Periodic Traveling Waves in an Integro-Difference Equation With a Nonmonotone Growth Function and Strong Allee Effect

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May 9, 2021

Abstract

We derive sufficient conditions for the existence of periodic traveling wave solutions for a class of integro-difference equation with piecewise constant growth function exhibiting a period two cycle and a strong Allee effect. We also prove the convergence of solutions with compactly supported initial data to translations of the traveling wave under appropriate conditions.

Key words: Integro-difference equation, period two cycle, Allee effect, periodic traveling wave.

AMS Subject Classification: 92D40, 92D25.

1 Introduction

Integro-difference equations are of great interest in the studies of invasions of populations with discrete generations and separate growth and dispersal stages. They have been used to predict changes in gene frequency [8, 9, 10, 14, 17], and applied to ecological problems [2, 3, 4, 5, 7, 11, 12, 13]. Previous rigrous studies on integro-difference equations have assumed that

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the growth function is nondecreasing [17, 18], or is nonmonotone without strong Allee effect [10, 16]. The results show existence of constant spreading speeds and travelng waves with fixed shapes and speeds. Sullivan et al. [15] demonstrated numerically that an integro-difference equation with a nonmonotone growth function exhibiting a strong Allee effect can generate traveling waves with fluctuating speeds. In this paper we give a sufficient condition for the existence of periodic traveling waves with a periodic speed for such an equation with a specific growth function.

We consider the following integro-difference equation

$$u_{n+1}(x) \;=\; Q[u_n](x) \;:= (k * (g \circ u_n))(x) \;=\; \int_{-\infty}^{\infty} k(x-y)\, g\big(u_n(y)\big)\, \mathrm{d}y, \tag{1.1}$$

where

$$g(u) = \begin{cases} 0, & \text{if } u < a, \\ 1, & \text{if } a \le u \le b, \\ m, & \text{if } u > b, \end{cases} \tag{1.2}$$

with $0 < a < m < b < 1$. $g(u)$ is a piecewise constant nonmonotone growth function exhibiting a strong Allee effect [1]. Specifically, it has a stable fixed point at zero and a stable period two cycle $(1, m)$ with a the Allee threshold value.

Piecewise constant growth functions and uniform distributions have been used in the studies of integro-difference equations; see for example [6, 11, 13, 15]. We rigorously construct periodic traveling waves with periodic speeds for (1.1). To the best of our knowledge, this is the first time that traveling waves with oscillating speeds have been analytically established for scalar spatiotemporal equations with constant parameters. We also show the convergence of solutions with compactly supported initial data to translations of the traveling wave under appropriate conditions. Equation (1.1) may be viewed as a symbolic model for integro-difference equations with a growth function exhibiting a strong Allee effect and a period two cycle. The results obtained this paper provide important insights into integro-difference equations with general growth functions.

2 Periodic traveling waves

In this section, we generalize the theory of traveling wave solutions of 1.1 to the theory of periodic traveling waves.

Let $C(\mathbb{R})$ denote the space of continuous functions $\mathbb{R} \to \mathbb{R}$ equipped with the supremum-norm $||u||_\infty = \sup_{x\in\mathbb{R}} |u(x)|$ for any $u \in C(\mathbb{R})$.

For the remainder of the paper, we assume the following hypothesis on the dispersal kernel:

Hypothesis 2.1. The disperal kernel k is Lebesgue measurable, and

i.) k is non-negative, and $\int_{-\infty}^{\infty} k(x)\, dx = 1$;

ii.) $k(x) = k(-x)$ for all $x \in \mathbb{R}$;

iii.) the support of k is connected;

iv.) for all $y \in \mathbb{R}$, for all $\mu \in (0, 1)$, the function $f(x) = k(x) - \mu k(x - y)$ has at most one zero-crossing on $\mathbb{R}$.

