z}\|P_{\Sigma}) = \mathrm{D}_{\mathrm{KL}}(P_{\Sigma,Z}\|P_{\Sigma}P_{Z}) \leq \mathrm{H}(P_{Z}) \leq \log|\mathrm{Supp}(\mathcal{Z})| \leq \log(2)S.$$

Overall, we obtain

$$p \leq 2 \cdot \widehat{\boldsymbol{MaxS}}(T) + \frac{\log(2)4ST}{u}, \text{ and } p \leq \widehat{\boldsymbol{MaxS}}(T) + \sqrt{\frac{\log(2)ST}{u}},$$

where the second inequality uses the concavity of the square root function. This concludes the proof. ∎

4.3.3   Second step - completing the proof of Theorem 10.

Theorem 10 follows immediately from Theorem 12 and the following lemma (which is the only lemma that restricts $\boldsymbol{suc}$):

Lemma 2. *Let $PC := PC(N, \mathcal{D}, \mathcal{M}, tr, post, suc)$ be a permutation challenge game, such that tr is u-uniform, and suc compares the output of A_1 to some function of the secret. Denote by $MaxS(T)$ the optimal success probability (with respect to suc) of a non-preprocessing, non-adaptive algorithm that makes at most T queries.*

Let MID be the corresponding game defined above (see Definition 3). Denote by $\widehat{MaxS}(T)$ the optimal success probability (with respect to suc) of an adversary to MID with running time T. Then

$$\widehat{MaxS}(T) \leq MaxS(T) + \frac{T^2}{2u}.$$

Proof. Let $\widehat{A} = (\widehat{A}_0, \widehat{A}_1)$ be an adversary to MID with running time $T = T_1 + T_2$, and denote its success probability by $\widehat{q}$. We describe a non-preprocessing algorithm A for PC with success probability $q \geq \widehat{q} - \frac{T^2}{2u}$ that makes T queries.

A simulates $\widehat{A}_0$ to obtain the constraints I, O, which also determine the outer queries of $\widehat{A}_1$ (recall that a MID game has no inner queries). It ignores the constraints of $\widehat{A}$, requests its outer queries from $\mathcal{O}$, sends the responses to $\widehat{A}_1$ and outputs the same value.

Let $g : \mathcal{D} \to \mathcal{D}'$ be the function of the secret computed by suc ($\mathcal{D}'$ is its output space). Denote by $P : \mathcal{D}' \times \mathcal{D}' \to \{0, 1\}$ the success predicate of suc.

Below, we define the algorithm $suc_{A^{\mathcal{O}(\sigma, d)}}(d)$ in the above case, where $\widehat{A}$ is simulated by A (we omit the explicit interaction with A for simplicity).

1. Receive I, O from $\widehat{A}_0$.

2. Pass I, O to $\widehat{A}_1$ and receive the outer queries $\mathcal{U} = (m_1, \ldots, m_{T_2})$.

3. Sample $d \sim \mathcal{D}$.

4. For all $1 \leq i \leq T_2$:

    (a) Denote $u_i \leftarrow tr(d, m_i)$.

    (b) If $u_i = u_j$ for some $j < i$, set $v_i \leftarrow v_j$. Otherwise, sample $v_i \sim [N] \setminus \{v_1, \ldots, v_{i-1}\}$ uniformly.

5. Pass $(post(d, v_1), \ldots, post(d, v_{T_2}))$ to $\widehat{A}_1$ and receive the answer $\tilde{d} \in D'$.

6. Return $P(g(d), \tilde{d})$.

Below, we define the algorithm $suc_{\widehat{A}_1^{\mathcal{O}(\sigma, d)}}(d)$, in case that $\widehat{A}$ runs in the MID game. It uses two additional flags, W_1, W_2, that do not affect the output, and are only defined for the sake of the analysis. The differences from the previous algorithm are underlined.

1. Receive I, O from $\widehat{A}_0$.

2. Pass I, O to $\widehat{A}_1$ and receive the outer queries $\mathcal{U} = (m_1, \ldots, m_{T_2})$.

3. Sample $d \sim \mathcal{D}$.

4. <u>Set $W_1 \leftarrow 0, W_2 \leftarrow 0$.</u>

5. For all $1 \leq i \leq T_2$:

    (a) Denote $u_i \leftarrow tr(d, m_i)$.

(b) If $u_i = I_k \in I$, set $v_i \leftarrow O_k$ and $W_1 \leftarrow 1$. Continue to m_{i+1}.

(c) If $u_i = u_j$ for some $j < i$, set $v_i \leftarrow v_j$. Otherwise, sample $v_i \sim [N] \setminus \{v_1, \ldots, v_{i-1}\}$ uniformly.

(d) If $v_i \in O$, re-sample $v_i \sim [N] \setminus (O \cup \{v_1, \ldots, v_{i-1}\})$ uniformly, and set $W_2 \leftarrow 1$.

6. Pass $(\boldsymbol{post}(d, v_1), \ldots, \boldsymbol{post}(d, v_{T_2}))$ to $\widehat{A}_1$ and receive the answer $\tilde{d} \in D'$.

7. Return $\mathrm{P}(g(d), \tilde{d})$.

Observe that the algorithms are equivalent, as long as $W_1 = W_2 = 0$ at the end of the execution of $\boldsymbol{suc}_{\widehat{A}_1^{\mathcal{O}(\sigma,d)}}(d))$. Thus,

$$
\begin{aligned}
q &= \Pr[\boldsymbol{suc}_{A^{\mathcal{O}(\Sigma,D)}}(D)] \\
&\geq \Pr[\boldsymbol{suc}_{\widehat{A}_1^{\mathcal{O}(\Sigma,D)}}(D)) \cap (W_1 = W_2 = 0)] \\
&\geq \Pr[\boldsymbol{suc}_{\widehat{A}_1^{\mathcal{O}(\Sigma,D)}}(D))] - \Pr[W_1 = 1] - \Pr[W_2 = 1] \\
&= \widehat{q} - \Pr[W_1 = 1] - \Pr[W_2 = 1].
\end{aligned}
$$

By a union bound over the T_2 outer queries, we have $\Pr[W_1 = 1] \leq \frac{T_1 T_2}{u} \leq \frac{T^2}{4u}$, where the probability is taken over the choice of d. By another union bound over the T_2 outer queries, $\Pr[W_2 = 1] \leq \frac{T^2}{4u}$, where the probability is taken over the choice of $c_1, c_2, \ldots$. Overall, $q \geq \widehat{q} - \frac{T^2}{2u}$, as claimed. $\blacksquare$

5 Applications to Problems in the Generic Group Model

In this section we describe our applications to the DLOG, DDH, and sqDDH problems. All these applications (as well as the application to the Even-Mansour cryptosystem presented in Section 6) are obtained via Theorem 10. This theorem can indeed be used, since in all these applications, $\boldsymbol{suc}$ compares the output of $(A_1)_z$ to some function of the secret (as defined in Section 3.3). We stress again that none of the uses of this theorem directly analyzes the preprocessing setting. For simplicity, we count the input group elements of the generic group algorithm A_1 as part of its T queries. Otherwise, we have to add to T a small additive factor based on its input size.

