This document is to try to write all my examples down.
TODO:

1. Remove the example labels. It makes it look like a textbook.
2. Reduce the number of unneeded equations, and condense the discussion into paragraphs.
3. Include more figures.
4. Don't forget the condition on the maximum of w_2

1 Examples

In this section, we construct the periodic traveling wave solution for several well-known disperal kernels in population biology, namely the uniform, Laplace, and normal distributions. For the uniform and Laplace kernels, we were able to construct a piecewise expression for the mean wave speed in terms of the model parameters.

Example 1.1. The Laplace kernel,

$$k(x) = \frac{1}{2}e^{-|x|} \tag{1.1}$$

The reader can easily verify that the Laplace kernel satisfies hypotheses (H1) - (H3). The proof that is also satisfies (H4) is left in the appendix. The periodic traveling waves are given by

$$w_1(x) = \begin{cases} 1 - \frac{1}{2}e^{x} & x \leq 0 \\ \frac{1}{2}e^{-x} & x > 0 \end{cases} \tag{1.2}$$

and

$$w_2(x) = \begin{cases} m + C_1 e^{x} & x < \beta \\ 1 - C_2 e^{x} - C_3 e^{-x} & \beta < x < \alpha \\ C_4 e^{-x} & \alpha < x \end{cases} \tag{1.3}$$

where $\alpha = w_1^{-1}(a)$, $\beta = w_1^{-1}(b)$, with

$$w_1^{-1}(p) = \begin{cases} -\ln(2p) & p \leq \frac{1}{2} \\ \ln(2-2p) & p > \frac{1}{2} \end{cases}$$

The constants C_1, C_2, C_3, and C_4 are continuous functions of the growth parameters, with $C_2, C_3, C_4 \geq 0$, and they are given by

$$C_1 = \begin{cases} b(1-m) - a & a, b < \frac{1}{2} \\ \frac{1-m-4a(1-b)}{4(1-b)} & a < \frac{1}{2} < b \\ -\frac{1-b+m(1-a)}{4(1-a)(1-b)} & \frac{1}{2} < a, b \end{cases} \tag{1.4}$$

$$C_2 = \begin{cases} a & a < \frac{1}{2} \\ \frac{1}{4(1-a)} & a > \frac{1}{2} \end{cases} \tag{1.5}$$

$$C_3 = \begin{cases} \frac{1-m}{4b} & b < \frac{1}{2} \\ (1-m)(1-b) & b > \frac{1}{2} \end{cases} \tag{1.6}$$

$$C_4 = \begin{cases} \frac{b-a(1-m)}{4ab} & a, b < \frac{1}{2} \\ \frac{1-4a(1-m)(1-b)}{4a} & a < \frac{1}{2} < b \\ 1 - a - (1-m)(1-b) & \frac{1}{2} < a, b \end{cases} \tag{1.7}$$

To find c^*, we can now condition on the values of $w_2(\alpha)$ and $w_2(\beta)$.

$$c^* = \begin{cases} \frac{1}{2}\ln\left(\frac{C_4}{a}\right) & a < C_4 e^{-\alpha} \\ \frac{1}{2}\ln\left(\frac{1-a+\sqrt{(1-a)^2-4C_2C_3}}{2C_2}\right) & C_4 e^{-\alpha} < a < m + C_1 e^{\alpha} \\ \frac{1}{2}\ln\left(\frac{a-m}{C_1}\right) & a > m + C_1 e^{\alpha} \end{cases} \tag{1.8}$$

Since the form of $w_2(x)$, and thus of c^*, depends on the values of a and b; thus we will split into three cases for further analysis.