Let w_1 and w_2 be two functions belonging to $C(\mathbb{R})$ defined by

$$w_1(x) = \int_x^{\infty} k(y)\, dy \tag{2.1}$$

and

$$w_2(x) = \int_{-\infty}^{\infty} k(y) g(w_1(x - y))\, dy \tag{2.2}$$

Assuimption i. implies w_1 and w_2 have well-defined limits at $\pm\infty$ given by $w_1(\infty) = w_2(\infty) = 0$, $w_1(-\infty) = 1$, and $w_2(-\infty) = m$. Furthermore, w_1 is monotonically decreasing, while w_2 may be non-monotonic. Assumption iii. guarantees a unique right-inverse satisfying $w_1(w_1^{-1}(p)) = p$ for $0 < p < 1$.

Lemma 2.2. If condition (x) is satisfied then $Q[w_2](x) = w_1(x - 2c^*)$ for a unique $c^* \in \mathbb{R}$.

Proof. Let $\alpha, \beta \in \mathbb{R}$ such that $w_1(\alpha) = a$ and $w_1(\beta) = b$. By the definition of w_1, it is decreasing and satisfies $w_1(-\infty) = 1$ and $w_1(\infty) = 0$, hence α and β are unique. Applying the growth function g yields

$$g(w_1(x)) = \begin{cases} m & x < \beta \\ 1 & \beta \le x \le \alpha \\ 0 & x > \alpha \end{cases} \tag{2.3}$$

Applying the convolution operator yields a formula for $w_2(x)$:

$$\begin{aligned} w_2(x) &= \int_{-\infty}^{\infty} k(y) g(w_1(x-y))\, dy \\ &= \int_{x-\alpha}^{x-\beta} k(y)\, dy + \int_{x-\beta}^{\infty} m k(y)\, dy \\ &= \int_{x-\alpha}^{\infty} k(y)\, dy - \int_{x-\beta}^{\infty} (1-m) k(y)\, dy \end{aligned} \tag{2.4}$$

Taking the derivative with respect to x, we find

$$\frac{dw_2}{dx} = -k(x-\alpha) + (1-m)k(x-\beta) \tag{2.5}$$

It follows from assumption iv. of 2.1 that dw_2/dx has at most one zero-crossing. Hence, w_2 has at most one turning point. This leaves two cases: either w_2 has no turning points and is monotonically decreasing, or it has one turning point, to the left of which w_2 is increasing and to the right it is decreasing. In both cases, w_2 has a continuous right inverse on the open interval $(0, m)$.

Since $0 < a < m$, we can take $x_0 = w_2^{-1}(a)$ so that w_2 exceeds the Allee threshold to the left of x_0 and is below the Allee threshold on the right. Applying the growth function,

$$g(w_2(x)) = \begin{cases} 1 & x \leq w_2^{-1}(a) \\ 0 & x > w_2^{-1}(a) \end{cases} \quad \text{almost everywhere} \tag{2.6}$$

Applying the integro-difference operator, we obtain

$$Q[w_2](x) = \int_{x-w_2^{-1}(a)}^{\infty} k(y)\, dy = w_1(x - w_2^{-1}(a)). \tag{2.7}$$

By defining $c^* = \frac{1}{2} w_2^{-1}(a)$, and applying the integro-difference operator, we can see that w_1 and w_2 are both solutions to the periodic traveling wave equation with mean speed c^*:

$$Q^2[w_1](x) = w_1(x - 2c^*); \quad Q^2[w_2](x) = w_2(x - 2c^*) \tag{2.8}$$

□

Theorem 2.3. If $||w_2||_\infty \leq b$, then the sequence $(u_n)_{n=0}^{\infty}$ defined by

$$u_{2n}(x) = w_1(x - 2nc^*), \quad u_{2n+1}(x) = w_2(x - 2nc^*) \tag{2.9}$$

satisfies $u_{n+1} = Q[u_n]$ for all $n \geq 0$.

Proof. By induction on n. For $n = 0$, we have $u_0(x) = w_1(x)$ and $u_1(x) = Q[w_1](x) = w_2(x)$ by definition.