5.1 The discrete-log problem

As was written in the introduction, the bound we prove for the DLOG problem is the following.

Theorem 1. Let $A = (A_0, A_1)$ be a non-adaptive (S, T)-algorithm for the DLOG problem in the generic group model, over a group G with a prime number N of elements. Denote by $\boldsymbol{MaxS}_{\mathrm{DLOG}}(T)$ the optimal success probability of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success probability of A is at most

$$
2 \cdot \boldsymbol{MaxS}_{\mathrm{DLOG}}(T) + \frac{4 \log(2) ST}{N} + \frac{T^2}{N} \leq \frac{3T^2}{N} + \frac{4 \log(2) ST}{N}.
$$

Proof (of Theorem 1). We would like to use Theorem 10 via the corresponding permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{DL}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ defined in Section 3.3. For this purpose it remains to show that $\boldsymbol{tr}$ is N-uniform, where $\boldsymbol{tr}(d, (a, b)) = a \cdot d + b \mod N$. Since for every

query the translation function defines a polynomial of degree 1 in d, Lemma 1 implies that $\boldsymbol{tr}$ is N-uniform.

Finally, as was mentioned in the introduction, the bound on $\mathrm{online}(A_1)$ on the right-hand-side is by Shoup's theorem [Sho97] which bounds $\boldsymbol{MaxS}_{\mathrm{DLOG}}(T) \leq \frac{T^2}{N}$ for DLOG. ∎

5.2 The DDH and sqDDH problems

As was written in the introduction, the main theorem we prove for DDH and sqDDH is the following.

Theorem 2. Let $A = (A_0, A_1)$ be a non-adaptive (S, T) algorithm for the DDH (resp. sqDDH) problem in the GGM, over a group G with a prime number N of elements. Denote by $\boldsymbol{MaxS}_{\mathrm{DDH}}(T)$ the optimal success probability of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success probability of A is at most

$$\boldsymbol{MaxS}_{\mathrm{DDH}}(T) + \sqrt{\frac{2\log(2)ST}{N}} + \frac{T^2}{N} \leq \frac{1}{2} + \frac{2T^2}{N} + \sqrt{\frac{2\log(2)ST}{N}}.$$

Proof (of Theorem 2). We treat DDH and sqDDH separately.

DDH. We would like to use Theorem 10 via the corresponding permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{DDH}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ defined in Section 3.3. For this purpose, it remains to show that $\boldsymbol{tr}$ is $(N/2)$-uniform.

Recall that

$$\boldsymbol{tr}((d_1, d_2, d_3, k), (a_1, a_2, a_3, b)) = \begin{cases} a_1 \cdot d_1 + a_2 \cdot d_2 + a_3 \cdot d_3 + b \mod N & \text{if } k = 0, \\ a_1 \cdot d_1 + a_2 \cdot d_2 + a_3 \cdot (d_1 d_2) + b \mod N & \text{if } k = 1. \end{cases}$$

Since for every query, the translation function defines a polynomial of degree 1 or 2 in c_1, c_2, c_3, Lemma 1 implies that $\boldsymbol{tr}$ is $(N/2)$-uniform, as asserted.

sqDDH. We apply Theorem 10 via the corresponding permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{sqDDH}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ defined in Section 3.3. For this purpose, it remains to show that $\boldsymbol{tr}$ is $(N/2)$-uniform.

Recall that

$$\boldsymbol{tr}((d_1, d_2, k), (a_1, a_2, b)) = \begin{cases} a_1 \cdot d_1 + a_2 \cdot d_2 + b \mod N & \text{if } k = 0, \\ a_1 \cdot d_1 + a_2 \cdot (d_1)^2 + b \mod N & \text{if } k = 1. \end{cases}$$

Since for every query, the translation function defines a polynomial of degree 1 or 2 in d_1, d_2, Lemma 1 implies that $\boldsymbol{tr}$ is $(N/2)$-uniform, as asserted.

Finally, the inequality in the assertion holds by Shoup's theorem [Sho97] which bounds $\boldsymbol{MaxS}_{\mathrm{DDH}}(T) \leq \frac{1}{2} + \frac{T^2}{N}$ for DDH, and a similar argument which yields the same bound for sqDDH (see [CDG18]). ∎

6 Application to the Even-Mansour Cryptosystem

In this section we present our application to the EM cryptosystem. We first briefly review previous results on EM, and then we present the proof of Theorem 3.

6.1 Previous results on the security of EM

Recall that the Even-Mansour cryptosystem [EM97] is defined as $EM(m) = k_2 \oplus \sigma(k_1 \oplus m)$, where $k_1, k_2 \in \{0,1\}^n$ are n-bit keys and $\sigma : \{0,1\}^n \to \{0,1\}^n$ is a publicly known permutation. Even and Mansour showed that if σ is chosen uniformly at random, then any attack which makes T_2 encryption/decryption queries to EM and T_1 queries to σ or σ^{-1}, has a success probability of $O(T_1 T_2 / 2^n)$.

Right after the introduction of the EM cryptosystem, Daemen [Dae91] presented a matching attack: Pick an arbitrary $\alpha \in \{0,1\}^n$, query σ to obtain $T_1/2$ pairs of values $(\sigma(x_i), \sigma(x_i \oplus \alpha))$, and store the pairs $(\sigma(x_i) \oplus \sigma(x_i \oplus \alpha), x_i)$ in a sorted table. Then, query EM to obtain $T_2/2 = N/T_1$ pairs of values $(EM(m_i), EM(m_i \oplus \alpha))$, and check, for each j, whether $EM(m_j) \oplus EM(m_j \oplus \alpha)$ appears in the table. If a collision of the form $EM(m_j) \oplus EM(m_j \oplus \alpha) = \sigma(x_i) \oplus \sigma(x_i \oplus \alpha)$ is found, then it is likely that $k_1 = m_j \oplus x_i$ or $k_1 = m_j \oplus x_i \oplus \alpha$. At this stage, the second key k_2 can be retrieved easily. The attack algorithm is based on the fact that XOR with a secret key preserves XOR differences, and on the birthday paradox. The memory complexity of Daemen's attack is $\tilde{O}(T_1)$.

