Periodic Traveling Waves in an Integro-Difference Equation With a Nonmonotone Growth Function and Strong Allee Effect

Michael Nestor, Bingtuan Li *

Department of Mathematics, University of Louisville,
Louisville, KY 40292.

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Abstract

We derive sufficient conditions for the existence of periodic traveling wave solutions for a class of integro-difference equation with piecewise constant growth function exhibiting a period two cycle and a strong Allee effect. We also prove the convergence of solutions with compactly supported initial data to translations of the traveling wave under appropriate conditions.

Key words: Integro-difference equation, period two cycle, Allee effect, periodic traveling wave.

Todo: Proof of Theorem 2.2.

AMS Subject Classification: 92D40, 92D25.

1   Introduction

Integro-difference equations are of great interest in the studies of invasions of populations with discrete generations and separate growth and dispersal stages. They have been used to predict changes in gene frequency [8, 9,

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10, 14, 17], and applied to ecological problems [2, 3, 4, 5, 7, 11, 12, 13]. Previous rigrous studies on integro-difference equations have assumed that the growth function is nondecreasing [17, 18], or is nonmonotone without strong Allee effect [10, 16]. The results show existence of constant spreading speeds and travelng waves with fixed shapes and speeds. Sullivan et al. [15] demonstrated numerically that an integro-difference equation with a nonmonotone growth function exhibiting a strong Allee effect can generate traveling waves with fluctuating speeds. In this paper we give a sufficient condition for the existence of periodic traveling waves with a periodic speed for such an equation with a specific growth function.

We consider the following integro-difference equation

$$u_{n+1}(x) = Q[u_n](x) := (k * (g \circ u_n))(x) = \int_{-\infty}^{\infty} k(x-y) g(u_n(y)) \, \mathrm{d}y,$$

(1.1)

where

$$g(u) = \begin{cases} 0, & \text{if } u < \alpha, \\ 1, & \text{if } \alpha \leq u \leq \beta, \\ \mu, & \text{if } u > \beta, \end{cases}$$

(1.2)

with $0 < \alpha < \mu < \beta < 1$. $g(u)$ is a piecewise constant nonmonotone growth function exhibiting a strong Allee effect [1]. Specifically, it has a stable fixed point at zero and a stable period two cycle $(1, \mu)$ with α the Allee threshold value.

Piecewise constant growth functions and uniform distributions have been used in the studies of integro-difference equations; see for example [6, 11, 13, 15]. We rigorously construct periodic traveling waves with periodic speeds for (1.1). To the best of our knowledge, this is the first time that traveling waves with oscillating speeds have been analytically established for scalar spatiotemporal equations with constant parameters. We also show the convergence of solutions with compactly supported initial data to translations of the traveling wave under appropriate conditions. Equation (1.1) may be viewed as a symbolic model for integro-difference equations with a growth function exhibiting a strong Allee effect and a period two cycle. The results obtained this paper provide important insights into integro-difference equations with general growth functions.

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2 Periodic traveling waves

In this section, we construct a periodic traveling wave solution to the recurrence (1.1). We will assume the dispersal kernel k satisfies the following hypotheses:

(H1) k is a non-negative Lebesgue-integrable function with $\int_{-\infty}^{\infty} k(x)\,dx = 1$;

(H2) $k(x) = k(-x)$ for all $x \in \mathbb{R}$;

(H3) the support of k is connected;

(H4) for all $\lambda \in (0,1)$, for all $y \in \mathbb{R}$, the function $x \mapsto k(x) - \lambda k(x-y)$ has at most one zero-crossing in $\mathbb{R}$.

Under these assumptions, the cumulative density function $F(x) = \int_{-\infty}^{x} k(y)\,dy$ has a well-defined right inverse F^{-1} satisfying $F(F^{-1}(p)) = p$ for all $0 < p < 1$. Furthermore, it satisfies the symmetry relation $F(-x) = 1 - F(x)$.

Let w_1 and w_2 be two functions belonging to $C(\mathbb{R})$ defined by

$$w_1(x) = \int_x^{\infty} k(y)\,dy \tag{2.1}$$

and

$$w_2(x) = \int_{-\infty}^{\infty} k(y)g(w_1(x-y))\,dy \tag{2.2}$$

Hypothesis (H1) implies w_1 and w_2 have well-defined limits at $\pm\infty$ given by $w_1(\infty) = w_2(\infty) = 0$, $w_1(-\infty) = 1$, and $w_2(-\infty) = \mu$. Furthermore, w_1 is monotonically decreasing, while w_2 may be non-monotonic.