Case 1. $a < \frac{1}{2}$, $b < \frac{1}{2}$.

$$w_2(x) = \begin{cases} m + (b(1-m) - a)e^x & x < -\ln(2b) \\ 1 - ae^x - \frac{1-m}{4b}e^{-x} & -\ln(2b) < x < -\ln(2a) \\ \frac{b-a(1-m)}{4ab}e^{-x} & x > -\ln(2a) \end{cases} \tag{1.9}$$

We have $w_2(\alpha) = \frac{b-a(1-m)}{2b}$ and $w_2(\beta) = m + \frac{b(1-m)-a}{2b}$. Thus,

$$c^* = \begin{cases} \frac{1}{2}\ln\left(\frac{b-a(1-m)}{4a^2b}\right) & a < \frac{b-a(1-m)}{2b} \\ \frac{1}{2}\ln\left(\frac{1-a+\sqrt{(1-a)^2-\frac{a(1-m)}{b}}}{2a}\right) & \frac{b-a(1-m)}{2b} < a < \frac{b(1+m)-a}{2b} \\ \frac{1}{2}\ln\left(\frac{a-m}{b(1-m)-a}\right) & a > \frac{b(1+m)-a}{2b} \end{cases} \tag{1.10}$$

Case 2. $a < \frac{1}{2}$, $b > \frac{1}{2}$.

$$w_2(x) = \begin{cases} m + \frac{1-m-4a(1-b)}{4(1-b)} e^x & x < \ln(2-2b) \\ 1 - ae^x - (1-m)(1-b)e^{-x} & \ln(2-2b) < x < -\ln(2a) \\ \frac{1-4a(1-m)(1-b)}{4a} e^{-x} & x > -\ln(2a) \end{cases} \tag{1.11}$$

We have $w_2(\alpha) = \frac{1-4a(1-m)(1-b)}{2}$ and $w_2(\beta) = \frac{1+m-4a(1-b)}{2}$. Thus,

$$c^* = \begin{cases} \frac{1}{2} \ln\left(\frac{1-4a(1-m)(1-b)}{4a^2}\right) & a < \frac{1-4a(1-m)(1-b)}{2} \\ \frac{1}{2} \ln\left(\frac{1-a+\sqrt{(1-a)^2-4a(1-m)(1-b)}}{2a}\right) & \frac{1-4a(1-m)(1-b)}{2} < a < \frac{1+m-4a(1-b)}{2} \\ \frac{1}{2} \ln\left(\frac{4(a-m)(1-b)}{1-m-4a(1-b)}\right) & a > \frac{1+m-4a(1-b)}{2} \end{cases} \tag{1.12}$$

Case 3. $a > \frac{1}{2}$, $b > \frac{1}{2}$.

$$w_2(x) = \begin{cases} m - \frac{1-b+m(1-a)}{4(1-a)(1-b)} e^x & x < \ln(2-2b) \\ 1 - \frac{1}{4(1-a)} e^x - (1-m)(1-b)e^{-x} & \ln(2-2b) < x < \ln(2-2a) \\ (1-a-(1-m)(1-b))e^{-x} & x > \ln(2-2a) \end{cases} \tag{1.13}$$

Thus, $w_2(\alpha) = \frac{1-a-(1-m)(1-b)}{2(1-a)}$ and $w_2(\beta) = m - \frac{1-b+m(1-a)}{2(1-a)}$

$$c^* = \begin{cases} \frac{1}{2} \ln\left(\frac{1-a-(1-m)(1-b)}{a}\right) & a < \frac{1-a-(1-m)(1-b)}{2(1-a)} \\ \frac{1}{2} \ln\left(2(1-a)\left[1-a+\sqrt{\frac{(1-a)^3-(1-m)(1-b)}{1-a}}\right]\right) & \frac{1-a-(1-m)(1-b)}{2(1-a)} < a < \frac{b-1+m(1-a)}{2(1-a)} \\ \frac{1}{2} \ln\left(\frac{4(m-a)(1-a)(1-b)}{1-b+m(1-a)}\right) & a > \frac{b-1+m(1-a)}{2(1-a)} \end{cases} \tag{1.14}$$

Example 1.2. Consider the Gaussian kernel with zero mean and unit variance given by

$$k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The kernel is symmetric and has connected support, hence it satisfies hypotheses (H1)-(H3); the proof for hypothesis (H4) is left in the appendix.