In 2000, Biryukov and Wagner [BW00] presented an alternative attack with complexities of $T_1 = T_2 = O(2^{n/2})$, and in 2012, Dunkelman, Keller and Shamir [DKS15] showed that its memory complexity can be reduced to $\tilde{O}(1)$, using an adaptive collision-search procedure. The authors of [DKS15] also generalized the attack of [BW00] to the entire tradeoff curve $T_1 T_2 = O(2^n)$ and asked whether its memory complexity can be reduced to $\tilde{O}(1)$ for the entire curve. This was partially addressed by Fouque, Joux and Mavromati [FJM14], who showed that the memory complexity can be reduced to $o(T_1)$ using an adaptive procedure, though a reduction to $\tilde{O}(1)$ has not been found yet.

While Daemen's attack is clearly non-adaptive, the Biryukov-Wagner attack and its enhancements heavily use adaptivity via Floyd's algorithm. Like in the case of DLOG, for about 35 years it hasn't been known whether adaptivity is indeed essential for reducing the memory complexity below the $\tilde{O}(T_1)$ complexity of Daemen's algorithm.

Daemen's algorithm can be naturally viewed as an algorithm with preprocessing, as the public knowledge of σ allows performing all queries to it offline. The Biryukov-Wagner algorithm and its variants do not use preprocessing. Fouque, Joux and Mavromati [FJM14] showed that if preprocessing with a space of $S = 2^{n/3}$ is allowed, then an attack with online complexity of $T = \tilde{O}(2^{n/3})$ can be mounted. In the other direction, Coretti, Dodis and Guo [CDG18] showed that any distinguishing attack on EM with preprocessing has a success probability of $\frac{1}{2} + \tilde{O}(\sqrt{S(T_1 + T_2)T_2/2^n} + T_1 T_2/2^n)$, matching the attack of [FJM14] in the case of constant success probability.

6.2 Our bound for EM

In this section we prove Theorem 3 stated in the introduction. We assume that adversary can query only the public and encryption oracles and not the decryption oracle. A direct corollary due to the symmetry of EM is that the same results hold if the adversary can query only the public and decryption oracles.

Theorem 3. Let $A = (A_0, A_1)$ be a key-recovery, non-adaptive (S, T)-adversary for the Even-Mansour cryptosystem, which can query only the public and encryption oracles and not the decryption oracle. Denote by $\boldsymbol{MaxS}_{\mathrm{EM}}(T)$ the optimal success probability of a non-preprocessing, non-adaptive algorithm that makes at most T queries. Then, the success prob-

ability of A is at most

$$2 \cdot \boldsymbol{MaxS}_{\mathrm{EM}}(T) + \frac{4\log(2)S(T+1)}{N} + \frac{T^2}{N} \leq \frac{3T^2}{N} + \frac{4\log(2)S(T+1)}{N}.$$

Moreover, the theorem also holds for the single-key variant where $k_1 = k_2$.

Proof (of Theorem 3). We would like to use Theorem 10 via the corresponding permutation challenge game $\boldsymbol{PC} := \boldsymbol{PC}_{EM-KR}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ defined in Section 3.4. First, observe that $\boldsymbol{suc}$ compares the output of $(A_1)_z$ to the secret key, which is a function of the secret. It remains to show that $\boldsymbol{tr}$ is N-uniform, where $\boldsymbol{tr}((k_1, k_2), m) = m \oplus k_1$.

Fix $m \in [N]$ and let $j \in [N]$. Then,

$$\boldsymbol{tr}((k_1, k_2)), m) = j \Leftrightarrow m \oplus k_1 = j \Leftrightarrow k_1 = j \oplus m.$$

Therefore,

$$\mathop{\mathbb{E}}_{P_{K_1, K_2}} \left[\mathbb{1}(\boldsymbol{tr}((K_1, K_2), m) = j) \right] = \mathop{\mathbb{E}}_{P_{K_1, K_2}} \left[\mathbb{1}(K_1 = j \oplus m) \right] = \tfrac{1}{N},$$

as required. Note that this also holds for the single-key scheme with $k_1 = k_2$. ∎

A Adaptivity in our Model

The untranslated queries of a non-adaptive (online) algorithm in our model may depend on the advice, but not on the challenge. We argue that it is reasonable to forbid the non-adaptive queries from depending on the challenge since the translation function should be viewed as adding an adaptivity round to the model (compared to standard problems such as function inversion).

We demonstrate this below for the DLOG problem. In particular, we argue that even if the (untranslated) queries of the adversary do not depend on the challenge $\sigma(d)$, its oracle queries will still depend on the challenge, making it comparable to a non-adaptive algorithm in a standard model (with challenge-dependent queries). Moreover, if the (untranslated) queries of the adversary depend on the challenge, then it is comparable to an adaptive algorithm in a standard model.

We first state the definition of adaptivity from [CK19], which was given in the context of the function inversion problem.

Definition 4 (Adaptivity ([CK19], Definition 2)). *We say that an oracle algorithm is k-round adaptive if the algorithm's oracle queries consist of k sets, such that each set of queries depends on the advice string, the input, and the replies to the previous rounds of queries. We call a 1-round adaptive algorithm non-adaptive. Finally, we say that an algorithm is strongly non-adaptive if it issues a single set of queries that only depends on the algorithm's input, but not on the advice string. In all of the above cases, when referring to the number of queries made by the algorithm, we account for the sum over all rounds.*

Observe that Definition 4 considers the queries to the oracle σ.

Consider the DLOG problem, for a secret DLOG d, where the algorithm receives the inputs $\sigma(1), \sigma(d), z$ (such that z is the advice string). For simplicity, we ignore the input $\sigma(1)$, as it may be given as part of z.

Suppose first that the algorithm makes a query $(a, b) = (f_1(z), f_2(z))$ (for some functions f_1, f_2) that is independent of $\sigma(d)$. This query is translated to the oracle query $d \cdot f_1(z) +$

$f_2(z) \bmod N$ which depends on d. This case is not covered by Definition 4, since the value d is not part of the input and the query cannot be computed only from the input of the algorithm.

However, the query indirectly depends on $\sigma(d)$, since $d = \sigma^{-1}(\sigma(d))$.

In contrast, consider, for example, the function inversion problem for a permutation with advice, where the algorithm receives $z, \sigma(d)$ and its goal is to output d. Suppose that it makes the query $a = f(z)$ (for some function f), which is passed directly to the oracle σ. Clearly, there is no dependence at all on $\sigma(d)$. In fact, unlike in our model, such an adversary is not interesting at all (e.g., all such queries could be included in z, reducing the number of online queries to zero and making the problem degenerate and trivial).

Next, suppose that the adversary makes a query of the form $(a, b) = (f_1(\sigma(d), z), f_2(\sigma(d), z))$ (for some functions f_1, f_2), which is translated to the oracle query $d \cdot f_1(\sigma(d), z) + f_2(\sigma(d), z) \bmod N$. This query directly depends on both d and $\sigma(d)$.

Once again, we cannot directly rely on the adaptivity definition of Definition 4 to determine the number of adaptivity rounds in such an algorithm, since d is not part of the input. Generally, it is logical to consider such an algorithm as adaptive since it has an additional adaptivity round compared to the previous case.

