T-61.3050 Machine Learning: Basic Principles Solutions of Homework problems: Exercise Session 3

Md Mohsin Ali Khan(Student No: 336790)

September 29, 2013

1 Utility Theory:

Consider the actions α_{25}, α_{50} , α_{75} as ordering 25, 50 and 75 pairs of Skis respectively. Let us denote the purchase cost of per pair of Ski when action is α_i, by PC_i. Consider the states D_{30}, D_{40}, D_{50}, D_{60} when the demand of Ski is 30, 40, 50 and 60 pairs respectively. Let us denote the probability of different state by $p(D_k)$. It is given that $p(D_{30}) = .2$, $p(D_{40}) = .4$, $p(D_{50}) = .2$ and $p(D_{60}) = .2$. Let us define utility of an action α_i when state is D_k by $U_{i,k}$

$$U_{i,k} = \begin{cases} (i \times 75) - (i \times PC_i) & \text{when } k = i \\ (i \times 75) - (i \times PC_i) - (k - i) \times 5 & \text{when } k > i \\ (k \times 75) - (i \times PC_i) + (i - k) \times 25 & \text{when } k < i \end{cases}$$

From utility theory we know, Expected utility of an action α_i is:,

$$E[\alpha_i] = \sum_k U_{i,k} p(D_k)$$

Table 1: Utility Table

Actions(α_i) ↓	D_{30}	D_{40}	D_{50}	D_{60}	E[α_i]↓
α_{25}	600	550	500	450	530
α_{50}	500	1000	1500	1450	1090
α_{75}	375	875	1375	1875	1075

$$\begin{aligned} E[\alpha_{25}] &= 600 \times .2 + 550 \times .4 + 500 \times .2 + 450 \times .2 = 530 \\ E[\alpha_{50}] &= 500 \times .2 + 1000 \times .4 + 1500 \times .2 + 1450 \times .2 = 1090 \\ E[\alpha_{75}] &= 375 \times .2 + 875 \times .4 + 1375 \times .2 + 1875 \times .2 = 1075 \end{aligned}$$

We find that the Expected utility is maximized at action α_{50}, that is ordering 50 pairs Skis. So the optimum decision is to order 50 pairs of Ski

2 Posterior Distribution:

The posterior distribution of θ is $p(\theta|N) = \frac{p(N|\theta)p(\theta)}{p(N)}$. Now, the likelihood, $p(N|\theta) = \binom{N}{m}\theta^m(1-\theta)^{N-m}$ which is given. The prior $p(\theta)$ is $Uni(0,1)$. So, we can write $p(\theta) = 1$

So the posterior distribution of θ in case of uniform prior is:

$$\begin{aligned} p(\theta|N) &= \frac{\binom{N}{m}\theta^m(1-\theta)^{N-m}}{p(N)} \\ p(\theta|N) &\propto \binom{N}{m}\theta^m(1-\theta)^{N-m} \text{ ,As } p(N) \text{ is constant for a given } N \end{aligned}$$

And the posterior distribution of θ in case of prior as beta distribution is :

$$\begin{aligned} p(\theta|N) &= \frac{\binom{N}{m}\theta^m(1-\theta)^{N-m}p(\theta)}{p(N)} \\ p(\theta|N) &= \frac{\binom{N}{m}\theta^m(1-\theta)^{N-m}Beta(\theta|\alpha,\beta)}{p(N)} \\ p(\theta|N) &= \frac{\binom{N}{m}\theta^m(1-\theta)^{N-m}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{p(N)} \\ p(\theta|N) &= \frac{\binom{N}{m}\theta^{m+\alpha-1}(1-\theta)^{N-m+\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}{p(N)} \\ p(\theta|N) &\propto \binom{N}{m}\theta^{m+\alpha-1}(1-\theta)^{N-m+\beta-1} \text{ ,As } p(N) \text{ and } \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \text{ are constant} \end{aligned}$$

3 Bayes Estimation:

Let us assume that the sext Y of five parts has been investigated and X number of parts have been found with defect.The posterior distribution of θ is $p(\theta|Y) = \frac{p(Y|\theta)p(\theta)}{p(Y)}$. Now, the likelihood, $p(Y|\theta) = \binom{5}{X}\theta^X(1-\theta)^{5-X}$. The prior $p(\theta)$ has $Beta(1,9)$ distribution. So, we can write $p(\theta) = \frac{\Gamma(1+9)}{\Gamma(1)\Gamma(9)}\theta^{1-1}(1-\theta)^{9-1} = \frac{\Gamma(10)}{\Gamma(1)\Gamma(9)}(1-\theta)^8$ .So the posterior distribution of θ in case of prior as $Beta(1,9)$ is:

$$
\begin{aligned}
p(\theta|N) &= \frac{\binom{5}{X}\theta^X(1-\theta)^{5-X}\frac{\Gamma(10)}{\Gamma(1)\Gamma(9)}(1-\theta)^8}{p(N)} \\
p(\theta|N) &= \frac{\binom{5}{X}\theta^X(1-\theta)^{5-X+8}\frac{\Gamma(10)}{\Gamma(1)\Gamma(9)}}{p(N)} \\
p(\theta|N) &= \frac{\binom{5}{X}\theta^{(X+1)-1}(1-\theta)^{(14-X)-1}\frac{\Gamma(10)}{\Gamma(1)\Gamma(9)}}{p(N)} \\
p(\theta|N) &= \frac{\binom{5}{X}\theta^{(X+1)-1}(1-\theta)^{(14-X)-1}\frac{\Gamma(X+1+14-X)}{\Gamma(X+1)\Gamma(14-X)}C}{p(N)} \text{ (As } \frac{\Gamma(10)}{\Gamma(1)\Gamma(9)} \text{ is a constant)} \\
p(\theta|N) &= C_1\theta^{(X+1)-1}(1-\theta)^{(14-X)-1}\frac{\Gamma(X+1+14-X)}{\Gamma(X+1)\Gamma(14-X)} \text{ (As } p(N), \binom{5}{X} \text{ is a constant)} \\
p(\theta|N) &= C_1 Beta(\theta|X+1, 14-X)
\end{aligned}
$$

The Expectation of θ from a beta function $Beta(\theta|\alpha,\beta)$ is $E[\theta] = \frac{\alpha}{\alpha+\beta}$ $(Ref : http : //en.wikipedia.org/wiki/Beta_distribution)$. In Baye's estimation, we calculate the Expectation of the probability distribution of θ. So here we see the Bayes estimation of the posterior probability is $\frac{X+1}{X+1+14-X}$ ($C_1 = 1$, as Beta is a distribution). So we can write:

The Bayes estimate of the proportion of defective parts is $\frac{X+1}{15}$