T-61.3050 Exercise session 3/2012
Home Assignment Feedback

Home assignments

1. The owner of a ski shop must order skis for the upcoming season. Orders must be placed in quantities of 25 pairs of skis. The cost per pair of skis is \$50 if 25 are ordered, \$45 if 50 are ordered, and \$40 if 75 are ordered. The skis will be sold at \$75 per pair. Any skis left over at the end of the year can be sold (for sure) at \$25 a pair. If the owner runs out of skis during the season, he will suffer a loss of "goodwill"'among unsatisfied customers. He rates this loss at \$5 per unsatisfied customer. For simplicity, the owner feels that demand for the skis will be 30, 40, 50 or 60 pair of skis, with probabilities 0.2, 0.4, 0.2 and 0.2 respectively. What should be his optimum decision?

   ANSWER OUTLINE:

   Similarly to demo 1 we can calculate the utility table. We have four possible actions: Buying 0, 25, 50 or 75 pairs of skis, and the unknown demand is 30, 40, 50 or 60 pair of skis. We can calculate the utilities as:

$$U(B, D) = 75 * min(D, B) - Ps(i) * B + 25 * max(B - D, 0) - 5 * max(D - B, 0), \qquad (1)$$

   where B is the amount bought and D is the demand. This results in the following utilities:

		#pairs demanded			
		30	40	50	60
#bought	0	150	-200	-250	-300
	25	600	550	500	450
	50	500	1000	1500	1450
	75	375	875	1375	1875

   Then we calculate the expected utilities by multiplying each row with the vector of probabilites:

		#pairs demanded				
		30	40	50	60	
#bought	0	150×0.2	-200×0.4	-250×0.2	-300×0.2	-220
	25	600×0.2	550×0.4	500×0.2	450×0.2	530
	50	500×0.2	1000×0.4	1500×0.2	1450×0.2	1090
	75	375×0.2	875×0.4	1375×0.2	1875×0.2	1075

   We can see that the best action is to buy 50 pairs of skis, since that has the highest expected utility

2. Estimating the sex ratio θ based on observations. Assume we have m (out of N) successful occurrences $x = 1$ of a Bernoulli process, where $x = 1$ denotes a newborn boy. This is modeled as a binomial distribution $p(m|N, \theta) = \text{Bin}(m|N, \theta) = \binom{N}{m}\theta^m(1-\theta)^{N-m}$. The task is to derive the posterior distribution for θ parameter for two cases of prior distribution: 1) uniform prior $p(\theta) = \text{Uni}(0, 1)$ and 2) beta distribution $p(\theta) = \text{Beta}(\theta|\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$. The density function of beta distribution is given by

$$\text{Beta}(\theta|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

Comment the effect of increased prior information for second case (increased α and β) on posterior distribution (assume N is fixed).

ANSWER OUTLINE:
In case of the uniform prior $p(\theta) = \text{Uni}(0, 1)$, we have

$$\begin{aligned}
p(\theta|m) &= \frac{p(m|\theta)p(\theta)}{p(m)} \\
&\propto p(m|\theta)p(\theta) \\
&= \binom{N}{m}\theta^m(1-\theta)^{N-m} \\
&\propto \theta^m(1-\theta)^{N-m} \\
&\propto \text{Beta}(\theta|m+1, N-m+1)
\end{aligned}$$

Note that $p(\theta|m)$ is a continuous distribution over θ in the interval $[0, 1]$, so it would be wrong to write that it is proportional to $\text{Bin}(m|N, \theta)$, which is by definition a discrete distribution over $\{0, 1, 2, \ldots, N\}$. Also note that $\text{Uni}(0, 1)$ can be expressed as $\text{Beta}(\theta|1, 1)$

In case of the Beta prior, $p(\theta) = \text{Beta}(\theta|\alpha, \beta)$, we have

$$\begin{aligned}
p(\theta|m) &= \frac{p(m|\theta)p(\theta)}{p(m)} \\
&\propto p(m|\theta)p(\theta) \\
&= \binom{N}{m}\theta^m(1-\theta)^{N-m}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&\propto \theta^m(1-\theta)^{N-m}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&\propto \text{Beta}(\theta|m+\alpha, N-m+\beta)
\end{aligned}$$

From this expression, we can see that the effect of the Beta prior on the posterior distribution corresponds to seeing α boys and β girls, in addition to the m boys and $N-m$ girls in the sample, biasing the posterior distribution of θ in either one or the other direction. Also, in case α and β are large compared to m and $N-m$, the posterior mainly depends on the prior (and mainly on the likelihood, in case α and β are small compared to m and $N-m$)

3. A large shipment of parts is received, out of which 5 are tested for defects. The number of defective parts, X, is assumed to have a $Binomial(5, \theta)$ distribution. From past shipments, it is known that θ (proportion of defective parts in large shipment) has a $Beta(1, 9)$ prior distribution. Find the Bayes estimate of the proportion of defective parts in this large shipment.

ANSWER OUTLINE:
Following home assignment 2 the posterior distribution is given by:

$$P(\theta|X) = \text{Beta}(\theta|m+\alpha, N-m+\beta) = \text{Beta}(\theta|X+1, 5-X+9) = \text{Beta}(\theta|X+1, 14-X) \quad (2)$$

The Bayes estimate is given by the expectation over the posterior distribution. Here we solve it for a Beta distribution with parameters α and β.

$$\begin{aligned}
\theta_{Bayes} = E[\theta]_{P(\theta|X)} &= \int_0^1 \theta P(\theta|X) = \int_0^1 \theta \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} && (3)\\
&= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha}(1-\theta)^{\beta-1} && (4)\\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{\alpha}(1-\theta)^{\beta-1} && (5)
\end{aligned}$$

To calculate this integral we can see that it is the same form as a Beta distribution, but without the normalizing constant. We can use the property that the Beta-distribution will integrate to 1, and thus we can get the unnormalized integral:

$$\begin{aligned}
\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1} = 1 && (6)\\
\int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1} &= \frac{1}{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} && (7)
\end{aligned}$$

We insert this result and get:

$$\begin{aligned}
\theta_{Bayes} &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{\alpha}(1-\theta)^{\beta-1} && (8)\\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} && (9)\\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)} && (10)
\end{aligned}$$

Since, for integer values $\Gamma(n) = (n-1)!$

$$\begin{aligned}
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)} &= \frac{(\alpha+\beta-1)!(\alpha)!}{(\alpha-1)!(\alpha+\beta)!} && (11)\\
&= \frac{\alpha}{\alpha+\beta} && (12)
\end{aligned}$$

Finally, plugging in the numbers:

$$\frac{\alpha}{\alpha+\beta} = \frac{X+1}{X+1+14-X} = \frac{X+1}{15} \quad (13)$$

We can thus see that the prior acts biases us strongly towards believing that the failure rate is fairly close to 1/10, and our 5 samples is not enough to dominate it.