For the inductive step, assume $u_{2n}(x) = w_1(x - 2nc^*)$ and $u_{2n+1}(x) = w_2(x - 2nc^*)$ for some $n \geq 0$. By Lemma 2.2 and the translation invariance property of Q, we have

$$u_{2n+2}(x) = w_1(x - 2nc^* - 2c^*) = Q[w_2](x - 2nc^*) = Q[u_{2n+1}](x) \quad (2.10)$$

Likewise,

$$u_{2n+3}(x) = w_2(x - 2nc^* - 2c^*) = Q[w_1](x - 2nc^* - 2c^*) = Q[u_{2n+2}](x). \quad (2.11)$$

□

Theorem 2.4. For all $\epsilon > 0$, the sequence $(u_n)_{n=0}^{\infty}$ defined in Theorem 2.3 satisfies

$$\lim_{n\to\infty} \inf_{x<2n(c^*-\epsilon)} u_{2n}(x) = 1 \quad (2.12)$$

and

$$\lim_{n\to\infty} \inf_{x<2n(c^*-\epsilon)} u_{2n+1}(x) = m \quad (2.13)$$

and

$$\lim_{n\to\infty} \sup_{x>n(c^*+\epsilon)} u_n(x) = 0. \quad (2.14)$$

The next theorem concerns the spreading behavior of solutions to the IDE (1.1) with compactly supported initial data. We also assume the disperal kernel is compactly supported.

Theorem 2.5. Suppose $||w_2||_\infty \leq b$ and $k(x)$ has compact support. Assume the following hold:

i.) c^* is positive,

ii.) the initial condition u_0 has compact support;

iii.) the set $A = \{x \in \mathbb{R} : a \leq u_0(x) \leq b\}$ is connected and sufficiently large.

Then the sequence $(u_n)_{n=0}^{\infty}$ defined by $u_{n+1} = Q[u_n]$, $n \geq 0$, satisfies

$$\lim_{n\to\infty} \inf_{|x|<2nc} u_{2n}(x) = 1, \quad \forall c \in (0, c^*) \quad (2.15)$$

and

$$\lim_{n\to\infty} \inf_{|x|<2nc} u_{2n+1}(x) = m, \quad \forall c \in (0, c^*) \quad (2.16)$$

and

$$\lim_{n\to\infty} \sup_{|x|>nc} u_n(x) = 0, \quad \forall c \in (c^*, \infty). \tag{2.17}$$

Proof. Let $(-\sigma, \sigma)$ be the support of k, for $0 < \sigma < \infty$. Since A is connected, we may also assume by the translation invariance property that $A = [-r, r]$ for some $r \geq 0$. Furthermore, since u_0 has compact support we must have $u_0(x) = 0$ outside a bounded interval $(r^-, r^+) \supset A$, where $r^- < -r \leq r < r^+$.

Next, we will show that $0 < u_0(x) < a$ for $x \in (r^-, r^+) \setminus A$. If we suppose to the contrary that $u_0(x) > b$ for some x in this set, then suppose without loss of generality $x \in (r, r^+)$ such that $u_0(x) > b$. It follows by the intermediate value theorem that $u_0(x') = b$ for some $x' \in (x, r^+)$. But then $r \in A$, $x \notin A$, and $x' \in A$ with $r < x < x'$, violating the connectedness of A.

