Periodic traveling waves in integrodifference equations with Allee effect and overcompensation

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Remark 0.1. Source file for this manuscript and figures are available at github.com/mdnestor/ide-traveling-waves.

1 Introduction

This manuscript concerns periodic traveling wave solutions to the integrodifference equation

$$u_{t+1}(x) = Q[u_t](x) = \int_{-\infty}^{\infty} k(x-y) f(u_t(y)) dy \tag{1.1}$$

when the growth function f has both strong Allee effect (stability at zero) and overcompensation (non-monotonicity) with a stable 2-cycle.

Traveling waves in integrodifference equations with Allee effect were analyzed in [5] and [6], the former proving existence and stability conditions and the latter giving a formula for the sign of wave speed under certain conditions. On the other hand, [4] and [1] investigated integrodifference equations with overcompensatory growth, and gave numerical evidence for a period-doubling cascade of stacked traveling waves (see Figure 2). Studies incorporating both effects include [2], [8], [3], and [7], where fluctuating wave speeds and non-spreading solutions have been observed. In particular, [8] and [7] analyzed a piecewise-constant growth function with both Allee effect and overcompensation, and the latter proved existence of a bistable 2-periodic traveling wave solution.

This manuscripts focuses on the special case of the combination of Allee effect and overcompensation when the non-monotonicity induces a stable 2-cycle in the growth function. Specifically:

(G1) $f(0) = 0$ and $f'(0) < 1$, so that $u = 0$ is a stable fixed point (strong Allee effect);

(G2) there exists $a \in (0, 1)$ (the Allee threshold) such that $f(a) = a$ and $f'(a) > 1$, so that $u = a$ is an unstable fixed point;

(G3) $f(1) = 1$ and $f'(1) < -1$, so that $u = 1$ is an unstable fixed point (overcompensation);

(G4) $f(u) < u$ for all $u \in (0, a)$, $f(u) > u$ for all $u \in (a, 1)$, and $f(u) < u$ for all $u \in (1, \infty)$, so that $\{0, a, 1\}$ are the only fixed points;

(G5) there exist $u^- \in (a, 1)$ and $u^+ \in (1, \infty)$ such that $f(u^+) = u^-$, $f(u^-) = u^+$, $f^2(u^+) = u^+$, and $f^2(u^-) = u^-$.

(G6) $|(f^2)'(u^\pm)| < 1$, so that u^+ and u^- form a stable 2-cycle.

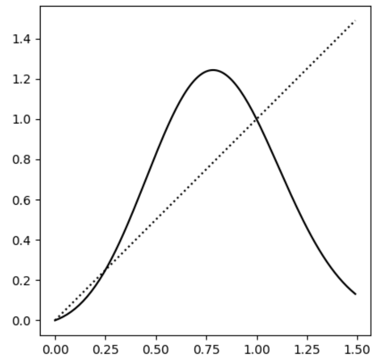


Figure 1: Example growth function satisfying our hypotheses.

1.1 Traveling waves

A solution $(u_t)_{t=0}^{\infty}$ to equation (1.1) is called a traveling wave (or traveling front) if there exists $w \in C(\mathbb{R}, [0, \infty))$ and $c \in \mathbb{R}$ called the wave speed (or wave velocity) such that

$$u_t(x) = w(x - tc) \tag{1.2}$$

for all t and x. Additionally we assume w has limits at $\pm\infty$, which are necessarily fixed points of f. Equivalently, a traveling wave is a solution to the equation

$$Q[w](x) = w(x - c) \tag{1.3}$$

for some c. More generally, a p-periodic traveling wave is a solution to the equation

$$Q^p[w](x) = w(x - pc) \tag{1.4}$$

for some c such that w has limits at $\pm\infty$ (which are necessarily p-periodic points of f). A periodic traveling wave is called bistable if both $w(+\infty)$ and $w(-\infty)$ belong to a stable cycle of f, and monostable if only one belongs to a stable cycle.

