J. Math. Biology (1983) 16: 199–220

Journal of Mathematical Biology

Existence and Stability of Travelling Wave Solutions of a Nonlinear Integral Operator

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Abstract. In this paper, we establish the existence and stability property of travelling wave solutions of a nonlinear integral operator in the inferior case.

Key words: Nonlinear integral operator – Travelling wave – Wave speed – Asymptotic speed of propagation – Stability

1. Introduction

In this paper we study the nonlinear integral operator

$$Q[u](x) = \int_{-\infty}^{+\infty} K(x-y)g(u(y))\,dy \tag{1.1}$$

defined on the set of functions $\mathscr{C} = \{u: 0 \leqslant u \leqslant 1, u \text{ piecewise continuous}\}$. $K(x)$ is assumed to be a nonnegative probability density, $g \in C^1[0, 1]$ is nondecreasing and $g(0) = 0$, $g(1) = 1$. Such an operator arises from a model in Population Genetics [15], [16] where, based on further properties of $g(u)$, there are two special cases of interest. The first (intermediate) case is when $g(u) > u$ in $(0, 1)$ while the second (inferior) case is when there exists a constant $\alpha \in (0, 1)$ such that $g(u) < u$ in $(0, \alpha)$ and $g(u) > u$ in $(\alpha, 1)$. For the building of the model as well as the discussion of its validity, we refer the reader to [16].

Let $u_0 \in \mathscr{C}$ and u_n be defined by the recursion,

$$u_{n+1} = Q[u_n]. \tag{1.2}$$

We are interested in the behavior of $u_n(x)$ as $n \to \infty$. For the intermediate case, the problem has been extensively studied, [3], [4], [5], [9], [10], [11], [13], [14], [15], [16]. Here we only concern ourselves with the inferior case where the only results known are those in [16].

It should be pointed out that there are other models on Population Genetics which involve semilinear parabolic equations rather than nonlinear integral operators or integro-differential equations [3], [4], [8]. The existence of travelling waves may be proved by the usual phase-plane analysis method for ordinary differential equations. For an account of these in the inferior case, consult the works of Aronson and Weinberger, Fife and McLeod, Hadeler and Rothe.

The organization of the paper is as follows. In Section 2, we define the wave speed of the operator Q and show that it is the asymptotic speed of propagation for initial data u_0 that are bounded away from one. We also obtain bounds on this wave speed using results from the intermediate case. Travelling wave solutions of the operator Q are then defined. Finally, our method of defining the wave speed leads us naturally into defining a dual wave speed which is then shown to be the asymptotic speed of propagation for initial data bounded away from zero.

Section 3 contains our first main result (Theorem 4) on the stability property of the travelling wave. More precisely, we show that if the initial data u_0 vaguely resembles the travelling wave, then u_n, defined recursively through (1.2), will be trapped between two suitable translates of the wave as $n \to \infty$. The actual existence of the travelling wave (Theorem 5) as well as its behavior at $\pm\infty$ are proved in Section 4. As a consequence of Theorems 4 and 5, the two wave speeds constructed have to be equal. Section 5 contains some straightforward generalizations of our results to the case where $g(u) - u$ has more than one internal zero.

We close this section by listing the hypotheses of $K(x)$ and $g(u)$ to be assumed throughout the entire paper, except those of $g(u)$ in Section 5.

(i) $K(x) \geqslant 0$. If $B_1 = \inf\{x : K(x) > 0\}$, $B_2 = \sup\{x : K(x) > 0\}$, then $K(x) > 0$ in (B_1, B_2). We allow $B_1 = -\infty$ or $B_2 = \infty$ so that $K(x)$ need not have compact support.

(ii) $K(x)$ is continuous in $\mathbb{R}$ except possibly at B_1, B_2 where $\lim_{x \downarrow B_1} K(x) = p_1$, $\lim_{x \uparrow B_2} K(x) = p_2$. Also $K(x)$ may be written in the form

$$K(x) = K_a(x) - p_1 \chi_{(-\infty, B_1]} - p_2 \chi_{[B_2, \infty)},$$

where $K_a(x)$ is absolutely continuous and χ_S is the indicator function of the set S.

(iii) $\int_{\mathbb{R}} K(x)\,dx = 1$.

(iv) $\int_{\mathbb{R}} e^{\mu x} K(x)\,dx$ is finite for all real μ. (1.3)

(v) $\int_x^\infty K(y)\,dy \leqslant \text{const}\, K(x)$ for large x and $\int_{-\infty}^x K(y)\,dy \leqslant \text{const}\, K(x)$ for small x.

(vi) $g \in C^1[0, 1]$.

(vii) $g(0) = 0$, $g(1) = 1$.

(viii) There exists a constant $\alpha \in (0, 1)$ such that $g(u) < u$ in $(0, \alpha)$ and $g(u) > u$ in $(\alpha, 1)$.

(ix) $g'(u) \geqslant 0$ in $[0, 1]$. If (1.4)

$$\sigma_1 = \inf\{u : g(u) > 0\}, \qquad \sigma_2 = \sup\{u : g(u) < 1\},$$

then $g'(u) > 0$ in (σ_1, σ_2).

(x) $g'(0) < 1$, $g'(1) < 1$.

(xi) $g'(u) \leqslant g'(\alpha)$ in $[0, 1]$.

(xii) $g'(0)u \leqslant g(u) \leqslant g'(1)(u - 1) + 1$ in $[0, 1]$.

Remark 1. Conditions (v) are used only in the proof of Lemma 4.1. They are satisfied, for example, if B_1, B_2 are finite.

Remark 2. Condition (xi) implies that

$$\max_{[0,1]} \frac{g(u)}{u} < g'(\alpha), \qquad \max_{[0,1]} \frac{1 - g(1 - u)}{u} < g'(\alpha)$$

and that

$$g(u) \geqslant g'(\alpha)(u-\alpha)+\alpha \quad \text{in } [0,\alpha], \qquad g(u) \leqslant g'(\alpha)(u-\alpha)+\alpha \quad \text{in } [\alpha,1].$$

It may be replaced by assuming these inequalities instead.

Remark 3. Condition (xii) is used only in the proof of Proposition 5.

For the rest of the paper, when the domain of integration is not specified, it is assumed to be all of $\mathbb{R}$. The letter A will be used as a generic constant.

2. Wave Speeds, Travelling Waves, and Asymptotic Speed of Propagation

Results in the first part of this section (until Proposition 3) may be found in [16]. We begin by defining the wave speed c_+^* of the operator Q in the inferior case.

Let $\phi(x)$ be a continuous nonincreasing function such that $\phi(-\infty) \in (\alpha, 1)$ and $\phi(x)=0$ for $x \geqslant 0$. For any real number c, define the operator

$$R_c[u](x) = \max\{\phi(x), Q[u](x+c)\} \tag{2.1}$$

on $\mathscr{C}$. R_c has the order preserving property in the sense that if $u \leqslant v$, then $R_c[u] \leqslant R_c[v]$.

We now define a sequence of functions $a_n(c,x)$ in $\mathscr{C}$ by the recursion $a_0=\phi$, $a_{n+1}=R_c[a_n]$. From (2.1), $R_c[a_0] \geqslant a_0$ and from the order preserving property of R_c, $a_n(c,x)$ is nondecreasing in n for each fixed c and x. Since a_n are bounded, it follows that $a_n(c,x)$ increases to a limit function $a(c,x)$ as $n \to \infty$.

We now define the wave speed of the operator Q to be the number

$$c_+^* = \sup\{c : a(c,\infty)=1\}$$

which is finite according to Lemma 2.2. Also $B_1 < c_+^* < B_2$. The following lemma summarizes the properties of a_n, a and c_+^*.

Lemma 2.1. (i) *$a_n(c,x)$ is nonincreasing in x and continuous in x and c.*

(ii) *Let $\alpha_0=\phi(-\infty)$, $\alpha_{n+1}=g(\alpha_n)$. Then $a_n(c,-\infty)=\alpha_n$ and $a_n(c,\infty)=0$ for every n.*

(iii) *$a(c,\infty)=1$ if and only if $a_n(c,0)>\phi(-\infty)$ for some n and if and only if $c<c_+^*$.*

Proof. See Section 5 of [16].

So far, our construction has been dependent on our choice of the function $\phi(x)$. However, it can be shown (see [16], Section 5) that if a different $\hat{\phi}(x)$ is chosen with the same properties as $\phi(x)$ and if we define $\hat{R}_c$, $\hat{a}_n$ and $\hat{a}$ in the obvious manner, then $\hat{a}(c,+\infty)=a(c,+\infty)$. Thus c_+^* is independent of the choice of ϕ. Writing out the recursion $a_{n+1}=R_c[a_n]$ explicitly and passing to the limit, we have

$$a(c,x) = \max\left\{\phi(x), \int K(x+c-y)g(a(c,y))\,dy\right\}. \tag{2.2}$$

Letting $x \to \infty$, in (2.2) we see that $a(c,\infty)=g(a(c,\infty))$ so that $a(c,\infty) \in \{0,\alpha,1\}$. Hence from Lemma 2.1, $a(c,\infty)=0$ or α if $c \geqslant c_+^*$ and $a(c,-\infty)=1$ for all values of c.

Remark 2.1. c_+^* is the wave speed in the positive direction. We can define the wave speed in the negative direction c_-^* by starting with the function $\phi(x)$, $\phi(x)$ continuous and nondecreasing, $\phi(\infty) \in (\alpha, 1)$, $\phi(x) = 0$ for $x \leqslant 0$ and letting $\bar{a}_0 = \phi$, $\bar{a}_{n+1} = R_c[\bar{a}_n]$. Then $\bar{a}_n(c, x)$ will increase to a limit function $\bar{a}(c, x)$ and $c_-^* \equiv \inf\{c : \bar{a}(c, -\infty) = 1\}$. The next two theorems justify the name wave speed.

Theorem 1a. *Let $u_0 \in \mathscr{C}$ satisfy the hypothesis $u_0(x) = 0$ for $x \geqslant A$, $u_0(x) \leqslant \theta < 1$ in $\mathbb{R}$. Let u_n be defined by the recursion* (1.2). *Then*

$$\limsup_{n \to \infty} \max_{x \geqslant nc} u_n(x) \leqslant \alpha \tag{2.3}$$

for every $c > c_+^*$.