Lemma 2.1. If $\sup_{x\in\mathbb{R}} w_2(x) \leq \beta$, then there exists unique $c^* \in \mathbb{R}$ such that $Q[w_2](x) = w_1(x - 2c^*)$ for all $x \in \mathbb{R}$.

Proof. Let $x_\alpha = \sup\{x \in \mathbb{R} \mid w_1(x) \geq \alpha\}$ and $x_\beta = \sup\{x \in \mathbb{R} \mid w_1(x) \geq \beta\}$. It follows that $w_1(x) > \beta$ for $x < x_\beta$, $\alpha \leq w_1(x) \leq \beta$ for $x_\beta \leq x \leq x_\alpha$, and $w_1(x) < \alpha$ for $x > x_\alpha$. Thus, composing $w_1(x)$ with the growth function g yields

$$g(w_1(x)) = \begin{cases} \mu & x < x_\beta \\ 1 & x_\beta \leq x \leq x_\alpha \\ 0 & x > x_\alpha \end{cases} \tag{2.3}$$

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Applying the convolution operator yields a formula for $w_2(x)$:

$$
\begin{aligned}
w_2(x) &= \int_{-\infty}^{\infty} k(y)g(w_1(x-y))\,dy \\
&= \int_{x-x_\alpha}^{x-x_\beta} k(y)\,dy + \mu \int_{x-\beta}^{\infty} k(y)\,dy \\
&= \int_{x-x_\alpha}^{\infty} k(y)\,dy - (1-\mu) \int_{x-x_\beta}^{\infty} k(y)\,dy
\end{aligned}
\tag{2.4}
$$

Taking the derivative with respect to x, we find

$$
\frac{dw_2}{dx} = -k(x-x_\alpha) + (1-\mu)k(x-x_\beta)
\tag{2.5}
$$

From hypothesis (H4), we can conclude that (2.5) has at most one zero-crossing, hence $w_2(x)$ has at most one turning point. This leaves two cases:

Case 1. If w_2 has no turning points, it must be monotonically decreasing.

Case 2. If w_2 has a single turning point x^*, then $w_2(x)$ is increasing on $(-\infty, x^*)$ and decreasing on (x^*, ∞), with $\mu \le w_2(x^*) \le \beta$.

In both cases, $w_2(x)$ has a well-defined right inverse on the open interval $(0, \mu)$. Since $0 < \alpha < \mu$, let

$$
c^* = \frac{1}{2} \sup\{x \in \mathbb{R} \mid w_2(x) \ge \alpha\}
\tag{2.6}
$$

It follows that $w_2(x) \ge \alpha$ for $x \le 2c^*$ and $w_2(x) < \alpha$ for $x > 2c^*$. Along with the assumption that $\sup_{x \in \mathbb{R}} w_2(x) \le \beta$, we can compose with g as follows:

$$
g(w_2(x)) = \begin{cases} 1 & x \le 2c^* \\ 0 & x > 2c^* \end{cases}
\tag{2.7}
$$

Taking convolution with k, we obtain

$$
Q[w_2](x) = \int_{x-x_0}^{\infty} k(y)\,dy = w_1(x - 2c^*).
\tag{2.8}
$$

$\square$

Theorem 2.2. If $\sup_{x \in \mathbb{R}} w_2(x) \le \beta$, then the recurrence (1.1) has a periodic traveling wave solution given by

$$
W_n(x) = \begin{cases} w_1(x - nc^*) & n \text{ even} \\ w_2(x - nc^*) & n \text{ odd} \end{cases}
\tag{2.9}
$$

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Proof. DO THIS □

Theorem 2.3. For all $\epsilon > 0$, the sequence $(u_n)_{n=0}^{\infty}$ defined in Theorem 2.2 satisfies

$$\lim_{n\to\infty} \inf_{x<2n(c^*-\epsilon)} u_{2n}(x) = 1 \tag{2.10}$$

and

$$\lim_{n\to\infty} \inf_{x<2n(c^*-\epsilon)} u_{2n+1}(x) = m \tag{2.11}$$

and

$$\lim_{n\to\infty} \sup_{x>n(c^*+\epsilon)} u_n(x) = 0. \tag{2.12}$$

The next theorem concerns the spreading behavior of solutions to the IDE (1.1) with compactly supported initial data. We also assume the disperal kernel is compactly supported.