1.2 Stacked waves and dynamical stabilization

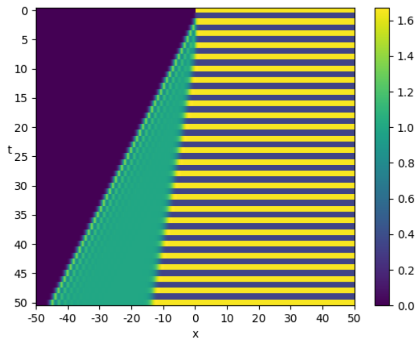
A stacked wave refers to a solution of (1.1) which consists of multiple superimposed traveling waves with distinct velocities that disperse as $t \to \infty$; see Figure 2. Such solutions were observed in [4] and rigorously analyzed in [1]. They were shown to occur for the logistic growth function

$$f(u) = ru(1 - u) \tag{1.5}$$

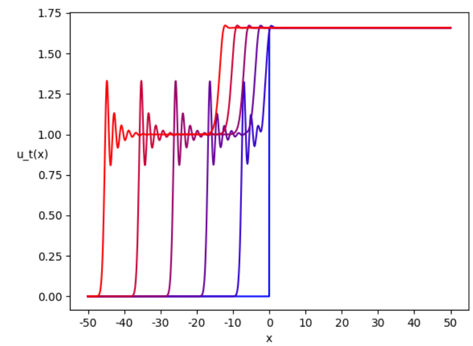
and the Ricker growth function

$$f(u) = u \exp(r(1 - u)) \tag{1.6}$$

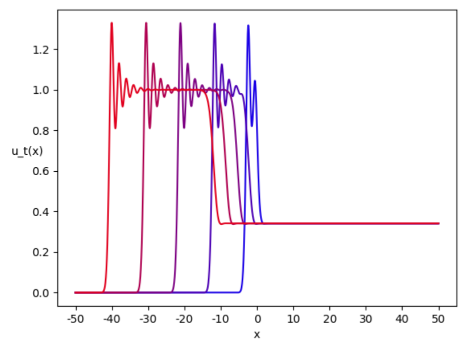
which both exhibit period-doubling cascades as the growth parameter r increases. A necessary (but not sufficient) condition for the existence of a stacked wave solution is a pair of traveling waves w_1 and w_2 with corresponding wave speeds c_1 and c_2 such that $w_1(+\infty) = w_2(-\infty)$ and $c_1 < c_2$.



(a)



(b)



(c)

Figure 2: Convergence to stacked traveling waves using the Ricker function (1.6) with $r = 2.4$, Laplace dispersal kernel, and initial data $u_0(x) = u^+H(x)$. (a) shows the space-time heatmap of the solution. (b) shows the solution curves on time steps $t = 0, 10, 20, ...$ while (c) shows the solution curves on $t = 5, 15, 25, ...$, and the color changes from blue to red as t increases.

For instance, [4] considered the logistic growth function with a stable 2-cycle paired with the Laplace kernel, and initial data of the form (2.11). The solution was shown numerically to quickly converge to a pair of stacked waves w_1 and w_2 with $w_1(-\infty) = 0$, $w_1(+\infty) = w_2(-\infty) = 1$ and $w_2(+\infty) = u^+$. This phenomenon has been referred to as "dynamical stabilization" because we have

$$u_t(x) \approx 1, \quad \forall x \in [c_1 t, c_2 t]$$

so that the solution attains the value of an unstable fixed point ($u = 1$) on an interval which grows in length as $t \to \infty$.

2 Model and simulations

We used the following 2-parameter growth function

$$f(u) = u \exp\left(r(1-u)(u-a)\right) \tag{2.1}$$

where $r \geq 0$ and $0 < a < 1$. This map is a reparameterization of the growth map used in [3],

$$f(u) = u \exp\Big(R(1-u)(\frac{u}{a}-1)\Big) \tag{2.2}$$

via $R = ra$.

This map has a period-doubling cascase for each choice of a; that is, a sequence $(r_{a,n})_{n=0}^{\infty}$ (depending on a) such that, whenever $r_{a,n} < r < r_{a,(n+1)}$ then the growth map has a stable 2^n-cycle and unstable 2^k-cycles for $k = 0, ..., (n-1)$. Our conditions are therefore satisfied whenever

$$r_{a,1} < r < r_{a,2} \tag{2.3}$$

For ease of calculation (2.9) can be expressed as

$$f(u) = u \exp(rp(u)); \quad p(u) = (1-u)(u-a) \tag{2.4}$$

Its first derivative is

$$f'(u) = (1 + rup'(u)) \exp(rp(u)) = (1 + r(a+1)u - 2ru^2) \exp\left(r(1-u)(u-a)\right) \tag{2.5}$$

We can determine its monotonicity and the stability of the fixed points by computing

1. $f'(0) = \exp(-ar) \in (0,1)$,
2. $f'(a) = 1 + ra(1-a) \in (1, \infty)$, and
3. $f'(1) = 1 - r(1-a)$.