Proof. Let $\phi(x)$ be a continuous nonincreasing function such that $\phi(-\infty) \in (\theta, 1)$, $\phi(x) = 0$ for $x \geqslant 0$ and construct $a_n(c, x)$ as before with $a_0 = \phi$. It is clear from our hypothesis on $u_0(x)$ that there exists a constant $\tau > 0$ such that $\phi(x - \tau) \geqslant u_0(x)$. Let $v_n(x) = a_n(c_+^*, x - nc_+^* - \tau)$ so that

$$v_{n+1}(x) = \max\{\phi(x - (n+1)c_+^* - \tau),\ Q[v_n](x)\} \geqslant Q[v_n](x). \tag{2.4}$$

Now $v_0(x) = \phi(x - \tau) \geqslant u_0(x)$. Assuming that $v_n(x) \geqslant u_n(x)$, then since Q is order preserving, we have $Q[v_n](x) \geqslant Q[u_n](x) = u_{n+1}(x)$. Combining this with (2.4), we conclude that $v_{n+1} \geqslant u_{n+1}$. Thus

$$u_n(x) \leqslant a_n(c_+^*, x - nc_+^* - \tau) \leqslant a(c_+^*, x - nc_+^* - \tau)$$

for all n. Since $a(c_+^*, \infty) = 0$ or α, the result of the theorem follows. Q.E.D.

Remark 2.2. It is clear from the above proof that if $a(c_+^*, \infty) = 0$, then the inequality in (2.3) may be replaced by $\lim_{n \to \infty} \max_{x \geqslant nc} u_n(x) = 0$. This will be proved in Theorem 1a′. A similar theorem also holds in the negative direction.

Theorem 1b. *Let $u_0 \in \mathscr{C}$ and define u_n by the recursion* (1.2). *Let $c_-^* < c_1 < c_2 < c_+^*$. For any $\sigma > \alpha$, there exists a constant $r_\sigma > 0$ such that if $u_0(x) \geqslant \sigma$ on an interval of length equal to r_σ, then*

$$\lim_{n \to \infty} \min_{x \in [nc_1, nc_2]} u_n(x) = 1. \tag{2.5}$$

Proof. Theorem 6.2 of [16].

Remark 2.3. If $\liminf_{x \to -\infty} u_0(x) > \alpha$ and $c_2 < c_+^*$, then

$$\lim_{n \to \infty} \min_{x \leqslant nc_2} u_n(x) = 1.$$

Remark 2.4. Theorems 1a and 1b together say that c_+^* is the asymptotic speed of propagation for initial data $u_0(x)$ whose maximum on $\mathbb{R}$ is less than 1, $u_0(x) > \alpha$ on a sufficiently large interval and vanishes near infinity.

We now briefly describe results about the intermediate case. They will be used to obtain bounds on the wave speed c_+^*. Here $p > 0$ and $h(u)$ is assumed to satisfy the following conditions.

$$\begin{array}{ll} \text{(i)} & h(u) \in C^1[0,p]. \\ \text{(ii)} & h(0) = 0,\ h(p) = p. \\ \text{(iii)} & h(u) > u \text{ in } (0,p). \\ \text{(iv)} & 0 \leqslant h(u) \leqslant h'(0)u \text{ in } [0,p]. \\ \text{(v)} & h'(u) \geqslant 0 \text{ in } [0,p]. \end{array} \tag{2.6}$$

Note that (iii) and (iv) together imply that $h'(0) > 1$. We define the operator H on the set of functions $\mathscr{H} = \{u : 0 \leqslant u \leqslant p,\ u \text{ piecewise continuous}\}$ by the relation

$$H[u](x) = \int K(x-y)h(u(y))\,dy. \tag{2.7}$$

We suppress the dependence of $h(u)$, $\mathscr{H}$ and H on the constant p. This should not cause any confusion.

There are two numbers $\rho_-^* < \rho_+^*$ analogous to the wave speeds c_+^*, c_-^* of the operator Q in the inferior case such that the following two theorems hold. Their proofs may be found in [15], [16].

Theorem 2a. *Let $u_0 \in \mathscr{H}$ and let u_n be defined recursively by $u_{n+1} = H[u_n]$. Suppose that $u_0(x) = 0$ for $x \geqslant A$. Then for any $\rho > \rho_+^*$, we have $\lim_{n\to\infty} \max_{x \geqslant n\rho} u_n(x) = 0$.*

Theorem 2b. *Let $u_0 \in \mathscr{H}$ and let u_n be defined recursively by $u_{n+1} = H[u_n]$. Suppose that u_0 is nontrivial in the sense that its integral is positive. Then for any $\rho_-^* < \rho_1 < \rho_2 < \rho_+^*$, we have $\lim_{n\to\infty} \min_{x\in[n\rho_1, n\rho_2]} u_n(x) = p$.*

Remark 2.5. In the inferior case, it is possible that $c_+^* < c_-^*$ (see [16] for an example) in which case Theorem 1b is not valid. However, in the intermediate case, $\rho_-^* < \rho_+^*$ is always satisfied.

Remark 2.6. In Theorem 1b, $u_0(x)$ has to be greater than α on a sufficiently large interval before $u_n(x)$ will propagate. In Theorem 2b, $u_n(x)$ will propagate as long as $u_0(x)$ is positive on an interval. The former is often called the threshold phenomenon while the latter is often called the hairtrigger effect.

Remark 2.7. ρ_+^* and ρ_-^* are the wave speeds of the operator H in the positive and negative direction respectively. They may be defined by the same procedure as in the inferior case, through the sequence a_n and a, with Q replaced by H and the obvious modification when $p \neq 1$.

In the inferior case, the construction of the wave speed makes it hard to compute c_+^* or c_-^* (even for their signs!). However, in the intermediate case, because of condition (iv) of (2.6), ρ_+^* and ρ_-^* have explicit formulas and they depend on $h(u)$ only through $h'(0)$. We define the function $\Phi(\mu, \beta)$ depending on the parameter $\beta > 0$ by

$$\Phi(\mu,\beta) = \frac{1}{\mu}\log\left\{\beta \int e^{\mu x} K(x)\,dx\right\}, \qquad \mu \neq 0. \tag{2.8}$$

If $\beta = h'(0)$, we simply write $\Phi(\mu)$ for $\Phi(\mu, h'(0))$. We now summarize the properties of $\Phi(\mu,\beta)$ in the following proposition.

Proposition 1. (i) *If $\beta > 1$, then $\Phi(\mu,\beta)$ has a unique local maximum at $\mu_*(\beta) < 0$ and a unique local minimum at $\mu^*(\beta) > 0$. $\Phi(\mu,\beta)$ is decreasing in the intervals $(\mu_*(\beta), 0)$*

and $(0, \mu^*(\beta))$ *and is increasing in the intervals* $(-\infty, \mu_*(\beta))$ *and* $(\mu^*(\beta), \infty)$. *Also* $\Phi(0^+, \beta) = \infty$, $\Phi(0^-, \beta) = -\infty$, $\lim_{\mu \to \infty} \Phi(\mu, \beta) = B_2$ *and* $\lim_{\mu \to -\infty} \Phi(\mu, \beta) = B_1$. *For* $\mu > 0$, $\Phi(\mu, \beta) > B_1$ *while for* $\mu < 0$, $\Phi(\mu, \beta) < B_2$. *We simply write* μ^*, μ_* *if* $\beta = h'(0)$.

(ii) *If* $\beta < 1$, *then* $\Phi(\mu, \beta)$ *is increasing in* $\mu \neq 0$. $\Phi(0^+, \beta) = -\infty$, $\Phi(0^-, \beta) = \infty$ *and* $\lim_{\mu \to \infty} \Phi(\mu, \beta) = B_2$, $\lim_{\mu \to -\infty} \Phi(\mu, \beta) = B_1$.

(iii) $\rho_+^* = \Phi(\mu^*)$, $\rho_-^* = \Phi(\mu_*)$.

(iv) *For each fixed* $\mu > 0$, $\Phi(\mu, \beta)$ *is an increasing function of* β. *In particular,* $\rho_+^*(\beta_1) > \rho_+^*(\beta_2)$ *if* $\beta_1 > \beta_2 > 1$.

Proof. (i) and (iii) may be found in Section 1 of [10], (ii) follows easily from the same argument and (iv) is evident from (2.8). Q.E.D.

Proposition 2. *Let* Q^1, Q^2 *be two operators defined on* $\mathscr{C}$ *such that each* Q^i *is of the form* Q *or* H *with* $p = 1$. *Suppose that* $Q^1[u] \leqslant Q^2[u]$ *for* $u \in \mathscr{C}$. *Let* c^1, c^2 *be their wave speed in the positive direction, then* $c^1 \leqslant c^2$.

Proof. If $c^1 > c^2$, let $c \in (c^2, c^1)$ and γ be sufficiently close to 1. Define $u_0(x) = \gamma$ for $x \leqslant 0$, $u_0(x) = 0$ for $x > 0$. Let $u_0^i = u_0$, $i = 1, 2$ and define u_n^i recursively by $u_{n+1}^i = Q^i[u_n]$, $i = 1, 2$. From our hypothesis on Q^1, Q^2 and an inductive argument, we can see that $u_n^1 \leqslant u_n^2$ for all n. However, according to Theorem 1a or Theorem 2a, $u_n^2(x + nc) \leqslant \alpha$ as $n \to \infty$ while according to Remark 2.3 or Theorem 2b, $u_n^1(x + nc) \to 1$ as $n \to \infty$. This is impossible and thus $c^1 \leqslant c^2$. Q.E.D.

We now come to the question of travelling waves. A nonconstant function $w_c(x)$ is a travelling wave solution of the operator Q with speed c if $u_n(x) = w_c(x - nc)$ is a solution to the recursion (1.2); i.e.

$$w_c(x) = \int K(x + c - y) g(w_c(y))\, dy. \tag{2.9}$$

This definition is of course also meaningful for the operator H and we denote by $z_c(x)$ the travelling wave solution of speed c for the operator H. Note that if $w_c(x)$ or $z_c(x)$ is a travelling wave, then so is $w_c(x + \tau)$ or $z_c(x + \tau)$ for any constant τ.