Theorem 2.4. Suppose $||w_2||_{\infty} \leq b$ and $k(x)$ has compact support. Assume the following hold:

  i.) c^* is positive,

 ii.) the initial condition u_0 has compact support;

iii.) the set $A = \{x \in \mathbb{R} : a \leq u_0(x) \leq b\}$ is connected and sufficiently large.

Then the sequence $(u_n)_{n=0}^{\infty}$ defined by $u_{n+1} = Q[u_n]$, $n \geq 0$, satisfies

$$\lim_{n\to\infty} \inf_{|x|<2nc} u_{2n}(x) = 1, \quad \forall c \in (0, c^*) \tag{2.13}$$

and

$$\lim_{n\to\infty} \inf_{|x|<2nc} u_{2n+1}(x) = m, \quad \forall c \in (0, c^*) \tag{2.14}$$

and

$$\lim_{n\to\infty} \sup_{|x|>nc} u_n(x) = 0, \quad \forall c \in (c^*, \infty). \tag{2.15}$$

Proof. Let $(-\sigma, \sigma)$ be the support of k, for $0 < \sigma < \infty$. Since A is connected, we may also assume by the translation invariance property that $A = [-r, r]$ for some $r \geq 0$. Furthermore, since u_0 has compact support we must have $u_0(x) = 0$ outside a bounded interval $(r^-, r^+) \supset A$, where $r^- < -r \leq r < r^+$.

Next, we will show that $0 < u_0(x) < a$ for $x \in (r^-, r^+) \setminus A$. If we suppose to the contrary that $u_0(x) > b$ for some x in this set, then suppose

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without loss of generality $x \in (r, r^+)$ such that $u_0(x) > b$. It follows by the intermediate value theorem that $u_0(x') = b$ for some $x' \in (x, r^+)$. But then $r \in A$, $x \notin A$, and $x' \in A$ with $r < x < x'$, violating the connectedness of A.

The preceding argument shows

$$g(u_0(x)) = \begin{cases} 1 & |x| < r \\ 0 & |x| \geq r \end{cases} \tag{2.16}$$

To prove the theorem, we will show the spreading property is satisfied for all $r \geq 2\sigma$. From $r \geq \sigma$ it follows,

$$u_1(x) = w_1(|x| - r) \tag{2.17}$$

Then,

$$g(u_1(x)) = \begin{cases} m & |x| < r + \beta \\ 1 & r + \beta \leq |x| \leq r + \alpha \\ 0 & |x| > r + \alpha \end{cases} \tag{2.18}$$

where α and β are defined as in Lemma 2.1, and satisfy $-\sigma < \beta < \alpha < \sigma$. Note that $g(u_1(x)) = g(Q[w](x))$ for all $x > -r - \beta$. Thus, $r \geq 2\sigma$ guarantees $r + \beta \geq \sigma$. It follows that

$$u_2(x) = Q[w_1](|x| - r) = w_2(|x| - r) \tag{2.19}$$

Applying g, we get

$$g(u_2(x)) = \begin{cases} 1 & |x| \leq r + 2c^* \\ 0 & |x| > r + 2c^* \end{cases} \tag{2.20}$$

Since $c^* > 0$ we have $r + 2c^* > r \geq \alpha$, so

$$u_3(x) = w_1(|x| - r - 2c^*) \tag{2.21}$$

The preceding argument can be repeated inductively to obtain

$$u_{2n+1}(x) = w_1(|x| - r - 2nc^*) \tag{2.22}$$

and

$$u_{2n+2}(x) = w_2(|x| - r - 2nc^*) \tag{2.23}$$

for all $n \geq 0$. $\qquad\square$

Remark 2.5. This theorem indicates that a solution with proper compactly supported initial data coverges to translations of periodic traveling waves with profiles $w_1(x)$ and $w_2(x)$ in the positive direction and profiles $w_1(-x)$ and $w_2(-x)$ in the negative direction.

3 Examples

In this section, we construct the periodic traveling wave solution for several well-known disperal kernels in population biology, namely the uniform, Laplace, and normal distributions. For the uniform and Laplace kernels, we were able to construct a piecewise expression for the mean wave speed in terms of the model parameters.