Specifically, we show below that such a query depends on σ beyond the direct dependency via the inputs $\sigma(d)$ and z. An algorithm making a query with such a complex dependency on σ should arguably be considered adaptive.

Suppose that we fix the inputs $\sigma(d) = v$ and z and change σ to a different σ' that is associated with the same advice string z. Now, the corresponding secret DLOG changes to $d' = (\sigma')^{-1}(v)$, and hence the oracle query changes to $d' \cdot f_1(\sigma(d), z) + f_2(\sigma(d), z) \bmod N$, as claimed above.

In contrast, consider, for example, the function inversion problem for a permutation, where the algorithm receives $\sigma(d)$ as input. Suppose that it makes a query $a = f(\sigma(d), z)$ (for some function f), which is passed directly to the oracle σ. Clearly, if we fix $\sigma(d)$ and z, the oracle query remains the same. Hence, the algorithm is non-adaptive by our argument, which is also consistent with Definition 4.

B Limitations of Previous Techniques in Dealing with Non-Adaptive Algorithms

We consider the three main generic techniques for proving cryptanalytic time-space tradeoffs: compression arguments, the pre-sampling technique, and concentration inequalities. The pre-sampling technique and the concentration inequalities technique are closely related, as demonstrated in [GLLZ21]. Our techniques are related to both, as they are also based on concentration inequalities. Yet, the previously used concentration inequalities are somewhat similar to martingales, which are generally not strong enough to obtain "dimension-free" inequalities such as those obtained from Shearer's inequality (see [Von10, Section 4]). Furthermore, it seems that compression arguments are not sufficiently strong to obtain our results as well (though, we cannot give a short intuitive argument for this).

We now argue in more detail that previous techniques are inherently limited and (up to logarithmic factors), cannot distinguish between adaptive and non-adaptive algorithms, focusing on the DLOG problem for simplicity. Since this argument is obvious for the pre-sampling technique, we focus on techniques based on concentration inequalities and compression arguments.

We remark that we cannot completely rule out the possibility that a simple extension of these techniques would give a significant improvement, but this seems highly unlikely.

B.1 Concentration inequality-based proofs in the multi-instance model

The MI model.

We summarize the main ideas of the technique that is based on concentration inequalities. It reduces proving security against adversaries with advice to analyzing multi-instance (MI) security against adversaries without advice. In the (basic) MI DLOG game, an adversary plays a sequence of DLOG instances. Initially, a uniform permutation is chosen, and it remains fixed throughout the MI DLOG game. In each instance, a new uniform DLOG secret is independently chosen and the adversary is allowed to issue T new queries in order to solve it. The adversary wins the multi-instance game if all instances are solved correctly. The main theorem asserts that if the success probability (of any adversary) in the MI DLOG game with S instances is at most δ^S for some $0 < \delta < 1$, then the success probability of any adversary with S bits of advice is at most $O(\delta)$.

For simplicity, we focus on the setting of a constant δ. Below we describe a non-adaptive adversary to the MI game with success probability of at least $2^{-\tilde{O}(N/T^2)}$ regardless of the number of instances. Asymptotically, this is also the success probability of the best general (adaptive) adversary, hence this technique cannot distinguish between adaptive and non-adaptive adversaries.

Considering time-space tradeoffs in the proprocessing model, our adversary shows that this technique cannot prove a bound that is better than $ST^2 \geq \tilde{\Omega}(N)$ (which is tight for adaptive algorithms [CK18]), while Theorem 1 proves a much better bound.

The MI adversary.

Assume that $T^2 = o(N)$, as otherwise, a non-preprocessing algorithm already achieves a constant advantage. In addition, the MI adversary can guess each secret DLOG with probability N^{-1}, hence it can achieve a success probability of at least $2^{-\tilde{O}(N/T^2)}$ with $\tilde{O}(N/T^2)$ instances. We may thus assume that the number of instances S is at least $\tilde{\Omega}(N/T^2)$.

In the MI game, we have a sequence of DLOG instances, with corresponding secrets $d_1, d_2, \ldots$. Consider an MI adversary B that for each instance, executes the same sequence of queries $(a_j, 0)_{j \in [T]}$. We select a generator $g \in \mathbb{Z}_N^*$ and choose $a_i = g^{-i} \mod N$ for $1 \leq i \leq T/2$ and $a_i = g^{(i-T/2) \cdot T} \mod N$ for $T/2 < i \leq T$ (somewhat mimicking the baby-step-giant-step algorithm). These queries are translated to oracle queries of the form $a_j d_i \mod N$.

Observe that if a collision in the translated queries of the form $a_j d_i \mod N = a_{j'} d_{i'} \mod N$ occurs for $d_i \neq d_{i'}$, then $d_i = a_{j'} \cdot (a_j)^{-1} \cdot d_{i'} \mod N$ (recall that $a_j \neq 0$ by assumption). Hence, $d_{i'}$ determines d_i and vice versa. Such collisions can be detected by B, as they lead to collisions at the output of σ.

Thus, for each instance, if its (translated) queries do not collide with the (translated) queries of a previous instance, B guesses its secret. Otherwise, using the collision, B computes the secret of the instance deterministically using the previous guess.

Analysis sketch.

Let us show that for $S \geq \tilde{\Omega}(N/T^2)$, the success probability of B is at least $2^{-\tilde{O}(N/T^2)}$. B guesses d_i correctly for all $1 \leq i \leq \tilde{O}(N/T^2)$ with probability $2^{-\tilde{O}(N/T^2)}$. In this case, B succeeds if each of the remaining secrets can be determined from them. Hence, it remains to show that each of the remaining secrets can be determined with high probability.

Notably, after $\tilde{\Omega}(N/T^2)$ instances — corresponding to a total of $\tilde{\Omega}(N/T)$ queries — the following property holds with very high probability due to our choice of non-adaptive queries: every interval of the form $g^i, g^{i+1}, \ldots, g^j$, where $i, j \in [N-1]$ and $j-i \bmod N-1 = T/2$, contains at least one queried point. The proof is by a Chernoff bound for each interval (exploiting the fact that each interval is hit by a query of an instance with probability $\Omega(T^2/N)$), followed by a union bound over all such intervals. Once this property holds, the queries of every new instance are guaranteed to collide with the queries of a (successfully guessed) previous one, allowing its recovery.

B.2   Compression arguments

Compression arguments for proving time-space lower bounds for the DLOG problem have the following structure: one starts with a DLOG adversary with advice $A = (A_0, A_1)$ that possesses a (too good to be true) time-space tradeoff. This adversary is then used to devise encoding and decoding procedures for the random permutation oracle. These procedures compress the oracle beyond what is possible information-theoretically, leading to a contradiction.