The proof of the following proposition may be found in [15], [16].

Proposition 3. There exists travelling wave solutions $z_c(x)$ of the operator H such that $z_c(x)$ is nonincreasing, $z_c(-\infty) = p$, $z_c(\infty) = 0$ if and only if $c \geqslant \rho_+^*$. Furthermore, for $c > \rho_+^*$, $z_c(x) \sim A_c e^{-\mu_c x}$ as $x \to \infty$ while for $c = \rho_+^*$, $z_c(x) \sim A_c x e^{-\mu^* x}$ as $x \to \infty$. Here A_c is a positive constant and μ_c is the unique smallest positive number such that $\Phi(\mu_c) = c$.

Remark 2.8. There are also travelling wave solutions of the operator H facing left and they exist if and only if $c \leqslant \rho_-^*$. Travelling waves are unique up to translation. The questions of uniqueness are discussed in [5] and [10].

Remark 2.9. The only if part of Proposition 3 is an immediate consequence of Theorem 2b.

Corollary. *There are nonincreasing travelling wave solutions* $q_c(x)$ *of the operator* Q *such that* $q_c(-\infty) = 1$, $q_c(\infty) = \alpha$ *and nondecreasing travelling wave solutions of the operator* Q *connecting* 0 *and* α *if and only if* $c \geqslant \kappa_1 \equiv \inf_{\mu > 0} \Phi(\mu, g'(\alpha))$.

Proof of Corollary. Let $h(u) = g(u + \alpha) - \alpha$ in $[0, 1 - \alpha]$. Then $h(u)$ satisfies the conditions of (2.6) with $p = 1 - \alpha$, $h'(0) = g'(\alpha)$. According to Proposition 3, there are nonincreasing travelling wave solutions $\bar{q}_c(x)$ of the operator $H[u] = K * h(u)$ such that $\bar{q}_c(-\infty) = 1 - \alpha$, $\bar{q}_c(\infty) = 0$ if and only if $c \geqslant \kappa_1$. The first half of the corollary follows from the observation that $q_c(x) = \bar{q}_c(x) + \alpha$. The second half may be proved similarly by considering the function $h(u) = \alpha - g(\alpha - u)$ in $[0, \alpha]$.

Q.E.D.

Let $m = \max_{[0,1]} g(u)/u$ and let P be the point with coordinate $(1/m, 1)$. Define the function $\bar{h}(u) = mu$ in the interval $[0, 1/m)$, $\bar{h}(u) = 1$ in the interval $(1/m, 1]$ and smooth out $\bar{h}(u)$ near P so that $\bar{h}'(u) \geqslant 0$, $g(u) \leqslant \bar{h}(u) \leqslant 1$ in $[0, 1]$. It is clear that $\bar{h}(u)$ has the properties listed in (2.6) with $p = 1$. Let $\bar{H}$ be the operator $\bar{H}[u] = K * \bar{h}(u)$ and let κ_2 be the wave speed of $\bar{H}$ in the positive direction. The following lemma provides a bound on c_+^*.

Lemma 2.2. $c_+^* \leqslant \kappa_2 < \kappa_1$.

Proof. $\kappa_2 < \kappa_1$ follows from condition (xi) of (1.4), Remark 2 of Section 1 and part (iv) of Proposition 1. The inequality $c_+^* \leqslant \kappa_2$ follows from Proposition 2 and the fact that $g(u) \leqslant \bar{h}(u)$ implies that $Q[u] \leqslant \bar{H}[u]$. Q.E.D.

Theorem 1a′. *In Theorem 1a, the inequality in* (2.3) *may be replaced by*

$$\lim_{n \to \infty} \max_{x \geqslant nc} u_n(x) = 0 \qquad \textit{for every} \qquad c > c_+^*.$$

The conditions on u_0 may be replaced by requiring $u_0(x) > \alpha$ near $-\infty$, $u_0(x) < \alpha$ near ∞.

Proof. According to Remark 2.2, it suffices to show that $a(c_+^*, \infty) = 0$. If $a(c_+^*, \infty) = \alpha$, then $a(c_+^*, x) \in [\alpha, 1]$ for all x. As in the proof of the corollary of Proposition 3, we define the operator $H[u] = K * h(u)$, where $h(u) = g(u + \alpha) - \alpha$ in $[0, 1 - \alpha]$. Let κ_1 be the wave speed in the positive direction for H. Let $v_0(x) = a(c_+^*, x) - \alpha$ and define $v_n(x) \in [0, 1 - \alpha]$ recursively by $v_{n+1}(x) = H[v_n](x + c_+^*)$. From (2.2), we have $Q[a(c_+^*, \cdot)](x + c_+^*) \leqslant a(c_+^*, x)$ so that

$$\begin{aligned} v_1(x) &= \int K(x + c_+^* - y)[g(a(c_+^*, y)) - \alpha]\, dy \\ &= Q[a(c_+^*, \cdot)](x + c_+^*) - \alpha \\ &\leqslant v_0(x). \end{aligned}$$

Since $K(x) \geqslant 0$ and $h'(u) \geqslant 0$, the operator H is order preserving. By an inductive argument, we can show that $0 \leqslant v_{n+1} \leqslant v_n \leqslant 1 - \alpha$ for all n. This is impossible because $c_+^* < \kappa_1$ and according to Theorem 2b and the definition of v_n, the sequence $v_n(x)$ converges to $1 - \alpha$ as $n \to \infty$. The last assertion will be proved in Section 4 after Theorem 5. Q.E.D.

One can define a dual wave speed for the operator Q. We start by taking a continuous nonincreasing function $\psi(x)$ such that $\psi(x) = 1$ for $x \leqslant 0$ and $\psi(\infty) \in (0, \alpha)$. For any real number c, define the operator,

$$S_c[u](x) = \min\{\psi(x), Q[u](x + c)\} \tag{2.10}$$

on the set $\mathscr{C}$. Let $b_0(c,x) = \psi(x)$ and define b_n recursively by $b_{n+1} = S_c[b_n]$. We should be able to obtain results analogous to Lemma 2.1, Theorems 1a, 1b and 1a′. In fact, let $\tilde{g}(u) = 1 - g(1-u)$, $K_1(x) = K(-x)$ and $\tilde{Q}$ be the operator $\tilde{Q}[u] = K_1 * \tilde{g}(u)$ defined on $\mathscr{C}$. Let $\tilde{\phi}(x) = 1 - \psi(-x)$, $\tilde{a}_0(c,x) = \tilde{\phi}(x)$ and define $\tilde{a}_n$ recursively by $\tilde{a}_{n+1}(c,x) = \tilde{R}_c[a_n(c,\cdot)](x)$, where $\tilde{R}_c$ is the operator, $\tilde{R}_c[u](x) = \max\{\tilde{\phi}(x), \tilde{Q}[u](x+c)\}$. There is a simple relation between b_n and $\tilde{a}_n$.

Lemma 2.3. *$\tilde{a}_n(-c,x) = 1 - b_n(c,-x)$ for all n and c.*

Proof. The proof is by induction and the lemma is certainly true when $n = 0$. Assuming that it is true up to n, we have according to our definitions,

$$\begin{aligned}\tilde{a}_{n+1}(-c,x) &= \max\left\{1 - \psi(-x), \int K_1(x-c-y)[1 - g(1 - \tilde{a}_n(-c,y))]\,dy\right\}\\ &= 1 - \min\left\{\psi(-x), \int K_1(x-c-y)g(b_n(c,-y))\,dy\right\}\\ &= 1 - \min\left\{\psi(-x), \int K(-x+c-y)g(b_n(c,y))\,dy\right\}\\ &= 1 - b_{n+1}(c,-x).\end{aligned}$$

Thus the lemma is true for all n. Q.E.D.

Since $\tilde{\phi}(x)$ and $\tilde{g}(u)$ have the desired properties, our results at the beginning of the section hold for the sequence $\tilde{a}_n$ and its limit function $\tilde{a}$. From Lemma 2.3, $b_n(c,x)$ will therefore decrease to a limit function $b(c,x) = 1 - \tilde{a}(-c,-x)$.

We now define our dual wave speed in the negative direction as,

$$\tilde{c}_-^* = \inf\{c : b(c,-\infty) = 0\}. \tag{2.11}$$

Lemma 2.4. (i) *$b_n(c,x)$ is nonincreasing in x and continuous in x and c.*

(ii) *Let $\gamma_0 = \psi(\infty)$ and define γ_n recursively by $\gamma_{n+1} = g(\gamma_n)$. Then $b_n(c,-\infty) = 1$ and $b_n(c,\infty) = \gamma_n$ for all n.*

(iii) *$b(c,-\infty) = 0$ if and only if $b_n(c,0) < \psi(\infty)$ for some n and if and only if $c > \tilde{c}_-^*$.*

Remark 2.10. The wave speed of the operator $\tilde{Q}$ in the positive direction is $-\tilde{c}_-^*$. We have from above,

$$\begin{aligned}\sup\{c : \tilde{a}(c,\infty) = 1\} &= \sup\{c : b(-c,-\infty) = 0\}\\ &= -\inf\{c : b(c,-\infty) = 0\}\\ &= -\tilde{c}_-^*.\end{aligned}$$

Using this fact and Theorems 1a, 1b and 1a′, we shall derive corresponding theorems which say that $\tilde{c}_-^*$ is the asymptotic speed of propagation in the negative direction for initial data that are bounded away from zero and equal 1 near $-\infty$. Note that $b(\tilde{c}_-^*, -\infty) = 1$.

Theorem 3a. *Let $u_0 \in \mathscr{C}$ satisfy the hypotheses that $u_0(x) = 1$ for $x \leqslant A$, $u_0(x) \geqslant \theta > 0$ in $\mathbb{R}$. Let u_n be a solution to the recursion* (1.2). *Then*

$$\lim_{n \to \infty} \min_{x \leqslant nc} u_n(x) = 1 \tag{2.12}$$

for every $c < \tilde{c}_-^*$.