The preceding argument shows

$$g(u_0(x)) = \begin{cases} 1 & |x| < r \\ 0 & |x| \geq r \end{cases} \tag{2.18}$$

To prove the theorem, we will show the spreading property is satisfied for all $r \geq 2\sigma$. From $r \geq \sigma$ it follows,

$$u_1(x) = w_1(|x| - r) \tag{2.19}$$

Then,

$$g(u_1(x)) = \begin{cases} m & |x| < r + \beta \\ 1 & r + \beta \leq |x| \leq r + \alpha \\ 0 & |x| > r + \alpha \end{cases} \tag{2.20}$$

where α and β are defined as in Lemma 2.2, and satisfy $-\sigma < \beta < \alpha < \sigma$. Note that $g(u_1(x)) = g(Q[w](x))$ for all $x > -r-\beta$. Thus, $r \geq 2\sigma$ guarantees $r + \beta \geq \sigma$. It follows that

$$u_2(x) = Q[w_1](|x| - r) = w_2(|x| - r) \tag{2.21}$$

Applying g, we get

$$g(u_2(x)) = \begin{cases} 1 & |x| \leq r + 2c^* \\ 0 & |x| > r + 2c^* \end{cases} \tag{2.22}$$

Since $c^* > 0$ we have $r + 2c^* > r \geq \alpha$, so

$$u_3(x) = w_1(|x| - r - 2c^*) \tag{2.23}$$

The preceding argument can be repeated inductively to obtain

$$u_{2n+1}(x) = w_1(|x| - r - 2nc^*) \tag{2.24}$$

and

$$u_{2n+2}(x) = w_2(|x| - r - 2nc^*) \tag{2.25}$$

for all $n \geq 0$. □

Remark 2.6. This theorem indicates that a solution with proper compactly supported initial data coverges to translations of periodic traveling waves with profiles $w_1(x)$ and $w_2(x)$ in the positive direction and profiles $w_1(-x)$ and $w_2(-x)$ in the negative direction.

3 Examples

In this section, we construct the population density functions w_1 and w_2 for several choices of dispersal kernel.

Example 3.1. The Laplace kernel,

$$k(x) = \frac{1}{2}e^{-|x|} \tag{3.1}$$

The traveling waves are given by

$$w_1(x) = \begin{cases} 1 - \frac{1}{2}e^{x} & x \leq 0 \\ \frac{1}{2}e^{-x} & x > 0 \end{cases} \tag{3.2}$$

and

$$w_2(x) = \int_{-\infty}^{w_1^{-1}(b)} mk(x-y)\,dy + \int_{w_1^{-1}(b)}^{w_1^{-1}(a)} k(x-y)\,dy \tag{3.3}$$

with

$$w_1^{-1}(p) = \begin{cases} -\log(2p) & p \leq \frac{1}{2} \\ \log(2-2p) & p > \frac{1}{2} \end{cases}$$

Then

$$w_2(x) = \begin{cases} \int_{-\infty}^{-\log(2b)} \frac{m}{2}e^{-|x-y|}\,dy + \int_{-\log(2b)}^{-\log(2a)} \frac{1}{2}e^{-|x-y|}\,dy & a < \frac{1}{2}, b < \frac{1}{2} \\ \int_{-\infty}^{\log(2-2b)} \frac{m}{2}e^{-|x-y|}\,dy + \int_{\log(2-2b)}^{-\log(2a)} \frac{1}{2}e^{-|x-y|}\,dy & a < \frac{1}{2}, b \geq \frac{1}{2} \\ \int_{-\infty}^{\log(2-2b)} \frac{m}{2}e^{-|x-y|}\,dy + \int_{\log(2-2b)}^{\log(2-2a)} \frac{1}{2}e^{-|x-y|}\,dy & a \geq \frac{1}{2}, b \geq \frac{1}{2} \end{cases}$$

Now make the substitution $y' = x - y$.

$$w_2(x) = \begin{cases} \int_{x+\log(2b)}^{\infty} \frac{m}{2} e^{-|y'|}\, dy' + \int_{x+\log(2b)}^{x+\log(2a)} \frac{1}{2} e^{-|y'|}\, dy' & a < \frac{1}{2}, b < \frac{1}{2} \\ \int_{x-\log(2-2b)}^{\infty} \frac{m}{2} e^{-|y'|}\, dy' + \int_{x-\log(2-2b)}^{x+\log(2a)} \frac{1}{2} e^{-|y'|}\, dy' & a < \frac{1}{2}, b \geq \frac{1}{2} \\ \int_{x-\log(2-2b)}^{\infty} \frac{m}{2} e^{-|y'|}\, dy' + \int_{x-\log(2-2b)}^{x-\log(2-2a)} \frac{1}{2} e^{-|y'|}\, dy' & a \geq \frac{1}{2}, b \geq \frac{1}{2} \end{cases}$$