Specifically, on input σ, the encoding procedure first runs $A_0(\sigma)$ and writes the advice z of length S bits as part of the encoding string. Then (somewhat similarly to the MI game), it runs A_1 sequentially using z on multiple DLOG instances, and the goal is to gain beyond S bits in the encoding length, establishing a contradiction.[6]

In each instance, the encoding procedure encodes the query answers of A_1, so that the decoding procedure can repeat the same steps. Assuming that A_1 is deterministic, this is sufficient (otherwise, shared randomness is used). A saving in the encoding length is obtained when A_1 answers the instance correctly, as this answer gives information about σ (of at most $\log N$ bits) that does not need to be encoded.

For simplicity, we assume that A answers correctly with constant probability. We shall show that after answering $\tilde{\Omega}(N/T^2)$ instances, a saving cannot be obtained (with high probability). Thus, the total saving is bounded by $\tilde{O}(N/T^2)$ bits and the technique cannot be used to prove a time-space tradeoff that is better than $ST^2 \geq \tilde{\Omega}(N)$. Once again, this time-space tradeoff is tight for adaptive algorithms, and this technique cannot distinguish between adaptive and non-adaptive algorithms (up to logarithmic factors).

It remains to show that after answering $\tilde{\Omega}(N/T^2)$ instances, a saving cannot be obtained (with high probability). The argument here is similar to the one used for concentration inequalities. At the threshold of $\tilde{\Omega}(N/T^2)$ instances, about $\tilde{\Omega}(N/T)$ queries were already made. Therefore, it is very likely that at least one of the T queries for the new instance will collide with a previous one. Such a collision query gives no new information about σ, hence encoding it in a standard way, using about $\log N$ bits, results in a loss that nullifies the encoding advantage from running A_1.

One may try to encode collisions in a special way to mitigate the loss by encoding the specific query indices that collide. However, at the threshold of $\tilde{\Omega}(N/T^2)$ instances (where $\tilde{\Omega}(N/T)$ queries have already been issued), encoding the query pair seems to require more than $\log T + (\log N - \log T) = \log N$ bits, once again nullifying the encoding advantage from running A_1.

---

[6]These DLOG instances have non-uniformly distributed secrets, on which the success of A_1 is not guaranteed. However, this problem is addressed using the random self-reducibility property of the DLOG problem, which allows to execute A_1 on instances where the secret is uniformly distributed.

C   Non-adaptive sqDDH Algorithm

We sketch the details of the non-adaptive variant of the adaptive sqDDH algorithm of [CK18]. Essentially, the algorithm replaces the adaptive random walk of [CK18] with a sequence that can be computed non-adaptively.

First, define a biased pseudorandom predicate $P : [N]^2 \to \{0,1\}$ that evaluates to 1 with probability about T^{-1}. We call a pair of the form $(\sigma(x), \sigma(x^2))$ (where $x \in [N]$) special if $P(\sigma(x), \sigma(x^2)) = 1$.

Second, define a balanced pseudorandom predicate $Q : [N]^2 \to \{0,1\}$ that evaluates to 1 with probability $1/2$.

Third, define a pseudorandom function $g : [N]^2 \to [S]$ that evaluates to each $i \in [S]$ with probability about S^{-1}.

Given σ, for each $v \in [S]$, define the set

$$\mathcal{L}_v = \{(\sigma(x), \sigma(x^2)) \in [N]^2 \colon P(\sigma(x), \sigma(x^2)) = 1 \wedge g(\sigma(x), \sigma(x^2)) = v\},$$

which contains roughly $\frac{N}{ST}$ pairs. Moreover, define the majority value of Q on $\mathcal{L}_v$ by

$$\mathrm{Maj}_v = \mathrm{Majority}\{Q(\sigma(x), \sigma(x^2)) \colon (\sigma(x), \sigma(x^2)) \in \mathcal{L}_v\}.$$

A standard probabilistic argument shows that with high probability, Maj_v agrees with a pair chosen uniformly form $\mathcal{L}_v$ on the value of Q with probability about $\frac{1}{2} + \sqrt{\frac{ST}{N}}$. Namely, with high probability,

$$\Pr_{(\sigma(x), \sigma(x^2)) \sim \mathcal{L}_v} [Q(\sigma(x), \sigma(x^2)) = \mathrm{Maj}_v] \geq \frac{1}{2} + \Omega\left(\sqrt{\frac{ST}{N}}\right).$$

The preprocessing algorithm A_0 works by calculating and storing Maj_v for each $v \in [S]$ in the advice string.

The online algorithm A_1 defines a sequence of $T/2$ pairs in $[N]^2$ that can be computed using non-adaptive queries and preserve the sqDDH relation. For example, for some pseudorandom function $f : [N] \to [N]$, starting from the secret (d_1, d_2), define the sequence $(\sigma(d_1), \sigma(d_2)), (\sigma(d_1 \cdot f(1)), \sigma(d_2 \cdot f(1)^2)), (\sigma(d_1 \cdot f(2)), \sigma(d_2 \cdot f(2)^2)) \ldots$.

With high probability, one of these $T/2$ pairs evaluates to 1 on P (otherwise, A_1 guesses the answer). Denote the first such pair by (y_1, y_2). A_1 computes $v := g(y_1, y_2)$ and $b := Q(y_1, y_2)$. It then returns 1 if $b = \mathrm{Maj}_v$, and 0 otherwise.

The main observation in the analysis is that on a YES instance, $b = \mathrm{Maj}_v$ with probability about $\frac{1}{2} + \sqrt{\frac{ST}{N}}$ (as noted above). On the other hand, on a NO instance, $b = \mathrm{Maj}_v$ with probability of about $\frac{1}{2}$, since Q is balanced and uncorrelated with pairs computed for NO instances.

D   A Variant of Shearer's Lemma for Bijections

In this appendix we present an elementary proof of a slightly weaker version of Theorem 7, in which the constant 2 is replaced by 9. Specifically, we prove the following.

Theorem 13. *Let $\mathcal{X}$ be a set of size N. Let $Q_X = Q_{X_1, \ldots, X_N}$ be the uniform distribution over bijections from $[N]$ to $\mathcal{X}$, and let $P_X = P_{X_1, \ldots, X_N}$ be another distribution over such bijections.*

Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be subsets of $[N]$, such that each $i \in [N]$ belongs to at most k of them. Then

$$9k \cdot \mathrm{D}_{KL}(P_X \| Q_X) \geq \sum_{j \in [m]} \mathrm{D}_{KL}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}),$$

where $P_{X_{\mathcal{U}}}$ is the distribution of the vector $X_{\mathcal{U}} := (X_i \mid i \in \mathcal{U})$ with respect to P (and analogously for Q).