Example 3.1. The Laplace kernel,

$$k(x) = \frac{1}{2}e^{-|x|} \tag{3.1}$$

The reader can easily verify that the Laplace kernel satisfies hypotheses (H1) - (H3). The proof that is also satisfies (H4) is left in the appendix. The periodic traveling waves are given by

$$w_1(x) = \begin{cases} 1 - \frac{1}{2}e^x & x \leq 0 \\ \frac{1}{2}e^{-x} & x > 0 \end{cases} \tag{3.2}$$

and

$$w_2(x) = \begin{cases} m + C_1 e^x & x < \beta \\ 1 - C_2 e^x - C_3 e^{-x} & \beta < x < \alpha \\ C_4 e^{-x} & \alpha < x \end{cases} \tag{3.3}$$

where $\alpha = w_1^{-1}(\alpha)$, $\beta = w_1^{-1}(\beta)$, with

$$w_1^{-1}(p) = \begin{cases} -\ln(2p) & p \leq \frac{1}{2} \\ \ln(2 - 2p) & p > \frac{1}{2} \end{cases}$$

The constants C_1, C_2, C_3, and C_4 are continuous functions of the growth parameters, with $C_2, C_3, C_4 \geq 0$, and they are given by

$$C_1 = \begin{cases} \beta(1 - \mu) - \alpha & \alpha, \beta < \frac{1}{2} \\ \frac{1 - \mu - 4\alpha(1 - \beta)}{4(1 - \beta)} & \alpha < \frac{1}{2} < \beta \\ -\frac{1 - \beta + \mu(1 - \alpha)}{4(1 - \alpha)(1 - \beta)} & \frac{1}{2} < a, b \end{cases} \tag{3.4}$$

$$C_2 = \begin{cases} \alpha & \alpha < \frac{1}{2} \\ \frac{1}{4(1 - \alpha)} & \alpha > \frac{1}{2} \end{cases} \tag{3.5}$$

$$C_3 = \begin{cases} \frac{1 - \mu}{4\beta} & \beta < \frac{1}{2} \\ (1 - \mu)(1 - \beta) & \beta > \frac{1}{2} \end{cases} \tag{3.6}$$

$$C_4 = \begin{cases} \frac{\beta - \alpha(1-m)}{4\alpha\beta} & \alpha, \beta < \frac{1}{2} \\ \frac{1 - 4\alpha(1-\mu)(1-\beta)}{4\alpha} & \alpha < \frac{1}{2} < \beta \\ 1 - \alpha - (1-\mu)(1-\beta) & \frac{1}{2} < a, b \end{cases} \tag{3.7}$$

To find c^*, we can now condition on the values of $w_2(\alpha)$ and $w_2(\beta)$.

$$c^* = \begin{cases} \frac{1}{2}\ln\left(\frac{C_4}{a}\right) & a < C_4 e^{-\alpha} \\ \frac{1}{2}\ln\left(\frac{1 - a + \sqrt{(1-a)^2 - 4C_2 C_3}}{2C_2}\right) & C_4 e^{-\alpha} < a < m + C_1 e^{\alpha} \\ \frac{1}{2}\ln\left(\frac{a-m}{C_1}\right) & a > m + C_1 e^{\alpha} \end{cases} \tag{3.8}$$

Since the form of $w_2(x)$, and thus of c^*, depends on the values of α and β; thus we will split into three cases for further analysis.

Case 1. $\alpha < \frac{1}{2}$, $\beta < \frac{1}{2}$.

$$w_2(x) = \begin{cases} \mu + (\beta(1-\mu) - \alpha)e^x & x < -\ln(2\beta) \\ 1 - \alpha e^x - \frac{1-\mu}{4\beta}e^{-x} & -\ln(2\beta) < x < -\ln(2\alpha) \\ \frac{\beta - \alpha(1-m)}{4\alpha\beta}e^{-x} & x > -\ln(2\alpha) \end{cases} \tag{3.9}$$