Let $\Phi(x) = \int_{-\infty}^{x} k(y)\,dy$ denote the cumulative density function of the standard normal distribution, and Φ^{-1} be its inverse. The periodic traveling wave solutions $w_1(x)$ and $w_2(x)$ are given by

$$w_1(x) = \Phi(-x) \tag{1.15}$$

and

$$w_2(x) = m - \Phi(x - \Phi^{-1}(a)) + (1-m)\Phi(x - \Phi^{-1}(b)) \tag{1.16}$$

where $\alpha = \Phi^{-1}(a)$ and $\beta = \Phi^{-1}(b)$.

w_2 has a unique global maximum at $x^* = \frac{\alpha+\beta}{2} + \frac{1}{\alpha-\beta}\ln(1-m)$. Thus, by Theorem **??**, w_1 and w_2 are a periodic traveling wave solution if $w_2(x^*) \le b$.

Example 1.3. Consider the uniform dispersal kernel given by

$$k(x) = \begin{cases} \frac{1}{2} & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \tag{1.17}$$

Then w_1 is given by

$$w_1(x) = \begin{cases} 1, & x \in (-\infty, -1), \\ \frac{1}{2} - \frac{1}{2}x, & x \in [-1, 1], \\ 0, & x \in (1, \infty), \end{cases} \tag{1.18}$$

with inverse $w_1^{-1}(p) = 1 - 2p$ for $0 < p < 1$. Let $\alpha = 1 - 2a$ and $\beta = 1 - 2b$. Then

$$w_2(x) = \begin{cases} m, & x \in (-\infty, \beta - 1), \\ \frac{1-m}{2}x + m + b - mb, & x \in [\beta - 1, \alpha - 1), \\ -\frac{m}{2}x + m + b - mb - a, & x \in [\alpha - 1, \beta + 1), \\ -\frac{1}{2}x - a + 1, & x \in [\beta + 1, \alpha + 1], \\ 0, & x \in (\alpha + 1, \infty). \end{cases} \tag{1.19}$$

Observe that w_2 has a global maximum at $x = \alpha - 1$ so that $||w_2||_\infty = w_2(\alpha - 1) = m + (b-a)(1-m)$. By Theorem **??**, the pair w_1 and w_2 are a solution to equation (**??**) if $m - a < m(b - a)$.

We can also explicitly calculate the speed of the wave given by

$$c^* = \begin{cases} 1 - 2a & \text{if } a \le b/2, \\ 1 - b + \frac{b-2a}{m} & \text{if } a > b/2. \end{cases} \tag{1.20}$$

Remark 1.4. $w_1(x)$ is positive for $x < 1$ and zero for $x \geq 1$, and $w_2(x)$ is positive for $x < 2 - 2a$ and zero for $x \geq 2 - 2a$. Thus, (**??**) has a traveling wave with wave profiles $w_1(x)$ and $w_2(x)$, intermediate wave speeds $c_1 = 1 - 2a$ and $c_2 = 2c^* - c_1$, and average wave speed c^*. It is easily seen that $c_1 = c_2$ if $a \leq b/2$, and $|c_1 - c_2| = (2\alpha - \beta)(1 - \frac{1}{m}) > 0$ if $a > b/2$. So for $a > b/2$, the traveling wave is periodic with two different intermediate wave speeds. Furthermore, the difference between these two intermediate speeds is increasing in a, decreasing in b, and increasing in m. This behavior is illustrated with two difference choices of parameters in Figure **??**.

The regions in the parameter space where oscillating spreading speed exists can be determined as follows: for any fixed choice of (n_1, n_2), with $0 < n_1 < n_2$, let R be the set of pairs $(a, b) \in \mathbf{R}^2$ such that the hypothesis of Theorem 2.1 holds. Then R is a triangle in the a-b plane with endpoints at $(0, n_2)$, (n_1, n_1), and (n_1, n_2), depicted in Figure **??**. The line $b = 2a$ partitions R into two non-empty sets $R_1 = \{(a, b) \in R : a \leq b/2\}$ and $R_2 = \{(a, b) \in R : a > b/2\}$ such that the traveling has constant speed if $(a, b) \in R_1$ and oscillating speed if $(a, b) \in R_2$.

2 Appendix

Lemma 2.1. The Laplace kernel (1.1) satisfies hypothesis (H4).