So, $u = 0$ is always stable and $u = a$ is always unstable. Assuming $r > 0$, we have

$$f'(1) < 0 \quad \iff \quad r < \frac{1}{1-a} \quad \implies f \text{ is monotone on } [0,1] \implies u = 1 \text{ is stable} \tag{2.6}$$

Therefore,

$$u = 1 \text{ is stable} \quad \Leftarrow \quad |f'(1)| < 1 \quad \iff \quad r < \frac{2}{1-a} \tag{2.7}$$

We have,

$$r_{a,0} = 0; \quad r_{a,1} = \frac{2}{1-a} \tag{2.8}$$

Computing $r_{a,2}$ is not straightforward and involves solving a transcendental equation $f^2(u) = u$ where $f^2 = f \circ f$ is the second-iterate map,

$$\begin{aligned} f^2(u) &= f(u) \exp\left(rp(f(u))\right) \\ &= u \exp\left(r(1-u)(u-a) + r(1 - u \exp\left(r(1-u)(u-a)\right))(u \exp\left(r(1-u)(u-a)\right) - a)\right) \end{aligned} \tag{2.9}$$

Locating the 2-periodic points thus amounts to solving the equation

$$f^2(u) = u \quad \iff \quad p(u) + p(u\exp(rp(u))) = 0 \tag{2.10}$$

In practice, assuming $r_{a,1} < r < r_{a,2}$, a numerical scheme can be used to estimate $u^{\pm}$ by perturbing the unstable fixed point.

The initial data is given by the step function

$$u_0(x) = \begin{cases} 0 & x < 0 \\ u^+ & x \geq 0 \end{cases} \tag{2.11}$$

The model with initial condition (2.11) exhibits two distinct behaviors, depending on the model parameters, as illustrated below in Figure 3. Either:

Outcome 1. a single bistable 2-periodic traveling wave connecting $u = 0$ to $u = u^{\pm}$ (as shown in the left column of Figure 3);

Outcome 2. a pair of monostable traveling waves that form a stacked wave train, connecting $u = 0$ to $u = 1$ and $u = 1$ to $u = u^{\pm}$, respectively (as shown in the right column of Figure 3).

This observation naturally leads to several questions:

Q1. for what regions of the parameter space does dynamical stability/a stacked traveling front occur, vs a periodic traveling wave?

Q2. can we prove existence/stability of the wave in either case?

Q3. is there a general formula, as in [6], that could guarantee/forbid the existence of this wave? For example, if there is a wave w_1 with $w_1(-\infty) = 0$ and $w_1(+\infty) = 1$ and a periodic traveling wave w_2 with $w_2(-\infty) = 1$ and $w_2(+\infty) = u^+$, with speeds c_1 and c_2 respectively, is $c_1 > c_2$ a necessary/sufficient condition?

References

[1] F. Lutscher A. Bourgeois V. LeBlanc. "Dynamical stabilization and traveling waves in integrodifference equations". In: (2020). URL: https://doi.org/10.3934/dcdss.2020117.

[2] B. Li G. Otto W. Fagan. "Density dependence in demography and dispersal generates fluctuating invasion speeds". In: (2017). URL: https://www.pnas.org/doi/10.1073/pnas.1618744114.

[3] B. Li G. Otto W. Fagan. "Nonspreading solutions and patch formation in an integro-difference model with a strong Allee effect and overcompensation". In: (2022). URL: https://link.springer.com/article/10.1007/s12080-022-00544-y.

[4] Mark Kot. "Discrete-time travelling waves: ecological examples". In: (1992). URL: https://link.springer.com/article/10.1007/BF00173295.

[5] R. Lui. "Existence and stability of traveling wave solutions of a nonlinear integral operator". In: (1983).