We need several tools in order to prove the theorem. Let us begin with the following technical lemma:

Lemma 3. *Let $N \in \mathbb{N}$ and let $\ell \leq \frac{N}{4}$. Let $p_1, \ldots, p_\ell \in (0, 1)$ with $\sum_i p_i \leq 1$, and let us denote $\epsilon_i = p_i - \frac{1}{N}$. Denote also $p' = \frac{1 - \sum_i p_i}{N - \ell}$ and $\epsilon' = p' - \frac{1}{N}$. Then the following holds:*

$$Np' \log\left(Np'\right) - N\epsilon' \leq 4 \sum_i \left(p_i \log(Np_i) - \epsilon_i\right).$$

Proof. The proof utilizes the following inequalities (recall that logarithms are in base e):

  (i) For all $x > 0$ it holds that $x \log x - x + 1 \geq 0$.

  (ii) For all $0 < x \leq 2$ it holds that $x \log x - x + 1 \leq (x - 1)^2 \leq |x - 1|$.

  (iii) For all $x < 5$ it holds that $x \log x - x + 1 \geq \frac{1}{4}(x - 1)^2$.

  (iv) For all $x \geq 5$ it holds that $x \log x - x + 1 \geq |x - 1|$.

All these inequalities can be easily proved by basic analytical tools.

We prove the statement by case-analysis. Let us first assume that

$$\sum_{i: Np_i \geq 5} |\epsilon_i| \geq \frac{1}{2} \sum_{i: Np_i < 5} |\epsilon_i|.$$

We obtain

$$\sum_i |\epsilon_i| \leq 3 \sum_{i: Np_i \geq 5} |\epsilon_i|.$$

Since $\ell \leq \frac{N}{4}$ and following the definition of ϵ', we may also obtain

$$\left|N\epsilon'\right| \leq \frac{4}{3} \sum_i |\epsilon_i| \leq 4 \sum_{i: Np_i \geq 5} |\epsilon_i|. \tag{4}$$

Therefore, we may bound:

$$
\begin{aligned}
Np' \log\left(Np'\right) - N\epsilon' &= (Np') \log(Np') - Np' + 1 \\
&\leq \left|N\epsilon'\right| && \text{ineq. (ii) with } x = Np' \\
&\leq 4 \sum_{i: Np_i \geq 5} |\epsilon_i| && \text{Equation (4)} \\
&\leq \frac{4}{N} \sum_{i: Np_i \geq 5} (Np_i \log(Np_i) - Np_i + 1) && \text{ineq. (iv) with } x = Np_i \\
&= 4 \sum_{i: Np_i \geq 5} (p_i \log(Np_i) - \epsilon_i) \\
&\leq 4 \sum_i (p_i \log(Np_i) - \epsilon_i) && \text{ineq. (i) with } x = Np_i,
\end{aligned}
$$

as required.

Next, we assume

$$\sum_{i:Np_i\geq 5}|\epsilon_i| < \frac{1}{2}\sum_{i:Np_i<5}|\epsilon_i|,$$

implying that

$$\left|N\epsilon'\right| \leq \frac{4}{3}\sum_i |\epsilon_i| \leq 2\sum_{i:Np_i<5}|\epsilon_i|. \tag{5}$$

Therefore we may bound:

$$
\begin{aligned}
Np'\log\left(Np'\right) - N\epsilon' = (Np')\log(Np') - Np' + 1 & \\
\leq \left(N\epsilon'\right)^2 & \qquad \text{ineq. (ii) with } x = Np' \\
\leq \left(2\sum_{i:Np_i<5}|\epsilon_i|\right)^2 & \qquad \text{Equation (5)} \\
\leq 4\ell\sum_{i:Np_i<5}\epsilon_i^2 & \qquad \text{Cauchy–Schwarz inequality} \\
\leq \frac{16\ell}{N}\sum_{i:Np_i<5}(p_i\log(Np_i) - \epsilon_i) & \qquad \text{ineq. (iii) with } x = Np_i \\
\leq 4\sum_{i:Np_i<5}(p_i\log(Np_i) - \epsilon_i) & \qquad 4\ell \leq N \\
\leq 4\sum_i (p_i\log(Np_i) - \epsilon_i) & \qquad \text{ineq. (i) with } x = Np_i,
\end{aligned}
$$

as required. ■

Lemma 3 allows proving a variant of Shearer's inequality for indicator vectors.

Lemma 4 (Variant of Shearer's inequality for indicator vectors)**.** *Let N be a positive integer. Let $Q_X = Q_{X_1,\ldots,X_N}$ be the uniform distribution function over the set $\binom{[N]}{1} := \{v \in \{0,1\}^N \mid \sum_i v_i = 1\}$ of indicator vectors, and let $P_X = P_{X_1,\ldots,X_N}$ be another distribution function over the same set. Let $\mathcal{U}_1,\ldots,\mathcal{U}_m \subseteq [N]$ be such that for each $i \in [N]$, $|\{j \in [m] \mid i \in \mathcal{U}_j\}| \leq k$. Then*

$$9k \cdot \mathrm{D}_{KL}(P_X\|Q_X) \geq \sum_{j\in[m]}\mathrm{D}_{KL}(P_{X_{\mathcal{U}_j}}\|Q_{X_{\mathcal{U}_j}}).$$

Proof. Denote $S = \mathrm{D}_{\mathrm{KL}}(P_X\|Q_X)$. By applying Property (8) we obtain that for every $\mathcal{U} \subseteq [N]$ we have $\mathrm{D}_{\mathrm{KL}}(P_{X_{\mathcal{U}}}\|Q_{X_{\mathcal{U}}}) \leq S$. Since the number of sets $\mathcal{U}_j$ with $|\mathcal{U}_j| > \frac{N}{4}$ is at most $4k$, we may assume that $|\mathcal{U}_j| \leq \frac{N}{4}$ and prove that

$$\sum_{j\in[m]}\mathrm{D}_{\mathrm{KL}}(P_{X_{\mathcal{U}_j}}\|Q_{X_{\mathcal{U}_j}}) \leq 5kS.$$

Moreover, since $\mathrm{D}_{\mathrm{KL}}(P_{X_i}\|Q_{X_i}) \geq 0$ following Property (4), we may assume that $|\{j \in [m] \mid i \in \mathcal{U}_j\}| = k$ for all $i \in [N]$.