Proof. Let $v_0(x) = 1 - u_0(-x)$ and define v_n recursively by $v_{n+1} = \tilde{Q}[v_n]$, then an inductive argument shows that $v_n(x) = 1 - u_n(-x)$ for all n. From our hypothesis on $u_0(x)$, the function $v_0(x)$ satisfies the assumptions of Theorem 1a with respect to $\tilde{Q}$. Since $-\tilde{c}_-^*$ is the wave speed in the positive direction for $\tilde{Q}$, we have from Theorem 1a′,

$$\lim_{n \to \infty} \max_{x \geqslant nc} v_n(x) = 0$$

for every $c > -\tilde{c}_-^*$. Using our relation between v_n and u_n, this is equivalent to (2.12). Q.E.D.

The proof of the following theorem is similar to that of Theorem 3a. Note that $\tilde{g}(u) = 0$ at $0, 1 - \alpha$ and 1. $-\tilde{c}_+^*$ is the wave speed of the operator $\tilde{Q}$ in the negative direction.

Theorem 3b. *Let* $u_0 \in \mathscr{C}$ *and define* u_n *by the recursion* (1.2). *Let* $\tilde{c}_-^* < c_1 < c_2 < \tilde{c}_+^*$. *For any* $0 < \sigma < \alpha$, *there exists a constant* r_σ *such that if* $u_0(x) \leqslant \sigma$ *on an interval of length equal to* r_σ, *then*

$$\lim_{n \to \infty} \max_{x \in [nc_1, nc_2]} u_n(x) = 0. \tag{2.13}$$

Remark 2.11. If $\limsup_{x \to \infty} u_0(x) < \alpha$ and $c_2 > \tilde{c}_-^*$, then $\lim_{n \to \infty} \max_{x \geqslant nc_2} u_n(x) = 0$.

The following properties of the functions $a(c, x)$ and $b(c, x)$ will be needed in Section 4. For each $c \geqslant c_+^*$, $a(c, -\infty) = 1$, $a(c, \infty) = 0$, $Q[a(c, \cdot)](x + c) \leqslant a(c, x)$ and for each $c \leqslant \tilde{c}_-^*$, $b(c, -\infty) = 1$, $b(c, \infty) = 0$, $Q[b(c, \cdot)](x + c) \geqslant b(c, x)$. The last inequality follows from the fact that b satisfies $b = S_c[b]$.

The important question now is whether $c_+^* = \tilde{c}_-^*$? Clearly $c_+^* \leqslant \tilde{c}_-^*$ for if $c \in (\tilde{c}_-^*, c_+^*)$, we let $u_0(x) = 1$ for $x \leqslant 0$, $u_0(x) = \gamma \in (0, \alpha)$ for $x > 0$. Define u_n recursively by (1.2). Then from Remark 2.3 $u_n(x + nc) \to 1$ as $n \to \infty$ while from Remark 2.11 $u_n(x + nc) \to 0$ as $n \to \infty$. This contradiction makes $c_+^* > \tilde{c}_-^*$ impossible. Note that if we take $u_0(x) = 0$ for $x > 0$, $u_0(x) = \gamma \in (\alpha, 1)$ for $x \leqslant 0$, then Theorem 3a is not applicable so that we cannot conclude that $\tilde{c}_-^* \leqslant c_+^*$.

3. Stability of the Travelling Waves

Proposition 4. *Let* $w_1, w_2 \in \mathscr{C}$ *be nonincreasing,* $w_i(-\infty) = 1$, $w_i(\infty) = 0$ *and they satisfy the condition that for any compact set* Γ, *where* $w_i(x) \in (0, 1)$, *there exists a positive constant* θ, *such that*

$$w_i(x) - w_i(y) \leqslant -\theta(x - y) \qquad \textit{for} \quad x > y, x, y \quad \textit{in} \quad \Gamma. \tag{3.1}$$

Let c_1, c_2 *be two numbers such that*

$$w_1(x) \leqslant Q[w_1](x + c_1) \tag{3.2a}$$

and

$$w_2(x) \geqslant Q[w_2](x + c_2), \tag{3.2b}$$

with equality holding in (3.2) *at places where* $w_i(x) = 0$ *or* 1. *Let* u_0 *be such that* $\liminf_{x\to-\infty} u_0(x) > \alpha$, $\limsup_{x\to\infty} u_0(x) < \alpha$ *and let* u_n *be defined recursively by* (1.2). *Then there exist constants* $x_1, x_2, q_0, q'_0, \mu_1, \mu_2$, *the last four positive, such that*

$$w_1(x - x_1) - q_0 e^{-\mu_1 n} \leqslant u_n(x + nc_1), \tag{3.3a}$$

$$u_n(x + nc_2) \leqslant w_2(x - x_2) + q'_0 e^{-\mu_2 n} \tag{3.3b}$$

for all n.

Proof. We only prove (3.3a), the other being similar. We first assume that $w_1(x) \in (0, 1)$ in $\mathbb{R}$. In this case our assumptions on the equality of (3.2) are of course vacuous.

From our hypothesis on u_0, there exists constants z_0, q_0, the latter positive, such that

$$w_1(x - z_0) - q_0 \leqslant u_0(x) \tag{3.4}$$

and

$$\alpha < 1 - q_0 < \liminf_{x\to-\infty} u_0(x). \tag{3.5}$$

Define $g(u) = 0$ for $u < 0$ and $v_n(x) = w_1(x - z_n) - q_n$ where z_n and q_n are two sequences of numbers to be determined later so that

$$Q[v_n](x + c_1) \geqslant v_{n+1}(x) \qquad \text{for all } n. \tag{3.6}$$

Assuming (3.6) for the moment, we show that it implies the inequality

$$v_n(x) \leqslant u_n(x + nc_1). \tag{3.7}$$

We proceed by induction and when $n = 0$, (3.7) is just (3.4). Assuming that (3.7) is true up to n, then by (3.6) and the order preserving property of the operator Q, we have,

$$v_{n+1}(x) \leqslant Q[v_n](x + c_1) \leqslant Q[u_n(\cdot + nc_1)](x + c_1) = u_{n+1}(x + (n + 1)c_1).$$

Hence (3.7) is valid for all n.

We now choose z_n and q_n so that (3.6) holds. At the same time, the sequence z_n will be nonincreasing, convergent and therefore bounded. The sequence q_n will decrease to zero as $n \to \infty$.

Writing (3.6) out explicitly, we have

$$w_1(x - z_{n+1}) - \int K(x + c_1 - y) g(w_1(y - z_n) - q_n)\, dy \leqslant q_{n+1}. \tag{3.8}$$

From (3.2a), left-hand side of (3.8) is bounded above by

$$w_1(x - z_{n+1}) - w_1(x - z_n) - \int K(x + c_1 - y)[g(w_1(y - z_n) - q_n) - g(w_1(y - z_n))]\, dy. \tag{3.9}$$

Hence it suffices to show that (3.9) is bounded above by q_{n+1}.

Consider the function $g(w - q) - g(w)$ for $0 \leqslant q \leqslant q_0$. Since $g'(1) < 1$, there exists a constant $\theta_1 \in (0, 1)$ such that $g(w - q) - g(w) \geqslant -\theta_1 q$ when $w = 1$ and

$q \in [0, q_0]$. By continuity, this also holds for w in a left neighborhood of 1. Near zero we increase θ_1 until $g'(0) < \theta_1$ and the same inequality holds by the mean value theorem as long as $w - q \geqslant 0$. If $w < q$, then since $\lim_{w \downarrow 0}(g(w)/w) = g'(0)$, we have for sufficiently small w,

$$g(w-q) - g(w) = -g(w) \geqslant -\theta_1 w \geqslant -\theta_1 q \qquad \text{for} \qquad q \in [0, q_0].$$

We therefore conclude that there are constants $\theta_1 \in (0,1)$ and $\delta > 0$ such that

$$g(w-q) - g(w) \geqslant -\theta_1 q \quad \text{for} \quad w \in [1-\delta, 1] \quad \text{or} \quad w \in [0, \delta] \quad \text{and} \quad q \in [0, q_0].$$

Now choose $\varepsilon > 0$ such that $\varepsilon g'(\alpha) + \theta_1 < 1$ and define $\theta = \varepsilon g'(\alpha) + \theta_1$, $\mu_1 = -\ln\theta$, $q_n = q_0 e^{-\mu_1 n}$ for $n = 1, 2, \ldots$. Also choose $\eta > 0$ so large that

$$\int_{\eta + c_1}^{\infty} K(y)\,dy \leqslant \varepsilon, \qquad \int_{-\infty}^{-\eta + c_1} K(y)\,dy \leqslant \varepsilon$$

and let $E_\gamma = w^{-1}(\gamma)$ for $\gamma \in (0,1)$. This is well defined since $0 < w_1(x) < 1$ in $\mathbb{R}$.

From (3.1), there exist constants $\theta_2 > 0$ such that

$$w_1(\xi_1) - w_1(\xi_2) \leqslant -\theta_2(\xi_1 - \xi_2)$$

if $\xi_1 > \xi_2$ are in $[E_{1-\delta} - 2\eta,\ E_\delta + 2\eta]$. Define

$$\theta_3 = \min\left\{\theta_2, \frac{w_1(E_\delta + \eta) - w_1(E_\delta + 2\eta)}{\eta}\right\}$$

and $\{z_n\}$ by the recursion

$$z_{n+1} = \frac{(\theta - g'(\alpha))e^{-\mu_1 n}}{\theta_3} + z_n, \qquad n = 0, 1, \ldots. \tag{3.10}$$

Note that $z_{n+1} < z_n$. We shall now show that under such definition of $\{z_n\}$ and $\{q_n\}$, (3.9) is bounded above by q_{n+1}.