We have $w_2(\alpha) = \frac{\beta - \alpha(1-\mu)}{2\beta}$ and $w_2(\beta) = \mu + \frac{\beta(1-\mu) - \alpha}{2\beta}$. Thus,

$$c^* = \begin{cases} \frac{1}{2}\ln\left(\frac{\beta - \alpha(1-\mu)}{4\alpha^2\beta}\right) & \alpha < \frac{\beta - \alpha(1-\mu)}{2\beta} \\ \frac{1}{2}\ln\left(\frac{1 - a + \sqrt{(1-a)^2 - \frac{\alpha(1-m)}{b}}}{2\alpha}\right) & \frac{\beta - \alpha(1-\mu)}{2\beta} < a < \frac{\beta(1+\mu) - \alpha}{2\beta} \\ \frac{1}{2}\ln\left(\frac{a-m}{\beta(1-\mu) - \alpha}\right) & a > \frac{\beta(1+\mu) - \alpha}{2\beta} \end{cases}$$
$$\tag{3.10}$$

Case 2. $\alpha < \frac{1}{2}$, $\beta > \frac{1}{2}$.

$$w_2(x) = \begin{cases} \mu + \frac{1 - \mu - 4\alpha(1-\beta)}{4(1-\beta)}e^x & x < \ln(2 - 2\beta) \\ 1 - \alpha e^x - (1-\mu)(1-\beta)e^{-x} & \ln(2 - 2\beta) < x < -\ln(2\alpha) \\ \frac{1 - 4\alpha(1-\mu)(1-\beta)}{4\alpha}e^{-x} & x > -\ln(2\alpha) \end{cases} \tag{3.11}$$

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We have $w_2(\alpha) = \frac{1-4\alpha(1-\mu)(1-\beta)}{2}$ and $w_2(\beta) = \frac{1+\mu-4\alpha(1-\beta)}{2}$. Thus,

$$
c^* = \begin{cases}
\frac{1}{2}\ln\left(\frac{1-4\alpha(1-\mu)(1-\beta)}{4\alpha^2}\right) & a < \frac{1-4\alpha(1-\mu)(1-\beta)}{2} \\[2mm]
\frac{1}{2}\ln\left(\frac{1-a+\sqrt{(1-a)^2-4a(1-\mu)(1-\beta)}}{2a}\right) & \frac{1-4\alpha(1-\mu)(1-\beta)}{2} < a < \frac{1+\mu-4\alpha(1-\beta)}{2} \\[2mm]
\frac{1}{2}\ln\left(\frac{4(a-m)(1-b)}{1-\mu-4\alpha(1-\beta)}\right) & a > \frac{1+\mu-4\alpha(1-\beta)}{2}
\end{cases}
$$
$$(3.12)$$

Case 3. $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$.

$$
w_2(x) = \begin{cases}
\mu - \frac{1-\beta+\mu(1-\alpha)}{4(1-\alpha)(1-\beta)}e^x & x < \ln(2-2\beta) \\[2mm]
1 - \frac{1}{4(1-\alpha)}e^x - (1-\mu)(1-\beta)e^{-x} & \ln(2-2\beta) < x < \ln(2-2\alpha) \\[2mm]
(1-\alpha-(1-\mu)(1-\beta))e^{-x} & x > \ln(2-2\alpha)
\end{cases}
$$
$$(3.13)$$

Thus, $w_2(\alpha) = \frac{1-\alpha-(1-\mu)(1-\beta)}{2(1-a)}$ and $w_2(\beta) = \mu - \frac{1-\beta+\mu(1-\alpha)}{2(1-\alpha)}$

$$
c^* = \begin{cases}
\frac{1}{2}\ln\left(\frac{1-\alpha-(1-\mu)(1-\beta)}{a}\right) & a < \frac{1-\alpha-(1-\mu)(1-\beta)}{2(1-a)} \\[2mm]
\frac{1}{2}\ln\left(2(1-a)\left[1-a+\sqrt{\frac{(1-a)^3-(1-\mu)(1-\beta)}{1-\alpha}}\right]\right) & \frac{1-\alpha-(1-\mu)(1-\beta)}{2(1-a)} < a < \frac{\beta-1+\mu(1-\alpha)}{2(1-\alpha)} \\[2mm]
\frac{1}{2}\ln\left(\frac{4(m-a)(1-\alpha)(1-\beta)}{1-\beta+\mu(1-\alpha)}\right) & a > \frac{\beta-1+\mu(1-\alpha)}{2(1-\alpha)}
\end{cases}
$$
$$(3.14)$$

Example 3.2. Consider the Gaussian kernel with zero mean and unit variance given by

$$
k(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}
$$

The kernel is symmetric and has connected support, hence it satisfies hypotheses (H1)-(H3); the proof for hypothesis (H4) is left in the appendix.