For every $i \in [N]$, we notice that $Q_{X_i}(1) = \frac{1}{N}$ (i.e., the probability of the event $X_i = 1$ when sampled with respect to Q is $1/N$) and denote $p_i = P_{X_i}(1)$ and $\epsilon_i = p_i - \frac{1}{N}$, to obtain by definition $S = \sum_i p_i \log(Np_i)$. Additionally, for every set $\mathcal{U} \subseteq [N]$ we denote

$$p'(\mathcal{U}) = \frac{1 - \sum_{i \in \mathcal{U}} p_i}{N - |\mathcal{U}|} \quad \text{and} \quad \epsilon'(\mathcal{U}) = p'(\mathcal{U}) - \frac{1}{N} = -\frac{\sum_{i \in \mathcal{U}} \epsilon_i}{N - |\mathcal{U}|}.$$

Let $\mathcal{U} \subseteq [N]$ be an index set. By the definition of KL-divergence, we may obtain:

$$D_{\mathrm{KL}}(P_{X_\mathcal{U}} \| Q_{X_\mathcal{U}}) = \sum_{v \in \mathrm{Supp}(X_\mathcal{U})} P_{X_\mathcal{U}}(v) \cdot \log\left(\frac{P_{X_\mathcal{U}}(v)}{Q_{X_\mathcal{U}}(v)}\right)$$

$$= \sum_{i \in \mathcal{U}} p_i \log(Np_i) + \left(1 - \sum_{i \in \mathcal{U}} p_i\right) \log\left(\frac{1 - \sum_{i \in \mathcal{U}} p_i}{1 - \frac{|\mathcal{U}|}{N}}\right)$$

$$= \sum_{i \in \mathcal{U}} p_i \log(Np_i) + (N - |\mathcal{U}|)p'(\mathcal{U}) \log(Np'(\mathcal{U})).$$

Notably, $\sum_{j \in [m]} \sum_{i \in \mathcal{U}_j} p_i \log(Np_i) = kS$, so it remains to bound

$$\sum_{j \in [m]} (N - |\mathcal{U}_j|)p'(\mathcal{U}_j) \log(Np'(\mathcal{U}_j)).$$

Since $|\{j \in [m] \mid i \in \mathcal{U}_j\}| = k$ is assumed to be constant, we obtain that

$$\sum_{j \in [m]} (N - |\mathcal{U}_j|)\epsilon'(\mathcal{U}_j) = -\sum_i k\epsilon_i = 0.$$

Therefore, we may bound:

$$\sum_{j \in [m]} (N - |\mathcal{U}_j|)p'(\mathcal{U}_j) \log(Np'(\mathcal{U}_j)) = \sum_{j \in [m]} (N - |\mathcal{U}_j|)\left(p'(\mathcal{U}_j) \log(Np'(\mathcal{U}_j)) - \epsilon'(\mathcal{U}_j)\right)$$

$$\leq \sum_{j \in [m]} N\left(p'(\mathcal{U}_j) \log(Np'(\mathcal{U}_j)) - \epsilon'(\mathcal{U}_j)\right)$$

$$\leq 4 \sum_{j \in [m]} \sum_{i \in \mathcal{U}_j} (p_i \log(Np_i) - \epsilon_i)$$

$$= 4kS,$$

where the first inequality follows from the fact that $(p'(\mathcal{U}_j) \log(Np'(\mathcal{U}_j)) - \epsilon'(\mathcal{U}_j)) \geq 0$ (implied by ineq. (i) from the proof of Lemma 3), and the second inequality follows from Lemma 3. ∎

In order to prove Theorem 13, we need another lemma which shows that under an additional assumption, conditioning does not decrease KL-divergence.

Lemma 5. *Let P and Q be two distributions on pairs (X, Y). Suppose that X and Y are independent with respect to Q (i.e., $Q_{X,Y} = Q_X Q_Y$). Then*

$$D_{KL}(P_Y \| Q_Y) \leq D_{KL}(P_{Y|X} \| Q_{Y|X}).$$

Proof. Since $Q_{X,Y} = Q_X Q_Y$, we obtain

$$Q_Y(y) = \sum_x P_X(x) Q_{Y|X=x}(y).$$

Moreover, by the law of total probability, we obtain

$$P_Y(y) = \sum_x P_X(x) P_{Y|X=x}(y).$$

Therefore, we obtain

$$\mathrm{D_{KL}}(P_Y \| Q_Y) = \mathrm{D_{KL}} \left(\left(\sum_x P_X(x) P_{Y|X=x}(y) \right) \| \left(\sum_x P_X(x) Q_{Y|X=x}(y) \right) \right).$$

Finally, we may apply Property (5) inductively to obtain

$$\mathrm{D_{KL}}(P_Y \| Q_Y) \le \sum_x P_X(x)\, \mathrm{D_{KL}}(P_{Y|X=x} \| Q_{Y|X=x}) =: \mathrm{D_{KL}}(P_{Y|X} \| Q_{Y|X}).$$

$\blacksquare$

Now we are ready to prove Theorem 13.

Proof (Proof of Theorem 13). First we set some notations, given a bijection $X = X_1, \ldots, X_N :$ $[N] \to \mathcal{X}$, indices $i, j \in [N]$ and a set $\mathcal{S} \subseteq [N]$:

$$
\begin{aligned}
X_{i \to j} &= \begin{cases} 1 & X_i = j, \\ 0 & \text{otherwise}; \end{cases} \\
X_{\to j} &= X^{-1}(j), \\
X_{\mathcal{S} \to j} &= \begin{cases} X^{-1}(j) & X^{-1}(j) \in \mathcal{S}, \\ * & \text{otherwise}. \end{cases}
\end{aligned}
$$

Note that $X = X_1, \ldots, X_N$ and the sequence $X_{\to 1}, \ldots, X_{\to N}$ are equivalent in the sense that they both encode the same information about the permutation. Similarly, X_i is equivalent to $(X_{i \to 1}, \ldots, X_{i \to N})$, and $X_{\mathcal{S}}$ is equivalent to $(X_{\mathcal{S} \to 1}, \ldots, X_{\mathcal{S} \to N})$.

By Property (7), we obtain:

$$\mathrm{D_{KL}}(P_X \| Q_X) = \mathrm{D_{KL}}(P_{X_{\to 1}} \| Q_{X_{\to 1}}) + \mathrm{D_{KL}}\left(P_{X_{\to 2}, \ldots, X_{\to N} | X_{\to 1}} \| Q_{X_{\to 2}, \ldots, X_{\to N} | X_{\to 1}} \right). \tag{6}$$

Similarly, given a set $\mathcal{U}$, we obtain:

$$
\begin{aligned}
\mathrm{D_{KL}}(P_{X_\mathcal{U}} \| Q_{X_\mathcal{U}}) = \;& \mathrm{D_{KL}}(P_{X_{\mathcal{U} \to 1}} \| Q_{X_{\mathcal{U} \to 1}}) + \\
& + \mathrm{D_{KL}}\left(P_{X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} | X_{\mathcal{U} \to 1}} \| Q_{X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} | X_{\mathcal{U} \to 1}} \right).
\end{aligned}
\tag{7}
$$

By Lemma 4, we have

$$\sum_{j \in [m]} \mathrm{D_{KL}}(P_{X_{\mathcal{U}_j \to 1}} \| Q_{X_{\mathcal{U}_j \to 1}}) \le 9k \cdot \mathrm{D_{KL}}(P_{X_{\to 1}} \| Q_{X_{\to 1}}). \tag{8}$$