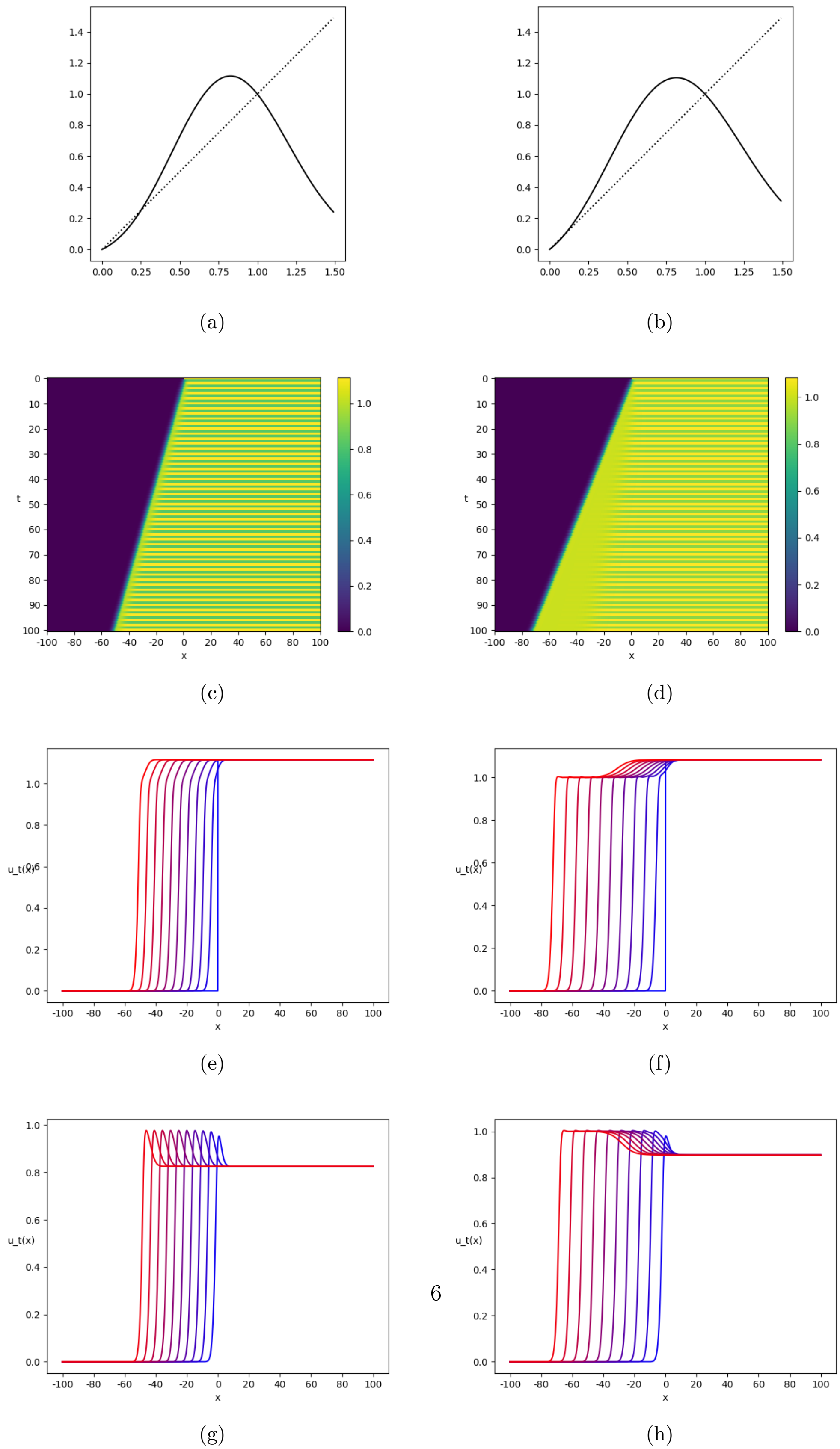


Figure 3: The left and right columns show our growth function (2.9) with parameters ($a = 0.25, r = 3.0$) and ($a = 0.1, r = 2.3$) respectively. (a) and (b) show the plot of the growth curve, while (c) and (d) show convergence to a pair of stacked waves vs. a single 2-periodic traveling wave, respecively.

[6] M. G. Neubert M. H. Wang M. Kot. “Integrodifference equations, Allee effects, and invasions”. In: (2002).

[7] B. Li M. Nestor. “Periodic traveling waves in an integro-difference equation with non-monotonic growth and strong allee effect”. In: (2020). URL: https://arxiv.org/abs/2202.00234.

[8] G. Otto. “Non-spreading solutions in a integro-difference model incorporating Allee and overcompensation effects”. In: (2017).