Let $\Phi(x) = \int_{-\infty}^{x} k(y)\,dy$ denote the cumulative density function of the standard normal distribution, and Φ^{-1} be its inverse. The periodic traveling wave solutions $w_1(x)$ and $w_2(x)$ are given by

$$
w_1(x) = \Phi(-x) \tag{3.15}
$$

and

$$
w_2(x) = m - \Phi(x - \Phi^{-1}(a)) + (1-m)\Phi(x - \Phi^{-1}(b)) \tag{3.16}
$$

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where $\alpha = \Phi^{-1}(a)$ and $\beta = \Phi^{-1}(b)$.

w_2 has a unique global maximum at $x^* = \frac{\alpha+\beta}{2} + \frac{1}{\alpha-\beta} \ln(1-m)$. Thus, by Theorem 2.2, w_1 and w_2 are a periodic traveling wave solution if $w_2(x^*) \le b$.

Example 3.3. Consider the uniform dispersal kernel given by

$$k(x) = \begin{cases} \frac{1}{2} & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \tag{3.17}$$

Then w_1 is given by

$$w_1(x) = \begin{cases} 1, & x \in (-\infty, -1), \\ \frac{1}{2} - \frac{1}{2}x, & x \in [-1, 1], \\ 0, & x \in (1, \infty), \end{cases} \tag{3.18}$$

with inverse $w_1^{-1}(p) = 1 - 2p$ for $0 < p < 1$. Let $\alpha = 1 - 2a$ and $\beta = 1 - 2b$. Then

$$w_2(x) = \begin{cases} m, & x \in (-\infty, \beta-1), \\ \frac{1-m}{2}x + m + b - mb, & x \in [\beta-1, \alpha-1), \\ -\frac{m}{2}x + m + b - mb - a, & x \in [\alpha-1, \beta+1), \\ -\frac{1}{2}x - a + 1, & x \in [\beta+1, \alpha+1], \\ 0, & x \in (\alpha+1, \infty). \end{cases} \tag{3.19}$$

Observe that w_2 has a global maximum at $x = \alpha - 1$ so that $\|w_2\|_\infty = w_2(\alpha-1) = m + (b-a)(1-m)$. By Theorem 2.3, the pair w_1 and w_2 are a solution to equation (2.9) if $m - a < m(b-a)$.

We can also explicitly calculate the speed of the wave given by

$$c^* = \begin{cases} 1 - 2a & \text{if } a \le b/2, \\ 1 - b + \frac{b-2a}{m} & \text{if } a > b/2. \end{cases} \tag{3.20}$$

Remark 3.4. $w_1(x)$ is positive for $x < 1$ and zero for $x \ge 1$, and $w_2(x)$ is positive for $x < 2 - 2a$ and zero for $x \ge 2 - 2a$. Thus, (1.1) has a traveling wave with wave profiles $w_1(x)$ and $w_2(x)$, intermediate wave speeds $c_1 = 1 - 2a$ and $c_2 = 2c^* - c_1$, and average wave speed c^*. It is easily seen that $c_1 = c_2$ if $a \le b/2$, and $|c_1 - c_2| = (2\alpha - \beta)(1 - \frac{1}{m}) > 0$ if $a > b/2$. So for $a > b/2$, the traveling wave is periodic with two different intermediate wave speeds. Furthermore, the difference between these two intermediate speeds is increasing in a, decreasing in b, and increasing in m. This behavior is illustrated with two difference choices of parameters in Figure ??.

The regions in the parameter space where oscillating spreading speed exists can be determined as follows: for any fixed choice of (n_1, n_2), with $0 < n_1 < n_2$, let R be the set of pairs $(a, b) \in \mathbf{R}^2$ such that the hypothesis of Theorem 2.1 holds. Then R is a triangle in the a-b plane with endpoints at $(0, n_2)$, (n_1, n_1), and (n_1, n_2), depicted in Figure **??**. The line $b = 2a$ partitions R into two non-empty sets $R_1 = \{(a, b) \in R : a \leq b/2\}$ and $R_2 = \{(a, b) \in R : a > b/2\}$ such that the traveling has constant speed if $(a, b) \in R_1$ and oscillating speed if $(a, b) \in R_2$.