Conversely, we may define $\sigma_p = \text{sign}(p - \frac{1}{2})$. Then, we have $w_1^{-1}(p) = \sigma_p \log(\sigma_p(2p+1)+1)$. The above equation then simplifies to

$$w_2(x) = \int_{-\infty}^{\sigma_a \log(\sigma_a(2a+1)+1)} mk(x-y)\, dy + \int_{\sigma_b \log(\sigma_b(2p+1)+1)}^{\sigma_a \log(\sigma_a(2a+1)+1)} k(x-y)\, dy \tag{3.4}$$

We will now split into further cases.

Case 1: $a < b < \frac{1}{2}$. Then

$$w_2(x) = \int_{x-\log(2b)}^{\infty} \frac{m}{2} e^{-|y|}\, dy + \int_{x-\log(2b)}^{x-\log(2a)} \frac{1}{2} e^{-|y|}\, dy$$

We can expand this out using min and max functions. Let $x_a = x - \log(2a)$ and $x_b = x - \log(2b)$. Then

$$w_2(x) = \int_{\min(x_b,0)}^{0} \frac{m}{2} e^{y}\, dy + \int_{\max(x_b,0)}^{\infty} \frac{m}{2} e^{-y}\, dy + \int_{\min(x_b,0)}^{\min(x_a,0)} \frac{1}{2} e^{y}\, dy + \int_{\max(x_b,0)}^{\max(x_a,0)} \frac{1}{2} e^{-y}\, dy$$

We can now take the integral:

$$\frac{m}{2} - \frac{m}{2} e^{\min(x_b,0)} - \frac{m}{2} e^{-\max(x_b,0)} + \frac{1}{2} e^{\min(x_a,0)} - \frac{1}{2} e^{\min(x_b,0)} - \frac{1}{2} e^{-\max(x_a,0)} + \frac{1}{2} e^{-\max(x_b,0)}$$

This is equal to

$$\frac{m}{2} - \frac{m}{2} \min(e^{x_b},1) - \frac{m}{2} \min(e^{-x_b},1) + \frac{1}{2} \min(e^{x_a},1) - \frac{1}{2} \min(e^{x_b},1) - \frac{1}{2} \min(e^{-x_a},1) + \frac{1}{2} \min(e^{-x_b},1)$$

$$= \frac{m}{2} - \frac{m+1}{2} \min(e^{x_b},1) - \frac{m+1}{2} \min(e^{-x_b},1) - \frac{1}{2} \min(e^{x_a},1) - \frac{1}{2} \min(e^{-x_a},1)$$

$$= \frac{m}{2} - \frac{m+1}{2} \left(\min(e^{x_b},1) - \min(e^{-x_b},1)\right) - \frac{1}{2} \left(\min(e^{x_a},1) - \min(e^{-x_a},1)\right)$$

$$= \frac{m}{2} - \frac{m+1}{2} \left(\min(2be^{-x},1) - \min(\frac{1}{2b}e^{x},1)\right) - \frac{1}{2} \left(\min(2ae^{-x},1) - \min(\frac{1}{2a}e^{x},1)\right)$$

It can be written in three cases

$$w_2(x) = \begin{cases} m - \dfrac{am+a-b}{4ab}e^x & x < \log(2a) \\ m+1-\dfrac{m+1}{4b}e^x - ae^{-x} & \log(2a) \le x \le \log(2b) \\ (mb+b-a)e^{-x} & x > \log(2b) \end{cases}$$