Let n be a positive integer and define $\Gamma_n = [E_{1-\delta} + z_n,\ E_\delta + z_n]$. From our choice of θ_1 above, we have

$$g(w_1(y - z_n) - q_n) - g(w_1(y - z_n)) \geqslant -\theta_1 q_n \qquad \text{if} \qquad y \notin \Gamma_n. \tag{3.11}$$

Let $\Gamma'_n \supset \Gamma_n$ be the interval $\Gamma'_n = [E_{1-\delta} + z_n - \eta, E_\delta + z_n + \eta]$. If $x \notin \Gamma'_n$, then $\int_{\Gamma_n} K(x + c_1 - y)\,dy \leqslant \varepsilon$ so that from (3.11) and $z_{n+1} < z_n$, (3.9) is bounded above by

$$-\int_{\Gamma_n} K(x + c_1 - y) g'(\alpha)(-q_n)\,dy + \theta_1 q_n \leqslant \theta q_n = q_{n+1}.$$

If $x \in \Gamma'_n$, then $x - z_n \in [E_{1-\delta} - \eta, E_\delta + \eta]$ and there are two cases to consider. Case 1 is when $x - z_n \leqslant x - z_{n+1} \leqslant E_\delta + 2\eta$ and

$$w_1(x - z_{n+1}) - w_1(x - z_n) \leqslant -\theta_2(z_n - z_{n+1}) \leqslant -\theta_3(z_n - z_{n+1}).$$

Case 2 is when $x - z_{n+1} > E_\delta + 2\eta$ in which case $\eta \leqslant z_n - z_{n+1}$ and we have

$$\frac{w_1(x - z_{n+1}) - w_1(x - z_n)}{z_n - z_{n+1}} \leqslant \frac{w_1(E_\delta + 2\eta) - w_1(E_\delta - \eta)}{\eta} \leqslant -\theta_3.$$

In both cases (3.9) is bounded above by the quantity $-\theta_3(z_n - z_{n+1}) + g'(\alpha)q_n$ which equals q_{n+1} by (3.10). Altogether, we have proved (3.8) and (3.6) assuming that $0 < w_1(x) < 1$ in $\mathbb{R}$. Since $\{z_n\}$ is convergent, (3.7) implies (3.3a).

We now indicate how to handle the case when $w_1(x) = 0$ or 1 at some points. We assume the worst, that $w_1(x) = 1$ for $x \leqslant E_1$ and $w_1(x) = 0$ for $x \geqslant E_0$. From our hypotheses, equality holds in (3.2a) when $x \leqslant E_1$ and $x \geqslant E_0$. Letting $x = E_0$ and $x = E_1$ in (3.2a), we see that B_1, B_2 are finite.

We assume that z_0, q_0, θ_1 have been chosen and that $g(w-q) - g(w) \geqslant -\theta_1 q$ whenever $q \in [0, q_0]$ and $w \geqslant \delta_2$ or $w \leqslant \delta_1$. We shall prove at the end that we may take $\delta_1 = \sigma_1$, $\delta_2 = \sigma_2$.

Define ε, θ, μ_1, $\{q_n\}$ as before and choose $\eta > 0$ such that

$$\int_{B_1}^{B_1+\eta} K(y)\,dy = \varepsilon = \int_{B_2-\eta}^{B_2} K(y)\,dy.$$

From (3.1), there exists $\bar{\theta}_2 > 0$ such that (3.1) holds for $x > y$ in $[E_1 + \frac{1}{2}\eta, E_0 - \frac{1}{2}\eta]$. Define $\bar{\theta}_3$ by

$$\bar{\theta}_3 = \min\left\{\bar{\theta}_2, \frac{w_1(E_0 - \eta) - w_1(E_0 - \frac{1}{2}\eta)}{\frac{1}{2}\eta}\right\}$$

and the decreasing sequence $\{z_n\}$ by the recursion (3.10) with θ_3 being replaced by $\bar{\theta}_3$.

Let n be a positive integer and define $\Gamma_n \subset \Gamma_n'$ by

$$\Gamma_n = [E_{\delta_2} + z_n, E_{\delta_1} + z_n] \quad \text{and} \quad \Gamma_n' = [B_1 + \eta + E_{\delta_2} + z_n - c_1, B_2 - \eta + E_{\delta_1} + z_n - c_1].$$

Note that (3.11) holds. If $x \notin \Gamma_n'$ then with a simple calculation we can show that $\int_{\Gamma_n} K(x + c_1 - y)\,dy \leqslant \varepsilon$. The proof for the case $x \notin \Gamma_n'$ now remains unchanged as before. In the opposite case, we first show that if $x \in \Gamma_n$, then $x - z_n \in [E_1 + \eta, E_0 - \eta]$. Consider the relation $w(x) = \int_{x+c_1-B_2}^{x+c_1-B_1} K(x + c_1 - y) g(w(y))\,dy$ which must hold at $x = E_1$ or E_0. This implies that $E_1 + c_1 - B_1 \leqslant E_{\sigma_2}$ and $E_0 + c_1 - B_2 \geqslant E_{\sigma_1}$. Since $E_{\sigma_1} = E_{\delta_1}$, $E_{\sigma_2} = E_{\delta_2}$, we have after some calculations proved that $E_1 + \eta \leqslant x - z_n \leqslant E_0 - \eta$ whenever $x \in \Gamma_n'$.

To continue, if $x - z_{n+1} \leqslant E_0 - \eta/2$, then $w_1(x - z_{n+1}) - w_1(x - z_n) \leqslant -\bar{\theta}_2(z_n - z_{n+1})$ while if $x - z_{n+1} > E_0 - \eta/2$, then $z_n - z_{n+1} \geqslant \eta/2$ so that

$$\frac{w_1(x - z_{n+1}) - w_1(x - z_n)}{z_n - z_{n+1}} \leqslant \frac{w_1(E_0 - \eta/2) - w_1(E_0 - \eta)}{\eta/2} \leqslant -\bar{\theta}_3.$$

Thus $w_1(x - z_{n+1}) - w_1(x - z_n) \leqslant -\bar{\theta}_3(z_n - z_{n+1})$ if $x \in \Gamma_n'$ and the rest of the proof proceeds as before.

Finally, we show that we may take $\delta_1 = \sigma_1$ and $\delta_2 = \sigma_2$ in the choice of θ_1. Recall from (3.5), q_0 was chosen so that $u_0(x) + q_0 > 1$ for x sufficiently small. From our hypothesis, $\liminf_{x \to -\infty} u_0(x) > \alpha$, the fact that $g(u) > u$ in $(\alpha, 1)$ and $K(x)$ has compact support, we see that $u_n(x)$ increases to 1 for x near $-\infty$ as $n \to \infty$. Thus if we replace $u_0(x)$ by $u_{n_0}(x)$, q_0 may be taken to be small if n_0 is sufficiently large. Now consider again the function $g(w - q) - g(w)$ when $q \in [0, q_0]$. Since $g(\sigma_2) = 1$, we have $g(\sigma_2 - q) - g(1) = g(\sigma_2 - q) - g(\sigma_2) \geqslant -\theta_1 q$ for some constant $\theta_1 \in (0, 1)$, $q \in [0, q_0]$ and q_0 sufficiently small. This implies that

$g(w - q) - g(w) \geqslant -\theta_1 q$ if $\sigma_2 \leqslant w \leqslant 1$ and therefore we may take $\delta_2 = \sigma_2$. If $w \leqslant \sigma_1$, then $g(w - q) - g(w) = 0 \geqslant -\theta_1 q$ for q in $[0, q_0]$, and we may take $\delta_1 = \sigma_1$.

(3.3a) is now proved with n replaced by $n_0 + n$ on the right-hand side of the inequality. This reduces to (3.3a) by increasing q_0. Q.E.D.

Theorem 4. *Let $w_c(x)$ be a nonincreasing travelling wave solution of the operator Q with speed c such that $w_c(-\infty) = 1$, $w_c(\infty) = 0$. Let $u_0 \in \mathscr{C}$ satisfy the hypotheses of Proposition 4. Then there exist constants $x_1, x_2, \bar{q}$ and μ, the last two positive, such that*

$$w_c(x - x_1) - \bar{q}e^{-\mu n} \leqslant u_n(x + nc) \leqslant w_c(x - x_2) + \bar{q}e^{-\mu n} \qquad \textit{for all } n.$$

Proof. Condition (ii) of (1.3) implies that $w_c(x)$ is in $C^1(\mathbb{R})$ (see Lemma 5 in [10] and its corollary). We first show that $w_c'(x) < 0$ unless $w_c(x) = 0$ or 1. Assuming that the contrary occurs at a point x_0, and let I_0 be the values of y such that $x_0 + c - y$ is in the support of $K(x)$. Then $w_c(x_0) \neq 0, 1$ implies that I_0 contains an interval I where $\sigma_1 < w_c(y) < \sigma_2$. By continuity, if $w_c(x_0) \in (\sigma_1, \sigma_2)$, then, I may be taken as a neighborhood of x_0. Since $w_c(x)$ is in $C^1(\mathbb{R})$, we have

$$w_c'(x_0) = \int_{I_0} K(x_0 + c - y) g'(w_c(y)) w_c'(y)\, dy = 0$$

and the assumption (1.3)(i) on $K(x)$ and (1.4)(ix) on $g(u)$ imply that $w_c'(y) = 0$ in I. Hence we may as well assume from the start that $w_c(x_0) \in (\sigma_1, \sigma_2)$. Therefore I will be a neighborhood of x_0 on which $w_c(x) = w_c(x_0)$. Repeating the argument, we arrive at the contradiction that $w_c(\infty) = w_c(x_0) > 0$. Thus $w_c'(x) < 0$ if $w_c(x) \in (0, 1)$. This implies property (3.1) and we can then apply Proposition 4 twice, once with $w_1 = w_c$, $c_1 = c$ and again with $w_2 = w_c$, $c_2 = c$ to obtain the desired inequality with $\bar{q} = \max(q_0, q_0')$ and $\mu = \min(\mu_1, \mu_2)$. Q.E.D.