Proof. let $f = f_{m,y}$ be the scalar function of x with two parameters $y \in \mathbb{R}$ and $\mu \in (0, 1)$ defined by

$$f(x) = f_{m,y}(x) = \frac{1}{2}e^{-|x|} - \frac{\mu}{2}e^{-|x-y|}$$

If $y = 0$, then f has no zero-crossings, since $f_{m,0}(x) = \frac{1-\mu}{2}e^{-|x|}$ is strictly positive. If y is nonzero, then one can easily check the symmetry relation $f_{m,-y}(x) = f_{m,y}(-x)$. Since the number of zero-crossings are invariant with respect to a reflection about the vertical axis, we can assume without loss of generality $y > 0$.

Under this assumption, f is strictly increasing on $(-\infty, 0)$, and strictly decreasing on $(0, y)$. The behavior on (y, ∞) is determined by the sign of $e^{-y} - m$. There are three cases:

1. if $y < \ln \frac{1}{m}$, then f is decreasing on $(0, \infty)$, hence has no zero-crossings;
2. if $y > \ln \frac{1}{m}$, then f has a unique zero-crossing at $x = \frac{1}{2}(y - \ln(m))$;

3. if $y = \ln \frac{1}{m}$, then f vanishes on (y, ∞), hence it has no zero-crossings.

In each case, the number of zero-crossings does not exceed one. $\square$

Lemma 2.2. If $k(x)$ is given by the Laplace kernel, then w_1 and w_2 form a periodic traveling wave solution if $C_1 \leq 0$, or if $C_1 > 0$ and $w_2\left(\ln \sqrt{\frac{C_3}{C_2}}\right) \leq b$.

Proof. We can proceed in cases. If $C_1 \leq 0$, then $w_2(x)$ is monotone decreasing, hence $w_2(x) < w_2(-\infty) = m < b$ everywhere. Otherwise, if $C_1 > 0$, then $w_2(x)$ is increasing on $(-\infty, \beta)$ and decreasing on (α, ∞). Since $w_2(x)$ is concave-down on (β, α), this implies there is a unique global maximum somewhere in this interval. To find it, we can differentiate:

$$\left.\frac{dw_2}{dx}\right|_{\beta<x<\alpha} = C_3 e^{-x} - C_2 e^{x}$$

Setting this expression equal to zero and multiplying by e^x, we obtain $C_3 - C_2 e^{2x} = 0$, which has a unique solution at $x = \ln \sqrt{\frac{C_3}{C_2}}$. $\square$

Lemma 2.3. The Gaussian kernel satisfies hypothesis H4.

Proof. Let $y \in \mathbb{R}$ and $\mu \in (0,1)$. Then

$$\begin{aligned} k(x) - \mu k(x-y) &= \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{x^2}{2}} - \mu e^{-\frac{-(x-y)^2}{2}}\right) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\left(1 - \mu e^{\frac{2xy-y^2}{2}}\right) \end{aligned} \tag{2.1}$$

This expression has a unique zero at $x = \frac{y^2 - 2\ln(\mu)}{2y}$, so the number of zero-crossings is at most one. $\square$

Lemma 2.4. For the Gaussian kernel, $w_2(x)$ has a unique local extrema which is a global maxium at $x = \frac{2\ln(1-m)}{\alpha-\beta} + \alpha + \beta$.

Proof. The derivative of $w_2(x)$ is given by

$$\frac{dw_2}{dx} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\alpha)^2}{2}} + \frac{1-m}{\sqrt{2\pi}} e^{-\frac{(x-\beta)^2}{2}}$$

Setting this quantity equal to zero, we obtain the equation

$$e^{-\frac{(x-\alpha)^2}{2}} = (1-m) e^{-\frac{(x-\beta)^2}{2}}$$

Taking logarithm on both sides, and rearrange terms,

$$(x - \beta)^2 = 2\ln(1 - m) + (x - \alpha)^2$$

Distributing both sides and cancelling the quadratic term, we get the solution

$$x = \frac{2\ln(1 - m)}{\alpha - \beta} + \alpha + \beta$$

$\square$