Since w_2 is increasing for $x < \log(2a)$, we know the solution to $w_2(x) = a$ must lie on the second or third pieces. This can be determined by checking the sign of $w_2(\log(2b)) - a$. If it is positive, the horizontal line $y = a$ intersects the curve $y = w_2(x)$ on the third piece, and the solution can be found by solving $(mb+b-a)e^{-x} = a$. Otherwise, if the sign is negative, the solution occurs on the second piece as the solution to the equation

$$w_2(x) = a \iff m+1-\frac{m+1}{4b}e^x - ae^{-x} = a$$

First let us group like terms and multiply by $4b$:

$$(m+1)e^x + (m+1-a) - 4abe^{-x} = 0$$

Using the substitution $y = e^x$, this equation becomes a quadratic in y:

$$(m+1)y^2 + (m+1-a)y - 4ab = 0$$

The solution is

$$y = \frac{a-m-1 \pm \sqrt{(m+1-a)^2 + 16ab(m+1)}}{2m+2}$$

We know the global maximum of $w_2(x)$ lies somewhere in the open interval $(\log(2a), \log(2b))$, therefore we take the greater solution so that

$$c^* = \frac{1}{2}\log\left(\frac{a-m-1+\sqrt{(m+1-a)^2+16ab(m+1)}}{2m+2}\right)$$

Then we can obtain the critical wave speed in two cases:

$$c^* = \begin{cases} \dfrac{1}{2}\log\left(\dfrac{mb+b-a}{a}\right) & m \ge \frac{2ab+a-b}{b} \\ \dfrac{1}{2}\log\left(\dfrac{a-m-1+\sqrt{(m+1-a)^2+16ab(m+1)}}{2m+2}\right) & m < \frac{2ab+a-b}{b} \end{cases}$$

Case 2: $a < \frac{1}{2} < b$. Then

$$w_2(x) = \int_{x+\log(2-2b)}^{\infty} \frac{m}{2} e^{-|y|}\, dy + \int_{x+\log(2-2b)}^{x-\log(2a)} \frac{1}{2} e^{-|y|}\, dy$$

Then

$$w_2(x) = \frac{m}{2} - \frac{m+1}{2} \left(\min(\frac{e^{-x}}{2-2b}, 1) - \min((2-2b)e^x, 1) \right) - \frac{1}{2} \left(\min(2ae^{-x}, 1) - \min(\frac{1}{2a} e^x, 1) \right)$$

so

$$w_2(x) = \begin{cases} m - \dfrac{4a(m+1)(1-b) - 1}{4a} e^x & x < \log(2a) \\ m + 1 - (m+1)(a-b)e^x - ae^{-x} & \log(2a) \leq x \leq -\log(2-2b) \\ \dfrac{m+1-4a(1-b)}{4(1-b)} e^{-x} & x > -\log(2-2b) \end{cases}$$

Example 3.2. Consider the Gaussian kernel with mean 0 and variance 1 given by

$$k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

k is symmetric, strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$, hence conditions i.-iii. of Hypothesis 2.1 are satisfied. For condition iv., let $y \in \mathbb{R}$ and $\mu \in (0, 1)$. It can be shown that $k(x) - \mu k(x-y) = B(x)(1 - Ce^{-xy})$ where $B(x) = e^{-\frac{x^2}{2}}/\sqrt{2\pi}$ is a strictly positive function and $C > 0$. This quantity decreases monotonically from 1 to $-\infty$ with a unique zero-crossing at $x = \frac{y}{2} - \frac{\ln \mu}{y}$. Thus, Hypothesis 2.1 is satisfied.