Corollary 1. *There is only one wave speed c for which nonincreasing travelling wave $w_c(x)$, of the operator Q with the property $w_c(-\infty) = 1$, $w_c(\infty) = 0$ can exist.*

Proof of Corollary 1. Let $w_1(x)$, $w_2(x)$ be two such waves with speed c_1, c_2 respectively. Applying Theorem 4 with $w_c = w_1$, $u_0 = w_2$, we obtain the inequality

$$w_1(x - x_1) - \bar{q}e^{-\mu n} \leqslant w_2(x + n(c_1 - c_2)) \leqslant w_1(x - x_2) + \bar{q}e^{-\mu n}$$

for all n. Letting $n \to \infty$, we arrive at a contradiction on one of the two sides unless $c_1 = c_2$. Q.E.D.

Remark 2. From (3.10), z_n decreases to the limit $kq_0 + z_0$ as $n \to \infty$ where k is some constant. Thus if z_0 and q_0 are close to 0, so will the sequence $\{z_n\}$ and the constant x_1. This means that in Theorem 4, the translates may be made small on account of how close the initial data is to the travelling wave. In particular, if for some subsequence n_j, $u_{n_j}(x + n_j c)$ converges uniformly to $w_c(x)$ as $j \to \infty$, then the entire sequence of functions converges to $w_c(x)$ as $n \to \infty$.

Remark 3. Theorem 4 says that $u_n(x + nc)$ will be trapped between translates of the travelling wave if $u_0(x)$ vaguely resembles the wave. Vaguely resembles here means $u_0(x) > \alpha$ near $-\infty$ and $u_0(x) < \alpha$ near ∞. In a subsequent paper, we shall show that, under additional hypotheses on $K(x)$, $u_n(x + nc_+^*)$ actually converge to a travelling wave as $n \to \infty$.

4. Existence of Travelling Waves

Theorem 5. *There exists a nonincreasing travelling wave solution $w_c(x)$ of the operator Q with speed $c = c_+^* = \tilde{c}_-^*$ such that $w_c(-\infty) = 1$, $w_c(\infty) = 0$.*

Proof. In Section 2, we have constructed two wave speeds $c_+^* \leqslant \tilde{c}_-^*$. We first show that for each $c \in [c_+^*, \tilde{c}_-^*]$, there exists a nonincreasing travelling wave $w_c(x)$ of speed c with the desired properties. From Corollary 1 of Theorem 4, there can only be one such c so that $c_+^* = \tilde{c}_-^*$.

Recall from the end of Section 2 that for each $c \in [c_+^*, \tilde{c}_-^*]$, there are two nonincreasing functions $a(c,x)$, $b(c,x)$ such that $a(c,-\infty) = b(c,-\infty) = 1$, $a(c,\infty) = b(c,\infty) = 0$ and they satisfy respectively the inequalities,

$$a(c,x) \geqslant Q[a(c,\cdot)](x+c), \tag{4.1}$$

$$b(c,x) \leqslant Q[b(c,\cdot)](x+c). \tag{4.2}$$

Let $v_0(x) = a(c,x)$ and define v_n recursively by $v_{n+1} = Q[v_n](x+c)$. From (4.1) and an inductive argument, we see that $v_n(x)$ is nonincreasing in n and x. Thus $v_n(x)$ decreases to a limit function $w_c(x)$ as $n \to \infty$, $w_c(x)$ is nonincreasing and since $a(c,\infty) = 0$, we have $w_c(\infty) = 0$. From the dominated convergence theorem, $Q[v_n](x+c)$ converges to $Q[w_c](x+c)$ as $n \to \infty$. Since $Q[v_n](x+c) = v_{n+1}(x)$, $w_c(x)$ satisfies the relation $w_c(x) = Q[w_c](x+c)$. It therefore remains to show that $w_c(-\infty) = 1$ in order to finish the proof of our theorem.

To this end, let $u_0 = v_0$, $c_1 = c$ and $w_1(x) = b(c,x)$ in Proposition 4. From (4.2), $w_1(x)$ satisfies (3.2a). The function $b(c,x)$ has the property that

$$b(c,x) = \min\{\psi(x), Q[b](x+c)\}, \tag{4.3}$$

where $\psi(x) > 0$ in $\mathbb{R}$. Thus (3.2a) is an equality at places where $b(c,x) = 0$. All the hypotheses of $w_1(x)$ in Proposition 4 are satisfied except possibly the inequality (3.1). Assuming this for the moment, Proposition 4 then implies that there exist constants x_1, q_0, μ_1 such that

$$b(c, x - x_1) - q_0 e^{-\mu_1 n} \leqslant v_n(x) \qquad \text{for all } n.$$

Letting $n \to \infty$, we have $b(c, x - x_1) \leqslant w_c(x)$ and hence $w_c(-\infty) = b(c,-\infty) = 1$.

The rest of the proof is rather technical and is devoted to proving (3.1) for the function $b(x) \equiv b(c,x)$. We introduce the notation that for any function $f(x)$ satisfying the type of inequality in (3.1) on a compact set Γ, the constant θ will be denoted by the symbol $\theta(\Gamma, f)$. We shall assume that $\psi(x)$ in (4.3) is smooth, $\psi(x) = 1$ in $(-\infty, 0]$, $\psi'(x) < 0$ in $(0,\infty)$ and $\psi(\infty) \in (0,\alpha)$.

We first assume that the function $B(x) \equiv Q[b](x+c)$ satisfies (3.1) on any compact set Γ and show that this implies the same for $b(x)$. Henceforth, let $\Gamma = [\xi_1, \xi_2]$ be fixed, $0 < b(x) < 1$ in Γ and let ξ_0 be the first point in Γ where $\psi(x)$ crosses $B(x)$. If $\psi(x)$ never crosses $B(x)$ in Γ, then from (4.3), we have either $b(x) = \psi(x)$ or $b(x) = B(x)$ in Γ, and the result follows from those of $\psi(x)$ or $B(x)$. Also from (4.3), $B(x) > 0$ in Γ and since both $\psi(x)$ and $B(x)$ are nonincreasing, we have $0 < \psi(x)$, $B(x) < 1$ in the interval $[\xi_0, \xi_2]$.

There are now two cases to consider. Case 1 is when $b(x) = \psi(x) \leqslant B(x)$ in $[\xi_1, \xi_0]$. Then $0 < \psi(x) < 1$ in Γ and we let $\theta_1 = \theta(\Gamma, \psi)$, $\theta_2 = \theta([\xi_0, \xi_2], B)$. Case 2 is when $b(x) = B(x) \leqslant \psi(x)$ in $[\xi_1, \xi_0]$, in which case $0 < B(x) < 1$ in Γ and we let $\theta_1 = \theta(\Gamma, B)$, $\theta_2 = \theta([\xi_0, \xi_2], \psi)$. We shall show that $\theta(b, \Gamma) = \min(\theta_1, \theta_2)$.

Given $x > y$ in Γ, if $\psi(z)$ does not cross $B(z)$ in the interval (y, x), then the result is clear because x, y may be on the same side of ξ_0 with $b = \psi$ or $b = B$ in (y, x). If $\psi(z)$ crosses $B(z)$ at only one point $\xi \in (y, x)$, then from our definition of ξ_0, we have $\xi_0 \leqslant \xi \leqslant x$. If $y < \xi_0$, then in Case 1 we have

$$\begin{aligned} b(x) - b(y) &= B(x) - B(\xi) + \psi(\xi) - \psi(y) \\ &\leqslant -\theta_2(x - \xi) - \theta_1(\xi - y) \leqslant -\min(\theta_1, \theta_2)(x - y). \end{aligned}$$

In Case 2, we simply interchange ψ and B to obtain the same inequality. It is also clear what to do if $y \geqslant \xi_0$. The places where $\psi(z)$ and $B(z)$ cross in the interval (y, x) are at most countable and may be reduced to this simpler situation by introducing intermediate points between y and x.

Finally, we show that $B'(x) < 0$ whenever $B(x) \in (0, 1)$. Note that condition (ii) of (1.3) implies that $B(x)$ is in $C^1(\mathbb{R})$. From (4.3) $2b(x) = -|\psi(x) - B(x)| + \psi(x) + B(x)$ so that $b(x)$ is absolutely continuous and

$$B'(x) = \int K(x + c - y) g'(b(y)) b'(y)\, dy.$$

If $B'(x_0) = 0$ while $B(x_0) \neq 0, 1$, then the same argument given in the proof of Theorem 4 for the function $w_c(x)$ would imply that $b(x)$ is constant on an interval I where $b(x) \in (0, 1)$. Since $\psi(x)$ is decreasing in $(0, \infty)$, $b(x) < \psi(x)$ in the interior of I. Thus $B(x) = b(x)$ and $B'(x) = 0$ in I. Repeating the argument inductively, we conclude that $b(-\infty) < 1$. This contradiction completes the proof of our assertion about $B(x)$ as well as the theorem. Q.E.D.

Remark 1. An immediate consequence of Theorems 4 and 5 is the strengthening of Theorem 1a′ in that the result holds as long as the initial data satisfy the conditions

$$\liminf_{x \to -\infty} u_0(x) > \alpha \qquad \text{and} \qquad \limsup_{x \to \infty} u_0(x) < \alpha.$$

Let $w(x)$ be the travelling wave constructed in Theorem 5. We drop the dependence of $w(x)$ on c because $w_c(x)$ exist only when $c = c_+^*$. We shall in the remainder of this section show that $w(x)$ converges to its limits exponentially as $x \to \pm\infty$.

Proposition 5. *Let $w(x)$ be a travelling wave solution of the operator Q with speed c_+^* such that $w(-\infty) = 1$, $w(\infty) = 0$. Let μ^* be the unique positive root of the equation $\Phi(\mu, g'(0)) = c_+^*$ with $g'(0) > 0$. Then $w(x) \sim \mathrm{const}\, e^{-\mu^* x}$ as $x \to \infty$. Similarly, let $-\bar{\mu}^*$ be the unique negative root of the equation $\Phi(\mu, g'(1)) = c_+^*$ and $g'(1) > 0$, then $1 - w(x) \sim \mathrm{const}\, e^{\bar{\mu}^* x}$ as $x \to -\infty$.*

Proof. We only need to prove the first half of the proposition for the second half may be reduced to the first by a change of variable (see Section 2). Except for Lemma 4.1 below, the proof is identical to the one given in [5], where the authors considered the intermediate case. The idea is to use Laplace transform and then apply a Tauberian theorem of Ikehara. Condition (xii) of (1.4) is needed here to guarantee that the Laplace transform of the function $K * [g(w) - g'(0)w]$ does not vanish on the real axis. We refer the reader to [5] for further details. Q.E.D.