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A Example calculations

Lemma A.1. The Laplace kernel (3.1) satisfies hypothesis (H4).

Proof. let $f = f_{m,y}$ be the scalar function of x with two parameters $y \in \mathbb{R}$ and $\mu \in (0,1)$ defined by

$$f(x) = f_{m,y}(x) = \frac{1}{2}e^{-|x|} - \frac{\mu}{2}e^{-|x-y|}$$

If $y = 0$, then f has no zero-crossings, since $f_{m,0}(x) = \frac{1-\mu}{2}e^{-|x|}$ is strictly positive. If y is nonzero, then one can easily check the symmetry relation $f_{m,-y}(x) = f_{m,y}(-x)$. Since the number of zero-crossings are invariant with

respect to a reflection about the vertical axis, we can assume without loss of generality $y > 0$.

Under this assumption, f is strictly increasing on $(-\infty, 0)$, and strictly decreasing on $(0, y)$. The behavior on (y, ∞) is determined by the sign of $e^{-y} - m$. There are three cases:

1. if $y < \ln \frac{1}{m}$, then f is decreasing on $(0, \infty)$, hence has no zero-crossings;

2. if $y > \ln \frac{1}{m}$, then f has a unique zero-crossing at $x = \frac{1}{2}(y - \ln(m))$;

3. if $y = \ln \frac{1}{m}$, then f vanishes on (y, ∞), hence it has no zero-crossings.

In each case, the number of zero-crossings does not exceed one. $\qquad\square$

Lemma A.2. If $k(x)$ is given by the Laplace kernel, then w_1 and w_2 form a periodic traveling wave solution if $C_1 \leq 0$, or if $C_1 > 0$ and $w_2 \left(\ln \sqrt{\frac{C_3}{C_2}} \right) \leq b$.

Proof. We can proceed in cases. If $C_1 \leq 0$, then $w_2(x)$ is monotone decreasing, hence $w_2(x) < w_2(-\infty) = m < b$ everywhere. Otherwise, if $C_1 > 0$, then $w_2(x)$ is increasing on $(-\infty, \beta)$ and decreasing on (α, ∞). Since $w_2(x)$ is concave-down on (β, α), this implies there is a unique global maximum somewhere in this interval. To find it, we can differentiate:

$$\left. \frac{dw_2}{dx} \right|_{\beta < x < \alpha} = C_3 e^{-x} - C_2 e^{x}$$

Setting this expression equal to zero and multiplying by e^x, we obtain $C_3 - C_2 e^{2x} = 0$, which has a unique solution at $x = \ln \sqrt{\frac{C_3}{C_2}}$. $\qquad\square$

Lemma A.3. The Gaussian kernel satisfies hypothesis H4.

Proof. Let $y \in \mathbb{R}$ and $\mu \in (0, 1)$. Then

$$
\begin{aligned}
k(x) - \mu k(x - y) &= \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{x^2}{2}} - \mu e^{-\frac{(x-y)^2}{2}} \right) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(1 - \mu e^{\frac{2xy - y^2}{2}} \right)
\end{aligned}
\tag{A.1}
$$

This expression has a unique zero at $x = \frac{y^2 - 2\ln(\mu)}{2y}$, so the number of zero-crossings is at most one. $\qquad\square$

Lemma A.4. For the Gaussian kernel, $w_2(x)$ has a unique local extrema which is a global maxium at $x = \frac{2\ln(1-m)}{\alpha - \beta} + \alpha + \beta$.

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Proof. The derivative of $w_2(x)$ is given by

$$\frac{dw_2}{dx} = -\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\alpha)^2}{2}} + \frac{1-m}{\sqrt{2\pi}}e^{-\frac{(x-\beta)^2}{2}}$$

Setting this quantity equal to zero, we obtain the equation

$$e^{-\frac{(x-\alpha)^2}{2}} = (1-m)e^{-\frac{(x-\beta)^2}{2}}$$

Taking logarithm on both sides, and rearrange terms,

$$(x-\beta)^2 = 2\ln(1-m) + (x-\alpha)^2$$

Distributing both sides and cancelling the quadratic term, we get the solution

$$x = \frac{2\ln(1-m)}{\alpha - \beta} + \alpha + \beta$$

$\square$