In addition, notice that the conditioned variable $X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} \mid X_{\mathcal{U} \to 1}$, distributed with respect to Q, is independent of $X_{\to 1}$. Indeed, if $X_{\mathcal{U} \to 1} = i \in \mathcal{U}$ then $X_{\to 1} = i$ deterministically. Otherwise, $X_{\mathcal{U} \to 1} = *$, implying that $X_\mathcal{U}$ is a uniform sequence of $|\mathcal{U}|$ distinct elements other than 1 (since Q is the uniform distribution on bijections). For any $i \notin \mathcal{U}$, conditioning on $X_{\to 1} = i$ has no effect on this distribution. Therefore, we may apply Lemma 5 to obtain:

$$\mathrm{D_{KL}}\left(P_{X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} | X_{\mathcal{U} \to 1}} \| Q_{X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} | X_{\mathcal{U} \to 1}} \right) \le \mathrm{D_{KL}}\left(P_{X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} | X_{\to 1}} \| Q_{X_{\mathcal{U} \to 2}, \ldots, X_{\mathcal{U} \to N} | X_{\to 1}} \right).$$

The distribution $Q_{X_{\mathcal{U}\to2},\ldots,X_{\mathcal{U}\to N}|X_{\to1}}$ is equivalent to a uniform distribution over bijections on $N-1$ elements. Therefore, we may apply an inductive argument to obtain

$$\sum_{j\in[m]} \mathrm{D_{KL}} \left(P_{X_{\mathcal{U}_j\to2},\ldots,X_{\mathcal{U}_j\to N}|X_{\mathcal{U}_j\to1}} \| Q_{X_{\mathcal{U}_j\to2},\ldots,X_{\mathcal{U}_j\to N}|X_{\mathcal{U}_j\to1}} \right)$$

$$\leq \sum_{j\in[m]} \mathrm{D_{KL}} \left(P_{X_{\mathcal{U}_j\to2},\ldots,X_{\mathcal{U}_j\to N}|X_{\to1}} \| Q_{X_{\mathcal{U}_j\to2},\ldots,X_{\mathcal{U}_j\to N}|X_{\to1}} \right)$$

$$\leq 9k \cdot \mathrm{D_{KL}} \left(P_{X_{\to2},\ldots,X_{\to N}|X_{\to1}} \| Q_{X_{\to2},\ldots,X_{\to N}|X_{\to1}} \right).$$

Combining with Equations (6), (7), and (8), we obtain

$$\sum_{j\in[m]} \mathrm{D_{KL}}(P_{X_{\mathcal{U}_j}} \| Q_{X_{\mathcal{U}_j}}) \leq 9k \cdot \mathrm{D_{KL}}(P_X \| Q_X),$$

completing the proof. ∎

E   Improved Bound for Problems without Post-Processing

Recall Theorem 10. A slightly better bound can be achieved for problems with a trivial post-processing function $\boldsymbol{post}(d, j) = j$. This rule captures the DLOG and DDH problems, but does not capture EM key recovery.

Theorem 14. *Let* $\boldsymbol{PC} := \boldsymbol{PC}(N, \mathcal{D}, \mathcal{M}, \boldsymbol{tr}, \boldsymbol{post}, \boldsymbol{suc})$ *be a permutation challenge game, such that* $\boldsymbol{tr}$ *is u-uniform and* $\boldsymbol{post}(d, j) = j$ *is a trivial post-processing function. Let* (A_0, A_1) *be an* (S, T) *non-adaptive algorithm with preprocessing for* $\boldsymbol{PC}$. *Denote by* $\boldsymbol{MaxS}(T)$ *the optimal success probability (with respect to* $\boldsymbol{suc}$*) of a non-preprocessing, non-adaptive algorithm that makes at most* T *queries. Then, the success probability of* (A_0, A_1) *is at most*

$$\min\left(2 \cdot \boldsymbol{MaxS}(T) + \frac{4\log(2)ST}{u}, \boldsymbol{MaxS}(T) + \sqrt{\frac{\log(2)ST}{u}}\right).$$

Proof. By Theorem 12, the success probability of (A_0, A_1) is at most

$$\min\left(2 \cdot \widehat{\boldsymbol{MaxS}}(T) + \frac{4\log(2)ST}{u}, \widehat{\boldsymbol{MaxS}}(T) + \sqrt{\frac{\log(2)ST}{u}}\right),$$

where $\widehat{\boldsymbol{MaxS}}(T)$ is the optimal success probability of an adversary to $\boldsymbol{MID}$ with running time T. Thus, it remains to show that $\widehat{\boldsymbol{MaxS}}(T) \leq \boldsymbol{MaxS}(T)$ and complete the proof. Let $\widehat{A} = (\widehat{A}_0, \widehat{A}_1)$ be an adversary to $\boldsymbol{MID}$ with success probability $\widehat{\boldsymbol{MaxS}}(T)$. Let us describe a non-preprocess algorithm A to the problem:

1. A receives N and a non-adaptive oracle access by inner and outer queries to the permutation σ with respect to a secret d.

2. A runs $\widehat{A}_0(N)$ to obtain the sequences $I = (I_1, \ldots, I_{T_1})$ and $O = (O_1, \ldots, O_{T_1})$.

3. A runs $\widehat{A}_1$ and obtains its queries (recall that $\widehat{A}_1$ is allowed to make only outer queries), $m_1, \ldots, m_{T_2} \in \mathcal{M}$.

4. A makes the inner queries $I_1, \ldots, I_{T_1}$ and obtains $O'_1, \ldots, O'_{T_1}$. Additionally, A makes the outer queries $m_1, \ldots, m_{T_2}$. Since the post-processing function is assumed to be trivial, the outputs are of the form $r_i := \sigma(\boldsymbol{tr}(d, m_i))$.

5. Using some deterministic process that depends only on $\{O_1, \ldots, O_{T_1}, O'_1, \ldots, O'_{T_1}\}$, A constructs a permutation π with $\pi(O'_i) = O_i$ for all i.

6. A simulates $\widehat{A}_1$'s oracle, and returns $\pi(r_i)$ as the response to the query m_i. A returns $\widehat{A}_1$'s output.

Clearly, the permutation $\pi \circ \sigma$ is uniformly distributed over the permutations with $\forall i : I_i \mapsto O_i$. Moreover, the responses $\widehat{A}_1$ obtained are $\pi(r_i) = (\pi \circ \sigma)(\boldsymbol{tr}(d, m_i))$. Therefore, the distribution of the permutation $\pi \circ \sigma$ is exactly as it is supposed to be, and the responses to all queries are correct with respect to it. Therefore, the simulation is accurate and does not affect the success probability. Thus, the success probability of A is exactly $\widehat{\boldsymbol{MaxS}}(T)$. ∎

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