Lemma 4.1. *Under the hypothesis of Proposition 5, there exists a constant* $\delta > 0$ *such that* $\int e^{\lambda x} w(x)\,dx$ *converges for* $0 < \operatorname{Re}\lambda < \delta$.

Proof. Let $\gamma \in (g'(0), 1)$ and let $K_1(x)$ be a function that satisfies the following conditions. $K_1(x) = 0$ for $x \geqslant \eta, \eta$ a constant to be chosen later, $K_1(x) \geqslant K(x)$ for $x \leqslant \eta$ and $\int K_1(x)\,dx = (1+\gamma)/2\gamma > 1$. We shall write c for c_+^* and rewrite our integral equation in the form

$$w(x) - \frac{2\gamma}{\gamma+1}\int K_1(x+c-y)w(y)\,dy = \frac{\gamma-1}{\gamma+1}w(x) + \frac{2}{\gamma+1}\psi(x), \tag{4.4}$$

where

$$\psi(x) = \int [K(x+c-y)g(w(y)) - \gamma K_1(x+c-y)w(y)]\,dy. \tag{4.5}$$

Define the function

$$n(x) = \begin{cases} \dfrac{2\gamma}{\gamma+1}\displaystyle\int_x^\infty K_1(c+y)\,dy, & x \geqslant 0 \\ -\dfrac{2\gamma}{\gamma+1}\displaystyle\int_{-\infty}^x K_1(c+y)\,dy, & x < 0 \end{cases} \tag{4.6}$$

and integrate (4.4) from z to r. We interchange the order of integration in the second term and then integrate by parts to obtain

$$\frac{-2\gamma}{\gamma+1}\int_z^r \int K_1(x+c-y)w(y)\,dy\,dx = -\int_z^r w(x)\,dx + (n*w)(r) - (n*w)(z).$$

Note that from (4.6), $n'(x) = (-2\gamma/(\gamma+1))K_1(x+c)$ if $x \neq 0$ and $n(0^+) - n(0^-) = 1$. Integration of (4.4) therefore gives us

$$(n*w)(r) - (n*w)(z) = \frac{\gamma-1}{\gamma+1}\int_z^r w(x)\,dx + \frac{2}{\gamma+1}\int_z^r \psi(x)\,dx. \tag{4.7}$$

Write $\psi(x)$ as

$$\begin{aligned}\psi(x) = &\int [K(x+c-y) - K_1(x+c-y)]g(w(y))\,dy \\ &+ \int_{x+c-\eta}^\infty K_1(x+c-y)[g(w(y)) - \gamma w(y)]\,dy.\end{aligned} \tag{4.8}$$

From our choice of γ, the last term of (4.8) will be negative for sufficiently large values of x. Since $K_1(x) \geqslant K(x)$ for $x \leqslant \eta$, we have

$$\psi(x) \leqslant \int_{-\infty}^{x+c-\eta} K(x+c-y)g(w(y))\,dy \tag{4.9}$$

when x is sufficiently large. Now integrate (4.9) from z to r. On the right we interchange the order of integration so that,

$$\begin{aligned}
&\int_z^r \int_{-\infty}^{x+c-\eta} K(x+c-y)g(w(y))\,dy\,dx \\
&= \int_{-\infty}^{z+c-\eta} \int_z^r K(x+c-y)\,dx\,g(w(y))\,dy \\
&\quad + \int_{z+c-\eta}^{r+c-\eta} \int_{y+\eta-c}^{r} K(x+c-y)\,dx\,g(w(y))\,dy \\
&= \int_{-\infty}^{z+c-\eta} \int_{z+c-y}^{r+c-y} K(x)\,dx\,g(w(y))\,dy \\
&\quad + \int_{z+c-\eta}^{r+c-\eta} \int_{\eta}^{r+c-y} K(x)\,dx\,g(w(y))\,dy.
\end{aligned}$$

Choose η so large that $\gamma \int_\eta^\infty K(x)\,dx \leqslant (1-\gamma)/4$. From our assumptions about $K(x)$, we have for η, z sufficiently large, the inequality

$$\begin{aligned}
&\int_z^r \int_{-\infty}^{x+c-\eta} K(x+c-y)g(w(y))\,dy\,dx \\
&\leqslant \text{const} \int_{-\infty}^{z+c-\eta} K(z+c-y)g(w(y))\,dy + \frac{1-\gamma}{4} \int_{z+c-\eta}^{r+c-\eta} w(y)\,dy \\
&\leqslant Aw(z) + \frac{1-\gamma}{4} \int_{z+c-\eta}^{r} w(y)\,dy.
\end{aligned}$$

Substituting this in for $\int_z^r \psi(x)\,dx$ in (4.7), we obtain the inequality,

$$\begin{aligned}
&(n * w)(r) - (n * w)(z) \\
&\leqslant \frac{\gamma-1}{\gamma+1} \int_z^r w(x)\,dx + \frac{2A}{\gamma+1} w(z) + \frac{1-\gamma}{2(\gamma+1)} \int_{z+c-\eta}^{r} w(y)\,dy \\
&= \frac{\gamma-1}{2(\gamma+1)} \int_z^r w(x)\,dx + \frac{2A}{\gamma+1} w(z) + \frac{1-\gamma}{2(\gamma+1)} \int_{z+c-\eta}^{z} w(y)\,dy. \qquad (4.10)
\end{aligned}$$

Since $(n * w)(r) \to 0$ as $r \to \infty$ and the coefficient of the first integral in (4.10) is negative, we see that $\int_z^\infty w(x)\,dx$ converges. Letting $r \to \infty$ in (4.10), we have

$$-(n * w)(z) \leqslant \frac{\gamma-1}{2(\gamma+1)} \int_z^\infty w(x)\,dx + \frac{2A}{\gamma+1} w(z) + \frac{(1-\gamma)(\eta-c)}{2(\gamma+1)} w(z+c-\eta).$$

Rearranging,

$$\int_z^\infty w(x)\,dx \leqslant 2\left(\frac{1+\gamma}{1-\gamma}\right)(n * w)(z) + \frac{4A}{1-\gamma} w(z) + (\eta-c)w(z+c-\eta).$$

Since $n(x) = 0$ for $x \geqslant \eta - c$ and $w(x)$ is nonincreasing, it follows from above that

$$\int_z^\infty w(x)\,dx \leqslant 2\left(\frac{1+\gamma}{1-\gamma}\right)\|n\|_1 w(z+c-\eta) + \frac{4A}{1-\gamma}w(z) + (\eta - c)w(z+c-\eta)$$
$$\leqslant \theta w(z-v), \tag{4.11}$$

where

$$\theta = \left\{2\left(\frac{1+\gamma}{1-\gamma}\right)\|n\|_1 + \frac{4A}{1-\gamma} + (\eta - c)\right\} \qquad \text{and} \qquad v = \eta - c$$

are positive constants.

Now we shall show that if we let $w^{(0)} = w$ and define $w^{(k)}$ recursively by $w^{(k+1)}(x) = \int_x^\infty w^{(k)}(y)\,dy$, we have the estimate,

$$\int_z^\infty w^{(k)}(y)\,dy \leqslant \theta w^{(k)}(z-v) \tag{4.12}$$

for sufficiently large z independently of k. (4.11) is just (4.12) when $k = 0$ and the case $k > 1$ follows from (4.11) by successive integration. We also deduce from (4.12) that

$$w^{(k)}(z-v) = \int_{z-v}^{z} w^{(k-1)}(y)\,dy + \int_z^\infty w^{(k-1)}(y)\,dy$$
$$\leqslant v w^{(k-1)}(z-v) + \theta w^{(k-1)}(z-v)$$
$$= (v+\theta)w^{(k-1)}(z-v).$$

By an inductive argument, $w^{(k)}(z-v) \leqslant (v+\theta)^k$ and from (4.12), we obtain the estimate,

$$\int_z^\infty w^{(k)}(y)\,dy \leqslant \theta(v+\theta)^k. \tag{4.13}$$

We now use a formula analogous to Lemma 4.4 of [5],

$$\int_z^\infty (y-z)^k f(y)\,dy = k!\int_z^\infty f^{(k)}(y)\,dy$$

which is valid for function $f(x)$ such that $f^{(k)}(x)$ is integrable near infinity. Applying this formula to $w(x)$ and using (4.13), we have

$$\frac{1}{k!}\int_z^\infty (y-z)^k w(y)\,dy \leqslant \theta(v+\theta)^k \equiv \theta\delta^{-k}$$

which implies that $\int_z^\infty e^{\lambda(y-z)}w(y)\,dy$ converges whenever $0 < \operatorname{Re}\lambda < \delta$. Q.E.D.

Remark 2. If $g'(0) = 0$, then Lemma 4.1 is still valid but not Proposition 5 because $\Phi(\mu, \beta)$ is defined only for $\beta > 0$. In this case we have for any constant t,

$$w(x) = \int_{-\infty}^{t} K(x+c-y)g(w(y))\,dy + \int_t^\infty K(x+c-y)g(w(y))\,dy$$

$$\leqslant \int_{x+c-t}^{\infty} K(y)\,dy + g(w(t)).$$

Let $t = 2x/3$ and x sufficiently large, we have the inequality,

$$w(x) \leqslant \text{const}\, K\left(\frac{x}{3} + c\right) + \text{const}\, w^2\left(\frac{2x}{3}\right). \tag{4.14}$$

If $w(x)e^{\mu x} \to 0$ as $x \to \infty$ for some $\mu > 0$, then multiplying (4.14) by $e^{4\mu x/3}$, we see from condition (iv) of (1.3) that $w(x)e^{4\mu x/3} \to 0$ as $x \to \infty$. A bootstrap argument thus implies that $w(x)$ must decay faster than $e^{-\mu x}$ for any $\mu > 0$.