The periodic traveling wave solutions $w_1(x)$ and $w_2(x)$ are given by

$$w_1(x) = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}} \right) \tag{3.5}$$

and

$$w_2(x) = \frac{m}{2} + \frac{1-m}{2} \operatorname{erf}\left(\frac{x-\beta}{\sqrt{2}} \right) - \frac{1}{2} \operatorname{erf}\left(\frac{x-\alpha}{\sqrt{2}} \right) \tag{3.6}$$

where $\alpha = \sqrt{2}\operatorname{erf}^{-1}(1-2a)$, and $\beta = \sqrt{2}\operatorname{erf}^{-1}(1-2b)$, and erf is the error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy$.

w_2 has a unique global maximum at $x^* = \frac{\alpha+\beta}{2} + \frac{1}{\alpha-\beta} \ln(1-m)$. Thus, by Theorem 2.3, w_1 and w_2 are a periodic traveling wave solution if $w_2(x^*) \leq b$.

Example 3.3. Consider the uniform dispersal kernel given by

$$k(x) = \begin{cases} \frac{1}{2} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \tag{3.7}$$

Then w_1 is given by

$$w_1(x) = \begin{cases} 1, & x \in (-\infty, -1), \\ \frac{1}{2} - \frac{1}{2}x, & x \in [-1, 1], \\ 0, & x \in (1, \infty), \end{cases} \tag{3.8}$$

with inverse $w_1^{-1}(p) = 1 - 2p$ for $0 < p < 1$. Let $\alpha = 1 - 2a$ and $\beta = 1 - 2b$. Then

$$w_2(x) = \begin{cases} m, & x \in (-\infty, \beta - 1), \\ \frac{1-m}{2}x + m + b - mb, & x \in [\beta - 1, \alpha - 1), \\ -\frac{m}{2}x + m + b - mb - a, & x \in [\alpha - 1, \beta + 1), \\ -\frac{1}{2}x - a + 1, & x \in [\beta + 1, \alpha + 1], \\ 0, & x \in (\alpha + 1, \infty). \end{cases} \tag{3.9}$$

Observe that w_2 has a global maximum at $x = \alpha - 1$ so that $||w_2||_\infty = w_2(\alpha - 1) = m + (b - a)(1 - m)$. By Theorem 2.4, the pair w_1 and w_2 are a solution to equation (2.9) if $m - a < m(b - a)$.

We can also explicitly calculate the speed of the wave given by

$$c^* = \begin{cases} 1 - 2a & \text{if } a \leq b/2, \\ 1 - b + \frac{b-2a}{m} & \text{if } a > b/2. \end{cases} \tag{3.10}$$

Remark 3.4. $w_1(x)$ is positive for $x < 1$ and zero for $x \geq 1$, and $w_2(x)$ is positive for $x < 2 - 2a$ and zero for $x \geq 2 - 2a$. Thus, (1.1) has a traveling wave with wave profiles $w_1(x)$ and $w_2(x)$, intermediate wave speeds $c_1 = 1 - 2a$ and $c_2 = 2c^* - c_1$, and average wave speed c^*. It is easily seen that $c_1 = c_2$ if $a \leq b/2$, and $|c_1 - c_2| = (2\alpha - \beta)(1 - \frac{1}{m}) > 0$ if $a > b/2$. So for $a > b/2$, the traveling wave is periodic with two different intermediate wave speeds. Furthermore, the difference between these two intermediate speeds is increasing in a, decreasing in b, and increasing in m. This behavior is illustrated with two difference choices of parameters in Figure **??**.

The regions in the parameter space where oscillating spreading speed exists can be determined as follows: for any fixed choice of (n_1, n_2), with $0 < n_1 < n_2$, let R be the set of pairs $(a, b) \in \mathbf{R}^2$ such that the hypothesis of Theorem 2.1 holds. Then R is a triangle in the a-b plane with endpoints at $(0, n_2)$, (n_1, n_1), and (n_1, n_2), depicted in Figure **??**. The line $b = 2a$ partitions R into two non-empty sets $R_1 = \{(a, b) \in R : a \leq b/2\}$ and $R_2 = \{(a, b) \in R : a > b/2\}$ such that the traveling has constant speed if $(a, b) \in R_1$ and oscillating speed if $(a, b) \in R_2$.

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