5. Generalizations

This section contains some straightforward generalizations of ideas presented in the previous sections to the case when $g(u) - u$ has more than one internal zero. Our results, however, fall short of proving existence of travelling waves in general.

For simplicity, we shall assume that $g(u) = u$ at the points $0 < \alpha_0 < \alpha_1 < \alpha_2 < 1$ and that the triples $(0, \alpha_0, \alpha_1), (\alpha_1, \alpha_2, 1)$ each have all the properties analogous to the zeros $(0, \alpha, 1)$ in the inferior case. We first summarize the results which should be obvious by now.

Lemma 5.1. *There exist nonincreasing functions w_0, w_1 and two numbers c_0, c_1, such that $w_0(-\infty) = \alpha_1$, $w_0(\infty) = 0$, $w_1(-\infty) = 1$, $w_1(\infty) = \alpha_1$, and $u_n(x) = w_0(x - nc_0)$, $u_n(x) = w_1(x - nc_1)$ each satisfies the recursion* (1.2). *Furthermore, if $u_0 \in [\alpha_1, 1]$, $\liminf_{x\to-\infty} u_0(x) > \alpha_2$, $\limsup_{x\to\infty} u_0(x) < \alpha_2$, define u_n recursively by* (1.2), *then $u_n(x + nc) \to 1$ or α_1 as $n \to \infty$ depending on whether $c < c_1$ or $c > c_1$. Also $u_n(x + nc_1)$ will be trapped between two translates of the function $w_1(x)$ as $n \to \infty$. Similar statements hold for initial data $u_0 \in [0, \alpha_1]$ with the numbers c_1, 1, α_2, α_1 replaced by c_0, α_1, α_0, 0 respectively.*

We now define the wave speed c_+^* as before, starting with a continuous nonincreasing function $\phi(x)$, $\phi(x) = 0$ for $x \geqslant 0$ and $\phi(-\infty) \in (\alpha_2, 1)$. We obtain the sequence $a_n(c, x)$ which increases to the limit function $a(c, x)$ as $n \to \infty$. Similarly, we may define the dual wave speed $\tilde{c}_-^*$ and the function $b(c, x)$, starting with a continuous nonincreasing function $\phi(x)$, $\phi(x) = 1$ for $x \leqslant 0$ and $\phi(\infty) \in (0, \alpha_0)$.

Lemma 5.2. *Theorem 1a holds with α in* (2.3) *replaced by $a(c_+^*, \infty)$. Theorem 1b holds with α replaced by α_2. Theorem 3a holds with* 1 *in* (2.12) *replaced by $b(\tilde{c}_-^*, -\infty)$. Theorem 3b holds with α replaced by α_0.*

In what follows, we shall adopt the notations, $a(x) \equiv a(c_+^*, x)$, $b(x) \equiv b(\tilde{c}_-^*, x)$, Q_c denotes the operator $Q_c[u](x) = Q[u](x + c)$, $Q_+ \equiv Q_{c_+^*}$, $Q_- \equiv Q_{\tilde{c}_-^*}$ and Q_c^r denotes the rth iterates of the operator Q_c. The functions $a(x)$, $b(x)$ have the following properties, $Q_+[a] \leqslant a$, $a(-\infty) = 1$, $a(\infty) \in \{0, \alpha_0, \alpha_1, \alpha_2\}$ and $Q_-[b] \geqslant b$, $b(\infty) = 0$, $b(-\infty) \in \{1, \alpha_2, \alpha_1, \alpha_0\}$. In particular, $Q_+^n[a]$ is nonincreasing in n and $Q_-^n[b]$ is nondecreasing in n. Finally, we let

$$\kappa_{10} = \inf_{\mu > 0} \Phi(\mu, g'(\alpha_0)), \qquad \kappa_{11} = \inf_{\mu > 0} \Phi(\mu, g'(\alpha_2)).$$

Then assuming that

$$\max_{[0,1]} \frac{g(u)}{u} < \min\{g'(\alpha_0), g'(\alpha_2)\},$$

we have as in Lemma 2.2

$$c_+^* < \kappa_{10}, \qquad c_+^* < \kappa_{11}.$$

Lemma 5.3. *$c_+^* \leqslant c_1$ and $c_0 \leqslant \tilde{c}_-^*$.*

Proof. If $c_1 < c_+^*$, let $u_0(x) = 1$ for $x \leqslant 0$, $u_0(x) = \alpha_1$, for $x > 0$. Then according to Lemma 5.1, $u_n(x + nc_1)$ will be trapped between translates of $w_1(x)$ as $n \to \infty$. But $u_0(x)$ is larger than α_2 on a semi-infinite interval. According to Lemma 5.2, $u_n(x + nc_1) \to 1$ as $n \to \infty$. Thus $c_1 < c_+^*$ is impossible. The proof of $c_0 \leqslant \tilde{c}_-^*$ is similar and will be omitted. Q.E.D.

Lemma 5.4. *$a(\infty) \neq \alpha_0$, α_2 and $b(-\infty) \neq \alpha_0, \alpha_2$.*

Proof. $a(\infty) \neq \alpha_2$ follows for the same reason that $a(c_+^*, \infty) \neq \alpha$. The latter is shown in the proof of Theorem 1a′. If $a(\infty) = \alpha_0$, then define $u_0(x) = \alpha_1$ for $x \leqslant 0$, $u_0(x) = \alpha_0$ for $x > 0$. Without loss of generality, we may assume that $u_0(x) \leqslant a(x)$. Thus $u_n(x + nc_+^*) \leqslant Q_+^n[a](x)$ for all n. Since $c_+^* < \kappa_{10}$, and $u_n \in [\alpha_0, \alpha_1]$, we have as in the intermediate case, $u_n(x + nc_+^*) \to \alpha_1$ as $n \to \infty$. This is impossible because $Q_+^n[a]$ is nonincreasing in n. The proofs of $b(-\infty) \neq \alpha_0, \alpha_2$ are similar and will be omitted. Q.E.D.

Remark 5.1. We are unable to show that $a(\infty) \neq \alpha_1$.

Lemma 5.5. *If $a(\infty) = \alpha_1$, then $c_1 = c_+^*$ and if $b(-\infty) = \alpha_1$, then $\tilde{c}_-^* = c_0$.*

Proof. We only prove the first half, the rest is similar. According to Lemma 5.3, it suffices to show that $c_1 \leqslant c_+^*$. Since $a(x) \in [\alpha_1, 1]$ is larger than α_2 on a semi-infinite interval, according to Lemma 5.1, $Q_+^n[a] \to 1$ as $n \to \infty$ if $c_+^* < c_1$. This is impossible since $Q_+^n[a]$ is nonincreasing. Q.E.D.

Theorem 6. *Suppose that $w(x)$ is a nonincreasing travelling wave solution of the operator Q with speed c such that $w(-\infty) = 1$, $w(\infty) = 0$. Let u_0 be such that*

$$\liminf_{x \to -\infty} u_0(x) > \alpha_2 \qquad \text{and} \qquad \limsup_{x \to \infty} u_0(x) < \alpha_0$$

and define u_n recursively by (1.2). *Then there exist constants, x_1, x_2, q_0, μ, the last two positive, such that*

$$w(x - x_1) - q_0 e^{-\mu n} \leqslant u_n(x + nc) \leqslant w(x - x_2) + q_0 e^{-\mu n} \qquad \text{for all } n.$$

Consequently, there is only one value of c for which such a solution $w(x)$ can exist.

Proof. Analogous to the proof of Theorem 4.

Since $w_1(x)$ obviously lies above a function $u_0(x)$ that satisfies the hypotheses of Theorem 6, we have the inequality

$$w(x - x_1) - q_0 e^{-\mu n} \leqslant u_n(x + nc) \leqslant w_1(x + n(c - c_1)).$$

Letting $n \to \infty$, we see that $c \leqslant c_1$. Similarly, we can show that $c_0 \leqslant c$. We therefore assume that $c_0 < c_1$ for the remainder of this section. From Lemma 5.5 and $c_+^* \leqslant \tilde{c}_-^*$, there are now three possibilities:

(i) $a(\infty) = \alpha_1$, $b(-\infty) = 1$,
(ii) $a(\infty) = 0$, $b(-\infty) = \alpha_1$, and
(iii) $a(\infty) = 0$, $b(-\infty) = 1$.

Lemma 5.6. *Assuming that* $c_0 < c_1$, *then in case* (i) $c_0 < c_1 = c_+^* = \tilde{c}_-^*$, *while in case* (ii) $c_+^* = \tilde{c}_-^* = c_0 < c_1$.

Proof. It suffices to show that in case (i), $c_1 \geqslant \tilde{c}_-^*$. If $c_1 < \tilde{c}_-^*$, define $u_0(x) = 1$ for $x \leqslant 0$, $u_0(x) = \alpha_1$ for $x > 0$. Then on one hand, according to Lemma 5.1, $u_n(x + nc_1)$ is trapped between two translates of $w_1(x)$ as $n \to \infty$ while on the other hand, according to Lemma 5.2 and the corresponding result of Theorem 3a, we have $u_n(x + nc_1) \to 1$ as $n \to \infty$. Thus $c_1 < \tilde{c}_-^*$ is not possible. The proof of case (ii) is similar. Q.E.D.

Combining all these results, we arrive at

Theorem 7. *Let* $c_0 < c_1$. *Then* $c_+^* = \tilde{c}_-^* \equiv C$ *and* $C \in [c_0, c_1]$. *If* $C \in (c_0, c_1)$, *then there exists a nonincreasing travelling wave solution* $w(x)$ *of the operator* Q *with speed* C *such that* $w(-\infty) = 1$, $w(\infty) = 0$.

Proof. According to Lemma 5.6, it suffices to consider only the third possibility. In that case, $a(\infty) = 0$ and $b(-\infty) = 1$ and the same proof as in Theorem 5 may be used to obtain the existence of $w(x)$. As a consequence, $c_+^* = \tilde{c}_-^* \equiv C$ and Lemma 5.3 implies that $C \in [c_0, c_1]$. Finally, note that if $C \neq c_0, c_1$, then the third possibility must occur. Q.E.D.

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Received April 20/Revised August 17, 1982